

ZERO-CYCLE GROUPS ON ALGEBRAIC VARIETIES

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ABSTRACT. We compare various groups of 0-cycles on quasi-projective varieties over a field. As applications, we show that for certain singular projective varieties, the Levine-Weibel Chow group of 0-cycles coincides with the corresponding Friedlander-Voevodsky motivic cohomology. We also show that over an algebraically closed field of positive characteristic, the Chow group of 0-cycles with modulus on a smooth projective variety with respect to a reduced divisor coincides with the Suslin homology of the complement of the divisor. We prove several generalizations of the finiteness theorem of Saito and Sato for the Chow group of 0-cycles over p -adic fields. We also use these results to deduce a torsion theorem for Suslin homology which extends a result of Bloch to open varieties.

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1. INTRODUCTION

The goal of this work is to clarify the relation between various groups of 0-cycles associated to algebraic varieties and prove some new properties of these groups. One of the questions we want to address is whether one can isolate a class of algebraic varieties for which one could describe the a priori non- \mathbb{A}^1 -invariant motivic cohomology groups (such as the Levine-Weibel Chow group and the Chow group with modulus) in terms of the — more classical — \mathbb{A}^1 -invariant motivic cohomology groups such as the ones defined by Friedlander, Suslin and Voevodsky. We answer this question for a projective variety which has either normal crossing singularities or it is obtained as the double of a smooth variety along a reduced divisor. The interest in the latter case originates from the fact that it allows us to describe the Chow group of 0-cycles with modulus on smooth varieties with reduced modulus in terms of the singular homology à la Suslin-Voevodsky. In the following subsections, we describe our main results and their applications.

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1.1. Cycles with modulus vs. Suslin homology. Let k be a field and X a smooth projective variety over k . Let $D \subset X$ be an effective Cartier divisor and let $U = X \setminus D$. Let $\mathrm{CH}_0(X|D)$ denote the Chow group of 0-cycles with modulus (see [32] or [7]). These groups are supposed to provide the cycle theoretic description of the relative algebraic K -theory in the same way as the classical Chow groups do for the non-relative K -theory: for the case of 0-cycles this relationship has now been established in many important cases (see e.g., [6], [23], [24]). The relative K -theory is an important tool to study the ordinary K -theory of singular varieties by embedding them inside smooth varieties. It is therefore important to know the structure of the Chow groups with modulus under various special conditions on the base field and the variety.

It is not hard to see that there is a natural surjection $\lambda_{X|D}: \mathrm{CH}_0(X|D) \twoheadrightarrow H_0^{sus}(U)$, where the latter is the 0-th Suslin homology of U (also known as the Suslin-Voevodsky singular homology). It acts, in many ways, as a replacement of the classical Chow group of 0-cycles for non-proper varieties over k (see [52] for an early example of this philosophy).

Since $H_0^{sus}(U)$ is part of the well developed theory of motivic cohomology with compact support, there are many tools to study and compute it. On the other hand, the nature of the *relations* defining $\mathrm{CH}_0(X|D)$ in some cases can be used to grasp new information about $H_0^{sus}(U)$ using $\lambda_{X|D}$. It is therefore important to get conditions under which the above surjection is an isomorphism. We give a complete answer to this when k is algebraically closed and a partial answer in general.

Theorem 1.1. *Assume that k is algebraically closed and D is reduced¹. Then the map*

$$\lambda_{X|D}: \mathrm{CH}_0(X|D) \rightarrow H_0^{sus}(U)$$

has the following properties.

- (1) *It is an isomorphism if $\mathrm{char}(k) > 0$.*
- (2) *It is not always an isomorphism if $\mathrm{char}(k) = 0$.*

Theorem 1.2. *Assume that k is perfect and the irreducible components of D_{red} are regular. Then the following hold.*

- (1) *If $\mathrm{char}(k) = p > 0$, then*

$$\lambda_{X|D}: \mathrm{CH}_0(X|D)\left[\frac{1}{p}\right] \rightarrow H_0^{sus}(U)\left[\frac{1}{p}\right]$$

is an isomorphism.

- (2) *If $\mathrm{char}(k) = 0$, then*

$$\lambda_{X|D}: \mathrm{CH}_0(X|D)/m \rightarrow H_0^{sus}(U)/m$$

is an isomorphism for every integer $m \neq 0$.

Part (2) Theorem 1.1 may appear surprising, because one would expect $\mathrm{CH}_0(X|D)$ to behave like an \mathbb{A}^1 -invariant theory if D is reduced. One application of this is that it implies that the Chow groups with modulus defined by Russell [47] does not in general coincide with the above Chow group with modulus. It also implies that Russell's Chow groups can not satisfy Bloch's formula in terms of the top Zariski/Nisnevich cohomology of Milnor or Quillen K -sheaves.

¹ $\lambda_{X|D}$ is almost never an isomorphism with integral coefficients if D is not reduced.

