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ABSTRACT

Nineteen classical superintegrable systems in two-dimensional non-Euclidean spaces are shown to possess hidden symmetries leading to their linearization. They are the two Perlick systems [Ballesteros *et al.*, Classical Quantum Gravity **25**, 165005 (2008)], the Taub–NUT system [Ballesteros *et al.*, SIGMA **7**, 048 (2011)], and all the 17 superintegrable systems for the four types of Darboux spaces as determined by Kalnins *et al.* [J. Math. Phys. **44**, 5811–5848 (2003)].

I. INTRODUCTION

In Ref. 1, we have shown that all classical superintegrable systems (and their generalizations, not necessarily superintegrable) in two-dimensional real Euclidean space E_2[2] possess hidden symmetries leading to their linearization, as well as the Tremblay–Turbiner–Winternitz system,[3] and a superintegrable system that is separable in Cartesian coordinates and admits a third-order integral of motion as derived by Gravel in Ref. 4. Then, we have conjectured that superintegrable systems in two-dimensional non-Euclidean space can also be reduced to linear equations by means of their hidden symmetries. In this paper, we consider the two Perlick systems on two-dimensional non-Euclidean spaces,[5–8] the two-dimensional Taub–NUT system,[9–11] and all the superintegrable systems for the four types of Darboux spaces as determined in Refs. 12 and 13. We show that they are all intrinsically linear by determining their hidden Lie symmetries. As in Refs. 1 and 14–17, we also make use of the reduction method.[18] More details on superintegrable systems and their hidden linearity have been described in Ref. 1. In particular, it is regardless of the separability of the corresponding Hamilton–Jacobi equation as shown in Ref. 15 for the Kepler problem in Cartesian coordinates and in Ref. 16 for a superintegrable system in E_2 that does not allow separation of variables.[19]

II. PERLICK TYPE I

We consider the so-called Hamiltonian of Perlik type I,[8] i.e.,

$$H_I = \frac{(1+kr^2)^2}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + A\frac{1-kr^2}{r}, \tag{1}$$

that generates the Hamiltonian equations

$$\begin{cases} \dot{r} = p_r(1+kr^2)^2, \\ \dot{\theta} = \dfrac{p_\theta(1+kr^2)^2}{r^2}, \\ \dot{p}_r = \dfrac{\big((1-kr^2)p_\theta^2 - 2kr^4p_r^2 + Ar\big)(1+kr^2)}{r^3}, \\ \dot{p}_\theta = 0. \end{cases} \tag{2}$$

The last equation can be easily integrated to give $p_\theta = w = constant$. If we apply the reduction method developed in Ref. 18 to the three remaining equations of system (2) and choose θ as a new independent variable y, then we obtain the following two equations:

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}y} = \dfrac{p_r r^2}{w}, \\ \dfrac{\mathrm{d}p_r}{\mathrm{d}y} = \dfrac{(1-kr^2)w^2 - 2kr^4p_r^2 + Ar}{rw(1+kr^2)}. \end{cases} \tag{3}$$

If we derive p_r from the first equation of system (3) and replace it into the second equation, then we obtain the following second-order equation in r:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}y^2} = \frac{Ar^3 + (1-kr^2)w^2r^2 + 2w^2\left(\frac{\mathrm{d}r}{\mathrm{d}y}\right)^2}{w^2 r(1+kr^2)}. \tag{4}$$

This equation admits an eight-dimensional Lie symmetry algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$ and thus is linearizable. A two-dimensional Abelian intransitive subalgebra is that generated by the two operators

$$\Gamma_7 = \frac{\cos(y)r^2}{1+kr^2}\partial_r, \quad \Gamma_8 = \frac{\sin(y)r^2}{1+kr^2}\partial_r \tag{5}$$

that can be put into the canonical form[20] $\partial_{\tilde{r}}$, $\tilde{y}\partial_{\tilde{r}}$ by means of the transformation

$$\tilde{y} = \tan(y), \quad \tilde{r} = \frac{kr - r^{-1} - A/w^2}{\cos(y)}. \tag{6}$$

Then, Eq. (4) becomes the free-particle equation

$$\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}\tilde{y}^2} = 0.$$

Instead, if we only make the transformation of the dependent variable $u = kr - r^{-1} - A/w^2$, then Eq. (4) becomes the equation of the harmonic oscillator

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = -u.$$

III. PERLICK TYPE II

We consider the so-called Hamiltonian of Perlik type II,[8] i.e.,

$$H_{II} = \frac{(1-\lambda^2r^4)^2}{2(1+\lambda^2r^4-2\delta r^2)}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{Br^2}{1+\lambda^2r^4-2\delta r^2}, \tag{7}$$

that generates the Hamiltonian equations

$$
\begin{cases}
\dot{r} = \dfrac{p_r(1-\lambda^2 r^4)^2}{1+\lambda^2 r^4 - 2\delta r^2}, \\
\dot{\theta} = \dfrac{p_\theta(1-\lambda^2 r^4)^2}{r^2(1+\lambda^2 r^4 - 2\delta r^2)}, \\
\dot{p}_r = \dfrac{1-\lambda^2 r^4}{r^3(1+\lambda^2 r^4 - 2\delta r^2)^2}\big(2r^2 p_r^2(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4) \\
\qquad + p_\theta^2(1+\lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2) - 2Br^4\big), \\
\dot{p}_\theta = 0.
\end{cases}
\tag{8}
$$

The last equation can be easily integrated to give $p_\theta = w =$ constant. If we apply the reduction method developed in Ref. 18 to the three remaining equations of system (8) and choose θ as a new independent variable y, then we obtain the following two equations:

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}y} = \dfrac{p_r r^2}{w}, \\
\dfrac{\mathrm{d}p_r}{\mathrm{d}y} = \dfrac{1}{wr(1+\lambda^2 r^4 - 2\delta r^2)(1-\lambda^2 r^4)}\big(2r^2 p_r^2(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4) \\
\qquad + w^2(1+\lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2) - 2Br^4\big).
\end{cases}
\tag{9}
$$

If we derive p_r from the first equation of system (9) and replace it into the second equation, then we obtain the following second-order equation in r:

$$
\frac{\mathrm{d}^2 r}{\mathrm{d}y^2} = \frac{w^2\left(2\left(\frac{\mathrm{d}r}{\mathrm{d}y}\right)^2(\lambda^4 r^6 + 3\lambda^2 r^2 - \delta - 3\delta\lambda^2 r^4) + 1 + \lambda^4 r^8 + 6\lambda^2 r^4 - 4\delta\lambda^2 r^6 - 4\delta r^2\right) - 2Br^4}{wr(1+\lambda^2 r^4 - 2\delta r^2)(1-\lambda^2 r^4)},
\tag{10}
$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $B = 2w^2(\lambda^2 - \delta^2)$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $sl(2,\mathbb{R})$. If we solve Eq. (10) with respect to B and derive once with respect to y, then we obtain the following third-order equation:

$$
\frac{\mathrm{d}^3 r}{\mathrm{d}y^3} = \frac{\frac{\mathrm{d}r}{\mathrm{d}y}}{r^2(1-\lambda^2 r^4)}\left(3r(3+\lambda^2 r^4)\frac{\mathrm{d}^2 r}{\mathrm{d}y^2} - 12\left(\frac{\mathrm{d}r}{\mathrm{d}y}\right)^2 - 4r^2(1-\lambda^2 r^4)\right),
\tag{11}
$$

which admits a seven-dimensional Lie symmetry algebra and therefore is linearizable. Indeed, the new dependent variable $\tilde{r} = \frac{1+\lambda^2 r^4}{2r^2}$ transforms Eq. (11) into the linear equation

$$
\frac{\mathrm{d}^3 \tilde{r}}{\mathrm{d}y^3} = -4\frac{\mathrm{d}\tilde{r}}{\mathrm{d}y},
$$

which is a once-derived linear harmonic oscillator with frequency equal to 2.

A. Taub–NUT

The following Taub–NUT Hamiltonian[10,11]

$$
H_{\mathrm{TN}}(\eta) = \frac{1}{2}\frac{r}{\eta + r}\left(p_r^2 + \frac{1}{r^2}p_\varphi^2\right) - \frac{\alpha}{\eta + r}
\tag{12}
$$

yields the Hamiltonian equations

$$\begin{cases} \dot{r} = \dfrac{rp_r}{\eta + r}, \\ \dot{\varphi} = \dfrac{p_\varphi}{(\eta + r)r}, \\ \dot{p}_r = -\dfrac{2\left(\alpha r - p_\varphi^2\right)r + \eta\left(r^2 p_r^2 - p_\varphi^2\right)}{2(\eta + r)^2 r^2}, \\ \dot{p_\varphi} = 0. \end{cases} \tag{13}$$

The last equation can be easily integrated to give $p_\varphi = w_0 = constant$. If we apply the reduction method developed in Ref. 18 to the three remaining equations of system (13) and choose φ as a new independent variable y, then we obtain the following two equations:

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}y} = \dfrac{r^2}{w_0} p_r, \\ \dfrac{\mathrm{d}p_r}{\mathrm{d}y} = \dfrac{2\left(\alpha r - w_0^2\right)r + \eta\left(r^2 p_r^2 - w_0^2\right)}{2(\eta + r) r w_0}. \end{cases} \tag{14}$$

Solving the first equation for p_r and substituting into the second yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{3\eta + 4u}{2u(\eta + u)} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - \frac{u\left(2\alpha u^2 - \eta w_0^2 - 2w_0^2 u\right)}{2w_0^2(\eta + u)}, \tag{15}$$

with $u \equiv r$. Equation (15) admits a three-dimensional Lie symmetry algebra spanned by the following operators:

$$\begin{gathered} \Theta_1 = \partial_y, \quad \Theta_2 = \cos(y)\partial_y + \frac{u(\eta + u)}{\eta} \sin(y)\partial_u, \\ \Theta_3 = \sin(y)\partial_y - \frac{u(\eta + u)}{\eta} \cos(y)\partial_u. \end{gathered} \tag{16}$$

However, if $\alpha = 0$, then the equation admits an eight-dimensional Lie symmetry algebra. Therefore, if we solve Eq. (15) with respect to α and derive once with respect to y, then we get the following third-order equation:

$$u^2 \frac{\mathrm{d}^3 u}{\mathrm{d}y^3} + \frac{\mathrm{d}u}{\mathrm{d}y}\left[u^2 - 6u\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + 6\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right] = 0, \tag{17}$$

which is linearizable since it admits a seven-dimensional Lie symmetry algebra spanned by the following operators:

$$\begin{gathered} \Pi_1 = \partial_y, \quad \Pi_2 = \cos(y)\partial_y + u \sin(y)\partial_u, \quad \Pi_3 = \sin(y)\partial_y - u \cos(y)\partial_u, \\ \Pi_4 = u\partial_u, \quad \Pi_5 = u^2\partial_u, \quad \Pi_6 = u^2 \cos(y)\partial_u, \quad \Pi_7 = u^2 \sin(y)\partial_u. \end{gathered} \tag{18}$$

A two-dimensional Abelian intransitive subalgebra is that generated by the operators Π_6 and Π_7. If we put them into the canonical form $\partial_U, Y\partial_U$, then the transformation

$$Y = \frac{\sin(y)}{\cos(y)}, \quad U = -\frac{1}{u \cos(y)} \tag{19}$$

changes Eq. (15) into the following linear equation:

$$\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3} = -3\frac{Y^2}{1+Y^2}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2}. \tag{20}$$

Moreover, if we consider the transformation $v = -\frac{1}{u}$, then Eq. (15) becomes the once-derived linear harmonic oscillator with frequency equal to 1, i.e.,

$$\frac{\mathrm{d}^3 v}{\mathrm{d}y^3} = -\frac{\mathrm{d}v}{\mathrm{d}y}. \tag{21}$$

This shows the connection between the Taub–NUT Hamiltonian (12) and the harmonic oscillator.

