



A conformal Yamabe problem with potential on the Euclidean space

Giovanni Catino¹ · Filippo Gazzola¹ · Paolo Mastrolia²

Received: 21 November 2019 / Accepted: 7 January 2021 / Published online: 10 February 2021
© The Author(s) 2021

Abstract

We consider, in the Euclidean setting, a conformal Yamabe-type equation related to a potential generalization of the classical constant scalar curvature problem and which naturally arises in the study of Ricci solitons structures. We prove existence and nonexistence results, focusing on the radial case, under some general hypothesis on the potential.

Keywords Yamabe problem · Conformal problems · Constant scalar curvature · Ordinary differential equations

Mathematics Subject Classification 53C20 · 53C25 · 34A34

1 Introduction

In [4], the first and the third authors considered ”potential” generalizations of some canonical metrics on smooth complete Riemannian manifolds. In this paper, we focus our attention on one of those classes, namely *f*-Yamabe metrics. We recall that, given a *n*-dimensional Riemannian manifold (M, g) , where *g* is the metric, and a smooth function $f \in C^\infty(M)$, we say that the triple $(M, g, f) \in \mathcal{Y}_f$ if and only if it satisfies the condition

$$\nabla R = 2\text{Ric}(\nabla f, \cdot), \quad (1.1)$$

where Ric and *R* are, respectively, the Ricci and the scalar curvature of *g* and ∇ denotes the Levi–Civita connection associated with *g*. In a local orthonormal frame $\{e_i\}$, $i = 1, \dots, n$, (1.1) becomes

✉ Filippo Gazzola
filippo.gazzola@polimi.it
Giovanni Catino
giovanni.catino@polimi.it
Paolo Mastrolia
paolo.mastrolia@unimi.it

¹ Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, Milano, Italy

² Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, Milano, Italia

$$\nabla_{e_i} R = 2R_{ij}\nabla_{e_j} f,$$

where $R_{ij} = \text{Ric}(e_i, e_j)$. Note that we are using the Einstein summation convention over repeated indices. This equation is a meaningful generalization of the one for *constant scalar curvature metrics* and naturally arises in the study of Ricci solitons structures (for a general overview see [3]). Moreover, it is clear that any Ricci flat metric satisfy (1.1), for any function f and, more in general, any product of a Ricci flat metric with a metric with constant scalar curvature solves (1.1), for any function f which depends only on the first factor.

In the same spirit of the classical *Yamabe problem*, it is natural to address the following questions:

- (A) having fixed $f \in C^\infty(M)$, does there exist a metric g such that $(M, g, f) \in \mathcal{Y}_f$?
- (B) having fixed $f \in C^\infty(M)$ and a metric g , does there exist a conformal metric \tilde{g} in the conformal class $[g]$ such that $(M, \tilde{g}, f) \in \mathcal{Y}_f$?

More generally, one could ask the question

- (C) does there exist a metric g and a smooth function $f \in C^\infty(M)$ such that $(M, g, f) \in \mathcal{Y}_f$?

Clearly, the answer to (C) is positive, since it is always possible to construct a (complete) metric with constant (negative) scalar curvature ([1] and [2]). Furthermore, when f is constant, (B) boils down to the well-known Yamabe problem, which is completely solved when M is compact (see, e.g., [9]). We will refer to (B) as the *conformal f -Yamabe problem*. In this paper, we consider problem (B) (when f is not constant) on the Euclidean space \mathbb{R}^n endowed with the standard flat metric $g_{\mathbb{R}^n}$. In particular, in dimension four, we prove the following:

Theorem 1.1 *Let $f \in C^\infty(\mathbb{R}^4)$ be a, not constant, radial function satisfying $f'(r) \leq 0$ for all $r > 0$. Then, there exists a conformal metric $\tilde{g} \in [g_{\mathbb{R}^n}]$ such that $(\mathbb{R}^n, \tilde{g}, f) \in \mathcal{Y}_f$.*

This Theorem is a straightforward consequence of Theorem 5.1 below, combined with the ODE formulation in Sect. 2. It will be complemented with some nonexistence results provided in Sect. 4.

