Stochastic Logistic Shocks and Economic Growth

1st Davide La Torre
Dubai Business School
University of Dubai
Dubai, UAE
dlatorre@ud.ac.ae
and
Department of Economics,
Management, and Quantitative Methods
University of Milan
Milan, Italy
davide.latorre@unimi.it

2nd Danilo Liuzzi
Department of Economics and Management
University of Cagliari
Cagliari, Italy
danilo.liuzzi@unica.it

3rd Simone Marsiglio
School of Accounting,
Economics and Finance
University of Wollongong
Wollongong, Australia
simonem@uow.edu.au

***Abstract*—We present an alternative to the geometric Brownian motion in order to model random shocks in economics, by focusing on the stochastic logistic process, which is a natural generalization of the geometric Brownian motion. We describe some potential applications in the context of economic growth, and show that its degree of tractability is very similar to that of the geometric Brownian motion, and thus its use can effectively improve the limits (related to the presence of a constant drift) of the geometric Brownian motion to model uncertainty.**

***Index Terms*—Stochastic Logistic Process, Economic Growth, Solow Model, Ramsey model**

I. Introduction

Uncertainty nowadays is everywhere and its implications for economic decisions are more important than ever. The large degree of randomness surrounding demographic change because of international migration, and environmental problems related to climate change are just two examples of the growing need to extend macroeconomic analysis to consider the implications of uncertain events. Indeed, the study of stochastic problems has received growing attention lately, especially in the optimal growth literature which traditionally provides a simple and useful framework for analyzing the dynamic evolution of capital and other important macroeconomic variables. Standard neoclassical growth theory, founded on the pioneering works of Ramsey (1928) and Solow (1956), has been recently extended to consider the extent to which uncertainty about the evolution of certain production factors affects the model's outcome. Specifically, several works focus on the implications of randomness in the dynamics of the labor force (Smith, 2007; Marsiglio and La Torre, 2012b; Marsiglio, 2014a), technology (Smith, 2007; Bucci et al. 2011; Hiraguchi, 2013) and environmental issues (La Torre et al., 2017; Marsiglio and La Torre, 2018) discussing how the presence of uncertainty impacts on economic performance. As a matter of tractability and in order to maintain the analysis as simple as possible, the largest share of these studies assumes that the underlying stochastic process is a geometric Brownian motion. Such an approach requires the drift component of the random process to be constant[1], and thus does not allow to take into account important aspects like those related to the carrying capacity concept of any (economic) resource; see for example Beltratti et al. (1995) and Brida and Accinelli (2007) for a discussion in the context of natural resources and demographic growth, respectively. In order to allow for the drift component to be time varying and at the same time to maintain a certain degree of tractability we modify the stochastic component by considering the stochastic logistic process, which is a natural generalization of the geometric Brownian motion. We thus focus on the stochastic logistic process, which allows to describe with more depth the drift component of certain economic problems (like natural resources and demographic growth) and shares some nice features with the geometric Brownian motion (analytical solutions). The process is very well known in the mathematics and natural sciences literature, but to the best of our knowledge it has not been introduced in an economic growth setup yet.

This brief paper proceeds as follows. Section II presents the stochastic logistic process and summarizes its most important characteristics comparing them to those of the geometric Brownian motion. Section III describes a direct application in economic growth, in particular in the Solow (1956) model, which is suitable to model the dynamics of demography (or equivalently, under the standard assumption of full employment, labor force) and natural resources. In section IV we show that also in a Ramsey-type framework it is possible to derive an analytical solution, similarly to what happens when the shock component follows a geometric Brownnian motion. Section V as usual presents concluding remarks.

[1]Note that also in models with agents optimizing their behavior, even if the drift is not a priori imposed to be constant, the optimal solution (imposed by the necessity to solve in closed form an Hamilton-Jacobi-Bellman equation) generally requires that it actually is constant. See, among others, Marsiglio (2014).

II. THE STOCHASTIC LOGISTIC PROCESS

Let us denote by $L(t)$ the total population at the time t and suppose its evolution over time is driven by the following stochastic differential equation:

$$dL(t) = (n - bL(t))L(t)dt + \sigma L(t)dW \quad (1)$$

where $b, \theta, \sigma \geq 0$. and dW is the classical Wiener process. In the statistical literature this process is known as the geometric mean reversion or stochastic Verhulst diffusion. It takes its name from the Belgian demographer Verhulst (1938) who used the drift component:

$$dL(t) = (n - bL(t))L(t)dt \quad (2)$$

as a deterministic model of population growth. Verhulst's work extended a previous paper by Malthus (1798) who was among the first to observe the existence of two different phases in the growth of world population. Researchers from difference disciplines have studied extensively generalizations of the Verhulst's model and used them to forecast the annual population growth rate which is expected to vary from 1.8% between 1950-2000 to 0.9% between 2000-2050 and then to 0.2% between 2050 and 2100. The analysis of stochastic models subject to external shocks can help demographers to describe the effect of migration waves on developed economies.