1.2. Bloch's torsion theorem for Suslin homology. Let U be a quasi-projective variety of dimension $d \geq 0$ over an algebraically closed field k which admits a smooth compactification. Let $\ell \neq \text{char}(k)$ be a prime number. If U itself is complete, then Bloch's torsion theorem [8] (established independently from Roitman's theorem) affirms the existence of an isomorphism

$$\text{CH}_0(U)\{\ell\} \xrightarrow{\cong} H_{\text{ét}}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)).$$

This provides a simple description of the prime-to- p torsion in the Chow group of zero cycles in terms of an étale cohomology group. This isomorphism is very subtle because there is no obvious map in either direction and its construction is one of the main challenges.

When U is no longer complete, but admits a smooth compactification as indicated above, we can replace the Chow group $\text{CH}_0(U)$ by $H_0^{\text{sus}}(U)$, as explained in the previous paragraph. An extension of Roitman's torsion theorem for $H_0^{\text{sus}}(U)$, providing an isomorphism between the prime-to- p torsion part of $H_0^{\text{sus}}(U)$ and the corresponding torsion in the semi-abelian Albanese variety of U , has been proved by Spieß and Szamuely [52]. As a consequence of our main result, we also get the following theorem (see Remark 5.5 for a different approach). Let $H_{c,\text{ét}}^*(U, \mathcal{F})$ denote the étale cohomology of U with compact support (e.g., see [45]).

Theorem 1.3. *Let U be a quasi-projective variety of dimension $d \geq 0$ over an algebraically closed field such that U admits a smooth compactification. Then there is an isomorphism*

$$H_0^{\text{sus}}(U)\{\ell\} \xrightarrow{\cong} H_{c,\text{ét}}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)).$$

Remark 1.4. When U is complete, the p -adic analogue of Theorem 1.3 can be deduced from [21, Cor. 3.18] using [17, Thm. 8.1]. We hope to address this question in the non-complete case in a future project. It is also an interesting question if it is possible to eliminate the condition in Theorem 1.3 that U admits a smooth compactification. We expect this to be the case.

1.3. Finiteness of 0-cycle groups. To motivate our next set of results, suppose that k is a p -adic field and $m \geq 1$ is an integer such that $p \nmid m$. A conjecture of Colliot-Thélène predicted that if X is a smooth projective variety over k , then $\text{CH}_0(X)/m$ is finite. Such a finiteness result was previously known over finite and algebraically closed fields (without any restriction on m).

The conjecture of Colliot-Thélène was affirmatively answered by Saito and Sato [48] (see also Bloch's appendix to [11] and [12] for simplified proofs). It is natural to expect that a similar finiteness result would hold even if X is not complete provided we replace $\text{CH}_0(X)/m$ by $H_0^{\text{sus}}(X)/m$. In the same spirit, one would expect that $\text{CH}_0(X|D)/m$ is finite for X projective, if we interpret $\text{CH}_0(X|D)$ as a Chow group with compact support for $X \setminus |D|$.

We give positive answers to both questions. Furthermore, we show that the finiteness result of Saito and Sato also holds for singular schemes. The question of finiteness of Suslin homology with finite coefficients arose during a conversation of authors with Colliot-Thélène on his conjecture.

For a reduced quasi-projective scheme X over a field, let $\text{CH}_0^{LW}(X)$ denote the Levine-Weibel Chow group of X (see § 2).

Theorem 1.5. *Let X be an integral quasi-projective scheme over k which is regular in codimension one. Let $D \subset X$ be an effective Cartier divisor. We then have the following.*

- (1) $H_0^{sus}(X)/m$ is finite if X is smooth.
- (2) $\mathrm{CH}_0(X|D)/m$ is finite if X is smooth and complete.
- (3) $\mathrm{CH}_0^{LW}(X)/m$ is finite if X is complete.

We show that the above finiteness result also holds over finite and algebraically closed fields (see § 6).

1.4. Levine-Weibel Chow group vs. motivic cohomology. Let k be a field of exponential characteristic $p \geq 1$. Let Λ be a commutative ring, and assume that p is invertible in Λ or that k admits resolution of singularities. Let X be a reduced quasi-projective k -scheme of dimension $d \geq 0$.

Let $H^{2d}(X, \Lambda(d))$ denote the Friedlander-Voevodsky motivic cohomology of X (see § 2 for a reminder of the definition). It was shown in [37] that there is a canonical map

$$(1.1) \quad \lambda_X: \mathrm{CH}_0^{LW}(X) \otimes_{\mathbb{Z}} \Lambda \rightarrow H^{2d}(X, \Lambda(d)),$$

where Λ is as above. Note that, a priori, the source and the target of λ_X are very different objects: on one side we have a group of cycles, defined by generators and relations (in particular, no topology is involved), while on the other side we have a group defined either as a Hom-group in a motivic category, or as the cdh-hypercohomology of a certain complex of sheaves.