2 ODE formulation of the conformal f -Yamabe problem

Let (M, g) be a smooth n -dimensional Riemannian manifold, $n \geq 2$, and let $f \in C^\infty(M)$. It is well known (see for instance [5]) that if $\tilde{g} = e^{2w}g \in [g]$ for some $w \in C^\infty(M)$, then the following formulas hold:

$$\begin{aligned} \widetilde{\text{Ric}} &= \text{Ric} - (n - 2)\nabla^2 w + (n - 2)dw \otimes dw - (\Delta w)g - (n - 2)|\nabla w|^2 g, \\ \widetilde{R} &= e^{-2w}(R - 2(n - 1)\Delta w - (n - 1)(n - 2)|\nabla w|^2) \end{aligned}$$

where ∇^2 is the Hessian and $\Delta = g^{ij}\nabla_{ij}^2$ is the Laplace–Beltrami operator of g . A computation shows that $(M, \tilde{g}, f) = (M, e^{2w}g, f) \in \mathcal{Y}_f$ if and only if the function w solves the system of PDEs

$$\begin{aligned} &\nabla\Delta w + (n - 2)\nabla^2 w(\nabla w, \cdot) - \left(2\Delta w + (n - 2)|\nabla w|^2 - \frac{1}{n - 1}R\right)\nabla w - \frac{1}{2(n - 1)}\nabla R \\ &= -\frac{1}{n - 1}\text{Ric}(\nabla f, \cdot) + \frac{n - 2}{n - 1}\nabla^2 w(\nabla f, \cdot) + \frac{1}{n - 1}\left(\Delta w + (n - 2)|\nabla w|^2\right)\nabla f \\ &\quad - \frac{n - 2}{n - 1}\langle\nabla w, \nabla f\rangle\nabla w. \end{aligned} \tag{2.1}$$

In particular, since $(\mathbb{R}^n, g_{\mathbb{R}^n})$ is Ricci flat, then $(\mathbb{R}^n, \tilde{g}, f) = (\mathbb{R}^n, e^{2w}g_{\mathbb{R}^n}, f) \in \mathcal{Y}_f$ if and only if w solves the system of PDEs

$$\begin{aligned} &\nabla\Delta w + (n - 2)\nabla^2 w(\nabla w, \cdot) - \left(2\Delta w + (n - 2)|\nabla w|^2\right)\nabla w \\ &= \frac{n - 2}{n - 1}\nabla^2 w(\nabla f, \cdot) + \frac{1}{n - 1}\left(\Delta w + (n - 2)|\nabla w|^2\right)\nabla f - \frac{n - 2}{n - 1}\langle\nabla w, \nabla f\rangle\nabla w. \end{aligned} \tag{2.2}$$

To fully exploit the symmetries of the Euclidean space, it is reasonable to start our analysis by considering radial solutions $w = w(r)$ of (2.2) for a given radial function $f = f(r)$, where r denotes the distance function from the origin. In this case, in standard polar coordinates, one has

$$\begin{aligned} g_{\mathbb{R}^n} &= dr^2 + r^2g_{\mathbb{S}^{n-1}}, \\ dw &= w'(r)dr \quad \text{and} \quad df = f'(r)dr, \\ \nabla dw &= \nabla^2 w = w''(r)dr^2 + rw'(r)g_{\mathbb{S}^{n-1}}, \\ \Delta w &= w''(r) + \frac{n - 1}{r}w'(r) \end{aligned}$$

and a computation shows that system (2.2) boils down to the following second-order non-linear ODE for the function $u(r) := w'(r)$

$$\begin{aligned} u''(r) + \frac{n - 1}{r}u'(r) - (n - 2)u(r)^3 - \frac{2(n - 1)}{r}u(r)^2 - \frac{n - 1}{r^2}u(r) + (n - 4)u(r)u'(r) \\ = \left[u'(r) + \frac{u(r)}{r}\right]h(r), \end{aligned} \tag{2.3}$$

where $h(r) := f'(r)$. Note that if $n = 2$, then the cubic term disappears in (2.3).

We then impose the initial conditions

$$u(0) = 0, \quad u'(0) = \alpha \neq 0 \tag{2.4}$$

which require some explanation since (2.3) is singular at $r = 0$. Assume that $u \in C^2[0, \infty)$ satisfies (2.4), then $u(r) = \alpha r + O(r^2)$ as $r \rightarrow 0$ and, in turn,

$$\frac{n - 1}{r}u'(r) - \frac{n - 1}{r^2}u(r) = O(1), \quad \frac{u(r)}{r} = O(1), \quad \frac{u(r)^2}{r} = o(1) \quad \text{as } r \rightarrow 0$$

which shows that, by combining suitably the terms in (2.3), we obtain finite limits as $r \rightarrow 0$. The existence and uniqueness of a solution of (2.3)-(2.4) can then be proved rigorously by

adapting the arguments of Proposition 1 in [11]: one needs to combine the Ascoli-Arzelà Theorem with the Schauder fixed point Theorem in order to obtain existence of a solution. Then, the solution is unique as long as it can be continued [6, Proposition 4.2].

Before stating our existence and nonexistence results, let us discuss heuristically the structure of (2.3).