Let $f[L(s), s; L(t), t]$ be the probability density of $L(s)$ at time s, conditional upon its value $L(t)$ at time t. This is driven by the Fokker-Planck equation, which for (1) takes the form:

$$\frac{\partial f}{\partial t} = -\frac{\partial \left[f(n - bL)L\right]}{\partial L} + \frac{1}{2}\sigma^2 \left(\frac{\partial^2 \left[fL^2\right]}{\partial L^2}\right). \quad (3)$$

The steady state density $f[L(\infty), \infty, s; L(t), t]$ is obtained by setting $\frac{\partial f}{\partial t} = 0$, yielding a second order ordinary differential equation for f:

$$0 = -\frac{\partial \left[f(n - bL)L\right]}{\partial L} + \frac{1}{2}\sigma^2 \left(\frac{\partial^2 \left[fL^2\right]}{\partial L^2}\right) \quad (4)$$

with the regular solution:

$$f[L(\infty), \infty, s; L(t), t] = L^{d-1}e^{-cL}(c)^d/\Gamma(d); \quad (5)$$

which we recognise as the scale Gamma distribution. The mean of this distribution is: $\frac{v-1}{c} = (\theta - \frac{\sigma^2}{2b})$ and the variance $\frac{\theta\sigma^2}{2b} - \frac{\sigma^4}{4b^2}$. These are strictly positive under $d = \frac{2n}{\sigma^2} - 1 > 0$. The following result compares the expected value of the solution to (1) with the solution to (2).

Theorem. Let $L(t)$ be the solution to (1) and $\bar{L}(t)$ be the solution to (2). Then $\mathbb{E}(L(t)) \leq \bar{L}(t)$ for all $t \geq 0$.

Proof. Since $L(t)$ is solution to (1), this implies that

$$L(t) - L(0) = \int_0^t (n - bL(s))L(s)ds + \int_0^t \sigma L(s)dW(s).$$

Taking the expected values of both sides and recalling that $\mathbb{E}(\int_0^t \sigma L(s)dW(s)) = 0$, we obtain

$$\mathbb{E}(L(t)) = L(0) + \int_0^t \mathbb{E}\left[(n - bL(s))L(s)\right] ds =$$

$$\int_0^t n\mathbb{E}(L(s))ds - \int_0^t b\mathbb{E}(L^2(s))ds$$

which is equivalent to

$$\frac{d\mathbb{E}(L(t))}{dt} = n\mathbb{E}(L(t)) - b\mathbb{E}(L^2(t)), \quad E(L(0)) = L(0)$$

Using Jensen's inequality it is straightforward to prove that

$$\frac{d\mathbb{E}(L(t))}{dt} = n\mathbb{E}(L(t)) - b\mathbb{E}(L^2(t)) \leq$$

$$n\mathbb{E}(L(t)) - b\mathbb{E}(L(t))^2$$

and, using a classical comparison theorem, the thesis follows.

Several empirical estimates of the two parameters n and b have been provided by several authors showing that the values the two parameters take vary dramatically from country to country and from region to region. In particular, the largest variability across these estimates can be found between industrialized and developing areas. For example, Marsiglio and La Torre (2012) use a fractal-based method to estimate these parameters across different continents, showing that n tends to be higher and b lower in industrialized (Europe, North America, Australia) than in developing areas (Africa, Asia, South America). Clearly such an aggregate variability is more strongly reflected in the variability at single county level. In the case of UK, for instance, La Torre and Marsiglio (2010) obtained the values $n = 0.15565694$ and $b = 0.00000869$. Figure 1 shows the evolution of the British population using the above estimated parameters and the stochastic logistic equation.