It is well known that this map can not be an isomorphism for arbitrary varieties: for instance, $H^{2d}(X, \Lambda(d))$ does not see the difference between X and its semi-normalization. On the other hand, one can easily show for a curve X with a cusp singularity that $\mathrm{CH}_0^{LW}(X) \otimes_{\mathbb{Z}} \Lambda$ changes after passing to the semi-normalization of X .

On the other hand, it is a result of Voevodsky that for regular X , both groups are isomorphic, as a (very) special case of the identification of motivic cohomology with Bloch's higher Chow groups. It is therefore reasonable to ask if there is a specific class of singular varieties or, rather, a specific class of singularities, for which λ_X is an isomorphism (for instance, we have already observed that semi-normality is a necessary condition).

In this paper, we prove the following positive result when X has normal crossings singularities (see Definition 2.12).

Theorem 1.6. *Assume that k is an infinite perfect field and X is a normal crossing projective variety of dimension $d \geq 0$ over k . Then λ_X is an isomorphism in the following cases.*

- (1) $p > 1$ and $\Lambda = \mathbb{Z}[\frac{1}{p}]$.
- (2) $m \in k^\times$ and $\Lambda = \mathbb{Z}/m$.

We prove a similar isomorphism when X is the join of a smooth projective variety along a reduced divisor if k is algebraically closed, see Theorem 3.5.

Note that $H^{2d}(X, \Lambda(d))$ is not known to have a presentation in terms of generators and relations in general. In [12, Thm. 7.1], a variant (denoted by $\mathrm{CH}_0^{EKW}(X)$ in this paper) of the Levine-Weibel Chow group was defined for X as in Theorem 1.6 in terms of generators and relations. It was shown that $\mathrm{CH}_0^{EKW}(X) \otimes_{\mathbb{Z}} \Lambda$ is isomorphic to $H^{2d}(X, \Lambda(d))$. Theorem 1.6 implies that under its hypothesis, there is an isomorphism

$$\mathrm{CH}_0^{EKW}(X) \otimes_{\mathbb{Z}} \Lambda \xrightarrow{\cong} \mathrm{CH}_0^{LW}(X) \otimes_{\mathbb{Z}} \Lambda.$$

As a consequence, it follows that $\mathrm{CH}_0^{EKW}(X) \otimes_{\mathbb{Z}} \Lambda$ can be defined in terms of only one type of relations instead of two types of relations used in [12].

We end the discussion of our main results with yet another consequence. Let X be a smooth projective variety of dimension $d \geq 2$ over an infinite perfect field and let $D \subset X$ be a simple normal crossing divisor. Let $\iota: D \hookrightarrow X$ be the inclusion. By the classical moving lemma for Chow groups, one knows that $\mathrm{CH}_1(X)$ is the quotient of the free abelian group $\mathcal{Z}_1(X)_D$ of integral 1-dimensional cycles on X which do not meet D_{sing} by a suitable rational equivalence relation. We have the obvious pull-back map $\iota^*: \mathcal{Z}_1(X)_D \rightarrow \mathcal{Z}_1(D, D_{\mathrm{sing}})$ (see § 2.1) defined by taking the scheme theoretic intersection with D . But it is not clear if it descends to a map between the Chow groups. As a consequence of Theorem 1.6, we can conclude the following.

Corollary 1.7. *Let X and D be as above, and let $\Lambda = \mathbb{Z}[\frac{1}{p}]$ if $p > 1$ or $\Lambda = \mathbb{Z}/m$ with $m \in k^\times$. Then the map $\iota^*: \mathcal{Z}_1(X)_D \rightarrow \mathcal{Z}_1(D, D_{\mathrm{sing}})$ descends to a homomorphism*

$$\iota^*: \mathrm{CH}_1(X) \otimes_{\mathbb{Z}} \Lambda \rightarrow \mathrm{CH}_0^{LW}(D) \otimes_{\mathbb{Z}} \Lambda.$$

1.5. Overview of proofs. To prove the first part of Theorem 1.1, the key ingredients are the Roitman torsion theorems of [35] and [52], and the Lefschetz hyperplane section theorem for Serre's Albanese variety from [20]. Using these, the theorem is deduced from Proposition 4.2. The latter is a general result of independent interest and estimates the kernel of the map from the Chow group with modulus to Suslin homology. We prove the second part of Theorem 1.1 by constructing a specific example.

The proof of Theorem 1.2 uses three main tricks. The first is that in order to compare Chow group with modulus with Suslin homology, we need to replace the integral normal curves used in defining the Suslin homology by some embedded curves. To take care of this, we pass from X to a projective space over it where the curve is automatically embedded and then use the covariant functoriality of the Chow group with modulus. The second trick is to perform a series of monoidal transformations on X in order to make the given integral curve transverse with respect to the modulus divisor D . We achieve this using a result of Jannsen (see the appendix of [48]). The last trick is to apply a result of Miyazaki [46] which allows us to assume that the pull-back of the divisor under blow-ups is reduced.

