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A PROPAGATION OF SINGULARITIES THEOREM AND A WELL-POSEDNESS RESULT FOR THE KLEIN-GORDON EQUATION ON ASYMPTOTICALLY ANTI-DE SITTER SPACETIMES WITH GENERAL BOUNDARY CONDITIONS

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Introduction

The n-dimensional anti-de Sitter spacetime (AdS) is a maximally symmetric solution of the vacuum Einstein equations with a negative cosmological constant [Wal84]. A geometric feature of AdS spacetime is that its conformal boundary is a timelike submanifold, a feature which in particular entails that AdS spacetime is not globally hyperbolic in the usual boundaryless sense. From a PDE viewpoint, the main consequence is that initial value problems for hyperbolic partial differential equations on this background do not yield a unique solution, unless complemented by suitable boundary conditions assigned on the conformal boundary [BGP07, DFM18]. A natural extension of AdS spacetime is the class of the so-called asymptotically AdS spacetimes (aAdS), sharing the same behavior of AdS in a neighborhood of conformal infinity. Compared to the AdS case, on these backgrounds, the analysis of partial differential equations is more difficult due to the lack of isometries of the metric.

In the last two decades the Klein-Gordon equation on anti-de Sitter and asymptotically anti-de Sitter spacetimes was studied in several works [Bac11, DF16, DFM18, EK13, Gal10, Hol12, HS16b, HS16a, HW14, KW15, Vas12, Wro17, GW20, War12, KY09] using different methods. As for the asymptotically anti-de Sitter case, the results obtained include propagation of singularities theorems [Vas12, GW20], well-posedness results under several regularity hypotheses [Vas12, Wro17, GW20, War12, EK13] – in the last cited paper also a non-linear Klein-Gordon equation was considered – and the study of the properties of the propagators and of the associated fundamental solutions [Wro17, GW20]. However, the analysis in the aforementioned works is limited to the case of Dirichlet, Neumann and Robin boundary conditions, but these kind of boundary conditions are not the only interesting ones for the Klein-Gordon equation on aAdS spacetimes. Indeed, classical and quantum field theory on asymptotically anti-de Sitter spacetimes has been the target of significant attention in mathematical and theoretical physics, mainly because of the AdS/CFT correspondence, relating a field theory on the interior of the spacetime with a suitable dual one on its conformal boundary. Therefore it makes sense, in general, to consider also boundary conditions of dynamical type, like Wentzell boundary conditions which have been studied in [DFJA18, Zah18] on AdS spacetime. Dynamical boundary conditions have been considered also for other non-globally hyperbolic spacetimes, for example in the analysis of the Casimir effect [JAW21]. As shown in [GW20], in an asymptotically AdS spacetime M, given $u \in \mathcal{H}^1_{loc}(M)$, it is possible to define two trace maps γ_+ and γ_- the first encoding the Neumann data and the second the Dirichlet ones, see Sections 1.4.2 and 2.2. Robin boundary conditions can be imposed requiring that $\gamma_+ u = f\gamma_- u$ for a suitable smooth function f on ∂M. From the analysis carried out in [DFJA18], we also know that boundary conditions of Wentzell type can be imposed in this way, with the function f replaced by a suitable second order differential operator acting on the boundary. Using the notion of boundary triple, it has been shown in [DDF19] that there exists a large class of boundary conditions relating Neumann and Dirichlet data via pseudodifferential operators for which there exist advanced and retarded fundamental solutions for the Klein-Gordon operator. However, the approach followed in [DDF19] does not allow to establish any estimate on the wavefront set of the propagators, since the framework employed is not well-suited to prove a propagation of singularities theorem.

This thesis, which is based on [DM21b, DM21a], is devoted to the study of the Klein-Gordon equation on aAdS spacetimes with boundary conditions implemented by a suitable class of pseudodifferential operators. The core idea of this work is to use techniques proper of b-calculus, as advocated in [Vas08, War12, Wro17, GW20], in order to overcome the hurdle posed by the singular metric of asymptotically anti-de Sitter spacetimes, along with the ideas introduced in [DDF19]. As it will become manifest from our analysis, we can distinguish two notable cases, namely that of pseudodifferential operators of order $k \leq 0$ – which can be seen as a natural extension of the results of [GW20] – and the case of pseudodifferential operators of order $0 < k \leq 2$, which is a novel interesting case, including for example the aforementioned Wentzell boundary conditions. In particular, we prove two propagation of singularities theorems - Theorem 3.0.2 for $k \leq 0$ and Theorem 3.0.1 for $0 < k \leq 2$, with the latter taking into account the singularities introduced by the boundary conditions. The other main achievement of this work is the well-posedness result given by Proposition 4.1.2, generalizing the statements obtained in [Vas08, Wro17] for Dirichlet boundary conditions and in [GW20] for Robin ones to a very general class of boundary conditions implemented by pseudodifferential operators. In particular, we discuss the existence of advanced and retarded fundamental solutions for the Klein-Gordon operator with prescribed boundary conditions, see Theorem 4.2.1 and Remark 4.2.2, and we characterize the wavefront set of the advanced and of the retarded fundamental solutions. To obtain this result, we need to restrict the class of pseudodifferential operators encoding the boundary conditions, see Hypotesis 4.1.1. However, for applications, the hypotheses we assume are mild constraints, since the most interesting cases of boundary conditions are included. At last, as a concrete application of the theory we developed, we build the fundamental solutions for a massless Klein-Gordon equation on a static aAdS spacetime with admissible static boundary conditions, as per Definition 4.2.1, a result that we can see as a natural extension of the analysis carried out in [DDF19].

The structure of this work is the following. In the first chapter we introduce the analytic and geometric notions we need, in particular globally hyperbolic, asymptotically anti-de Sitter spacetimes, b-calculus and boundary triples. In the second chapter we give the weak formulation of the dynamical problem we are interested in. The third chapter is devoted to proving two propagation of singularities theorems, one for boundary conditions encoded by pseudodifferential operators of order $k \leq 0$ – Theorem 3.0.2 – and the other for the case $0 < k \leq 2$ – Theorem 3.0.1. Much of this chapter is focused on proving suitable microlocal estimates needed to prove the two propagation of singularities theorems. In the last chapter we establish a well-posedness result for the Klein-Gordon equation with boundary conditions implemented by a suitable class of pseudodifferential operators – Proposition 4.1.2 – and we study the existence and uniqueness of the propagators and of the fundamental solutions – Theorem 4.2.1 – proceeding then to characterize them and to give a concrete example of fundamental solutions for the massless Klein-Gordon equation.

Contents

Chapter 1

Geometric and Analytic Preliminaries

The goal of this chapter is to fix the notation and to introduce the analytic and geometric notions which play a pivotal rôle in the following, namely globally hyperbolic spacetimes with timelike boundary, manifolds of bounded geometry, b-calculus and boundary triples. We assume that the reader is acquainted with the basic notions of Lorentzian geometry, *e.g.* [O'N83].

1.1 Globally hyperbolic asymptotically anti-de Sitter spacetimes

In this section we introduce the class of Lorentzian manifolds we use in the following, namely globally hyperbolic asymptotically anti-de Sitter spacetimes. This is the class of backgrounds for which one can expect that a mixed initial/boundary value problem for partial differential equations ruled by a normally hyperbolic operator is well-posed [DDF19, DM21b].

1.1.1 Globally hyperbolic spacetimes with timelike boundary

We begin introducing a distinguished class of backgrounds, that of globally hyperbolic spacetimes with timelike boundary, following [HFS20] and [DM21b].

Definition 1.1.1. Let (M, g) be a connected, oriented, time oriented, smooth Lorentzian manifold of dimension $\dim M = n \geq 2$ with non-empty boundary $\iota : \partial M \to M$. We say that (M, g)

1. has a **timelike boundary** if $(\partial M, \iota^* g)$ is a smooth, Lorentzian manifold,
2. is **globally hyperbolic** if it does not contain any closed causal curve and if, for every $p, q \in M$, $J^+(p) \cap J^-(q)$ is either empty or compact. Here $J^\pm$ stand for the causal future (+) and past (-).

If both conditions are met, we say that (M, g) is a *globally hyperbolic spacetime with timelike boundary.*

For the sake of simplicity, in the following we assume that ∂M is connected. A key notion in globally hyperbolic spaces is that of Cauchy surface. Prior to recalling its definition, we remember that a subset $\Sigma \subset M$ of a Lorentzian manifold (M, g), is called *achronal* if $S \cap I^+(S) = \emptyset$, $I^+(S)$ being the chronological future of S.

Definition 1.1.2. Let (M, g) be a Lorentzian manifold. A Cauchy surface $\Sigma \subset M$ is an achronal subset of M such that every inextensible, piecewise smooth curve intersects Σ only once.

It is well known that a Cauchy problem for the Klein-Gordon equation is well posed on globally hyperbolic spacetimes without timelike boundary [BGP07, CB08] . Indeed, the following theorem holds true for normally hyperbolic operators – a class of linear second order scalar operators whose principal symbol is constructed only out of the metric, namely P is normally hyperbolic if $\sigma(P)(\xi) = -g^{-1}(\xi, \xi)$ for every $\xi \in T^*M$.

Theorem 1.1.1 ([BGP07], Thm. 3.2.11)**.** Let (M, g) be a globally hyperbolic spacetime with empty boundary and let $\Sigma \subset \mathcal{M}$ be any of its spacelike Cauchy surfaces, together with its future pointing unit normal vector $\mathbf{n}$. Consider a normally hyperbolic operator $P : C^\infty(M) \longrightarrow C^\infty(M)$ and a covariant derivative $\bigtriangledown$ on M such that $\Box_{\bigtriangledown} + A = P$, with $A \in C^\infty(M)$. Then, for all $f \in C^\infty(M)$ and $u_0, u_1 \in C^\infty(\Sigma)$, the problem

$$\begin{cases} Pu = f & \textit{on } M \\ u = u_0 & \textit{on } \Sigma \\ \bigtriangledown_{\mathbf{n}} u = u_1 & \textit{on } \Sigma \end{cases} \tag{1.1}$$

admits a unique solution $u \in C^\infty(M)$. Furthermore, if we set $\Omega = supp(u_0) \cup supp(u_1) \cup supp(f)$, then $supp(u) \subset J_M(\Omega)$, with $J(\Omega) := J^+(\Omega) \cup J^(\Omega)$.

As a byproduct of this theorem, the Green functions and the fundamental solutions exist and are unique [BGP07, Thm. 3.3.1, Prop. 3.4.2].

However, there are other interesting cases in which the underlying manifold has a timelike boundary, one example being the conformal compactification of anti-de Sitter spacetime. A complete characterization of globally hyperbolic spacetimes with timelike boundary is given by the following theorem.

Theorem 1.1.1. *[[HFS20], Thm. 1.1] Let (M, g) be a globally hyperbolic spacetime with timelike boundary of dimension $\dim M = n \geq 2$. Then it is isometric to a Cartesian product $\mathbb{R} \times \Sigma$ where Σ is an $(n-1)$-dimensional Riemannian manifold. The associated line element reads*

$$ds^2 = -\beta d\tau^2 + \kappa_\tau,$$

where $\beta \in C^\infty(\mathbb{R} \times \Sigma; (0, \infty))$ while $\tau : \mathbb{R} \times \Sigma \to \mathbb{R}$ plays the rôle of a time coordinate. In addition $\mathbb{R} \ni \tau \mapsto \kappa_\tau$ is a family of Riemmannian metrics, smoothly dependent on τ and such that, calling $\Sigma_\tau \doteq \{\tau\} \times \Sigma$, each $(\Sigma_\tau, \kappa_\tau)$ is a Cauchy surface with non-empty boundary.

Remark 1.1.1. An important consequence of this theorem is that, calling $\iota_{\partial M} : \partial M \to M$ the natural embedding map, then $(\partial M, h)$ where $h = \iota^*_{\partial M} g$ is a globally hyperbolic spacetime. In particular the associated line element reads

$$ds^2|_{\partial M} = -\beta|_{\partial M} d\tau^2 + \kappa_\tau|_{\partial M}.$$

On globally hyperbolic spacetime with timelike boundary, one can prove that the fundamental solutions and the Green's operators of a Cauchy problem exist – but in general they are not unique. This holds true if the manifold M can be embedded in a suitable way as an open subset of another globally hyperbolic spacetime without timelike boundary on which the Klein-Gordon operator can be extended [BGP07]. We shall see an example in the next section, in which we introduce the prototype of a globally hyperbolic spacetime with timelike boundary, namely AdS spacetime. In general, to obtain a well-posed Cauchy problem for the wave equation on a globally hyperbolic spacetime with timelike boundary, we need to complement the Cauchy data with suitable boundary conditions [DDF19, DFM18, DFJA18, GM21]. For example, in quantum field theory, the Casimir effect between uncharged conductive plates in vacuum can be studied introducing timelike boundaries in correspondence of the plates, on which we assign the boundary conditions [JAW21]. The well posedness of mixed Cauchy/boundary value problems for the wave equation on generic globally hyperbolic spacetimes with timelike boundary was investigated recently in [DDF19] – for the case of static spacetimes – and in [GM21]. Both papers consider the cases of Dirichlet, Neumann and Robin boundary conditions. The former also investigates the case of boundary conditions of Wentzell type, which are also known as dynamical boundary conditions, whereby the boundary data solves a non-homogeneous, boundary Klein-Gordon equation, with the source term fixed by the normal derivative of the scalar field at the boundary [Uen73, DFJA18]. The same class of problems for the Klein-Gordon equation on asymptotically anti-de Sitter spacetimes – see Section 1.1.3 – was studied in [Vas12, Hol12, War12, Wro17] – for Dirichlet and Neumann boundary conditions – and in [GW20] for the case of Robin boundary conditions.

1.1.2 Anti-de Sitter spacetime

The prototype of a globally hyperbolic spacetime with timelike boundary is the conformal compactification of $PAdS$, the Poicarè patch of anti-de Sitter (AdS) spacetime, considered as a manifold of its own. AdS is a maximally symmetric solution to the vacuum Einstein's equations with a negative cosmological constant [CB08]. Anti-de Sitter space is well known for its rôle in the AdS/CFT correspondence [Wit98, Mal98].

The n-dimensional anti-de Sitter spacetime AdS can be realized as an isometric embedding in $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\tilde{\eta} = dX_0^2 + dX_1^2 - \sum_{j=2}^{n-1} dX_j^2 \tag{1.2}$$

where $(X_0, \cdots, X_{n+1})$ are the standard Cartesian coordinates in $\mathbb{R}^{n+1}$. AdS_n is the set of points $X \in \mathbb{R}^{n+1}$ such that

$$\tilde{\eta}^{\mu\nu} X_\mu X_\nu = l^2. \tag{1.3}$$

where $l \neq 0$ is a real number related to the dimension n and to the cosmological constant Λ by

$$l^2 = -\frac{n(n+1)}{\Lambda} \tag{1.4}$$

Equation (1.3) identifies a hyperboloid in $\mathbb{R}^{n+1}$. The symmetry group of AdS is the Lorentz group $SO(2, n-1)$, the group of linear transformations of $\mathbb{R}^{n+1}$ preserving the quadratic form (1.3). A commonly employed global chart of AdS is given by the following parametrization of the hyperboloid in Equation (1.3) [DFM18]

$$\begin{cases} X_0 = l\ cosh(\rho)\ cos(\tau) \\ X_1 = l\ cosh(\rho)\ sin(\tau) \\ X_i = l\ sinh(\rho)\ e_i\ , \quad i = 2, \cdots, n \end{cases} \tag{1.5}$$

where $\tau \in (0, 2\pi)$, $\rho \in \mathbb{R}^+$ and $e_i = e_i(\theta, \phi_1, \cdots, \phi_{n-3})$ parametrizes a point on S^{n-2} in terms of the angular coordinates. In this global chart, the metric of AdS reads

$$g = l^2[cosh^2(\tau)d\tau^2 - d\rho^2 - sinh^2(\rho)d\Omega^{n-2}] \tag{1.6}$$

where $d\Omega^{n-2}$ is the metric of S^{n-2}. Now we show that a Cauchy problem in AdS is not well posed, following [BGP07]. AdS is conformally related to the Einstein static universe (ESU) which is a globally hyperbolic spacetime with empty boundary. The n-dimensional ESU can be realized as the n-dimensional cylinder

$$(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = K, \quad K \in \mathbb{R}^+$$

in $\mathbb{R}^n_1$ equipped with the metric $ds^2 = -(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2$. AdS admits closed timelike curves, therefore it is not globally hyperbolic. Let $\mathbb{R} \times S^{n-1}_+$ and $\mathbb{R} \times S^{n-1}_-$ be the two subsets of ESU with $x_n > 0$ and $x_n < 0$ respectively. One can prove that the conformal compactification of AdS is diffeomorphic to $\mathbb{R} \times S^{n-1}_+$ [BGP07]. The non-uniqueness of the fundamental solutions is better seen using the notion of fundamental solution at a point.

Definition 1.1.3. Let M be a time-oriented Lorentzian manifold. A fundamental solution of a normally hyperbolic operator $P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ at a point $p \in M$ is a distribution $F \in \mathcal{D}'(M)$ such that $PF = \delta_p$. In other words, for all $\phi \in \mathcal{D}(M)$ it holds $F[P^*\phi] = \phi(p)$. If $supp(F) \subset J^\pm(p)$, then we call F an advanced (+) or retarded (−) fundamental solution at p.

Consider a point p in the conformal embedding of AdS into ESU – the upper half of the cylinder in Figure 1.1 – and another point q in $ESU \setminus AdS$ – the lower half of the cylinder.

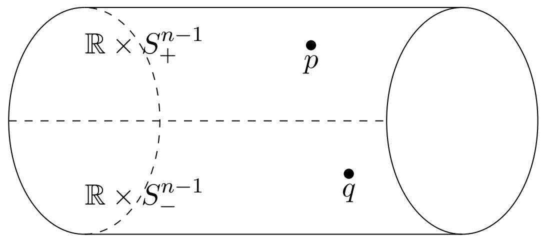


Figure 1.1: Einstein cylinder. AdS can be conformally embedded in the upper half of the cylinder.

Let $F_+(p)$ and $F_+(q)$ be the advanced fundamental solutions at p and q respectively. Then both $F_+(p)$ and $F_+(p) + F_+(q)$ are advanced fundamental solutions in p. Indeed, for every test function ϕ in $\mathcal{C}^\infty(AdS)$ it holds $F_+(p)[P^*\phi] + F_+(q)[P^*\phi] = \phi(p) + \phi(q)$. Since $supp(\phi) \subseteq AdS$, we have $\phi(q) = 0$ and thus $F_+(p) + F_+(q)$ is still an advanced fundamental solution at $p \in AdS$.

Often in the literature, instead of using a global chart of AdS, it is employed a local chart covering half of anti-de Sitter spacetime called Poincaré patch of AdS, $PAdS$ for short, which sometimes is considered as a manifold of its own. $PAdS$ coordinates $x \in \mathbb{R}^+, y_i \in \mathbb{R}$ are defined as [DFJA18]

$$\begin{cases} X_0 = \dfrac{l}{x} y_0 \\ X_1 = \dfrac{x}{2}\left[1 + \dfrac{1}{x^2}\left(-y_0^2 + \sum_{i=1}^{n-1} y_i^2 + l^2\right)\right] \\ X_i = \dfrac{l}{x} y_{i-1} \ , \ i = 2, \cdots, n \\ X_{d+1} = \dfrac{x}{2}\left[1 + \dfrac{1}{x^2}\left(-y_0^2 + \sum_{i=1}^{n-1} y_i^2 - l^2\right)\right] \end{cases} \tag{1.7}$$

with $X_0, \cdots, X_{n+1}$ the Cartesian coordinates of the embedding space $\mathbb{R}^{n+1}$. In this chart, the metric reads

$$g_{PAdS} = \frac{l^2}{x^2}\left(-dy_0^2 + dx^2 + \sum_{i=i}^{n-1} dy_i^2\right) \tag{1.8}$$

showing that $PAdS$ is conformal to the interior of the n-dimensional half-Minkowski spacetime $(\mathring{\mathbb{H}}^n, \eta_n)$ with $\eta_n = x^2 g_{PAdS}$ and that the conformal (time-like) boundary of $PAdS$ can be attached at $x = 0$.

1.1.3 Asymptotically anti-de Sitter spacetimes

We end this section introducing a class of spacetimes sharing with AdS their behavior at the conformal boundary and already considered in [GW20, Vas12]. The following definition of asymptotically anti-de Sitter spacetime, whose constraints on the metric are reminiscent of the metric of $PAdS$, was first given in [GW20].

Definition 1.1.4. Let M be an n-dimensional manifold with non-empty boundary ∂M. Suppose that $\mathring{M} = M \setminus \partial M$ is equipped with a smooth Lorentzian metric g and that

a) If $x \in \mathcal{C}^\infty(M)$ is a boundary function, then $\widehat{g} = x^2 g$ extends smoothly to a Lorentzian metric on M.

b) The pullback $h = \iota^*_{\partial M}\widehat{g}$ via the natural embedding map $\iota_{\partial M} : \partial M \to M$ individuates a smooth Lorentzian metric.

c) $\widehat{g}^{-1}(dx, dx) = 1$ on ∂M.

Then (M, g) is called an n-dimensional *asymptotically anti-de Sitter (AdS) spacetime*. In addition, if $(M, \widehat{g})$ is a globally hyperbolic spacetime with timelike boundary, *cf.* Definition 1.1.1, then we call (M, g) a n-dimensional *globally hyperbolic asymptotically AdS spacetime*.

Conditions a), b) and c) are determined up to a conformal multiple, since we have the freedom to multiply the boundary function x by any nowhere vanishing $\Omega \in C^\infty(M)$. Such freedom plays no rôle in our investigation and we do not consider it further.

Remark 1.1.2. The class of asymptotically AdS spacetimes we introduced in Definition 1.1.4 is more general than the one commonly employed in theoretical physics, given in [AD00]. The differences are that h_x in Equation (1.9) is not required to be an Einstein metric and ∂M does not need to be diffeomorphic to $\mathbb{R} \times \mathbb{S}^{n-2}$.

Given a point $p \in \partial M$, applying the collar neighbourhood theorem and with a convenient choice of the boundary function x – remember that in Definition 1.1.4 there is the freedom to multiply the original boundary function by a smooth nowhere vanishing one – we can find a neighbourhood $U \subset \partial M$ containing p and $\varepsilon > 0$ such that on $U \times [0, \varepsilon)$ the metric reads

$$g = \frac{-dx^2 + h_x}{x^2} \tag{1.9}$$

with h_x a family of Lorentzian metrics depending smoothly on x and such that $h_0 \equiv h$ [GW20].

Remark 1.1.3. From now on, with a slight abuse of notation, we denote with x both the boundary function of an asymptotically AdS spacetime (M, g) and the coordinate normal to ∂M.

1.2 Manifolds of bounded geometry

In this section we briefly introduce the notion of manifold of bounded geometry, needed to define Sobolev spaces in the case of a manifold with non-empty boundary. For simplicity we give an extrinsic definition using an embedding in another manifold without boundary, as in [DDF19]. One can also give an intrinsic definition not requiring any extrinsic data, see for example [Sch01]. We begin giving the definition of manifold of bounded geometry in the case of a manifold without boundary.

Definition 1.2.1. A Riemannian manifold (N, h) with empty boundary is of bounded geometry if

a) The injectivity radius $r_{inj}(N)$ is strictly positive,

b) N is of totally bounded curvature, namely for all $k \in \mathbb{N} \cup \{0\}$ there exists a constant $C_k > 0$ such that $\|\bigtriangledown^k R\|_{L^\infty(M)} < C_k$.

In order to extend the definition of manifold of bounded geometry to manifolds with boundary, we need a preliminary notion.

Definition 1.2.2. Let (N, h) be a Riemannian manifold of bounded geometry and let (Y, ι_Y) be a codimension $k = 1$, closed, embedded smooth submanifold with an inward pointing, unit normal vector field ν_Y. The submanifold $(Y, \iota_Y^* g)$ is of bounded geometry if

a) The second fundamental form II of Y in N and all its covariant derivatives along Y are bounded,

b) There exists $\varepsilon_Y > 0$ such that the map $\phi_{\nu_Y} : Y \times (-\varepsilon_Y, \varepsilon_Y) \to N$ defined as $(x, z) \mapsto \phi_{\nu_Y}(x, z) \doteq exp_x(z\nu_{Y,x})$ is injective.

The map ϕ_{ν_Y} introduced in Definition 1.2.2 is called *normal exponential map* and sometimes it is denoted $exp^\perp$. Now we are ready to introduce the class of Riemannian manifolds with boundary and of bounded geometry.

Definition 1.2.3. Let (N, h) be a Riemannian manifold with $\partial N \neq \emptyset$. We say that (N, h) is of bounded geometry if there exists a Riemannian manifold of bounded geometry (N', h') of the same dimension as N such that

a) $N \subset N'$ and $h = h'|_N$

b) $(\partial N, \iota^* h')$ is a bounded geometry submanifold of N', where $\iota : \partial N \to N'$ is the embedding map.

Remark 1.2.1. One can prove that Definition 1.2.3 is independent from the choice of N'. An equivalent intrinsic definition which does not require to introduce the manifold N' is given in [Sch01].

Before introducing Sobolev spaces on a Riemannian manifold (N, h) with boundary and of bounded geometry, we need to recall the main results of [ANN19, Sec. 2.4]. In the following, we denote with $r_{inj}(N)$ and $r_{inj}(\partial N)$, the injectivity radius of N and ∂N respectively while $\delta > 0$ is such that the normal exponential map $exp^{\perp} : \partial N \times [0, \delta) \to N$ is injective. With these data let

$$\begin{cases} k_p : & B_r^{n-1}(0) \times [0, r) \to N \quad \text{\textit{if}} \;\; p \in \partial N \\ & (x, t) \mapsto exp^{\perp}(exp_p^{\partial N}(x), t) \\ k_p : & B_r^n(0) \to N \qquad\qquad\quad \text{\textit{if}} \;\; p \in \mathring{N} \\ & v \mapsto exp_p^N(v) \end{cases}, \tag{1.10}$$

where we are implicitly identifying $T_p \partial N$ with $\mathbb{R}^{n-1}$, whenever $p \in \partial N$. In addition we introduce the sets

$$U_p(r) \doteq \begin{cases} k_p(B_r^{n-1}(0) \times [0, r)) \subset N & \text{\textit{if}} \;\; p \in \partial N \\ k_p(B_r^n(0)) & \text{\textit{if}} \;\; p \in \mathring{N} \end{cases} \tag{1.11}$$

where $r < \mathtt{min}\left\{\frac{1}{2} r_{inj}(N), \frac{1}{4} r_{inj}(\partial N), \frac{1}{2} r_\delta\right\}$.

Definition 1.2.4. Let (N, h) be a Riemannian manifold with boundary and of bounded geometry of dimension $\dim N = n$. Let

$$r < \mathtt{min}\left\{\frac{1}{2} r_{inj}(N), \frac{1}{4} r_{inj}(\partial N), \frac{\delta}{2}\right\}$$

For each $p \in \partial N$, we call Fermi coordinate chart the map $k_p : B_r^{n-1}(0) \times [0, r) \to W_p(r)$ with associated coordinates $(x, z) : U_p(r) \to \mathbb{R}^{n-1} \times [0, \infty)$.

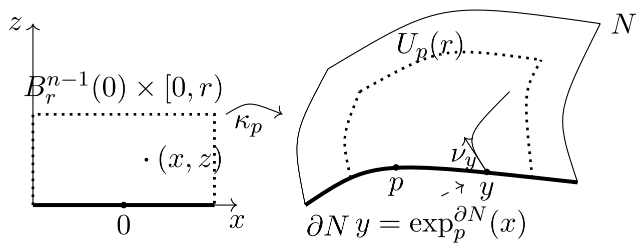


Figure 1.2: Fermi coordinates introduced in Definition 1.2.4

In view of Equations (1.10) and (1.11), if $p \in \mathring{N}$, we can always consider geodesic neighborhoods not intersecting ∂N and endowed with normal coordinates. This allows us to introduce a distinguished covering, defined as follows.

Definition 1.2.5. Let (N, h) be a Riemannian manifold with boundary and of bounded geometry. Let $0 < r < \mathtt{min}\left\{\frac{1}{2}r_{inj}(N), \frac{1}{4}r_{inj}(\partial N), \frac{\delta}{2}\right\}$. A subset $\{p_\gamma\}_{\gamma \in I}$, $i \subseteq \mathbb{N}$, is an *r-covering subset* of N if

a) For each $R > 0$, there exists $K_R \in \mathbb{N}$ such that, for each $p \in N$, the set $\{\gamma \in I \mid dist(p_\gamma, p) < R\}$ has at most K_R elements.

b) For each $\gamma \in I$, we have either $p_\gamma \in \partial N$ or $dist(p_\gamma, \partial N) \geq r$.

c) $N \subset \bigcup_{\gamma \in I} U_{p_\gamma}(r)$, *cf.* Equation (1.11).

To define Sobolev spaces over manifolds of bounded geometry, we make use of Sobolev spaces over r-covering subsets. In order to obtain a global definition, we glue together these local Sobolev spaces using a partition of unity which is compatible with a given r-covering set.

Definition 1.2.6. Under the same assumptions as in Definition 1.2.5, a partition of unity $\{\phi_\gamma\}_{\gamma \in I}$ of N is called an r-uniform partition of unity associated with the r-covering set $\{p_\gamma\}$ if

a) The support of each ϕ_γ is contained in U_{p_γ}, *cf.* Equation (1.11),

b) For each multi-index α, there exists $C_\alpha > 0$ such that $|\partial^\alpha \phi_\gamma| \leq C_\alpha$ for all $\gamma \in I$. Here the derivatives ∂^α are computed either in the normal geodesic or in the Fermi coordinates on U_{p_γ} depending on whether p lies in $\mathring{N} = N \setminus \partial N$ or in ∂N.

Now we have all ingredients we need to define Sobolev spaces on a Riemannian manifold (N, h) with boundary and of bounded geometry. Let $\{\phi_\gamma\}$

be a uniform partition of unity associated with the r-covering set p_γ as per Definition 1.2.6. For every $k \in \mathbb{N}$ we call k-th Sobolev space, $H^k(N)$, the collection of all distributions $u \in \mathcal{D}'(N)$ such that

$$\|u\|^2_{H^k(N)} \doteq \sum_\gamma \|(\phi_{p_\gamma} u) \circ k_{p_\gamma}\|^2_{H^k} < \infty \tag{1.12}$$

where $\|\cdot\|_{H^k}$ is the standard Sobolev space norm either on $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Remark 1.2.2. As in the case of a manifold without boundary [GS13], it turns out that, regardless of the chosen r-covering and of the associated r-uniform partition of unity, $H^k(N)$ is equivalent to $W^{2,k}(N)$ which is the completion of

$$\mathcal{E}^k(N) \doteq \{f \in C^\infty(N) \mid f, \nabla f, \dots, (\nabla)^k f \in L^2(N)\},$$

with respect to the norm

$$\|f\|_{W^{2,k}(N)} = \left(\sum_{i=0}^k \|(\nabla)^i f\|_{L^2(N)}\right)^{\frac{1}{2}}.$$

Here ∇ is the covariant derivative built out of the Riemannian metric h, while $(\nabla)^i$ indicates the i-th covariant derivative. This notation is employed to disambiguate with $\nabla^i = h^{ij}\nabla_j$.

The previous analysis can be extended to the case of Lorentzian manifolds as well. For the sake of simplicity we focus on the case without boundary. Following [GOW17] we start from (N, h) a Riemannian manifold of bounded geometry such that $\dim N = n$.
In addition we call $BT^m_{m'}(B_n(0, \frac{r_{inj}(N)}{2}), \delta_E)$, the space of all bounded tensors on the ball $B_n(0, \frac{r_{inj}(N)}{2})$ centered at the origin of the Euclidean space $(\mathbb{R}^n, \delta_E)$ where δ_E stands for the flat metric. For every $m, m' \in \mathbb{N} \cup \{0\}$, we denote with $BT^m_{m'}(N)$ the space of all rank (m, m') tensors T on N such that, for any $p \in M$, calling $T_p \doteq (\exp_p \circ e_p)^* T$ where $e_p : (\mathbb{R}^n, \delta) \to (T_pN, h_p)$ is a linear isometry, the family $\{T_p\}_{p\in M}$ is bounded on $BT^m_{m'}(B_n(0, \frac{r_{inj}(N)}{2}), \delta_E)$.