3 Heuristic preliminaries

We first notice that if $n \geq 3$, then there exist exactly two singular (negative) solutions of (2.3) of the type cr^{-1} given by

$$u_1(r) = -\frac{1}{r}, \quad u_2(r) = -\frac{2}{r}, \tag{3.1}$$

regardless of the explicit form of h . This fact suggests that the “interesting dynamics” for (2.3) occurs when $u(r) < 0$ and global solutions of (2.3) are more likely to be prevalently negative. If $n = 2$, then the functions $u(r) = cr^{-1}$ are singular solutions of (2.3) for any $c \neq 0$; in particular, there exist infinitely many positive singular solutions and the dynamics appears much more chaotic.

It is quite useful to consider the two functions defined for all $(r, y) \in \mathbb{R}_+ \times \mathbb{R}$:

$$P(r, y) = \left((n-2)y^2 + \frac{2(n-1)}{r}y + \frac{n-1}{r^2} + \frac{h(r)}{r} \right) y,$$

$$Q(r, y) = h(r) - \frac{n-1}{r} + (4-n)y.$$

Then, (2.3) may be written in normal form as

$$u''(r) = Q(r, u(r))u'(r) + P(r, u(r)). \tag{3.2}$$

Depending on $h \in C^0[0, \infty)$, we define the two regions

$$I_h := \{r \geq 0; (n-2)rh(r) < n-1\}, \quad I^h = \{r \geq 0; (n-2)rh(r) > n-1\}.$$

Clearly, I_h contains a right neighborhood of $r = 0$ and is therefore nonempty for all h , while I^h is empty if $rh(r) \leq \frac{n-1}{n-2}$ for all r , in particular if $h(r) \leq 0$. It is also straightforward that:

- if $r \in I^h$, then $P(r, y) = 0$ if and only if $y = 0$; moreover, P has the same sign as y ;
- if $r \in I_h$, then we may write

$$P(r, y) = (n-2) \left[y + \frac{n-1 + \sqrt{n-1 - (n-2)rh(r)}}{(n-2)r} \right] \left[y + \frac{n-1 - \sqrt{n-1 - (n-2)rh(r)}}{(n-2)r} \right] y,$$

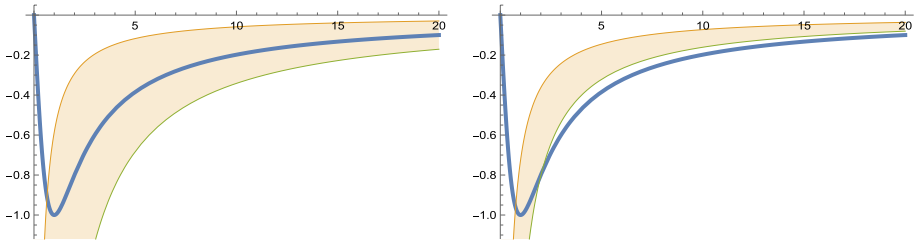


Fig. 1 Plot of $\bar{u}_1(r)$ in (3.3) (thick line) and of Γ (shaded region) when $n = 3$ (left) and $n = 8$ (right)

and hence, $P(r, y)$ vanishes if and only if one of the following facts occurs:

$$y = 0, \quad y = \varphi(r) := -\frac{n - 1 + \sqrt{n - 1 - (n - 2)rh(r)}}{(n - 2)r},$$

$$y = \psi(r) := \frac{1 - n + \sqrt{n - 1 - (n - 2)rh(r)}}{(n - 2)r}.$$

Note that $\psi(r) > \varphi(r)$ for all $r \in I_h$ but, while $\varphi(r) < 0$ for all $r \in I_h$, the sign of $\psi(r)$ may vary and it is the opposite of the sign of $rh(r) + n - 1$; in particular, $\psi(r) < 0$ in a right neighborhood of $r = 0$.

One expects a crucial role for the existence results to be played by the signs of Q and P . However, the overall picture is not completely clear. To see this, consider the trivial case $h \equiv 0$ for which the function

$$\bar{u}_a(r) = -\frac{2ar}{1 + ar^2} \tag{3.3}$$

solves (2.3)-(2.4) with $\alpha = -2a$. First, notice that if $a < 0$ (so that $u'_a(0) > 0$), then u_a blows up as $r \rightarrow 1/\sqrt{|a|}$. Therefore, u_a is a global solution of (2.3) if and only if $a > 0$. Simple computations then show that

$$\begin{aligned} \text{if } n = 3, 4, 5 \text{ then } \exists \rho > 0, \quad \varphi(r) < \bar{u}_1(r) < \psi(r) \quad \forall r > \rho, \\ \text{if } n \geq 6 \text{ then } \exists \rho > 0, \quad \bar{u}_1(r) < \varphi(r) \quad \forall r > \rho. \end{aligned}$$

These facts are illustrated in Fig. 1, where the shaded region is

$$\Gamma := \{(r, y) \in \mathbb{R}_+ \times \mathbb{R}_-; \varphi(r) < y < \psi(r)\}.$$

In the left picture, we see that the graph of \bar{u}_1 (thick line) eventually lies inside Γ while in the right picture it eventually lies outside. Therefore, the function $P(r, \bar{u}_1(r))$ does not always have the same sign as $r \rightarrow \infty$.