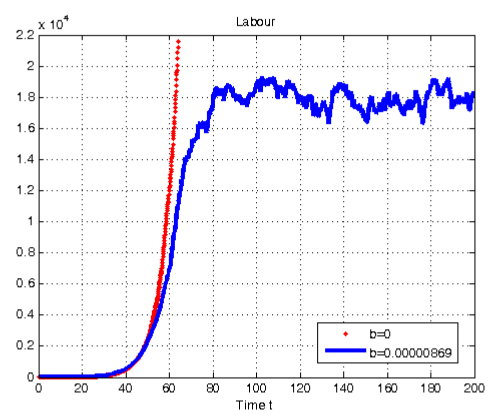

Fig. 1. Evolution of population L.

III. STOCHASTIC LOGISTIC SHOCKS IN THE SOLOW MODEL

Since modeling demography or natural resources in a Solow framework does not make any substantial difference we focus on population as a matter of expositional simplicity. However, by interpreting population as natural resources (and per capita capital as the input ratio) exactly the same comments apply.

Consider an economy in which the production function takes a Cobb-Douglas form, that is $Y(t) = AK(t)^\alpha L(t)^{1-\alpha}$ where A measures the total factor productivity, $K(t)$ is the capital stock and $L(t)$ is the labor force, meaning the capital and labor are the only inputs in the production of the final consumable good $Y(t)$. Consider the following stochastic version of the canonical Solow (1956) model:

$$\dot{K}(t) = sAK(t)^\alpha L(t)^{1-\alpha} - \delta K(t) \quad (6)$$
$$dL(t) = L(t)(n - bL(t))dt + \sigma L(t)dW(t), \quad (7)$$

where s and δ are the saving and depreciation rates respectively, while $L(t)$ is thus a random variable driven by a stochastic logistic process. Applying Ito's lemma yields the stochastic differential equation for per capita capital, $k(t) = \frac{K(t)}{L(t)}$:

$$\frac{dk(t)}{k(t)} = [sAk(t)^{\alpha-1} \quad (8)$$
$$- \quad \delta + bL(t) - n + \sigma^2]dt - \sigma dW(t)$$
$$dk = \{b + sAk^\alpha - (\delta + n - \sigma^2)k\}dt - \sigma kdz \quad (9)$$

The following figures show the result of a numerical simulation of the previous equations, by relying on the demographic parameter values discussed in the previous section and standard macroeconomic parameter values (Barro and Sala–i–Martin, 2004). Specifically, we set: $s = 0.2$, $A = 1$, $\alpha = 0.33$, $\delta = 0.05$, $n = 0.15565694$, $b = 0.00000869$, $\sigma = 0.5$, $K(0) = 1$ and $L(0) = 1$. More precisely, Figure 2 shows the evolution of capital K and Figure 3 the evolution of capital and population in the capital and population plane.

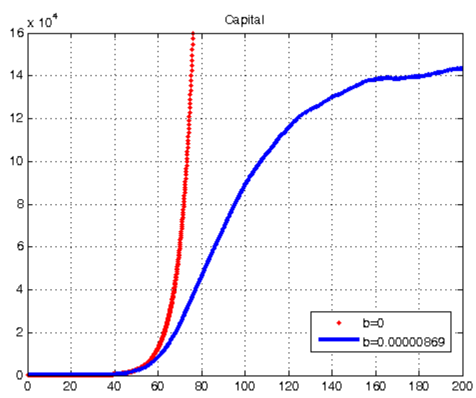

Fig. 2. The evolution of physical capital K

IV. Stochastic Logistic Shocks in the Ramsey Model

Now consider a Ramsey setting, in which the representative agent determines how much to consume given the dynamics of capital and the productive factor in order to maximize its (expected) intertemporal wellbeing. The welfare function is the discounted sum (ρ is the discount factor) of total utilities, given by the product between instantaneous utilities and the altruism-weighted population size, where $\epsilon \in [0, 1]$ is a measure of altruism; the instantaneous utility function depends only on per capita consumption, $c(t)$, and takes the following iso-elastic form: $u[c(t)] = \frac{c(t)^{1-\phi}}{1-\phi}$, with $\phi > 0, \neq 1$ being the inverse of the intertemporal elasticity of substitution. The dynamic resource constraint implies that output is entirely allocated between investment (inclusive of replacement investment) and consumption as follows: $Y(t) = \dot{K}(t) - \delta K(t) - c(t)L(t)$. Therefore, given the initial conditions $K(0) > 0$ and $L(0) > 0$, the representative agent's intertemporal optimization problem is as follows:

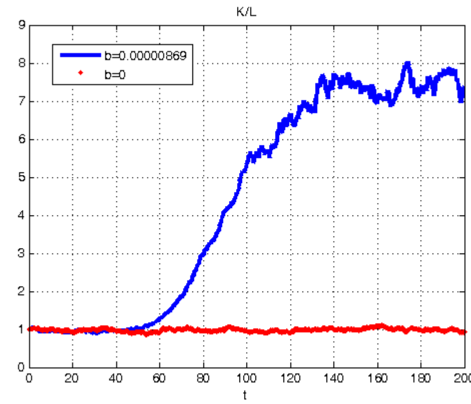

Fig. 3. Capital and labour phase portrait

$$\max_{c(t)} \mathbb{E}[W] = \mathbb{E}\left[\int_0^\infty \frac{c(t)^{1-\phi}}{1-\phi} L(t)^{1-\epsilon} e^{-\rho t} dt\right] \quad (10)$$

such that

$$\dot{K}(t) = AK(t)^\alpha L(t)^{1-\alpha} - \delta K(t) - c(t)L(t)$$
$$dL(t) = L(t)(n - bL(t))dt + \sigma L(t)dW(t)$$

In the welfare function given by expression (10) the term $1 - \epsilon$ represents the degree of altruism, which ultimately determines the type of welfare function employed. In fact, if $\epsilon = 1$ ($\epsilon = 0$), the welfare is defined according to the average (total) utilitarianism (see Marsiglio and La Torre, 2012; Marsiglio, 2014).

The following proposition states that it is possible to explicitly determine the optimal paths of per capita consumption and capital under a specific parameters configuration.

Theorem. Assume $\phi = \alpha = \epsilon$; then the Hamilton-Jacobi-Bellmann (HJB) equation associated with the problem (10) exhibits the following closed form solution:

$$J(K(t), L(t)) = \frac{1}{1-\alpha}\left[\frac{\alpha}{\rho + (1-\alpha)\delta}\right]^\alpha K(t)^{1-\alpha}$$
$$- \frac{1}{\rho(1-\alpha)} + g(L) \quad (11)$$

The optimal path of per capita consumption and (aggregate) capital are given, respectively, by:

$$\frac{c(t)}{k(t)} = \frac{\rho + (1-\phi)\delta}{\phi} \quad (12)$$

$$K(t) = K(0)e^{\frac{A-\delta-\rho}{\phi}t} \quad (13)$$

Proof. Let J be the value function associated with this stochastic optimization problem. The HJB equation can be written as:

$$\begin{aligned} 0 &= \max_c \left\{ \frac{c^{1-\phi}}{1-\phi} L^{1-\epsilon} - \rho J \right. \\ &\left. + J_K \dot{K} + J_L L(n-bL) + \frac{J_{LL}\sigma^2 L^2}{2} \right\} \end{aligned} \quad (14)$$

The first order necessary optimality condition requires:

$$c = [J_K L^{\epsilon}]^{-\frac{1}{\phi}}, \quad (15)$$

which substituted into (14) yields:

$$\begin{aligned} 0 &= \frac{[J_K L^{\epsilon}]^{-\frac{1-\phi}{\phi}}}{1-\phi} L^{1-\epsilon} - \rho J + J_K(AK^{\alpha}L^{1-\alpha} - \delta K \\ &- [J_K L^{\epsilon}]^{-\frac{1}{\phi}} L) + J_L L(n-bL) + \frac{J_{LL}\sigma^2 L^2}{2} \end{aligned} \quad (16)$$

Under the hypothesis $\epsilon = \phi$, this equation can be simplified as:

$$\begin{aligned} 0 &= \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} - \rho J \quad (17) \\ &+ J_K(AK^{\alpha}L^{1-\alpha} - \delta K) + J_L L(n-bL) + \frac{J_{LL}\sigma^2 L^2}{2} \end{aligned}$$

We now look for a solution to (17) of this form:

$$J(K,L) = \Phi_K K^{\theta_K} - \frac{1}{\rho(1-\phi)} + g(L)$$

where Φ_K and θ_K are parameters which have to be determined and g is an unknown function. If $\theta_K = 1-\alpha$, it is then possible to split equation (17) into two differential equations which involve, respectively, K and L. Since only the first equation is relevant to determine the optimal paths of c and K, we do not consider the other involving L in the following calculations. By substitution we get:

$$0 = \frac{\phi}{1-\phi}\left[(1-\alpha)\Phi_K\right]^{\frac{\phi-1}{\phi}} K^{\frac{-\alpha(\phi-1)}{\phi}} \quad (18)$$

$$-\rho\Phi_K K^{1-\alpha} - \delta(1-\alpha)\Phi_K K^{1-\alpha}$$

If $\alpha = \phi$, then the equation is satisfied whenever:

$$\Phi_K = \frac{1}{1-\alpha}\left[\frac{\alpha}{\rho + (1-\alpha)\delta}\right]^{\alpha}$$

Exactly the same parametric configuration, $\phi = \alpha = \epsilon$, stating that the capital share is equal to both the inverse of the intertemporal elasticity of substitution and the altruism parameter is obtained by Marsiglio and La Torre (2012a, 2012b) in a two-sector model where demographic shocks are driven by a geometric Brownian motion.