Definition 1.2.7. A smooth Lorentzian manifold (M, g) is of bounded geometry if there exists a Riemannian metric $\widehat{g}$ on M such that:

a) $(M, \widehat{g})$ is of bounded geometry.

b) $g \in BT^0_2(M, \widehat{g})$ and $g^{-1} \in BT^2_0(M, \widehat{g})$.

From now on all manifolds we will consider shall be of bounded geometry. This hypothesis is vital every time we need to invoke a partition of unity argument.

Remark 1.2.3. A spacetime whose underlying manifold is of bounded geometry cannot contain singularities, due to the regularity assumptions on the metric and on the scalar curvature.

The reader interested in manifolds of bounded geometry can find more details in [Sch01, ANN19, GS13, GOW17, DDF19].

1.3 b-calculus

To prove the well-posedness of a mixed boundary value/Cauchy problem on asymptotically anti-de Sitter spacetimes for a very general class of boundary conditions implemented by pseudodifferential operators, we make use of boundary calculus, following the road first paved by [Vas12]. In this section we define b-pseudodifferential operators using a Schwartz kernel approach, following [Gri01] and [Mel93]. Before discussing together b-pseudodifferential operators, we need to introduce some basic notions of b-geometry.

1.3.1 Introduction to b-geometry

b-geometry was fist introduced by R. Melrose and P. Piazza in [Mel93, MP92] as a framework to study differential calculus and differential operators on manifolds with boundary and it has been used by many authors in different context, e.g. [Vas08, VGP14, APG17, GW20]. In this section we focus on the construction of b-geometry in the case of a manifold with non-empty boundary. The readers interested in the general case, in which the underlying manifold is a manifold with corners, can refer to [MP92]. In this section M denotes a connected, orientable, smooth manifold of dimension $\dim M = n \geq 2$, with boundary ∂M and we call $\iota : \partial M \hookrightarrow M$ the natural embedding map. The main object in b-geometry is the space of vectors of M tangent to the boundary ∂M, defined as

$$\mathcal{V}_b(M) \doteq \{X \in \Gamma(TM) \mid X|_{\partial M} \in \Gamma(T\partial M)\} \tag{1.13}$$

This space can be characterized as the collection of the sections of a vector bundle as follows. First, we note that for any open set $U \subset M$ such that $U \cap \partial M = \emptyset$, then $\mathcal{V}_b(M)$ coincides with $\Gamma(TM)$. Consider now an open subset U whose intersection with the boundary is non-empty. By the tubular neighborhood theorem there is an open neighborhood U_ε of $\iota(\partial M \cap U)$ which is diffeomorphic to $[0, \varepsilon) \times \iota(\partial M \cap U)$. Calling x the coordinate of the projection on the first factor, then any element X of $\mathcal{V}_b(U)$ can be written as

$$X|_{U_\varepsilon} = f\frac{\partial}{\partial x} + Y_x$$

where f is a smooth real-valued function over U such that $f|_{\partial M \cap U_\varepsilon} = 0$ and $Y_x \in \Gamma(TU_\varepsilon)$ is a family of vector fields depending smoothly on x such that $Y_0 \in \Gamma(T(\partial M \cap U_\varepsilon))$. Since f is a smooth function vanishing at the boundary, we can write $f = x\alpha$ with $\alpha \in \mathcal{C}^\infty(U_\varepsilon)$. Therefore the vector field X takes the form

$$X|_{U_\varepsilon} = \alpha\left(x\frac{\partial}{\partial x}\right) + Y_x$$

from which we can see that it is convenient to consider, in addition to the usual basis of $T(M \cap U_\varepsilon)$, also $x\frac{\partial}{\partial x}|_p$ as a basis vector at a point $p \in U_\varepsilon$. These observations suggest to introduce a new bundle, called the b-tangent bundle ${}^bTM$, whose base space is M and whose fiber over a point p is defined as follows

$$ {}^bT_pM := \begin{cases} T_pM & \text{if } p \notin U_\varepsilon \\ , span_{\mathbb{R}}\left\{x\frac{\partial}{\partial x}, T_pU_{\partial M}\right\} & \text{if } p \in U_\varepsilon. \end{cases} \tag{1.14}$$

Remark 1.3.1. The definition of ${}^bT_pM$ given above does not depend on the choice of ε. Indeed, if a point p does not lie in $\iota(\partial M)$, then we can find an open neighborhood U_p not intersecting the boundary such that ${}^bT_pM|_{U_p}$ is diffeomorphic to $TM|_{U_p}$.

Note that the restriction map

$$\pi : \mathcal{V}_b(M) \to \Gamma(T\partial M) \quad X \mapsto X|_{\partial M}$$

is not injective, as on the boundary it associates any b-vector of the form $ax\partial_x$, $a \in \mathcal{C}^\infty(\partial M)$, to the zero section of TM. The dual of ${}^bTM$ is ${}^bT^*M$, the b-cotangent bundle. Proceeding as before, making use of the tubular neighborhood theorem, one can show that for every $p \in \mathring{M}$, ${}^bT^*_pM$ coincides with T^*_pM, while for a point $p \in \partial M$, we have ${}^bT^*_pM = span_{\mathbb{R}}\left\{T^*_p\partial M, \frac{dx}{x}\right\}$. We also observe that there is a natural non injective map $\pi : T^*M \to {}^bT^*M$, built as follows. Consider a tubular neighbourhood of ∂M and a chart U centered at point $p \in \partial M$, inducing the local trivializations of T^*M and ${}^bT^*_pM$ whose local coordinates are (x, y_i, ξ, η_i) and (x, y_i, ζ, η_i) respectively. In these coordinates the action of the map π is:

$$\pi(x, y_i, \xi, \eta_i) = (x, y_i, x\xi, \eta_i)$$

In particular, since $p \in \partial M$, we have $x = 0$ and therefore $\pi(0, y_i, \xi, \eta_i) = (0, y_i, 0, \eta_i)$ for every $\xi \in \mathbb{R}$. If we consider, instead, a chart U' centered at a point $q \in \mathring{M}$, the map π is a diffeomorphism. We call *compressed b-cotangent bundle*

$$ {}^b\dot{T}^*M \doteq \pi[T^*M], \tag{1.15} $$

which is a subset of ${}^bT^*M$. This space will play a pivotal rôle in the following, when we shall define the b-geometry analogue of characteristic set of a PDE. Further details can be found in [Mel81, Vas12]. The last geometric structure that we shall need in this work is the b-cosphere bundle which is realized as the quotient manifold obtained via the action of the dilation group on $T_b^*M \setminus \{0\}$, namely

$$ {}^bS^*M \doteq {}^bT^*M \setminus \{0\} \,/\mathbb{R}^+. \tag{1.16} $$

We remark that, if we consider a local chart $U \subset M$ such that $U \cap \partial M \neq \emptyset$ and the local coordinates (x, y_i, ζ, η_i), $i = 1, \ldots, n-1 = \dim \partial M$, on ${}^bT^*_UM \doteq {}^bT^*M|_U$, we can build a natural counterpart on ${}^bS^*_UM$, namely $(x, y_i, \widehat{\zeta}, \widehat{\eta}_i)$ where $\widehat{\zeta} = \frac{\zeta}{\rho}$ and $\widehat{\eta}_i = \frac{\eta_i}{\rho}$ with $\rho = |\eta_{n-1}|$. The b-cosphere bundle is useful to prove some estimates we need in order to obtain a propagation of singularities theorem for the Klein-Gordon operator with general boundary conditions.

1.3.2 b-differential operators

Using b-vector fields in $\mathcal{V}_b(M)$, we can build b-differential operators in the same way we build differential operators using ordinary vector fields. The space $\mathit{Diff}_b^k(M)$ is the space of *b-differential operators of order* $k \in \mathbb{N}$, namely the space of the linear maps $P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ given by a finite sum up to k-fold products of elements of $\mathcal{V}_b(M)$ and $\mathcal{C}^\infty(M)$. In local coordinates (x, y) of M, with $y = y_1, \cdots, y_n$, we can write a b-differential operator P as

$$ P = \sum_{|\alpha| \leq k} a_\alpha(x, y) \left(x\frac{\partial}{\partial x} \right)^{\alpha_0} \left(\frac{\partial}{\partial y_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial y_{n-1}} \right)^{\alpha_{n-1}} \tag{1.17} $$

where $\alpha = (\alpha_0, \cdots, \alpha_{n-1})$ is a multi-index and $\{a_\alpha\}_{|\alpha| \leq k}$ are smooth functions up to the boundary. The *b-principal symbol* is the polynomial

$$ {}^b\sigma_k(P) = \sum_{|\alpha| = k} a_\alpha(x, y) \tilde{\zeta}^{\alpha_0} \eta_1^{\alpha_1} \cdots \eta_{n-1}^{\alpha_{n-1}} \tag{1.18} $$

with $(x, y, \tilde{\zeta}, \eta)$ local coordinates on ${}^bT^*M$. As we will see, the microlocalization of $\mathit{Diff}_b^k(M)$ is $\Psi_b^k(M)$, the algebra of the properly supported b-pseudo differential operators of order k. A classical pseudodifferential operator on a manifold M without boundary is an operator whose integral kernel over $M \times M$ conormal with respect to the diagonal $\Delta = \{(p,p) \in M \times M \mid p \in M\}$ and such that the principal symbol is determined by the singular behavior at Δ. However, we cannot slavishly extend this definition to b-pseudodifferential operators on manifolds with boundary, because in addition to the singularities on the interior of the diagonal $diag(\mathring{M} \times \mathring{M})$, we may encounter also singularities in the corners of $M \times M$. For simplicity in the following we focus on the case $M = \mathbb{R}_+$, which will be enough for our purposes, since the corner of $\mathbb{R}_+ \times \mathbb{R}_+$ is the only one we encounter in integral kernels over $aAdS$ spacetimes. We need to give a precise description of the singular behaviour approaching the interior of the diagonal Δ, at the boundary $\partial(\mathbb{R}_+ \times \mathbb{R}_+) \setminus (0,0)$ and at the corner $(0,0)$. To overcome this hurdle, we introduce the notion of blow-up, which allows to deal with the singularities of integral kernels at the boundaries and at the corners.

1.3.3 Polyhomogeneous conormal functions and blow-ups

In this section we introduce the notion of blow-up of a manifold M with boundary. As a motivation for the need of b-geometry and blow-ups we begin with the study of polyhomogeneous conormal functions, a class of functions which can diverge at the boundary of M as a monomial or as a logarithm. To give a precise description of the singularities, it is convenient to introduce the notion of index set.

Definition 1.3.1. An index set is a discrete subset $F \subset \mathbb{C} \times \mathbb{N}_0$ such that:

a) $F \cap \{(z,p) \in \mathbb{C} \times \mathbb{N}_0 \mid Re(z) < N,\, N \in \mathbb{R}\}$ is a finite set.

b) If $(z,p) \in F$ and $p \leq q$, then $(z,q) \in F$.

Now we can give a precise definition of a polyhomogeneous conormal function.

Definition 1.3.2. Let M be a smooth manifold with boundary ∂M and let F be an index set. A smooth function $u \in \mathring{M}$ is called polyhomogeneous conormal function with respect to F if, on a tubular neighborhood $[0,\varepsilon) \times \partial M$, u satisfies the following asymptotic expansion:

$$u(x,y) \sim \sum_{(z,p)\in F} a_{z,p}(y) x^z log^p(x) \text{ as } x \to 0^+$$

with $a_{z,p} \in \mathcal{C}^\infty(\partial M)$.

We followed the same notation of the previous section denoting with x the boundary coordinate. We have written $y = y_1, \dots, y_n$ to denote the other coordinates collectively. We are interested in studying the singularities of the integral kernel of an operator, namely we are dealing with a function defined on $M \times M$, which is a manifold with corners. We extend the previous definitions as follows:

Definition 1.3.3. An index family $\mathfrak{F}$ for a manifold with corners is an assignment of an index set to each boundary.

For the sake of simplicity we treat only the case of $M = \mathbb{R}_+$, which is sufficient to introduce b-calculus on manifolds with boundary. In this case, denoting with x_1, x_2 the coordinates of the two factors of $M \times M$, we write the index family over $M \times M = \mathbb{R}^2_+$ as $\mathfrak{F} = (E, F)$, with E the index set associated to the x_1 axis and F the one associated with the x_2 axis. In this case the underlying manifold is $\mathbb{R}^2_+$, hence the definition of polyhomogeneous conormal function can be extended as follows.

Definition 1.3.4. Let (E, F) be an index family for $\mathbb{R}^2_+$. A function u over $\mathbb{R}^2_+$ is *polyhomogeneous conormal* with respect to (E, F) if for every $x_2 \in \mathbb{R}$ it admits an asymptotic expansion in $x_1 \in \mathbb{R}$ as in Definition 1.3.2 with index set F such that the coefficients $a_{z,p}$ are polyhomogeneous conormal functions on $\mathbb{R}_+$ with index set E.

The last requirement in this definition entails that for each x_1 the coefficients $a_{z,p}$ have the same singular behavior. A function which does not satifsfies this definition is $u(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, defined over $\mathbb{R}^2_+$, as its asymptotic expansion as $x_1 \to 0^+$ is

$$\sqrt{x_1^2 + x_2^2} \sim \sum_{i=0}^{\infty} a_i(y) x^i = x_2 + \frac{1}{2}\frac{x_1^2}{x_2} - \frac{1}{8}\frac{x_1^4}{x_2^3} + \cdots$$

whose coefficients $a_i(x_2)$ are more singular as we increase the order in x_1 of the expansion. However, if we see u as a function of x_1 and x_1/x_2, then we have

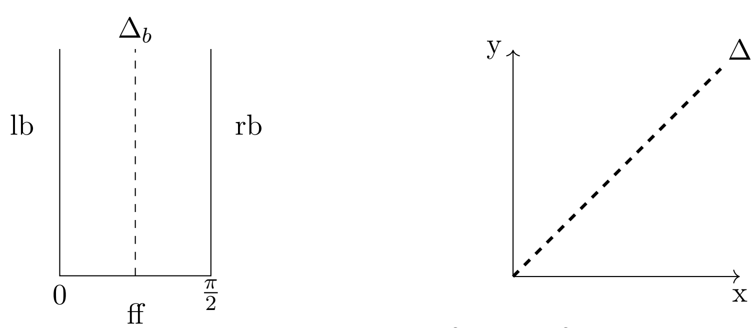
$$\sqrt{x_1^2 + x_2^2} \sim x_2 + \frac{1}{2} x_2 \left(\frac{x_1}{x_2}\right)^2 - \frac{1}{8} x_2 \left(\frac{x_1}{x_2}\right)^4 + \cdots = x_2 \sum_{i=0}^{\infty} c_i \left(\frac{x_1}{x_2}\right)^{2i}$$

Now the coefficients of the expansion are the real values c_i. Therefore in these new variables u is a polyhomogeneous conormal function. This example suggests that, to understand the singular behavior, we may need to find a suitable set of coordinates. We formalize this fact with the notion of asymptotic type.

Definition 1.3.5. Let M be any smooth manifold and let N be a compact, connected manifold with corners. Let u be a function over M and consider a diffeomorphism $\beta : \mathring{N} \to M$ such that $\beta^* u = u \circ \beta$ is polyhomogeneous conormal on N. Then we say that u *is of asymptotic type* β and we call β a singular coordinate change, which resolves the function β.

The simplest example of singular coordinate change for the function $u(x_1, x_2) : \mathbb{R}^2_+,\ (x_1, x_2) \mapsto \sqrt{x_1^2 + x_2^2}$ considered before is obtained considering polar coordinates (r, θ) restricted to $N = [0, +\infty) \times [0, \pi/2]$, with $\beta(r, \theta) = (rsin\theta, rcos\theta)$. We have $u \circ \beta(r, \theta) = r$, which is a conormal polyhomogenous function as per Definition 1.3.4.

The reason polar coordinates allow us to describe the asymptotic behavior of u is actually the fact that polar coordinates over N are blowing up the origin of $\mathbb{R}^2_+$. Indeed, since β is a diffeomorphism on the interior of $\mathbb{R}^2_+$, $\beta^{-1}(p)$ is a point if $p \in \mathring{N}$, but this is not true for $p = (0, 0)$, which is mapped to the interval $\{0\} \times [0, \pi/2]$. Considering $\beta^* u$ we spread out the values of u near the origin to the interval $\{0\} \times [0, \pi/2]$, with the asymptotic expansion depending only on the first factor. We say that $M^2_b = \mathbb{R}_+ \times [0, \pi/2]$ is obtained from $M \times M = \mathbb{R}^2_+$ by 'blowing up' the point $(0, 0)$. Sometimes, to denote that in this case the origin blowed-up we employ the notation $M^2_b = [M \times M, (0, 0)]$ used in [Gri01]. β is called the *blow-down map*. The boundary hypersurfaces of M^2_b are the left boundary $lb = \{\theta = \frac{\pi}{2}\}$, the right boundary $rb = \{\theta = 0\}$ and the front face $ff = \{r = 0\}$. The blow-up of the diagonal $\Delta = \{\theta = \frac{\pi}{4}\}$ is the space $\Delta_b = \overline{\beta^{-1}(\Delta^\circ)}$, called the lifted diagonal or b-diagonal.



(a) Picture of M^2_b. $\Delta_b = \{\theta = \frac{\pi}{4}\}$ is the b-diagonal.

(b) $M^2 = \beta(M^2_b)$. $\Delta = \{\theta = \frac{\pi}{4}\}$ is the diagonal.

Figure 1.3

In general this procedure of blowing up a point – the origin in the case of polar coordinates – goes under the name of blow-up. On $[\mathbb{R}^2_+, (0,0)]$ there are two distinguished coordinate systems which are usually employed: projective coordinates and rational polar coordinates [Gri01].

Rational polar coordinates

Let $x, x' \in \mathbb{R}_+ \cup \{0\}$ be the Cartesian coordinates over $\mathbb{R}^2_+$. The rational polar coordinates ρ, τ are defined as

$$\begin{cases} \rho = x + x' \\ \tau = \dfrac{x - x'}{x + x'} \end{cases} \tag{1.19}$$

The boundary functions for ff, lb and rb are $\rho, 1+\tau$ and $1-\tau$ respectively. The blow-down map is given by $\beta(\rho,\tau) = \left(\frac{1}{2}\rho(1+\tau), \frac{1}{2}\rho(1-\tau)\right)$. When representing a blow-up as in Figure 1.4a we are using these coordinates. These are the coordinates we shall employ in the following.

Projective coordinates

Another distinguished set of coordinates is the one given by the following two local charts. Let θ be the angular variable of standard polar coordinates over $\mathbb{R}^2_+$. If $\theta \neq \frac{\pi}{2}$ we define the coordinates (ξ_1, η_1) as:

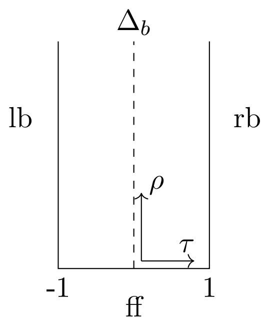
$$\begin{cases} \xi_1 = x \\ \eta_1 = \dfrac{y}{x} \end{cases}$$

Instead, if $\theta \neq 0$, we can use the coordinates (ξ_2, η_2) given by

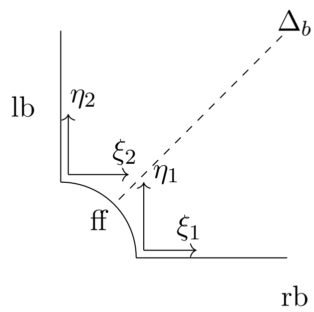
$$\begin{cases} \xi_2 = \dfrac{x}{y} \\ \eta_2 = y \end{cases}$$

In these charts, the blow-down maps β takes the form $\beta_1(\xi_1, \eta_1) = (\xi_1, \xi_1, \eta_1)$ and $\beta_2(\xi_2, \eta_2) = (\xi_2\eta_2, \eta_2)$.

As we shall see later, the space of b-pseudodifferential operators is the space of distributions whose singularities are located on the b-diagonal and satisfy certain constraints. In order to give a precise definition, we need to introduce the notion of conormal distribution on a manifold, beginning from the particular case of a polyhomogeneous conormal distribution.



(a) Rational polar coordinates.



(b) Projective coordinates.

Figure 1.4

Definition 1.3.6 (Polyhomogeneous conormal distribution of order m on a manifold)**.** Let Z be a manifold and $T \subset Z$ a submanifold. A distribution $u \in \mathcal{D}(Z)$ is one-step conormal or polyhomogeneous conormal with respect to T if $\exists m \in \mathbb{R}$ such that:

1) u is smooth on $Z \setminus T$.

2) In any local coordinate system $\phi : U \subset Z \to \mathbb{R}^n$ mapping $T \cap U$ to $\mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$ there is a representation

$$u(t,z) = \int_{\mathbb{R}^{n-k}} e^{iz\zeta} a(t,\zeta) d\zeta \tag{1.20}$$

where $t = (\phi_1, \cdots, \phi_k)$, $z = (\phi_{k+1}, \cdots, \phi_n)$ and a is a smooth function on $(T \cap U) \times \mathbb{R}^{n-k}$ with asymptotics[1]

$$a(t,\zeta) \sim \sum_{j=0}^{\infty} a_{m-j}(t,\zeta) \tag{1.21}$$

as $|\zeta| \to \infty$, where a_l is homogeneous of degree l in ζ, for each l.

3) If T is a (sub)manifold with corners, $u(t,z)$ is smooth up to ∂T in the variable t.

Remark 1.3.2. Condition (1.23) is the Fourier transform in the directions transveral to T.

[1]Here the meaning of the asymptotic expansion is the following: For any N, if $a^{(N)}$ is the sum up to the term a_{-N}, then $|a(y,\zeta) - a^{(N)}(y,\zeta)| \leq |\zeta|^{-N-1}$. Similar estimates must hold true also for the derivatives in y and ζ.

Weakening the second condition of the definition above we obtain the larger class of non-polyhomogeneous conormal distributions. Before giving the precise definition, we need to recall the notion of symbol, as introduced by Hörmander [H00̈].

Definition 1.3.7. Let (z, ξ) be coordinates over ${}^bT^*M$. A symbol a is a smooth function over ${}^bT^*M$ such that, given a compact exhaustion $\{K_i\}$ of M, satisfies

$$|\partial_z^\alpha \partial_\xi^\beta a(z,\xi)| \leq C_{K_i,\alpha,\beta}(1+|\xi|)^{m-|\beta|} \tag{1.22}$$

on $K_i \times \mathbb{R}^n$.

In the following we denote with $S^m({}^bT^*M)$ the space of symbols of order m over the b-cotangent bundle of M. We are ready to give the general definition of conormal distribution.

Definition 1.3.8 (Conormal distribution of order m on a manifold)**.** Let Z be a manifold and $T \subset Z$ a submanifold. A distribution $u \in \mathcal{D}(Z)$ is conormal with respect to T if $\exists m \in \mathbb{R}$ such that:

1) u is smooth on $Z \setminus T$.

2) In any local coordinate system $\phi : U \subset Z \to \mathbb{R}^n$ mapping $T \cap U$ to $\mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$ there is a representation

$$u(t,z) = \int_{\mathbb{R}^{n-k}} e^{iz\zeta} a(t,\zeta) d\zeta \tag{1.23}$$

where $t = (\phi_1, \cdots, \phi_k)$, $z = (\phi_{k+1}, \cdots, \phi_n)$ while a is a symbol as per Definition 1.3.7.

3) If T is a (sub)manifold with corners, $u(t,z)$ is smooth up to ∂T in the variable t.

With these data a ΨDO on a manifold M without corners is a distribution on $M \times M$ which is conormal with respect to

$$\Delta = \{(p,p) \mid p \in M\} \subset M \times M \tag{1.24}$$

In the case of a manifold with corner, in addition to the singularities on the diagonal, we must take care of the boundary behavior. In particular, b-pseudodifferential operators are characterized by their behavior at *ff*, *lf*, *rf* and Δ_b.

1.3.4 Half b-densities

To build the small calculus of b-pseudodifferential operators, it is convenient to introduce the notion of half b-density. The reason is that it is simpler to introduce the space of b-pseudodifferential operators seeing their integral kernels as half-b-densities over M_b^2 instead than as distributions over M^2 [Gri01, Mel93]. In general, given a vector space V of dimension n, the space of s-densities over V, with $s \in \mathbb{R}$, is

$$\Omega^s V = \{\mu \in \Lambda^n V^* \setminus \{0\} \to \mathbb{R} \mid \mu(t\lambda) = |t|^s \mu(\lambda)\ \forall \lambda \in \Lambda^n V^*,\ t \neq 0\}$$

Here $\Lambda^n V^*$ is the space of the n-differential forms over V. Note that, since $dim(\Lambda^n V^*) = 1$, an element $\mu \in \Omega^s V$ is fixed by its value when evaluated against an n-differential form $\lambda \neq 0$. This entails that the space of s-densities is of dimension one. The definition of s-density immediately yields the following canonical isomorphisms

$$\begin{aligned} \Omega^s V \otimes \Omega^t V \simeq \Omega^{s+t} V, \quad \forall\ s, t \in \mathbb{R} \\ \Omega^0 V \simeq \mathbb{R} \\ \Omega^{-s} V \simeq (\Omega^s V)^* \\ \Omega^s (V \oplus W) \simeq \Omega^s V \otimes \Omega^s W\ \forall s \in \mathbb{R} \end{aligned} \tag{1.25}$$

where V and W are finite dimensional vector spaces.

Let M be a smooth compact manifold of dimension n, with or without boundary. Consider a point $p \in M$ and let $\Omega_p^s := \Omega^s(T_p^* M)$ and ${}^b\Omega_p^s := \Omega^s({}^bT_p^* M)$ denote the spaces of s-densities and s-b-densities over M at the point p. Then:

- $\Omega^s M = \sqcup_{p \in M} \Omega(T_p^* M)$ is the bundle of s-densities over M.
- ${}^b\Omega^s M = \sqcup_{p \in M} \Omega({}^bT_p^* M)$ is the bundle of s-b-densities over M.

These two bundles are related as follows

$$\mu \in \mathcal{C}^\infty(M; {}^b\Omega^s M) \Leftrightarrow x^s \mu \in \mathcal{C}^\infty(M; \Omega^s M) \tag{1.26}$$

In particular, calling

$$\dot{\mathcal{C}}^\infty(M, {}^b\Omega^s M) = \{u \in \mathcal{C}^\infty(M, {}^b\Omega^s M) \mid u \textit{ vanishes to all orders at } \partial M\} \tag{1.27}$$

equation (1.26) yields that $\dot{\mathcal{C}}^\infty(M, {}^b\Omega^s M) \simeq \dot{\mathcal{C}}^\infty(M, \Omega^s M)$.

In local coordinates $x, y_1, \cdots, y_{n-1}$ near the boundary of M, a local basis element of the bundle $\Omega^s M$ is of the form

$$\left(a\partial_x \wedge \partial_{y_1} \wedge \cdots \wedge \partial_{y_{n-1}}\right) |dxdy_1 \cdots d_{y_{n-1}}|^s$$

while an element of ${}^b\Omega^s M$ is of the form

$$\left(ax\partial_x \wedge \partial_{y_1} \wedge \cdots \wedge \partial_{y_{n-1}}\right) \left|\frac{dx}{x} dy_1 \cdots d_{y_{n-1}}\right|^s$$

with $a \in \mathcal{C}^\infty(M)$.

Now let us focus on half-densities. Equation (1.25) entails that there exists a well-defined product bewteen half-densities, namely a map:

$$\mathcal{C}^\infty(M; \Omega^{\frac{1}{2}} M) \times \mathcal{C}^\infty(M; \Omega^{\frac{1}{2}} M) \to \mathcal{C}^\infty(M; \Omega^1 M) \tag{1.28}$$

Furthermore, this product can be extended to smooth sections of the complexified 1/2-densties bundles, yielding the sesquilinear pairing

$$\langle \mu, \nu \rangle = \int_M \mu\overline{\nu} \quad \textit{for every } \mu, \nu \in \mathcal{C}^\infty(M; \Omega^{\frac{1}{2}} M) \tag{1.29}$$

The completion of $\mathcal{C}^\infty(M, \Omega^{\frac{1}{2}} M)$ with respect to this inner product yields the space of the square-integrable half-densities $L^2(M, \Omega^{\frac{1}{2}} M)$. The pairing (1.29) also allows us to define the space of distributional half densities $\mathcal{C}^{-\infty}(M, \Omega^{\frac{1}{2}} M)$, namely the space of the the continuous linear maps from $\mathcal{C}^\infty(M, \Omega^{\frac{1}{2}} M)$ to $\mathbb{C}$. Making use of the relation (1.26), we can extend the product map in Equation (1.28) to

$$\mathcal{C}^\infty(M, {}^b\Omega^{\frac{1}{2}} M) \times \dot{\mathcal{C}}^\infty(M, {}^b\Omega^{\frac{1}{2}} M) \to \mathcal{C}^\infty(M, {}^b\Omega M)$$

and the pairing (1.29) as a map

$$\mathcal{C}^\infty(M, {}^b\Omega^{\frac{1}{2}} M) \times \dot{\mathcal{C}}^\infty(M, {}^b\Omega^{\frac{1}{2}} M) \to \mathbb{C}.$$

The Schwartz kernel theorem can be formulated for half-b-densities over manifolds with boundary as follows [Mel93].

Proposition 1.3.1. Let M be a compact manifold with boundary ∂M and consider the blow-up $M_b^2 = [M^2, (0,0)]$. The continuous linear operators

$$\dot{\mathcal{C}}^\infty(M, {}^b\Omega^{\frac{1}{2}} M) \to \mathcal{C}^{-\infty}(M, {}^b\Omega^{\frac{1}{2}} M)$$

are in one-to-one correspondence with the elements of the space of distributional sections $\mathcal{C}^{-\infty}(M_b^2, {}^b\Omega^{\frac{1}{2}} M_b^2)$. In particular, operators on distributional densities can be identified with the lifts of their kernels to M_b^2:

$$\left\{A_K : \dot{\mathcal{C}}^{\infty}(M, {}^b\Omega^{\frac{1}{2}} M) \to \mathcal{C}^{-\infty}(M, {}^b\Omega^{\frac{1}{2}} M)\right\} \leftrightarrow \left\{K \in \mathcal{C}^{-\infty}(M_b^2; {}^2\Omega^{\frac{1}{2}} M_b^2)\right\}$$
$$\langle A_K\phi, \psi\rangle = \langle K, (\beta^2)^*(\psi \boxtimes \phi)\rangle \tag{1.30}$$

where $(\beta^2)^*K$ is the pullback of K in both entries with respect to the blow-down map β. The interested reader can find more about s-b-densities in [Mel93, Chapter 4].

1.3.5 Kernels of b-differential operators as b-half-densities

Before defining b-pseudodifferential operators, let us focus on b-differential operators. We start by studying the identity $\mathbb{I}$ as a b-differential operator on half-b-densities over a manifold with boundary M. Given a half-b-density $\alpha \left|\frac{dx}{x} dy\right|^{\frac{1}{2}}$ over M, the action of the identity – in local coordinates $x, x', y_1, \cdots, y'_{n-1}$ of $M \times M$, with x, x' the boundary functions of the two factors – can be expressed as the formal integral

$$\mathbb{I}\left(\alpha \left|\frac{dx}{x} dy\right|^{\frac{1}{2}}\right) = \int_M \delta(x - x')\delta(y - y')\phi(x', y')dx'dy' \left|\frac{dx}{x} dy\right|^{\frac{1}{2}} \tag{1.31}$$

From this expression we can read the kernel of the identity in the sense of Equation (1.30):

$$K_{\mathbb{I}} = x'\delta(x - x')\delta(y - y') \left|\frac{dx'}{x'} dy' \frac{dx}{x} dy'\right|^{\frac{1}{2}} \tag{1.32}$$

This kernel is degenerate on the boundary of $M \times M$, since $x'\delta(x-x')\delta(y-y')$ vanishes for $x' = 0$. Lifting this kernel to M_b^2 making use of the projective coordinates $s = x/x'$ and $t = x'$, yields

$$K_{\mathbb{I}} = \delta(s - 1)\delta(y - y') \left|\frac{ds}{s} dy' \frac{dt}{t} dy'\right|^{\frac{1}{2}} \tag{1.33}$$

which is no longer degenerate at the boundary of M_b^2. In general, the lifting of the kernels of b-differential operators removes the degeneracy of the kernels on the boundary. Before stating the main proposition concerning b-differential operators as b-half-densities, we need to introduce the notion of smooth Dirac section ${}^b\Omega^{\frac{1}{2}} M_b^2$.