IV. DARBOUX I

Three superintegrable systems were determined in Ref. 12, where the problem of superintegrability for the Hamiltonian

$$H_{DI} = \frac{1}{4u}\left(p_u^2 + p_v^2\right) + V(u,v) \tag{22}$$

was addressed, namely, finding the potentials $V(u,v)$ such that H_{DI} admits at least two extra quadratic integrals. We show that all of three systems have hidden symmetries that make them linear.

A. Case (1)

The Hamiltonian

$$\mathcal{H}_{DI1} = \frac{1}{4u}\left(p_u^2 + p_v^2\right) + b_1\frac{4u^2+v^2}{4u} + \frac{b_2}{u} + \frac{b_3}{uv^2} \tag{23}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{u} = \dfrac{p_u}{2u}, \\ \dot{v} = \dfrac{p_v}{2u}, \\ \dot{p}_u = \dfrac{v^2(p_u^2+p_v^2) + b_1 v^2(v^2-4u^2) + 4b_2 v^2 + 4b_3}{4u^2v^2}, \\ \dot{p}_v = \dfrac{4b_3 - b_1 v^4}{2uv^3}. \end{cases} \tag{24}$$

If we apply the reduction method developed in Ref. 18 and choose v as a new independent variable y, then we obtain the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{p_u}{p_v}, \\ \dfrac{\mathrm{d}p_u}{\mathrm{d}y} = \dfrac{y^2(p_u^2+p_v^2) + b_1 y^2(y^2-4u^2) + 4b_2 y^2 + 4b_3}{2uy^2 p_v}, \\ \dfrac{\mathrm{d}p_v}{\mathrm{d}y} = \dfrac{4b_3 - b_1 y^4}{y^3 p_v}. \end{cases} \tag{25}$$

The last equation can be easily integrated, i.e.,

$$p_v = \pm\frac{\sqrt{2w_0 b_1 y^2 - b_1 y^4 - 8w_0 b_3 y^2 - 4b_3}}{y}, \tag{26}$$

with w_0 being an arbitrary constant. Moreover, if we derive p_u from the first equation of system (25) and replace it into the second equation, then we obtain the following second-order equation in u:

$$\frac{d^2u}{dy^2} = \frac{1}{2u}\left(\frac{du}{dy}\right)^2 + \frac{u(b_1y^4 - b_3)\frac{du}{dy} + y^3(w_0b_1 - 2b_1u^2 + 2b_2 - 4w_0b_3)}{yu(2w_0b_1y^2 - b_1y^4 - 8w_0b_3y^2 - 4b_3)}, \tag{27}$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $b_2 = 0$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $sl(2,\mathbb{R})$. If we solve Eq. (27) with respect to b_2 and derive once with respect to y, then we obtain the following linear third-order equation:

$$\frac{d^3u}{dy^3} = \frac{(b_1y^4 - 4b_3)\left(\frac{du}{dy} - y\frac{d^2u}{dy^2}\right)}{y^2(2w_0b_1y^2 - b_1y^4 - 8w_0b_3y^2 - 4b_3)}. \tag{28}$$

B. Case (2)

The Hamiltonian

$$\mathcal{H}_{DI2} = \frac{1}{4u}(p_u^2 + p_v^2) + \frac{a_1}{u} + \frac{a_2v}{u} + a_3\frac{u^2 + v^2}{u} \tag{29}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{u} = \dfrac{p_u}{2u}, \\ \dot{v} = \dfrac{p_v}{2u}, \\ \dot{p}_u = \dfrac{p_u^2 + p_v^2 + 4a_1 + 4a_2v^2 - 4a_3(u^2 - v^2)}{4u^2}, \\ \dot{p}_v = -\dfrac{a_2 + 2a_3v}{u}. \end{cases} \tag{30}$$

If we apply the reduction method developed in Ref. 18 and choose v as a new independent variable y, then we obtain the following three equations:

$$\begin{cases} \dfrac{du}{dy} = \dfrac{p_u}{p_v}, \\ \dfrac{dp_u}{dy} = \dfrac{4a_1 + 4a_2y - 4a_3u^2 + 4a_3y^2 + p_u^2 + p_v^2}{2up_v}, \\ \dfrac{dp_v}{dy} = -2\dfrac{a_2 + 2a_3y}{p_v}. \end{cases} \tag{31}$$

The last equation can be easily integrated, i.e.,

$$p_v = \pm 2\sqrt{a_2w_0 - a_2y - a_3y^2}, \tag{32}$$

with w_0 being an arbitrary constant. Moreover, if we derive p_u from the first equation of system (31) and substitute it into the second equation, then we obtain the following second-order equation in u:

$$\frac{d^2u}{dy^2} = \frac{(a_2w_0 - a_2y - a_3y^2)\left(\frac{du}{dy}\right)^2 + (a_2 + 2a_3y)u\frac{du}{dy} - a_3u^2 + a_1 + a_2w_0}{2u(a_2w_0 - a_2y - a_3y^2)}, \tag{33}$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $a_1 + a_2 w_0 = 0$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathrm{sl}(2,\mathbb{R})$. If we solve Eq. (33) with respect to a_1 and derive once with respect to y, then we obtain the following linear third-order equation:

$$\frac{\mathrm{d}^3 u}{\mathrm{d}y^3} = \frac{3(a_2 + 2a_3 y)}{2(a_2 w_0 - a_2 y - a_3 y^2)} \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}. \tag{34}$$

C. Case (3)

The Hamiltonian

$$\mathcal{H}_{DI3} = \frac{1}{4u}\left(p_u^2 + p_v^2\right) + \frac{a}{u} \tag{35}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{u} = \dfrac{p_u}{2u}, \\ \dot{v} = \dfrac{p_v}{2u}, \\ \dot{p}_u = \dfrac{4a + p_u^2 + p_v^2}{4u^2}, \\ \dot{p}_v = 0. \end{cases} \tag{36}$$

The last equation can be easily integrated, i.e., $p_v = w_0$, with w_0 being an arbitrary constant. If we apply the reduction method developed in Ref. 18 and choose v as new independent variable, then system (36) reduces to the following two equations:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{p_u}{w_0}, \\ \dfrac{\mathrm{d}p_u}{\mathrm{d}y} = \dfrac{4a + p_u^2 + w_0^2}{2uw_0}. \end{cases} \tag{37}$$

If we derive p_u from the first equation of system (37) and replace it into the second equation, then we obtain the following second-order equation in u:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{1}{2u}\left(\frac{\mathrm{d}u}{\mathrm{d}v}\right)^2 + \frac{4a + w_0^2}{2w_0^2 u}, \tag{38}$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $4a + w_0^2 = 0$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathrm{sl}(2,\mathbb{R})$. If we solve Eq. (38) with respect to a and derive once with respect to y, then we obtain the following linear third-order equation:

$$\frac{\mathrm{d}^3 u}{\mathrm{d}y^3} = 0. \tag{39}$$

V. DARBOUX II

Four superintegrable systems were determined in Ref. 13, where the problem of superintegrability for the Hamiltonian

$$H_{DII} = \frac{w_1^2}{w_1^2 + 1}\left(w_3^2 + w_4^2\right) + V(w_1, w_2) \tag{40}$$

was addressed, namely, finding the potentials $V(w_1, w_2)$ such that H_{DII} admits at least two extra quadratic integrals.

A. Case (A)

The Hamiltonian

$$\mathcal{H}_{DIIA} = \frac{w_1^2}{w_1^2+1}\left(w_3^2 + w_4^2 + a_1\left(\frac{w_1^2}{4} + w_2^2\right) + a_2 w_2 + \frac{a_3}{w_1^2}\right) \tag{41}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = 2\dfrac{w_1^2 w_3}{w_1^2+1}, \\ \dot{w}_2 = 2\dfrac{w_1^2 w_4}{w_1^2+1}, \\ \dot{w}_3 = -w_1\dfrac{4w_3^2 + 4w_4^2 + a_1 w_1^4 + 2a_1 w_1^2 + 4a_1 w_2^2 + 4a_2 w_2 - 4a_3}{2(w_1^2+1)^2}, \\ \dot{w}_4 = -w_1^2\dfrac{2a_1 w_2 + a_2}{w_1^2+1}. \end{cases} \tag{42}$$

If we apply the reduction method developed in Ref. 18 and choose w_1 as a new independent variable y, then we obtain the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}w_2}{\mathrm{d}y} = \dfrac{w_4}{w_3}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = -\dfrac{4w_3^2 + 4w_4^2 + a_1 y^4 + 2a_1 y^2 + 4a_1 w_2^2 + 4a_2 w_2 - 4a_3}{2yw_3(w_1^2+1)}, \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}y} = -\dfrac{2a_1 w_2 + a_2}{w_3}. \end{cases} \tag{43}$$

If we solve the second equation with respect to a_3 and then derive once with respect to y, then we obtain the following second-order equation in $w_3(y)$:

$$w_3'' = -\frac{w_3'(yw_3' + 3w_3) + a_1 y}{yw_3}, \tag{44}$$

which admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and therefore is linearizable. In this case, Lie canonical transformation is

$$\tilde{w}_3 = \frac{y^2 w_3}{2} + \frac{a_1 y^4}{8}, \quad \tilde{y} = y^2 \Longrightarrow \frac{\mathrm{d}^2\tilde{w}_3}{\mathrm{d}\tilde{y}^2} = 0, \tag{45}$$

and consequently,

$$w_3 = \pm\frac{\sqrt{8C_2 y^2 + 8C_1 - a_1 y^4}}{2y}, \tag{46}$$

with C_1, C_2 being arbitrary integration constants, although only one is really arbitrary since there is a relationship between them and a_3. Then, if we solve the first equation in (43) with respect to w_4 and replace it into the third equation, we obtain the following linear second-order equation in $w_2(y)$:

$$w_2'' = \frac{-(a_1 y^4 + 8C_1)w_2' + 4a_1 y^3 w_2 + 2a_2 y^3}{a_1 y^5 - 8C_1 y - 8C_2 y^3}. \tag{47}$$

Its general solution is

$$w_2 = (a_1 y^2 - 4C_2)C_3 + \sqrt{a_1 y^4 - 8C_1 - 8C_2 y^2}\,C_4 - \frac{a_2}{2a_1}, \tag{48}$$

with C_3, C_4 being arbitrary integration constants.