We prove Theorem 1.3 by first reducing it to the case of surfaces. We solve the latter case by translating the problem to a torsion theorem for the Chow group of a singular surface, namely, the double of X along D . We solve this problem using the methods of the torsion theorems of [35].

We deduce part (1) of Theorem 1.5 from a general finiteness result (see Proposition 6.2) that we prove in this paper. This result applies to all projective schemes over certain fields. Using this, the finiteness of the Suslin homology follows using the localization sequence for motivic cohomology with compact support. The part (2) of Theorem 1.5 follows from Theorem 1.2 and part (1) of Theorem 1.5 after we make D reduced using a theorem of Miyazaki [46], and a divisor with normal crossings using a sequence of blow ups with smooth centers. We deduce part (3) of Theorem 1.5 from its part (1) and an independent result Theorem 6.5 which compares the Levine-Weibel Chow group and the Suslin homology.

The strategy to prove Theorem 1.6 is to construct a surjective map $\mathrm{CH}_0^{EKW}(X) \rightarrow \mathrm{CH}_0^{LW}(X)$ such that its composite with λ_X is the one shown to be an isomorphism by Esnault, Kerz and Wittenberg. In order to construct such a map, our key step is Theorem 8.3, which in turn relies on a moving lemma from [4].

We recollect various cycle and motivic cohomology groups in § 2. We recall and prove a result about the Albanese maps in § 3. These are used in proving Theorem 1.1. The main result of this section is Theorem 3.5 which proves an isomorphism between the Levine-Weibel Chow group and motivic cohomology of a special kind of singular variety. The results of § 1.1 are proven in § 4. Theorem 1.5 is shown in § 6. In § 7, we prove Theorem 1.6 for surfaces and in § 8, we complete the proof of the general case.

Notations and conventions. If A is a Noetherian ring, we shall write \mathbf{Sch}_A for the category of schemes which are separated and of finite type over $\mathrm{Spec}(A)$. We let \mathbf{Sm}_A denote the full subcategory of \mathbf{Sch}_A whose objects are smooth schemes over $\mathrm{Spec}(A)$. For $X \in \mathbf{Sch}_A$, we shall write X_{reg} (resp. X_{sm}) for the regular locus of X (resp. for the smooth locus of $X \rightarrow \mathrm{Spec}(A)$). By a regular (or smooth) closed point of X we will mean a closed point lying in X_{reg} (resp. in X_{sm}). We will write X_{sing} for the singular locus of X (i.e., the complement $X \setminus X_{\mathrm{reg}}$) with reduced closed subscheme structure. We let $X_{(i)}$ denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = i$.

We will be mostly interested in the case where $A = k$ is a perfect field. If p is the exponential characteristic of k , we shall let Λ be the ring \mathbb{Z} if k admits resolution of singularities, or the ring $\mathbb{Z}[\frac{1}{p}]$ otherwise ([15, Def. 3.4]). For any abelian group M , we shall write $M\{p'\}$ (resp. $M\{p\}$) for the subgroup of elements in M which are torsion of order prime to p (resp. a power of p). For a commutative ring R , we let $M_R = M \otimes_{\mathbb{Z}} R$.

In this paper, we shall describe the setting of a subsection in the beginning of the subsection. Further assumptions will be explicitly stated whenever they are used in the middle of a subsection.

2. PRELIMINARIES: CYCLES GROUPS AND MOTIVIC COHOMOLOGY

We begin by fixing some notations and clarifying our terminology. Throughout this paper, we shall encounter several different groups of 0-cycles: the groups will often share the same set of generators, but will be subject to different notions of rational equivalence. In a nutshell, our main results will be a comparison between the different notions, by means of suitable moving lemmas, and to deduce new properties of these 0-cycle group. We shall draw some consequences. In this preliminary section, we introduce the main players, recalling some definitions already existing in the literature. New groups of cycles, tailored on the specific geometry of the varieties we are interested in, will be defined later in the paper.

2.1. Cycle groups on singular schemes. Let k be a field of exponential characteristic $p \geq 1$. Let $X \in \mathbf{Sch}_k$ be a reduced k -scheme and $Y \subset X$ a closed subscheme. For any $r \geq 0$, we write $\mathcal{Z}_r(X, Y)$ for the free abelian group on the set of integral r -dimensional cycles in X (i.e., integral closed subschemes of dimension r) which do not meet Y . Let $\mathcal{R}_r(X)$ be the subgroup of $\mathcal{Z}_r(X) := \mathcal{Z}_r(X, \emptyset)$ generated by the cycles which are rationally equivalent to zero (see for example [14, §1]). We write $\mathrm{CH}_r(X)$ for the quotient $\mathcal{Z}_r(X)/\mathcal{R}_r(X)$ and refer to it as the (Fulton's) Chow group of r -cycles on X modulo rational equivalence. We shall also use the notation $\mathrm{CH}^{d-r}(X)$ for $\mathrm{CH}_r(X)$ if X has pure dimension d .