Definition 1.3.9. A smooth Dirac section of order k of ${}^b\Omega^{\frac{1}{2}} M_b^2$, with respect to $\triangle_b$, is a distribution in $\mathcal{C}^{-\infty}(M_b^2; {}^b\Omega^{\frac{1}{2}} M_b^2)$ which has support contained in $\triangle_b$ and such that in local coordinates it admits the form

$$\sum_{0 \leq p+|\alpha| \leq k} a_{p,\alpha}(x', y) D_s^p \delta(s-1) D_y^\alpha \delta(y - y') \tag{1.34}$$

with $a_{p,\alpha} \in \mathcal{C}^\infty(M)$ for every $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{n-1}$, where $n = dim\ M$.

Proposition 1.3.2 ([Mel93], Lemma 4.21)**.** Under the isomorphism given by Proposition 1.3.1 the space $\mathrm{Diff}_b^k(M; {}^b\Omega^{\frac{1}{2}} M)$ is mapped isomorphically onto the space of all smooth Dirac sections of order k, as per Definition 1.3.9, with respect to $\triangle_b$.

1.3.6 The small calculus of b-pseudodifferential operators

Now we are ready to microlocalize the space of b-differential operators, seen as operators over half-b-densities.

Definition 1.3.10 (Small b-calculus)**.** The (small) space $\Psi_b^m(M; {}^b\Omega^{\frac{1}{2}} M)$ of b-pseudodifferential operators of order $m \in \mathbb{R}$, acting on half-b-densities, is the space of continuous linear operators which, by means of Equation (1.3.1), correspond to conormal sections of order m associated to the lifted diagonal and vanishing to all orders at $lb \cup rb$.

The meaning of this definition is that the kernel κ associated to a b-pseudodifferential operator of order m must satisfy the following properties:

a) $\kappa|_{M_b^2 \setminus \triangle_b} \in \mathcal{C}^\infty(M_b^2 \setminus \triangle_b; {}^b\Omega^{\frac{1}{2}} M_b^2)$

b) In a neighborhood of $\triangle_b \setminus ff$

$$\kappa(z, z') = (2\pi)^{-n-1} \int e^{i(z-z')\zeta} a(z', \zeta) d\zeta |dz dz'|^{\frac{1}{2}} \tag{1.35}$$

with (z, z') coordinates of M_b^2.

c) In a neighborhood of $\triangle_b \cap \mathit{ff}$, using rational polar coordinates instead of the boundary functions x, x', it holds

$$\kappa(\rho,\tau,y,y') = (2\pi)^{-n-1}\int e^{i\tau\lambda+i(y-y')\eta} b(\rho,y',\lambda,\eta)d\lambda d\eta \left|\frac{d\rho}{\rho}d\tau dy dy'\right|^{\frac{1}{2}} \tag{1.36}$$

d) The Taylor series of κ vanishes at every order at $lb \cup rb$.

The amplitudes a and b in the Fourier integral above are symbols of the same order m of the b-pseudodifferential operators. If the symbols satisfy the full asymptotic Equation (1.21) we obtain the smaller space of one-step polyhomogeneous b-pseudodifferential operators.

Remark 1.3.3. One can prove that the expressions in Equations (1.35), (1.36) are actually coordinate invariant [Mel93].

As in the case of pseudodifferential operators, also b-differential operators admit a principal symbol map.

Proposition 1.3.3 ([Mel93], Proposition 4.23)**.** The local symbols in Equations (1.35) and (1.36) fix the symbol map, giving the short exact sequence

$$0 \longrightarrow \Psi_b^{m-1}(M; {}^b\Omega^{\frac{1}{2}}M) \hookrightarrow \Psi_b^m(M; {}^b\Omega^{\frac{1}{2}}M) \xrightarrow{{}^b\sigma_m} S^m({}^bT^*M) \longrightarrow 0$$

In particular the short exact of this proposition entails that there exits an isomorphism

$$\Psi_b^m(M)/\Psi_b^{m-1}(M) \cong S^m({}^bT^*M)/S^{m-1}({}^bT^*M).$$

Remark 1.3.4. This isomorphism and the definition of classical symbol over ${}^bT^*M$ yield that $\Psi_b^m(M) \subset \Psi_b^n(M)$ if $m < n$.

The kernel approach over half-b-densities presented above is a way to define b-pseudodifferential operators which is well-suited to study the properties of this space. In the following we revert to reading b-pseudodifferential operators as operators acting on smooth functions. In the following, as in [Vas12] and [GW20], we work with properly supported b-pseudodifferential operators and with $\Psi_b(M)$ we shall denote this space. Since we are dealing with properly supported distributions, we can view an element of $\Psi_b^m(M)$ as a map $\dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$, which can be extended to a continuous endomorphism over $\mathcal{C}^\infty(M)$ [GW20]. Here, with $\dot{\mathcal{C}}^\infty(M)$ we denote the space of smooth functions vanishing at the boundary ∂M with all their derivatives. The pairing between $\dot{\mathcal{C}}^\infty(M)$ and $\mathcal{C}^{-\infty}(M)$, which we denote with $\langle u, v\rangle$, allows us to extend a pseudodifferential operator to an endomorphism on $\mathcal{C}^{-\infty}(M)$.

Remark 1.3.5. Fixed a positive $\mathcal{C}^\infty$ density μ over M, $u \in L^2_{loc}(M, \mu)$ determines an element of $\mathcal{C}^{-\infty}(M)$ via

$$\langle u, \phi \rangle = \int_M u \cdot \overline{\phi} d\mu$$

where $\phi \in \dot{\mathcal{C}}^\infty(M)$ has compact support. Sometimes, in concrete applications, it may be more convenient to define locally the small b-calculus using local quantization maps [Vas12] – namely using Fourier integral operators defined over local charts of M – and then to define a global quantization map using a partition of unity. The result is a map $Op : S^m({}^bT^*M) \to \Psi_b^m(M)$ which associates a b-pseudodifferential operator to a symbol. We can see the map Op as a non-canonical inverse of the symbol map, whose form is fixed case by case as it is more convenient. For example, a local quantization map can be defined as follows. Let U be a local chart with coordinates (x, y) and let $a \in S^m({}^bT^*M)$ be a classical symbol as per Definition 1.3.7 with support in $K \subset {}^bT^*_K M$, $K \subset U$ compact. We can define the quantization map Op, associating to $a \in S^m(M)$ the b-pseudodifferential operator $Op(a) \in \Psi_b^m(M)$, as the following oscillatory integral

$$Op(a)u(x,y) = \\ = \frac{1}{(2\pi)^n} \int_U e^{i[(x-x')\sigma + (y-y')\cdot\eta]} \Phi\left(\frac{x-x'}{x}\right) a(x,y,x\sigma,\eta) u(x',y') dx' dy' d\sigma d\eta \tag{1.37}$$

with $u \in \mathcal{C}^\infty(M)$. The integral in x' is over the interval $[0, \infty)$, while those in the other variables are over the real line. In addition $\Phi \in C_0^\infty(\Omega)$, where $0 \in \Omega \subset \mathbb{R}$, Φ is identically 1 near 0 and it localizes to a neighborhood of the diagonal $\{x = x'\}$. Symbolically we can write $Op(a) = a(x, y, xD_x, D_y)$. In order to reflect the form of a coordinate system on ${}^bT^*M$, we can make the change of variable $\xi = x\sigma$, which leads to the following expression of the quantization map:

$$Op(a)u(x,y) = \\ = \frac{1}{(2\pi)^n} \int e^{i[\frac{(x-x')}{x}\xi + (y-y')\cdot\eta]} \Phi\left(\frac{x-x'}{x}\right) a(x,y,\xi,\eta) u(x',y') \frac{dx'}{x} dy' d\xi d\eta \tag{1.38}$$

1.3.7 Properties of b-pseudodifferential operators

In this section we recall some useful properties of b-differential operators that we use in the following and we introduce the notion of operatorial wavefront set, following [Vas12], [GW20] and [DM21a]. We begin reviewing some properties of the principal symbol of a b-pseudodifferential operator. Given two pseudodifferential operators $A \in \Psi_b^m(M)$ and $B \in \Psi_b^n(M)$, the principal symbol of the composition AB is $\sigma_{b,m+n}(AB) = \sigma_{b,m}(A) \cdot \sigma_{b,n}(B)$, while their commutator $[A, B] \in \Psi_b^{m+n-1}(M)$ has principal symbol $\sigma_{b,m+n-1}([A, B]) = \{\sigma_{b,m}(A), \sigma_{b,n}(B)\}$. The adjoint of a pseudodifferential operator of order m with respect to a measure μ over the manifold M is again a pseudodifferential operators of order m. In particular the adjoint with respect to the volume form of a Lorentzian manifold is such that [MS11]

$$\sigma(A^*)(z, \zeta) \sim \sum_{\alpha=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\zeta^\alpha \bigtriangledown_z^\alpha \overline{a(z, \zeta)} \tag{1.39}$$

At the level of principal symbol, the equality $\sigma_m(A^*) = \overline{\sigma_m(A)}$ holds true. The principal symbol of a b-pseudodifferential operator is invariant under conjugation by a power of the boundary function, namely given $A \in \Psi_b^m(M)$, then $x^{-s} A x^s \in \Psi_b^m(M)$ $s \in \mathbb{R}$, while $\sigma_{b,m}(x^{-s} A x^s) = \sigma_{b,m}(A)$.
Through the notion of symbol we can endow the space of b-pseudodifferential operators with the structure of metric space as follows. First, for $m \in \mathbb{R}$, we equip the space of symbols $S^m({}^bT^*M)$ with the structure of a Fréchet space defining the following family of seminorms

$$\|a\|_N = \sup_{(z,\zeta) \in K_i \times \mathbb{R}^n} \max_{|\alpha|+|\beta| \leq N} \frac{|\partial_z^\alpha \partial_\zeta^\beta a(z, \zeta)|}{\langle \zeta \rangle^{m-|\beta|}}$$

where $\{K_i\}$ is a compact exhaustion of M – that is an increasing sequence $\{K_i\}$ with each $K_i \Subset M$ such that $\cup_{i=0}^{\infty} K_i = M$ – and $\langle \zeta \rangle = 1 + |\zeta|$. A metric over $S^m({}^bT^*M)$ can be defined in the following way: Given two symbols $a, b \in S^m({}^bT^*M)$, their distance is

$$d(a, b) = \sum_{N \in \mathbb{N}} 2^{-N} \frac{\|a - b\|_N}{1 + \|a - b\|_N} \tag{1.40}$$

The distance $\overline{d}$ between two elements $A, B \in \Psi_b^m(M)$ is defined as the one between their symbols $a, b \in S^m(M)$, namely $\overline{d} : \Psi_b^m(M) \times \Psi_b^m(M) \to [0, +\infty)$ is such that $\overline{d}(A, B) = d(a, b)$, the right hand side being as per Equation (1.40). In particular, we say that a family of b-pseudodifferential operators in $\Psi_b(M)$ is bounded if the subset of the symbols associated with the family of ΨDOs is bounded.

To study the behavior of b-pseudodifferential operators at the boundary ∂M, it is useful to introduce an object describing the mapping properties of a b-ΨDO in terms of the decay at the boundary. This information is captured by the indicial family associated to a b-pseudodifferential operator.

Definition 1.3.11. Let $A \in \Psi_b^m(M)$. For a fixed boundary function x and $v \in \mathcal{C}^\infty(\partial M)$ the indicial family $\widehat{N}(A)(s)$ is defined as:

$$\widehat{N}(A)(s) = x^{-is} A\left(x^{is} u\right)|_{\partial M}$$

where $u \in \mathcal{C}^\infty(M)$ is any function restricting to v at the boundary.

The indicial family is an algebra homomorphism, hence it satisfies

$$\widehat{N}(AB)(s) = \widehat{N}(A)(s) \circ \widehat{N}(B)(s). \tag{1.41}$$

At last we discuss the microlocal properties of b-pseudodifferential operators. The starting point is the notion of elliptic b-pseudodifferential operator.

Definition 1.3.12. A b-pseudodifferential operator $A \in \Psi_b^m(M)$ is elliptic at a point $q_0 \in {}^bT^*M \setminus 0$ if there exists $b \in S^{-m}({}^bT^*M)$ such that

$$\sigma_{b,m}(A) \cdot b - 1 \in S^{-1}({}^bT^*M)$$

in a conic neighborhood of q_0. We call $ell_b(A)$ the conic subset of ${}^bT^*M \setminus 0$ in which A is elliptic.

Remark 1.3.6. Using the coordinates (z, ζ) of ${}^bT^*M$, the condition above is equivalent to $|\sigma_{b,m}(A)| \geq \varepsilon|\zeta|^m$ for $|\zeta| \geq C_\varepsilon$ for every $\varepsilon > 0$.

As in the case of pseudodifferential operators, also the wavefront set of b-pseudodifferential operators is defined in terms of its symbol [Jos]:

Definition 1.3.13. If $P \in \Psi_b^m(M)$, then $(z_0, \zeta_0) \notin WF_b'(P)$ if its symbol $p(z, \zeta)$ is such that

$$|\partial_z^\alpha \partial_\zeta^\beta p(z, \zeta)| \leq C_{m,\alpha,\beta} \langle\zeta\rangle^{-N} \quad \forall\ N$$

for z in a neighborhood of z_0 and ζ in a conic neighborhood of ζ_0. Here $\langle\zeta\rangle = (1 + |\zeta|)$.

Remark 1.3.7. From this definition the following fact follows immediately: If $WF_b'(P) = \emptyset$, then $P \in \Psi_b^{-\infty}(M)$. Also the converse holds true.

The notion of wavefront set can be extended also to a family of b-pseudodifferential operators [GW20].

Definition 1.3.14. Suppose that $\mathcal{A}$ is a bounded subset of $\Psi_b^m(M)$ and $q \in {}^bT^*M$. We say that $q \notin WF_b'(\mathcal{A})$ if there exists $B \in \Psi_b(M)$ which is elliptic at q such that $\{BA : A \in \mathcal{A}\}$ is a bounded subset of $\Psi_b^{-\infty}(M)$.

Remark 1.3.8. This definition reduces to the one given before in the case the set of b-ΨDOs consists of a single operator.

The usual properties of the operator wavefront set also hold true for a family of ΨDOs. Given two bounded families $\mathcal{A}$ and $\mathcal{B}$ the following relations, true in the case in which $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$, are still valid:

$$WF_b'(\mathcal{A} + \mathcal{B}) \subset WF_b'(\mathcal{A}) \cup WF_b'(\mathcal{A}) \qquad WF_b'(\mathcal{A}\mathcal{B}) \subset WF_b'(\mathcal{A}) \cap WF_b'(\mathcal{A})$$

Another important property of bounded families of pseudodifferential operators is the following: If $B \in \Psi_b(M)$ with $WF_b'(B) \cap WF_b'(\mathcal{A}) = \emptyset$, then $\{AB : A \in \mathcal{B}\}$ is bounded in $\Psi^{-\infty}(M)$.

Definition 1.3.15. Let $S \subset \Psi_b^m(M)$ be a closed subspace. We say that a bounded linear map $M : S \to \Psi_b^k(M)$ is microlocal if $WF_b'(M(A)) \subset WF_b'(A)$ for all $A \in S$.

We also can microlocalize the notion of parametrix [Vas08].

Definition 1.3.16 (microlocal parametrix)**.** Let $A \in \Psi_b^m(M)$ be elliptic in an open cone centered at a point $q \in {}^bT^*M \setminus \{0\}$. Then there exists a microlocal parametrix $G \in \Psi_b^{-m}(M)$ for A at q, so that GA and AG are microlocally the identity operator near q, namely $q \notin WF_b'(GA - \mathbb{I})$ and $q \notin WF_b'(AG - \mathbb{I})$.

A useful consequence of this definition is that, given a compact subset $K \subset {}^bT^*M$ and an operator $A \in \Psi_b^m(M)$ which is elliptic in K, then there exists a b-pseudodifferential operator $G \in \Psi_b^{-m}(M)$ such that $K \cap WB_b'(GA - \mathbb{I}) = K \cap WB_b'(AG - \mathbb{I}) = \emptyset$. In pacticular, the operators $E_1 = GA - \mathbb{I}$ and $E_2 = AG - \mathbb{I}$ are in $\Psi_b^{-\infty}(K)$, namely they are smoothing b-pseudodifferential operators.

1.4 b-calculus and twisted Sobolev spaces

1.4.1 Twisted Sobolev spaces

A key ingredient of our analysis will be the Dirichlet form, following the strategy employed in [GW20]. To this end it is necessary to introduce a twisted version of the standard Sobolev spaces to account for the behaviour of the fields at the boundary, as acknowledged in [War12]. To motivate the necessity of introducing twisting Sobolev spaces, we start with a short motivational example, considering the simplest case of globally hyperbolic asymptotically AdS spacetime: $PAdS_2$, the Poincaré patch of the two dimensional anti-deSitter spacetime. As a manifold on its own $PAdS_2$ is diffeomorphic to $\mathbb{R} \times [0, \infty)$ and the metric reads $g = x^{-2}\eta_2$ where η is the two-dimensional Minkowski metric, see Section 1.1.2. Let $\phi : PAdS_2 \to \mathbb{R}$ be a scalar field obeying the Klein-Gordon equation

$$\left(\Box_g - m^2\right)\phi = 0 \Longrightarrow x^2\left(\Box_\eta - \frac{m^2}{x^2}\right)\phi = 0. \tag{1.42}$$

Equation (1.42) can be solved by separation of variables using the ansatz $\phi(t, x) = F(t)H(x)$, which leads to the following systems of ordinary differential equations

$$\begin{cases} \dfrac{d^2F}{dt^2} = \lambda F \\ \left(\dfrac{d^2}{dx^2} + \dfrac{m^2}{x^2}\right) H = \lambda H \end{cases} \tag{1.43}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter of the problem. For future convenience, we introduce the parameter $\nu = \frac{1}{2}\sqrt{1 + 4m^2} > 0$, which is related to the indicial roots of the operators. In the theoretical physics literature this constraint is known as the Breitenlohner-Freedman bound [BF82]. Solving the two ODEs above yields

$$\begin{aligned} F(t) &= C_1 e^{\lambda t} + C_2 e^{-\lambda t} \\ H(x) &= C_3\sqrt{x}J_\nu(-i\sqrt{\lambda}x) + C_4\sqrt{x}Y_\nu(-i\sqrt{\lambda}x) \end{aligned}$$

with J_ν, Y_ν the standard Bessel functions of first and second kind and $C_1, C_2, C_3, C_4 \in \mathbb{R}$ to be determined imposing boundary conditions at $x = 0$ and at $x \to +\infty$. Imposing a Dirichlet boundary condition, we obtain $C_4 = 0$ and therefore we call $H_1(x) = \sqrt{x}J_\nu(-i\sqrt{\lambda}x)$ the Dirichlet solution. Instead, imposing a boundary condition of Neumann type yields $C_3 = 0$, thus we call $H_2(x) = \sqrt{x}Y_\nu(-i\sqrt{\lambda}x)$ the Neumann solution. The behavior of the solutions near the boundary is described by the indicial roots $\nu_\pm$ of Equation (1.42), which are given by $\nu_\pm = \frac{1}{2} \pm \nu$. Consider a relatively compact subset $U \subset PAdS_2$ such that $U \cap \partial PAdS_2 \neq \emptyset$ and let us introduce the Dirichlet form

$$\mathcal{E}_D(\phi_1, \phi_2) = -\int_U g(d\phi_1, d\phi_2)d\mu_g = -\int_U \eta(d\phi_1, d\phi_2)dxdt, \tag{1.44}$$

with ϕ_1, ϕ_2 arbitrary solutions of Equation (1.42) and $g(\cdot,\cdot)$ the metric induced pairing between 1-forms. A direct inspection unveils that, choosing as ϕ_1 the solution of Equation (1.42) with Dirichlet boundary conditions, then $\eta(d\phi_1, d\phi_1) \sim x^{1+2\nu}$ close to the boundary $x = 0$. Hence the x-integral in Equation (1.44) is always convergent. Consider now as ϕ_2 the solution of Equation (1.42) with Neumann boundary conditions. The x-integral is always divergent since $\eta(d\phi_2, d\phi_2) \sim x^{-1-2\nu}$ as $x \to 0^+$. For this reason the Dirichlet form in Equation (1.44) is not the right choice to study boundary conditions other than Dirichlet one. In order to bypass this hurdle, we introduce the twisted derivatives

$$\widetilde{Q}_0\phi = x^{\frac{1}{2}-\nu}\frac{\partial}{\partial t}\Big(x^{-\frac{1}{2}+\nu}\phi\Big) = x^{\nu_-}\frac{\partial}{\partial t}\Big(x^{-\nu_-}\phi\Big) \tag{1.45}$$

$$\widetilde{Q}_1\phi = x^{\frac{1}{2}-\nu}\frac{\partial}{\partial x}\Big(x^{-\frac{1}{2}+\nu}\phi\Big) = x^{\nu_-}\frac{\partial}{\partial x}\Big(x^{-\nu_-}\phi\Big) \tag{1.46}$$

with ϕ a generic solution of Equation (1.42). Note that twisting by x raised to a power of the indicial root ν_- only affects the derivative in the x direction, but not that in the time direction. Now we define a new energy form making use of the twisted differentials associated with $\widetilde{Q}_0$ and $\widetilde{Q}_1$.

$$\mathcal{E}_0(\phi_1, \phi_2) = -\int_U g(d_{\widetilde{Q}}\phi_1, d_{\widetilde{Q}}\phi_2)d\mu_g = -\int_U \eta(d_{\widetilde{Q}}\phi_1, d_{\widetilde{Q}}\phi_2)dxdt. \tag{1.47}$$

Here $d_{\widetilde{Q}}$ is the twisted differential defined as

$$d_{\widetilde{Q}}\phi = x^{\nu_-}d\Big(x^{-\nu_-}\phi\Big). \tag{1.48}$$

After a short computation, we find that for $x \to 0^+$ $\eta(d_{\widetilde{Q}}\phi_1, d_{\widetilde{Q}}\phi_1) \sim x^{2\nu-1}$ while $\eta(d_{\widetilde{Q}}\phi_1, d_{\widetilde{Q}}\phi_1) \sim x^{3-2\nu}$. Hence for $0 < \nu < 1$ the integral is convergent in both cases. Motivated by this example we introduce the space of twisted differential operators on a generic globally hyperbolic asymptotically AdS spacetime (M, g)

$$\mathbf{Diff}^1_\nu(M) = \{x^{\nu_-}Dx^{-\nu_-} \mid D \in \mathbf{Diff}^1(M)\}$$

where $\nu_- = \frac{n-1}{2} - \nu$, $\nu > 0$ and $\mathbf{Diff}^1(M)$ is the set of first order differential operators on (M, g). In the next chapter we shall see that ν_- corresponds to the lowest indicial root of the Klein-Gordon operator on the spacetime (M, g).

Remark 1.4.1. Since $\mathbf{Diff}^1_\nu(M) \subset x^{-1}\mathbf{Diff}^1_b(M)$ [GW20, Lemma 3.1], it follows that $\mathbf{Diff}^1_\nu(M)$ is finitely generated.

Following [GW20] we introduce the following L^2 space

$$\mathcal{L}^2(M) \doteq L^2(M, x^2 d\mu_g) \tag{1.49}$$

and the corresponding twisted Sobolev space

$$\mathcal{H}^1(M) \doteq \Big\{ u \in \mathcal{L}^2(M) \mid Qu \in \mathcal{L}^2(M)\ \forall Q \in \mathbf{Diff}^1_\nu(M) \Big\}, \tag{1.50}$$

whose norm is

$$\|u\|^2_{\mathcal{H}^1(M)} = \|u\|^2_{\mathcal{L}^2(M)} + \sum_{i=1}^{n} \|Q_i u\|^2_{\mathcal{L}^2(M)} \tag{1.51}$$

where $\{Q_i\}_{i=1\ldots n}$ is a generating set of $\mathbf{Diff}^1_\nu(M)$. In addition we also define $\mathcal{L}^2_{loc}(M)$, the space of locally square integrable functions over M with respect to the measure $x^2 d\mu_g$, the corresponding local first order Sobolev space being $\mathcal{H}^1_{loc}(M)$. With $\dot{\mathcal{H}}^1_{loc}(M)$ we denote the closure of $\dot{\mathcal{C}}_{loc}(M)$ in $\mathcal{H}^1_{loc}(M)$. The topological duals of the last two Sobolev spaces we introduced are $\dot{\mathcal{H}}^{-1}_{loc}(M)$ and $\mathcal{H}^{-1}_{loc}(M)$ respectively. In addition we define

$$\mathcal{H}^1_0(M) = \mathcal{H}^1_{loc}(M) \cap \mathcal{E}'(M), \tag{1.52}$$

where we denote with $\mathcal{E}'(M)$ the topological dual space of $\dot{C}^\infty(M)$. Similarly one can define $\mathcal{H}^{-1}_0(M)$.

On twisted Sobolev spaces on asymptotically anti-de Sitter spacetimes, there is a distinguished trace map $\gamma_- : \mathcal{H}^1_{loc}(M) \to \mathcal{L}^2_{loc}(\partial M)$, which can be built using an asymptotic expansion.

Theorem 1.4.1 ([GW20], Lemma 3.3)**.** *Let $\nu > 0$, $2r = n - 2$ and let $\mathbb{R}^n_+ \doteq \mathbb{R}^{n-1} \times [0, \infty)$. If $u \in \mathcal{H}^1(\mathbb{R}^n_+)$, then the restriction of u to $\mathbb{R}^{n-1} \times [0, \varepsilon)$ for any $\varepsilon > 0$ admits an asymptotic expansion*

$$u = x^{\nu_-} u_- + x^{r+1} H^1_b([0, \varepsilon); L^2(\mathbb{R}^{n-1})) \tag{1.53}$$

where $u_- \in H^\nu(\mathbb{R}^{n-1})$ while x is the coordinate along $[0, \infty)$. Furthermore, the application $u \mapsto \gamma_- u \doteq u_-$ is a continuous map from $\mathcal{H}^1(\mathbb{R}^n_+) \to H^\nu(\mathbb{R}^{n-1})$.

We can extend this result to a generic globally hyperbolic, asymptotically AdS spacetime using a partition of unity argument, obtaining a continuous map

$$\gamma_- : \mathcal{H}^1_0(M) \to \mathcal{H}^\nu(\partial M) \tag{1.54}$$

Similarly, we can extend γ_- on $\mathcal{H}^1_{loc}(M)$. The definition of γ_- depends on the choice of the boundary function x, therefore we assume that, chosen a boundary function, we continue to use the same. A useful property of the trace map γ_- is the following, see [Gan18].

Lemma 1.4.1. *Let $u \in \mathcal{H}^1_{loc}(M)$ be compactly supported. Then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|\gamma_- u\|^2_{L^2(\partial M)} \leq \varepsilon \|u\|^2_{\mathcal{H}^1_{loc}(M)} + C_\varepsilon \|u\|^2_{\mathcal{L}^2_{loc}(M)}$$

To conclude this section, we observe that we can generalize the notion of twisted derivatives and the definition of twisted Dirichlet form using different twisting factors. The starting point consists of defining a convenient class of twisting function to use instead of the boundary function.

Definition 1.4.1. We call smooth twisting function any $F \in x^{\nu_-}\mathcal{C}^\infty(M)$, such that $x^{-\nu_-}F > 0$ is strictly positive on M.

For any $B \in \mathbf{Diff}^1(M)$, it holds that $FBF^{-1} \in \mathbf{Diff}^1_\nu(M)$ and, conversely, any $Q \in \mathbf{Diff}^1_\nu(M)$ is of the form $Q = FBF^{-1}$ for some $B \in \mathbf{Diff}^1(M)$ with F a twisting function [GW20].

1.4.2 Interaction with b-calculus and wavefront sets

In this section we recall some results from [GW20], [Vas08] and [Vas10] concerning the interplay between properly supported b-ΨDOs and twisted differential operators. Thanks to Theorem 1.1.1, M is isometric to $\mathbb{R} \times \Sigma$ and therefore we can introduce a time coordinate $t \in \mathbb{R}$. Let F be any twisting function as per Definition 1.4.1. In the following we denote with Q_0 the operator $Q_0 := F\partial_x F^{-1} \in \mathbf{Diff}^1_\nu(M)$. We begin to study the interaction between b-pseudodifferential operators and $\mathbf{Diff}_\nu(M)$ with the following lemma, concerning the interplay between Q_0 and $\Psi^m_b(M)$.

Lemma 1.4.2 (Lemma 3.7 of [GW20])**.** *Let $A \in \Psi^m_b(M)$ have compact support in $U \subset M$. There exist two pseudodifferential operators $A_1 \in \Psi^{m-1}_b(M)$ and $A_0 \in \Psi^m_b(M)$ such that*

$$[Q_0, A] = A_1 Q_0 + A_0$$

where $\sigma_{b,m-1}(A_1) = -i\partial_\zeta \sigma_{b,m}(A)$ and $\sigma_{b,m}(A_0) = -i\partial_x \sigma_{b,m}(A)$, $\sigma_{b,m}$ being the principal symbol map, while (x, ζ) are the local coordinates on ${}^bT^*M$

introduced in Section 1.3.1. Also, the maps $A \mapsto A_0$ and $A \mapsto A_1$ are microlocal in the sense of Definition 1.3.15. Furthermore,

$$Q_0 A = A' Q_0 + A''$$

for some $A', A'' \in \Psi_b^m(M)$. The maps $A \mapsto A'$ and $A \mapsto A''$ are microlocal.

In Chapters 2 and 3 the space $\Psi_b^0(M)$ plays a pivotal rôle, since we shall employ bounded families of b-pseudodifferential operators in some proofs. For this reason, it is convenient to know how b-pseudodifferential operators of order zero act on twisted Sobolev spaces. The following two results answer to this question.

Lemma 1.4.3 (Lemma 3.8 of [GW20], Lemma 3.2 of [Vas08])**.** *Let $A \in \Psi_b^0(M)$. Then A is a continuous linear map*

$$\mathcal{H}^1_{loc/0}(M) \to \mathcal{H}^1_{loc/0}(M), \quad \dot{\mathcal{H}}^1_{loc/0}(M) \to \dot{\mathcal{H}}^1_{loc/0}(M),$$

which extends per duality to a continuous map

$$\dot{\mathcal{H}}^{-1}_{0/loc}(M) \to \dot{\mathcal{H}}^{-1}_{0/loc}(M), \quad \mathcal{H}^{-1}_{0/loc}(M) \to \mathcal{H}^{-1}_{0/loc}(M).$$

Remark 1.4.2. We recall that the spaces $\mathcal{H}_0^m(M)$ and $\dot{\mathcal{H}}_0^m(M)$, $m = \pm 1$, are defined in Equation (1.52).

An interesting consequence of this lemma is the following bound.

Proposition 1.4.1. Let $A \in \Psi_b^0(M)$ have compact support in $U \subset M$. Then there exists $\chi \in C_0^\infty(U)$ such that

$$\|Au\|_{\mathcal{H}^k(M)} \leq C \|\chi u\|_{\mathcal{H}^k(M)}$$

for every $u \in \mathcal{H}^k_{loc}(M)$ with $k = \pm 1$.