4 Nonexistence results

We can prove the following (partial) nonexistence results.

Theorem 4.1 *If $n \geq 3$, $h(r) \geq -\frac{n-1}{r}$ for all $r > 0$, $\alpha > 0$, then the solution of (2.3)-(2.4) is not global.*

The proof of Theorem 4.1 is given in Sect. 7. As a by-product, the very same proof enables us to obtain

Theorem 4.2 *If $n \geq 3$, $h(r) \geq -\frac{n-1}{r}$ for all $r > 0$, then any global solution u of (2.3)-(2.4) is necessarily strictly negative.*

Indeed, Theorem 4.1 excludes the existence of positive solutions u satisfying $u'(0) > 0$. If $u'(0) < 0$, then the solution of (2.3) is initially negative and, if it becomes positive, one can argue as in Sect. 7 in order to show finite space blow up.

Concerning nonexistence of negative solutions, a weaker result holds. First of all, we put together the three static terms

$$\begin{aligned} & (n-2)u(r)^3 + \frac{2(n-1)}{r}u(r)^2 + \frac{n-1}{r^2}u(r) \\ &= (n-2)\left(u(r) + \frac{n-1 + \sqrt{n-1}}{(n-2)r}\right)\left(u(r) + \frac{n-1 - \sqrt{n-1}}{(n-2)r}\right)u(r). \end{aligned} \tag{4.1}$$

This shows that the static term changes sign whenever the graph of u crosses one of the two hyperbolas:

$$h_1(r) = -\frac{n-1 + \sqrt{n-1}}{(n-2)r}, \quad h_2(r) = -\frac{n-1 - \sqrt{n-1}}{(n-2)r}.$$

Then rewrite (2.3) as

$$\begin{aligned} \frac{1}{r^{n-1}}\left(r^{n-1}u'(r)\right)' &= (n-2)\left(u(r) + \frac{n-1 + \sqrt{n-1}}{(n-2)r}\right)\left(u(r) + \frac{n-1 - \sqrt{n-1}}{(n-2)r}\right)u(r) \\ &\quad - (n-4)u(r)u'(r) + \left[u'(r) + \frac{u(r)}{r}\right]h(r). \end{aligned}$$

If we assume that

$$h(r) \geq 0 \quad \forall r \geq 0, \quad n \geq 4, \tag{4.2}$$

and that

$$\exists R > 0 \text{ such that } u(R) = h_1(R), \quad u'(R) \leq 0, \tag{4.3}$$

then the above equation tells us that $r \mapsto r^{n-1}u'(r)$ is decreasing for $r > R$. In particular, we have that $u'(r) < 0$ and $u(r) < h_1(r)$ for all $r > R$. Finally, this yields the existence of $\gamma > 0$ such that

$$\frac{1}{r^{n-1}}\left(r^{n-1}u'(r)\right)' \leq \gamma u(r)^3 \quad \forall r > R.$$

By arguing as in the proof of Theorem 4.1 (see Lemma 7.2), one obtains that

$$\exists \bar{R} > R \quad \text{s.t.} \quad \lim_{r \rightarrow \bar{R}} u(r) = -\infty.$$

Summarizing, we have

Proposition 4.3 *If (4.2) holds, then there exists no global solution of (2.3)-(2.4) which satisfies (4.3).*

5 Existence results

We start with a simple but interesting example. If

$$h(r) = -\frac{\alpha r}{2} \left((n-2)\alpha r^2 + n + 2 \right)$$

then $u(r) = \alpha r$ solves (2.3)-(2.4). We point out that, in any case, the solution u is *global and unbounded*. Moreover, if $\alpha > 0$, then $h(r) < 0$ for all r and the solution of (2.3)-(2.4) is positive, while if $\alpha < 0$ then h changes sign and the solution is negative.

Theorem 4.1 suggests that (2.3) is more likely to have negative solutions whenever h itself is negative. We prove that this is the case, at least in dimension $n = 4$.

