Fields of Definition of Elliptic Fibrations on Covers of Certain Extremal Rational Elliptic Surfaces

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1 Introduction

One main distinction of K3 surfaces, among others, is that they form the only class of surfaces that might admit more than one elliptic fibration with section, which is not of product type [17, Lemma 12.18]. It is therefore a natural problem to classify such fibrations. This has been done in the past three decades, via different methods by several authors, see for instance [15, 14, 7, 2, 3, 6] and [1]. Recently, the second and third authors have proposed a new method to classify elliptic fibrations on K3 surfaces which arise as double cover of rational elliptic surfaces. We refer the reader to [5] and [6] for more details.

Let X be a K3 surface obtained as a double cover of an extremal rational elliptic surface defined over a number field k. The purpose of this paper is to determine

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fields of definition of the distinct elliptic fibrations on X, i.e., fields over which the classes of the fiber and of at least one section are defined (see Definition 1). We also determine, in some examples, an upper bound for the degree of the field over which the Mordell–Weil group admits a set of generators. Extremal rational elliptic surfaces have been classified by Miranda and Persson in [10]. There are sixteen configurations of singular fibers on such surfaces. We restrict further our attention to smooth double covers of extremal rational elliptic surfaces with distinct reducible fibers, i.e. such that there are no two reducible fibers of the same Kodaira type. Given a genus 1 fibration on such a K3 surface, we show that it admits a section over a field that depends on the action of the cover involution on its fibers (see Theorem 1). We illustrate this last result for K3 surfaces that arise as a double cover branched over two smooth fibers of the extremal rational elliptic surfaces with one unique reducible fiber and also on smooth double covers of the surface with fiber configurations either (III^*, I_2) or (III^*, III). Remark that among those sixteen configurations of singular fibers on extremal rational elliptic surfaces only four of them have a unique reducible fiber, namely $(I_9, 3I_1)$, (II^*, II), $(II^*, 2I_1)$ and $(I_4^*, 2I_1)$. As only the configuration of reducible fibers plays a role in our arguments, we narrow these down to three classes and study those extremal rational elliptic surfaces, denoted by R_9, R_2, and R_4 and the corresponding K3 surfaces X_9, X_2, and X_4, respectively. We denote by R_3 an extremal rational elliptic surface with fibers either (III^*, I_2) or (III^*, III) and its generic K3 cover X_3. Notice that the surface X_4 also occurs as a double cover of R_3 and hence, X_3 and X_4 belong to the same family of K3 surfaces. A reason to explore elliptic fibrations on X_i, $i = 2, 3, 4, 9$ is that they have different behavior with respect to the cover involution of $X_i \to R_i$. Fibrations that are preserved by this involution are easier to describe via linear systems of curves on a rational surface, and one can also exhibit a Weierstrass equation for those as pointed out in [1] and [6]. In particular, on X_3 and X_4, which can be identified, we have two different involutions (induced by the covers $X_4 \to R_4$ and $X_3 \to R_3$) and the behavior of each fibration on $X_3 \simeq X_4$ with respect to these two involutions can be different.

This paper is organized as follows. In Sect. 2 we introduce the notations which remain in force during the paper and lay down our setting. Section 3 is devoted to the study of rational curves on the K3 surface X obtained as a double cover of a rational elliptic surface R. More precisely, motivated by the work done in [5] and [6], we study the behavior of the image by the quotient map $\pi : X \to R$ of rational curves on X and we determine the rational curves on X coming from a section defined over k of the elliptic fibration $\mathcal{E}_R$. While Sect. 3 is of geometric nature, Sect. 4 is dedicated to study the arithmetic of extremal rational elliptic surfaces defined over k. In particular, we obtain the quite intriguing fact that with a possible unique exception all extremal rational elliptic surfaces can be obtained, over the ground field, as a blow-up of base points of a pencil of genus one curves in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, Lemma 4. Section 5 is dedicated to the study of K3 surfaces coming from double covers of extremal rational elliptic surfaces. We prove in Theorem 1 that a genus 1 fibration on X admits a section over a field which depends on the action of the cover involution on the fibers of the genus 1 fibration. Finally, in Sects. 6 and 7 we

illustrate the previous result. More precisely, in Sect. 6 we give a classification of elliptic fibrations on the surface X_9 given by a generic double cover of an extremal rational elliptic surface R_9 with an I_9. We present a fiber class corresponding to each fibration on X_9 using sections and components of the reducible fibers of the fibration induced by the elliptic fibration on R_9. We also study the Mordell–Weil groups of each fibration and the fields of definition of the fibrations and their Mordell–Weil groups. Section 7 has similar results for the K3 covers of the rational elliptic surfaces R_2, R_3 and R_4, with reducible fibers (II^*), (III^*, I_2) and (I_4^*), respectively.

1.1 Relation to the Literature

Fields of definition of the Mordell–Weil group of non-isotrivial elliptic surfaces were studied independently by Swinnerton–Dyer in [19] and Kuwata in [8] via different methods than the ones presented here. While the first focused on elliptic surfaces fibered over $\mathbb{P}^1$, the latter dealt with basis of arbitrary genus. Nevertheless, both works are concerned with more general elliptic surfaces than the scope of this paper. In Kuwata's work he supposes that each component of the reducible fibers is defined over the ground field k. Let E be the generic fiber of an elliptic surface defined over k with base curve C. He proves that there is an explicitly computable number m and an explicitly computable extension L/k such that $mE(\bar{k}(C)) = mE(L(C))$. Our work differs from Kuwata's in several ways. Firstly, while he focusses on one unique elliptic fibration on a surface, we consider one elliptic fibration which we assume is defined over some number field k and use it as a point of start to study the other elliptic fibrations present on the surface. Thus in our work, one elliptic fibration is defined over the ground field, while the others not necessarily. For that reason we are concerned with different fields of definition, namely the one of the elliptic fibration and that of the Mordell–Weil group. Secondly, we focus on an specific class of surfaces, namely K3 surfaces. The further assumption that the K3 is a double cover of an extremal rational elliptic surface guarantees that the fields of definition will be much smaller than those for arbitrary elliptic surfaces. Indeed, fields of definition of the Mordell–Weil group of an elliptic surface can be quite large, for instance in [19] Swinnerton–Dyer constructed an elliptic surface for which the field of definition of the Mordell–Weil group has degree $2^7 \cdot 3^4 \cdot 5$, and the degrees of the fields of definition in Kuwata's work are also much larger than the bounds obtained here. Finally, it is worth to mention that Kuwata's work deals with fields of arbitrary characteristic while we focus on number fields. We expect that our work allows generalizations to that setting and the restriction has been made for the matter of simplicity but also because some of our work builds up on Miranda and Persson's work in [10], and on two of the author's paper [5]. Both settings are restricted to characteristic zero.

2 Preliminaries and Setting

Let R be a rational elliptic surface, i.e. a smooth projective rational surface endowed with a relatively minimal genus one fibration. We assume throughout this article that such a fibration admits a section. We denote by

$$\mathcal{E}_R : R \to \mathbb{P}^1 \tag{1}$$

the elliptic fibration on R. Let $d : C \to \mathbb{P}^1$ be a double cover of $\mathbb{P}^1$ branched on $2n$ points $p_i, i = 1, \ldots, 2n$. Then the fiber product $R \times_{\mathbb{P}^1} C$ is endowed with an elliptic fibration $R \times_{\mathbb{P}^1} C \to C$, induced by $\mathcal{E}_R$. We call the fibers $\mathcal{E}_R^{-1}(p_i), i = 1, \ldots, 2n$, the branch fibers. If all the branch fibers are smooth, then the fiber product $R \times_{\mathbb{P}^1} C$ is smooth, and we denote it by X. Otherwise, $R \times_{\mathbb{P}^1} C$ is singular and we denote by X its smooth model such that the elliptic fibration $\mathcal{E}_X : X \to C$, induced by $\mathcal{E}_R$, is relatively minimal.

$$
\begin{array}{ccc}
X & \dashrightarrow R \times_{\mathbb{P}^1} C \longrightarrow & R \\
& \searrow^{\mathcal{E}_X} \quad \Big\downarrow & \Big\downarrow \mathcal{E}_R \\
& C \xrightarrow{\;2:1\;} & \mathbb{P}^1
\end{array}
$$

Assume that R, the fibration $\mathcal{E}_R$ and the zero section O are all defined over a given number field k, which we fix once and for all. If the morphism d is defined over k then so is the fiber product, its possible desingularization X and the inherited elliptic fibration $\mathcal{E}_X$.

The surface $R \times_{\mathbb{P}^1} C$ is naturally endowed with an involution, namely the cover involution of the map $R \times_{\mathbb{P}^1} C \to R$ induced by the $2:1$ map $d : C \to \mathbb{P}^1$. It extends to an involution $\tau \in \text{Aut}(X)$ which is the cover involution of the generically $2:1$ cover $X \to R$. we denote by π the quotient map $\pi : X \to X/\tau \simeq_{\text{bir}} R$.

From now on we make the following assumptions.

- $d : C \to \mathbb{P}^1$ is defined over k,
- $n = 1$, i.e. $d : C \to \mathbb{P}^1$ is branched in two points. Hence $C \simeq \mathbb{P}^1$,
- the (two) branch fibers are reduced.

As a consequence of the previous assumptions we have that X is a K3 surface over k (see [17, Example 12.5]), the involution τ is non-symplectic, i.e. it does not preserve the symplectic form defined on X, since the quotient of a K3 by a symplectic involution is again a K3 surface (see [13]), and both $\mathcal{E}_X$ and its zero section are defined over k. Moreover, if the branch fibers are smooth, the reducible fibers of $\mathcal{E}_X$ occur in pairs that are exchanged by τ.

Notation 1 We denote by τ^* the involution induced by τ on $\text{NS}(X)$.

We recall that, due to their geometry, i.e. trivial canonical class and regularity, K3 surfaces might admit more than one elliptic fibration, all with basis $\mathbb{P}^1$, see for instance [17, Lemma 12.18]. Let X be as above, then it admits an elliptic fibration $\mathcal{E}_X$ and at least another elliptic fibration different from $\mathcal{E}_X$ [3, §8.1] and [6, Proposition 2.9]. One can divide the elliptic fibrations on X in three different classes, depending on the action of τ on its fibers. In particular, let η be an elliptic fibration on X then, by [5, Section 4.1], it is

- of type 1 with respect to τ, if τ preserves all the fibers of η;
- of type 2 with respect to τ, if τ does not preserve all the fibers of η, but maps a fiber of η to another one. In this case τ is induced by an involution of the basis of $\eta : X \to \mathbb{P}^1$. It fixes exactly two fibers and τ^* preserves the class of a fiber of η;
- of type 3, if τ maps fibers of η to fibers of another elliptic fibration. In this case τ^* does not preserve the class of the generic fiber of η.

The distinct elliptic fibrations on X are not necessarily defined over k. Moreover, different fibrations might be defined over different fields. The aim of this paper is to take a first step into understanding how the action of the involution τ on the fibers of a given fibration might influence its field of definition. Throughout this paper we adopt the following definition.

Definition 1 Given X as above and an elliptic fibration η on X, then the smallest field extension of k over which the class of a fiber of η is defined and η admits a section is called the field of definition of the fibration η. We denote it by k_η. We denote by $k_{\eta,\mathrm{MW}}$ the smallest field extension of k_η over which the Mordell–Weil group of η admits a set of generators.

Remark 1 The reader should be aware that in Definition 1 our starting data is a K3 surface X constructed as a base change of a rational elliptic surface R. Thanks to this construction X inherits an elliptic fibration from R which is defined over a number field k. All other fields of definition that appear in this paper are (possibly trivial) field extensions of k. In this sense, the field of definition is unique, but when considering X without this preliminary data then the field is no longer necessarily unique. Indeed, one could for instance obtain the same X as a double cover of another rational elliptic surface R' defined over a different field k'.

3 Rational Curves on K3 Surfaces

Let X be a K3 surface as in Sect. 2. In this section we study the behavior of the image by the quotient map π of the rational curves on X. As in the case of elliptic curves, this behavior depends on the action of the cover involution τ on the rational curve.

Lemma 1 *Let C be a smooth rational curve on X and $D = \pi(C)$ its image on R. Denote by m the intersection number $C \cdot \tau(C)$. Then D is of one of the following types.*

(i) *A fiber component of $\mathcal{E}_R$ on R.*
(ii) *A section of $\mathcal{E}_R$.*
(iii) *An m-section of $\mathcal{E}_R$, where $m > 0$.*

Moreover, if π is branched over two smooth fibers of $\mathcal{E}_R$ then (i) implies $m = 0$.

Proof Let C be a smooth rational curve on X and $D = \pi(C)$. By the adjunction formula we have that $C^2 = -2$. We consider the following cases $\tau(C) = C$ and $\tau(C) \neq C$.

1. $\tau(C) = C$. In this case, the involution can either act as the identity on C or as an involution of C. If the former holds then D is a (-2)-curve on R and therefore it is a component of a fiber of $\mathcal{E}_R$. If τ acts as an involution on C then since $\pi_*(C) = 2D$, we have that $2D^2 = C^2 = -2$. Hence $D^2 = -1$, and in particular D is a section of $\mathcal{E}_R$.
2. $\tau(C) = C' \neq C$. Then $m \geq 0$, then $\pi^*(D)^2 = 2D^2 = (C + C')^2 = -4 + 2m$. Hence $D^2 = m - 2$. By the adjunction formula we have that $D(-K_R) = m$. To conclude it is enough to recall that the class of a fiber of the elliptic fibration on R is given by $-K_R$. Thus, D is an m-section of $\mathcal{E}_R$ if $m > 0$, or a fiber component of $\mathcal{E}_R$ if $m = 0$.