Remark 1.4.3. A similar bound holds true if $u \in \dot{\mathcal{H}}^k_{loc}(M)$.

Now we introduce a family of subspaces of $\mathcal{H}^k(M)$, for $k = -1, 0, 1$, enjoying additional regularity properties with respect to the action of b-pseudodifferential operators of fixed order. These spaces allow us to get a better control on estimates like that of Proposition 1.4.1.

Definition 1.4.2. Let $k = -1, 0, 1$ and let $m \geq 0$. Given $u \in \mathcal{H}^k_{loc}(M)$, we say that $u \in \mathcal{H}^{k,m}_{loc}(M)$ if $Au \in \mathcal{H}^k_{loc}(M)$ for all $A \in \Psi_b^m(M)$. Furthermore, we define $\mathcal{H}^{k,\infty}(M)$ as:

$$\mathcal{H}^{k,\infty}(M) \doteq \bigcap_{m=0}^{\infty} \mathcal{H}^{k,m}(M) \tag{1.55}$$

Remark 1.4.4. The spaces $\dot{\mathcal{H}}^{k,m}_{loc}(M)$, $\mathcal{H}^{k,m}(M)$ and $\mathcal{H}^{k,m}_0(M)$ are defined in a similar way. Furthermore, as observed in [Vas08], whenever m is finite, it is enough to check that both u and Au lie in $\mathcal{H}^k_{loc}(M)$ for a single elliptic operator $A \in \Psi^m_b(M)$. As a consequence, for $u \in \mathcal{H}^{k,m}_0(M)$ with $m \geq 0$, we can define the following norm:

$$\|u\|_{\mathcal{H}^{k,m}(M)} = \|u\|_{\mathcal{H}^k(M)} + \|Au\|_{\mathcal{H}^k(M)} \tag{1.56}$$

where A is any elliptic b-pseudodifferential operator in $\Psi^m_b(M)$.

Definition 1.4.3. Let $k = \pm 1$ and $m < 0$. Let $A \in \Psi^{-m}_b(M)$ be a fixed pseudo-differential operator of positive order. We call $\mathcal{H}^{k,m}_{loc}(M)$ the set of distributions $u \in \mathcal{D}'(M)$ of the form

$$u = u_1 + Au_2$$

where $u_1, u_2 \in \dot{\mathcal{H}}^k_{loc}(M)$.

Remark 1.4.5. In the same spirit of Remark 1.4.4, we can define $\dot{\mathcal{H}}^{k,m}_{loc}(M)$ and $\mathcal{H}^{k,m}(M)$ in a similar way. Furthermore, when $m < 0$ is finite, it is enough to check that both u and Au lie in $\mathcal{H}^k_{loc}(M)$ for a single elliptic operator $A \in \Psi^{-m}_b(M)$.

We can extend the trace γ_- defined in Equation (1.54) to these spaces as stated by the following lemma, whose proof can be found in [Vas08, Rem. 3.16], see also [GW20, Sec. 3.4].

Lemma 1.4.4. *Let $m < 0$ and let $\mathcal{H}^{k,m}(M)$ be as in Definition 1.4.3. Then γ_- as per Equation* (1.54) *extends to a continuous map*

$$\gamma_- : \mathcal{H}^{1,m}_{loc}(M) \to \mathcal{H}^{\nu+m}_{loc}(\partial M).$$

The notion of wavefront set can be refined for elements in $\mathcal{H}^{k,m}_{loc}(M)$ as follows, the definition for the other spaces such as $\dot{\mathcal{H}}^{k,m}_{loc}(M)$ and $\mathcal{H}^{k,m}(M)$ being analogous.

Definition 1.4.4. Let $k = 0, \pm 1$ and let $u \in \mathcal{H}^{k,m}_{loc}(M)$, $m \in \mathbb{R}$. Given $q \in {}^bT^*M \setminus \{0\}$, we say that $q \notin WF^{k,m}_b(u)$ if there exists $A \in \Psi^m_b(M)$ such that $q \in ell_b(A)$ and $Au \in \mathcal{H}^k_{loc}(M)$, where ell_b stands for the elliptic set as per Definition 1.3.12. When $m = +\infty$, we say that $q \notin WF^{k,\infty}_b(M)$ if there exists $A \in \Psi^0_b(M)$ such that $q \in ell_b(A)$ and $Au \in \mathcal{H}^{k,\infty}_{loc}(M)$.

Definition 1.4.4 is microlocal in the following sense:

$$WF_b^{k,m}(Au) \subset WF_b^{k,m-s}(u) \cup WF_b'(A)$$

for each $A \in \Psi_b^s(M)$, $s \geq 0$. In energy estimates it is useful to have at our disposal a quantitative version of this property, in the form of a bound. The following lemmas, collecting some results in [GW20] and [Vas08], answer to this need.

Lemma 1.4.5. *Let $\mathcal{A}$ be a bounded family in $\Psi_b^s(M)$ and let $G \in \Psi_b^s(M)$ be such that $WF_b'(\mathcal{A}) \subset ell_b(G)$. Suppose that $\mathcal{A}$ and G have compact support in $U \subset M$. Let $m \in \mathbb{R}$ and $k = \pm 1$. Then there exist $\chi \in \mathcal{C}_0^\infty(U)$ and a constant $C > 0$ such that*

$$\|Au\|_{\mathcal{H}^k(M)} \leq C \left(\|Gu\|_{\mathcal{H}^k(M)} + \|\chi u\|_{\mathcal{H}^{k,m}(M)} \right)$$

for every $u \in \mathcal{H}_{loc}^{k,m}(M)$ with $WF_b^{k,s}(u) \cap WF_b'(G) = \emptyset$ and for every $A \in \mathcal{A}$.

Lemma 1.4.6 (Lemma 3.13 [GW20])**.** *Let $\mathcal{A}$ be a bounded family of pseudodifferential operators in $\Psi_b^s(M)$ and let $G \in \Psi_b^{s-1}(M)$ be such that*

$$WF_b'(\mathcal{A}) \subset ell_b(G).$$

Suppose that $\mathcal{A}$ and G have compact support in $U \subset M$. Let $m \in \mathbb{R}$ and let $k = \pm 1$. Then there exist $\chi \in \mathcal{C}_0^\infty(U)$ and a constant $C > 0$ such that

$$\|Au\|_{\mathcal{L}^2(M)} \leq C \left(\|Gu\|_{\mathcal{H}^k(M)} + \|\chi u\|_{\mathcal{H}^{k,m}(M)} \right)$$

for every $u \in \mathcal{H}_{loc}^{k,m}(M)$ with $WF_b^{k,s-1}(u) \cap WF_b'(G) = \emptyset$ and for every $A \in \mathcal{A}$.

These two lemmas play a pivotal rôle in the following, when we employ energy estimates to prove the propagation of singularity theorem.

1.5 Boundary triples

The notion of boundary triple is a useful tool to parametrize the self-adjoint extensions of second order differential operators. In this section we discuss some basics facts on boundary triples following [DDF19, DM21b] and references therein.

Definition 1.5.1. Let H be a separable Hilbert space over $\mathbb{C}$ and let $P : D(P) \subset H \to H$ be a closed, linear and symmetric operator. A boundary triple for the adjoint operator P^* is a triple (h, γ_0, γ_1), where h is a separable Hilbert space over $\mathbb{C}$ and $\gamma_0, \gamma_1 : D(P^*) \to h$ are two linear maps satisfying

1) For every $f, f' \in D(P^*)$ it holds

$$(P^* f|f')_H - (f|P^* f')_H = (\gamma_1 f|\gamma_0 f')_h - (\gamma_0 f|\gamma_1 f')_h \tag{1.57}$$

2) The map $\gamma : D(P^*) \to h \times h$ defined by $\gamma(f) = (\gamma_0 f, \gamma_1 f)$ is surjective.

Remark 1.5.1. The notion of boundary triple is inspired by the theory of Sturm-Liouville operators on a half-line [BL12] and we can view Equation (1.57) as a generalization of Lagrange's identity for a boundary value problem. Therefore we can identify h as the space of boundary data. In the case at hand, we set $h = L^2(\partial M, dg^*)$, where $dg^* = \iota_M^* g$ is the pullback of the metric on the boundary.

One of the advantages of this framework is the fact that boundary triples allow to characterize the self-adjoint extensions of a linear, closed and symmetric operator on a Hilbert space in terms of boundary conditions on h as stated by the following proposition from [Mal92].

Proposition 1.5.1. Let P be a linear, closed and symmetric operator on H. Then an associated boundary triple (h, γ_0, γ_1) exists if and only if P^* has equal deficiency indices. In addition, if $\Theta : D(\Theta) \subseteq h \to h$ is a closed and densely defined linear operator, then $P_\Theta \doteq P^*|_{ker(\gamma_1 - \Theta\gamma_0)}$ is a closed extension of P with domain

$$D(P_\Theta) \doteq \{f \in D(P^*)|\gamma_0(f) \in D(\Theta),\ \gamma_1(f) = \Theta\gamma_0(f)\}.$$

The map $\Theta \mapsto P_\Theta$, associating a self-adjoint operators Θ over the boundary space h to a self-adjoint extension of P is one-to-one.

Boundary triples also allow to characterize the spectral properties of the self-adjoint extensions of an Hermitian operator. First, we need to introduce the notions of γ-field and Weyl function.

Definition 1.5.2. Let $P : D(S) \subseteq H \to H$ be a closed, symmetric operator and let (h, γ_0, γ_1) be an associated boundary triple. Moreover, consider the self-adjoint extension P_0 of P defined by $P_0 \doteq P^*|_{ker\gamma_0}$. We call γ-field and Weyl function respectively the maps $\Gamma : \rho(S_0) \to D(P^*)$ and $M : \rho(S_0) \to h$ such that

$$\Gamma(\lambda) \doteq [\gamma_0|_{\mathcal{N}_\lambda(P^*)}]^{-1}, \quad M(\lambda) \doteq \gamma_1 \circ \Gamma(\lambda)$$

where $\rho(S_0)$ is the resolvent of P_0.

The following theorem allows us to study the spectrum of P_Θ, through the knowledge of those of P_0, Θ and $M(\lambda)$.

Theorem 1.5.1. *Let $P : D(S) \subseteq H \to H$ be a closed, symmetric operator and let (h, γ_0, γ_1) be an associated boundary triple. Let P_Θ be a self-adjoint extension of P individuated by means of a self-adjoint operator $\Theta : D(\Theta) \subset h \to h$. Let ρ, σ_p and σ_c indicate respectively resolvent, point spectrum and continuous spectrum of an operator. Then, for every $\lambda \in \rho(P_0)$, $P_0 = D(P^*)|_{ker\gamma_0}$, it holds:*

1) $\lambda \in \rho(P_\Theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$, where M is the Weyl function.

2) $\lambda \in \sigma_i(P_\Theta)$, $i = p, c$ if and only if $0 \in \sigma_i(\Theta - M(\lambda))$.

Remark 1.5.2. The main consequence of this proposition is that the computation of the spectrum of P_Θ, a self-adjoint extension of P, is tantamount to the evaluation of the spectra of P_0 and of $\Theta - M(\lambda)$.

Chapter 2

The wave equation on aAdS spacetimes

2.1 The wave equation on aAdS spacetimes

Let us consider a set of special coordinates $(x, y_0, \cdots, y_{n-1})$ on a coordinate patch of the form $[0, \varepsilon) \times U$, with x a boundary function. In these coordinates, the Klein-Gordon equation for a scalar field u can be written as

$$\left[(-x\partial_x)^2 + (n-1)(x\partial_x) + xE(x\partial_x) + x^2\Box_{h_x} + \left(\frac{n-1}{2}\right)^2 - \nu^2\right] u = 0 \tag{2.1}$$

with h_x as Equation (1.9). In this expression $E := -\partial_x \, ln|\hat{g}|$ is a smooth function. For future convenience, it is useful to rewrite Equation (2.1) also in terms of twisted derivatives. A short computation yields

$$x^2 \left(Q_0^* Q_0 + Q_\alpha \hat{g}^{\alpha\beta} Q_\beta\right) u + S_F u + \mu^2 u = 0 \tag{2.2}$$

with $\mu := \left(\frac{n-1}{2}\right)^2 - \nu^2$ and $S_F := x^{-\nu_-} P(x^{\nu_-})$ with P the Klein-Gordon operator as in eq. (2.1). We note that in the case of a massless scalar field, $\nu = \frac{n-1}{2}$, the singular potential in the equation above vanishes. In this case $Q_0 = \partial_x$ and the Sobolev space $\mathcal{H}^1(M)$ coincides with $H^1(M)$ defined in terms of ordinary derivatives $\partial_x, \partial_{y_1}, \cdots$. Furthermore, the trace γ_- becomes the usual Lions trace, see [GW20, DDF19]. As we shall see, these observations simplify the analysis, allowing to prove also the support properties of the propagators, see Theorem 4.3.1.

2.2 Asymptotic expansion and traces

To employ the formalism of boundary triples, in addition to the trace map γ_- individuated in Equation (1.54), we need another trace map γ_+. To this end, following [GW20], we introduce a family of functional spaces enjoying additional regularity with respect to the Klein-Gordon operator.

Definition 2.2.1. Let (M, g) be a globally hyperbolic, asymptotically anti-de Sitter spacetime and let P be the Klein-Gordon operator as in Equation (2.1). For all $m \in \mathbb{R}$, we define the Fréchet spaces

$$\mathcal{X}^m(M) = \{u \in \mathcal{H}^{1,m}_{loc}(M) \mid Pu \in x^2\mathcal{H}^{0,m}_{loc}(M)\}, \tag{2.3}$$

with respect to the seminorms

$$\|u\|_{\mathcal{X}^m(M)} = \|\phi u\|_{\mathcal{H}^{1,m}(M)} + \left\|x^{-2}\phi Pu\right\|_{\mathcal{H}^{0,m}(M)}, \tag{2.4}$$

where ϕ is a suitable smooth and compactly supported function.

Remark 2.2.1. We observe that when K is a relatively compact subset of M we can introduce the space $\mathcal{H}^{k,m}(K)$, with $k = 0, 1$ and $m > 0$, see Remark 1.4.4, and can define

$$\mathcal{X}^m(K) = \{u \in \mathcal{H}^{1,m}(K) \mid x^{-2}Pu \in \mathcal{H}^{0,m}(K)\},$$

endowed with the norm

$$\|u\|_{\mathcal{X}^m(K)} = \|u\|_{\mathcal{H}^{1,m}(K)} + \left\|x^{-2}Pu\right\|_{\mathcal{H}^{0,m}(K)} \tag{2.5}$$

The reason we introduced these spaces is that, for every $m \in \mathbb{R} \cup \{\infty\}$, given a function in $\mathcal{X}^m(M)$, we can improve the expansion of Theorem 1.4.1 as follows.

Lemma 2.2.1 (Lemma 4.6 in [GW20])**.** *Let $(\mathbb{R}^n_+, g)$ be an asymptotically AdS spacetime such that, with respect to the standard Cartesian coordinates, the line elements reads*

$$g = \frac{-dx^2 + h_{ab}dy^a dy^b}{x^2}. \tag{2.6}$$

Consider an admissible twisting function F, as per Definition 1.4.1 such that at $x = 0$ $x^{-\nu_-}F = 1$, where $\nu_- = \frac{1}{2} - \nu$ is the indicial root. If $u \in \mathcal{H}^{1,k}_0(\mathbb{R}^n_+)$ and $Pu \in x^2\mathcal{H}^{0,k}_0(\mathbb{R}^n_+)$ for $k \geq 0$, then, for any $\varepsilon > 0$ the restriction of u to $\mathbb{R}^{n-1} \times [0, \varepsilon)$ admits an asymptotic expansion

$$u = Fu_- + x^{\nu_+}u_+ + x^{r+2}H^{k+2}_b([0, \varepsilon); H^{k-3}(\mathbb{R}^{n-1})) \tag{2.7}$$

where $2r = n - 2$, $u_- \in H^{\nu+k}(\mathbb{R}^{n-1})$ and $u_+ \in H^{-1-2\nu+k}(\mathbb{R}^{n-1})$.

This lemma allows us to define the trace map γ_+ on $\mathcal{X}^\infty(M)$ as

$$\gamma_+ u = x^{1-2\nu}\partial_x(F^{-1}u)|_{\partial X} \tag{2.8}$$

Since in a special coordinate patch, the restriction of u to the boundary can be written as

$$u = x^{\nu_-}Fu_- + x^{\nu_+}u_+ + u_2, \quad u_2 \in x^2\mathcal{H}^{2,\infty}_{loc}([0,\varepsilon)\times\mathbb{R}^{n-1}). \tag{2.9}$$

a direct inspection shows that in these coordinates $\gamma_+ u = 2\nu u_+$. In the next section, we shall extend the trace γ_+ to $\mathcal{X}^k(M)$ for any $k \in \mathbb{R}$.

Remark 2.2.2. The second term of the expansion, of the form $x^{\nu_+}u_+$ is the leading term of the asymptotic behavior of a solution of the Klein-Gordon equation with Dirichlet boundary conditions on an anti-de Sitter spacetime. For this reason, we call γ_+ the Dirichlet trace map, see [DFM18] as well as the example concerning the Klein-Gordon equation in AdS_2 we discussed at the beginning of Section 1.4.

2.3 Weak formulation of the problem

In this section we use the analytic tools introduced in Chapter 1 to give a weak formulation for the Klein-Gordon equation (2.1) on asymptotically anti-de Sitter spacetimes with a boundary condition implemented by pseudodifferential operators. We consider only the case $\nu \in (0,1)$. We do not study the case $\nu = 0$ because it requires each time a separate analysis. The values of the mass for which $\nu \geq 1$ do not require a boundary condition [DDF19, DFM18, GW20].

2.3.1 The twisted Dirichlet energy form

Consider a twisting function F as per Definition 1.4.1. Motivated by the study of the Klein-Gordon equation in $PAdS_2$, see Section 1.4, we define the twisted differential

$$d_F \doteq F \circ d \circ F^{-1},$$

whose action on smooth functions vanishing at ∂M together with all its derivatives is

$$d_F : \dot{C}^\infty(M) \to \dot{C}^\infty(M; T^*M), \quad v \mapsto d_F v = Fd(F^{-1}v) = dv + vF^{-1}dF$$

Given $u, v \in \mathcal{L}^2_{loc}(M)$, we define the twisted Dirichlet form by:

$$\mathcal{E}_0(u,v) = -\int\limits_M g(d_F u, d_F \overline{v}) d\mu_g, \tag{2.10}$$

where $d\mu_g$ is the metric induced volume form. Note that, if $u, v \in \mathcal{H}^1_{loc}(M)$ with $supp(u) \cap supp(v)$ compact, then $\mathcal{E}_0(u,v)$ is finite. Using the twisted differential, we can rewrite the Klein-Gordon operator as follows

$$P = -(d_F)^\dagger d_F + F^{-1}P(F), \tag{2.11}$$

where $(d_F)^\dagger$ is the formal adjoint of d_F with respect to the inner product on $L^2(M; d\mu_g)$. As observed in [War12], twisted differentials can be used to regularize the energy form in the case in which the multiplication by $S_F = F^{-1}P(F) \in \mathcal{C}^\infty(\mathring{M})$ is a bounded operator from $\mathcal{L}^2(M)$ to $x^2\mathcal{L}^2(M)$, where x is the boundary function. For this reason we consider only a particular class of twisting functions:

Definition 2.3.1. A twisting function F as in Definition 1.4.1 is called *admissible* if $S_F \doteq F^{-1}PF \in x^2L^\infty(M)$ where P is the Klein-Gordon operator.

Suppose that $\nu \in (0,1)$ and let $u, v \in \mathcal{X}^\infty(M)$. Then, if F is an admissible twisting function, the following Green formula holds true:

$$\int_M Pu \cdot \overline{v}\, d\mu_g = \mathcal{E}_0(u,v) + \int_M S_F u \cdot \bar{v}\, d\mu_g + \int_{\partial M} \gamma_+ u \cdot \gamma_- \bar{v}\, d\mu_h, \tag{2.12}$$

with $d\mu_h$ the volume form induced by h – the pull-back of g to ∂M. As a matter of fact we can extend the domain of the trace map γ_+, and therefore that of Equation (2.12), as discussed in [GW20, Lemma 4.8]:

Lemma 2.3.1. *The map γ_+ as per Equation* (2.8) *can be extended to a bounded map*

$$\gamma_+ : \mathcal{X}^k(M) \to \mathcal{H}^{k-\nu}_{loc}(\partial M), \quad \forall k \in \mathbb{R}$$

and, if $u \in \mathcal{X}^k(M)$, the Green's formula in Equation (2.12) *holds true for every $v \in \mathcal{H}^{1,-k}_0(M)$.*

2.3.2 Boundary conditions and the associated Dirichlet form

In this section we illustrate the weak formulation of the Klein-Gordon equation with boundary conditions implemented by pseudodifferential operators. Formally, we look for $u \in H^1_{loc}(M)$ such that

$$Pu = f, \quad \text{and} \quad \gamma_+ u = \Theta \gamma_- u, \tag{2.13}$$

where P is the Klein-Gordon operator, $f \in \mathcal{H}^{-1}_{loc}(M)$ and $\Theta \in \Psi^k(\partial M)$. In order for this problem to be defined in a strong sense, we also need that $Pu \in x^2 L^2_{loc}(M)$. To avoid focusing on this issue we give a weak formulation. Let $\Theta \in \Psi^k(\partial M)$ and define the energy form

$$\mathcal{E}_\Theta(u,v) = \mathcal{E}_0(u,v) + \int_M S_F u \cdot \overline{v} d\mu_g + \int_{\partial M} \Theta\gamma_- u \cdot \gamma_- \overline{v} d\mu_h, \tag{2.14}$$

where $u \in \mathcal{H}^{1,m+k}_{loc}(M)$, $v \in \mathcal{H}^{1,m+k}_0(M)$ while F is an admissible twisting function, whose existence is assumed a priori. Then, we introduce the operator $P_\Theta : \mathcal{H}^{1,m+k}_{loc}(M) \to \dot{\mathcal{H}}^{-1,m+k}_{loc}(M)$, $m \in \mathbb{R}$ defined as

$$\langle P_\Theta u, v\rangle = \mathcal{E}_\Theta(u,v) \tag{2.15}$$

The weak formulation of the problem in Equation (2.13) is given by:

$$\langle P_\Theta u, v\rangle = \langle f, v\rangle \tag{2.16}$$

Remark 2.3.1. In this work, for simplicity, we denote with $\langle , \rangle$ different pairings, since the exact meaning can be understood from the context without risk of confusion. For example, in Equation (2.15), the brackets $\langle , \rangle$ denote the pairing between $\mathcal{H}^1(M)$ and $\dot{\mathcal{H}}^{-1}(M)$.

We end this chapter establishing three microlocal estimates for the Dirichlet form introduced above. The first one is the following bound, which does not depend on the boundary conditions.

Lemma 2.3.2 ([GW20], Lemma 5.2)**.** *Let $U \subset M$ be a coordinate patch such that $U \cap \partial M \neq \emptyset$ and let $m \leq 0$. Let $\mathcal{A} = \{A_r \mid r \in (0,1)\}$ be a bounded subset of $\Psi^s_b(M)$, $s \in \mathbb{R}$ with compact support in U, such that*

$$A_r \in \Psi^m_b(M) \text{ for each } r \in (0,1)$$

Let $G_1 \in \Psi^{s-1/2}_b(M)$ be elliptic on $WF'_b(\mathcal{A}) \subset {}^bT^*M \setminus \{0\}$, with compact support in U. Then there exist $C_0 > 0$ and $\chi \in \mathcal{C}^\infty_0(U)$ such that

$$\mathcal{E}_0(A_r u, A_r u) \leq \mathcal{E}_0(u, A_r^* A_r u) + C_0 \left(\|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} \right)$$

for every $r \in (0,1)$ and every $u \in \mathcal{H}^{1,m}(M)$, provided that

$$WF^{1,s-1/2}_b(u) \cap WF'_b(G_1) = \emptyset$$

Now we prove an estimate for the boundary value problem associated with a pseudodifferential operator $\Theta \in \Psi^k(\partial M)$. We can control two different classes:

- $\Theta \in \Psi^k(\partial M)$ with $k \leq 0$,
- $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$.

The two cases must be analyzed separately, therefore most of the microlocal estimates in this chapter and in the next one are given in two versions, one for $k \leq 0$ and the other for $0 < k \leq 2$. In the following two lemmas we bound the difference between a generic positive-definite sesquilinear pairing form $\mathcal{Q}$ and the Dirichlet energy form $\mathcal{E}_0$ introduced in Equation (2.10). We shall employ these estimates in the next chapter, to prove a propagation of singularities theorem.

Remark 2.3.2. In the next two lemmas – and in the remainder of this work – we need to consider an extension of Θ to a b-pseudodifferential operator $\Psi_b^k(M)$. To this end, we consider a collar neighbourhood of ∂M, with x the local coordinate subordinated to the normal direction to the boundary, so that $\partial M = \{x = 0\}$. Let $\chi \equiv \chi(x) \in C_0^\infty(M)$ such that $\chi = 1$ in a neighbourhood of $x = 0$ a function playing the rôle of microlocal cutoff. Given a pseudodifferential operator $\Theta \in \Psi^k(\partial M)$, the map $\Theta \to \Theta_\chi \doteq \chi\Theta$, identifies an element of $\Psi_b^k(M)$ which we call an extension of Θ to $\Psi_b^k(M)$. Since all our results are independent of the choice of χ, with a slight abuse of notation, we denote the extension of Θ with the same symbol.

Lemma 2.3.3. *Let $U \subset M$ be a coordinate patch such that $U \cap \partial M \neq \emptyset$ and let $m \leq 0$. Suppose $\Theta \in \Psi_b^k(\partial M)$ with $k \leq 0$. Let $\mathcal{A} = \{A_r \mid r \in (0,1)\}$ be a bounded subset of $\Psi_b^s(M)$, $s \in \mathbb{R}$, with compact support in U, such that*

$$A_r \in \Psi_b^m(M) \text{ for each } r \in (0,1)$$

Let $G_0 \in \Psi_b^s(M)$ be elliptic on $WF_b'(\mathcal{A})$ and let $G_1 \in \Psi_b^{s-1/2}(M)$ be elliptic on $WF_b'(\mathcal{A})$, both with compact support in U. In addition let $\mathcal{E}_0$ and $\mathcal{Q}$ be respectively the twisted Dirichlet form and a generic positive-definite sesquilinear pairing both defined on $\mathcal{H}^1_{loc}(M)$. Then there exists $C_0 > 0$ and $\chi \in \mathcal{C}_0^\infty(U)$ such that

$$\mathcal{E}_0(A_r u, A_r u) - \varepsilon\mathcal{Q}(A_r u, A_r u) \leq$$
$$C_0\Big(\|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)}\Big)$$

for every $r \in (0,1)$ and every $u \in \mathcal{H}^{1,m}(M)$, provided that the following conditions are met:

$$WF_b^{-1,s}(P_\Theta u) \cap WF_b'(G_0) = \emptyset$$

$$WF_b^{1,s-1/2}(u) \cap WF_b'(G_1) = \emptyset$$

Proof. We start by considering an extension of $\Theta \in \Psi^k(\partial M)$ to the whole M as per Remark 2.3.2. With a slight abuse of notation, we use the symbol Θ in both cases.

In order to bound

$$\mathcal{E}_0(A_r u, A_r u) - \varepsilon \mathcal{Q}(A_r u, A_r u), \tag{2.17}$$

it is convenient to rewrite this expression as

$$\begin{aligned} &\mathcal{E}_0(A_r u, A_r u) - \mathcal{E}_0(u, A_r^* A_r u) + \\ &+\mathcal{E}_0(u, A_r^* A_r u) - \mathcal{E}_\Theta(u, A_r^* A_r u) + \\ &+\mathcal{E}_\Theta(u, A_r^* A_r u) - \varepsilon \mathcal{Q}(A_r u, A_r u) \end{aligned} \tag{2.17}$$

Applying Lemma 2.3.2, we can bound the first line of Equation (2.17) as

$$\mathcal{E}_0(A_r u, A_r u) - \mathcal{E}_0(u, A_r^* A_r u) \leq C_0 \left(\|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} \right)$$

The third line can be controlled as follows: Calling $f = P_\Theta u$, we can write $\mathcal{E}_\Theta(u, A_r^* A_r u) = \langle A_r f, A_r u \rangle$. Using the pairing between $\mathcal{H}^1(M)$ and $\dot{\mathcal{H}}^{-1}(M)$,

$$\langle A_r f, A_r u \rangle \leq \|A_r f\|_{\dot{\mathcal{H}}^{-1}(M)} \|A_r u\|_{\mathcal{H}^1(M)}.$$

Since for $C \geq 1/2$, $ab \leq C(a^2 + b^2)$ for any $a, b \in \mathbb{R}$, then it holds that

$$\|A_r f\|_{\dot{\mathcal{H}}^{-1}(M)} \|A_r u\|_{\mathcal{H}^1(M)} \leq C \left(\|A_r f\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|A_r u\|^2_{H^1(M)} \right) \tag{2.18}$$

Using Lemma 1.4.6 one obtains

$$\begin{aligned} &\|A_r u\|^2_{\mathcal{H}^1(M)} - \varepsilon \mathcal{Q}(A_r u, A_r u) \leq \\ &\leq C_1 \left(\|G_0 u\|_{\mathcal{H}^1(M)} + \|\chi u\|_{\mathcal{H}^{1,m}(M)} \right)^2 - \varepsilon \mathcal{Q}(A_r u, A_r u) \end{aligned}$$

where $G_0 \in \Psi_b^s(M)$. Applying again the inequality $ab \leq C(a^2 + b^2)$, the second term in Equation (2.18) is bounded by

$$\|A_r u\|^2_{H^1(M)} \leq C \left(\|G_0 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} \right) - \varepsilon \mathcal{Q}(A_r u, A_r u), \tag{2.19}$$

where $G_0 \in \Psi_b^s(M)$. We estimate the first term of (2.18) using an analogue procedure, this time with the help of Lemma 1.4.5:

$$\|A_r f\|_{\dot{\mathcal{H}}^{-1}(M)} \leq C_2 \left(\|\chi f\|_{\dot{\mathcal{H}}^{-1}(M)} + \|G_0 u\|_{\dot{\mathcal{H}}^{-1,m}(M)} \right), \tag{2.20}$$

where $G_0 \in \Psi_b^s(M)$. Combining Equations (2.19) and (2.20), we obtain the bound

$$\begin{aligned} &|\langle A_r f, A_r u\rangle| \leq \varepsilon \mathcal{Q}(A_r u, A_r u) + \\ &+ C \left(\|G_0 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi f\|_{\dot{\mathcal{H}}^{-1}(M)} + \|G_0 u\|_{\dot{\mathcal{H}}^{-1,m}(M)} \right) \end{aligned} \tag{2.21}$$

At last, we control the second line in Equation (2.17).

$$\begin{aligned} &\mathcal{E}_\Theta(u, A_r^* A_r u) - \mathcal{E}_0(u, A_r^* A_r u) = \\ &= \langle x^{-2} S_F u, A_r^* A_r u\rangle + \langle \Theta \gamma_- u, \gamma_-(A_r^* A_r u)\rangle_{\partial M} \end{aligned} \tag{2.22}$$

Using that $S_F \in x^2 \mathcal{C}^\infty(M)$, *cf.* Definition 2.3.1, it holds

$$\begin{aligned} \langle x^{-2} S_F u, A_r^* A_r u\rangle &= \int_U x^{-2} S_F u \overline{A_r^* A_r u}\; x^2 d\mu_g \leq \\ &\leq \max_{x \in \pi_1 \circ \mathrm{supp}(A_r)} \left| x^{-2} S_F \right| \cdot |\langle u, A_r^* A_r u\rangle|. \end{aligned}$$

In order to control $\langle u, A_r^* A_r u\rangle = \|A_r u\|^2_{\mathcal{L}^2(M)}$, we use the same algebraic trick as above. On account of Lemma 1.4.6, it holds

$$|\langle x^{-2} S_F u, A_r^* A_r u\rangle| \leq C_0 \left(\|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} \right).$$

At last, we focus on the boundary term $\langle \Theta \gamma_- u, \gamma_-(A_r^* A_r u)\rangle|_{\partial M}$. We recall that for every $B \in \Psi_b^m(M)$, it holds

$$\gamma_-(Bu) = \left(x^{-\nu_-} B u \right)\big|_{\partial M} = \widehat{N}(B)(-i\nu_-)(\gamma_- u),$$

where $\widehat{N}(B)$ is the indicial family of B. Extending Θ as explained at the beginning of the proof, we can write $\Theta f = \widehat{N}(\Theta)(-i\nu_-) f$ for every $f \in Dom(\Theta) \cap L^2(\partial M)$. We also note that, using Equation (1.41), it holds

$$\widehat{N}(A_r^* A_r)(-i\nu_-) = \widehat{N}(\widetilde{A}_r)(-i\nu_-)^* \widehat{N}(A_r)(-i\nu_-)$$

where $\widetilde{A}_r = x^{2\nu_-} A_r x^{-2\nu_-}$ and where the adjoint is computed with respect to the L^2-paring induced by the metric h on ∂M. Using these data, we can rewrite the boundary term as

$$\begin{aligned} &\langle \Theta \gamma_- u, \gamma_-(A_r^* A_r u)\rangle_{\partial M} = \langle \widehat{N}^*(\widetilde{A}_r \Theta)(-i\nu_-)\gamma_- u, \widehat{N}(A_r)(-i\nu_-)\gamma_- u\rangle_{\partial M} = \\ &= \langle \gamma_-(\widetilde{A}_r \Theta) u, \gamma_-(A_r u)\rangle_{\partial M} = \langle \gamma_-(\Theta \widetilde{A}_r u + [\widetilde{A}_r, \Theta] u), \gamma_- A_r u\rangle_{\partial M} \end{aligned}$$

Using Cauchy-Schwartz inequality and Lemma 1.4.1 it holds

$$|\langle \gamma_- \Theta \widetilde{A}_r u, \gamma_- A_r u \rangle_{\partial M}| \leq C_1 \|\Theta \widetilde{A}_r u\|_{\mathcal{L}^2(M)} + C_2 \|A_r u\|^2_{\mathcal{L}^2(M)},$$

where C_1 and C_2 are suitable constants. If $\Theta \in \Psi^k(\partial M)$ with $k \leq 0$ it holds

$$\|\Theta \widetilde{A}_r u\|^2_{\mathcal{L}^2(M)} \leq \|\Theta \widetilde{A}_r u\|^2_{\mathcal{H}^1(M)} \leq \|\chi \widetilde{A}_r u\|^2_{\mathcal{H}^1(M)}$$

Thus, proceeding as in the previous case, using Lemma 1.4.1 we arrive at the same estimate for $|\langle \Theta \gamma_- u, \gamma_-(A_r^* A_r u)\rangle|_{\partial M}|$. Combining all the bounds together with Equation (2.17), we obtain the sought thesis. □

We conclude this chapter formulating a counterpart of Lemma 2.3.3 for the case in which $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$. As in the previous case, we extend Θ to an operator over M as per Remark 2.3.2. We also observe that each Θ identifies per duality a map from $\mathcal{H}^1_{loc}(M)$ to $\dot{\mathcal{H}}^{-1}_{loc}(M)$.