Theorem 5.1 *In dimension $n = 4$, if $h(r) \leq 0$ for all $r > 0$, then (2.3) admits infinitely many negative global solutions. More precisely, for any $\alpha < 0$ the solution of (2.3)-(2.4) is global and it satisfies $-\frac{2}{r} < u(r) < 0$ for all $r > 0$.*

6 Remarks and open problems

We discuss some open problem related to conformal f -Yamabe metrics and to solutions of equation (2.1).

- (1) In Theorem 4.1, we stated a partial nonexistence results for radial solutions in the Euclidean space, while Theorem 5.1 provides a general existence result. It would be interesting to prove a sharp condition on the potential function f (or on its derivative) ensuring existence of global solutions to (2.3).
- (2) It is well known [8] that global positive solution the Yamabe equation

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{R}^n$$

have to be radial (and thus classified). We could ask the same question for (general) solutions to (2.2). For a given $f \in C^\infty(\mathbb{R}^n)$, are there any nonradial solutions w ? If f is radial, are all solutions to (2.2) radial?

- (3) In this paper we studied conformal f -Yamabe metrics for $(\mathbb{R}^n, g_{\mathbb{R}^n})$. What about other rotationally symmetric spaces? In particular, what we can say for the hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}^n})$ or the round sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$?
- (4) In the existence result (Theorem 5.1), the dimension $n = 4$ seems to be peculiar, at least from the analytic point of view. Is there a geometric interpretation of this fact?

7 Proof of Theorem 4.1

Throughout this proof, we will need the following particular class of test functions.

Definition 7.1 Let $\rho > 0$. We say that a nonnegative function $\phi \in C_c^2[0, \infty)$ satisfies the ρ -property if

$$\phi(r) = 1 \quad \text{for } r \in [0, \rho] \quad \text{and} \quad \phi(r) = 0 \quad \text{for } r \geq 2\rho, \tag{7.1}$$

and if

$$\int^{2\rho} \frac{|\phi''(r) + \frac{2(n-1)}{r}\phi'(r)|^{3/2}}{\sqrt{\phi(r)}} dr < \infty. \tag{7.2}$$

It is clear that such functions exist; to see this, it suffices to replace any ϕ satisfying (7.1) with a power ϕ^k for k sufficiently large so that (7.2) will be satisfied.

For the proof of Theorem 4.1, we first observe that, since $\alpha > 0$ the solution of (2.3)-(2.4) is positive and strictly increasing in a right neighborhood of $r = 0$, say in some maximal interval $(0, R)$. Clearly, among u and u' the first one which can vanish is u' . But if $u'(R) = 0$ then, using the lower bound for h , we see that (2.3) yields

$$\begin{aligned} u''(R) &\geq (n-2)u(R)^3 + \frac{2(n-1)}{R}u(R)^2 + \left(\frac{n-1}{R^2} + \frac{h(R)}{R}\right)u(R) \\ &\geq (n-2)u(R)^3 + \frac{2(n-1)}{R}u(R)^2 > 0, \end{aligned}$$

giving a contradiction. Therefore, u' cannot vanish and two cases may occur:

$$(i) R = \infty, \quad (ii) R < \infty \text{ and } \lim_{r \uparrow R} u(r) = +\infty. \tag{7.3}$$

The proof will be complete if we show that (ii) occurs. At this point, we distinguish two cases.

• **Case $n \in \{3, 4\}$.** In order to prove (ii) in (7.3), we argue for contradiction by assuming that $R = \infty$ so that $u, u' > 0$ for all $r > 0$. From the assumptions and (2.3), we then infer that (recall $n \leq 4$)

$$u''(r) + \frac{2n-2}{r}u'(r) > (n-2)u(r)^3 > 0 \quad \forall r > 0. \tag{7.4}$$

To reach a contradiction, we need the following estimate, inspired to the method developed by Mitidieri-Pohožaev [10] (see also the proof of [7, Proposition 5]).

Lemma 7.2 Assume that $w \in C^2[0, \infty)$. Then, for any $\varepsilon > 0$, for any $\rho > 0$ and for all ϕ satisfying the ρ -property, we have

$$\begin{aligned} \int_0^{2\rho} r^{2n-2} \left| w''(r) + \frac{2n-2}{r}w'(r) \right| \phi(r) dr &\leq \frac{\varepsilon}{3} \int_0^{2\rho} r^{2n-2} |w(r)|^3 \phi(r) dr \\ &\quad + \frac{2}{3\sqrt{\varepsilon}} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi''(r) + \frac{2n-2}{r}\phi'(r)|^{3/2}}{\sqrt{\phi(r)}} dr. \end{aligned}$$