B. Case (B)

The Hamiltonian

$$\mathcal{H}_{DIIB} = \frac{w_1^2}{w_1^2+1}\left(w_3^2 + w_4^2 + b_1(w_1^2+w_2^2) + \frac{b_2}{w_1^2} + \frac{b_3}{w_2^2}\right) \tag{49}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = 2\dfrac{w_1^2 w_3}{w_1^2+1}, \\ \dot{w}_2 = 2\dfrac{w_1^2 w_4}{w_1^2+1}, \\ \dot{w}_3 = -2w_1\dfrac{w_2^2w_3^2 + w_2^2w_4^2 + b_1w_1^4w_2^2 + 2b_1w_1^2w_2^2 + b_1w_2^4 - b_2w_2^2 + b_3}{w_2^2(w_1^2+1)^2}, \\ \dot{w}_4 = -2w_1^2\dfrac{b_1w_2^4 - b_3}{w_2^3(w_1^2+1)}. \end{cases} \tag{50}$$

If we apply the reduction method developed in Ref. 18 and choose w_1 as a new independent variable y, then we obtain the following three equations:

$$\begin{cases} \dfrac{dw_2}{dy} = \dfrac{w_4}{w_3}, \\ \dfrac{dw_3}{dy} = -\dfrac{w_2^2w_3^2 + w_2^2w_4^2 + b_1y^4w_2^2 + 2b_1y^2w_2^2 + b_1w_2^4 - b_2w_2^2 + b_3}{w_2^2w_3y(y^2+1)}, \\ \dfrac{dw_4}{dy} = -\dfrac{b_1w_2^4 - b_3}{w_2^3w_3}. \end{cases} \tag{51}$$

If we solve the second equation with respect to b_2 and then derive once with respect to y, then we obtain the following second-order equation in $w_3(y)$:

$$w_3'' = -\frac{w_3'(yw_3' + 3w_3) + 4b_1y}{yw_3}, \tag{52}$$

which is exactly the linearizable equation (44) if a_1 is replaced with $4b_1$, and consequently,

$$w_3 = \pm\frac{\sqrt{8C_2y^2 + 8C_1 - 4b_1y^4}}{2y}, \tag{53}$$

with C_1, C_2 being arbitrary integration constants.

Another reduction would also lead to linearity. If we choose w_2 as a new independent variable y, then we obtain the following three equations:

$$\begin{cases} \dfrac{dw_1}{dy} = \dfrac{w_3}{w_4}, \\ \dfrac{dw_3}{dy} = -\dfrac{y^2w_3^2 + y^2w_4^2 + b_1w_1^4y^2 + 2b_1w_1^2y^2 + b_1y^4 - b_2y^2 + b_3}{w_1w_4y^2(w_1^2+1)}, \\ \dfrac{dw_4}{dy} = -\dfrac{b_1y^4 - b_3}{y^3w_4}, \end{cases} \tag{54}$$

and we can easily integrate the third equation, i.e.,

$$w_4 = \pm\frac{1}{y}\sqrt{2b_1w_0y^2 - b_1y^4 - 2b_3w_0y^2 - b_3}, \tag{55}$$

with w_0 being an arbitrary integration constant. Then, if we solve the first equation in (54) with respect to w_3 and replace it into the second equation, we obtain the following second-order equation in $w_1(y)$:

$$w_1'' = \frac{-w_1'^2}{w_1(w_1^2+1)} + \frac{(b_1y^4 - b_3)w_1'}{y(2(b_1-b_3)w_0y^2 - b_1y^4 - b_3)} - y^2\frac{b_1(w_1^4 + 2w_1^2 + 2w_0) - b_2 - 2b_3w_0}{w_1(w_1^2+1)(2(b_1-b_3)w_0y^2 - b_1y^4 - b_3)}, \tag{56}$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $b_2 = 2(b_1-b_3)w_0 - b_1$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathrm{sl}(2,\mathbb{R})$. If we solve Eq. (56) with respect to b_2 and derive once with respect to y, then we obtain the following third-order equation:

$$w_1''' = -3\frac{w_1'w_1''}{w_1} + 3\frac{b_1y^4 - b_3}{2(b_1-b_3)w_0y^2 - b_1y^4 - b_3}\left(\frac{w_1''}{y} + \frac{w_1'^2}{yw_1} - \frac{w_1'}{y^2}\right), \tag{57}$$

which admits a seven-dimensional Lie symmetry algebra and therefore is linearizable. Indeed, the new dependent variable $U = \frac{1}{2w_1^2}$ and independent variable $Y = b_1y^2 + w_0(b_3 - b_1)$ transform Eq. (57) into the linear equation

$$\frac{\mathrm{d}^3U}{\mathrm{d}Y^3} = -\frac{3Y\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}}{Y^2 - b_3^2w_0^2 + b_1b_3 + 2b_1b_3w_0^2 - b_1^2w_0^2}.$$

C. Case (C)

The Hamiltonian

$$\mathcal{H}_{DIIC} = \frac{w_3^2 + w_4^2 + a_1 + \frac{a_2}{w_1^2} + \frac{a_3}{w_2^2}}{w_1^2 + w_2^2 + \frac{1}{w_1^2} + \frac{1}{w_2^2}} \tag{58}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = 2\dfrac{w_1^2w_2^2w_3}{(w_1^2w_2^2+1)(w_1^2+w_2^2)}, \\ \dot{w}_2 = 2\dfrac{w_1^2w_2^2w_4}{(w_1^2w_2^2+1)(w_1^2+w_2^2)}, \\ \dot{w}_3 = 2w_1w_2^2\dfrac{(a_1w_2^2 + a_3 + (w_3^2+w_4^2)w_2^2)(w_1^4-1) + (w_2^4 + 1 + 2w_1^2w_2^2)a_2}{(w_1^2w_2^2+1)^2(w_1^2+w_2^2)^2}, \\ \dot{w}_4 = 2w_1^2w_2\dfrac{(a_1w_1^2 + a_2 + (w_3^2+w_4^2)w_1^2)(w_2^4-1) + (w_1^4 + 2w_1^2w_2^2 + 1)a_3}{(w_1^2w_2^2+1)^2(w_1^2+w_2^2)^2}. \end{cases} \tag{59}$$

Before applying the reduction method,[18] we introduce the following transformations of the dependent variables in order to avoid the mishandling of formulas with square roots by either REDUCE or MAPLE, i.e.,

$$w_1 = \sqrt{r_1}, \quad w_2 = \sqrt{r_2}, \quad w_3 = \sqrt{r_3}, \quad w_4 = \sqrt{r_4}, \tag{60}$$

and then choose r_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\dfrac{r_1 r_3}{y r_4}}, \\ \dfrac{\mathrm{d}r_3}{\mathrm{d}y} = \sqrt{\dfrac{r_3}{y r_1 r_4}} \dfrac{(a_1 y + a_3 + (r_3 + r_4)y)(r_1^2 - 1) + (y^2 + 1 + 2r_1 y)a_2}{(r_1 y + 1)(r_1 + y)}, \\ \dfrac{\mathrm{d}r_4}{\mathrm{d}y} = \dfrac{(a_1 r_1 + a_2 + (r_3 + r_4)r_1)(y^2 - 1) + (r_1^2 + 1 + 2r_1 y)a_3}{y(r_1 y + 1)(r_1 + y)}. \end{cases} \tag{61}$$

From the Hamiltonian $\mathcal{H}_{DIIC}$, i.e.,

$$\mathcal{H}_{DIIC} = \frac{r_3 + r_4 + a_1 + \frac{a_2}{r_1} + \frac{a_3}{y}}{r_1 + y + \frac{1}{r_1} + \frac{1}{y}} = h_0, \tag{62}$$

we can derive

$$r_3 = \frac{(r_1 y + 1)(r_1 + y)h_0 - r_1 r_4 y - a_3 r_1 - a_2 y - a_1 y r_1}{y r_1}, \tag{63}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (61) becomes

$$\frac{\mathrm{d}r_4}{\mathrm{d}y} = \frac{a_3 + (y^2 - 1)h_0}{y^2}, \tag{64}$$

which can be easily integrated, i.e.,

$$r_4 = \frac{w_0 y - a_3 + (y^2 + 1)h_0}{y^2}, \tag{65}$$

with w_0 being an arbitrary constant. Finally, we are left with the first equation in (61), i.e.,

$$\frac{\mathrm{d}r_1}{\mathrm{d}y} = \frac{\sqrt{h_0 r_1^2 - (a_1 + w_0)r_1 - a_2 + h_0}}{\sqrt{h_0 y^2 + w_0 y - a_3 + h_0}}, \tag{66}$$

which can be solved by quadratures. However, if we solve it with respect to a_2 and derive once by y, then the following linear second-order equation is obtained:

$$2\big(a_3 - w_0 y - (y^2 + 1)h_0\big)\frac{\mathrm{d}^2 r_1}{\mathrm{d}y^2} - (w_0 + 2h_0 y)\frac{\mathrm{d}r_1}{\mathrm{d}y} + 2h_0 r_1 - w_0 - a_1 = 0. \tag{67}$$

D. Case (D)

The Hamiltonian

$$\mathcal{H}_{DIID} = \frac{w_1^2}{w_1^2 + 1}\left(w_3^2 + w_4^2 + d\right) \tag{68}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = 2\dfrac{w_1^2 w_3}{w_1^2+1}, \\ \dot{w}_2 = 2\dfrac{w_1^2 w_4}{w_1^2+1}, \\ \dot{w}_3 = -2w_1\dfrac{w_3^2+w_4^2+d}{(w_1^2+1)^2}, \\ \dot{w}_4 = 0. \end{cases} \tag{69}$$

The last equation yields $w_4 = w_0$. If we apply the reduction method developed in Ref. 18 and choose w_2 as a new independent variable y, then we obtain the following two equations:

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}y} = \dfrac{w_3}{w_0}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = -\dfrac{w_3^2+w_0^2+d}{w_0 w_1 (w_1^2+1)}. \end{cases} \tag{70}$$

Then, if we solve the first equation in (70) with respect to w_3 and replace it into the second equation, we obtain the following second-order equation in $w_1(y)$:

$$w_1'' = -\frac{w_0^2 w_1'^2 + w_0^2 + d}{w_0^2 w_1 (w_1^2+1)}, \tag{71}$$

which admits a three-dimensional symmetry algebra sl$(2,\mathbb{R})$, unless $d = -w_0^2$, in which case it admits an eight-dimensional Lie symmetry algebra sl$(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra sl$(2,\mathbb{R})$. If we solve Eq. (71) with respect to d and derive once with respect to y, then we obtain the following third-order equation:

$$w_1''' = -\frac{3w_1' w_1''}{w_1}, \tag{72}$$

which admits a seven-dimensional Lie symmetry algebra and therefore is linearizable. Indeed, the new dependent variable $r_1 = w_1^2$ transforms Eq. (72) into the linear equation

$$r_1''' = 0. \tag{73}$$

VI. DARBOUX III

Five superintegrable cases were determined in Ref. 13, where the problem of superintegrability for the Hamiltonian

$$H_{DIII} = \frac{e^{2u}\left(p_u^2 + p_v^2\right)}{4e^{u+1}} \tag{74}$$

was addressed.