Lemma 2.3.4. *Let $U \subset M$ be a coordinate patch such that $U \cap \partial M \neq \emptyset$ and let $m \leq 0$. Let $\Theta \in \Psi^k_b(\partial M)$ with $0 < k \leq 2$ and let $\mathcal{A} = \{A_r \mid r \in (0,1)\}$ be a bounded subset of $\Psi^s_b(M)$, $s \in \mathbb{R}$, with compact support in U, such that*

$$A_r \in \Psi^m_b(M) \text{ for each } r \in (0,1).$$

Let $G_0 \in \Psi^s_b(M)$ and $G_1 \in \Psi^{s-1/2}_b(M)$ be elliptic on $WF'_b(\mathcal{A})$, both with compact support in U. Then there exist $C_0 > 0$ and $\chi \in \mathcal{C}^\infty_0(U)$ such that

$$\mathcal{E}_0(A_r u, A_r u) - \varepsilon \mathcal{Q}(A_r u, A_r u) \leq C_0 \Big(\|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \\ + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|G_0 \Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi \Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} \Big)$$

for every $r \in (0,1)$ and every $u \in \mathcal{H}^{1,m+k}_{loc}(M)$, provided that the following conditions are met:

$$WF^{-1,s}_b(P_\Theta u) \cap WF'_b(G_0) = \emptyset,$$

$$WF^{-1,s}_b(\Theta u) \cap WF'_b(G_0) = \emptyset,$$

$$WF^{1,s-1/2}_b(u) \cap WF'_b(G_1) = \emptyset.$$

Proof. The proof is analogous to that of Lemma (2.3.3), hence we do not enter into the details. We point out that the only key difference is the estimate of the boundary term $\langle \Theta \gamma_- u, \gamma_-(A_r^* A_r u)\rangle|_{\partial M}$. Thanks to [Vas08, Lemma 3.18], which holds true also for $\dot{\mathcal{H}}^{-1}$ – see the discussion below

Definition 3.14 and the proof of Lemma 3.14 in [Vas08], we can control the boundary terms as

$$\begin{aligned}\langle\gamma_-(\widetilde{A}_r\Theta)u,\gamma_-(A_ru)\rangle|_{\partial M}&\leq C_0\left(\|\widetilde{A}_r\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)}+\|A_ru\|^2_{\mathcal{H}^1(M)}\right)\\ \leq C\left(\|\chi u\|^2_{\mathcal{H}^{1,m}(M)}+\|G_1u\|^2_{\mathcal{H}^1(M)}\right.&\left.+\|G_0\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)}+\|\chi\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)}\right)\end{aligned} \tag{2.23}$$

Observe that in this chain of inequalities, we also employed that $\Psi_b^{m+k}(M)\subseteq\Psi_b^m(M)$ if $k\geq 0$. □

Remark 2.3.3. Using that, for $\varepsilon>0$, $\Psi_b^m(M)\subset\Psi_b^{m+\varepsilon}(M)$, the previous results hold true also for $G_1\in\Psi_b^{s-1/2}(M)$, similarly to what happens in [GW20].

Remark 2.3.4. In the previous lemmas we considered only $k\leq 2$, because if we would have allowed k to be larger than 2, we would have not been able to prove in general a result similar to Lemma 2.3.4. At the level of applications, this is a mild constraint, since interesting examples of boundary conditions, such as the Robin ones discussed in [GW20] or those of Wentzell type, see [DFJA18, Zah18], satisfy the constraint $k\leq 2$.

Chapter 3

Propagation of singularities theorems

This chapter is devoted to proving a propagation of singularities theorem for the Klein-Gordon operator subject to boundary conditions implemented by a b-pseudodifferential operator $\Theta \in \Psi^k(\partial M)$, with $k \leq 2$. Prior to proving the microlocal estimates needed to establish the sought result, we need to introduce the notion of compressed characteristic set of the Klein-Gordon operator over the compressed b-cotangent bundle ${}^b\dot{T}^*M$.

3.0.1 The compressed characteristic set

We begin studying the characteristic set of the principal symbol of the Klein-Gordon operator. Recall that the principal symbol of $x^{-2}P$ is $\widehat{p} \doteq \widehat{g}(X, X)$, with $X \in \Gamma(T^*M)$. The associated characteristic set is

$$\mathcal{N} = \left\{(q, k_q) \in T^*M \setminus \{0\} \mid \widehat{g}^{ij}(k_q)_i(k_q)_j = 0\right\}, \tag{3.1}$$

while the compressed characteristic set is

$$\dot{\mathcal{N}} = \pi[\mathcal{N}] \subset {}^b\dot{T}(M), \tag{3.2}$$

where π is the projection map from T^*M to the compressed cotangent bundle, *cf.* Equation (1.15). We equip $\dot{\mathcal{N}}$ with the subspace topology inherited from ${}^bT^*M$. To prove the propagation of singularities theorem, it is convenient to individuate in the compressed b-cotangent bundle the following three conic subsets:

- The *elliptic* region

$$\mathcal{E}(M) = \{q \in {}^b\dot{T}^*M \setminus \{0\} \ : \ \pi^{-1}(q) \cap \mathcal{N} = \emptyset\}, \tag{3.3}$$

where $\pi : T^*M \to {}^b\dot{T}^*M$.

- The *glancing* region

$$\mathcal{G}(M) = \{q \in {}^b\dot{T}^*M \setminus \{0\} \ : \ Card(\pi^{-1}(q) \cap \mathcal{N}) = 1\}, \tag{3.4}$$

where $Card$ refers to the cardinality of a set.

- The *hyperbolic* region

$$\mathcal{H}(M) = \{q \in {}^b\dot{T}^*M \setminus \{0\} \ : \ Card(\pi^{-1}(q) \cap \mathcal{N}) = 2\}. \tag{3.5}$$

We observe that we can characterize the three regions $\mathcal{E}(M), \mathcal{G}(M)$ and $\mathcal{H}(M)$ also using local coordinates. Let $\widetilde{q} \in {}^bT^*\partial M$ be such that $\widetilde{q} = (0, y_i, 0, \eta_i)$, $i = 1, \dots, n-1$, where we used the same coordinates introduced in Section 1.1.3. Then $\pi^{-1}(\widetilde{q}) = (0, y_i, \xi, \eta_i)$, with $(\xi, \eta_i) \in T^*_{(0,y_i)}M$ and therefore $\pi^{-1}(\widetilde{q}) \simeq \mathbb{R}$. Equations (3.1) and (1.9) yield that $\pi^{-1}(\widetilde{q}) \cap \mathcal{N}$ corresponds to the points whose coordinates solve the equation $\xi^2 + h^{ij}\eta_i\eta_j = 0$. This entails that a point $\widetilde{q} \in {}^bT^*M$ lies in $\mathcal{H}(M)$ when $h^{ij}\eta_i\eta_j < 0$, in $\mathcal{G}(M)$ when $h^{ij}\eta_i\eta_j = 0$ and in $\mathcal{E}(M)$ when $h^{ij}\eta_i\eta_j > 0$.

Definition 3.0.1. Let $I \subset \mathbb{R}$ be an interval. A continuous map $\gamma : I \to \dot{\mathcal{N}}$ is called a *generalized broken bicharacteristic* (GBB) if for every $s_0 \in I$ the following conditions hold true:

a) If $q_0 = \gamma(s_0) \in \mathcal{G}$, then for every $\omega \in \Gamma^\infty({}^bT^*M)$,

$$\frac{d}{ds}(\omega \circ \gamma) = \{\widehat{p}, \pi^*\omega\}(\eta_0), \tag{3.6}$$

where $\eta_0 \in \mathcal{N}$ is the unique point for which $\pi(\eta_0) = q_0$, with $\pi : T^*M \to {}^bT^*M$ the projection map introduced in Equation (1.15), and $\{,\}$ are the Poisson brackets on T^*M.

b) If $q_0 = \gamma(s_0) \in \mathcal{H}$, then there exists $\varepsilon > 0$ such that $0 < |s - s_0| < \varepsilon$ implies $x(\gamma(s)) \neq 0$, where x is the global boundary function.

Remark 3.0.1. The first condition is basically telling us that in the glancing region generalized broken bicharacteristics are integral curves of the Hamilton vector field associated with the principal symbol $\widehat{p}$. The second condition entails that, at hyperbolic points, GBBs reflect instantaneously. In particular a GBB coming from $\mathring{M}$ propagates along the boundary only at glancing points. Also note that, since $\gamma \in C^0(I; \dot{\mathcal{N}})$, the component of the co-vector tangent to the boundary is conserved.

The space of GBBs enjoys some notable properties that we need in the proof of the propagation of singularities theorem. We collect them in the following lemma, which summarizes the results from [Leb97] and [Vas08].

Lemma 3.0.1. *Let $\mathcal{R}_K[a,b]$ be the space of the generalized broken bicharacteristics $\gamma : [a,b] \to K$ where $K \subset \dot{\mathcal{N}}$ is compact. Let γ_n be a sequence in $\mathcal{R}_K[a,b]$ converging uniformly to a curve γ. Then $\gamma : [a,b] \to K$ is a generalized broken bicharacteristic. In addition, if $\mathcal{R}_K[a,b]$ is not empty, then it is compact in the uniform topology.*

In the following we focus on the boundary ∂M. Let us consider a chart $U \subset M$ such that $U \cap \partial M \neq \emptyset$. Following the conventions introduced in Section 1.1.3, we consider on $T^*_U M$ the local coordinates (x, y_i, ξ, η_i), $i = 1, \dots, n-1$. We identify the time coordinate with y_{n-1}, while η_{n-1} is the associated dual coordinate. In these coordinates the following lemma holds true.

Lemma 3.0.2. *If $q_0 \in {}^bT^*_U M \setminus \{0\}$, there exists a conic neighborhood V of q_0 in which one of the following facts is true:*

1) If $q_0 \in {}^b\dot{T}^*M$, there exists $\varepsilon > 0$ such that $\sigma^2 < \varepsilon^2(\beta\eta^2_{n-1} + \kappa^{ij}\eta_i\eta_j)$ and $\kappa^{ij}\eta_i\eta_j > \beta\eta^2_{n-1}$.

2) If $q_0 \notin {}^b\dot{T}^*M$, there exists $C > 0$ such that $|\eta_{n-1}| < C|\sigma|$

The proof is identical to that of [GW20, Lemma 6.2], except that we have to take into account the specific form of the metric on ∂M, see the discussion in Remark 1.1.1.

Remark 3.0.2. For simplicity, in the following we work with pseudodifferential operators whose compact support is contained in a fixed local chart. However, using a partition of unity argument, our results are also valid in the general case in which the support of the ΨDO is not contained in one coordinate patch.

In order to prove a propagation of singularities theorem, we need some preliminary microlocal estimates. Each one of the three regions of ${}^b\dot{T}^*M$ individuated at the beginning of this section requires a separate analysis. In each case the two scenarios in which $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$ or with $k \leq 0$ are to be discussed individually. At last we remember that we denote an extension of $\Theta \in \Psi^k(\partial M)$ to the whole manifold M with the same symbol, see Remark 2.3.2.

3.0.2 Estimates in the elliptic region

We start with the elliptic region, proving a microlocal regularity result. We consider a coordinate neighbourhood $U \subset M$ and we indicate with $T^*_U M \doteq T^*M|_U$ and ${}^bT^*_U M \doteq {}^bT^*M|_U$. In addition, using the coordinates introduced in Section 1.1.3, in analogy to the twisted derivatives as per Equations (1.45) and (1.46), we introduce the operators

$$Q_0 = F\nabla_x F^{-1}, \quad Q_i = F\nabla_i F^{-1},\ i = 1, \dots, n-1 \tag{3.7}$$

where F is an admissible twisting function, as per Definition 2.3.1. The main result of this section is the following regularity result, concerning the case of boundary conditions implemented by pseudodifferential operators of order $0 \leq k \leq 2$.

Proposition 3.0.1 (microlocal elliptic regularity). Let $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$ and let $u \in \mathcal{H}^{1,m+k}_{loc}(M)$ for $m \leq 0$ and consider a point $q_0 \in {}^bT^*_U M$. If $s \in \mathbb{R} \cup \{+\infty\}$, then

$$q_0 \in WF^{1,s}_b(u) \setminus \left(WF^{-1,s}_b(P_\Theta u) \cup WF^{-1,s}_b(\Theta u)\right)$$

entails $q_0 \in \dot{\mathcal{N}}$, where is the compressed characteristic set defined in Equation (3.2).

Proof. We follow the strategy of [GW20, Th. 3] with the due difference that we need to control the contribution due to Θ. Hence we proceed by induction with respect to s, proving that $q \notin WF^{1,s+1/2}_b(u)$ and $q \notin WF^{-1,s}_b(P_\Theta u) \cup WF^{-1,s}_b(\Theta u)$ entails $q \notin WF^{1,s}_b(u)$.

The statement holds true for $s \leq m + k + 1/2$ since $u \in \mathcal{H}^{k,m+k}(M)$. To proceed in the inductive procedure, observe that, since we want to study properties of the wavefront set at a point $q_0 \in {}^bT^*_U M$ it is convenient to evaluate the energy form, *cf.* Equation (2.10) with the arguments replaced by Au, with $A \in \Psi^s_b(M)$ elliptic at q_0 and with compact support in $U \cap \{x < \delta\}$ where $\delta > 0$. To control such energy form we consider a family $\{J_r \in \Psi^{m-s-1}_b(M) \,|\, r \in (0,1)\}$, bounded in $\Psi^0_b(M)$ converging to the identity in $\Psi^1_b(M)$ as $r \to 0$. We approximate A using the family $\mathcal{A} = \{A_r = J_r A\}$. As shown in [Vas10], it holds

$$\begin{gathered}\mathcal{E}_0(A_r u, A_r u) \geq \|Q_0 A_r u\|^2 + \\ +(1 - C\delta)\langle \kappa^{ij} Q_i A_r u, Q_j A_r u\rangle - (1 + C\delta)\|\beta^{\frac{1}{2}} Q_{n-1} A_r u\|^2,\end{gathered} \tag{3.8}$$

where κ_{ij} and β are the components of the metric as in Theorem 1.1.1, while C is a positive constant. In addition we have adopted the convention that y_{n-1} corresponds to the time coordinate τ on the boundary, see Remark 1.1.1 while η_{n-1} is the associated momenta on the b-cotangent bundle. It is convenient to distinguish two cases, corresponding to those of Lemma 3.0.2 First, let us assume that $q \in {}^b\dot{T}^*M$. We can rewrite the last two terms of Equation (3.8) as

$$\langle\left[(1-C\delta)\kappa^{ij}Q_i^*Q_j - (1+C\delta)\beta Q_{n-1}^*Q_{n-1}\right] A_r u, A_r u\rangle + \\ +\langle\left((1-C\delta)\left(Q_j^*\kappa^{ij}\right)Q_i - (1+C\delta)\left(Q_{n-1}^*\beta\right)Q_{n-1}\right) A_r u, A_r u\rangle \tag{3.9}$$

Now we focus on the operator $(1-C\delta)\kappa^{ij}Q_i^*Q_j-(1+C\delta)\beta Q_{n-1}^*Q_{n-1}$, whose symbol $(1-C\delta)\kappa^{ij}\eta_i\eta_j - (1+C\delta)\beta\eta_{n-1}^2$ is of order 2. Since, whenever $q \in {}^b\dot{T}^*M$, it holds $\kappa^{ij}\eta_i\eta_j > (1+\varepsilon)\beta\eta_{n-1}^2$, *cf.* Lemma 3.0.2,

$$(1-C\delta)\kappa^{ij}\eta_i\eta_j - (1+C\delta)\beta\eta_{n-1}^2 = \\ (1-C\delta)(\kappa^{ij}\eta_i\eta_j - \beta\eta_{n-1}^2) - 2C\delta\beta\eta_{n-1}^2 > \Big(\varepsilon(1-C\delta) - 2C\delta\Big)\beta\eta_{n-1}^2$$

Then, for C and δ small enough, it holds:

$$(1-C\delta)\kappa^{ij}\eta_i\eta_j - (1+C\delta)\beta\eta_{n-1}^2 > \frac{\varepsilon}{2}\beta\eta_{n-1}^2$$

This inequality yields that $(1-C\delta)\kappa^{ij}\eta_i\eta_j-(1+C\delta)\beta\eta_{n-1}^2$ is a positive and elliptic symbol at q. Therefore, we can take an approximate square root $R \in \Psi_b^1(M)$ of the operator $(1-C\delta)\kappa^{ij}Q_i^*Q_j-(1+C\delta)\beta Q_{n-1}^*Q_{n-1}$, namely a pseudodifferential operator with principal symbol given by $\sigma_{b,1}(R) = (1-C\delta)\kappa^{ij}\eta_i\eta_j - (1+C\delta)\beta\eta_{n-1}^2$ and such that

$$R^2 = (1-C\delta)\kappa^{ij}Q_i^*Q_j - (1+C\delta)\beta Q_{n-1}^*Q_{n-1} + S,$$

with $S \in \Psi_b^{-\infty}(M)$. To summarize, we can recast

$$(1-C\delta)\langle\kappa^{ij}Q_iA_ru, Q_jA_ru\rangle - (1+C\delta)\|\beta^{\frac{1}{2}}Q_{n-1}A_ru\|^2$$

as

$$\langle RA_ru, RA_ru\rangle + \langle TA_ru, A_ru\rangle \tag{3.10}$$

with $T = S + (1-C\delta)\left(Q_i^*\kappa^{ij}\right)Q_j - (1+C\delta)\left(Q_{n-1}^*\beta\right)Q_{n-1} \in \Psi_b^1(M)$. Since $T \in \Psi_b^1(M)$ it descends that $|\langle TA_ru, A_ru\rangle|$ is uniformly bounded for $r \in (0,1)$. Let $\Lambda_+ \in \Psi_b^{1/2}(M)$ be an elliptic pseudodifferential operator and let $\Lambda_- \in \Psi_b^{-1/2}(M)$ be a parametrix. Then $\mathbb{I} = \Lambda_-\Lambda_+ + E$, with $E \in \Psi_b^{-\infty}(M)$ and we can write:

$$\langle TA_r u, \mathbb{I}A_r u\rangle = \langle \Lambda_-^* TA_r u, \Lambda_+ A_r u\rangle + \langle TA_r u, EA_r u\rangle \tag{3.11}$$

By Cauchy-Schwartz and triangular inequalities, it descends

$$|\langle TA_r u, \mathbb{I}A_r u\rangle|^2 \leq \|\Lambda_-^* TA_r u\|\cdot\|\Lambda_+ A_r u\| + \|TA_r u\|\cdot\|EA_r u\| \tag{3.12}$$

Thanks to Lemma 1.4.6 and to the hypotheses on u and on the family $\{A_r\}$, all norms on the right hand side are uniformly bounded for $r \in (0,1)$. In particular it holds

$$|\langle TA_r u, \mathbb{I}A_r u\rangle|^2 \leq C\left(\|G_1 u\|^2_{\mathcal{H}^1(M)} + \|G_2 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)}\right) \tag{3.13}$$

where $G_1 \in \Psi_b^{s-1/2}(M)$ is such that $WF'(\Lambda_-^* TA_r) \cup WF'(\Lambda_+ A_r) \subset ell_b(G_1)$ and $G_2 \in \Psi_b^{s-1}(M)$ is such that $WF'(TA_r) \subset ell_b(G_2)$. Therefore from Equation (3.8) one obtains

$$0 \leq (1 - C\delta)\|Q_0 A_r u\|^2_{\mathcal{L}^2(M)} + \|RA\|^2_{\mathcal{L}^2(M)} \leq \mathcal{E}_0(A_r u, A_r u) - \langle TA_r u, A_r u\rangle \tag{3.14}$$

Note that the Dirichlet form $\mathcal{E}_0(A_r u, A_r u)$ is uniformly bounded for $r \to 0$ thanks to Lemma 2.3.4, which can be applied thanks to our hypotheses on u and Θ.
Thus, we draw the same conclusion for $(1 - C\delta)\|Q_0 A_r u\|^2_{\mathcal{L}^2(M)} + \|RA\|^2_{\mathcal{L}^2(M)}$. Hence one can find subsequences $A_{r_k} u$, $Q_0 A_{r_k} u$ and $RA_{r_k} u$, weakly convergent in $\mathcal{L}^2(M)$ and such that $r_k \to 0$ as $k \to \infty$. Since they converge to Au, $Q_0 Au$ and RAu in $\mathcal{D}'(M)$, in particular the weak limits lie in $\mathcal{L}^2(K)$ with K a compact subset of M such that $K \cap (U \cap \{x < \delta\}) \neq \emptyset$. This entails that $Au \in \mathcal{H}^1(K)$, and hence that $q \notin WF_b^{1,s}(u)$.

As for the second case of Lemma 3.0.2, first we note that for u supported in $\{x < \delta\}$, the following relation holds true

$$\|Q_0 u\|^2_{\mathcal{L}^2(M)} \geq \delta^{-2}\|xQ_0 u\|^2_{\mathcal{L}^2(M)}$$

Hence it holds

$$\mathcal{E}_0(A_r u, A_r u) \geq \delta^{-2}\langle xQ_0 A_r u, xQ_0 A_r u\rangle +$$
$$\langle [(1 - C\delta)\kappa_{ij} Q^i Q^j - (1 + C\delta)\beta Q^2_{n-1}]A_r u, A_r u\rangle + \langle TA_r u, A_r u\rangle,$$

where T accounts for lower order terms. We can rewrite the right hand side as

$$\begin{aligned}\langle[\delta^{-2}(xQ_0)^*(xQ_0)-(1+C\delta\beta)Q^2_{y_{n-1}}]A_ru,A_ru\rangle+\\ \langle(1-C\delta)(\kappa)_{ij}Q^iQ^jA_ru,A_ru\rangle+\langle TA_ru,A_ru\rangle\end{aligned}\tag{3.15}$$

The operator $\delta^{-2}(xQ_0)^*(xQ_0)-(1+C\delta)\beta Q^2_{n-1}$ has symbol $\zeta^2/(2\delta^2)-(1+C\delta)\beta\eta^2_{n-1}$, that is elliptic near V since, on account of Lemma 3.0.2, there must exist a constant c such that

$$\frac{\zeta^2}{2\delta^2}-(1+C\delta)\beta\eta^2_{n-1}>c\beta\eta^2_{n-1}$$

Hence, we can define, modulo lower order terms, its square root as a pseudodifferential operator and then we proceed exactly like in the previous case. We conclude by stressing that the underlying boundary conditions come into play in the proof through the application of Lemma 2.3.4. □

The counterpart of the previous proposition in the case $\Theta\in\Psi^k(\partial M)$, $k\leq 0$, is the following.

Proposition 3.0.2 (microlocal elliptic regularity). Let $u\in\mathcal{H}^{1,m}_{loc}(M)$ for $m\leq 0$ and let $q_0\in {}^bT^*_UM$. If $s\in\mathbb{R}\cup\{+\infty\}$ and if $\Theta\in\Psi^k(\partial M)$ with $k\leq 0$, then $WF^{1,s}_b(u)\setminus\dot{\mathcal{N}}\subseteq WF^{-1,s}_b(P_\Theta u)$.

The proof is identical to that of Proposition 3.0.1, barring the fact that we must use Lemma 2.3.3 instead of Lemma 2.3.4. For this reason we omit the proof of this statement.

3.0.3 Estimates in the hyperbolic region

Now we focus on the hyperbolic region $\mathcal{H}(M)$. In comparison to the previous case, we must adopt a different strategy based on a positive commutator argument. In this section, we still use the coordinates introduced in Section 1.1.3 with the convention that y_{n-1} coincides with τ, *cf.* Theorem 1.1.1 and Remark 1.1.1, while η_{n-1} is the associated momentum on the b-cotangent bundle.

In the microlocal estimates of this section, a key rôle will be played by a multiple of $Im\mathcal{E}_0(u,A^*Au)$, with $u\in\mathcal{H}^{1,m}_{loc}(M)$ and $A\in\Psi^0_b(M)$ with principal symbol $\sigma_{b,0}(A)=a$ of compact support. For this reason, we begin by studying this term. A direct computation yields

$$\begin{aligned}2iIm\mathcal{E}_0(u,A^*Au)=\langle\widehat{g}^{ij}Q_ju,[Q_i,A^*A]u\rangle-\langle[\widehat{g}^{ij}Q_j,A^*A]u,Q_iu\rangle+\\ +\langle Q_0u,[Q_0,A^*A]u\rangle-\langle[Q_0,A^*A]u,Q_0u\rangle+\langle[Q_i\widehat{g}^{ij}Q_j,A^*A]u,u\rangle,\end{aligned}\tag{3.16}$$

where the operators Q_i, $i = 0, \dots, n-1$ are defined as in Equation (3.7). For future convenience, it is useful to compute explicitly the commutators in the first two terms in the second line, getting

$$\begin{aligned} &\langle Q_0 u, [Q_0, A^*A]u\rangle - \langle [Q_0, A^*A]u, Q_0 u\rangle = \\ &= \langle Q_0 u, Q_0 A_1 u\rangle - \langle Q_0 A_1 u, u\rangle + \langle Q_0 u, A_0 u\rangle - \langle A_0 u, Q_0 u\rangle, \end{aligned} \tag{3.17}$$

where $A_0 \in \Psi_b^0(M)$, $A_1 \in \Psi_b^{-1}(M)$ have principal symbol respectively $a_0 = -i\partial_x a^2$ and $a_1 = -i\partial_\zeta a^2$.

We now study the relevant wavefront sets in the hyperbolic regions. As in the previous section, we divide the analysis in two parts depending on whether $\Theta \in \Psi^k(\partial M)$ with $k \leq 0$ or with $0 < k \leq 2$. We recall that Θ denotes both the pseudodifferential operator implementing the boundary conditions and its extension to M, see Remark 2.3.2.

Proposition 3.0.3. Let $\Theta \in \Psi_b^k(M)$, $0 < k \leq 2$. Let $u \in \mathcal{H}_{loc}^{1,m+k}(M)$ with $m \leq 0$ and suppose that $q_0 \notin WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)$. If there exists a conic neighborhood $W \subset T^*M \setminus \{0\}$ of q_0 such that $W \cap \{\zeta < 0\} \cap WF_b^{1,s}(u) = \emptyset$ then $q_0 \notin WF_b^{1,s}(u)$.

Proposition 3.0.4. Let $\Theta \in \Psi_b^k(M)$ for some $k \leq 0$. Let $u \in \mathcal{H}_{loc}^{1,m}(M)$ for some $m \leq 0$ and suppose that $q_0 \notin WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)$. If there exists a conic neighborhood $W \subset {}^bT^*M \setminus \{0\}$ of q_0 such that $W \cap \{\zeta < 0\} \cap WF_b^{1,s}(u) = \emptyset$ then $q_0 \notin WF_b^{1,s}(u)$.

The proof of both Proposition 3.0.3 and 3.0.4 is similar to that of Proposition 3.0.1 and 3.0.2 respectively, the main difference consisting in replacing Lemma 2.3.3 and 2.3.4 with suitable counterparts valid in the hyperbolic region. For this reason, we postpone the proofs of Propositions 3.0.3 and 3.0.4 to the end of the section, discussing first the hyperbolic estimates we need.

Remark 3.0.3. Let $\Theta \in \Psi_b^k(M)$ for $0 < k \leq 2$ and let $Z \subset W$ with $q_0 \in Z$. Since $q_0 \notin WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)$, if Z is small enough then, by the elliptic regularity theorem, see Proposition 3.0.1,

$$\left(WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u) \right) \cap Z = \emptyset.$$

Hence we can conclude that $Z \cap WF_b^{1,s}(u) \subset \dot{\mathcal{N}}$. In particular, this fact means that on the set $Z \cap \{\zeta < 0\} \cap WF_b^{1,s}(u)$ it holds $x \neq 0$ and a point $q_0 \in WF_b^{1,s}(u)$ can be seen as a limit of points in the wavefront set, each of which does not lie on the boundary. An analogous statement holds true for the case in which $k \leq 0$.

Let U be a coordinate patch such that $U \cap \partial M \neq \emptyset$ and let $q_0 \in \mathcal{H}(M) \cap {}^bT^*_U M$. Following [Vas08] we introduce the smooth scalar function μ on ${}^bT^*M$, defined as $\mu = -\zeta = -x\xi$. This function is homogeneous of degree 0 and it has the notable property that in a neighborhood of q_0 the sign of $H_{\widehat{p}}(\pi^*\mu)$ does not change. With $H_{\widehat{p}}$ we denote the Hamiltonian vector field associated to the principal symbol $\widehat{p}$ of $x^{-2}P$.