Moreover, if π is branched over two different smooth fibers, $\tau(C) = C$ implies that τ is an involution of C, and thus D is a section of the elliptic fibration $\mathcal{E}_R$. Hence if D is a component of a fiber one must have $\tau(C) \neq C$, i.e., case 2. with $m = 0$. □

The next lemma deals with rational curves on X that come from sections defined over k of the elliptic fibration $\mathcal{E}_R$. As sections do not split on the double cover we show that their inverse image is as irreducible curve defined over k.

Lemma 2 *Let P_R be a section of $\mathcal{E}_R : R \to \mathbb{P}^1$ that is defined over k, then $P_X := \pi^{-1}(P_R)$ is an irreducible smooth rational curve of X and $\tau(P_X) = P_X$. In particular P_X is defined over k.*

Proof If P_R is a section of an elliptic fibration on a rational surface then it meets the branch locus of $R \times_{\mathbb{P}^1} \mathbb{P}^1 \to R$, which is given by two fibers, in two points. Thus its inverse image is a $2:1$ cover of a rational curve branched in two points, i.e. either an irreducible smooth rational curve or the union of two smooth rational curves meeting in two points. If the inverse image of P_R is the union of two curves, say P_1 and P_2, we have $\pi^*(P_R) = P_1 + P_2$. Since the inverse image of a fiber F_R, which is not a branch fiber, consists of two disjoint fibers, we have $\pi^*(F_R) = (F_1 + F_2)$. But then we would have $\pi^*(F_R)\pi^*(P_R) = 2 = (F_1 + F_2)(P_1 + P_2) = 2(F_1 P_1) + 2(F_1 P_2)$, where we used that F_1 and F_2 are linearly equivalent, since they are fibers of the same fibration on X. This would imply that either P_1 or P_2 is a component of a fiber, which is not possible, because they intersect in two points which lie in two different fibers, namely the ramification fibers. We conclude that $\pi^{-1}(P_R)$ is

a smooth rational curve. Even if one has to blow up some points to obtain X from $R \times_{\mathbb{P}^1} \mathbb{P}^1$, the strict transform of the inverse image of P_R, which we denote by P_X, remains irreducible and thus $\tau(P_X) = P_X$. Since the double cover map d is assumed to be defined over k and so are the points that one has to possibly blow up, we have that P_X is also defined over k.   □

4 Extremal Rational Elliptic Surfaces

In what follows we analyze the arithmetic of extremal rational elliptic surfaces defined over k. Let us recall that an extremal rational elliptic surface has Mordell–Weil rank equal to 0, and thus only finitely many sections, i.e. (-1)-curves.

Lemma 3 *Let R be an extremal rational elliptic surface defined over k. Assume that all reducible fibers of the elliptic fibration are distinct. Then the Néron–Severi group $\mathrm{NS}(R)$ admits generators defined over a field extension of k of degree at most 2.*

Proof There are two main ingredients in the proof of the statement. The first one is the Shioda–Tate formula which tells us that

$$\mathrm{NS}(R)/T \simeq \mathrm{MW}(\mathcal{E}_R), \tag{2}$$

where $T = \langle O, F \rangle \oplus \sum_{\substack{v \in \text{reducible fibers} \\ i \in S_v}} \Theta_{v,i}$, with $\Theta_{v,i}$ denoting the n_v components of the reducible fiber $\mathcal{E}_R^{-1}(v)$, $S_v = \{1, \cdots, n_v - 1\}$ and, since the surface is extremal, $\mathrm{MW}(\mathcal{E}_R)$ is a finite group. The second is the fact that the absolute Galois group $G_{\bar{k}}$ acts on $\mathrm{NS}(R)$ preserving the intersection pairing.

Recall that both the zero section O and the class of a smooth fiber F are defined over k. A reducible fiber with exactly two components has each component defined over k since the component that intersects the zero section is preserved $G_{\bar{k}}$. Thus in what follows we can focus on reducible fibers with at least three components. By the hypothesis on the reducible fibers being distinct, there are at most two such fibers, say F_{v_1} and F_{v_2}, see the table in [10, Thm. 4.1]. Assume w.l.o.g that F_{v_1} is the fiber with more reducible components. Each reducible fiber is globally defined over k because, by assumption, it is unique. Hence its trivial component is also defined over k. Since the latter intersects at most two other components, these are $G_{\bar{k}}$-conjugate and as a pair they form a $G_{\bar{k}}$-orbit. The same happens to all other components that are not defined over k. Let k_R/k be the quadratic extension over which the fiber components of F_{v_1} are defined. We show that each section is defined over k_R. The Mordell–Weil group is globally defined over k since its elements are precisely the (-1)-curves in the Néron–Severi group. Moreover because each section C intersects transversally a unique fiber component of F_{v_1}, the point of intersection is mapped by any element in $G_{\bar{k}}$ to another point of intersection of a component of F_{v_1} and a section. Since a component of a fiber is mapped by $G_{\bar{k}}$ either to itself or to a unique other fiber component defined over k_R, the intersection point is also defined

over k_R. Thus C is a rational curve with a k_R-point and hence it is also defined over k_R. It remains to show that the components of F_{v_2} are defined over k_R. This follows from the fact that after contracting the sections and certain fiber components of F_{v_1} we reach either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ cf. Lemma 4. The components of F_{v_2} are thus rational curves with k_R-points that correspond to the contracted curves, and hence are defined over k_R as well.   ☐

Example 1 The extremal rational elliptic surface with Weierstrass equation

$$y^2 = x^3 - 3(t^2 - 3)(t - 2)^2 x + t(2t^2 - 9)(t - 2)^3 \tag{3}$$

has reducible fibers of types I_1^* and I_4. Its Mordell–Weil group is $\mathbb{Z}/4\mathbb{Z}$ with two sections defined over $\mathbb{Q}$, namely $[0, 1, 0]$ and $[t^2 - 2t, 0, 1]$, and two conjugate sections, namely $[(t-3)(t-2), \pm 3\sqrt{3}(t-2)^2, 1]$, which are defined over a quadratic extension. The reader can find this example as X_{141} in [10, Table 5.2].

The next example shows that the hypothesis on the distinct reducible fibers is indispensable in Lemma 3.

Example 2 The extremal rational elliptic surface with Weierstrass equation

$$y^2 = x^3 + (3t^4 + 24t)x + 2t^6 + 40t^3 - 16 \tag{4}$$

has four reducible fibers of type I_3. Its Mordell–Weil group is defined over a biquadratic extension $\mathbb{Q}(i, \sqrt{3})$. This corresponds to the surface X_{3333} in [10, Table 5.3]. See also Remark 2(iii).

Notation 2 In what follows, we keep the notation introduced in Lemma 3 and denote by k_R the extension of k over which the Néron–Severi group $\mathrm{NS}(R)$ admits a set of generators given by fiber components and sections of the elliptic fibration on R, and by G_R the Galois group $\mathrm{Gal}(k_R/k)$. We keep the subscript R for the Galois group to reinforce the dependence on the surface. By Lemma 3, if the Kodaira types of the reducible fibers of $\mathcal{E}_R$ are different then k_R/k has degree at most 2.

Remark 2

(i) Certain configurations of reducible fibers force the Galois group G_R to be trivial. Thus such surfaces always admit a set of generators for their Néron–Severi group over the ground field k. This holds for instance for any rational elliptic surface over k which has reducible fiber configurations (II^*), (III^*, I_2), (III^*, III) or (I_4^*); see the proof of Lemma 4 or [16, Cor. 4.4].

(ii) Five out of sixteen configurations of reducible fibers on extremal rational elliptic surfaces, namely $(2I_0^*)$, $(2I_5)$, $(2I_4, 2I_2)$, $(I_2^*, 2I_2)$ and $(4I_3)$ do not satisfy the hypothesis of Lemma 3, see [10, Theorem 4.1].

(iii) Extremal rational elliptic surfaces with repeated reducible fibers have their Néron–Severi group defined, in general, over extensions of larger degree. For instance, a rational elliptic surface with reducible fiber configuration $(2I_5)$ has, in general, its Néron–Severi group defined over an extension of degree four,

with cyclic Galois group (see the proof of Lemma 4), while a surface R_0 with $(2I_0^*)$ has, in general, $\mathrm{NS}(R_0)$ defined over an extension of the ground field with Galois group given by the dihedral group of order 12. Indeed, the Galois group is generated by an involution which preserves each section and switches the two I_0^*-fibers and by $\mathfrak{S}_3$ which preserves the fibers, and permutes the non-trivial elements of $\mathrm{MW}(R_0) = (\mathbb{Z}/2\mathbb{Z})^2$.

4.1 Minimal Models for Extremal RES Over k

We recall that every rational elliptic surface defined over and algebraically closed field of characteristic zero can be obtained as the blow-up of the base points of a pencil of generically smooth cubics [4, §5.6.1] or [9, Lemma IV.1.2.]. This fact clearly does not hold, in general, over a number field k. For instance, the blow-up of the base point of the anti-canonical linear system of a k-minimal del Pezzo surface of degree one is a rational elliptic surface defined over k which does not admit a blow down to $\mathbb{P}^2$ as it is clearly not even k-rational. On the other hand, if one restricts our attention to extremal rational elliptic surfaces then one can show that they are always k-rational, with possible exception given by those with reducible fiber configuration $(2I_0^*)$.[1] Still this is not enough to assure that they can be obtained as a blow-up of the projective plane. Indeed, we provide an example in Proposition 2 for which this does not hold. Nonetheless, we obtain a quite intriguing fact, namely that with a possible exception of surfaces with configuration $(2I_0^*)$, all extremal rational elliptic surfaces can be obtained, over the ground field, as a blow-up of base points of a pencil of genus one curves in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, in Lemma 4. Despite its simple proof, this intriguing fact is not in the literature and likely not known to many experts.

Since an extremal rational elliptic surface has finite Mordell–Weil group, it has only finitely many curves of negative self-intersection [9, Proposition VIII.1.2]. The Galois group G_R acts on $\mathrm{NS}(R)$ preserving the intersection pairing. Since, by hypothesis, the zero section of the fibration $\mathcal{E}_R$ is defined over k it is always preserved by G_R.

From now on we will use the following notation for the irreducible components of a reducible fiber: the component which intersects the zero section will be denoted by C_0; in a fiber of type I_n, the components C_i, $i \in \mathbb{Z}/n\mathbb{Z}$ are numbered requiring that $C_i C_j = 1$ if and only if $|i - j| = 1$.

Lemma 4 *Let R be an extremal rational elliptic surface defined over k with at most one non-reduced fiber. Then R is k-isomorphic to the blow-up of the base points of a pencil of cubic curves in $\mathbb{P}^2$ or a pencil of curves of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, such surfaces are always k-rational.*

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[1]These can be k-birational to a k-minimal Châtelet surface depending on whether the elliptic fibration has a 2-torsion section over k or not.

Proof We recall that the Galois group G_R preserves the zero section, maps a fiber of a certain Kodaira type to a fiber of the same Kodaira type and maps sections to sections. The consequences are the following:

(i) If $\mathrm{MW}(\mathcal{E}_R) = \{0\}$ or $\mathrm{MW}(\mathcal{E}_R) = \mathbb{Z}/2\mathbb{Z}$, then G_R maps each section to itself;
(ii) If every reducible fiber is of different Kodaira type, then G_R maps the zero component of each fiber to itself;
(iii) If both (i) and (ii) are satisfied, then G_R is trivial since in that case the fiber with most components is a non-reduced fiber of type II^*, III^* or I_4^* (see the table in [10, Thm. 4.1]) and each component is preserved by the Galois group because the zero section and the two torsion sections are preserved and defined over k;
(iv) If there is a fiber which is preserved by G_R as, for example, in case (ii) and it is either of type I_n or of type IV^*, then G_R restricted to that fiber and to $\mathrm{MW}(\mathcal{E}_R)$ acts trivially or as the hyperelliptic involution because it has to preserve the intersection properties of the components of the reducible fiber of type I_n.

Using these properties of G_R, one is able to find an explicit contraction γ defined over k, which maps a rational elliptic surface R either to $\mathbb{P}^2$ or to $\mathbb{P}^1 \times \mathbb{P}^1$ for all the extremal rational elliptic surfaces R with reducible fiber configuration different from $(2I_0^*)$.

Fibrations (II^*, II)**,** $(II^*, 2I_1)$**,** (III^*, III)**,** (III^*, I_2, I_1)**,** $(I_4^*, 2I_1)$: G_R is trivial because (iii) in the previous list is satisfied. One first contracts all the sections, then contracts the image of the components of the fibers II^*, III^*, I_4^*, respectively, that are the (-1)-curves after the previous contractions. One iterates this process in order to contract 9 curves. The composition of all these contractions is a map $R \to \mathbb{P}^2$, defined over k.

Fibrations (IV^*, IV)**,** (IV^*, I_3, I_1): by (iv), G_R acts trivially or coincides with the hyperelliptic involution. After contracting all the sections, one obtains three (-1)-curves in the image of the IV^*-fiber. One is preserved by G_R, the other two might be exchanged by it. After contracting these three curves, one is in a similar situation, i.e. there are three (-1)-curves, forming two or three orbits for G_R. After contracting also these three curves, one obtains a k-rational map from R to $\mathbb{P}^2$.