We now recall the definition of the Levine-Weibel Chow group of 0-cycles on X from [42] and its modified version from [4]. Let $Y \subset X$ be a nowhere dense closed subscheme containing X_{sing} .

Definition 2.1. Let C be a reduced k -scheme which is of pure dimension one (from now on, such C will be simply called a *curve*). We shall say that a pair (C, Z) is a *good curve relative to Y* if there exists a finite morphism $\nu: C \rightarrow X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

- (1) No component of C is contained in Z .
- (2) $\nu^{-1}(Y) \cup C_{\text{sing}} \subseteq Z$.
- (3) ν is local complete intersection at every point $x \in C$ such that $\nu(x) \in Y$.

Let (C, Z) be a good curve relative to Y and let $\{\eta_1, \dots, \eta_r\}$ be the set of generic points of C . Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \dots, \eta_r\}$. Let $k(C)$ denote the ring of total quotients of C and write $\mathcal{O}_{C,Z}^\times$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \emptyset$. As C is Cohen-Macaulay, $\mathcal{O}_{C,Z}^\times$ is the subgroup of group of units in the ring of total quotients $k(C)^\times$ consisting of those $f \in \mathcal{O}_{C,x}$ which are regular and invertible for every $x \in Z$.

Given any $f \in \mathcal{O}_{C,Z}^\times \hookrightarrow k(C)^\times$, we denote by $\text{div}_C(f)$ (or $\text{div}(f)$ in short) the divisor of zeros and poles of f on C , which is defined as follows. If C_1, \dots, C_r are the irreducible components of C , and f_i is the factor of f in $k(C_i)$, we set $\text{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \text{div}(f_i)$, where $\text{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [14]. As f is an invertible regular function on C along Z , $\text{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to Y , we have a push-forward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X, Y)$. We shall write $\mathcal{R}_0(C|Z, X|Y)$ for the subgroup of $\mathcal{Z}_0(X, Y)$ generated by the set $\{\nu_*(\text{div}(f)) | f \in \mathcal{O}_{C,Z}^\times\}$.

Definition 2.2. (1) Let $\mathcal{R}_0(X, Y)$ denote the subgroup of $\mathcal{Z}_0(X, Y)$ generated by the image of the map $\mathcal{R}_0(C|Z, X|Y) \rightarrow \mathcal{Z}_0(X, Y)$, where (C, Z) runs through all good curves relative to Y . We let $\text{CH}_0^{BK}(X, Y) = \frac{\mathcal{Z}_0(X, Y)}{\mathcal{R}_0(X, Y)}$ and call it the *lci Chow group* of zero cycles of X . We shall write $\mathcal{R}_0(X, X_{\text{sing}})$ as $\mathcal{R}_0(X)$ and $\text{CH}_0^{BK}(X, X_{\text{sing}})$ as $\text{CH}_0^{BK}(X)$.

(2) Let $\mathcal{R}_0^{LW}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu: C \hookrightarrow X$ is a closed immersion. We let $\text{CH}_0^{LW}(X) = \frac{\mathcal{Z}_0(X, X_{\text{sing}})}{\mathcal{R}_0^{LW}(X)}$ and call it the *Levine-Weibel Chow group* of zero cycles on X . We shall refer to good curves relative to X_{sing} as *lci curves* and embedded good curves on X as *Cartier curves*.

By definition, there is a canonical surjection $\text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0^{BK}(X)$.

Remark 2.3. The Chow group $\text{CH}_0^{LW}(X)$ was discovered by Levine and Weibel [42] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $\text{CH}_0^{BK}(X)$ was introduced in [4]. \square

Note that there is always a map

$$(2.1) \quad \text{CH}_0^{BK}(X) \rightarrow \text{CH}_0(X)$$

which is far, in general, from being an isomorphism if $X_{\text{sing}} \neq \emptyset$. This is already clear when $\dim(X) = 1$, see [42] or [4, Lem. 3.12].