A. Case (A)

The Hamiltonian

$$\mathcal{H}_{DIIIA} = \frac{w_3^2 + w_4^2 + a_1 w_1 + a_2 w_2 + a_3}{4 + w_1^2 + w_2^2} \tag{75}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = \dfrac{2w_3}{w_1^2 + w_2^2 + 4}, \\ \dot{w}_2 = \dfrac{2w_4}{w_1^2 + w_2^2 + 4}, \\ \dot{w}_3 = \dfrac{2a_2 w_1 w_2 + a_1(w_1^2 - w_2^2 - 4) + 2a_3 w_1 + 2w_1(w_3^2 + w_4^2)}{(w_1^2 + w_2^2 + 4)^2}, \\ \dot{w}_4 = \dfrac{2a_1 w_1 w_2 - a_2(w_1^2 - w_2^2 - 4) + 2a_3 w_2 + 2w_2(w_3^2 + w_4^2)}{(w_1^2 + w_2^2 + 4)^2}. \end{cases} \tag{76}$$

We apply the reduction method[18] by choosing w_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}y} = \dfrac{w_3}{w_4}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = \dfrac{2a_2 w_1 y + a_1(w_1^2 - y^2 - 4) + 2a_3 w_1 + 2w_1(w_3^2 + w_4^2)}{2w_4(w_1^2 + y^2 + 4)}, \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}y} = \dfrac{2a_1 w_1 y - a_2(w_1^2 - y^2 - 4) + 2a_3 y + 2w_2(w_3^2 + w_4^2)}{2w_4(w_1^2 + y^2 + 4)}. \end{cases} \tag{77}$$

From the Hamiltonian $\mathcal{H}_{DIIIA}$, i.e.,

$$\mathcal{H}_{DIIIA} = \frac{w_3^2 + w_4^2 + a_1 w_1 + a_2 y + a_3}{4 + w_1^2 + y^2} = h_0, \tag{78}$$

we can derive

$$w_3 = \pm\sqrt{h_0(w_1^2 + y^2) + 4h_0 - a_1 w_1 - a_2 y - a_3 - w_4^2}, \tag{79}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (77) becomes

$$\frac{\mathrm{d}w_4}{\mathrm{d}y} = \frac{2h_0 y - a_2}{2w_4}, \tag{80}$$

which can be easily integrated, i.e.,

$$w_4 = \pm\sqrt{a_2(w_0 - y) + h_0 y^2}, \tag{81}$$

with w_0 being an arbitrary constant. Finally, we are left with the first equation in (77), i.e.,

$$\frac{\mathrm{d}w_1}{\mathrm{d}y} = \frac{\sqrt{h_0(w_1^2 + 4) - a_1 w_1 - a_2 w_0 - a_3}}{\sqrt{a_2(w_0 - y) + h_0 y^2}}, \tag{82}$$

which can be solved by quadratures. However, if we solve it with respect to a_3 and derive once by y, then the following linear second-order equation is obtained:

$$2\big((w_0 - y)a_2 + h_0 y^2\big)\frac{\mathrm{d}^2 w_1}{\mathrm{d}y^2} + (a_2 - 2h_0 y)\frac{\mathrm{d}w_1}{\mathrm{d}y} + 2h_0 w_1 - w_0 - a_1 = 0. \tag{83}$$

B. Case (B)

The Hamiltonian

$$\mathcal{H}_{DIIIB} = \frac{w_3^2 + w_4^2 + \frac{b_1}{w_1^2} + \frac{b_2}{w_2^2} + b_3}{4 + w_1^2 + w_2^2} \tag{84}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = \dfrac{2w_3}{w_1^2 + w_2^2 + 4}, \\ \dot{w}_2 = \dfrac{2w_4}{w_1^2 + w_2^2 + 4}, \\ \dot{w}_3 = 2\dfrac{\left(b_2 + b_3 w_2^2 + (w_3^2 + w_4^2)w_2^2\right)w_1^4 + \left(w_2^2 + 4 + 2w_1^2\right)b_1 w_2^2}{(w_1^2 + w_2^2 + 4)^2 w_1^3 w_2^2}, \\ \dot{w}_4 = 2\dfrac{\left(b_1 + b_3 w_1^2 + (w_3^2 + w_4^2)w_1^2\right)w_2^4 + \left(2(w_2^2 + 2) + w_1^2\right)b_2 w_1^2}{(w_1^2 + w_2^2 + 4)^2 w_1^2 w_2^3}. \end{cases} \tag{85}$$

Before applying the reduction method,[18] we introduce the following transformations of dependent variables, in order to render the next calculations more amenable to computer algebraic software such as REDUCE, i.e.,

$$w_1 = \sqrt{r_1}, \; w_2 = \pm\sqrt{r_2}, \tag{86}$$

and then choose r_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}y} = \dfrac{\sqrt{r_1}w_3}{\sqrt{y}w_4}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = \dfrac{\left(b_2 + b_3 y + (w_3^2 + w_4^2)y\right)r_1^2 + (y + 4 + 2r_1)b_1 y}{2yr_1\sqrt{yr_1}(r_1 + y + 4)w_4}, \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}y} = \dfrac{\left(b_1 + b_3 r_1 + (w_3^2 + w_4^2)r_1\right)y^2 + (2(y+2) + r_1)b_2 r_1}{2y^2 r_1 (r_1 + y + 4)w_4}. \end{cases} \tag{87}$$

From the Hamiltonian $\mathcal{H}_{DIIIB}$, i.e.,

$$\mathcal{H}_{DIIIB} = \frac{w_3^2 + w_4^2 + \frac{b_1}{r_1} + \frac{b_2}{y} + b_3}{4 + r_1 + y} = h_0, \tag{88}$$

we can derive

$$w_3 = \pm\sqrt{\frac{\left(h_0(y + 4 + r_1) - b_3 - w_4^2\right)yr_1 - b_2 r_1 - b_1 y}{yr_1}}, \tag{89}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (87) becomes

$$\frac{\mathrm{d}w_4}{\mathrm{d}y} = \frac{b_2 + h_0 y^2}{2w_4 y^2}, \tag{90}$$

which can be easily integrated, i.e.,

$$w_4 = \pm\sqrt{\frac{h_0 y(y - w_0) - b_2(1 + w_0 y)}{y}}, \tag{91}$$

with w_0 being an arbitrary constant. Finally, we are left with the first equation in (87), i.e.,

$$\frac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\frac{b_1 + (b_3 - b_2 w_0) r_1 - h_0 r_1 (r_1 + w_0 + 4)}{b_2(1 + w_0 y) + h_0 y (w_0 - y)}}, \tag{92}$$

which can be solved by quadratures. However, if we solve it with respect to b_1 and derive once by y, then the following linear second-order equation is obtained:

$$\frac{\mathrm{d}^2 r_1}{\mathrm{d}y^2} = -\frac{(b_2 w_0 + h_0 w_0 - 2h_0 y)\frac{\mathrm{d}r_1}{\mathrm{d}y} + 2h_0 r_1 + b_2 w_0 - b_3 + h_0 w_0 + 4h_0}{2(b_2(1 + w_0 y) + h_0 y (w_0 - y))}. \tag{93}$$

C. Case (C)

The Hamiltonian

$$\mathcal{H}_{DIIIC} = \frac{w_1^2 w_3^2 - w_2^2 w_4^2 + c_1(w_1 + w_2) + c_2 \frac{w_1 + w_2}{w_1 w_2} + c_3 \frac{w_1^2 - w_2^2}{w_1^2 w_2^2}}{(w_1 + w_2)(2 + w_1 - w_2)} \tag{94}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = \dfrac{2 w_1^2 w_3}{(w_1 + w_2)(2 + w_1 - w_2)}, \\ \dot{w}_2 = -\dfrac{2 w_2^2 w_4}{(w_1 + w_2)(2 + w_1 - w_2)}, \\ \dot{w}_3 = 2\dfrac{(w_3^2 - w_4^2) w_1 w_2^2 - 2 w_3^2 w_1 w_2 - w_1^2 w_3^2 - w_2^2 w_4^2}{(w_1 + w_2)^2 (2 + w_1 - w_2)^2} + \dfrac{c_1}{(2 + w_1 - w_2)^2} \\ \quad + c_2 \dfrac{2w_1 - w_2 + 2}{(2 + w_1 - w_2)^2 w_1^2 w_2} + 2c_3 \dfrac{w_1^2 - 2w_1 w_2 + w_1 + w_2^2 - 2w_2}{(2 + w_1 - w_2)^2 w_1^3 w_2^2}, \\ \dot{w}_4 = -2\dfrac{(w_3^2 - w_4^2) w_1^2 w_2 - w_3^2 w_1^2 - (2w_1 + w_2) w_2 w_4^2}{(w_1 + w_2)^2 (2 + w_1 - w_2)^2} - \dfrac{c_1}{(2 + w_1 - w_2)^2} \\ \quad + c_2 \dfrac{w_1 - 2w_2 + 2}{(2 + w_1 - w_2)^2 w_1 w_2^2} + 2c_3 \dfrac{w_1^2 - 2w_1 w_2 + 2w_1 + w_2^2 - w_2}{(2 + w_1 - w_2)^2 w_1^2 w_2^3}. \end{cases} \tag{95}$$

We apply the reduction method[18] by choosing w_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}y} = -\dfrac{w_1^2 w_3}{y^2 w_4}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = -\dfrac{(w_3^2 - w_4^2) w_1 y^2 - 2 w_3^2 w_1 y - w_1^2 w_3^2 - y^2 w_4^2}{(w_1 + y)(2 + w_1 - y) y^2 w_4} - c_1 \dfrac{w_1 + y}{2(2 + w_1 - y) y^2 w_4} \\ \quad - c_2 \dfrac{(2w_1 - y + 2)(w_1 + y)}{2(2 + w_1 - y) w_1^2 y^3 w_4} - c_3 \dfrac{(w_1^2 - 2w_1 y + w_1 + y^2 - 2y)(w_1 + y)}{(2 + w_1 - y) w_1^3 y^4 w_4}, \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}y} = \dfrac{w_1^2 y (w_3^2 - w_4^2) - w_1^2 w_3^2 - 2 w_1 w_4^2 y - w_4^2 y^2}{(w_1 + y)(2 + w_1 - y) w_4 y^2} + c_1 \dfrac{w_1 + y}{2(2 + w_1 - y) w_4 y^2} \\ \quad - c_2 \dfrac{(w_1 + y)(w_1 - 2y + 2)}{2(2 + w_1 - y) w_1 w_4 y^4} - c_3 \dfrac{(w_1^2 - 2w_1 y + 2w_1 + y^2 - y)(w_1 + y)}{(2 + w_1 - y) w_1^2 w_4 y^5}. \end{cases} \tag{96}$$