Fibrations $(I_9, 3I_1)$**,** $(I_8, I_2, 2I_1)$**,** (I_6, I_3, I_2): by (iv), G_R acts trivially or coincides with the hyperelliptic involution. First one contracts all the sections. Then one contracts some curves in the image of the fibers of type I_9, I_8 and I_6 respectively, but not in the other reducible fibers. For the fiber I_9 one contracts the images of the components C_0, preserved by G_R, and of C_3 and C_6, which are either fixed or switched by G_R; after that one contracts the images of the curves C_2 and C_7, which are also either fixed or switched by G_R. For the fiber I_8 one contracts the images of components C_0 and C_4, which are preserved by G_R, and of the components C_2 and C_6, which are either fixed or conjugate under G_R. For the fiber of type I_6 one contracts the images of components C_0 and C_3, which are preserved by G_R. In all the cases one obtains a k-rational map from R to $\mathbb{P}^1 \times \mathbb{P}^1$.

Fibrations of type $(4I_3)$ **and** $(2I_4, 2I_2)$: in both these cases there are many sections, namely 9 sections in case $(4I_3)$ and 8 in case $(2I_4, 2I_2)$. Since the torsion

sections are disjoint and G_R preserves $\text{MW}(\mathcal{E}_R)$, one can contract simultaneously all the sections. This produces a k-rational map to $\mathbb{P}^2$ in case $(4I_3)$ and to $\mathbb{P}^1 \times \mathbb{P}^1$ in case $(2I_4, 2I_2)$.

Fibration of type $(2I_5, 2I_1)$: we have $G_R \subseteq \mathbb{Z}/4\mathbb{Z}$, and if $G_R = \mathbb{Z}/4\mathbb{Z}$ then the action of the generator of G_R is the following. $t_0 \to t_0$, $C_0^{(1)} \leftrightarrow C_0^{(2)}$, $t_1 \to t_3 \to t_4 \to t_2 \to t_1$, $C_1^{(1)} \to C_1^{(2)} \to C_4^{(1)} \to C_4^{(2)}$, $C_2^{(1)} \to C_2^{(2)} \to C_3^{(1)} \to C_3^{(2)}$, where $C_i^{(j)}$ the i-th component of the j-th fiber of type I_5. To obtain a k-rational map to $\mathbb{P}^2$, one first contracts all the sections, and then one contracts the components $C_1^{(1)}$, $C_1^{(2)}$, $C_4^{(1)}$, $C_4^{(2)}$, which form an orbit if $G_R = \mathbb{Z}/4\mathbb{Z}$.

Fibration of type $(I_2^*, 2I_2)$ **and** (I_1^*, I_4, I_1): one contracts first the four sections and then the images of the four simple components of the fiber of type I_i^*. This gives a k-rational map to $\mathbb{P}^1 \times \mathbb{P}^1$.   □

Proposition 1 *Let R be a semi-stable extremal rational elliptic surface defined over k and m the order of the Mordell–Weil group. Then the following holds.*

 (i) *If m is odd and R has a unique reducible fiber then R admits a contraction over k to $\mathbb{P}^1 \times \mathbb{P}^1$.*
 (ii) *If m is odd and R has at least two reducible fibers then R admits a contraction over k to $\mathbb{P}^2$.*
(iii) *If m is even then R admits a contraction over k to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof The result follows by the proof of the previous lemma. Indeed, if R is a semi-stable extremal elliptic fibration and m is odd, then the fibration on R is one of the following: $(I_9, 3I_1)$, $(2I_5, 2I_1)$, $(4I_3)$. The first fibration corresponds to case (i) and can be contracted to $\mathbb{P}^1 \times \mathbb{P}^1$, for every action of G_R. The other two fibrations correspond to the case (ii) and it was already proved that they can be contracted to $\mathbb{P}^2$.

If m is even (case (iii)), then the fibration on R is one of the following: $(I_8, I_2, 2I_1)$, (I_6, I_3, I_2), $(2I_4, 2I_2)$ and in the proof of the previous lemma is shown that all of them can be contracted to $\mathbb{P}^2$.   □

Remark 3 The converse of the different cases in Proposition 1 is not always true; some of the surfaces treated in Lemma 4 can be contracted, over an algebraically closed field, to both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Whether or not these surfaces can be contracted to both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ over k as well depends on the action of G_R, and in particular on the action of the hyperelliptic involution on the reducible fibers. See Proposition 2 and Fig. 1, where we show this for a surface with fibers $(I_9, 3I_1)$.

5   Double Covers of Extremal Rational Elliptic Surfaces

In the rest of this article we consider K3 surfaces that are double covers of extremal rational elliptic surfaces defined over k and branched on two smooth $G_{\bar{k}}$-conjugate fibers. Let X be such a surface. Recall that since the extremal rational elliptic

surfaces considered here[2] are rigid, their K3 double covers have a 2-dimensional moduli space, as each branch point is allowed to vary in $\mathbb{P}^1$.

In this section we show that the field over which a genus one fibration on X admits a section depends on the action of the cover involution on the fibers of the genus one fibration.

Notation 3 Let R and X be as above and $t_1, \cdots, t_m \in \mathbb{P}^1_k$ points over which the reducible fibers of R are located. Since the base change map $X \to R$ is branched only over smooth fibers, there are two distinct points above each t_i. Then τ restricted to the pair of fibers of $\mathcal{E}_X$ above each t_i is a field homomorphism, which we denote by σ_i. We denote by k_τ the Galois field extension of k whose Galois group is generated by $\sigma_1, \cdots, \sigma_m$. By construction k_τ/k is an extension of even degree dividing 2^m. We denote by $k_{R,\tau}$ the compositum of the fields k_R and k_τ.

Lemma 5 *Let R be an extremal rational elliptic surface as above and X a generic member of the 2-dimensional family given by double covers of R branched in two smooth fibers. Then* $\mathrm{NS}(X)$ *admits a set of generators over $k_{R,\tau}$.*

Proof Since the Néron–Severi group has rank 10 and the Mordell–Weil group has rank zero, it follows from the Shioda–Tate formula that the reducible fibers of an extremal rational elliptic surface R have in total 8 components contributing to the set of generators of $\mathrm{NS}(R)$. Since X is a double cover of an extremal rational elliptic surface R branched on smooth fibers, the reducible fibers of the inherited fibration $\mathcal{E}_X$ contribute with 16 components to a set of generators of $\mathrm{NS}(X)$. If X is generic among such surfaces then it lies in a 2-dimensional family and hence $\mathrm{NS}(X)$ has rank 18 and is generated by fiber components, the zero section and a smooth fiber of $\mathcal{E}_X$. All such curves are defined at most over $k_{R,\tau}$. □

Theorem 1 *Let R be an extremal rational elliptic surface defined over k such that its reducible fibers are all of distinct Kodaira types. Let X be a K3 surface obtained as a double cover of R branched on two smooth fibers conjugate under $G_{\bar{k}}$, τ the cover involution and η a genus 1 fibration on X. Then the following hold.*

 (i) If η is of type 1 *w.r.t. τ then η is defined over k_R and admits a section over $k_{R,\tau}$.*
(ii) If η is of type 2 *w.r.t. τ then it is defined and admits a section over k.*

Proof For (ii) notice that because the branch locus is smooth there is only one fibration of type 2, namely the one induced by the elliptic fibration on R. Indeed, different fibrations of type 2 correspond to different contractions of (-1)-curves in X/τ that are components of non-relatively minimal elliptic fibrations. Since the branch locus is smooth there are no (-1)-curves to be contracted and, in particular, $X/\tau \simeq R$. Since the double cover morphism is defined over k so is the induced elliptic fibration on X and the zero section inherited from R. If η is of type 1 then each fiber is the pull-back of a conic[3] in R [5, Theorem 4.2]. Let C be such a conic.

Since $\mathrm{NS}(R)$ is generated by curves defined over k_R then the class of C has a divisor C_0 whose components are defined over k_R. Moreover, as the fibers of η are fixed by τ, the pull-back C_0 is also defined over k_R. Its class moves in X giving the elliptic fibration η.   □

The fibrations of type 3 are certainly more difficult to study by using the geometry related with R. Indeed, even if X is a double cover of R, the fibrations of type 3 are not easily related with the geometry of R, by definition, since they are not preserved by the cover involution. But, one is still able to prove that certain fibrations of type 3 are defined on certain fields, if one is able to find components of their reducible fiber is a proper way, as observed in the next Remark.

Remark 4 Since the irreducible components of reducible fibers and of the sections of the elliptic fibration on K3 surface are rational curves, they are rigid in their class. So if their class is defined over a certain field, say $k_{R,\tau}$, and they are irreducible curves, then they are defined over $k_{R,\tau}$. Suppose now that the Néron–Severi group is defined over $k_{R,\tau}$ and it is generated by a certain set of classes of irreducible rational curves. If the union of some of these curves is a reducible fiber F of a fibration η, then the reducible fiber F and its class are defined over $k_{R,\tau}$. In particular the fibration η is defined on $k_{R,\tau}$ and if also a section of η can be found among the generators of the Néron–Severi, then η is an elliptic fibration on $k_{R,\tau}$.

So, in order to prove that a fibration of type 3 defined on a K3 surface satisfying the assumptions of Theorem 1, is defined over $k_{R,\tau}$, it suffices to find among the generators of $\mathrm{NS}(X)$ a configuration of (-2)-curves which corresponds to a reducible fiber of η.

Remark 5 We believe that it is always possible to find a fibration of type 3 as in the previous remark, at least for the K3 surfaces X as in Theorem 1. We are able to prove this for all the elliptic fibrations of type 3 on the surfaces considered in Sects. 6.3 and 7.3 of this paper. Hence for all the surfaces considered in this paper, we have that the fields of definition of the elliptic fibrations on the K3 surfaces X as in Theorem 1 are at most biquadratic extensions of k, by the explicit description of the elliptic fibration and the Remark 4.

Remark 6 Certain sections on elliptic K3 surfaces as above might be defined over a smaller subfield of $k_{R,\tau}$ that contains k. See, for instance, the fifth column of lines 2, 3, 4, 9, 11, and 12 in Table 2.

Following the geometric classification of extremal rational elliptic surfaces by Miranda and Persson [10, Theorem 4.1], we notice that, among those surfaces, only four of them have only one reducible fiber, namely $(I_9, 3I_1)$, (II^*, II), $(II^*, 2I_1)$, and $(I_4^*, 2I_1)$. From a lattice theoretic point of view the surfaces with singular fibers (II^*, II) and $(II^*, 2I_1)$ are the same since, from that perspective, only the reducible fibers matter. Moreover, they share the same properties of interest to us, namely reducible fibers and fields of definition of components of fibers and thus we denote both of them by R_2. In the following sections, we study those extremal rational elliptic surfaces, denoted by R_9, R_2, and R_4 and their corresponding K3

surfaces X_9, X_2, and X_4, respectively. We also study the surface R_3 which has two reducible fibers (III^*, III) and its generic K3 cover X_3. The justification for considering R_3 as well is the fact that the surface X_4 occurs also as double cover of R_3 and hence X_3 and X_4 belong to the same family of K3 surfaces.

5.1 Arithmetic Models of Extremal Rational Elliptic Surfaces

Over algebraically close fields, all rational elliptic surfaces can be obtained by the blow up of the base points of a pencil of genus 1 curves in the projective plane. Over a number field k, this no longer holds true. Nevertheless, if one restricts attention to extremal rational elliptic surfaces, we have shown in Lemma 4 that, with one possible exception, they can be obtained as a blow up of a pencil of genus 1 curves in the plane or in the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$. The realization of the blow down of an extremal rational elliptic surface R to either rational minimal model is connected to, but not always determined by, the Galois group G_R introduced in Notation 2. More precisely, given singular fiber configurations on an extremal rational elliptic surface might entail more than one possible action of the Galois group $G_{\bar{k}}$ on its fiber components and hence, with a few exceptions, it does not make sense anymore to speak about **the** extremal rational elliptic surface with a given configuration as one does over algebraically closed fields. In what follows we keep the notation R_i and X_i for **a** surface with fiber configuration described in the previous paragraph. We study what are the possible actions of $G_{\bar{k}}$ on each configuration. We show, in Propositions 2 and 5 respectively, that R_9 might admit two possible actions, while R_2, R_3, R_4 always admit a unique action.

6 The Surfaces R_9 and X_9

Let R_9 be an extremal rational elliptic surface with one reducible fiber of type I_9 and X_9 a K3 surface obtained by a double cover of R_9 branched in two smooth $G_{\bar{k}}$-conjugate fibers. In this section, we classify all the possible fibrations of the K3 surface X_9 and determine their types with respect to the cover involution τ_9, a field over which the class of a fiber is defined and a field over which the Mordell–Weil group is defined.