Assume now that X is complete of pure dimension $d \geq 0$ and let X_1, \dots, X_r be the irreducible components of X . For any 0-cycle $\alpha \in \mathcal{Z}_0(X, X_{\text{sing}})$, we let α_i be the restriction of α on X_i and let $\text{deg}(\alpha) = (\text{deg}(\alpha_1), \dots, \text{deg}(\alpha_r)) \in \mathbb{Z}^r$. This defines a homomorphism $\text{deg}: \mathcal{Z}_0(X, X_{\text{sing}}) \rightarrow \mathbb{Z}^r$. It is well known and easy to check from the

definition of $\mathcal{R}_0^{LW}(X)$ that $\deg(\mathcal{R}_0^{LW}(X)) = 0$. One therefore has a degree homomorphism $\deg: \text{CH}_0^{LW}(X) \rightarrow \mathbb{Z}^r$. Using a projective bundle trick (e.g., see the proof of Theorem 4.5), one deduces that \deg actually factors through $\text{CH}_0^{BK}(X)$. We shall let $\text{CH}_0^{LW}(X)_{\deg 0}$ and $\text{CH}_0^{BK}(X)_{\deg 0}$ denote the kernels of these degree maps. Note that \deg does not factor through $\text{CH}_0(X)$ unless X is integral.

2.2. Singular homology. Let k be a field and U a smooth k -scheme. Recall that the Suslin-Voevodsky singular homology of U (also called Suslin homology) is defined as follows. Let Δ^n be the algebraic n -simplex and denote by $C_n(U)$ the free abelian group generated by those integral closed subschemes Z of $U \times \Delta^n$ for which the projection $Z \rightarrow \Delta^n$ is finite and surjective. By pulling back via the structure maps of the cosimplicial scheme Δ_k^\bullet , one equips $C_\bullet(U)$ with the structure of a simplicial abelian group. We denote the associated chain complex (via the Dold-Kan correspondence) also by $C_\bullet(U)$. The n -th singular homology group $H_n^{sus}(U)$ of U is the n -th homology of this complex.

When $n = 0$, another, explicit, description is available (and will be taken as a definition for us). For $i: C \hookrightarrow U$ an integral curve, let $\phi: C^N \rightarrow C$ be its normalization and let \overline{C}^N be the unique projective normal compactification of C^N . We denote by C_∞ the complement $\overline{C}^N \setminus C^N$, and consider it as a *reduced* Cartier divisor on \overline{C} . Write $G(\overline{C}^N, C_\infty)$ for $\text{Ker}(\mathcal{O}_{\overline{C}^N, C_\infty}^\times \rightarrow \mathcal{O}_{C_\infty}^\times)$, where $\mathcal{O}_{\overline{C}^N, C_\infty}$ denotes as before the semilocal ring of \overline{C}^N at the union of the closed points supporting C_∞ . Note that $G(\overline{C}^N, C_\infty) \subset k(C)^\times = k(\overline{C}^N)^\times$.

For $f \in G(\overline{C}^N, C_\infty)$, we can consider its divisor $\text{div}_{\overline{C}^N}(f) \in \mathcal{Z}_0(\overline{C}^N, C_\infty)$. Since ϕ is finite and i is a closed immersion, we can consider the composite

$$\mathcal{Z}_0(\overline{C}^N, C_\infty) = \mathcal{Z}_0(C^N) \xrightarrow{\phi_*} \mathcal{Z}_0(C) \xrightarrow{i_*} \mathcal{Z}_0(U).$$

We write $\mathcal{R}_0^S(C, U)$ for the subgroup of $\mathcal{Z}_0(U)$ generated by the set $\{i_*\phi_*(\text{div}(f)) \mid f \in G(\overline{C}^N, C_\infty)\}$. Let $\mathcal{R}_0^S(U)$ denote the subgroup of $\mathcal{Z}_0(U)$ generated by $\mathcal{R}_0^S(C, U)$ where C runs through all integral curves in X .

Theorem 2.4. ([49, Thm. 5.1]) *There is a canonical isomorphism*

$$\frac{\mathcal{Z}_0(U)}{\mathcal{R}_0^S(U)} \xrightarrow{\cong} H_0^{sus}(U).$$

In the sequel, we shall use the left hand side of this isomorphism as the definition of $H_0^{sus}(U)$. There is a degree homomorphism $\deg: H_0^{sus}(U) \rightarrow \mathbb{Z}^r$, where r is the number of connected components of U . This is the product of maps on Suslin homology induced by the projection of each connected component of U to $\text{Spec}(k)$. We let $H_0^{sus}(U)_{\deg 0}$ be the kernel of \deg .

Remark 2.5. The j -th Suslin homology group of U with coefficients in Λ has the following motivic interpretation which will be explored further later in the text.

$$H_j^{sus}(U)_\Lambda = \text{Hom}_{\mathbf{DM}_{\text{Nis}}^{\text{eff}}(k, \Lambda)}(\Lambda[j], M(U)),$$

where $\mathbf{DM}_{\text{Nis}}^{\text{eff}}(k, \Lambda)$ is Voevodsky's category of effective motives with Λ -coefficients and $M(U)$ is the motive of U .

(2) It is clear from the above description that $H_0^{sus}(U) \cong \text{CH}_0(U)$ if U is complete.