From the Hamiltonian $\mathcal{H}_{DIIIC}$, i.e.,

$$\mathcal{H}_{DIIIC} = \frac{w_1^2 w_3^2 - y^2 w_4^2 + c_1(w_1 + y) + c_2 \frac{w_1 + y}{w_1 y} + c_3 \frac{w_1^2 - y^2}{w_1^2 y^2}}{(w_1 + y)(2 + w_1 - y)} = h_0, \tag{97}$$

we can derive

$$w_3 = \pm\frac{\sqrt{w_1^2 w_4^2 y^4 - w_1 y(w_1+y)(c_1 w_1 y + c_2) + c_3(y^2 - w_1^2) + h_0(w_1+y)(2+w_1-y)w_1^2 y^2}}{w_1^2 y}, \tag{98}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (96) becomes

$$\frac{\mathrm{d}w_4}{\mathrm{d}y} = \frac{2w_4^2 y^4 - c_1 y^3 + c_2 y + 2c_3 + 2y^3 h_0(1-y)}{2w_4 y^5}, \tag{99}$$

which can be easily integrated, i.e.,

$$w_4 = \pm\frac{\sqrt{w_0 y^2 + c_1 y^3 + c_2 y + c_3 + h_0 y^3(y-2)}}{y^2}, \tag{100}$$

with w_0 being an arbitrary constant. Finally, we are left with the first equation in (96), i.e.,

$$\frac{\mathrm{d}w_1}{\mathrm{d}y} = -\frac{\sqrt{w_0 w_1^2 - c_1 w_1^3 - c_2 w_1 + c_3 + h_0 w_1^3(w_1+2)}}{\sqrt{w_0 y^2 + c_1 y^3 + c_2 y + c_3 + h_0 y^3(y-2)}}, \tag{101}$$

which could be solved by quadratures. If we introduce new parameters in order to simplify this equation, i.e.,

$$c_1 = C_1 + 2h_0, \qquad c_2 = C_2 C_1, \quad c_3 = C_3 C_1, \quad h_0 = H_0 C_1, \quad w_0 = W_0 C_1, \tag{102}$$

and the new dependent variable $u = -w_1$, then Eq. (101) becomes

$$u'(y) \equiv \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\sqrt{C_2 u + C_3 + H_0 u^4 + W_0 u^2 + u^3}}{\sqrt{C_2 y + C_3 + H_0 y^4 + W_0 y^2 + y^3}}. \tag{103}$$

If we solve this first-order equation with respect to C_3 and derive once by y, then a second-order equation is obtained. If we solve this second-order equation with respect to W_0 and derive once by y, then a third-order equation is obtained. Finally, if we solve this third-order equation with respect to H_0 and derive once by y, then the following fourth-order equation is obtained:

$$u^{(iv)} = -\frac{\alpha_1 u''^3 + \alpha_2 u''^2 - \alpha_3 u'' u''' - \alpha_4 u'' + \alpha_5 u'''^2 - 6\alpha_7 u''' + \alpha_8}{3(C_2 - uy)(uu'' - 2u'^2 - 2u' - u'' y)(u^2 - y^2)}, \tag{104}$$

with

$$\begin{aligned}
\alpha_1 &= 9(u-y)\left[C_2(3u+5y) - 2u^2 y - 5uy^2 - y^3\right],\\
\alpha_2 &= C_2(36u + 54y + 36uu' - 54u'^2 y) - 36u^2 u'^2 y + 72uu'^2 y^2 + 18u'^2 y^3\\
&\quad + 18u^3 u' - 72u^2 u' y - 18uu' y^2 + 36u' y^3 - 18u^2 y - 72uy^2,\\
\alpha_3 &= 3(u-y)\left[C_2(13uu' + 15u' y + 5u + 7y) - 12u^2 u' y - 15uu' y^2 - u' y^3 + u^3 - 5u^2 y - 8uy^2\right],\\
\alpha_4 &= 18u'(u'+1)\left[C_2(u'^2 - 1) - 4uu'^2 y + u'^2 y^2 + 3u^2 u' - 3u' y^2 - u^2 + 4uy\right],\\
\alpha_5 &= 5(u+y)(u-y)^2(C_2 - uy),\\
\alpha_7 &= u'(u'+1)\left[C_2(3uu' + 5u' y - 5u - 3y) - 2u^2 u' y - 5uu' y^2 - u' y^3 + u^3 + 5u^2 y + 2uy^2\right],\\
\alpha_8 &= 36u'^2(u'-1)(u'+1)^2(u - u' y).
\end{aligned}$$

It admits a fourth-dimensional Lie symmetry algebra $2A_2$ generated by the following operators:

$$\Gamma_1 = \frac{3}{u-y}\big(-(C_2^2+uy^3)\partial_y + (C_2^2+u^3y)\partial_u\big), \tag{105}$$

$$\Gamma_2 = \frac{u+y}{u-y}\big(-(C_2+y^2)\partial_y + (u^2+C_2)\partial_u\big), \tag{106}$$

$$\Gamma_3 = \frac{1}{u-y}\big((uy-2C_2-y^2)\partial_y + (2C_2+u^2-uy)\partial_u\big), \tag{107}$$

$$\Gamma_4 = -\frac{1}{3(u-y)}\big((C_2-2uy-y^2)\partial_y + (u^2-C_2+2uy)\partial_u\big). \tag{108}$$

In order to follow the classification of the fourth-dimensional Lie symmetry algebra in Ref. 22 and the fourth-order equations, admitting them as derived in Ref. 23, we choose another representation of the operators that generate $2A_2$, i.e.,

$$X_1 = \Gamma_1 - 3C_2\Gamma_3, \quad X_2 = \Gamma_2, \quad X_3 = \frac{1}{3}\Gamma_3 + \Gamma_4, \quad X_4 = \frac{1}{3}\Gamma_3 - 2\Gamma_4. \tag{109}$$

We thank Nicola Ciccoli for his invaluable help on this issue. A two-dimensional Abelian intransitive subalgebra of the Lie symmetry algebra $2A_2$ is that generated by X_1 and X_2, and the corresponding canonical transformations[20] are

$$\tilde{y} = \frac{u+y}{3(uy-C_2)}, \quad \tilde{u} = -\frac{1}{uy-3C_2}.$$

Then, Eq. (104) turns into the following fourth-order equation:

$$\frac{d^4\tilde{u}}{d\tilde{y}^4} = \left(3\frac{d^2\tilde{u}}{d\tilde{y}^2} + 5\tilde{y}\frac{d^4\tilde{u}}{d\tilde{y}^4}\right)\frac{\frac{d^3\tilde{u}}{d\tilde{y}^3}}{3\tilde{y}\frac{d^2\tilde{u}}{d\tilde{y}^2}}. \tag{110}$$

If we make the substitution

$$\frac{d^2\tilde{u}}{d\tilde{y}^2} = R(\tilde{y}),$$

then Eq. (110) becomes the following second-order equation:

$$\frac{d^2R}{d\tilde{y}^2} = \left(3R + 5\tilde{y}\frac{dR}{d\tilde{y}}\right)\frac{\frac{dR}{d\tilde{y}}}{3\tilde{y}R}, \tag{111}$$

which admits an eight-dimensional Lie symmetry algebra $sl(3,\mathbb{R})$ and therefore is linearizable.[20] Indeed, the transformation $Y = R^{2/3}$, $U = \frac{\tilde{y}^2}{2}R^{2/3}$ yields

$$\frac{d^2U}{dY^2} = 0 \Rightarrow U = A_1Y + A_2,$$

with A_1, A_2 being arbitrary constants. Consequently, the general solution of Eq. (111) is

$$R = \frac{2A_2\sqrt{2A_2}}{(\tilde{y}^2-2A_1)\sqrt{\tilde{y}^2-2A_1}}, \tag{112}$$

which integrated twice yields the general solution of Eq. (110), i.e.,

$$\tilde{u} = \frac{A_2}{A_1}\sqrt{2A_2(\tilde{y}^2-2A_1)} + A_3\tilde{y} + A_4, \tag{113}$$

with A_3, A_4 being arbitrary constants. Finally, the general solution of Eq. (104) is

$$u = \frac{\beta_2 y^2 + \beta_1 y + \beta_0 + \sqrt{\gamma_4 y^4 + \gamma_3 y^3 + \gamma_2 y^2 + \gamma_1 y + \gamma_0}}{9A_1(A_4^2 A_1 + 4A_2^3)y^2 + 6A_1^2 A_3 A_4 y + A_1^2 A_3^2 - 2A_2^3}, \tag{114}$$

with

$$\begin{aligned}
\beta_2 &= -3A_1^2 A_3 A_4,\\
\beta_1 &= 9A_1^2 A_4^2 C_2 - 3A_1^2 A_4 + 2A_2^3 - A_1^2 A_3^2 + 36A_2^3 A_1 C_2,\\
\beta_0 &= A_1^2 A_3(3C_2 A_4 - 1),\\
\gamma_4 &= -18A_1^2 A_3^2 + 9A_4^2 A_1 + 36A_2^3,\\
\gamma_3 &= -36A_1^2 A_3,\\
\gamma_2 &= 18A_4^2 A_1 C_2 - 6A_4 A_1 + 72A_2^3 C_2 - 36A_1^2 A_3^2 C_2 - 18A_1^2,\\
\gamma_1 &= -36A_1^2 C_2 A_3,\\
\gamma_0 &= -18A_1^2 A_3^2 C_2^2 + 9A_1 A_4^2 C_2^2 - 6A_1 C_2 A_4 + A_1 + 36A_2^3 C_2^2.
\end{aligned}$$

We would like to remark that if we solve the fourth-order equation (104) with respect to C_2 and derive once by y, then a fifth-order equation is obtained, which admits an eight-dimensional Lie symmetry algebra and can be transformed into a third-order linearizable equation since it admits a seven-dimensional Lie symmetry algebra, quite similar to the fourth-order equation that we discuss in detail here.

D. Case (D)

The Hamiltonian

$$\mathcal{H}_{DIIID} = \frac{w_1^2 w_3^2 - w_2^2 w_4^2 + d_1 w_1 + d_2 w_2 + d_3(w_1^2 + w_2^2)}{(w_1 + w_2)(2 + w_1 - w_2)} \tag{115}$$

yields the Hamiltonian equations

$$\begin{cases}
\dot{w}_1 = \dfrac{2w_1^2 w_3}{(w_1 + w_2)(2 + w_1 - w_2)},\\
\dot{w}_2 = -\dfrac{2w_2^2 w_4}{(w_1 + w_2)(2 + w_1 - w_2)},\\
\dot{w}_3 = \dfrac{1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2}\big(2\big(w_1 w_2^2(w_3^2 - w_4^2) - 2w_1 w_2 w_3^2 - w_2^2 w_4^2 - w_1^2 w_3^2\big)\\
\qquad + d_1(w_1^2 + w_2^2 - 2w_2) + 2d_2 w_2(w_1 + 1) + 2d_3(w_2^2 - w_1^2 + 2w_1 w_2^2 - 2w_1 w_2)\big),\\
\dot{w}_4 = \dfrac{1}{(w_1 + w_2)^2(2 + w_1 - w_2)^2}\big(2\big(w_1^2 w_2(w_4^2 - w_3^2) + 2w_1 w_2 w_4^2 + w_2^2 w_4^2 + w_1^2 w_3^2\big)\\
\qquad + 2d_1 w_1(1 - w_2) - d_2(w_1^2 + w_2^2 + 2w_1) - 2d_3(w_2^2 - w_1^2 + 2w_1^2 w_2 + 2w_1 w_2)\big).
\end{cases} \tag{116}$$