Proof In this proof, we will use the Young inequality in the following form:

$$\forall \varepsilon > 0 \quad \forall a, b \geq 0 \quad ab \leq \frac{\varepsilon a^3}{3} + \frac{2b^{3/2}}{3\sqrt{\varepsilon}}. \tag{7.5}$$

Fix $\varepsilon > 0$ and $\rho > 0$. We use a PDE approach and introduce the radial $C^2(\mathbb{R}^{2n-1})$ -function v such that $v(x) = w(|x|)$ for all x : note that the space dimension is here $2n - 1$ and that $\Delta v(x) = w''(|x|) + \frac{2n-2}{|x|}w'(|x|)$. Then, multiply Δv by some function $\Phi(x) = \phi(|x|)$, where ϕ satisfies the ρ -property. Since $\Phi \equiv 1$ in B_ρ , two integrations by parts and (7.5) yield

$$\begin{aligned} \int_{B_{2\rho}} \Delta v \Phi &= \int_{B_{2\rho}} v \Delta \Phi = \int_{B_{2\rho} \setminus B_\rho} v \Delta \Phi = \int_{B_{2\rho} \setminus B_\rho} v \Phi^{1/3} \frac{\Delta \Phi}{\Phi^{1/3}} \\ &\leq \frac{\varepsilon}{3} \int_{B_{2\rho}} |v|^3 \Phi + \frac{2}{3\sqrt{\varepsilon}} \int_{B_{2\rho} \setminus B_\rho} \frac{|\Delta \Phi|^{3/2}}{\Phi^{1/2}}, \end{aligned}$$

and back to the radial form of v and Φ this proves the statement. □

Take a function ϕ_1 satisfying the 1-property and observe that the function

$$\phi_\rho(r) := \phi_1\left(\frac{r}{\rho}\right) \quad \forall \rho > 1$$

satisfies the ρ -property. Therefore, for all $\varepsilon > 0$, from (7.4) and Lemma 7.2 we infer that

$$\begin{aligned} (n-2) \int_0^\rho r^{2n-2} u(r)^3 dr &\leq (n-2) \int_0^{2\rho} r^{2n-2} u(r)^3 \phi_\rho(r) dr \\ &\leq \int_0^{2\rho} r^{2n-2} \left(u''(r) + \frac{2n-2}{r} u'(r) \right) \phi_\rho(r) dr \\ &\leq \frac{\varepsilon}{3} \int_0^{2\rho} r^{2n-2} u(r)^3 \phi_\rho(r) dr + \frac{2}{3\sqrt{\varepsilon}} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi_\rho''(r) + \frac{2n-2}{r} \phi_\rho'(r)|^{3/2}}{\sqrt{\phi_\rho(r)}} dr. \end{aligned}$$

Take $0 < \varepsilon < 3(n-2)$, then the latter inequality yields

$$\left(n - 2 - \frac{\varepsilon}{3} \right) \int_0^\rho r^{2n-2} u(r)^3 dr \leq \frac{2}{3\sqrt{\varepsilon}} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi_\rho''(r) + \frac{2n-2}{r} \phi_\rho'(r)|^{3/2}}{\sqrt{\phi_\rho(r)}} dr.$$

With the change of variable $r = \rho t$, this becomes

$$\left(n - 2 - \frac{\varepsilon}{3} \right) \int_0^1 t^{2n-2} u(\rho t)^3 dt \leq \frac{2}{3\sqrt{\varepsilon}} \frac{1}{\rho^3} \int_1^2 t^{2n-2} \frac{|\phi_1''(t) + \frac{2n-2}{t} \phi_1'(t)|^{3/2}}{\sqrt{\phi_1(t)}} dt,$$

Since u is increasing on \mathbb{R}_+ , we have $u(\rho t) \geq u(t)$ for all $\rho > 1$ so that the left-hand side of this inequality is positive and increasing for $\rho \geq 1$. By letting $\rho \rightarrow \infty$, the right-hand side tends to 0 and this leads to a contradiction which rules out case (i). Hence, case (ii) occurs and the solution u of (2.3)-(2.4) with $\alpha > 0$ cannot be continued to all the interval $[0, \infty)$. This completes the proof of Theorem 4.1 in the case $n = 3, 4$.