6.1 Negative Curves on R_9

Recall that the configuration I_9 is given by 9 smooth rational curves meeting in a cycle with dual graph $\tilde{A}_8$ (see [9, Table I.4.1][4]). The singular fibers of R_9 are $I_9 + 3I_1$ and the Mordell–Weil group is $\mathbb{Z}/3\mathbb{Z} = \{O, t_1, t_2\}$, where O is the zero section and t_1 and t_2 are 3-torsion sections. The Néron–Severi group of R_9 contains also the classes of the irreducible components of the unique reducible fiber, denoted by $C_0, C_1, \ldots, C_8$. The intersections which are not trivial are the following

$$C_i^2 = -2; \qquad\qquad C_i C_j = 1 \text{ iff } |i - j| = 1; \qquad\qquad (5)$$

$$OC_0 = t_1 C_3 = t_2 C_6 = 1; \quad O^2 = t_1^2 = t_2^2 = -1. \qquad\qquad (6)$$

The following result tells us that R_9 can always be obtained as the blow-up of the eight base points on a pencil of curves of bi-degree (2,2) in $\mathbb{P}^1 \times \mathbb{P}^1$, and that if the Galois group G_{R_9} fixes each 3-torsion section then R_9 can also be obtained as the blow-up of the nine base points of a pencil of cubics in $\mathbb{P}^2$ (see also Lemma 4). Both blow-ups occur in multiple points, i.e., points with assigned multiplicities.

Proposition 2 *If for every $g \in G_{R_9} = Gal(k_{R_9}/k)$ we have $g(t_1) = t_1$, then $G_{R_9} = \{id\}$ and R_9 can be contracted both to $\mathbb{P}^2$ and to $\mathbb{P}^1 \times \mathbb{P}^1$. If there exists at least one $g \in G_{R_9}$ such that $g(t_1) \neq t_1$, then $g(t_1) = t_2$, $G_{R_9} = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ and g is the elliptic involution ι_{R_9} restricted to the fiber I_9. In this case R_9 can be contracted to $\mathbb{P}^1 \times \mathbb{P}^1$ but not to $\mathbb{P}^2$.*

Proof Let F be the class of a fiber of $\mathcal{E}_{R_9}$. Since F is preserved by G_{R_9}, for each $g \in G_{R_9}$ we have $1 = t_1 F = g(t_1)g(F)$ and thus $g(t_1)$ is necessarily a section. It is different from O as the latter is fixed by G_{R_9}. Hence either $g(t_1) = t_1$ or $g(t_1) = t_2$. We begin with $g(t_1) = t_1$. In that case $g(t_2) = t_2$ and since t_1 intersects the fiber component C_3 and t_2 intersects C_6, we have $g(C_3) = C_3$ and $g(C_6) = C_6$. Since each other fiber component intersects one among C_0, C_3 and C_6, it is also fixed by g. Hence G_{R_9} is trivial. We pass to the case $g(t_1) = t_2$. This implies that $g(C_3) = C_6$. The fiber components intersecting C_3 and C_6 must be switched by g and, a posteriori, so must C_1 and C_8. We have $g(C_i) = C_{9-i}$. Hence, in that case, G_{R_9} has order 2 and is generated by the elliptic involution.

Let us now consider the contraction of the (-1)-curves on R_9, i.e., the sections O, t_1 and t_2. The reader might find it helpful to follow Fig. 1 in parallel. First one contracts the three sections, which are all disjoint and form either 3 or 2 orbits for the action of G_{R_9}, depending on whether G_{R_9} is $\{id\}$ or $\mathbb{Z}/2\mathbb{Z}$. Let us denote by $\beta_1 : R_9 \to R'$ this contraction. The curves $\beta_1(C_0), \beta_1(C_3), \beta_1(C_6)$ are disjoint (-1)-curves of R' and form 2 or 3 orbits with respect to G_{R_9}. Secondly, we call $\beta_2 : R' \to R''$ the contraction of these three curves. The curves $\beta_2(\beta_1(C_i))$ for $i = 1, 2, 4, 5, 7, 8$ are (-1)-curves on R''. The curves $\beta_2(\beta_1(C_2))$ and $\beta_2(\beta_1(C_7))$

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[4]Though this table contains a typo, namely a fiber of I_n has dual graph $\tilde{A}_{n-1}$.

form 1 or 2 orbits with respect to G_{R_9}. Hence they can be contracted in order to obtain a minimal surface. Let us denote by $\beta_3 : R'' \to R'''$ this contraction. Then R''' is $\mathbb{P}^1 \times \mathbb{P}^1$, the curves $\beta_3(\beta_2(\beta_1(C_1)))$ and $\beta_3(\beta_2(\beta_1(C_5)))$ are curves of bidegree $(1, 0)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and the curves $\beta_3(\beta_2(\beta_1(C_4)))$ and $\beta_3(\beta_2(\beta_1(C_8)))$ are curves of bidegree $(0, 1)$. Hence the image of the reducible fiber I_9 is a reducible curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. There is another possible choice of curves to contract on R'' in order to obtain a minimal surface. If $G_{R_9} = \{id\}$, one can contract the curves $\beta_2(\beta_1(C_1))$, $\beta_2(\beta_1(C_4))$, $\beta_2(\beta_1(C_7))$ obtaining $\mathbb{P}^2$ as minimal surface. But these curves do not form an orbit for G_{R_9} if $G_{R_9} = \mathbb{Z}/2\mathbb{Z}$, hence this contraction is allowed only if G_{R_9} is trivial.   □

Figure 1 shows the contractions β_1, β_2, and β_3 of the fiber I_9 as in the proof of Proposition 2. Black lines represent curves defined over k. Lines of the same color (not black) represent curves that are conjugate under the action of G_{R_9} if $G_{R_9} \neq \{id\}$; of course if $G_{R_9} = \{id\}$ then all curves are defined over k. Dotted lines represent (-1)-curves, lines with label 0 represent curves with self-intersection 0, and all other lines represent (-2)-curves.

Remark 7 If one contracts R_9 to $\mathbb{P}^1 \times \mathbb{P}^1$, the elliptic involution defined on R_9 induces an involution of $\mathbb{P}^1 \times \mathbb{P}^1$, which is precisely the exchange of the two rulings. Indeed ι_{R_9} maps C_1 to C_8 and C_4 to C_5, so the automorphism induced by ι_{R_9} on $\mathbb{P}^1 \times \mathbb{P}^1$ maps the $(1, 0)$-curves $\beta_3(\beta_2(\beta_1(C_1)))$ and $\beta_3(\beta_2(\beta_1(C_5)))$ to the $(0, 1)$-curves $\beta_3(\beta_2(\beta_1(C_8)))$ and $\beta_3(\beta_2(\beta_1(C_4)))$.

Remark 8 Over the complex field all the rational elliptic fibrations can be contracted to a pencil of cubics in $\mathbb{P}^2$ and for each extremal rational elliptic fibration, the equation of an associated pencil is known [4, Proof of Theorem 5.6.2]. In particular, an equation of a pencil of cubics associated to the surface R_9 in $\mathbb{P}^2_{(z_0:z_1:z_2)}$ is given by

$$\mathcal{P}_9 := (z_0 z_1 z_2) + t(z_0^2 z_1 + z_1^2 z_2 + z_2^2 z_0). \tag{7}$$

The base points of $\mathcal{P}_9$ are $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$, each with multiplicity 3. After blowing up these points one obtains a rational elliptic surface, with a reducible fiber over $t = 0$. A birational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the blow-up of two points and the contraction of the line through these points. For example the maps

$$\alpha_1 : \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \to \mathbb{P}^2, \ \alpha_1((x_0 : x_1), (y_0 : y_1)) = (x_0 y_0 : x_0 y_1 : x_1 y_0) \tag{8}$$

$$\alpha_2 : \mathbb{P}^2_{(z_0:z_1:z_2)} \to \mathbb{P}^1 \times \mathbb{P}^1, \ \alpha_2(z_0 : z_1 : z_2) = ((z_0 : z_2), (z_0 : z_1)) \tag{9}$$

are birational inverse maps. They correspond to blowing up the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ in $\mathbb{P}^2$ and to contracting the line $z_0 = 0$.

We observe that the points $(0 : 1 : 0)$ and $(0 : 0 : 1)$ are base points of the pencil $\mathcal{P}_9$. The birational image of this pencil is a bidegree $(2, 2)$ pencil in $\mathbb{P}^1 \times \mathbb{P}^1$, given by
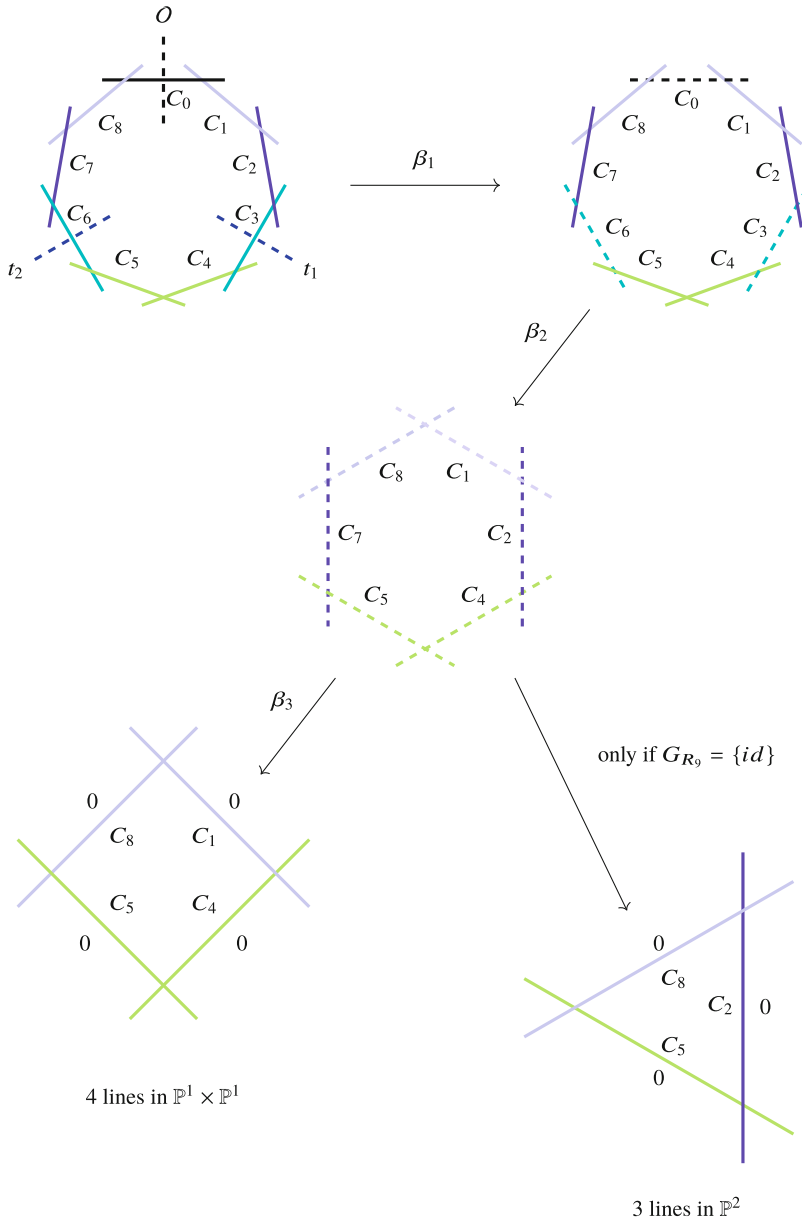
Fig. 1 Two ways to contract the fiber I_9

$$(x_0 x_1 y_0 y_1) + t(x_0^2 y_0 y_1 + x_0 x_1 y_1^2 + x_1^2 y_0^2), \tag{10}$$

which still corresponds to R_9. We conclude that the pencil (10), considered as pencil of curves of bidegree $(2, 2)$ over a certain field k, defines the rational elliptic surface R_9 over k. We already observed that the Galois action on R_9 corresponds to an involution exchanging the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and indeed, with the chosen coordinates, it is $((x_0 : x_1), (y_0 : y_1)) \mapsto ((y_1 : y_0), (x_1 : x_0))$.

The following example illustrates the two different Galois actions that occur in Proposition 2.

Example 3 In Remark 8, we saw that the pencil of cubics given by $\mathcal{P}_9 = (z_0 z_1 z_2) + t(z_0^2 z_1 + z_1^2 z_2 + z_2^2 z_0)$ gives rise to an R_9 surface. A Weierstrass equation for this surface is

$$y^2 = x^3 - (432t^3 + 10368)xt + 3456t^6 + 124416t^3 + 746496, \tag{11}$$

and the Mordell–Weil group consists of three sections defined over $\mathbb{Q}$, which are given by $[0, 1, 0]$ and $[12t^2, \pm864, 1]$. We conclude from 2 that in this case we have $G_{R_9} = \{id\}$.

Another example of an R_9 surface is given by the Weierstrass equation

$$y^2 = x^3 - 3(t^3 + 24)xt + 2(t^6 + 36t^3 + 216), \tag{12}$$

which has Mordell–Weil group given by the section $[0, 1, 0]$ and the two sections $[t^2 - 1, \pm3\sqrt{3}t, 1]$ [10, Table 5.3]. So the Mordell–Weil group of this surface is trivial over $\mathbb{Q}$, and defined over the quadratic extension $\mathbb{Q}(\sqrt{3})$. We conclude from Proposition 2 that in this case we have $G_{R_9} = \mathbb{Z}/2\mathbb{Z}$, and the surface can not be contracted to $\mathbb{P}^2$.

6.2 The K3 Surface X_9

Let X_9 be a K3 surface obtained by a generic base change of order 2 on the rational elliptic surface R_9 as described in Sect. 2.

Then the elliptic fibration $\mathcal{E}_{R_9} : R_9 \to \mathbb{P}^1$ induces an elliptic fibration $\mathcal{E}_{X_9} : X_9 \to \mathbb{P}^1$ on X_9. We denote by ι_{X_9} the elliptic involution on $\mathcal{E}_{X_9}$. We denote by τ_9 the cover involution of $\pi : X_9 \to R_9$.