We apply the reduction method[18] by choosing w_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases}
\dfrac{dw_1}{dy} = -\dfrac{w_1^2 w_3}{y^2 w_4},\\
\dfrac{dw_3}{dy} = -\dfrac{1}{2w_4 y^2(w_1 + y)(2 + w_1 - y)}\big(2\big(w_1 y^2(w_3^2 - w_4^2) - 2w_1 y w_3^2 - y^2 w_4^2 - w_1^2 w_3^2\big)\\
\qquad + d_1(w_1^2 + y^2 - 2y) + 2d_2 y(w_1 + 1) + 2d_3(y^2 - w_1^2 + 2w_1 y^2 - 2w_1 y)\big),\\
\dfrac{dw_4}{dy} = -\dfrac{1}{2w_4 y^2(w_1 + y)(2 + w_1 - y)}\big(2\big(w_1^2 y(w_4^2 - w_3^2) + 2w_1 y w_4^2 + y^2 w_4^2 + w_1^2 w_3^2\big)\\
\qquad + 2d_1 w_1(1 - y) - d_2(w_1^2 + y^2 + 2w_1) - 2d_3(y^2 - w_1^2 + 2w_1^2 y + 2w_1 y)\big).
\end{cases} \tag{117}$$

From the Hamiltonian $\mathcal{H}_{DIIID}$, i.e.,

$$\mathcal{H}_{DIIID} = \frac{w_1^2 w_3^2 - y^2 w_4^2 + d_1 w_1 + d_2 y + d_3(w_1^2 + y^2)}{(w_1 + y)(2 + w_1 - y)} = h_0, \tag{118}$$

we can derive

$$w_3 = \pm \frac{\sqrt{(h_0 - d_3)w_1^2 + (2h_0 - d_1)w_1 + (2h_0 - d_2)y - (d_3 + h_0)y^2 + w_4^2 y^2}}{w_1}, \tag{119}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (117) becomes

$$\frac{\mathrm{d}w_4}{\mathrm{d}y} = \frac{d_2 - 2h_0 + 2(d_3 + h_0)y - 2w_4^2 y}{2w_4 y^2}, \tag{120}$$

which can be easily integrated, i.e.,

$$w_4 = \pm \frac{\sqrt{(d_2 - 2h_0)y + (d_3 + h_0)y^2 + w_0}}{y}, \tag{121}$$

with w_0 being an arbitrary constant. Let us introduce new parameters that simplify the formula for w_3 and w_4, i.e.,

$$D_1 = 2h_0 - d_1, \qquad D_2 = d_2 - 2h_0, \qquad D_3 = d_3 + h_0, \tag{122}$$

and consequently,

$$w_3 = \pm \frac{\sqrt{(2h_0 - D_3)w_1^2 + D_1 w_1 + w_0}}{w_1}, \quad w_4 = \pm \frac{\sqrt{D_2 y + D_3 y^2 + w_0}}{y}. \tag{123}$$

Finally, we are left with the first equation in (117), i.e.,

$$\frac{\mathrm{d}w_1}{\mathrm{d}y} = -\frac{w_1\sqrt{(2h_0 - D_3)w_1^2 + D_1 w_1 + w_0}}{y\sqrt{D_3 y^2 + D_2 y + w_0}}, \tag{124}$$

which can be solved by quadratures. However, if we solve it with respect to D_1 and derive once by y, then the following second-order equation is obtained:

$$\begin{aligned} &2y^2 w_1(D_3 y^2 + D_2 y + w_0)\frac{\mathrm{d}^2 w_1}{\mathrm{d}y^2} - 3y^2(D_3 y^2 + D_2 y + w_0)\left(\frac{\mathrm{d}w_1}{\mathrm{d}y}\right)^2 \\ &\quad + (4D_3 y^2 + 3D_2 y + 2w_0)y w_1 \frac{\mathrm{d}w_1}{\mathrm{d}y} + (D_3 - 2h_0)w_1^4 + w_0 w_1^2 = 0, \end{aligned} \tag{125}$$

which admits a three-dimensional symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, unless $D_3 = 2h_0$, in which case it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$ and thus is linearizable. We now use the general method described in Ref. 21 and that may be applied to any second-order ordinary differential equation that admits a Lie symmetry algebra $\mathrm{sl}(2,\mathbb{R})$. If we solve Eq. (125) with respect to h_0 and derive once with respect to y, then we obtain the following third-order equation:

$$\frac{2yw_1^2}{3}(D_3y^2+D_2y+w_0)\frac{d^3w_1}{dy^3} = -4y(D_3y^2+D_2y+w_0)\left(\frac{dw_1}{dy}\right)^3 + 2w_1(4D_3y^2+3D_2y+2w_0)\left(\frac{dw_1}{dy}\right)^2 - 2(2D_3y+D_2)w_1^2\frac{dw_1}{dy} + \left(4yw_1(D_3y^2+D_2y+w_0)\frac{dw_1}{dy} - (4D_3y^2+3D_2y+2w_0)w_1^2\right)\frac{d^2w_1}{dy^2}, \tag{126}$$

which admits a seven-dimensional Lie symmetry algebra and therefore is linearizable. Indeed, the new dependent and independent variables, i.e.,

$$\tilde{w}_1 = -\frac{1}{w_1}, \qquad \tilde{y} = \frac{2D_3y+D_2}{y}, \tag{127}$$

transform Eq. (126) into the linear equation

$$\frac{d^3\tilde{w}_1}{d\tilde{y}^3} = \frac{3(4D_3w_0 - D_2^2 - 2w_0\tilde{y})}{2\left(w_0\tilde{y}^2 + (4D_3w_0 - D_2^2)(D_3-\tilde{y})\right)}\frac{d^2\tilde{w}_1}{d\tilde{y}^2}. \tag{128}$$

E. Case (E)

The Hamiltonian

$$\mathcal{H}_{DIIIE} = \frac{w_3^2+w_4^2+c}{4+w_1^2+w_2^2} \tag{129}$$

is a subcase of Hamiltonian $\mathcal{H}_{DIIIA}$, with $a_1 = a_2 = 0$ and $a_3 = c$. Consequently, its corresponding Hamiltonian equations, i.e.,

$$\begin{cases} \dot{w}_1 = \dfrac{2w_3}{w_1^2+w_2^2+4}, \\ \dot{w}_2 = \dfrac{2w_4}{w_1^2+w_2^2+4}, \\ \dot{w}_3 = \dfrac{2w_1(c+w_3^2+w_4^2)}{(w_1^2+w_2^2+4)^2}, \\ \dot{w}_4 = \dfrac{2w_2(c+w_3^2+w_4^2)}{(w_1^2+w_2^2+4)^2}, \end{cases} \tag{130}$$

can be reduced to the following linear equation in $w_1 = w_1(w_2)$:

$$(2w_0 - w_2^2)\frac{d^2w_1}{dw_2^2} - w_2\frac{dw_1}{dw_2} + w_1 = 0 \tag{131}$$

with

$$w_3 = \pm\sqrt{h_0(w_1^2+w_2^2) + 4h_0 - c - w_4^2} \tag{132}$$

and

$$w_4 = \pm\sqrt{h_0(w_2^2 - 2w_0)}. \tag{133}$$

VII. DARBOUX IV

Four superintegrable systems were determined in Ref. 13, where the problem of superintegrability for the Hamiltonian

$$H_{DIV} = -\sin^2(2u)\frac{p_u^2 + p_v^2}{2\cos(2u) + a} \tag{134}$$

was addressed.

A. Case (A)

The Hamiltonian

$$\mathcal{H}_{DIVA} = -4w_1^2 w_2^2 \frac{w_3^2 + w_4^2 + a_1 + a_2\left(\frac{1}{w_1^2} + \frac{1}{w_2^2}\right) + a_3(w_1^2 + w_2^2)}{(a+2)w_1^2 + (a-2)w_2^2} \tag{135}$$

yields the Hamiltonian equations

$$\begin{cases} \dot{w}_1 = -\dfrac{8w_1^2 w_2^2 w_3}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\ \dot{w}_2 = -\dfrac{8w_1^2 w_2^2 w_4}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\ \dot{w}_3 = \dfrac{8w_1 w_2^2}{\left(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)\right)^2}\big(aa_3(w_1^2 + w_2^2)^2 + aw_2^2(w_3^2 + w_4^2) + a_1 w_2^2(a-2) - 4a_2 \\ \quad + 2a_3(w_1^4 - w_2^4 - 2w_1^2 w_2^2) - 2w_2^2(w_3^2 + w_4^2)\big), \\ \dot{w}_4 = \dfrac{8w_1^2 w_2}{\left(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)\right)^2}\big(aa_3(w_1^2 + w_2^2)^2 + aw_1^2(w_3^2 + w_4^2) + a_1 w_1^2(a+2) + 4a_2 \\ \quad + 2a_3(w_1^4 - w_2^4 + 2w_1^2 w_2^2) + 2w_1^2(w_3^2 + w_4^2)\big). \end{cases} \tag{136}$$

In order to simplify the calculations, we make the following substitutions of the four dependent variables:

$$w_1 = \sqrt{r_1}, \quad w_2 = \sqrt{r_2}, \quad w_3 = \sqrt{r_3}, \quad w_4 = \sqrt{r_4}, \tag{137}$$

and consequently, system (136) is transformed into the following system:

$$\begin{cases} \dot{r}_1 = -\dfrac{16 r_1\sqrt{r_1 r_3}\, r_2}{a(r_1 + r_2) + 2(r_1 - r_2)}, \\ \dot{r}_2 = -\dfrac{16 r_2\sqrt{r_2 r_4}\, r_1}{a(r_1 + r_2) + 2(r_1 - r_2)}, \\ \dot{r}_3 = \dfrac{16\sqrt{r_1 r_3}\, r_2}{\left(a(r_1 + r_2) + 2(r_1 - r_2)\right)^2}\big(aa_3(r_1 + r_2)^2 + ar_2(r_3 + r_4) + a_1 r_2(a-2) - 4a_2 \\ \quad + 2a_3(r_1^2 - r_2^2 - 2r_1 r_2) - 2r_2(r_3 + r_4)\big), \\ \dot{r}_4 = \dfrac{16\sqrt{r_2 r_4}\, r_1}{\left(a(r_1 + r_2) + 2(r_1 - r_2)\right)^2}\big(aa_3(r_1 + r_2)^2 + +ar_1(r_3 + r_4) + a_1 r_1(a+2) + 4a_2 \\ \quad + 2a_3(r_1^2 - r_2^2 + 2r_1 r_2) + 2r_1(r_3 + r_4)\big). \end{cases} \tag{138}$$

We apply the reduction method[18] by choosing r_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\dfrac{r_1 r_3}{y r_4}}, \\ \dfrac{\mathrm{d}r_3}{\mathrm{d}y} = -\dfrac{\sqrt{r_1 r_3}}{r_1\sqrt{y r_4}\left(a(r_1 + y) + 2(r_1 - y)\right)}\big(aa_3(r_1 + y)^2 + ay(r_3 + r_4) + a_1 y(a-2) - 4a_2 \\ \quad + 2a_3(r_1^2 - y^2 - 2r_1 y) - 2y(r_3 + r_4)\big), \\ \dfrac{\mathrm{d}r_4}{\mathrm{d}y} = \dfrac{1}{y\left(a(r_1 + y) + 2(r_1 - y)\right)}\big(aa_3(r_1 + y)^2 + ar_1(r_3 + r_4) + a_1 r_1(a+2) + 4a_2 \\ \quad + 2a_3(r_1^2 - y^2 + 2r_1 y) + 2r_1(r_3 + r_4)\big). \end{cases} \tag{139}$$