2.3. Motivic cohomology. Let k be a perfect field of exponential characteristic $p \geq 1$. Let X be a quasi-projective scheme of dimension $d \geq 0$ over k . Recall from [44, Lecture 16] that given $T \in \mathbf{Sch}_k$ and integer $r \geq 0$, the presheaf $z_{\text{equi}}(T, r)$ on \mathbf{Sm}_k is defined by letting $z_{\text{equi}}(T, r)(U)$ be the free abelian group generated by integral closed subschemes $Z \subset U \times T$ such that Z is dominant and equidimensional of relative dimension r (any fiber is either empty or all its components have dimension r) over a component of U . It is known that $z_{\text{equi}}(T, r)$ is a sheaf on the big étale site of \mathbf{Sm}_k .

Let $C_* z_{\text{equi}}(T, r)$ be the chain complex of presheaves of abelian groups associated to the simplicial presheaf on \mathbf{Sm}_k given by $C_n z_{\text{equi}}(T, r)(U) = z_{\text{equi}}(T, r)(U \times \Delta_k^n)$. The simplicial structure on $C_* z_{\text{equi}}(T, r)$ is through the cosimplicial scheme Δ_k^\bullet , as before. Recall the following definition of motivic cohomology of singular schemes from [15, Defn. 9.2].

Definition 2.6. The motivic cohomology groups of X are defined as the hypercohomology $H^m(X, \mathbb{Z}(n)) = \mathbb{H}_{\text{cdh}}^{m-2n}(X, C_* z_{\text{equi}}(\mathbb{A}_k^n, 0)_{\text{cdh}})$. Here, $C_* z_{\text{equi}}(\mathbb{A}_k^n, 0)_{\text{cdh}}$ is the complex obtained by applying term wise the change of site functor from \mathbf{Sm}_k to \mathbf{Sch}_k .

This definition is rather untouchable. However, if one is willing to work with Λ -coefficients (or under the assumption of resolution of singularities), motivic cohomology groups have a description in terms of Hom-groups in Voevodsky's category of motives, and even on the complexes $C_* z_{\text{equi}}(\mathbb{A}_k^n, 0)_{\text{cdh}}$ we have a better understanding. More precisely, we have the following result (see [37, Thm. 3.10]).

Theorem 2.7. *There is a functorial isomorphism*

$$H^m(X, \Lambda(n)) \simeq \text{Hom}_{\mathbf{DM}(k, \Lambda)}(M(X), \Lambda(n)[m]),$$

where $\mathbf{DM}(k, \Lambda)$ is Voevodsky's non-effective category of motives for the cdh-topology (also known as the 'big' category of motives) with Λ -coefficients.

Remark 2.8. When X is smooth, the motivic cohomology groups are isomorphic to Bloch's higher Chow groups. More precisely, for any $m, n \in \mathbb{Z}$, there is a natural isomorphism $H^m(X, \mathbb{Z}(n)) = \text{CH}^n(X, 2n - m)$. See [56, Cor. 2]. \square

In this paper, we are specifically interested in the bi-degree $(2d, d)$. By [37, Lem. 7.12], there is a canonical group homomorphism

$$(2.2) \quad \gamma_X: \text{CH}_0^{BK}(X)_\Lambda \rightarrow H^{2d}(X, \Lambda(d)).$$

We remark here that the construction of γ_X uses [37, Thm. 5.1] whose proof holds verbatim with integral coefficients when k admits resolution of singularities. The map γ_X has the property that for a closed point $x \in X_{\text{reg}}$, the diagram

$$(2.3) \quad \begin{array}{ccc} \Lambda & \xrightarrow{\simeq} & \text{CH}_0(k(x))_\Lambda & \xrightarrow{\simeq} & H^0(k(x), \Lambda(0)) \\ & & \downarrow & & \downarrow \\ & & \text{CH}_0^{BK}(X)_\Lambda & \xrightarrow{\gamma_X} & H^{2d}(X, \Lambda(d)) \end{array}$$

commutes, where the vertical arrows are the push-forward maps (see [37, § 7], and [5, §5.1] where a similar construction is performed with motivic cohomology replaced by étale cohomology with finite coefficients). We let

$$\lambda_X^{LW}: \text{CH}_0^{LW}(X)_\Lambda \rightarrow \text{CH}_0^{BK}(X)_\Lambda \rightarrow H^{2d}(X, \Lambda(d))$$

denote the composite map.