By easy considerations it is possible to determine the expected value of k, which reads as

$$\begin{aligned} \mathbb{E}[k(t)|L(0),K(0)] &= \mathbb{E}\left[\frac{K(t)}{L(t)}|X(0)\right] \\ &= K(t)\mathbb{E}[X(t)|X(0)] \\ &= K(t)[X(0)\exp[-\alpha t] \\ &+ \theta(\exp[\alpha t]-1)] \\ &= X(0)K(0)\exp\left\{\left[\frac{A-\delta-\rho}{\phi} - \alpha\right]t\right\} \\ &+ \theta K(0)(\exp[\alpha t]-1) \\ &\quad \exp\left\{\left[\frac{A-\delta-\rho}{\phi}\right]t\right\} \end{aligned}$$

where $X(t) = \frac{1}{L(t)}$

$$\begin{aligned} \frac{dX}{X} &= \left(\frac{dL}{L}\right)^2 - \frac{dL}{L} \\ dX &= \{b - (n+\sigma^2)X\}dt - \sigma X dW(t) \\ &= \alpha(\theta - X)dt + \sigma X dW(t) \end{aligned} \quad (19)$$

and:

$$dW(t) = -dW(t); \alpha = (n+\sigma^2); \theta = b/(n+\sigma^2). \quad (20)$$

It is easy to prove that:

$$X(t) = C\exp[-(\alpha - \sigma^2/2)t + \sigma W(t)] +$$

$$\alpha\theta\int_0^t \exp[(\alpha-\sigma^2/2)(t-s) + \sigma(W(t)-W(s))]ds$$

which can be readily translated back into the parameters of the original problem. Using the fact that the expectation of an integral is (under certain regularity conditions which hold here) the integral of the expectations and $E_0[\exp[\sigma W(t)]] = \exp[\sigma^2 t/2]$:

$$\begin{aligned} \mathbb{E}[X(t)] &= \mathbb{E}[C\exp[-(\alpha-\sigma^2/2)t + \sigma W(t)] \\ &+ \alpha\theta\int_0^t \exp[(\alpha-\sigma^2/2)(t-s) \\ &+ \sigma(W(t)-W(s))]ds] \\ &= C\exp[-(\alpha-\sigma^2/2)t]\mathbb{E}[\exp[\sigma W(t)]] \\ &+ \alpha\theta\int_0^t \exp[(\alpha-\sigma^2/2)(t-s)] \\ &\quad \mathbb{E}[\exp[\sigma(W(t)-W(s))]ds \\ &= C\exp[-\alpha t] + \alpha\theta\int_0^t \exp[\alpha(t-s)]ds \\ &= C\exp[-\alpha t] + \alpha\theta\int_0^t \exp[\alpha u]du \\ &= C\exp[-\alpha t] + \theta(\exp[\alpha t]-1). \end{aligned}$$

where $u = t - s$. This gives a basic result for the capital labour ratio which drives output per capita:

$$\begin{aligned}\mathbb{E}[k(t)|L(0),K(0)] &= K(t)\mathbb{E}[X(t)|X(0)] \\ &= K(t)[X(0)\exp[-\alpha t] \\ &+ \theta(\exp[\alpha t]-1)] \\ &= K(0)X(0)\exp[-\alpha t+(sA-\delta)] \\ &+ \theta K(0)\exp\{(sA-\delta)t\}(\exp[\alpha t]-1)\end{aligned}$$

Can we make a plot of per capita capital and the expected per capita capital just to use the above expression and add more comments? Also, we could add in the figures another line to show the geometric Brownian motion case (setting $b = 0$). This allows us to add some more comments and respond to the referee.

V. Conclusion

We have presented and analyzed a stochastic logistic process, which is a natural generalization of the geometric Brownian motion. We have also described some applications in economic growth, namely to the Solow and the Ramsey models. In both contexts, we have modeled the behavior of population through a stochastic logistic process which better describes the evolution of population dynamics subject to uncertainty.

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