From the Hamiltonian $\mathcal{H}_{DIVA}$, i.e.,

$$\mathcal{H}_{DIVA} = -4r_1 y \frac{r_3 + r_4 + a_1 + a_2\left(\frac{1}{r_1} + \frac{1}{y}\right) + a_3(r_1 + y)}{(a+2)r_1 + (a-2)y} = h_0, \tag{140}$$

we can derive

$$r_3 = -\frac{4r_1r_4y + ((a+2)r_1 + (a-2)y)h_0 + 4a_1r_1y + 4a_2(r_1+y) + 4a_3r_1y(r_1+y)}{4r_1y}, \tag{141}$$

with h_0 being an arbitrary constant. Consequently, the third equation in (139) becomes

$$\frac{\mathrm{d}r_4}{\mathrm{d}y} = \frac{(a+2)h_0 + 4a_2 - 4a_3y^2}{4y^2}, \tag{142}$$

which can be easily integrated, i.e.,

$$r_4 = \frac{4r_0y - (a+2)h_0 - 4a_2 - 4a_3y^2}{4y}, \tag{143}$$

with r_0 being an arbitrary constant. Finally, we are left with the first equation in (139), i.e.,

$$\frac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\frac{(2-a)h_0 - 4a_2 - 4r_1(a_1 + r_0) - 4a_3r_1^2}{-(a+2)h_0 - 4a_2 - 4a_3y^2 + 4r_0y}}, \tag{144}$$

which could be easily solved by quadratures. If we introduce a new parameter $b_2 = -(a+2)h_0 - 4a_2$ such that $a_2 = -((a+2)h_0 + b_2)/4$ and then solve the first-order equation (144) with respect to a_1 and derive once by y, then a second-order equation is obtained. If we solve this second-order equation with respect to h_0 and derive once by y, then the following linear third-order equation is obtained:

$$(b_2 + 4r_0y - 4a_3y^2)\frac{\mathrm{d}^3r_1}{\mathrm{d}y^3} - 6(2a_3y - r_0)\frac{\mathrm{d}^2r_1}{\mathrm{d}y^2} = 0. \tag{145}$$

B. Case (B)

The Hamiltonian

$$\mathcal{H}_{DIVB} = -\frac{\sin^2(2w_1)\left(w_3^2 + w_4^2 + \frac{b_2}{\sinh^2(w_2)} + \frac{b_3}{\cosh^2(w_2)}\right) + b_1}{2\cos(2w_1) + a} \tag{146}$$

can be written in the following equivalent form with sinh and cosh replaced by exp:

$$\mathcal{H}_{DIVB} = -\frac{\sin^2(2w_1)\left(w_3^2 + w_4^2 + \frac{4b_2}{(e^{w_2} - e^{-w_2})^2} + \frac{4b_3}{(e^{w_2} + e^{-w_2})^2}\right) + b_1}{2\cos(2w_1) + a}. \tag{147}$$

We apply the reduction method[18] by choosing $r_2 = e^{w_2}$ as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}y} = \dfrac{w_3}{yw_4}, \\ \dfrac{\mathrm{d}w_3}{\mathrm{d}y} = \dfrac{N}{\sin(2w_1)w_4(y^4-1)^2[a + 2\cos(2w_1)]}, \\ \dfrac{\mathrm{d}w_4}{\mathrm{d}y} = \dfrac{4y^2\left[(b_2+b_3)y^8 + 4(b_2-b_3)y^6 + 6(b_2+b_3)y^4 + 4(b_2-b_3)y^2 + b_2 + b_3\right]}{w_4(y^4-1)^3}, \end{cases} \tag{148}$$

where

$$N = -\big[4(w_3^2+w_4^2)y^8+16(b_2+b_3)y^6-8(w_3^2+w_4^2-4b_2+4b_3)y^4+16(b_2+b_3)y^2+4(w_3^2+w_4^2)\big]\cos(2w_1)^2 \\ -2a\big[(w_3^2+w_4^2)y^8+4(b_2+b_3)y^6-2(w_3^2+w_4^2-4b_2-4b_3)y^4+4(b_2+b_3)y^2+w_3^2+w_4^2\big]\cos(2w_1) \\ +\big[-2(w_3^2+w_4^2)y^8-8(b_2+b_3)y^6+4(w_3^2+w_4^2-4b_2+4b_3)y^4-8(b_2+b_3)y^2-2(w_3^2+w_4^2)\big]\sin(2w_1)^2 \\ -2b_1y^8+4b_1y^4-2b_1. \tag{149}$$

We can solve the third equation with respect to w_4, i.e.,

$$w_4 = \pm\frac{2}{15\sqrt{15}(y^4-1)}\sqrt{3375(b_3-b_2)(y^8+1)-3375(b_3+b_2)(y^4+1)y^2-4(625b_2+81b_3)(y^4-1)^2w_0}. \tag{150}$$

Then, the first equation in (148) yields

$$w_3 = yw_4\frac{\mathrm{d}w_1}{\mathrm{d}y}, \tag{151}$$

which replaced into the second equation in (148) gives rise to a second-order equation in w_1 that we solve with respect to b_1. Then, we derive once with respect to y and the following third-order equation is obtained ($w_1 = u$):

$$\frac{\mathrm{d}^3u}{\mathrm{d}y^3} = -6\cot(2u)\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}-\frac{3}{y(y^4-1)}\frac{P_1(y)}{Q(y)}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+4\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3 \\ -\frac{6\cot(2u)}{y(y^4-1)}\frac{P_1(y)}{Q(y)}\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2+\frac{3}{y^2(y^4-1)^2}\frac{P_2(y)}{Q(y)}\frac{\mathrm{d}u}{\mathrm{d}y}, \tag{152}$$

where

$$P_1 = [(2500b_2+324b_3)w_0+3375(b_2-b_3)]y^{12}-[(7500b_2+972b_3)w_0+16\,875(b_2-b_3)]y^8-20\,250(b_2+b_3)y^6 \\ +[(7500b_2+972b_3)w_0-10\,125(b_2-b_3)]y^4-6750(b_2+b_3)y^2-(2500b_2+324b_3)w_0-3375(b_2-b_3), \tag{153}$$

$$P_2 = [(2500b_2+324b_3)w_0+3375(b_2-b_3)]y^{16}-[(10\,000b_2+1296b_3)w_0-20\,250(b_2-b_3)]y^{12} \\ -33\,750(b_2+b_3)y^{10}+[(15\,000b_2+1944b_3)w_0-47\,250(b_2-b_3)]y^8-67\,500(b_2+b_3)y^6 \\ -[(10\,000b_2+1296b_3)w_0-47\,250(b_2-b_3)]y^4-6750(b_2+b_3)y^2+(2500b_2+324b_3)w_0+3375(b_2-b_3), \tag{154}$$

$$Q = [(2500b_2+324b_3)w_0+3375(b_2-b_3)]y^8+3375(b_2+b_3)y^6-(5000b_2+648b_3)w_0y^4 \\ +3375(b_2+b_3)y^2+(2500b_2+324b_3)w_0+3375(b_2-b_3). \tag{155}$$

Equation (152) is linearizable since it admits a seven-dimensional Lie symmetry algebra. In fact, the two-dimensional Abelian intransitive subalgebra generated by the two operators

$$-\frac{\cos(2u)}{2\sin(2u)}\partial_u, \qquad \frac{1}{\sin(2u)}\partial_u \tag{156}$$

when put into the canonical form $\partial_{\tilde u}, \tilde y\partial_{\tilde u}$ yields the new dependent and independent variables, i.e.,

$$\tilde u = -\frac{1}{2}\cos(2u), \quad \tilde y = \frac{-B_2y^4-B_2+4B_3y^2+96W_0y^2}{6y^2}, \tag{157}$$

where we have introduced new constants B_2, B_3, W_0 such that

$$b_2 = \frac{B_2 - B_3 - 12W_0}{40\,500}, \quad b_3 = \frac{B_2 + B_3 + 12W_0}{40\,500}, \quad w_0 = \frac{3375(B_3 + 18W_0)}{4(353B_2 - 272B_3 - 3264W_0)}. \tag{158}$$

Then, Eq. (152) transforms into the linear equation

$$\frac{\mathrm{d}^3\tilde{u}}{\mathrm{d}\tilde{y}^3} = \frac{9}{2}\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\tilde{y}^2}\frac{B_2^2 + 48B_3W_0 + 1152W_0^2 - 72W_0\tilde{y}}{B_2^2B_3 + 24(B_2^2 + 2B_3^2)W_0 + 2304B_3W_0^2 + 27\,648W_0^3 - 3(B_2^2 + 48B_3W_0 + 1152W_0^2)\tilde{y} + 108W_0\tilde{y}^2}. \tag{159}$$

C. Case (C)

The original Hamiltonian

$$\mathcal{H}_{DIVC} = -\frac{w_3^2 + w_4^2 + \frac{c_1}{\cos^2(w_1)} + \frac{c_2}{\cosh^2(w_2)} + c_3\left(\frac{1}{\sin^2(w_1)} - \frac{1}{\sinh^2(w_2)}\right)}{\frac{a+2}{\sinh^2(2w_2)} + \frac{a-2}{\sin^2(2w_1)}} \tag{160}$$

can be written in the following equivalent form with sinh and cosh replaced by exp:

$$\mathcal{H}_{DIVC} = -\frac{w_3^2 + w_4^2 + \frac{c_1}{\cos^2(w_1)} + \frac{c_2}{\left(\frac{e^{w_2}+e^{-w_2}}{2}\right)^2} + c_3\left(\frac{1}{\sin^2(w_1)} - \frac{1}{\left(\frac{e^{w_2}-e^{-w_2}}{2}\right)^2}\right)}{\frac{a+2}{\left(\frac{e^{2w_2}-e^{-2w_2}}{2}\right)^2} + \frac{a-2}{\sin^2(2w_1)}}. \tag{161}$$