By definition the fibration $\mathcal{E}_{X_9}$ is of type 2 with respect to τ. So, by Theorem 1, the field of definition of the elliptic fibration and of a section of it is k.

Nevertheless there could be other sections or components of some reducible fibers which are not defined over k.

In what follows we describe the Galois group $G_{\mathcal{E}_{X_9},\mathrm{MW}}$, i.e., the Galois group of the field extension $k_{\mathcal{E}_{X_9}}/k$ over which all components of reducible fibers and sections of the fibration $\mathcal{E}_{X_9}$ are defined.

Proposition 3 *The Galois group $G_{\mathcal{E}_{X_9},\mathrm{MW}}$ of the elliptic fibration $\mathcal{E}_{X_9} : X_9 \to \mathbb{P}^1$ is as follows*

- $G_{\mathcal{E}_{X_9},\mathrm{MW}} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ *if, and only if, $G_{R_9} \simeq \mathbb{Z}/2\mathbb{Z}$ and the branch fibers of π are not defined over k_{R_9},*
- $G_{\mathcal{E}_{X_9},\mathrm{MW}} \simeq \mathbb{Z}/2\mathbb{Z}$ *if, and only if, $G_{R_9} \simeq \mathbb{Z}/2\mathbb{Z}$ and the branch fibers of π are defined over k_{R_9},*
- $G_{\mathcal{E}_{X_9},\mathrm{MW}} \simeq \mathbb{Z}/2\mathbb{Z}$ *if, and only if, $G_{R_9} = \{id\}$ and the branch fibers of π are not defined over k_{R_9},*
- $G_{\mathcal{E}_{X_9},\mathrm{MW}} = \{id\}$ *if, and only if, $G_{R_9} = \{id\}$ and the branch fibers of π are defined over k_{R_9}.*

Proof This follows from a simple analysis of whether the extensions k_{R_9} and k_{τ_9} are linearly disjoint or not. This depends of course on the branch locus of the base change map. See the discussion in Notation 2. □

The elliptic fibration $\mathcal{E}_{X_9}$ has two fibers of type I_9. Let us denote by Θ_i^j, for $i = 0, \ldots, 8$, $j = 1, 2$ the i-th component of the j-th fiber of type I_9. The sections of $\mathcal{E}_{R_9}$ induce sections of $\mathcal{E}_{X_9}$, and thus $\mathrm{MW}(\mathcal{E}_{X_9}) = \{O_{X_9}, T_1, T_2\}$. Thus

$$\pi(O_{X_9}) = O, \quad \pi(T_1) = t_1, \quad \pi(T_2) = t_2, \quad \pi(\Theta_i^j) = C_i, \ i = 0, \ldots, 8, \ j = 1, 2. \tag{13}$$

The automorphism τ_9 is the cover involution of π and thus

$$\tau_9(O_{X_9}) = O_{X_9}, \quad \tau_9(T_1) = T_1, \quad \tau_9(T_2) = T_2, \quad \tau_9(\Theta_i^1) = \Theta_i^2, \ i = 0, \ldots, 8. \tag{14}$$

Figure 2 summarizes the above.

Proposition 4 *The Néron–Severi group of X_9 has rank 18, signature $(1, 17)$, discriminant group $\mathbb{Z}/9\mathbb{Z}$ and its discriminant form is the opposite to the one of A_8. The transcendental lattice of X_9 is the unique (up to isometries) even lattice with signature $(2, 2)$, discriminant group $\mathbb{Z}/9\mathbb{Z}$ and discriminant form equal to the one of A_8.*

Proof The Néron–Severi group contains the 18 linearly independent classes O_{X_9}, T_1, T_2 and Θ_i^j, for $i = 1, \ldots, 8$, $j = 1, 2$. Hence it has rank at least 18. On the other hand the family of X_9 is a two dimensional family (because of the choice of two branch fibers of the double cover $X_9 \to R_9$). So the Néron–Severi has rank at most 18. We conclude that the 18 classes listed before form a basis of $\mathrm{NS}(X_9)$. The intersection form and the discriminant form of $\mathrm{NS}(X_9)$ can be explicitly computed and one can check that it has discriminant 9. In
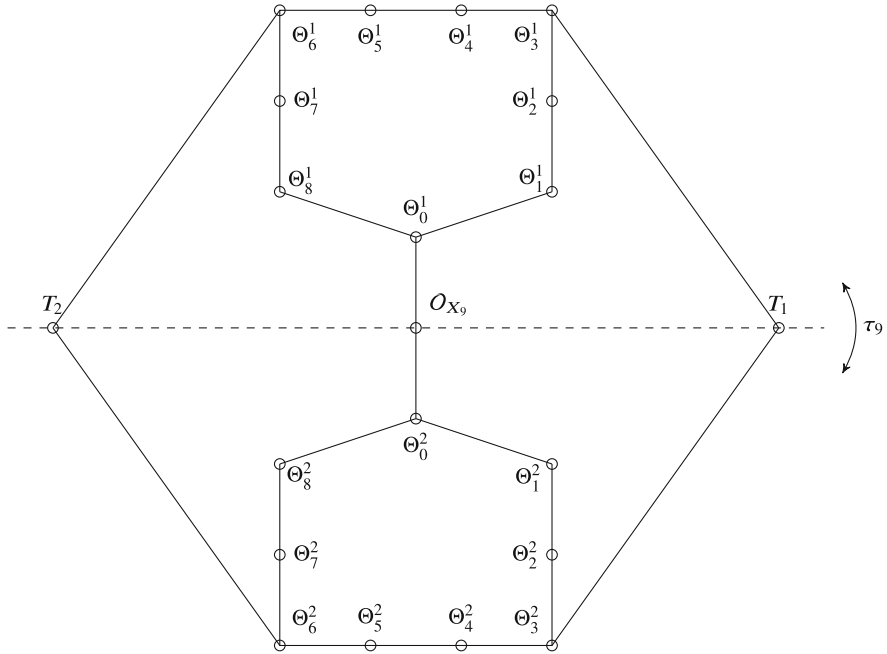
Fig. 2 Reducible fibers and sections of the fibration $\mathcal{E}_{X_9}$ on X_9

particular, a generator for the discriminant group is $2\left(\sum_{i=1}^{9} i\left(\Theta_i^1 - \Theta_i^2\right)\right)/9$ and its discriminant form is $\mathbb{Z}/9\mathbb{Z}\left(\frac{8}{9}\right)$, which is the opposite to the discriminant form of A_8. The discriminant form of the transcendental lattice is the opposite of the discriminant form of the Néron–Severi group. Hence the transcendental lattice T_{X_9} is an even lattice with signature $(2, 2)$ and discriminant form $\mathbb{Z}/9\mathbb{Z}\left(\frac{-8}{9}\right)$. The transcendental lattice is uniquely determined by these data by [12, Theorem 1.13.2]. We observe that the discriminant form of T_{X_9} is the same as the one of A_8 and that rank $(T_{X_9}) + 4 = $ rank (A_8). $\square$

Corollary 1 *The filed $k_{\mathcal{E}_9}$ coincides with $k_{R,\tau}$.*

Proof By Proposition 4 the classes of the reducible fibers and of the sections of $\mathcal{E}_{X_9}$ form a basis of NS(X). Each of these classes corresponds to a unique curve (since these are negative curves), which is a smooth rational curve. Hence the field where all these classes are defined coincides with the field where NS(X) is defined. The former is $k_{\mathcal{E}_9}$ by definition, the latter is k_{R_9,τ_9} by Lemma 5. $\square$

6.3 Classification of All the Possible Fibrations of the $K3$ Surface X_9

In order to find all elliptic fibrations on X_9, we use Nishiyama's method explained in [14]. As explained in [14, Section 6.1], if one is able to find a lattice T_0 which is negative definite, has the same discriminant form of the transcendental lattice of a K3 surface and its rank is the rank of the transcendental group plus four, then there is an operative method to classify the configuration of the reducible fibers of the elliptic fibrations on the surface. In our particular case, by Proposition 4, we put $T_0 = A_8$ and in order to classify the elliptic fibrations on X_9 (and in particular the lattice W of each of these elliptic fibration, with the notation of [14]) we have to find the orthogonal complements of primitive embeddings of the root lattice A_8 in the 24 possible lattices listed (by their root type) by Niemeier [11, Satz 8.3] (or [14, Theorem 1.7]). By [14, Lemmas 4.1 and 4.3] we know that A_8 embeds primitively uniquely, up to the action of the Weyl group, in A_m for $m \geq 8$, in D_n for $n \geq 9$, and in no other root lattice. The orthogonal complements of these embeddings in the 24 Niemeier lattices are then found in [14, Corollary 4.4], and this determines the reducible fibers and the rank of the Mordell–Weil group for each fibration. These results are summarized in Table 1. Note that line 1 is the fibration $\mathcal{E}_{X_9}$. Apart from the torsion part of the Mordell–Weil group, everything is found by Nishiyama's method as explained above. We compute the torsion parts in what follows.

6.3.1 Torsion of the Mordell–Weil Group for the Elliptic Fibrations Associated to X_9

By [18, Table 1], we can immediately conclude that the torsion of the fibrations in lines 2, 3, 4, 5, 8, 9, and 12 is trivial, and the torsion part of fibrations 6, 7, 10, and 11 is either $\mathbb{Z}/2\mathbb{Z}$ or trivial.

Fibration 11 comes from the orthogonal complement of the embedding of A_8 in a lattice N of rank 24 with root type A_{24}. We observe that $N/A_{24} = \mathbb{Z}/5\mathbb{Z}$ ([11, Satz 8.3] or [14, Theorem 1.7]). By Nishiyama [14, Lemma 6.6, iii)], the torsion of the elliptic fibration corresponding to this embedding of A_8 in N has to be contained in N/A_{24}, so this fibration does not have a 2-torsion section and the torsion part of the Mordell–Weil group is trivial.

Note that, in terms of the notation of our configuration of $2I_9$ (see Fig. 2), we find a fiber of type I_{16} composed of the following curves on X_9.

$$\Theta_0^1, \Theta_1^1, \Theta_2^1, \Theta_3^1, T_1, \Theta_3^2, \Theta_2^1, \Theta_1^2, \Theta_0^2, \Theta_8^2, \Theta_7^2, \Theta_6^2, T_2, \Theta_6^1, \Theta_7^1, \Theta_8^1. \tag{15}$$

Moreover, $\Theta_5^1, \Theta_4^1, \Theta_5^2, \Theta_4^2$ are sections for this fibration. Let Θ_4^1 be the 0-section, then the height $h(\Theta_5^2)$ of the section Θ_5^2 is $2 \cdot 2 + 0 - \frac{8(16-8)}{16} = 0$ [17, Chap. 11 §11.8], and therefore it is a torsion section [17, Theorem 11.5]. Since we know that the fibration in line 11 has trivial torsion, and the fibration in line 7 is the only other

Table 1 Elliptic fibrations of X_9

n^o	Niemeier	Embedding	Roots orth.	Reducible fibers	MW
1	$A_8^{\oplus 3}$	$A_8 \subset A_8$	$A_8^{\oplus 2}$	$2I_9$	$\mathbb{Z}/3\mathbb{Z}$
2	$E_8 \oplus D_{16}$	$A_8 \subset D_{16}$	$E_8 \oplus D_7$	$II^* + I_3^*$	$\mathbb{Z}$
3	$E_7^{\oplus 2} \oplus D_{10}$	$A_8 \subset D_{10}$	$E_7^{\oplus 2}$	$2III^*$	$\mathbb{Z}^2$
4	$E_7 \oplus A_{17}$	$A_8 \subset A_{17}$	$E_7 \oplus A_8$	$III^* + I_9$	$\mathbb{Z}$
5	D_{24}	$A_8 \subset D_{24}$	D_{15}	I_{11}^*	$\mathbb{Z}$
6	$D_{12}^{\oplus 2}$	$A_8 \subset D_{12}$	$D_{12} \oplus A_3$	$I_8^* + I_4$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$
7	$D_9 \oplus A_{15}$	$A_8 \subset D_9$	A_{15}	I_{16}	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$
8	$D_9 \oplus A_{15}$	$A_8 \subset A_{15}$	$D_9 \oplus A_6$	$I_5^* + I_7$	$\mathbb{Z}$
9	$E_6 \oplus D_7 \oplus A_{11}$	$A_8 \subset A_{11}$	$E_6 \oplus D_7 \oplus A_2$	$IV^* + I_3^* + I_3$	$\mathbb{Z}$
10	$D_6 \oplus A_9^{\oplus 2}$	$A_8 \subset A_9$	$D_6 \oplus A_9$	$I_2^* + I_{10}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$
11	A_{24}	$A_8 \subset A_{24}$	A_{15}	I_{16}	$\mathbb{Z}$
12	$A_{12}^{\oplus 2}$	$A_8 \subset A_{12}$	$A_{12} \oplus A_3$	$I_{13} + I_4$	$\mathbb{Z}$

one with reducible fiber of type I_{16}, we conclude that we found a representation of the fibration in line 7, and therefore the torsion part of the Mordell–Weil group of this fibration is $\mathbb{Z}/2\mathbb{Z}$.

Finally, we find that the torsion part of the Mordel–Weil groups of the fibrations in lines 6 and 10 are $\mathbb{Z}/2\mathbb{Z}$ in the same way as we did for line 7.