If we consider the b-cosphere bundle ${}^bS^*M$ as per Equation (1.16) together with the associated coordinates on ${}^bS^*_U M \doteq {}^bS^*M|_U$, we can introduce the function $\widehat{\omega} : {}^bS^*_U M \to \mathbb{R}$

$$\widehat{\omega}(q) = |x(q)|^2 + \sum_{i=1}^{n-2} |y_i(q) - y_i(q_0)|^2 + |\widehat{\zeta}(q) - \widehat{\zeta}(q_0)|^2 + \sum_{i=1}^{n-2} \left|\widehat{\eta}_i(q) - \widehat{\eta}_i(q_0)\right|^2, \tag{3.18}$$

which induces in turn a function $\omega : {}^bT^*_U M \setminus \{0\} \to \mathbb{R}$ defined as $\omega = \widehat{\omega} \circ \pi_S$ where $\pi_S : {}^bT^*M \setminus \{0\} \to {}^bS^*M$ is the natural projection map implementing the quotient in Equation (1.16). For the sake of simplicity of the notation, we have refrained from indicating that $\widehat{\omega}$ depends explicitly on the choice of q_0. In addition, on a conic neighborhood of q_0, consider the homogeneous smooth function ϕ

$$\phi = \mu + \frac{1}{\lambda^2 \delta}\omega \tag{3.19}$$

where λ and δ are positive parameters. By construction ϕ can be read as a π-invariant function on $T^*M \setminus \{0\}$ and, to localize it near q_0, consider $\chi_0, \chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\chi_1(s) = \begin{cases} 0 & \text{if } s \in (-\infty, 0) \\ 1 & \text{if } s \in [1, \infty) \end{cases},$$

while the derivative χ_1' is positive on $(0,1)$. At the same time we set

$$\chi_0(s) = \begin{cases} 0 & \textit{if } s \leq 0, \\ \exp\left(-s^{-1}\right) & \textit{if } s > 0. \end{cases} \tag{3.20}$$

Consider now

$$a \doteq \chi_0(2 - \phi/\delta)\chi_1(\hat{\zeta}/\delta + 2), \tag{3.21}$$

which is a smooth homogeneous function of degree zero in a conic neighborhood of q_0. On account of the properties of χ_0 and of χ_1 it holds that

$$\omega \leq \lambda^2\delta(2\delta - \eta) \leq 4\delta^2\lambda^2 \quad \text{and} \quad |\widehat{\zeta}| \leq 2\delta.$$

This entails that, for any $\lambda > 0$ and for $\delta > 0$ small enough, a has support inside a conic neighborhood of q_0. At last, we also localize in a conic neighborhood of q_0 with compact closure and such that $\widehat{g}^{ab}k_ak_b > 0$ where $k_a = (\zeta, \eta_i)$, $i = 1, \dots, n-1$ are coordinates on the fiber of the b-cotangent bundle . Let V_0 be a set satisfying these properties and consider a function $\psi_0 \in \mathcal{C}_0^\infty({}^bS^*M)$ such that $\psi_0 = 1$ on V_0 and whose support lies in a small neighborhood of V_0.

We can now choose a family of pseudodifferential operators for regularization purposes. Let $\{J_r | r \in (0,1)\}$ be a family of ΨDOs in $\Psi_b^{s+k+1/2}(M)$ such that $J_r \in \Psi_b^m(M)$ for $r \in (0,1)$ and whose principal symbol is $j_r = \psi_0 \rho^{s+1/2}(1 + r\rho)^{m-s-1/2}$. By construction J_r is elliptic in V_0. We build a family of regulators

$$A_r = AJ_r \tag{3.22}$$

with $A \in \Psi_b^0(M)$ with principal symbol a as in Equation (3.21). Note that since $A \in \Psi_b^0(M)$, A_r is bounded in $\Psi_b^{s+1/2}(M)$. We report now a notable result [GW20, Lemma 6.7].

Lemma 3.0.3. *Let $G \in \Psi_b^k(M)$. Given $\lambda > 0$ there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$*

$$i[A_r^* A_r, G] = B_r^* D_r B_r + F_r + T_r \tag{3.23}$$

where

- *$B_r \in \Psi_b^{s+1}(M)$, $r \in (0,1)$, has principal symbol $b_r = j_r b$ with*

$$b = \rho^{-1/2}\delta^{-1/2}[\chi_0'(2 - \phi/\delta)\chi_0(2 - \phi/\delta)]^{1/2}\chi_1(2 + \zeta/\delta),$$

- *$D_r \in \Psi_b^{k-2}(M)$, $r \in (0,1)$ and its principal symbol d_r satisfies*

$$\rho^{2-k}|(d_r)| \leq C_0(\lambda\delta + \delta + \lambda^{-1}),$$

  for some positive real constant C_0.

- *$T_r \in \Psi^{2s+k-1}(M)$, $r \in (0,1)$, is such that:*

$$WF_b'(\mathcal{T}) \subset \{|\widehat{\zeta}| \leq 2\delta, \omega^{1/2} \leq 2\lambda\delta\}$$

  where $\mathcal{T} = \{T_r | r \in (0,1)\}$ and WF_b' is the wavefront set of a family of pseudodifferential operators, see Definition 1.3.14.

- $F_r \in \Psi^{2s+k}(M)$, $r \in (0,1)$, *is such that*

$$WF_b'(\mathcal{F}) \subset \{-2\delta \leq \widehat{\zeta} \leq -\delta, \omega^{1/2} \leq 2\lambda\delta\},$$

where $\mathcal{F} = \{F_r \mid r \in (0,1)\}$ *is bounded in* $\Psi_b^{2s+k}(M)$.

As mentioned at the beginning of this section, we seek $Q \in \Psi_b^s(M)$ such that the norm of $QA_r u$ is bounded in $\mathcal{L}^2(M)$. This can be individuated as follows. Starting from Proposition 3.0.3 we observe that $Im\mathcal{E}_0(u, A_r^* A_r u)$ contains a term of the form

$$\langle Q_0 u, iQ_0 A_{1,r} u\rangle.$$

Focusing on $a_{1,r} = -i\partial_\sigma a_r^2$, the principal symbol of $A_{1,r}$, a straightforward computation shows that

$$iA_{1,r} = \widetilde{B}_r^* \widetilde{B}_r + F_r + T_r, \Longrightarrow a_{1,r} = \widetilde{b}_r^2 + f_r + t_r,$$

where $\widetilde{b}_r = \rho^{-1} b_r = j_r b$ is a symbol of order $m - 1/2$ which arises when we differentiate χ_0, with $\{B_r\}$ being a bounded family in $\Psi_b^s(M)$. The principal symbols $\{f_r\}$ are associated instead to the bounded family $\{F_r\}$ in $\Psi_b^s(M)$ which originates from the derivatives of χ_1 while t_r are principal symbols associated to the bounded family $\{T_r\}$ in $\Psi_b^{2s-1}(M)$, that includes the contribution by lower order terms.

We choose the sought operator Q as $\widetilde{B_r}$. In order to prove that $QA_r u$ is bounded in $\mathcal{L}^2(M)$ we analyze separately the usual two cases. We start from a boundary condition implemented by $\Theta \in \Psi^k(\partial M)$ with $k \leq 0$. In this case we can use [GW20, Lemma 6.8] with the due exception that one needs to replace in the proof Lemma 5.3 from [GW20] with Lemma 2.3.3.

Lemma 3.0.4. *There exist* $C_1, c, \lambda, \delta_0 > 0$, *a cutoff* $\chi \in \mathcal{C}_0^\infty(M)$ *and a compactly supported operator* $G_2 \in \Psi_b^s(M)$ *with*

$$WF_b'(G_2) \subset W \cap \{\ \zeta < 0\} = \emptyset,$$

such that

$$c\|\widetilde{B}_r u\|^2 \leq -2Im\mathcal{E}_0(u, A_r^* A_r u) + C\Big(\|G_0 u\|^2_{\mathcal{H}^1(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \\ + \|G_2 u\|^2_{\mathcal{H}^1(M)} + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)}\Big). \tag{3.24}$$

In the case where $\Theta \in \Psi^k(\partial M)$, with $0 < k \leq 2$, we can exploit Lemma 2.3.4 in place of Lemma 2.3.3 to obtain the estimate

$$
\begin{aligned}
c\|\widetilde{B}_r u\|^2 \leq & -2Im\mathcal{E}_0(u, A_r^* A_r u) + C\Big(\|G_0 u\|^2_{\mathcal{H}^1(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \\
& + \|G_2 u\|^2_{\mathcal{H}^1(M)} + \|G_0 P_\Theta u\|^2_{\mathcal{H}^{-1}(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\mathcal{H}^{-1,m}(M)} + \\
& + \|\chi \Theta u\|^2_{\mathcal{H}^{-1,m}(M)} + \|G_0 \Theta u\|^2_{\mathcal{H}^{-1}(M)}\Big).
\end{aligned} \tag{3.25}
$$

At last we give a bound for $Im\mathcal{E}_0(u, A_r^* A_r u)$. As above we divide the analysis in two cases, starting from a boundary condition implemented by $\Theta \in \Psi^k(\partial M)$, with $0 < k \leq 2$. For the reader's convenience we recall that $\mathcal{H}^{1,m+k} \subseteq \mathcal{H}^{1,m}$ if $k \geq 0$.

Lemma 3.0.5. *Given $\varepsilon > 0$, under the hypotheses of Lemma* (2.3.4)*, there exists $\lambda > 0$ and $\delta_0 > 0$ such that*

$$
\begin{aligned}
& Im\mathcal{E}_0(u, A_r^* A_r u) \leq \\
& \varepsilon\|\widetilde{B}_r u\|^2_{\mathcal{H}^1(M)} + C\Big(\|G_2 u\|^2_{\mathcal{H}^1(M)} + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|G_0 \Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \\
& + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|\chi \Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)}\Big),
\end{aligned}
$$

for every $\delta \in (0, \delta_0)$.

Proof. Let $\Lambda_{-1/2} \in \Psi_b^{-1/2}(M)$ be an elliptic pseudodifferential operator. Then, there exists $\Lambda_{1/2} \in \Psi_b^{1/2}(M)$ such that $\Lambda_{1/2}\Lambda_{-1/2} = \mathbb{I} + R$ with $R \in \Psi_b^{-1}(M)$. In order to account for the boundary conditions, we bound $\mathcal{E}_\Theta(u, A_r^* A_r u)$.

$$
\begin{aligned}
|\mathcal{E}_\Theta(u, A_r^* A_r u)| = & |\langle A_r P_\Theta u, A_r u\rangle| = |\langle A_r P_\Theta u, \Big(\Lambda_{1/2}\Lambda_{-1/2} + R\Big) A_r u\rangle| \\
\leq & |\langle A_r P_\Theta u, \Lambda_{1/2}\Lambda_{-1/2} A_r u\rangle| + |\langle A_r P_\Theta u, R A_r u\rangle|.
\end{aligned} \tag{3.26}
$$

We can control the first term similarly to the proof of Lemma 2.3.3:

$$
\begin{aligned}
|\langle A_r P_\Theta u, \Lambda_{1/2}\Lambda_{-1/2} A_r u\rangle| \leq & C\Big(\|\Lambda_{1/2} A_r^* P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\Lambda_{-1/2} A_r u\|^2_{\mathcal{H}^1(M)}\Big) \\
\leq & C\Big(\|G_0 f\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi f\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)}\Big),
\end{aligned} \tag{3.27}
$$

where $G_0 \in \Psi_b^{s+1}(M)$ while $G_1 \in \Psi_b^s(M)$. Focusing on the second term of Equation (3.26), we get

$$|\langle A_r P_\Theta u, R A_r u\rangle| \leq \Big(\|G_0 f\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi f\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)}\Big). \tag{3.28}$$

The next step consists of finding a bound for

$$|\mathrm{Im}\mathcal{E}_\Theta(u, A_r^* A_r u) - \mathrm{Im}\mathcal{E}_0(u, A_r^* A_r u)|.$$

A direct inspection of Equation (2.14) and of Equation (2.10) unveils that this last difference consists of two terms. The first is

$$\langle A_r x^{-2} S_F u, A_r u\rangle - \langle A_r u, A_r x^{-2} S_F u\rangle, \tag{3.29}$$

which can be rewritten as

$$\langle A_r^* [A_r, x^{-2} S_F] u, u\rangle - \langle u, A_r^* [A_r, x^{-2} S_F] u\rangle. \tag{3.30}$$

Observing that $A_r^*[A_r, x^{-2}S_F]$ is uniformly bounded in $\Psi_b^{2s}(M)$, we find that

$$|\mathrm{Im}\langle x^{-2} S_F u, A_r^* A_r u\rangle| \leq C\Big(\|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)}\Big). \tag{3.31}$$

The second term is instead

$$2\mathrm{Im}\langle \Theta\gamma_- u, \gamma_-(A_r^* A_r u)\rangle_{\partial M},$$

which can be rewritten in the form

$$\langle \Theta\gamma_- u, \gamma_-(A_r^* A_r u)\rangle_{\partial M} - \langle \gamma_-(A_r^* A_r u), \Theta\gamma_- u\rangle_{\partial M}. \tag{3.32}$$

Proceeding as in the first bound of the proof and using the properties of the indicial operator as in Lemma 2.3.3, we can write, modulo lower order terms

$$\begin{aligned} &|\langle \Theta\gamma_- u, \gamma_-(A_r^* A_r u)\rangle_{\partial X} - \langle \gamma_-(A_r^* A_r u), \Theta\gamma_- u\rangle_{\partial M}| \leq \\ &\leq 2|\langle \Theta\gamma_- u, \gamma_-(A_r^* \Lambda_{1/2}\Lambda_{-1/2} A_r u)\rangle_{\partial M}| \leq \\ &\leq \varepsilon \|\widetilde{B}_r u\|^2_{\mathcal{H}^1(M)} + C\Big(\|G_0 \Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)}\Big). \end{aligned} \tag{3.33}$$

Collecting all estimates, we obtain the sought result. □

We focus on the case where $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$.

Lemma 3.0.6. *Given $\varepsilon > 0$, under the hypotheses of Lemma (2.3.4), there exists $\lambda > 0$ and $\delta_0 > 0$ such that*

$$\mathrm{Im}\mathcal{E}_\Theta(u, A_r^* A_r u) \leq \varepsilon \|\widetilde{B}_r u\|^2_{\mathcal{H}^1(M)} + C\Big(\|G_2 u\|^2_{\mathcal{H}^1(M)} + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \\ + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)}\Big),$$

for every $\delta \in (0, \delta_0)$.

Proof. The first part of the proof is identical to that of Lemma 3.0.5. The difference lies in the estimates for the boundary terms, *cf.* Equation (3.32). This time, using the properties of the indicial family, we can rewrite the relevant terms as

$$\langle \widehat{N}(\widetilde{A}_r^* \widetilde{A}_r \Theta)(-i\nu_-)\gamma_- u, \gamma_- u\rangle_{\partial M} - \langle \widehat{N}(\Theta \widetilde{A}_r^* A_r)(-i\nu_-)\gamma_- u, \gamma_- u\rangle_{\partial M} = \\ = \langle \widehat{N}[\widetilde{A}_r^* \widetilde{A}_r, \Theta](-i\nu_-)\gamma_- u, \gamma_- u\rangle_{\partial M}$$

where $\widetilde{A}_r = x^{2\nu_-} A_r x^{-2\nu_-}$. Note that A_r and $\widetilde{A}_r$ have the same principal symbol, hence we can write $\widetilde{A}_r^* = A_r + N_r$, with N_r containing lower order terms. Hence:

$$[\widetilde{A}_r^* \widetilde{A}_r, \Theta] = [A_r^* A_r, \Theta] + [N_r^* A_r + A_r^* N_r + N_r^* N_r, \Theta] = \\ = [A_r^* A_r, \Theta] + \widetilde{A}_r^* \widetilde{A}_r - A_r^* A_r,$$

which yields

$$\langle [\widetilde{A}_r^* \widetilde{A}_r, \Theta]\gamma_- u, \gamma_- u\rangle_{\partial M} = \\ = \langle \gamma_-([A_r^* A_r, \Theta]u), \gamma_- u\rangle_{\partial M} + \langle \Theta\gamma_-(\widetilde{A}_r^* \widetilde{A}_r - A_r^* A_r)u], \gamma_- u\rangle_{\partial M}. \quad (3.34)$$

We use Lemma 3.0.3 to control the first term writing $[A_r^* A_r, \Theta] = \widetilde{B}_r^* \widetilde{D}_r \widetilde{B}_r + T_r$ with $\widetilde{D}_r \in \Psi_b^k(M)$ and $T_r \in \Psi_b^{2s+k-1}(M)$. Observe that $\widetilde{D}_r$ is related to $D_r \in \Psi_b^{k-2}(M)$ as in Lemma 3.0.3 since their respective principal symbols d_r and $\widetilde{d}_r$ are connected via the identity $\widetilde{d}_r = \rho^2 d_r$ where $\rho = |\eta_{n-1}|$. Hence

$$\langle \gamma_-([A_r^* A_r, \Theta]u), \gamma_- u\rangle_{\partial M} = \\ = \langle \gamma_-(\widetilde{D}_r B_r u), \gamma_-(\widetilde{B}_r u)\rangle_{\partial M} + \langle \gamma_-(T_r u), \gamma_- u\rangle_{\partial M},$$

where, in the second equality, we used the properties of the indicial family to bring $\widetilde{B}_r^*$ to the right hand side. Thus it descends that

$$\langle \gamma_-([A_r^* A_r, \Theta]u), \gamma_- u\rangle_{\partial M} = \langle \gamma_-(\widetilde{D}_r \widetilde{B_r} u), \gamma_-(\widetilde{B}_r u)\rangle_{\partial M},$$

modulo lower order terms bounded by $s-1/2$. Using the indicial family we obtain

$$\begin{aligned}&\langle\gamma_-(\widetilde{D}_r\widetilde{B}_r u),\gamma_-(\widetilde{B}_r u)\rangle_{\partial M}=\\&=\langle\widehat{N}(-i\nu_-)(\widetilde{D}_r)\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u,\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u\rangle_{\partial M}.\end{aligned}$$

Using that $\widehat{N}(-i\nu_-)(\widetilde{D}_r)\in\Psi_b^0(M)$, $\widehat{N}(-i\nu_-)(\widetilde{B}_r)\in\Psi_b^{m-1/2}(M)$ together with Equation (1.4.1), we obtain

$$\begin{aligned}&|\langle\widehat{N}(-i\nu_-)(\widetilde{D}_r)\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u,\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u\rangle_{\partial M}|\leq\\&\leq\|\widehat{N}(-i\nu_-)(\widetilde{D}_r)\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u\|^2_{\mathcal{L}^2(M)}+\|(\widetilde{B}_r)\gamma_- u\|^2_{\mathcal{L}^2(M)}\leq\\&\|\chi\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u\|^2_{\mathcal{L}^2(M)}+\|\widehat{N}(-i\nu_-)(\widetilde{B}_r)\gamma_- u\|^2_{\mathcal{L}^2(M)}\leq\\&\leq\varepsilon\|\widetilde{B}_r u\|^2_{\mathcal{H}^1(M)}+C(\|G_1 u\|_{\mathcal{H}^1(M)}+\|\chi u\|^2_{\mathcal{H}^{1,m}(M)}).\end{aligned}\tag{3.35}$$

Focusing on the second term in Equation (3.34), we write

$$\begin{aligned}&\langle\Theta\gamma_-((\widetilde{A}_r^*\widetilde{A}_r-A_r^*A_r)u),\gamma_- u\rangle_{\partial M}=\\&=\langle\Theta\gamma_-(x^{2\nu_-}[A_r^*A_r,x^{-2\nu_-}]u),\gamma_- u\rangle_{\partial M}.\end{aligned}$$

We can compute $[A_r^*A_r,x^{-2\nu_-}]$ thanks to Lemma (3.0.3) obtaining

$$[A_r^*A_r,x^{-2\nu_-}]=(2i\nu_-)\widetilde{B}_r^*\widetilde{B}_r+E_r+T_r.\tag{3.36}$$

Each term can be controlled as above, obtaining ultimately

$$\begin{aligned}&|\langle\Theta\gamma_-((\widetilde{A}_r^*\widetilde{A}_r-A_r^*A_r)u),\gamma_- u\rangle_{\partial X}|\leq\varepsilon\|\widetilde{B}_r u\|^2_{\mathcal{H}^1}+\\&C\Big(\|G_1 u\|^2_{\mathcal{H}^1(M)}+\|\chi u\|^2_{\mathcal{H}^{1,m}(M)}+\|C_2 u\|^2_{\mathcal{H}^1(M)}\Big),\end{aligned}\tag{3.37}$$

which entails the sought conclusion. □

Finally we can complete the proofs of Propositions 3.0.3 and 3.0.4. Here we focus only on the first case since the second one follows suit.

Proof of Proposition 3.0.3: We sketch the main steps since we can proceed exactly as in the elliptic case, *cf.* Proposition 3.0.1. Most notably we follow an induction procedure with respect to s. Notice in particular that the statement holds true for $s<m+\frac{1}{2}$ since $u\in\mathcal{H}^{1,m+k}_{loc}(M)\subset\mathcal{H}^{1,m}_{loc}(M)$. To continue in the inductive procedure we consider once more a family $J_r\in\Psi_b^{m-s-1}(M)$, $r\in(0,1)$, such that $J_r\to\mathbb{I}\in\Psi_b^0(M)$. Then $\widetilde{B}_r\to\widetilde{B}\in\Psi_b^s(M)$

as $r \to 0$. Using Lemma 3.0.4 and Lemma 3.0.5 or (3.0.6) depending on the order of Θ, one obtains that $\|\widetilde{B}_r u\|_{\mathcal{H}^1(M)}$ is uniformly bounded. Therefore, we can find a subsequence $\widetilde{B}_{r_k} u$, with $r_k \to 0$ for $k \to +\infty$, that is weakly convergent in $\mathcal{H}^1(M)$. Since $\widetilde{B}_r u \to \widetilde{B} u$ in $\mathcal{D}'(M)$, the weak limit lies in $\mathcal{H}^1(K)$ for a suitable compact subset $K \subset M$. By uniqueness of the limit and considering that $\widetilde{B}$ is elliptic at q_0, we obtain the thesis.

3.0.4 Estimates in the glancing region

Finally we turn our attention to the glancing region $\mathcal{G}(M)$. As in the hyperbolic region, we use a positive commutator argument and barring some geometrical aspects we proceed similarly to Propositions 3.0.3 and 3.0.4. For this reason, we introduce in some detail only the geometric framework, without reporting the detailed proofs of the microlocal estimates. In the following U will denote an open coordinate neighbourhood, while $V = U \cap \partial M \neq \emptyset$. As in the previous section we need to consider two scenarios depending on the class of boundary conditions, namely $\Theta \in \Psi^k(\partial M)$ with either $k \leq 0$ or $0 < k \leq 2$. Similarly to the preceding cases, we pick an extension of Θ to M as per Remark 2.3.2, indicating it with the same symbol. In the following y_{n-1} still refers to the time coordinate corresponding to τ in Theorem 1.1.1, while η_{n-1} is the corresponding momentum on the b-cotangent bundle. In addition q_0 refers to a point lying in a compact region K where

$$K \subset (\mathcal{G} \cap T^*_U M) \setminus WF_b^{-1,s+1}(P_\Theta u) \quad \text{if} \quad k \leq 0,$$

or

$$K \subset (\mathcal{G} \cap T^*_U M) \setminus \Big(WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)\Big) \quad \text{if} \quad 0 < k \leq 2.$$

In local coordinates q_0 reads $(0, (y_0)_i, 0, (\eta_0)_i)$, $i = 1, \ldots, n-1$, while it holds $\widehat{g}^{ij}(0, y_0)(\eta_0)_i(\eta_0)_j = 0$. Since $\eta_{n-1} \neq 0$, we can use the projective coordinates on ${}^bS^*M$ near $\pi_S(q_0)$, where $\pi_S : {}^bT^*M \setminus \{0\} \to {}^bS^*M$ is the quotient map. We denote the projection to the boundary with

$$\widetilde{\pi} : T^*_U M \to T^*V,$$

$$(x, y_i, \xi, \eta_i) \mapsto (y_i, \eta_i),$$

where $i = 1, \ldots, n-1$. As last ingredient we introduce the gliding vector field W, describing the evolution of a point in the directions tangent to the boundary. Consider thus a point on $\widetilde{\pi}(T^*_U M)$ whose coordinates are $(\eta_0, (y_0)_i)$, $i = 1, \dots, n-1$ and define

$$W(\eta_0, (y_0)_i) = \sum_{i=1}^{n-1} (\partial_{\eta_i} \widehat{p})(0, \eta_0, 0, (y_0)_i)\partial_{y_i} - (\partial_{y_i} \widehat{p})(0, \eta_0, 0, (y_0)_i)\partial_{\eta_i}, \quad (3.38)$$

where $\widehat{p}$ is the principal symbol of $x^{-2}P$, with P the Klein-Gordon operator, see Equation (2.1). Letting $\rho = |\eta_{n-1}|$, we observe that, in a neighbourhood of $(\eta_0, (y_0)_i)$, $\rho^{-1}W$ is a non degenerate vector field, since $\rho^{-1}Wy^{n-1} = 2sgn(\eta_{n-1})$. Thus we can use the straightening theorem [LGPV13] to find $2n-2$ homogeneous degree zero functions $\rho_1, \cdots, \rho_{2n-2} \in T^*_U M$ with linearly independent differentials such that $\rho^{-1}W\rho_1 = 1$ and $\rho^{-1}W\rho_i = 0$ for $i = 2, \cdots, 2n-2$. We also note that $\widehat{p}(0, \eta, 0, y_i)$ is annihilated by W. Since $d\widehat{p} \neq 0$, we can set $\rho_2(\eta_0, (y_0)_i) = \widehat{p}(0, \eta_0, 0, (y_0)i)$. Then we extend $\rho_1, \cdots, \rho_{2n-2}$ in such a way to be independent from (x, ξ), in order to obtain a local chart whose coordinate functions are $x, \widehat{\zeta}, \rho_1, \cdots, \rho_{2n-2}$.

With these data we can introduce two homogeneous functions ω_0 and ω on $K \cap V$, playing the same rôle as η and ω in the hyperbolic region:

$$\Omega_0 = \sum_{i=1}^{2n-2} (\rho_i - \rho_i(q_0))^2, \qquad \Omega = x^2 + \Omega_0,$$

where we omit to indicate the explicit dependence on q_0 for the sake of simplicity of the notation. In connection to these functions we introduce

$$\phi_0 = \rho_1 + \frac{\Omega_0}{\lambda^2\delta}, \quad \phi = \rho_1 + \frac{\Omega}{\lambda^2\delta}.$$

Using the same cutoff functions χ_0 and χ_1 introduced in Section 3.0.3, we localize near q_0 using a b-pseudodifferential operator A of order zero whose total symbol is

$$a = \chi_0(2 - \phi/\delta)\chi_1(1 + (\rho + \delta)/(\lambda\delta)).$$

The ensuing families $\mathcal{A}$, $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are defined as in Section 3.0.3. With these data, the following generalizations of [GW20, Prop. 6.11] hold true. Observe that, with a slight abuse of notation, we identify subsets of the b-cosphere bundle with their pre-image on the b-cotangent bundle.

Proposition 3.0.5. Let $\Theta \in \Psi^k_b(M)$ with $k \leq 0$ and let $u \in \mathcal{H}^{1,m}_{loc}(M)$ with $m \leq 0$. If $K \subset {}^bS^*_U M$ is compact and $K \subset (\mathcal{G} \cap T^*\partial M) \setminus WF_b^{-1,s+s}(P_\Theta u)$, then there exist $C_0, \delta_0 > 0$ such that for each $q_0 \in K$ and $\delta \in (0, \delta_0)$ the following holds true. Let $\alpha_0 \in \mathcal{N}$ be such that $\pi(\alpha_0) = q_0$. If the conditions

$$\alpha \in \mathcal{N}, \quad |\widetilde{\pi}(\alpha) - exp(-\delta W)(\widetilde{\alpha}_0)| \leq C_0\delta^2, \quad |x(\alpha)| \leq C_0\delta^2,$$

imply $\pi(\alpha) \notin WF_b^{1,s}(u)$, then $q_0 \notin WF_b^{1,s}(u)$.

In the case where $\Theta \in \Psi_b^k(M)$, $0 < k \leq 2$, the generalization of [GW20, Prop. 6.11] is the following:

Proposition 3.0.6. Let $\Theta \in \Psi_b^k(M)$ with $0 < k \leq 2$ and let $u \in \mathcal{H}^{1,m+k}_{loc}(M)$ with $m \leq 0$. If $K \subset {}^bS^*_U X$ is compact and

$$K \subset (\mathcal{G} \cap T^*\partial M) \setminus \left(WF_b^{-1,s+s}(P_\Theta u) \cup WF_b^{-1,s+s}(\Theta u) \right),$$

then there exist $C_0, \delta_0 > 0$ such that for each $q_0 \in K$ and $\delta \in (0, \delta_0)$ the following holds. Let $\alpha_0 \in \mathcal{N}$ be such that $\pi(\alpha_0) = q_0$. If the conditions

$$\alpha \in \mathcal{N}, \quad |\widetilde{\pi}(\alpha) - exp(-\delta W)(\widetilde{\alpha}_0)| \leq C_0\delta^2, \quad |x(\alpha)| \leq C_0\delta^2,$$

imply $\pi(\alpha) \notin WF_b^{1,s}(u)$, then $q_0 \notin WF_b^{1,s}(u)$.

We focus on the case $0 < k \leq 2$. The proof is based on two lemmas along with the counterpart of Lemma 3.0.4 for the glancing region. The proofs are similar to those of the hyperbolic case and they are adapted from those in [GW20], hence we omit them.

The first lemma we gives a bound on the difference between the $\mathcal{L}^2$-norm of $Q_0 A_r u$ and of a generic positive sesquilinear $\mathcal{Q}$ applied $A_r u$.

Lemma 3.0.7. *Let $U \subset M$ be a boundary coordinate patch and $m \leq 0$. Let $\mathcal{A} = \{A_r : r \in (0,1)\}$ be a bounded subset of $\Psi_b^s(M)$ with compact support in U such that $A_r \in \Psi_b^m(M)$ for each $r \in (0,1)$. Let $\delta > 0$ and let $V_\delta = \{q \in {}^bT^*_U M \setminus \{0\} : \widehat{g}^{ij}\eta_i\eta_j \leq \delta\beta^{-1}|\eta_{n-1}|^2\}$ and assume that $WF_b'(\mathcal{A}) \subset V_\delta$. Let $G_0 \in \Psi_b^s(M)$ and $G_1 \in \Psi_b^{s-1/2}(M)$ be elliptic on $WF_b'(\mathcal{A})$ and on $WF_b'(\Theta\mathcal{A})$ respectively, both with compact support in U. Then there exist C_ε and $\chi \in \mathcal{C}_0^\infty(U)$ such that, for every $u \in \mathcal{H}^{1,m+k}_{loc}(M)$,*

$$\begin{gathered}\|Q_0 A_r u\|^2_{\mathcal{L}^2(M)} - \varepsilon\mathcal{Q}(A_r u, A_r u) \leq \\ \leq 2\delta\|Q_{n-1}A_r u\|^2_{\mathcal{L}^2(M)} + C_\varepsilon\Big(\|\chi u\|^2_{\mathcal{H}^{1,m}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \\ + \|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|G_1 u\|^2_{\mathcal{H}^1(M)} + \|\chi\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_0\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)}\Big).\end{gathered}$$

The proof is the same of Lemma 6.10 in [GW20], except that we use Lemma (2.3.4) to control the boundary terms. This result can be used to generalize the proof of Lemma 6.12 in [GW20] to the case in hand:

Lemma 3.0.8. *There exist $C, c, \lambda, \delta_0 > 0$, a cutoff $\chi \in \mathcal{C}_0^\infty(M)$, G_0, G_1 as above and an operator $G_2 \in \Psi_b^s(M)$ with*

$$WF_b'(G_2) \subset W \cap \{-2\delta\lambda < \rho_1 < -\delta/2, \omega^{1/2} < 3\lambda\delta\},$$

such that

$$c\|\widetilde{B}_r u\|^2_{\mathcal{H}^1(M)} \leq$$
$$\leq -2Im\mathcal{E}_0(u, A_r^* A_r u) + C\Big(\|G_1 u\|^2_{\mathcal{H}^1(M)} + \|G_2 u\|^2_{\mathcal{H}^1(M)} + \|\chi u\|^2_{\mathcal{H}^{1,m}(M)} +$$
$$\|G_0 P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\chi P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|\chi \Theta u\|^2_{\dot{\mathcal{H}}^{-1,m}(M)} + \|G_0 \Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)}\Big).$$

The proof follows that of Lemma 6.12 in [GW20], using Lemma 3.0.7 to take into account the boundary conditions. In particular, given the data above, the main idea of the proof is similar to that of Proposition 3.0.3, barring some differences arising because of the geometric nature of the glancing region.