• **Case $n \geq 5$.**

The same arguments leading to (7.4) now yield

$$u''(r) + \frac{2n-2}{r} u'(r) + \frac{n-4}{2} \left(u(r)^2 \right)' > (n-2)u(r)^3 > 0 \quad \forall r > 0 \tag{7.6}$$

and we need to estimate one more term. The companion of Lemma 7.2 reads

Lemma 7.3 Assume that $w \in C^2[0, \infty)$. Then, for any $\delta > 0$, for any $\rho > 0$ and for all ϕ satisfying the ρ -property, we have

$$\int_0^{2\rho} r^{2n-2} (w(r)^2)' \phi(r) dr \leq \frac{2\sqrt{\delta}}{3} \int_0^{2\rho} r^{2n-2} |w(r)|^3 \phi(r) dr + \frac{1}{3\delta} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi'(r)|^3}{\phi(r)^2} dr.$$

Proof In this proof, we will use the Young inequality in the following form:

$$\forall \delta > 0 \quad \forall a, b \geq 0 \quad ab \leq \frac{2\sqrt{\delta} a^{3/2}}{3} + \frac{b^3}{3\delta}. \tag{7.7}$$

Fix $\delta > 0$ and $\rho > 0$; then, take ϕ satisfying the ρ -property. An integration by parts yields

$$\int_0^{2\rho} r^{2n-2} (w(r)^2)' \phi(r) dr = - \int_0^{2\rho} w(r)^2 (2(n-1)r^{2n-3} \phi(r) + r^{2n-2} \phi'(r)) dr$$

and, since $\phi \geq 0$,

$$\begin{aligned} \int_0^{2\rho} r^{2n-2} (w(r)^2)' \phi(r) dr &\leq - \int_0^{2\rho} r^{2n-2} w(r)^2 \phi'(r) dr \\ &\leq \int_0^{2\rho} \left(r^{4(n-1)/3} w(r)^2 \phi(r)^{2/3} \right) \cdot \left(r^{2(n-1)/3} \frac{|\phi'(r)|}{\phi(r)^{2/3}} \right) dr \\ \text{[by (7.7)]} &\leq \frac{2\sqrt{\delta}}{3} \int_0^{2\rho} r^{2n-2} |w(r)|^3 \phi(r) dr + \frac{1}{3\delta} \int_0^{2\rho} r^{2n-2} \frac{|\phi'(r)|^3}{\phi(r)^2} dr. \end{aligned}$$

Since $\phi' \equiv 0$ on $(0, \rho)$, this completes the proof. □

Take again a function ϕ_1 satisfying the 1-property and let $\phi_\rho(r) := \phi_1(r/\rho)$ for all $\rho > 1$ so that ϕ_ρ satisfies the ρ -property. Multiply (7.6) by ϕ_ρ and integrate over $(0, 2\rho)$ to obtain

$$\begin{aligned} (n-2) \int_0^\rho r^{2n-2} u(r)^3 dr &\leq (n-2) \int_0^{2\rho} r^{2n-2} u(r)^3 \phi_\rho(r) dr \\ &\leq \int_0^{2\rho} r^{2n-2} \left(u''(r) + \frac{2n-2}{r} u'(r) \right) \phi_\rho(r) dr \\ &\quad + \frac{n-4}{2} \int_0^{2\rho} r^{2n-2} (u(r)^2)' \phi_\rho(r) dr. \end{aligned}$$

Then, by Lemmas 7.2 and 7.3 we infer that

$$\begin{aligned} &\left(n-2 - \frac{\varepsilon}{3} - \frac{(n-4)\sqrt{\delta}}{3} \right) \int_0^\rho r^{2n-2} u(r)^3 dr \\ &\leq \frac{2}{3\sqrt{\varepsilon}} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi_\rho''(r) + \frac{2n-2}{r} \phi_\rho'(r)|^{3/2}}{\sqrt{\phi_\rho(r)}} dr \\ &\quad + \frac{n-4}{6\delta} \int_\rho^{2\rho} r^{2n-2} \frac{|\phi_\rho'(r)|^3}{\phi_\rho(r)^2} dr. \end{aligned}$$

Take $\varepsilon > 0$ and $\delta > 0$ sufficiently small in such a way that $C_{\varepsilon,\delta} := n - 2 - \frac{\varepsilon}{3} - \frac{(n-4)\sqrt{\delta}}{3} > 0$ and perform the change of variable $r = \rho t$ to obtain

$$C_{\varepsilon,\delta} \int_0^1 t^{2n-2} u(\rho t)^3 dt \leq \frac{2}{3\sqrt{\varepsilon}} \frac{1}{\rho^3} \int_1^2 t^{2n-2} \frac{|\phi_1''(t) + \frac{2n-2}{t} \phi_1'(t)|^{3/2}}{\sqrt{\phi_1(t)}} dt + \frac{n-4}{6\delta} \frac{1}{\rho^3} \int_1^2 t^{2n-2} \frac{|\phi_1'(t)|^3}{\phi_1(1)^2} dt.$$

Since u is increasing on \mathbb{R}_{+} , we have $u(\rho t) \geq u(t)$ for all $\rho > 1$ so that the left-hand side of this inequality is positive and increasing for $\rho \geq 1$. By letting $\rho \rightarrow \infty$, the right-hand side tends to 0 and this leads to a contradiction which rules out case (i). Hence, case (ii) occurs and the solution u of (2.3)-(2.4) with $\alpha > 0$ cannot be continued to all the interval $[0, \infty)$. This completes the proof of Theorem 4.1 also in the case $n \geq 5$.