We find the class of the fiber of the other elliptic fibrations, by giving the components of one reducible fiber in terms of the configuration of $2I_9$ (see Fig. 2). The rational curves orthogonal to the class of the fiber are necessarily components of other reducible fibers, hence we list all the irreducible components of at least one reducible fiber and some components of the others for each elliptic fibration.

$I_8^* + I_4$

$\Theta_1^2, \Theta_8^2, 2\Theta_0^2, 2O_{X_9}, 2\Theta_0^1, 2\Theta_8^1, 2\Theta_7^1, 2\Theta_6^1, 2\Theta_5^1, 2\Theta_4^1, 2\Theta_3^1, \Theta_2^1, T_1 + \Theta_6^2, \Theta_5^2, \Theta_4^2,$

$I_2^* + I_{10}$

$\Theta_8^1, \Theta_0^1, 2\Theta_1^1, 2O_{X_9}, 2\Theta_0^2, \Theta_8^2, \Theta_1^2 + T_2, \Theta_6^1, \Theta_5^1, \Theta_4^1, \Theta_3^1, T_1\Theta_3^2, \Theta_4^2, \Theta_5^2, \Theta_6^2, \Theta_6^1.$

For this configuration of $I_8^* + I_4$ we find the three sections Θ_7^2, Θ_3^2, and Θ_2^2. If we set Θ_7^2 as the 0-section, then Θ_3^2 has height 0 and hence it is a 2-torsion section. Since there is only one fibration with reducible fiber $I_8^* + I_4$ in our list, we conclude that this configuration represents the fibration in line 6. Hence the torsion part of the Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z}$. For the fibration in line 10 we have the same reasoning, after finding the sections $\Theta_4^1, \Theta_7^2, \Theta_6^2, \Theta_2^2$, setting Θ_2^1 as the 0-section and finding that Θ_7^2 is a 2-torsion section.

6.4 Determining the Type of Each Fibration of X_9

In what follows, we assume that the surface R_9 is general, i.e., its Galois group G_{R_9} is not trivial. The goal of this section is to find an example, for each fibration η in Table 1, and to determine for each example the following:

(a) The type with respect to the cover involution τ_9;
(b) an upper bound for the degree over k of a field of definition of the fibration, that is, a field over which the reducible fiber and a 0-section are defined;
(c) an upper bound for the degree over k of a field $k_{\eta,\mathrm{MW}}$ over which the Mordell–Weil group of the fibration admits a set of generators.

The results are summarized in Table 2, with the notations introduced in Sects. 4, 5, and 6.

For each fibration in Table 1 we find a configuration of (parts of the) reducible fibers in terms of the curves in Fig. 2. Note that for lines 6, 7, and 10, this is done in the previous section.

$II^* + I_3^*$
$\Theta_6^2, 2\Theta_7^2, 3\Theta_8^2, 4\Theta_0^2, 5O_{X_9}, 6\Theta_0^1, 4\Theta_8^1, 2\Theta_7^1, 3\Theta_1^1 \quad + \quad \Theta_4^2, \Theta_2^2, 2\Theta_3^2, 2T_1, 2\Theta_3^1,$
$2\Theta_4^1, \Theta_5^1.$

$2III^*$
$\Theta_5^1, 2\Theta_6^1, 3T_2, 4\Theta_6^2, 3\Theta_5^2, 2\Theta_4^2, \Theta_3^2, 2\Theta_7^2 \quad + \quad \Theta_1^2, 2\Theta_0^2, 3O_{X_9}, 4\Theta_0^1, 3\Theta_1^1, 2\Theta_2^1,$
$\Theta_3^1, 2\Theta_8^1.$

$III^* + I_9$
$\Theta_0^1, 2\Theta_8^1, 3\Theta_7^1, 4\Theta_6^1, 3\Theta_5^1, 2\Theta_4^1, \Theta_3^1, 2T_2 \quad + \quad \Theta_7^2, \Theta_8^2, \Theta_0^2, \Theta_1^2, \Theta_2^2, \Theta_3^2, \Theta_4^2, \Theta_5^2.$

$I_5^* + I_7$
$\Theta_5^1, T_2, 2\Theta_6^1, 2\Theta_7^1, 2\Theta_8^1, 2\Theta_0^1, 2O_{X_9}, 2\Theta_0^2, \Theta_1^2, \Theta_8^2 \quad + \quad \Theta_5^2, \Theta_4^2, \Theta_3^2, T_1, \Theta_3^1, \Theta_2^1.$

$IV^* + I_3^* + I_3$
$\Theta_6^2, 2T_2, 3\Theta_6^1, 2\Theta_5^1, \Theta_4^1, 2\Theta_7^1, \Theta_8^1 \quad + \quad O_{X_9}, \Theta_8^2, 2\Theta_0^2, 2\Theta_1^2, 2\Theta_2^2, 2\Theta_3^2, T_1, \Theta_4^2 \quad +$
$\Theta_1^1, \Theta_2^1.$

$I_{13} + I_4$
$T_2, \Theta_6^1, \Theta_5^1, \Theta_4^1, \Theta_3^1, T_1, \Theta_3^2, \Theta_2^2, \Theta_1^2, \Theta_0^2, \Theta_8^2, \Theta_7^2, \Theta_6^2 \quad + \quad \Theta_8^1, \Theta_0^1, \Theta_1^1.$

To find the elliptic fibrations described in lines 5 and 11 of the Table 1, we need to find another rational curve on X_9. We recall that the K3 surface X_9 has an infinite number of rational curves, and considering the elliptic fibration with fibers $IV^* + I_3^* + I_3$ we are able to describe one of them. Indeed, the divisor

$$F := \Theta_6^2 + 2T_2 + 3\Theta_6^1 + 2\Theta_5^1 + \Theta_4^1 + 2\Theta_7^1 + \Theta_8^1 \qquad (16)$$

corresponds to a reducible fiber of type IV^* of this elliptic fibration. In particular F is the class of the fiber of this fibration, and the divisor $O_{X_9} + \Theta_8^2 + 2\Theta_0^2 + 2\Theta_1^2 + 2\Theta_2^2 + 2\Theta_3^2 + T_1 + \Theta_4^2$ is linearly equivalent to F and corresponds to the fiber of type I_3^*. The remaining reducible fiber consists of three curves meeting in a triangle, one is Θ_1^1, one is Θ_2^1 and we denote the third one by M. Since $\Theta_1^1 + \Theta_2^1 + M$ is a fiber of the elliptic fibration, M is linearly equivalent to $F - \Theta_1^1 - \Theta_2^1$. In particular this implies that the intersections properties of M are the following: $M\Theta_1^1 = M\Theta_2^1 = M\Theta_7^2 = M\Theta_5^2 = 1$ and M is orthogonal to all the other curves appearing in Fig. 2.

Let us consider the following configuration of curves:

I_{16}
$\Theta_6^1, \Theta_5^1, \Theta_4^1, \Theta_3^1, \Theta_2^1, M, \Theta_5^2, \Theta_4^2, \Theta_3^2, \Theta_2^2, \Theta_1^2, \Theta_0^2, O_{X_9}, \Theta_0^1, \Theta_8^1, \Theta_7^1.$

The curves T_2 and Θ_7^2 are sections of this fibration. Assume that T_2 is the zero section, then Θ_7^2 is a section, orthogonal to the zero section and meeting the reducible fiber I_{16} in his fifth component. Thus the height of this section is $\frac{9}{16}$. As a consequence, the lattice spanned by the irreducible components of the reducible fiber of type I_{16}, the zero section T_2 and the section Θ_7^2 is a sublattice of NS(X_9), which has the same rank and the same discriminant of NS(X_9) and therefore coincides with NS(X_9). So there are no torsion sections for this elliptic fibration (otherwise one should add their contribution to obtain the Néron–Severi group). As a consequence the fibration whose class of the fiber is

$$\Theta_6^1 + \Theta_5^1 + \Theta_4^1 + \Theta_3^1 + \Theta_2^1 + M + \Theta_5^2 + \Theta_4^2 + \Theta_3^2 + \Theta_2^2 + \Theta_1^2 + \Theta_0^2 + O_{X_9} + \Theta_0^1 + \Theta_8^1 + \Theta_7^1 \qquad (17)$$

corresponds to the fibration in line 11 of Table 1.

Similarly, the fibration in line 5 of Table 1 with a reducible fiber of type I_{11}^* is given by

$$\Theta_4^2, \Theta_6^2, 2\Theta_5^2, 2M, 2\Theta_2^1, 2\Theta_3^1, 2\Theta_4^1, 2\Theta_5^1, 2\Theta_6^1, 2\Theta_7^1, 2\Theta_8^1, 2\Theta_0^1, 2O_{X_9}, 2\Theta_0^2, \Theta_1^2, \Theta_8^2. \qquad (18)$$

Corollary 2 *For each fibration in Table 1, there exists at least one elliptic fibration on X_9 with the properties given in the list which is defined over k_{R_9,τ_9}.*

Proof The result follows by 1 for the fibration of type 1 and 2. For the fibration of type 3, one wants to apply Remark 4. For all the listed fibrations with the exception of the 11, we are able to write the class of the fiber as a linear combination of Θ_i^j, O_{X_9} and T_k. All these curves are defined on k_{R_9,τ_9}, by 1. In the case of the fibration 11, we introduced another curve, M. Since its class is written as a linear combination of the classes generating NS(X), its class is defined over k_{R_9,τ_8}. Since it is a negative effective class, we deduce that it is supported either on an irreducible rational curve

Table 2 Types of the different elliptic fibrations of X_9 and fields of definition

n^o	Roots Orth.	Type	Sections	Field of Def. 0-section	Field of Def. all sections	$[k_{\eta,\mathrm{MW}} : k]$
1	$A_8^{\oplus 2}$	2	O_{X_9}, T_1, T_2	O_{X_9}/k	k_R	≤ 2
2	$E_8 \oplus D_7$	3	T_2, Θ_5^2	T_2/k_R	k_{R,τ_9}	≤ 4
3	$E_7^{\oplus 2}$	3	$T_1, \Theta_2^2, \Theta_4^1$	T_1/k_R	k_{R,τ_9}	≤ 4
4	$E_7 \oplus A_8$	3	$O_{X_9}, T_1, \Theta_1^1, \Theta_2^1$	O_{X_9}/k	k_{R,τ_9}	≤ 4
5	D_{15}	3	Θ_2^2, Θ_3^2	$\Theta_2^2/k_{R,\tau_9}$	k_{R,τ_9}	≤ 4
6	$D_{12} \oplus A_3$	3	$\Theta_2^2, \Theta_3^2, \Theta_7^2$	$\Theta_7^2/k_{R,\tau_9}$	k_{R,τ_9}	≤ 4
7	A_{15}	1	$\Theta_4^1, \Theta_5^1, \Theta_4^2, \Theta_5^2$	$\Theta_4^1/k_{R,\tau_9}$	k_{R,τ_9}	≤ 4
8	$D_9 \oplus A_6$	3	$\Theta_4^1, \Theta_2^2, \Theta_6^2, \Theta_7^2$	$\Theta_4^1/k_{R,\tau_9}$	k_{R,τ_9}	≤ 4
9	$E_6 \oplus D_7 \oplus A_2$	3	$\Theta_0^1, \Theta_3^1, \Theta_5^2, \Theta_7^2$	Θ_0^1/k_{τ_9}	k_{R,τ_9}	≤ 4
10	$D_6 \oplus A_9$	1	$\Theta_7^1, \Theta_7^2, \Theta_2^1$	$\Theta_7^1/k_{R,\tau_9}$	k_{R,τ_9}	≤ 4
11	A_{15}	3	$T_2, \Theta_7^2, \Theta_8^2, \Theta_6^2$	T_2/k_R	k_{R,τ_9}	≤ 4
12	$A_{12} \oplus A_3$	3	$O_{X_9}, \Theta_2^1, \Theta_7^1, \Theta_4^2, \Theta_5^2$	O_{X_9}/k	k_{R,τ_9}	≤ 4

or on the union of rational curves. Since it is a component of a fiber of a certain fibration, at least on the closure of the field of definition of the fibration, it is an irreducbile curve (where it is defined). Hence, M is an irreducible smooth rational curve defined over k_{R_9,τ_9} □

We gave an example for each fibration in Table 1. We choose a section for each of them to be the zero section and we determine their type with respect to τ. By using Proposition 3 we describe the properties of the fields $k_{\eta,MW}$, which follow by the previous Corollary. The results are listed in Table 2.

7 The Surfaces R_4, R_3, R_2 and the Surfaces X_4, X_3, X_2

In this section we establish an analogous study for the extremal rational surfaces R_i, for $i = 4, 3, 2$. We classify all the possible fibrations of the K3 surfaces X_i and determine their types with respect to the cover involutions τ_i, for $i = 4, 3, 2$.