3.0.5 Propagation of singularities theorems

At last, we combine all the microlocal estimates we obtained so far, to establish the following propagation of singularities theorem, generalizing that of [GW20]. Recall that Θ denotes both the pseudodifferential operator implementing the boundary condition and its extension to M, see Remark 2.3.2.

Theorem 3.0.1. *Let $\Theta \in \Psi^k(\partial M)$ with $0 < k \leq 2$. If $u \in \mathcal{H}^{1,m}_{loc}(M)$ for $m \leq 0$ and $s \in \mathbb{R} \cup \{+\infty\}$, then $WF_b^{1,s}(u) \setminus \Big(WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)\Big)$ is the union of maximally extended generalized broken bicharacteristics within the compressed characteristic set $\dot{\mathcal{N}}$, see Equation (3.2).*

In full analogy it also holds

Theorem 3.0.2. *Let $\Theta \in \Psi^k(M)$ with $k \leq 0$. If $u \in \mathcal{H}^{1,m}_{loc}(M)$ for $m \leq 0$ and $s \in \mathbb{R} \cup \{+\infty\}$, then it holds that $WF_b^{1,s}(u) \setminus WF_b^{-1,s+1}(P_\Theta u)$ is the union of maximally extended GBBs within the compressed characteristic set $\dot{\mathcal{N}}$, see Equation (3.2).*

The proof of both theorems is similar to that given in [Vas08], using the estimates we derived in the previous sections. For this reason we do not give all details, limiting ourselves to outlining the analysis of forward propagation in the hyperbolic region for the reader's convenience. We focus on the case of Theorem 3.0.1, concerning $\Theta \in \Psi^k(\partial M)$ of order $0 < k \leq 2$.

First, we note that in $\mathring{M}$, the statement can be reduced to Duistermaat and Hörmander's theorem of propagation of singularities [DH72, H03]. The non-trivial part is to prove that the thesis holds true on the boundary. We begin proving a local version of the theorem, extending a non maximal GBB to ∂M. Namely, calling U an open chart of M such that $U \cap \partial M \neq \emptyset$, we show that if $q_0 \in WF_b^{1,s}(u) \setminus WF_b^{-1,s+1}(P_\Theta u)$ with $q_0 \in {}^bT^*_U M$, then there exists a GBB $\gamma : [-\varepsilon_0, 0] \to \dot{N}$, with $\varepsilon_0 > 0$, such that $\gamma(0) = q_0$ and $\gamma(s) \in WF_b^{1,s}(u) \setminus \Big(WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)\Big)$ for $s \in [-\varepsilon_0, 0]$. We focus on the case in which $q_0 \in \mathcal{H}(M)$.

Given $q_0 \in {}^bT^*_U M$, we build a sequence of generalized broken bicharacteristics $\gamma_j : [-\varepsilon_0, 0] \to \dot{N}$ such that

$$\gamma_j(s) \in WF_b^{1,s}(u) \setminus \Big(WF_b^{-1,s+1}(P_\Theta u) \cup WF_b^{-1,s+1}(\Theta u)\Big)$$

and with the endpoint $\gamma_j(0) \doteq q_j \in {}^bT^*\mathring{M}$ converging to q_0 on the boundary. Thanks to Proposition 3.0.3, choosing increasingly smaller sets $W \subset T^*M \setminus \{0\}$ we can found the sought sequence of points $\{q_j\}_{j\in\mathbb{N}}$. Since every $q_j \in \mathring{M}$, Hörmander's theorem on propagation of singularities [H03] [DH72] guarantees existence of the sought sequence of GBBs. The assumption of forward propagation, that is $\xi(q_j) < 0$, ensures that there exists $\varepsilon_0 > 0$ such that for $s \in [-\varepsilon_0, 0]$, $\gamma_j(s) \notin {}^bT^*_Y M$, where $\xi = 0$.

Since generalized broken bicharacteristics $\mathcal{R}_K[-\varepsilon_0, 0]$ with K compact are themselves compact in the topology of uniform convergence, Lemma 3.0.1 allows to conclude that there exists a subsequence $\{\gamma_{j_k}\}$ uniformly converging to γ.

At last we extend the result to maximal GBBs. Given a subset $V \subset \dot{N}$ with $q \in V$ and $a, b \in \mathbb{R}$ containing 0, there is a natural partial order on the set $\mathcal{GBB}_q$ of broken generalized bicharacteristics $\gamma : (a, b) \to V$ such that $\gamma(0) = q$. Let $\gamma_1 : (a_1, b_1) \to V$ and $\gamma_2 : (a_2, b_2) \to V$ be two elements of $\mathcal{GBB}_q$, we say that $\gamma_1 \leq \gamma_2$ if $(a_1, b_1) \subset (a_2, b_2)$ and if the two curves agree over the common domain (a_1, b_1).

Since a non-empty totally ordered subset has an upper bound, we can extend the GBBs joining the domains of those in the chain. At this point we apply Zorn's lemma to conclude that the maximal element of any totally ordered subsets being the maximal extension of a GBBs. In the glancing region the main idea is still to build a sequence of curves approximating a GBB, although the details are different due to some technical hurdles related to the geometry of the glancing region. The reader can find the argument in [Leb97] and [Vas08].

Chapter 4

Well-posedness of the problem and fundamental solutions

In this chapter we focus on the well-posedness of the weak problem and we study the associated fundamental solutions and propagators, generalizing to the case of boundary conditions implemented by suitable pseudodifferential operators the results of [Vas12],[War12] and [GW20].

4.1 Well-posedness of the problem

We begin studying the well-posedness of the boundary value problem as in Equation (2.3.2). In comparison to the previous chapter, we need to restrict the class of admissible pseudodifferential operators $\Theta \in \Psi^k(\partial M)$. Prior to stating our additional hypothesis, we need a preliminary definition. In the following remember that with Θ we denote both the operator implementing the boundary condition and its extension to M as per Remark (2.3.2).

Definition 4.1.1. Let $\Theta \in \Psi^k_M(\partial M)$. We call it local in time if, for every u in the domain of Θ, $\tau(\text{supp}(\Theta u)) \subseteq \tau(\text{supp}(u))$ where $\tau : \mathbb{R} \times \Sigma \to \mathbb{R}$ is the time coordinate individuated in Theorem 1.1.1.

We assume that the operators $\Theta \in \Psi^k(\partial M)$ implementing the boundary conditions satisfy the following hypothesis.

Hypotesis 4.1.1. *We consider $\Theta \in \Psi^k(\partial M)$ with $k \leq 2$, only if it is* local *in time and if $\Theta = \Theta^*$.*

Remark 4.1.1. We assume $k \leq 2$ because in Proposition 4.1.2 below we need to invoke Theorem 3.0.1.

We note that, at the level of applications, this additional hypothesis is only a mild constraint, because most cases of interest – for example boundary conditions of Dirichlet, Neumann and Robin type – satisfy Hypotesis 4.1.1.

Following the analysis in [Vas12, GW20], we introduce a cutoff function playing an important rôle in the following theorems. Consider once again the function χ_0 introduced in Equation (3.20) and let $\chi_1 \in C^\infty(\mathbb{R})$ be an increasing function such that $\chi_1(s) = 0$ for all $s \in (-\infty, 0]$ while $\chi_1(s) = 1$ if $s \in [1, +\infty)$. Given a fixed $\tau_0, \tau_1 \in \mathbb{R}$ with $\tau_0 < \tau_1$, we define the smooth function $\chi : (\tau_0, \tau_1) \to \mathbb{R}$ as

$$\chi(s) \doteq \chi_0(-\delta^{-1}(s - \tau_1))\chi_1((s - \tau_0)/\varepsilon), \tag{4.1}$$

where $\delta \gg 1$ while $\varepsilon \in (0, \tau_1 - \tau_0)$. Note that, calling $\chi_0' = \frac{d\chi_0}{ds}$, it holds that, *cf.* [Vas12]

$$\chi \leq -\delta^{-1}(\tau_1 - \tau_0)^2\chi' \text{ with } \chi' = -\delta^{-1}\chi_0'(-\delta^{-1}(s - \tau_1)). \tag{4.2}$$

Let $u \in \mathcal{H}^{1,1}_{loc}(M)$ be such that its support lies in $[\tau_0 + \varepsilon, \tau_1] \times \Sigma$, *cf.* Definition 1.1.1.

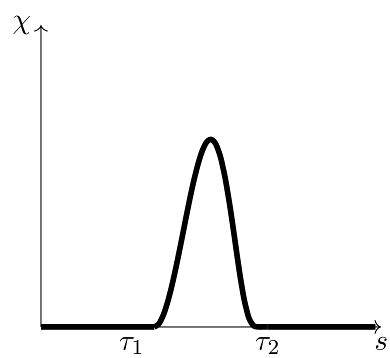


Figure 4.1: The function χ.

Using this cutoff function, as in [GW20], one can prove a twisted version of the Poincaré inequality obtained in [Vas12, Proposition 2.5]:

$$\|(-\chi')^{1/2}u\|^2_{\mathcal{L}^2(M)} \leq C\|(-\chi')^{1/2}d_F u\|^2_{\mathcal{L}^2(M)}, \tag{4.3}$$

where d_F is the twisted differential, as in Section 2.3.1.

The main step of our analysis lies in proving the following lemma, generalizing a result in [GW20] for the case of Robin boundary conditions. We discuss in details only the case with $\Theta \in \Psi^k(\partial M)$ of order $k > 0$, since the scenario in which $k \leq 0$, can be seen as a corollary of the well-posedness statement of [GW20].

Proposition 4.1.1. Let $\Theta \in \Psi^k(\partial M)$ be a pseudodifferential operator of positive order k such that its canonical extension to M abides to Hypothesis 4.1.1 and let $u \in \mathcal{H}^{1,1+k}_{loc}(M)$. Then there exists a compact subset $K \subset M$ and a real positive constant C such that

$$\|(-\phi')^{1/2}u\|_{\mathcal{H}^1(K)} \leq C\|P_\Theta u\|_{\mathcal{H}^{-1,1+k}(K)},$$

where $\phi = \chi \circ \tau$, χ being the same as in Equation (4.3).

Proof. The proof is a generalization of those in [Vas12] and [GW20] to the case of boundary conditions encoded by pseudodifferential operators. Therefore we shall sketch the common part of the proof, focusing on the terms introduced by the boundary conditions. Adopting the same conventions as at the beginning of the section, assume that $supp(u) \subset [\tau_0+\varepsilon, \tau_1]\times\Sigma$. We start by computing a twisted version of the energy form considered in [Vas12]. Consider $\langle -i[(V')^* P_\Theta - P_\Theta V']u, u\rangle$, with $V' = FVF^{-1} \in Diff_b^1(M)$ and $V \in \mathcal{V}_b(M)$ with compact support. Note that, since Θ is self-adjoint, *i.e.*, $\Theta = \Theta^*$, then $i[(V')^* P_\Theta - P_\Theta V']$ is a second order formally self-adjoint operator, the purpose of V'^* being to remove zeroth order terms. Let $V = -\phi W$ with $W = \nabla_{\widehat{g}}\tau$. Observe that we should also localize in space, working on a compact subset of a Cauchy surface. Hence the vector field V should be multiplied by a smooth, compactly supported test function ψ depending only on the spatial variables. However, to avoid burdening the notation, in the following we do not write explicitly the test function ψ. The vector field V belongs to $\mathcal{V}_b(M)$ because $\widehat{g}(dx, dt) = 0$. A direct computation shows that

$$\begin{aligned} \langle -i[(V')^* P_\Theta - P_\Theta V']u, u\rangle = 2Re\langle P_\Theta u, V'u\rangle = \\ = 2Re\mathcal{E}_0(u, V'u) + 2Re\langle S_F u, V'u\rangle + 2Re\langle \Theta\gamma_- u, \gamma_- V'u\rangle, \end{aligned} \tag{4.4}$$

where $\mathcal{E}_0$ is the twisted Dirichlet energy form, *cf.* Equation (2.10), $S_F \doteq F^{-1}PF \in x^2 L^\infty(M)$ as in Section 2.3.1, while γ_+ and γ_- are the trace maps introduced in Theorem 1.4.1, Lemma 2.2.1 and Lemma 2.3.1. We analyze each term in the above sum separately. Starting form the first one and proceeding as in [GW20], we rewrite

$$2Re\mathcal{E}_0(u, V'u) = \langle B^{ij} Q_i u, Q_j u\rangle,$$

where Q_i, $i = 1, \dots, n$ is a generating set of $\mathbf{Diff}^1_\nu(M)$, while the symmetric tensor B is

$$\begin{aligned} B = -(\phi \cdot div_{\widehat{g}} W + 2F\phi V(F^{-1}) + (n-2)\phi x^{-1} W(x))\widehat{g}^{-1} + \\ +\phi \mathcal{L}_W \widehat{g}^{-1} + 2T(W, \nabla_{\widehat{g}}\phi). \end{aligned} \tag{4.5}$$

Here $T(W, \nabla_{\widehat{g}}\phi)$ is the stress-energy tensor, with respect to $\widehat{g}$, of a scalar field associated with W and $\nabla_{\widehat{g}}\phi$, that is, denoting with $\odot$ the symmetric tensor product,

$$T(W, \nabla_{\widehat{g}}\phi) = (\nabla_{\widehat{g}}\phi) \odot W - \frac{1}{2}\widehat{g}(\nabla_{\widehat{g}}\phi, W) \cdot \widehat{g}^{-1}. \tag{4.6}$$

Focusing on this term and using that $\nabla_{\widehat{g}}\phi = \chi' \nabla_{\widehat{g}} \tau$, a direct computation yields:

$$T_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi) = \frac{1}{2}(\chi' \circ \tau)[2(\nabla_{\widehat{g}}\tau) \otimes (\nabla_{\widehat{g}}\tau) - \widehat{g}(\nabla_{\widehat{g}}\tau, \nabla_{\widehat{g}}\tau) \cdot \widehat{g}^{-1}]. \tag{4.7}$$

Since $\nabla_{\widehat{g}}\phi$ and $\nabla_{\widehat{g}}\tau$ are respectively past- and future-pointing timelike vectors, then $T_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi)$ is negative definite. Hence we can rewrite Equation (4.4) as

$$\begin{aligned} \langle -T^{ij}_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi) Q_i u, Q_j u\rangle = \langle -i[(V')^* P_\Theta - P_\Theta V']u, u\rangle + \\ +2Re\mathcal{E}_0(K^{ij} Q_i u, Q_j u) + 2Re\langle S_F u, V'u\rangle + 2Re\langle \Theta\gamma_- u, \gamma_- V'u\rangle, \end{aligned} \tag{4.8}$$

with

$$K = -(F\phi V(F^{-1}) + (n-2)\phi x^{-1} W(x))\widehat{g}^{-1} + \phi\mathcal{L}_W \widehat{g}^{-1}.$$

Since $-T_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi)^{ij}$ is positive definite, then

$$\mathcal{Q}(u,u) \doteq \langle -T_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi)^{ij} Q_i u, Q_j u\rangle \geq 0$$

. This can be seen by direct inspection from the explicit form

$$\begin{aligned} \mathcal{Q}(u,u) &= \int_M \phi' \left((\nabla_{\widehat{g}}\tau)^i (\nabla_{\widehat{g}}\tau)^j - \frac{1}{2}\widehat{g}((\nabla_{\widehat{g}}\tau)^i (\nabla_{\widehat{g}}\tau)^j) \right) Q_i u \ \overline{Q_j u} \ x^2 d\mu_g \\ &= \int_M H((-\phi')^2 d_F u, (-\phi')^{1/2} d_F \overline{u}) x^2 d\mu_g, \end{aligned} \tag{4.9}$$

where H is the sesquilinear pairing between 1-forms induced by the metric. Focusing then on the term $\langle K^{ij} Q_i u, Q_j u\rangle$, we observe that, as a consequence of our choice for the functions f and W, we have $V(x) = \widehat{g}(\nabla_{\widehat{g}}\tau, \nabla_{\widehat{g}} x) = 0$

on ∂M. In addition it holds that $x^{-1}W(x) = \mathcal{O}(1)$ near ∂M, and $\mathcal{L}_V \widehat{g}^{-1} = 2\,\nabla_{\widehat{g}}(\nabla_{\widehat{g}}\tau) = 2\widehat{\Gamma}^i_{\tau\tau}\partial_i$. These observations allow us to establish the following bound, *cf.* [Vas12] and [GW20]:

$$|\langle K^{ij}Q_i u, Q_j u\rangle| \leq C\|\phi^{1/2}d_F u\|_{\mathcal{L}^2(M)} \leq C\delta^{-1}(\tau_1-\tau_0)^2\|(-\phi')^{1/2}d_F u\|^2_{\mathcal{L}^2(M)}, \tag{4.10}$$

with C a suitable, positive constant. Now we focus on establishing a bound for the terms on the right hand side of Equation (4.8). We estimate the first one as follows:

$$\begin{aligned}|\langle -i[(V')^*P_\Theta - P_\Theta V']u,u\rangle| &\leq C\left(\|\phi^{1/2}FWF^{-1}P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \|\phi^{1/2}u\|^2_{\mathcal{H}^1(M)}\right)\\ &+C\left(\|\phi^{1/2}P_\Theta u\|^2_{\mathcal{L}^2(M)} + \|\phi^{1/2}FWF^{-1}u\|^2_{\mathcal{L}^2(M)}\right) \leq\\ &\leq C(\|FWF^{-1}P_\Theta u\|^2_{\dot{\mathcal{H}}^{-1}(M)} + \delta^{-1}(\tau_1-\tau_0)^2\|(-\phi')^{1/2}u\|^2_{\mathcal{H}^1(M)}+\\ &+\|P_\Theta u\|^2_{\mathcal{L}^2(M)} + \delta^{-1}(\tau_1-\tau_0)^2\|(-\phi')^{1/2}FWF^{-1}u\|^2_{\mathcal{L}^2(M)}\Big),\end{aligned} \tag{4.11}$$

where in the last inequality we used Equation (4.2). As for the second term in Equation (4.8), using that $S_F \in x^2L^\infty(M)$, we establish the bound

$$2|Re\langle S_F u, V'u\rangle| \leq \widetilde{C}\left(\|\phi^{1/2}\ u\|^2_{\mathcal{L}^2(M)} + \|\phi^{1/2}\ d_F u\|^2_{\mathcal{L}^2(M)}\right),$$

for a suitable constant $\widetilde{C} > 0$. Using Equation (4.2) and the Poincaré inequality, this last bound becomes

$$2|Re\langle S_F u, V'u\rangle| \leq C\delta^{-1}(\tau_1-\tau_0)^2\|(-\phi')^{1/2}d_F u\|^2_{\mathcal{L}^2(M)}. \tag{4.12}$$

At last we give a bound for the last term in Equation (4.4), containing the pseudodifferential operator Θ which implements the boundary conditions. Recalling Hypothesis 4.1.1, it is convenient to consider the following three cases separately

a) $\Theta \in \Psi^k(\partial M)$ with $k \leq 1$,

b) $\Theta \in \Psi^k(\partial M)$ with $1 < k \leq 2$.

Now we give a bound case by case.

a) It suffices to focus on $\Theta \in \Psi^1(\partial M)$ recalling that, for $k < 1$, $\Psi^k(\partial M) \subset \Psi^1(\partial M)$. If with a slight abuse of notation we denote with Θ both the operator on the boundary and its trivial extension to the whole manifold, we can write

$$\langle \Theta\gamma_- u, \gamma_- V'u\rangle = \langle \widehat{N}(\Theta)(-i\nu_-)\gamma_- u, \gamma_- V'u\rangle = \langle \gamma_-\Theta u, \gamma_- V'u\rangle,$$

where $\widehat{N}(\Theta)(-i\nu_-)$ is the indicial family of Θ, introduced in (1.3.7). We recall that any $A \in \Psi^s(\partial M)$, $s \in \mathbb{N}$, can be decomposed as $\sum_{i=1}^{n} Q_i A_i + B$, with $A_i, B \in \Psi^{s-1}(\partial M)$, while Q_i, $i = 2, \ldots, n-1$ is a generating set of $\mathbf{Diff}^1_\nu(\partial M)$. Hence we can rewrite Θ as

$$\Theta = \sum_i Q_i \Theta_i + \Theta' = \sum_i (\Theta_i Q_i + [Q_i, \Theta_i]) + \Theta',$$

where Θ_i, Θ' and $[Q_i, \Theta_i]$ are in $\Psi^0(\partial M)$. Therefore

$$|\langle \gamma_- \Theta u, \gamma_- V' u\rangle| \leq |\langle \gamma_- \left(\sum_i \Theta_i Q_i u\right), \gamma_- V' u\rangle| + \\ + |\langle \gamma_- \left(\left([Q_i, \Theta_i] + \Theta'\right) u\right), \gamma_- V' u\rangle|.$$

To begin with, we focus on the first term on the right hand side of this inequality. Using Equations (1.4.1) and (4.2) together with the Poincaré inequality (4.3) and Lemma 1.4.3,

$$|\langle \gamma_- \left(\sum_i \Theta_i Q_i u\right), \gamma_- V' u\rangle| \leq \\ \varepsilon \left(\sum_i \|\phi^{1/2} \Theta_i Q_i u\|^2_{\mathcal{H}^1(M)} + \|\phi^{1/2} F W F^{-1} u\|^2_{\mathcal{H}^1(M)}\right) + \\ + C_\varepsilon \left(\sum_i \|\phi^{1/2}\ Q_i u\|^2_{\mathcal{L}^2(M)} + \|\phi^{1/2} F W F^{-1} u\|^2_{\mathcal{L}^2(M)}\right) \leq \\ \leq C_\varepsilon \delta^{-1} (\tau_1 - \tau_0)^2 \|(-\phi')^{1/2} d_F u\|^2_{\mathcal{L}^2(M)},$$

for a suitable constant $C_\varepsilon > 0$. As for the second term, since $u \in \mathcal{H}^{1,1}_{loc}(M)$ we can proceed as above using that the operator $\Theta' + [Q_i, \Theta_i]$ is of order 0 and we can conclude that

$$\left|\langle \gamma_- \left(\left([Q_i, \Theta_i] + \Theta'\right) u\right), \gamma_- V' u\rangle\right| \leq \widetilde{C}_\varepsilon \|\phi^{1/2}\ u\|^2_{\mathcal{H}^1(M)} \\ \leq C_\varepsilon \delta^{-1} (\tau_1 - \tau_0)^2 \|(-\phi')^{1/2} d_F u\|^2_{\mathcal{L}^2(M)},$$

for suitable positive constants C_ε and $\widetilde{C}_\varepsilon$. Therefore, it holds a bound of the form

$$|Re\langle \Theta \gamma_- u, \gamma_- V' u\rangle| \leq C' \delta^{-1} (\tau_1 - \tau_0)^2 \|(-\phi')^{1/2} d_F u\|^2_{\mathcal{L}^2(M)}.$$

b) Since $\Psi_b^k(\partial M) \subset \Psi_b^{k'}(\partial M)$ if $k < k'$, it is enough to consider $\Theta \in \Psi_b^2(\partial M)$ and to observe that, we can decompose Θ as

$$\Theta = \sum_{i=1}^{n} Q_i \left(\sum_{j=1}^{n} Q_j A_{ij} \right) + B_i,$$

where $B_i \in \Psi_b^1(\partial M)$ while $A_{ij} \in \Psi_b^0(\partial M)$. At this point one can apply twice consecutively the same reasoning as in item a) to draw the sought conclusion. Here the key hypothesis it that $u \in \mathcal{H}^{1,1+k}_{loc}(M)$.

Finally, considering Equation (4.8) and collecting all bounds we proved, we obtain

$$\begin{aligned} \langle -T^{ij}_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi) Q_i u, Q_j u \rangle \leq C \| P_\Theta u \|^2_{\dot{\mathcal{H}}^{-1,1+k}(M)} \\ + C\delta^{-1}(\tau_1 - \tau_0)^2 \| (-\phi')^{1/2} d_F u \|^2_{\mathcal{L}^2(M)}. \end{aligned} \tag{4.13}$$

Since the inner product H defined by the left hand side of Equation (4.9) is positive definite, then for δ large enough

$$\langle -T^{ij}_{\widehat{g}}(W, \nabla_{\widehat{g}}\phi) Q_i u, Q_j u \rangle - C\delta^{-1}(\tau_1 - \tau_0)^2 \| (-\phi')^{1/2} d_F u \|^2_{\mathcal{L}^2(M)} \geq 0,$$

and the associated Dirichlet form $\widetilde{\mathcal{Q}}$ defined as

$$\begin{aligned} \widetilde{\mathcal{Q}}(u,u) = \int_M \Big[H((-\phi')^2 d_F u, (-\phi')^{1/2} d_F \overline{u}) + \\ - C\delta^{-1}(\tau_1 - \tau_0)^2 |(-\phi')^{1/2} d_F u|^2 \Big] x^2 d\mu_g, \end{aligned} \tag{4.14}$$

bounds $\| (-\phi')^{1/2} d_F u \|^2_{\mathcal{L}^2(M)}$. We conclude the proof by observing that, once we have an estimate for $\| (-\phi')^{1/2} d_F u \|^2_{\mathcal{L}^2(M)}$, with the Poincaré inequality we can also bound $\| (-\phi')^{1/2} u \|_{\mathcal{L}^2(M)}$. Therefore, considering the support of χ and u, there exists a compact subset $K \subset M$ such that

$$\| (-\phi')^{1/2} u \|_{\mathcal{L}^2(M)} \leq C \| (-\phi')^{1/2} P_\Theta u \|_{\dot{\mathcal{H}}^{-1,1+k}(K)}, \tag{4.15}$$

from which the sought thesis descends. □

Remark 4.1.2. If the pseudodifferential operator $\Theta \in \Psi^k(\partial M)$ implementing the boundary condition is of non-positive order, then in the previous statement u can be taken in $\mathcal{H}^{1,1}_{loc}(M)$ instead of $\mathcal{H}^{1,1+k}_{loc}(M)$. The same holds true for the statements of Corollary 4.1.1.

Making use of the previous lemma, we obtain the following statements concerning uniqueness and existence of the solutions for the Klein-Gordon equation associated to the operator P_Θ.

Corollary 4.1.1. *Let M be a globally hyperbolic, asymptotically anti-de Sitter spacetime and let $f \in \dot{\mathcal{H}}^{-1,1+k}(M)$ – with k the order of Θ – be vanishing whenever $\tau < \tau_0$, $\tau_0 \in \mathbb{R}$. Suppose in addition that Θ abides to the Hypothesis 4.1.1. Then there exists at most one $u \in \mathcal{H}^1_{loc}(M)$ such that $supp(u) \subset \{q \in M \mid \tau(q) \geq \tau_0\}$ and it is a solution of $P_\Theta u = f$.*

We omit the proof of this statement, since it is identical to that of [Vas12, Lemma 4.13].

Lemma 4.1.1. *Let M be a globally hyperbolic, asymptotically anti-de Sitter spacetime and let $f \in \dot{\mathcal{H}}^{-1,1}(M)$ be vanishing whenever $\tau < \tau_0$, $\tau_0 \in \mathbb{R}$. Then there exists $u \in \mathcal{H}^{1,-1+k}_{loc}(M)$ solution of the problem $P_\Theta u = f$ such that $\tau(supp(u)) \geq \tau_0$.*

Lemma 4.1.2. *Let M be a globally hyperbolic, asymptotically anti-de Sitter spacetime, let $\Theta \in \Psi^k(M)$ with $k \geq 0$ and let $f \in \dot{\mathcal{H}}^{-1,1+k}(M)$ be vanishing whenever $\tau < \tau_0$, $\tau_0 \in \mathbb{R}$. Then there exists $u \in \mathcal{H}^{1,-1+k}_{loc}(M)$ solution of the problem $P_\Theta u = f$ such that $\tau(supp(u)) \geq \tau_0$.*

The proofs of these two lemmas follow the one given in [Vas12, Prop. 4.15]. Therefore, we sketch only the main ideas, focusing for simplicity on the case $k \geq 0$. The first step consists of proving a local version of the lemma, namely that given a compact set $I \subset \mathbb{R}$, there exists $\sigma > 0$ such that for every $\tau_0 \in I$ there exists $u \in \mathcal{H}^{1,-1}(M)$ such that $supp(u) = \{p \in M \mid \tau(p) \geq 0\}$ and $P_\Theta u = f$ for $\tau < \tau_0 + \sigma$. The main point of this part of the proof consists of applying Lemma 4.1.1 to ensure that the adjoint of the Klein-Gordon operator, say P^*_Θ, is invertible over the set of smooth functions supported in suitable compact subsets of M – see [Vas12, Lem. 4.14] for further details. With this result in hand, one divides the time direction into sufficiently small intervals $[\tau_j, \tau_{j+1}]$ and uses a partition of unity along the time coordinate to build a global solution for $P_\Theta u = f$.

At last we extend our results for $u \in \mathcal{H}^{1,m}_{loc}(M)$ and for $f \in \dot{\mathcal{H}}^{-1,m+1}_{loc}(M)$. Let us consider $\Theta \in \Psi^k_b(\partial M)$, the proof for the positive cases being the same. If $m \geq 0$, Lemma 4.1.1 entails that Equation (2.13) admits a unique solution lying in $\mathcal{H}^1_{loc}(M)$. By the propagation of singularities theorem, *cf.* Theorem 3.0.2 and using Hypothesis 4.1.1, the solution lies in $\mathcal{H}^{1,m}_{loc}(M)$ and the following generalization of the bound in Lemma 4.1.1 holds true:

$$\|u\|_{\mathcal{H}^{1,m}(M)} \leq C\|f\|_{\dot{\mathcal{H}}^{-1,m+1}(M)}.$$

If $m < 0$ we can draw the same conclusion considering, as in [Vas12, Thm. 8.12],

$$P_\Theta u_j = f_j \tag{4.16}$$

where $f_j \in \dot{\mathcal{H}}^{-1,m+1}(M)$ is a sequence converging to f as $j \to \infty$. Each of these equations has a unique solution $u_j \in \mathcal{H}^1(M)$. In addition the propagation of singularities theorem, *cf.* Theorem (3.0.2) yields the bound

$$\|u_k - u_j\|_{\mathcal{H}^{1,m}(K)} \leq C\|f_k - f_j\|_{\dot{\mathcal{H}}^{-1,m+1}(L)}$$

for suitable compact sets $K, L \subset M$ and for every $j, k \in \mathbb{N}$. Since $f_j \to f$ in $\dot{\mathcal{H}}^{-1,m+1}(L)$, we can conclude that the sequence u_j is converging to $u \in \mathcal{H}^{1,m}(K)$. Considering $\{f_j\}$ such that each f_j vanishes if $\{\tau < \tau_0\}$, one obtains the desired support property of the solution. To conclude this analysis we summarize the final result which combines Corollary 4.1.1 and Lemmata 4.1.1 and 4.1.2.

Proposition 4.1.2. Let M be a globally hyperbolic, asymptotically anti-de Sitter spacetime and let $m, \tau_0 \in \mathbb{R}$ while $f \in \dot{\mathcal{H}}^{-1,m+1}_{loc}(M)$. Assume in addition that Θ abides to Hypothesis 4.1.1. If f vanishes for $\tau < \tau_0$, $\tau_0 \in \mathbb{R}$ being arbitrary but fixed, then there exists a unique $u \in \mathcal{H}^{1,m+k}_{loc}(M)$ such that

$$P_\Theta u = f, \tag{4.17}$$

where P_Θ is the operator in Equation (2.15).