Before applying the reduction method,[18] we introduce the following transformations of dependent variables, in order to render the next calculations more amenable to computer algebraic software such as REDUCE and MAPLE, i.e.,

$$w_1 = \arccos r_1,\ w_2 = \log r_2,\ w_3 = \sqrt{r_3},\ w_4 = \sqrt{r_4}, \tag{162}$$

and then choose r_2 as a new independent variable y that gives rise to the following three equations:

$$\begin{cases} \frac{\mathrm{d}r_1}{\mathrm{d}y} = -\frac{1}{y}\sqrt{\frac{r_3(1-r_1^2)}{r_4}}, \\ \frac{\mathrm{d}r_3}{\mathrm{d}y} = -\frac{2\sqrt{r_3}N_3}{y\sqrt{1-r1^2}\sqrt{r_4}r_1D}, \\ \frac{\mathrm{d}r_4}{\mathrm{d}y} = \frac{8N_4}{(y^4-1)D}, \end{cases} \tag{163}$$

where

$$\begin{aligned} N_3 = {} & 16y^4(c_1 - c_3)(a+2){r_1}^4 + [(14c_1 + 8c_2 + 10c_3)a + 36c_1 - 16c_2 - 20c_3]y^4 \\ & + \Big[2(y-1)^2(y+1)^2\left(y^2+1\right)^2(a-2)r_3 + 2(y-1)^2(y+1)^2\left(y^2+1\right)^2(a-2)r_4 \\ & + 8y^2\big((c_2 - c_3)(a-2)y^4 + ((-4c_1 - 2c_2 - 2c_3)a - 8c_1 + 4c_2 + 4c_3)y^2 + (c_2 - c_3)(a-2)\big)\Big]{r_1}^2 \\ & - (y-1)^2(y+1)^2\left(y^2+1\right)^2(a-2)r_3 - (y-1)^2(y+1)^2\left(y^2+1\right)^2(a-2)r_4 \\ & + (c_1 - c_3)(a-2)y^8 - 4(c_2 - c_3)(a-2)y^6 - 4(c_2 - c_3)(a-2)y^2 + (c_1 - c_3)(a-2), \end{aligned} \tag{164}$$

$$
\begin{aligned}
N_4 = {} & 8y^2\big[\big(y^4+1\big)r_3+\big(y^4+1\big)r_4+2y^2(c_2-c_3)\big](a+2){r_1}^4 \\
& +8y^2\big[-\big(y^4+1\big)(r_3+r_4)+(c_1-c_3)y^4+(-2c_2+2c_3)y^2+c_1-c_3\big](a+2){r_1}^2 \\
& +(c_2-c_3)(a-2)y^8+[(-8c_1-4c_2-4c_3)a-16c_1+8c_2+8c_3]y^6+6(c_2-c_3)(a-2)y^4 \\
& +[(-8c_1-4c_2-4c_3)a-16c_1+8c_2+8c_3]y^2+(c_2-c_3)(a-2),
\end{aligned} \tag{165}
$$

$$
D=-16y^4(a+2){r_1}^4+16y^4(a+2){r_1}^2+\big(y^4-1\big)^2(a-2). \tag{166}
$$

From the Hamiltonian $\mathcal{H}_{DIVC}$, i.e.,

$$
\mathcal{H}_{DIVC}=\frac{\left\{\begin{array}{c}\Big[4\big(y^4-1\big)^2(r_3+r_4)+16y^2\big((c_2-c_3)\big(y^4+1\big)-2(c_2+c_3)y^2\big)\Big]r_1^4-4c_1\big(y^4-1\big)^2\\ -\Big[4\big(y^4-1\big)^2(r_3+r_4)-4(c_1-c_3)\big(y^8+1\big)+16(c_2-c_3)y^2\big(y^4+1\big)+8(c_1-4c_2-5c_3)y^4\Big]r_1^2\end{array}\right\}}{(y^4-1)^2(a-2)-16y^4(a+2)r_1^2({r_1}^2-1)}=h_0, \tag{167}
$$

we can derive

$$
r_3=-r_4-\frac{c_1}{r_1^2}-\frac{4y^2c_2}{(y^2+1)^2}+\frac{(4r_1^2y^2+y^4-6y^2+1)c_3}{(y^2-1)^2(r_1^2-1)}+\frac{(a-2)h_0}{4r_1^2(r_1^2-1)}-\frac{4(a+2)h_0y^4}{(y^4-1)^2}, \tag{168}
$$

with h_0 being an arbitrary constant. Consequently, the third equation in (163) becomes

$$
\frac{\mathrm{d}r_4}{\mathrm{d}y}=8y\frac{(y-1)^4(y+1)^4c_2-(y^2+1)^4c_3+2h_0y^2(y^4+1)(a+2)}{(y^4-1)^3}, \tag{169}
$$

which can be easily integrated, i.e.,

$$
r_4=4c_2\frac{(y^2+1)^2-y^2}{(y^2+1)^2}+4c_3\frac{y^4-y^2+1}{(y^2-1)^2}-2(a+2)h_0\frac{y^8+1}{(y^4-1)^2}+w_0, \tag{170}
$$

with w_0 being an arbitrary constant. If we introduce new constants C_2, C_3, C_1, A as

$$
c_2=C_2+c_3,\;\; c_3=\frac{4h_0-w_0-C_3-4C_2+2ah_0}{8},\;\; c_1=\frac{-2ah_0-C_1+4h_0}{8},\;\; a=\frac{4C_2+9C_3+w_0-A-C_1}{4h_0}, \tag{171}
$$

then we are left with the following simplified expression of the first equation in (163):

$$
\frac{\mathrm{d}r_1}{\mathrm{d}y}=\frac{y^4-1}{2yr_1}\sqrt{\frac{8C_3r_1^4-Ar_1^2-C_1}{2C_3y^8+8C_2y^6+4w_0y^4+8C_2y^2+2C_3}}, \tag{172}
$$

which could be solved by quadratures. However, if we solve it with respect to A and derive once by y, then a second-order equation is obtained that admits a three-dimensional Lie symmetry algebra $\mathrm{sl}(2,\mathbb{R})$, and as a particular case, if C_1 is equal to zero, then it is linearizable since it admits an eight-dimensional Lie symmetry algebra $\mathrm{sl}(3,\mathbb{R})$. If we solve this second-order equation with respect to C_1 and derive once by y, then the following third-order equation is obtained ($r_1\equiv u$):

$$\frac{\mathrm{d}^3 u}{\mathrm{d}y^3} = -\frac{3}{u}\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} - \frac{3\left[C_3 y^{12} - (5C_3 + 2w_0)y^8 - 24C_2 y^6 - (3C_3 + 6w_0)y^4 - 8C_2 y^2 - C_3\right]}{y(y^4-1)(C_3 y^8 + 4C_2 y^6 + 2w_0 y^4 + 4C_2 y^2 + C_3)}\left[\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{1}{u}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}y^2}\right)^2\right] + \frac{3\left[C_3 y^{16} - (6C_3 + 2w_0)y^{12} - 40C_2 y^{10} - 2(7C_3 + 10w_0)y^8 - 80C_2 y^6 - 2(7C_3 + 5w_0)y^4 - 8C_2 y^2 + C_3\right]}{y^2(y^4-1)^2(C_3 y^8 + 4C_2 y^6 + 2w_0 y^4 + 4C_2 y^2 + C_3)}\frac{\mathrm{d}u}{\mathrm{d}y}, \tag{173}$$

which is linearizable since it admits a seven-dimensional Lie symmetry algebra. In fact, the two-dimensional Abelian intransitive subalgebra generated by the two operators

$$u^{-1}\partial_u, \qquad \frac{C_3 + 2C_2 y^2 + C_3 y^4}{u y^2}\partial_u \tag{174}$$

when put into the canonical form $\partial_{\tilde{u}}, \tilde{y}\partial_{\tilde{u}}$ yield the new dependent and independent variables, i.e.,

$$\tilde{u} = \frac{u^2}{2}, \qquad \tilde{y} = \frac{C_3 + 2C_2 y^2 + C_3 y^4}{y^2}, \tag{175}$$

which transform Eq. (173) into the linear equation

$$\frac{\mathrm{d}^3 \tilde{u}}{\mathrm{d}\tilde{y}^3} = \frac{3\tilde{y}}{\omega^2 - \tilde{y}^2}\frac{\mathrm{d}^2 \tilde{u}}{\mathrm{d}\tilde{y}^2}, \tag{176}$$

with $\omega^2 = 2(2C_2^2 + C_3^2 - C_3 w_0)$.

D. Case (D)

The Hamiltonian

$$\mathcal{H}_{DIVD} = -4w_1^2 w_2^2 \frac{w_3^2 + w_4^2 + d\left(\frac{1}{w_1^2} + \frac{1}{w_2^2}\right)}{(a+2)w_1^2 + (a-2)w_2^2} \tag{177}$$

is a subcase of Hamiltonian $\mathcal{H}_{DIVA}$, with $a_1 = a_3 = 0$ and $a_2 = d$. Consequently, its corresponding Hamiltonian equations, i.e.,

$$\begin{cases} \dot{w}_1 = -\dfrac{8w_1^2 w_2^2 w_3}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\ \dot{w}_2 = -\dfrac{8w_1^2 w_2^2 w_4}{a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)}, \\ \dot{w}_3 = \dfrac{8w_1 w_2^2}{\left(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)\right)^2}\left(a w_2^2(w_3^2 + w_4^2) - 4d - 2w_2^2(w_3^2 + w_4^2)\right), \\ \dot{w}_4 = \dfrac{8w_1^2 w_2}{\left(a(w_1^2 + w_2^2) + 2(w_1^2 - w_2^2)\right)^2}\left(a w_1^2(w_3^2 + w_4^2) + 4d + 2w_1^2(w_3^2 + w_4^2)\right), \end{cases} \tag{178}$$

can be reduced to the following system of three equations, after making the substitutions (137) and choosing r_2 as a new independent variable y:

$$\begin{cases} \dfrac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\dfrac{r_1 r_3}{y r_4}}, \\ \dfrac{\mathrm{d}r_3}{\mathrm{d}y} = -\dfrac{\sqrt{r_1 r_3}}{r_1\sqrt{y r_4}(a(r_1 + y) + 2(r_1 - y))}(a y(r_3 + r_4) - 4d - 2y(r_3 + r_4)), \\ \dfrac{\mathrm{d}r_4}{\mathrm{d}y} = \dfrac{1}{y(a(r_1 + y) + 2(r_1 - y))}(a r_1(r_3 + r_4) + 4d + 2r_1(r_3 + r_4)). \end{cases} \tag{179}$$

Then,

$$r_3 = -\frac{4r_1r_4y + ((a+2)r_1 + (a-2)y)h_0 + 4d(r_1+y)}{4r_1y} \tag{180}$$

and

$$r_4 = \frac{4r_0y - (a+2)h_0 - 4d}{4y}, \tag{181}$$

and the first equation in (179) becomes

$$\frac{\mathrm{d}r_1}{\mathrm{d}y} = \sqrt{\frac{(2-a)h_0 - 4d - 4r_1r_0)}{-(a+2)h_0 - 4\ d + 4r_0y}}, \tag{182}$$

which could be easily solved by quadratures. However, if we make the simplifying substitution $d = D - (a+2)h_0/4$, solve the first-order equation (182) with respect to h_0, and derive once with respect to y, then the following linear second-order equation is obtained:

$$2(r_0y - D)\frac{\mathrm{d}^2r_1}{\mathrm{d}y^2} + r_0\frac{\mathrm{d}r_1}{\mathrm{d}y} + r_0 = 0. \tag{183}$$

VIII. CONCLUSIONS

In this paper, 19 classical superintegrable systems in two-dimensional non-Euclidean spaces are shown to possess hidden symmetries leading to linearity. This fulfills the conjecture that we made in Ref. 1, namely that all classical superintegrable systems in two-dimensional space hide linearity regardless of the separation of variables of the corresponding Hamilton–Jacobi equation and of the order of the first integrals.

In some cases, we have used the Hamiltonian in order to derive one of the two momenta as a function of the other momentum and coordinates. None of the other two known first integrals have been used. In other cases, one of the equations of the Hamiltonian system could be integrated by quadrature, and that was all we needed in order to then find the hidden symmetries leading to the linear equation of either second or third order.

As we stated in Ref. 1, it remains an open-problem to see if linear equations are hidden in (maximally) superintegrable systems in $N > 2$ dimensions, regardless of the separability of the corresponding Hamilton–Jacobi equation and the degree of the known first integrals.

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DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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