7.1 The Rational Elliptic Surfaces R_4, R_3, and R_2

Let R_4 be an extremal rational elliptic surface with one reducible fiber of type I_4^*. Its Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z} = \{O, t_1\}$, where O is the zero section and t_1 is a 2-torsion section. Recall that a fiber of type I_4^* is given by 9 smooth rational curves meeting with dual graph $\tilde{D}_8$, see [9, Table I.4.1]. The Néron–Severi group of R_4 contains also the classes of the irreducible components of the reducible fiber, denoted by $C_0, C_1, \ldots, C_8$. The intersections which are not trivial are the following:

$$C_l^2 = -2,\, C_0 C_2 = C_6 C_8 = 1, \tag{19}$$

$$C_l C_j = 1 \quad \text{if and only if} \quad |l - j| = 1 \text{ and } \{l, j\} \subset \{2, 3, 4, 5, 6\}, \tag{20}$$

$$OC_0 = t_1 C_8 = 1, \quad \text{and} \quad O^2 = t_1^2 = -1. \tag{21}$$

Let R_3 be an extremal rational elliptic surface over k with one reducible fiber of type III^*. As R_3 is extremal, there is another reducible fiber which is either an I_2 or an III. Its Mordell–Weil group is $\mathbb{Z}/2\mathbb{Z} = \{O, t_1\}$, where O is the zero section and t_1 is a 2-torsion section. Recall that a fiber of type III^* is given by 8 smooth rational curves meeting with dual graph $\tilde{E}_7$, see [9, Table I.4.1]. The Néron–Severi group of R_3 contains also the classes of the irreducible components of the reducible fiber. Denoted by C_l the components of the III^* fiber and by D_l the ones of the other reducible fiber, the intersections which are not trivial are the following:

$$C_l^2 = -2,\, C_l C_j = 1 \text{ if and only if } |l - j| = 1 \text{ and } \{l, j\} \subset \{0, 1, 2, 3, 4, 5, 6\}, \tag{22}$$

$$C_3 C_7 = 1, \quad D_0 D_1 = 2, \quad D_j^2 = -2, \tag{23}$$

$$OC_0 = t_1 C_6 = OD_0 = t_1 D_1 = 1 \quad \text{and} \quad O^2 = t_1^2 = -1. \tag{24}$$

Let R_2 be an extremal rational elliptic surface over k with one reducible fiber of type II^*. The other singular fibers are either II or $2I_1$. Its Mordell–Weil group is $\{O\}$, i.e., it is trivial. Recall that a fiber of type II^* is given by 9 smooth rational curves meeting with dual graph $\tilde{E}_8$, see [9, Table I.4.1]. The Néron–Severi group of R_2 contains also the classes of the irreducible components of the reducible fiber, denoted by $C_0, C_1, \ldots, C_8$. The intersections which are not trivial are the following:

$$C_l^2 = -2,\, C_l C_j = 1 \text{ if and only if } |l - j| = 1 \text{ and } \{l, j\} \subset \{0, 1, 2, 3, 4, 5, 6, 7\}, \tag{25}$$

$$C_8 C_5 = 1, \quad OC_0 = 1, \quad O^2 = -1. \tag{26}$$

The following result shows that the surfaces R_i have trivial Galois group G_{R_i}, that is its Néron-Severi group admits a set of generators over k given by the zero

section, a smooth fiber and the non-trivial fiber components of the reducible fibers. It also presents their contractions of negative curves to minimal k-rational surfaces.

Proposition 5 *Let R be one the following surfaces: R_2, R_3, R_4. Then G_R is trivial. Moreover, the surfaces R_2, R_3 and R_4 can be contracted to $\mathbb{P}^2$; the surfaces R_3 and R_4 can be also contracted to $\mathbb{P}^1 \times \mathbb{P}^1$ and the surfaces R_2 and R_3 can be also contracted to $\mathbb{F}_2$, the Hirzebruch surface with a unique (-2)-curve.*

Proof The proof is similar to the one of Proposition 2. Indeed, for $R = R_2$ or R_3, each $g \in G_R$, $g(O) = O$ and if MW $= \{O, t_1\}$, $g(t_1)$ has to be a section different from O and hence $g(t_1) = t_1$. Thus for each R_i, $i = 2, 3$, the sections are preserved and this implies, arguing via the intersection of the components of the reducible fibers as in Proposition 2, that all the components of the unique reducible fibers are fixed.

Let us consider the surface of type R_3. We have three different possibilities, to obtain three different surfaces:

- Let us contract the sections O and t_1. Then we contract the images of C_0 and C_6 (which are now (-1)-curves); the images of C_1 and C_5; the images of C_2 and C_4. There remain the images of C_3, which is a curve with self-intersection 0, and of C_7, which is a curve with self-intersection -2. There are no (-1)-curves on this surface, so we obtain a minimal rational surface, with two independent classes in the Néron–Severi group which have self-intersection 0 and -2. Hence we obtained $\mathbb{F}_2$.
- Let us contract first the section O and then (in this order), the images of the components C_0, C_1, C_2, C_3. Now the image of C_7 is a (-1)-curve. We contract it. It remains a unique (-1)-curve, which is the section t_1. We contract it and then (in this order) the images of the components C_6 and C_5. We obtain a minimal rational surface whose Néron–Severi group is generated by one class (we contracted 9 curves), which is the image of C_4. This rational surface is necessarily $\mathbb{P}^2$.
- Let us contract first the section O and then (in this order), the images of the components C_0, C_1, C_2, C_3. Now the image of C_4 is a (-1)-curve. We contract it. Then we contract t_1 and the image of the component C_5. We obtain a minimal rational surface, whose Néron–Severi group is generated by the two classes which are the images of C_7 and C_5. Their self-intersection is 0 and they meet in a point, so we obtained $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us now consider the surface R_2 (see Fig. 3). There is a unique (-1)-curve, the section O. So we contract it, and then we contract (in this order) the images of the components $C_0, C_1, C_2, C_3, C_4, C_5$. Now both the images of C_6 and C_8 are (-1)-curves and they meet in a point.

- If one contracts the image of C_8, one obtains a minimal surface, whose generators of the Néron–Severi group are the images of C_7 and C_6 and this surface is $\mathbb{F}_2$ (because of the presence of a (-2)-curve, image of C_7).
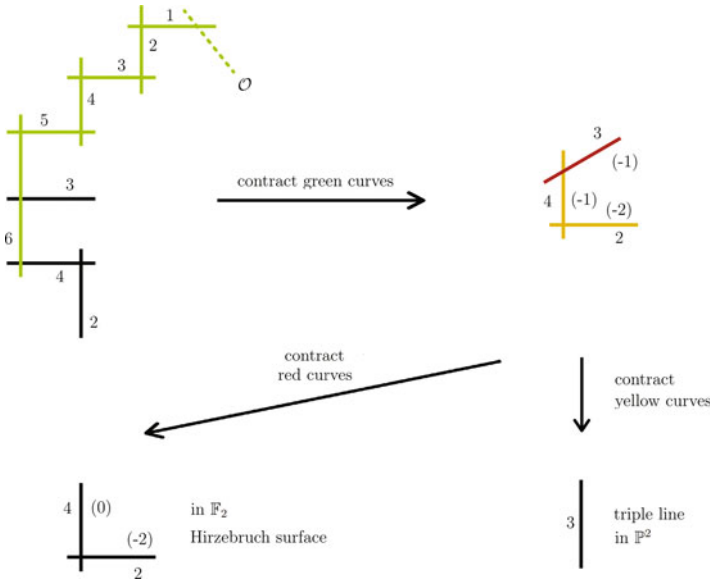
Fig. 3 Contractions of R_2 to $\mathbb{F}_2$ and to $\mathbb{P}^2$

- If one contracts the image of C_6, then one has to contract the image of C_7 and one obtains a minimal rational surface, whose Néron–Severi group has one generator (the image of C_8) and thus the surface is $\mathbb{P}^2$.

Let us consider the surface of type R_4. We contract first the section O and then (in this order) the images of the components C_0, C_2, C_3, C_4, C_5, C_6. Now we have three (-1) curves, i.e. the images of C_7, C_8 and t_1. The image of C_8 meets both the images of C_7 and of t_1: if one contracts the image of C_8, one obtains the minimal surface $\mathbb{P}^1 \times \mathbb{P}^1$; if one contracts the images of t_1 and C_7 one obtains $\mathbb{P}^2$.   □

7.2   The K3 Surfaces X_4, X_3, X_2

Let X_i be a K3 surface obtained by a generic base change of order 2 on the rational elliptic surface R_i for $i = 4, 3, 2$ as in Sect. 2. Let P_i and Q_i be the points corresponding to the branch fibers of the cover $X_i \to R_i$. We have the following result, analogous to Proposition 3.

Proposition 6 *The Galois group $G_{\mathcal{E}_{X_i}}$ of the elliptic fibration $\mathcal{E}_{X_i} : X_i \to \mathbb{P}^1$ is contained in $(\mathbb{Z}/2\mathbb{Z})$. It is trivial if and only if the points P_i and Q_i are defined over the ground field.*

Proof The group G_{R_i} is trivial by Proposition 5, so the unique Galois action is the one of the cover involution τ_i, which is trivial if and only if the branch fibers are defined over the ground field. □

The elliptic fibrations $\mathcal{E}_{X_i}$, $i = 4, 3, 2$, are induced by $\mathcal{E}_{R_i}$. We fix the following notation: each component C_l (resp. D_l) of a reducible fiber of $\mathcal{E}_{R_i}$ corresponds to two curves Θ_l^j (resp. Φ_l^j), $j = 1, 2$ on X_i which are components of two different reducible fibers on X_i. Moreover the zero section of $\mathcal{E}_{R_i}$ induces the zero section, O_{X_i}, of $\mathcal{E}_{X_i}$ and, if there is a torsion section t_1 on R_i, it induces a torsion section T_1 on X_i.

So we have the following curves on X_i:

$$\Theta_l^j \; j = 1, 2; \; O_{X_i}; \; T_1 \text{ if } i \neq 2; \; \Phi_l^j \; j = 1, 2, l = 0, 1 \text{ if } i = 3. \tag{27}$$

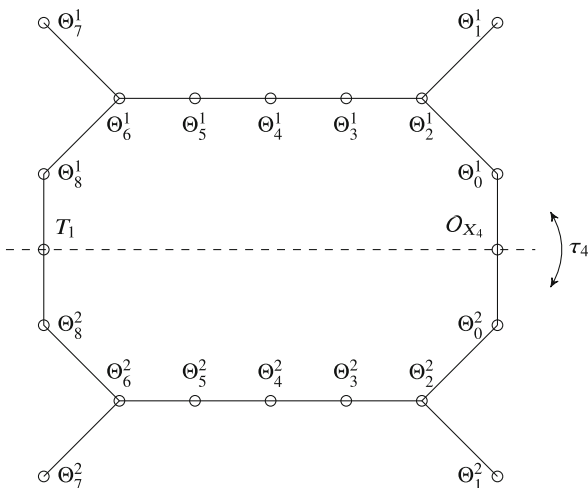Denote by $\pi_i : X_i \to R_i$ the double cover of R_i induced by the base change and by τ_i the cover involution. We have

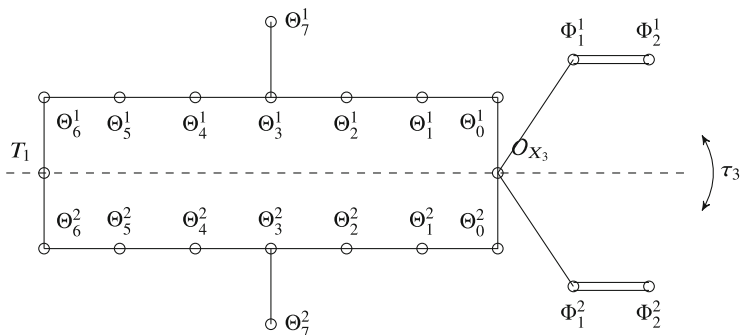$$\pi_i(O_{X_i}) = O, \;\; \pi_i(\Theta_l^1) = \pi_i(\Theta_l^2) = C_l, \tag{28}$$

$$\pi_i(T_1) = t_1 \text{ if } i \neq 2, \;\; \pi_i(\Phi_l^1) = \pi_i(\Phi_l^2) = D_l \text{ if } i = 3. \tag{29}$$

$$\tau_i(O_{X_i}) = O_{X_i}, \;\; \tau_i(\Theta_l^1) = \Theta_l^2, \;\; \tau_i(T_1) = T_1 \text{ if } i \neq 2, \;\; \tau_i(\Phi_l^1) = \Phi_l^2 \text{ if } i = 3. \tag{30}$$

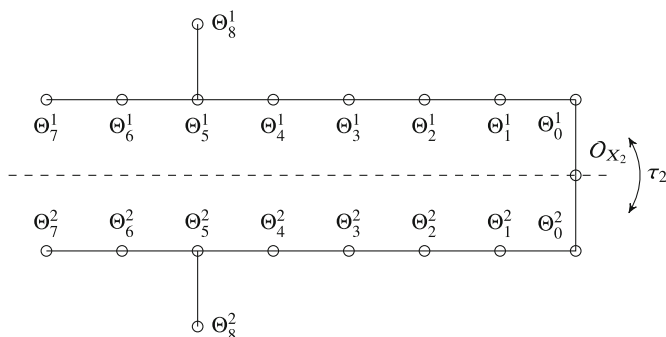Figures 4, 5, and 6 summarize the above. Note that in Fig. 5, Φ_2^1 and Φ_2^2 are both connected to T_1.

Fig. 4 Reducible fibers and sections of the fibration $\mathcal{E}_{X_4}$ on X_4

Fig. 5 Reducible fibers and sections of the fibration $\mathcal{E}_{X_3}$ on X_3

Fig. 6 Reducible fibers and sections of the fibration $\mathcal{E}_{X_2}$ on X_2

Proposition 7 *The Néron–Severi group of X_i has rank 18, signature $(1, 17)$, for every $i = 2, 3, 4$.*

Both lattices $\mathrm{NS}(X_4)$ and of $\mathrm{NS}(X_3)$ are isometric to $U \oplus D_8 \oplus E_8$ and their transcendental lattices are both isometric to $U \oplus U(2)$, which has the same discriminant group and form as D_8. In particular X_3 and X_4 lie in the same family of K3 surfaces, namely the family of $U \oplus D_8 \oplus E_8$-polarized K3 surfaces.