4.2 Fundamental solutions and propagators

In this section we focus on the study of the fundamental solutions associated with the Klein-Gordon operator P_Θ. As a preliminary step, we need to define the following subspaces of $\mathcal{H}^{k,m}(M)$, $k = 0, \pm 1$, $m \in \mathbb{N} \cup \{0\}$:

$$\mathcal{H}^{k,m}_-(M) = \{u \in \mathcal{H}^{k,m}(M) \mid \exists \tau_- \in \mathbb{R} \text{ such that } p \notin \text{supp}(u), \text{ if } \tau(p) < \tau_-\}, \tag{4.18a}$$

$$\mathcal{H}^{k,m}_+(M) = \{u \in \mathcal{H}^{k,m}(M) \mid \exists \tau_+ \in \mathbb{R} \text{ such that } p \notin \text{supp}(u) \text{ if } \tau(p) > \tau_+\}, \tag{4.18b}$$

$$\mathcal{H}^{k,m}_{tc}(M) \doteq \mathcal{H}^{k,m}_+(M) \cap \mathcal{H}^{k,m}_-(M), \tag{4.18c}$$

where the subscript tc stands for *timelike compact*. To take into account the boundary conditions, we also need to define the spaces

$$\mathcal{H}^{1,m}_{\pm,\Theta}(M) \doteq \{u \in H^{1,m}_\pm(M) \mid \gamma_+(u) = \Theta\gamma_-(u)\}, \tag{4.19}$$

where γ_-, γ_+ are the trace maps introduced in Theorem 1.4.1 and in Lemma 2.2.1, while Θ is a pseudodifferential satisfying Hypothesis 4.1.1.

Making use of Lemma 4.1.1 and Proposition 4.1.2 and proceeding as in [GW20] we obtain the following result on the advanced and retarded propagators $G^{\pm}_{\Theta}$ associated to the Klein-Gordon operator P_{Θ}.

Theorem 4.2.1. *Let P_{Θ} be the Klein-Gordon operator as per Equation where Θ abides to Hypothesis 4.1.1.*
Then there exist unique retarded $(+)$ and advanced $(-)$ propagators, that is continuous operators $G^{\pm}_{\Theta} : \dot{\mathcal{H}}^{-1,m+1}_{\pm}(M) \to \mathcal{H}^{1,m}_{\pm}(M)$ such that $P_{\Theta}G^{\pm}_{\Theta} = \mathbb{I}$ on $\dot{\mathcal{H}}^{-1,m+1}_{\pm}(M)$ and $G^{\pm}_{\Theta}P_{\Theta} = \mathbb{I}$ on $\mathcal{H}^{1,m}_{\pm,\Theta}(M)$. Furthermore, $G^{\pm}_{\Theta}$ is a continuous map from $\dot{\mathcal{H}}^{-1,\infty}_{0}(M)$ to $\mathcal{H}^{1,\infty}_{loc}(M)$ where the subscript 0 indicates that we consider only functions of compact support.

Remark 4.2.1. We need to restrict P_{Θ} to $\mathcal{H}^{1,m}_{\pm,\Theta}(M)$ because per construction an element in the range of $G^{\pm}_{\Theta}P_{\Theta}$ abides to the boundary conditions as in Equation (2.13).

In addition to the advanced and the retarded propagators, we can also define the *causal propagator* $G_{\Theta} : \dot{\mathcal{H}}^{-1,m+1}_{0}(M) \to \mathcal{H}^{1,m}_{loc}(M)$ as $G_{\Theta} = G^{+}_{\Theta} - G^{-}_{\Theta}$.

Remark 4.2.2. Since $G^{\pm}_{\Theta}$ are continuous maps, one can apply the Schwartz kernel theorem to infer that one can associate to them a bi-distribution $\mathcal{G}^{\pm}_{\Theta} \in \mathcal{D}'(\mathring{M} \times \mathring{M})$.

To conclude this section we study some properties of the causal propagator which are useful for applications in quantum field theory. For physical reasons we individuate two special classes of boundary conditions. We remember that the spacetime M is isometric to $\mathbb{R} \times \Sigma$ and ∂M to $\mathbb{R} \times \partial\Sigma$, see Theorem 1.1.1

Definition 4.2.1. Let $\Theta \in \Psi^k_b(M)$ with $k \leq 2$ and let $\Theta = \Theta^*$ We call Θ

- *physically admissible* if $WF_b^{-1,s+1}(\Theta u) \subseteq WF_b^{-1,s+1}(P_{\Theta}u)$ for all $u \in \mathcal{H}^{1,m}_{loc}(M)$ with $m \leq 0$ and $s \in \mathbb{R} \cup \{\infty\}$.

- a *static* boundary condition if $\Theta \equiv \Theta_K$ is the natural extension to $\Psi^k_b(M)$ of a pseudodifferential operator $K = K^* \in \Psi^k_b(\partial\Sigma)$ with $k \leq 2$.

Remark 4.2.3. Any static boundary condition is automatically local in time, as per Definition 4.1.1.

Proposition 4.2.1. Let P_Θ be the Klein-Gordon operator and let G_Θ be its associated causal propagator. Then the following is an exact sequence:

$$0 \to \mathcal{H}^{1,\infty}_{tc,\Theta}(M) \xrightarrow{P_\Theta} \dot{\mathcal{H}}^{-1,\infty}_{tc}(M) \xrightarrow{G_\Theta} \mathcal{H}^{1,\infty}_{\Theta}(M) \xrightarrow{P_\Theta} \dot{\mathcal{H}}^{-1,\infty}(M) \to 0\,. \quad (4.20)$$

Proof. To prove that the sequence is exact, we start by establishing that P_Θ is injective on $\mathcal{H}^{1,\infty}_{tc,\Theta}(M)$. This is a consequence of Theorem 4.2.1 which guarantees that, if $P_\Theta(h) = 0$ for $h \in \mathcal{H}^{1,\infty}_{tc,\Theta}(M)$, then $G^+P_\Theta(h) = h = 0$.

Secondly, on account of Theorem 4.2.1 and in particular of the identity $G^{\pm}_\Theta P_\Theta = \mathbb{I}$ on $\mathcal{H}^1_{\pm,\Theta}(M)$, it holds that $G_\Theta P_\Theta(f) = 0$ for all $f \in \mathcal{H}^{1,\infty}_{tc,\Theta}(M)$. Hence $\mathrm{Im}(P_\Theta) \subseteq \ker(G_\Theta)$. Assume that there exists $f \in \dot{\mathcal{H}}^{-1,\infty}_{tc}(M)$ such that $G_\Theta(f) = 0$. It descends that $G^+_\Theta(f) = G^-_\Theta(f) \in \mathcal{H}^{1,\infty}_{tc,\Theta}(M)$. Applying P_Θ it holds that $f = P_\Theta G^+_\Theta(f)$, that is $f \in P_\Theta[\mathcal{H}^{1,\infty}_{tc,\Theta}(M)]$.

The third step consists of recalling that, per construction, $P_\Theta G_\Theta = 0$ and that, still on account of Theorem 4.2.1, $\mathrm{Im}(G_\Theta) \subseteq \ker(P_\Theta)$. To prove the opposite inclusion, suppose that $u \in \ker(P_\Theta)$. Let $\chi \equiv \chi(\tau)$ be a smooth function such that there exists $\tau_0, \tau_1 \in \mathbb{R}$ such that $\chi = 1$ if $\tau > \tau_1$ and $\chi = 0$ if $\tau < \tau_0$. Since Θ is a static boundary condition and, therefore, it commutes with χ, it holds that $\chi u \in \mathcal{H}^{1,\infty}_{+,\Theta}(M)$. Hence setting $f \doteq P_\Theta \chi u$, a direct calculation shows that $G_\Theta f = u$

To conclude we need to show that the map P_Θ on the before last arrow is surjective. To this end, let $j \in \dot{\mathcal{H}}^{-1,\infty}(M)$ and let $\chi \equiv \chi(\tau)$ be as above. Let $h \doteq G^+_\Theta(\chi j) + G^-_\Theta((1-\chi)j)$. Per construction $h \in \mathcal{H}^{1,\infty}_\Theta(M)$ and $P_\Theta(h) = j$. □

Now we focus on the study of the singularities of the advanced and of the retarded propagators. To this end we introduce the space $\mathcal{W}^{-\infty}_b(M)$ of bounded operators from $\dot{\mathcal{H}}^{-1,-\infty}_0(M)$ to $\mathcal{H}^{1,\infty}_{loc}(M)$ and we give a definition of wavefront set complementary to that of Definition 1.3.14.

Definition 4.2.2 (Operatorial wavefront set $WF^{Op}_b(M)$)**.** Consider a continuous map $\Lambda : \dot{\mathcal{H}}^{-1,-\infty}_0(M) \to \mathcal{H}^{1,\infty}_{loc}(M)$. A point $(q_1, q_2) \in {}^bS^*M \times {}^bS^*M$ is not in the operatorial wavefront set $WF^{Op}_b(M)$ if there exists two b-pseudodifferential operators B_1 and B_2 in $\Psi^0_b(M)$ elliptic at q_1 and q_2 respectively, such that $B_1 \Lambda B_2^* \in \mathcal{W}^{-\infty}_b(M)$.

We can characterize the singularities of the advanced and of the retarded fundamental solutions in the cosphere bundle as follows.

Theorem 4.2.2. *Let Δ denote the b-diagonal in ${}^bS^*M \times {}^bS^*M$ and let Θ be physically admissible as per Definition 4.2.1. Then*

$$WF_b^{Op}(G^{\pm}_{\Theta}) \setminus \Delta \subset \{(q_1, q_2) \in {}^bS^*M \times {}^bS^*M \mid q_1 \dot{\sim} q_2,\ \pm t(q_1) > \pm t(q_2)\},$$

where $q_1 \dot{\sim} q_2$ means that q_1, q_2 are two points in $\dot{\mathcal{N}}$, cf. Equation (3.2) connected by a generalized broken bicharacteristic, cf. Definition 3.0.1.

This statement follows from Theorem 3.0.1 – in the case of Θ of order $0 < k \leq 2$ or Theorem 3.0.2 – if Θ is of order $k \leq 0$. We refer to [GW20] for the details.

Remark 4.2.4. We assume that the operator Θ is physically admissible as per Definition 4.2.1 because we do not want to alter the microlocal behavior of the system in $\mathring{M}$. Suppose not to place any restriction on the wavefront set of Θu. Then by the propagation of singularities theorem Theorem 3.0.1, in addition to the singularities propagating along the generalized broken bicharacteristics of the Klein-Gordon operator we should account also for those of Θu. However, in concrete applications, for example the construction of Hadamard two-point functions, one looks for bi-distributions with a prescribed form of the wave front set and whose antisymmetric part coincides with the difference between the advanced and retarded fundamental solutions associated to the Klein-Gordon operator with boundary condition implemented by Θ, see *e.g.* [DF16, DDF19, DW, Wro17, GW20].

4.3 Construction of fundamental solutions on static spacetimes in the massless case

We conclude this chapter giving an example of advanced and retarded fundamental solutions for the Klein-Gordon operator on a static, globally hyperbolic, asymptotically AdS spacetime. In particular, we shall focus on a massless scalar field, corresponding to the case $\nu = (n-1)/2$. The construction of the fundamental solutions via functional calculus in this particular case can be done along the lines of [DDF19]. Therefore we limit ourselves to describing the analytic and the geometric framework, referring to [DDF19] for the derivation and the technical details.

As a preliminary step, we specify the underlying geometric structure:

Definition 4.3.1. Let (M, g) be an n-dimensional Lorentzian manifold. We call it a static globally hyperbolic, asymptotically AdS spacetime if it abides to Definition 1.1.4 and, in addition,

1) There exists an irrotational, timelike Killing field $\chi \in \Gamma(TM)$, such that $\mathcal{L}_\chi(x) = 0$ where x is the global boundary function,

2) $(M, \hat{g})$ is isometric to a standard static spacetime, that is a warped product $\mathbb{R} \times_\beta S$ with line element $ds^2 = -\alpha^2 dt^2 + h_S$ where h_S is a t-independent Riemannian metric on S, while $\alpha \neq \alpha(t)$ is a smooth, positive function not depending on t.

On account of Theorem 1.1.1 we can assume, without loss of generality, that the timelike Killing field χ coincides with the vector field ∂_τ. Under this assumption the underlying line-element can be written as $ds^2 = -\beta d\tau^2 + \kappa$ where β and κ are τ-independent while S can be identified with the Cauchy surface Σ in Theorem 1.1.1. In view of this characterization of the metric, the associated Klein-Gordon equation $Pu = 0$ with $P = \Box_g$ reads

$$\left(-\partial_\tau^2 + E\right) u = 0, \tag{4.21}$$

where $E = \beta \Delta_\kappa$, with Δ_κ the Laplace-Beltrami operator associated to the the Riemannian metric κ. Let us consider a static boundary condition as per Definition 4.2.1, implemented by a pseudodifferential operator $K \in \Psi_b^k(\partial M)$ and let us employ the symbol Θ_K to recall that it is induced from K. Since the underlying spacetime is static, in order to construct the advanced and retarded fundamental solutions, we can focus our attention on $\mathcal{G}_{\Theta_K} \in \mathcal{D}'(\mathring{M} \times \mathring{M})$, the bi-distribution associated to the causal propagator G_{Θ_K}. It satisfies the following initial value problem, see also [DDF19]:

$$\begin{cases} (P_{\Theta_K} \otimes \mathbb{I})\mathcal{G}_{\Theta_K} = (\mathbb{I} \otimes P_{\Theta_K})\mathcal{G}_{\Theta_K} = 0 \\ \mathcal{G}_{\Theta_K}|_{\tau=\tau'} = 0 \\ \partial_\tau \mathcal{G}_{\Theta_K}|_{\tau=\tau'} = -\partial_{\tau'} \mathcal{G}_{\Theta_K}|_{\tau=\tau'} = \delta_{\dot{\Sigma}\times\dot{\Sigma}} \end{cases} \tag{4.22}$$

where δ is the Dirac distribution on the diagonal of $\mathring{M} \times \mathring{M}$. Starting from $\mathcal{G}_{\Theta_K}$ one can recover the advanced and retarded fundamental solutions $\mathcal{G}^\pm_{\Theta_K}$ via the identities:

$$\mathcal{G}^-_{\Theta_K} = \vartheta(\tau - \tau')\mathcal{G}_{\Theta_K} \quad \text{and} \quad \mathcal{G}^+_{\Theta_K} = -\vartheta(\tau' - \tau)\mathcal{G}_{\Theta_K}, \tag{4.23}$$

where ϑ is the Heaviside function. The existence and the properties of $\mathcal{G}_{\Theta_K}$ have been thoroughly analyzed in [DDF19] using the framework of boundary triples, *cf.* [Gru68] and Section 1.5.

To construct a boundary triple associated with E^*, let n be the unit, outward pointing, normal of $\partial\Sigma$ and let

$$\Gamma_0 \colon H^2(\Sigma) \ni f \mapsto \Gamma f \in H^{3/2}(\Sigma), \qquad \Gamma_1 \colon H^2(\Sigma) \ni f \mapsto -\Gamma \nabla_n f \in H^{1/2}(\Sigma)\,,$$

where $H^k(\Sigma)$ indicates the Sobolev space associated to the Riemannian manifold (Σ, κ) introduced at the end of Section 1.2. Here $\Gamma : H^s(\Sigma) \to H^{s-\frac{1}{2}}(\Sigma)$, $s > \frac{1}{2}$, is the continuous surjective extension of the restriction map from $C_0^\infty(\Sigma)$ to $C_0^\infty(\partial\Sigma)$, *cf.* [GS13, Th. 4.10 & Cor. 4.12]. In addition, since the inner product $(\,|\,)_{L^2(\partial\Sigma)}$ on $L^2(\partial\Sigma) \equiv L^2(\partial\Sigma; \iota_\Sigma^* d\mu_g)$, $\iota_\Sigma : \partial\Sigma \hookrightarrow \Sigma$, extends continuously to a pairing on $H^{-1/2}(\partial\Sigma) \times H^{1/2}(\partial\Sigma)$ as well as on $H^{-3/2}(\partial\Sigma) \times H^{3/2}(\partial\Sigma)$, there exist isomorphisms

$$\iota_\pm \colon H^{\pm 1/2}(\partial\Sigma) \to L^2(\partial\Sigma), \qquad j_\pm \colon H^{\pm 3/2}(\partial\Sigma) \to L^2(\partial\Sigma)\,,$$

such that, for all $(\psi, \phi) \in H^{1/2}(\partial\Sigma) \times H^{-1/2}(\partial\Sigma)$ and for all $(\widetilde{\psi}, \widetilde{\phi}) \in H^{3/2}(\partial\Sigma) \times H^{-3/2}(\partial\Sigma)$,

$$(\psi, \phi)_{(1/2,-1/2)} = (\iota_+\psi|\, \iota_-\phi)_{L^2(\partial\Sigma)}\,, \quad (\widetilde{\psi}, \widetilde{\phi})_{(3/2,-3/2)} = (j_+\widetilde{\psi}|\, j_-\widetilde{\phi})_{L^2(\partial\Sigma)}\,,$$

where $(,)_{(1/2,-1/2)}$ and $(,)_{(3/2,-3/2)}$ stand for the duality pairings between the associated Sobolev spaces.

Remark 4.3.1. Note that in the massless case, the two trace operators Γ_0 and Γ_1 coincide respectively with the restriction to $H^2(M)$ of the traces γ_- and γ_+ introduced in Theorem 1.4.1 and in Lemma 2.2.1.

Gathering all the above ingredients, we can state the following proposition, *cf.* [DDF19, Prop. 24 & Rmk 26]:

Proposition 4.3.1. Let E^* be the adjoint of a second order, elliptic, partial differential operator on a Riemannian manifold (Σ, κ) with boundary and of bounded geometry. Let

$$\gamma_0 \colon H^2(M) \ni f \mapsto \iota_+\Gamma_0 f \in L^2(\partial M)\,, \tag{4.24}$$

$$\gamma_1 \colon H^2(M) \ni f \mapsto j_+\Gamma_1 f \in L^2(\partial M). \tag{4.25}$$

Then $(L^2(\partial M), \gamma_0, \gamma_1)$ is a boundary triple for E^*.

Combining all these data together, particularly Proposition 1.5.1 and Proposition 4.3.1 we can state the following theorem, whose proof can be found in [DDF19, Thm 30].

Theorem 4.3.1. *Let (M, g) be a static, globally hyperbolic, asymptotically AdS spacetime as per Definition 4.3.1. Let $(\gamma_0, \gamma_1, L^2(\partial M))$ be the boundary triple as in Proposition 4.3.1 associated with E^*, the adjoint of the elliptic operator defined in Equation* (4.21) *and let K be a densely defined self-adjoint operator on $L^2(\partial\Sigma)$ which individuates a static and physically*

admissible boundary condition as per Definition 4.2.1. Let E_K be the self-adjoint extension of E defined as per Proposition 4.3.1 by $E_K \doteq E^*|_{D(E_K)}$, where $D(E_K) \doteq \ker(\gamma_1 - K\gamma_0)$. Furthermore, let assume that the spectrum of E_K is bounded from below.
Then, calling Θ_K the associated boundary condition, the advanced and retarded Green's operators $\mathsf{G}^{\pm}_{\Theta_K}$ associated to the wave operator $\partial_t^2 + E_K$ exist and they are unique. They are completely determined in terms of $\mathcal{G}^{\pm}_{\Theta_K} \in \mathcal{D}'(\mathring{M} \times \mathring{M})$. These are bidistributions such that $\mathcal{G}^{-}_{\Theta_K} = \vartheta(t-t')\mathcal{G}_{\Theta_K}$ and $\mathcal{G}^{+}_{\Theta_K} = -\vartheta(t'-t)\mathcal{G}_{\Theta_K}$ where $\mathcal{G}_{\Theta_K} \in \mathcal{D}'(\mathring{M} \times \mathring{M})$ is such that, for all $f_1, f_2 \in \mathcal{D}(\mathring{M})$

$$\mathcal{G}_{\Theta_K}(f_1, f_2) \doteq \int_{\mathbb{R}^2} dt dt' \left(f_1(t) \middle| (-E_K)^{-\frac{1}{2}} \sin\left[(-E_K)^{\frac{1}{2}}(t-t')\right] f_2(t') \right), \tag{4.26}$$

where $f_1(t), f_2(t) \in H^2(\Sigma)$ denote the evaluation of f_1 and f_2, regarded as elements of $C_c^\infty(\mathbb{R}, H^\infty(\Sigma))$ while $E_K^{-\frac{1}{2}} \sin[E_K^{\frac{1}{2}}(t-t')]$ is defined exploiting the functional calculus for E_K. Moreover it holds that

$$\mathsf{G}^{\pm}_{\Theta_K} : \mathcal{D}(\mathring{M}) \to C^\infty(\mathbb{R}, H^\infty_{\Theta_K}(\Sigma)),$$

where $H^\infty_{\Theta_K}(\Sigma) \doteq \bigcap_{k \geq 0} D(E^k_{\Theta_K})$. In particular,

$$\gamma_1(\mathsf{G}^{\pm}_{\Theta_K} f) = \Theta_K \gamma_0(\mathsf{G}^{\pm}_{\Theta_K} f) \qquad \forall f \in C_0^\infty(\mathring{M}). \tag{4.27}$$

Remark 4.3.2. Observe that, in Theorem 4.3.1 we have constructed the advanced and retarded fundamental solutions $\mathcal{G}^{\pm}_{\Theta}$ as elements of $\mathcal{D}'(\mathring{M} \times \mathring{M})$. Yet we can combine this result with Theorem 4.2.1 to conclude that there must exist unique advanced retarded propagators on the whole M whose restriction to $\mathring{M}$ coincide with $\mathcal{G}^{\pm}_{\Theta_K}$. With a slight abuse of notation we shall refer to these extended fundamental solutions with the same symbol.

The analysis carried out in this chapter has an immediate application to quantum field theory, in particular in the study of the propagators, which are singular bidistributions on aAdS spacetimes. A key property that the propagators must satisfy in order to be physically meaningful is the so called Hadamard condition, prescribing certain constraints on their wavefront sets. The well-posedness result Theorem 4.2.1, together with the microlocal propagation of singularities theorems 3.0.2 and 3.0.1 obtained in Chapter 3, allows to prove that, under certain assumptions, on aAdS spacetimes there are unique propagators satisfying the Hadamard condition for a very general class of boundary conditions, see [DM21b].

Bibliography

[AD00] A. Ashtekar and S. Das. Asymptotically Anti-de Sitter space-times: Conserved quantities. *Class. Quant. Grav.*, 17:L17–L30, 2000. arXiv:hep-th/9911230 [hep-th].

[ANN19] B. Ammann, N.Große, and V. Nistor. Well-posedness of the Laplacian on manifolds with boundary and bounded geometry. *Mathematische Nachrichten*, 292(6):1213–1237, 2019. arXiv:1611.00281 [math-AP].

[APG17] P. Antonini, G. De Philippis, and N. Gigli. The injectivity radius of Lie manifolds. 2017. arXiv:1707.07595 [math.DG].

[Bac11] Alain Bachelot. The Klein–Gordon equation in the anti-de Sitter cosmology. *Journal de Mathématiques Pures et Appliquées*, 96:527–554, 2011.

[BF82] P. Breitenlohner and D. Freedman. Stability in Gauged Extended Supergravity. *Annals of Physics*, 144:249–281, 1982.

[BGP07] C. Bär, N. Ginoux, and F. Pfäffle. *Wave equations on Lorentzian manifolds and quantization.* ESU, 2007.

[BL12] J. Behrndt and M. Langer. Operators Methods for Boundary Value Problems. In *Elliptic operators, Dirichlet-to-Neumann maps and quasi boundary triples*, page 298 pg. Cambridge University press, 2012. arXiv:math/0010314 [math.AP].

[CB08] Y. Choquet-Bruhat. *General Relativity and the Einstein Equations.* UOP Oxford, 2008.

[DDF19] C. Dappiaggi, N. Drago, and H. Ferreira. Fundamental solutions for the wave operator on static Lorentzian manifolds with timelike boundary. *Lett. Math. Phys.*, 109, 2019. arXiv:1804.03434 [math-ph].

[DF16] Claudio Dappiaggi and Hugo R. C. Ferreira. Hadamard states for a scalar field in anti–de Sitter spacetime with arbitrary boundary conditions. *Phys. Rev. D*, 94(12):125016, 2016.

[DFJA18] Claudio Dappiaggi, Hugo R. C. Ferreira, and Benito A. Juárez-Aubry. Mode solutions for a Klein-Gordon field in anti–de Sitter spacetime with dynamical boundary conditions of Wentzell type. *Phys. Rev. D*, 97:085022, Apr 2018.

[DFM18] Claudio Dappiaggi, Hugo R. C. Ferreira, and Alessio Marta. Ground states of a Klein-Gordon field with Robin boundary conditions in global anti–de Sitter spacetime. *Phys. Rev. D*, 98:025005, Jul 2018.

[DH72] J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta Mathematica*, 128:183–269, 1972.

[DM21a] C. Dappiaggi and A. Marta. A generalization of the propagation of singularities theorem on asymptotically anti-de Sitter spacetimes. *Math. Nach.*, 2021. arXiv:2006.00560 [math-ph].

[DM21b] C. Dappiaggi and A. Marta. Fundamental solutions and Hadamard states for a scalar field with arbitrary boundary conditions on an asymptotically AdS spacetimes. 2021. arXiv:2101.10290 [math-ph].

[DW] W. Dybalski and M. Wrochna. A mechanism for holography for non-interacting fields on anti-de Sitter spacetimes.

[EK13] Alberto Enciso and Niky Kamran. A singular initial-boundary value problem for nonlinear wave equations and holography in asymptotically anti-de Sitter spaces. *Journal de Mathématiques Pures et Appliquées*, 103:1053–1091, 2013.

[Gal10] Anahit Galstian. $L^p - L^q$-Decay Estimates for the Klein-Gordon Equation in the Anti-de Sitter Space-Time. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste: an International Journal of Mathematics*, 42:27–50, 2010.

[Gan18] O. Gannot. Elliptic boundary value problems for Bessel operators, with applications to anti-de Sitter spacetimes. *Comptes Rendus Mathematique*, 356, 2018.

[GM21] Nicolas Ginoux and Simone Murro. On the Cauchy problem for Friedrichs systems on globally hyperbolic manifolds with timelike boundary, 2021.

[GOW17] C. Gérard, O. Oulghazi, and M. Wrochna. Hadamard States for the Klein-Gordon equation on Lorentzian manifolds of bonded geometry. *Comm. Math. Phys.*, 2017. arXiv:1206.2020 [math.SG].

[Gri01] D. Grieser. Basics of the b-Calculus. In *Advances in Partial Differential Equations*, volume 125. Birkhäuser, Basel, 2001. arXiv:math/0010314 [math.AP].

[Gru68] G. Grubb. A characterization of the non-local boundary value problems associated with an elliptic operator. *Ann. Sc. Norm. Sup. Pisa*, 1968.

[GS13] N. Große and C. Schneider. Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces. *Mathematische Nachrichten*, 286(16):1586–1613, 2013.

[GW20] O. Gannot and M. Wrochna. Propagation of Singularities on AdS Spacetimes for General Boundary Conditions ant the Holographic Hadamard Condition. *J. Inst. Math. Juissieu*, 2020. arXiv:1812.06564 [math.AP].

[HÖ0] L. H. Hörmander. *The analysis of partial differential operators III.* Springer, 2000.

[HÖ3] L. H. Hörmander. *The analysis of partial differential operators I.* Springer, 2003.

[HFS20] L. Aké Hau, J. L. Flores, and M. Sánchez. Structure of globally hyperbolic spacetimes with timelike boundary. *Rev. Mat. Iberoamericana*, 2020. arXiv:1808.04412 [gr-qc].

[Hol12] G. Holzegel. Well-posedness for the massive wave equation on asymptotically anti-de Sitter spacetimes. *Journal of Hyperbolic Differential Equations*, 09(02):239–261, 2012.

[HS16a] Gustav Holzegel and Arick Shao. Unique continuation from infinity in asympotically Anti-de Sitter spacetimes II: Non-static boundaries. *Communications in Partial Differential Equations*, 42(12):1871–1922, 8 2016.

[HS16b] Gustav Holzegel and Arick Shao. Unique continuation from infinity in asymptotically Anti-de Sitter spacetimes. *Commun. Math. Phys.*, 347(3):723–775, 2016.

[HW14] Gustav H. Holzegel and Claude M. Warnick. Boundedness and growth for the massive wave equation on asymptotically anti-de Sitter black holes. *J. Funct. Anal.*, 266(4):2436–2485, 2014.

[JAW21] Benito A Juárez-Aubry and Ricardo Weder. Quantum field theory with dynamical boundary conditions and the Casimir effect: coherent states. *Journal of Physics A: Mathematical and Theoretical*, 54(10):105203, 2021.

[Jos] M. S. Joshi. Lectures on pseudo-differential operators.

[KW15] Carl Kent and Elizabeth Winstanley. Hadamard renormalized scalar field theory on anti–de Sitter spacetime. *Phys. Rev. D*, 91(4):044044, 2015.

[KY09] A. Galstian. K. Yagdjian. The Klein-Gordon equation in anti-de Sitter spacetime. *Rend. Semin. Mat. Univ. Politec. Torino*, (67):271 – 292, 2009.

[Leb97] G. Lebeau. Propagation des ondes dans les variétés à coins. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 30(4):429–497, 1997.

[LGPV13] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke. *Poisson Structures.* Springer, 2013. Comprehensive Studies in Mathematics.

[Mal92] M. Malamud. On a formula for the generalized resolvents of a non-densely defined Hermitian operator. *Ukr. Math. J.*, 44, 1992.

[Mal98] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[Mel81] R. B. Melrose. Transformation of Boundary Problems. *Acta Math.*, 1981.

[Mel93] R. B. Melrose. *The Atiyah-Patodi-Singer index theorem.* CRC Press, 1993.

[MP92] R. B. Melrose and P. Piazza. Analytic K-theory on manifolds with corners. *Advances in Mathematics*, 1992.

[MS11] P. Mckeag and Y. Safarov. Pseudodifferential operators on manifolds: a coordinate-free approach. In M. Demuth, B. Schulze, and I. Witt, editors, *Partial Differential Equations and Spectral Theory (Operator Theory: Advances and Applications*. Birkhäuser, 2011.

[O'N83] B. O'Neill. *Semi-Riemannian Geometry with Applications to Relativity.* San Diego Academic Press, 1983.

[Sch01] T. Schick. Manifolds with Boundary and of Bounded Geometry. *Mathematische Nachrichten*, 223(1):103–120, 2001.

[Uen73] T. Ueno. Wave equation with Wentzell's boundary condition and a related semigroup on the boundary, I. *Proc. Japan Acad.*, 1973.

[Vas08] A. Vasy. Propagation of singularities for the wave equation on manifolds with corners. *Annals of Mathematics*, 2008. arXiv:math/0405431 [math.AP].

[Vas10] A. Vasy. Diffraction at corners for the wave equation on differential forms. *Comm. Part. Diff. Eq.*, 2010. arXiv:0906.0738 [math.AP].

[Vas12] András Vasy. The wave equation on asymptotically anti de Sitter spaces. *Analysis & PDE*, 5(1):81 – 144, 2012.

[VGP14] E. Miranda V. Guillemin and A. R. Piresa. Symplectic and Poisson geometry on b-manifolds. *Advances in Mathematics*, 2014. arXiv:1206.2020 [math.SG].

[Wal84] Robert M Wald. *General relativity.* Chicago Univ. Press, 1984.

[War12] C. Warnick. The Massive Wave Equation in Asymptotically AdS Spacetimes. *Communications in Mathematical Physics*, 321, 02 2012.

[Wit98] Edward Witten. AdS / CFT correspondence and topological field theory. *JHEP*, page 012, 1998.

[Wro17] M. Wrochna. The holographic Hadamard condition on asymptotically anti-de Sitter spacetimes. *Lett. Math. Phys.*, (107):2291 – 2331, 2017.

[Zah18] Jochen Zahn. Generalized Wentzell boundary conditions and quantum field theory. *Annales Henri Poincare*, 19(1):163–187, 2018.