8 Proof of Theorem 5.1

For our convenience, we introduce the functions

$$h(y) = (y + 2)(y + 1)y, \quad H(y) = \int_0^y h(\xi)d\xi = \frac{(y + 2)^2 y^2}{4}.$$

Let $v(r) = ru(r)$, then v satisfies the equation

$$v''(r) + \left[\frac{n-3}{r} + \frac{n-4}{r}v(r) - h(r) \right] v'(r) = \frac{n-2}{r^2} h(v(r)). \tag{8.1}$$

If u satisfies (2.3)-(2.4) with $\alpha < 0$, then $u(r)$ and $u'(r)$ are strictly negative in a right neighborhood of $r = 0$. By definition of v , also $v(r)$ is strictly negative in a right neighborhood of $r = 0$. We claim that $-2 < v(r) < 0$ for all $r > 0$. If not, let $R > 0$ be the first time where

$$\text{either } v(R) = 0 \text{ or } v(R) = -2. \tag{8.2}$$

Multiply (8.1) by $r^2 v'(r)$ and integrate over $[0, R]$ to obtain

$$\int_0^R \left(r^2 v''(r)v'(r) + [n - 3 + (n - 4)v(r) - rh(r)]rv'(r)^2 \right) dr = 0$$

since $H(0) = H(-2) = 0$. An integration by parts then yields

$$\int_0^R [n - 4 + (n - 4)v(r) - rh(r)]rv'(r)^2 dr + \frac{R^2 v'(R)^2}{2} = 0.$$

If $n = 4$ and $h(r) \leq 0$, we get a contradiction which shows that R does not exist, and therefore, $-2 < v(r) < 0$ for all $r > 0$. This proves the claim. Hence, by (8.1), also v' and v'' remain bounded and the solutions exist. This concludes the proof of Theorem 5.1.

Acknowledgements The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The first author is supported by the PRIN project "Variational methods, with applications to problems in

mathematical physics and geometry.” The second author is supported by the PRIN project “Equazioni alle derivate parziali di tipo ellittico e parabolico: aspetti geometrici, disuguaglianze collegate, e applicazioni.”

Funding Open Access funding provided by Politecnico di Milano.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Aubin, T.: Métriques riemanniennes et courbure. *J. Differ. Geom.* **4**, 383–424 (1970)
2. Bland, J., Kalka, M.: Negative scalar curvature metrics on noncompact manifolds. *Trans. Am. Math. Soc.* **316**(2), 433–446 (1989)
3. Cao H.-D.: *Recent progress on Ricci solitons*, Recent advances in geometric analysis. In: *Adv. Lect. Math. (ALM)*, vol. 11, Int. Press, Somerville, MA, pp. 1–38 (2010)
4. Catino, G., Mastrolia, P.: A potential generalization of some canonical Riemannian metrics. *Ann. Glob. Anal. Geom.* **55**(4), 719–748 (2019)
5. Catino, G., Mastrolia, P., Monticelli, D.D., Rigoli, M.: Conformal Ricci solitons and related integrability conditions. *Adv. Geom.* **16**(3), 301–328 (2016)
6. Franchi, B., Lanconelli, E., Serrin, J.: Existence and uniqueness of nonnegative solutions of quasilinear equations in \mathbb{R}^n . *Adv. Math.* **118**, 177–243 (1996)
7. Gazzola, F., Grunau, H.C.: Radial entire solutions for supercritical biharmonic equations. *Math. Ann.* **334**, 905–936 (2006)
8. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68**(3), 209–243 (1979)
9. Lee, J.M., Parker, T.H.: The Yamabe problem. *Bull. Am. Math. Soc. (N.S.)* **17**(1), 37–91 (1987)
10. Mitidieri, E., Pohožaev, S.: A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. *Proc. Steklov Inst. Math.* **234**, 1–362 (2001). Translated from Russian
11. Ni, W.M., Serrin, J.: Existence and nonexistence theorems for ground states for quasilinear partial differential equations The anomalous case. *Accad. Naz. Lincei Atti dei Convegni* **77**, 231–257 (1986)
12. Catino, G., Mastrolia, P.: *A Perspective on Canonical Riemannian Metrics*. Number 336 in *Progress in Mathematics*. Birkhäuser Basel, 2020.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.