The lattice $\mathrm{NS}(X_2)$ is isometric to $U \oplus E_8 \oplus E_8$ and its transcendental lattice is isometric to $U \oplus U$, which has the same discriminant form of E_8.

Proof The curves in the Figs. 4, 5, and 6 (i.e. the curves Θ_l^j, O_{X_i}, T_1 if $i \neq 2$ and Φ_l^j if $i = 3$) generate $\mathrm{NS}(X_i)$. They are not all linearly independent, but if one extracts a basis, one obtains 18 independent generators of $\mathrm{NS}(X_i)$. Since one knows all the intersection properties of these generators, one can explicitly compute their intersection matrix. This identifies the lattice $\mathrm{NS}(X_i)$ and in particular its discriminant group and form. We observe that all the lattices that appear are 2-elementary, i.e., the discriminant group is $(\mathbb{Z}/2\mathbb{Z})^a$, $a \in \mathbb{N}$. So the transcendental lattice is a 2-elementary lattice with signature $(2, 2)$. The indefinite 2-elementary lattices are completely determined by their signature and their length, i.e., by a, and

by another invariant, often denoted by δ, which is zero in all the cases considered. This allows us to identify the transcendental lattices. □

7.3 Classification of All the Possible Fibrations on the $K3$ Surfaces X_4, X_3, and X_2

In the same way as we did for X_9 in Sect. 6.3, we classify elliptic fibrations on the surfaces $X_4 \simeq X_3$ and X_2 in what follows. By Proposition 7 we take $T = D_8$ for $X_3 \simeq X_4$, and $T = E_8$ for X_2 and apply Nishiyama's method. By Nishiyama [14, Lemmas 4.1 and 4.3] we know that D_8 only embeds primitively in D_n for $n \geq 8$, and E_8 only embeds primitively in E_8. The orthogonal complements of these embeddings in the 24 Niemeier lattices are then found in [14, Corollary 4.4]. Those results are summarized in Tables 3 and 4. We notice that the fibrations on X_2, X_3 and X_4 were already classified in [6, Table 2, case $k = 8$ and Table 1 case $k = 8, \delta = 0$] via different methods.

7.4 Determining the Type of Each Fibration of X_4, X_3, and X_2

As in Sect. 6.4 we determine the type of each fibration obtained in Sect. 7.3 (Tables 3 and 4) with respect to the cover involutions τ_i, for $i = 4, 3, 2$. We determine moreover the sections and their fields of definition. This study allows us to obtain an upper bound for the degree over k of a field of definition k_η of a given fibration

Table 3 Elliptic fibrations of X_3 and X_4

n^o	Niemeier	Embedding	Roots Orth.	Reducible fibers	MW
1	$E_8 \oplus D_{16}$	$D_8 \subset D_{16}$	$E_8 \oplus D_8$	$II^* + I_4^*$	$\{O\}$
2	$E_7^{\oplus 2} \oplus D_{10}$	$D_8 \subset D_{10}$	$E_7^{\oplus 2} \oplus A_1^{\oplus 2}$	$2III^* + 2I_2$	$\mathbb{Z}/2\mathbb{Z}$
3	D_{24}	$D_8 \subset D_{24}$	D_{16}	I_{12}^*	$\{O\}$
4	$D_{12}^{\oplus 2}$	$D_8 \subset D_{12}$	$D_{12} \oplus D_4$	$I_8^* + I_0^*$	$\mathbb{Z}/2\mathbb{Z}$
5	$D_8^{\oplus 3}$	$D_8 \subset D_8$	$D_8^{\oplus 2}$	$2I_4^*$	$\mathbb{Z}/2\mathbb{Z}$
6	$D_9 \oplus A_{15}$	$D_8 \subset D_9$	A_{15}	I_{16}	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

Table 4 Elliptic fibrations of X_2

n^o	Niemeier	Embedding	Roots Orth.	Reducible fibers	MW
1	$E_8^{\oplus 3}$	$E_8 \subset E_8$	$E_8^{\oplus 2}$	$2II^*$	$\{O\}$
2	$E_8 \oplus D_{16}$	$E_8 \subset E_8$	D_{16}	I_{12}^*	$\mathbb{Z}/2\mathbb{Z}$

η, and an upper bound for the degree over k of a field of definition $k_{\eta,\mathrm{MW}}$ of a set of generators of the Mordell–Weil group of the fibration.

By Proposition 5 we know that the Galois group G_{R_i} is trivial and all the fiber components of R_i are defined over k, for $i = 4, 3, 2$. In order to determine the field of definition of the sections, the only action that is taken into account is the one of the cover involutions τ_i, for $i = 4, 3, 2$.

To determine the type of each fibration in Table 3 (resp. Table 4) with respect to τ_4 (resp. τ_3 and τ_2), we find a configuration of (parts of the) reducible fibers in terms of the curves in Fig. 4 (resp. Figs. 5 and 6). The fibration in line 5 (resp. line 2 and line 1) is represented in Fig. 4 (resp. Figs. 5 and 6).

The configurations associated to the fibers in lines 1, 2, 3, 4, and 6 in Table 3 for the K3 surface X_4 are listed below:

$\underline{II^* + I_4^*}$
$\Theta_0^1, 2\Theta_2^1, 3\Theta_3^1, 4\Theta_4^1, 5\Theta_5^1, 6\Theta_6^1, 4\Theta_8^1, 2T_1, 3\Theta_7^1 \;\;+\;\; \Theta_0^2, \Theta_1^2, 2\Theta_2^2, 2\Theta_3^2, 2\Theta_4^2,$
$2\Theta_5^2, 2\Theta_6^2, \Theta_7^2,$

$\underline{2III^* + 2I_2}$
$\Theta_3^1, 2\Theta_4^1, 3\Theta_5^1, 4\Theta_6^1, 3\Theta_8^1, 2T_1, \Theta_8^2, 2\Theta_7^1 \;\;+\;\; \Theta_0^1, 2O_{X_4}, 3\Theta_0^2, 4\Theta_2^2, 3\Theta_3^2, 2\Theta_4^2, \Theta_5^2,$
$2\Theta_1^2 \;\;+\;\; \Theta_1^1 \;\;+\;\; \Theta_7^2,$

$\underline{I_{12}^*}$
$\Theta_7^1, \Theta_8^1, 2\Theta_6^1, 2\Theta_5^1, 2\Theta_4^1, 2\Theta_3^1, 2\Theta_2^1, 2\Theta_0^1, 2O_{X_4}, 2\Theta_0^2, 2\Theta_2^2, 2\Theta_3^2, 2\Theta_4^2, 2\Theta_5^2,$
$2\Theta_6^2, \Theta_8^2, \Theta_7^2,$

$\underline{I_8^* + I_0^*}$
$\Theta_8^1, \Theta_7^1, 2\Theta_6^1, 2\Theta_5^1, 2\Theta_4^1, 2\Theta_3^1, 2\Theta_2^1, 2\Theta_0^1, 2O_{X_4}, 2\Theta_0^2, 2\Theta_2^2, \Theta_1^2, \Theta_3^1 \;\;+\;\; \Theta_8^2, \Theta_7^2,$
$2\Theta_6^2, \Theta_5^2,$

$\underline{I_{16}}$
$\Theta_8^1, \Theta_6^1, \Theta_5^1, \Theta_4^1, \Theta_3^1, \Theta_2^1, \Theta_0^1, O_{X_4}, \Theta_0^2, \Theta_2^2, \Theta_3^2, \Theta_4^2, \Theta_5^2, \Theta_6^2, \Theta_8^2, T_1.$

The configurations associated to the fibers in lines 1, 3, 4, 5, and 6 in Table 3 for the K3 surface X_3 are listed below:

$\underline{II^* + I_4^*}$
$\Theta_0^2, 2O_{X_3}, 3\Theta_0^1, 4\Theta_1^1, 5\Theta_2^1, 6\Theta_3^1, 4\Theta_4^1, 2\Theta_5^1, 3\Theta_7^1$
$\qquad\qquad\qquad\qquad + \;\; \Phi_2^1, \Phi_2^2, 2T_1, 2\Theta_6^2, 2\Theta_5^2, 2\Theta_4^2, 2\Theta_3^2, \Theta_7^2, \Theta_2^2,$

$\underline{I_{12}^*}$
$\Phi_1^1, \Phi_1^2, 2O_{X_3}, 2\Theta_0^1, 2\Theta_1^1, 2\Theta_2^1, 2\Theta_3^1, 2\Theta_4^1, 2\Theta_5^1, 2\Theta_6^1, 2T_1, 2\Theta_6^2, 2\Theta_5^2, 2\Theta_4^2,$
$2\Theta_3^2, \Theta_2^2, \Theta_7^2,$

$I_8^* + I_0^*$

$\Theta_2^1, \Theta_7^1, 2\Theta_3^1, 2\Theta_4^1, 2\Theta_5^1, 2\Theta_6^1, 2T_1, 2\Theta_6^2, 2\Theta_5^2, 2\Theta_4^2, 2\Theta_3^2, \Theta_2^2, \Theta_7^2$
$+ \; \Theta_0^2, \Theta_0^1, 2O_{X_3}, \Phi_1^1, \Phi_1^2,$

$2I_4^*$

$\Phi_1^1, \Phi_1^2, 2O_{X_3}, 2\Theta_0^1, 2\Theta_1^1, 2\Theta_2^1, 2\Theta_3^1, \Theta_4^1, \Theta_7^1$
$+ \; \Theta_2^2, \Theta_7^2, 2\Theta_3^2, 2\Theta_4^2, 2\Theta_5^2, 2\Theta_6^2, 2T_1, \Theta_6^1,$

I_{16}

$\Theta_3^1, \Theta_4^1, \Theta_5^1, \Theta_6^1, T_1, \Theta_6^2, \Theta_5^2, \Theta_4^2, \Theta_3^2, \Theta_2^2, \Theta_1^2, \Theta_0^2, O_{X_3}, \Theta_0^1, \Theta_1^1, \Theta_2^1.$

Finally, in terms of the configuration of $2II^*$ we find a fiber of type I_{12}^* representing the fibration in line 2 of Table 4 by including every curve in Fig. 6 except Θ_7^1, Θ_7^2; the latter are sections for this fibration.

Note that all the reducible fibers listed above only appear once in Table 3 (resp. Table 4), hence we know that they represent the corresponding fibrations in those tables. Therefore, using these configurations, we can determine the type of the corresponding fibration with respect to τ_4 (resp. τ_3 and τ_2), and find sections for the corresponding fibration. By choosing a 0-section, we determine whether the different sections are fixed by τ_4 (resp. τ_3 and τ_2) or not. The results are listed in Table 5 (resp. Tables 6 and 7).

Table 5 Types of the different elliptic fibrations of X_4 with respect to τ_4 and fields of definition

n^o	Roots Orth.	Type	Sections	Field of Def. 0-section	Field of Def. all sections	$[k_{\eta,\mathrm{MW}} : k]$
1	$E_8 \oplus D_8$	3	O_{X_4}	O_{X_4}/k	k	1
2	$E_7^{\oplus 2} \oplus A_1^{\oplus 2}$	3	Θ_2^1, Θ_6^2	Θ_2^1/k_{τ_4}	k_{τ_4}	≤ 2
3	D_{16}	1	T_1	T_1/k	k	1
4	$D_{12} \oplus D_4$	3	T_1, Θ_4^2	T_1/k	k_{τ_4}	≤ 2
5	$D_8^{\oplus 2}$	2	O_{X_4}, T_1	O_{X_4}/k	k	1
6	A_{15}	1	$\Theta_1^1, \Theta_7^1, \Theta_1^2$	Θ_1^1/k_{τ_4}	k_{τ_4}	≤ 2

Table 6 Types of the different elliptic fibrations of X_3 with respect to τ_3 and fields of definition

n^o	Roots Orth.	Type	Sections	Field of Def. 0-section	Field of Def. all sections	$[k_{\eta,\mathrm{MW}} : k]$
1	$E_8 \oplus D_8$	3	Θ_1^2	Θ_1^2/k_{τ_3}	k_{τ_3}	≤ 2
2	$E_7^{\oplus 2} \oplus A_1^{\oplus 2}$	2	O_{X_3}, T_1	O_{X_3}/k	k	1
3	D_{16}	3	Θ_1^2	Θ_1^2/k_{τ_3}	k_{τ_3}	≤ 2
4	$D_{12} \oplus D_4$	1	Θ_1^1, Θ_1^2	Θ_1^2/k_{τ_3}	k_{τ_3}	≤ 2
5	$D_8^{\oplus 2}$	3	Θ_5^1, Θ_1^2	Θ_5^1/k_{τ_3}	k_{τ_3}	≤ 2
6	A_{15}	1	$\Theta_7^1, \Theta_7^2, \Phi_1^1,$	Θ_7^1/k_{τ_3}	k_{τ_3}	≤ 2

Table 7 Types of the different elliptic fibrations of X_2 with respect to τ_2 and fields of definition

n^o	Roots Orth.	Type	Sections	Field of Def. 0-section	Field of Def. all sections	$[k_{\eta,\mathrm{MW}} : k]$
1	$E_8^{\oplus 2}$	2	O	O/k	k	1
2	D_{16}	1	Θ_7^1, Θ_7^2	Θ_7^1/k_{τ_2}	k_{τ_2}	≤ 2

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