We let $M^c(X) \in \mathbf{DM}(k, \Lambda)$ denote the motive of X with compact support. One knows that $M^c(X) \simeq M(X)$ if X is complete. Under our assumption on Λ , it follows from [55] and [30, Chap. 5] that for a closed immersion $Z_1 \subset Z_2$ of schemes, there is a distinguished triangle (called the localization sequence) in $\mathbf{DM}(k, \Lambda)$:

$$(2.4) \quad M^c(Z_1) \rightarrow M^c(Z_2) \rightarrow M^c(Z_2 \setminus Z_1) \rightarrow M^c(Z_1)[1].$$

Using the duality between the motivic homology and the motivic cohomology with compact support, one gets a functorial isomorphism $H_n^{sus}(Z)_\Lambda \simeq H_c^{2d-n}(Z, \Lambda(d))$ if Z is a smooth scheme of pure dimension d , see [15] and [30, Chap. 5], where the motivic cohomology groups with compact support is defined as

$$H_c^m(Z, \Lambda(n)) \simeq \mathrm{Hom}_{\mathbf{DM}(k, \Lambda)}(M^c(Z), \Lambda(n)[m]).$$

In what follows, we will frequently use these identifications. We shall sometimes write $H^m(Z, \mathbb{Z}(n))$ as $H^{m,n}(Z)$.

2.4. Cycles with modulus. Let k be a field. Given any integral normal curve C over k and an effective Cartier (or Weil) divisor E on C , we say that a rational function $f \in k(C)^\times$ on C has modulus E if $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$. As in § 2.2, we let $G(C, E)$ denote the subgroup of $k(C)^\times$ given by such rational functions. Note that E is not necessarily reduced.

Let $X \in \mathbf{Sch}_k$ be an integral scheme, and let D be an effective Cartier divisor on X . Write as before $\mathcal{Z}_0(X, D)$ for the free abelian group on the set of closed points in $X \setminus D$. If C is an integral normal curve over k equipped with a finite morphism $\phi_C: C \rightarrow X$ such that $\phi_C(C) \not\subset D$, we have a well defined group homomorphism

$$\tau_C: G(C, \phi_C^*(D)) \rightarrow \mathcal{Z}_0(X, D), \quad f \mapsto (\phi_C)_*(\mathrm{div}(f)),$$

where $\phi_C^*(D)$ denote the pullback of D to C . Note that $\phi_C^*(D)$ is not necessarily reduced, even when D is. Recall the following definition, that originally appeared in [32].

Definition 2.9 (Kerz-Saito). We define the Chow group $\mathrm{CH}_0(X|D)$ of 0-cycles of X with modulus D as the cokernel of the homomorphism

$$\mathrm{div}: \bigoplus_{\phi_C: C \rightarrow X} G(C, \phi_C^*D) \rightarrow \mathcal{Z}_0(X, D),$$

where the sum runs over the set of finite morphisms $\phi_C: C \rightarrow X$ from integral normal curves such that $\phi_C(C) \not\subset D$.

The Chow groups of zero cycles with modulus are known to be covariantly functorial (pushforward) for proper maps $f: X' \rightarrow X$ with respect to Cartier divisors D' on X' and D on X such that $f^*(D)$ is defined and satisfies $f^*(D) \leq D'$ as effective Cartier divisors. See [7, Lem. 2.4] or [36, Prop. 2.10]. Similarly, it is easy to construct a pullback along finite flat maps. See again [7, Lem. 2.4] or [36, Prop. 2.12].

Lemma 2.10. *Assume that X is complete and $U = X \setminus D$ is a smooth k -scheme. Then the canonical bijection $\mathcal{Z}_0(X, D) \xrightarrow{\cong} \mathcal{Z}_0(U)$ induces a surjective homomorphism*

$$(2.5) \quad \lambda_{X|D}: \mathrm{CH}_0(X|D) \twoheadrightarrow H_0^{sus}(U).$$

Proof. This is an easy exercise using Theorem 2.4. □

The map (2.5) is not an isomorphism if D is not reduced. If $\dim(X) = 1$ and X is a complete normal curve over k , the group $\mathrm{CH}_0(X|D)$ agrees with the relative Picard group $\mathrm{Pic}(X, D)$ of line bundles \mathcal{L} on X equipped with a trivialization $\mathcal{L}|_D \cong \mathcal{O}_D$ along D (see, e.g., [7, §4]), while $H_0^{sus}(U)$ agrees with the relative Picard group

$\text{Pic}(X, D_{\text{red}})$, by [44, Thm. 7.16]. So, the two groups do not agree when D is not reduced. Theorem 1.1 of this paper describes the nature of $\lambda_{X|D}$ when D is reduced.

2.5. Normal crossing varieties. Let X be an integral Noetherian scheme of finite Krull dimension $d \geq 0$. Let $D \subset X$ be an effective Cartier divisor and let $D_i \subset D$, $i \in I$ be the irreducible components of D . Note that $D = \emptyset$ if $d = 0$. Recall the following definition (e.g., see [10, 2.4]).

Definition 2.11. (1) We say that D is a *simple normal crossing divisor* on X if the local ring $\mathcal{O}_{X,x}$ is regular for each point $x \in D$ and for each $J \subset I$ nonempty, the scheme theoretic intersection $D_J = \bigcap_{i \in J} D_i$ is a regular scheme which is either empty or of pure dimension $d + 1 - |J|$.

(2) We say that D is a *normal crossing divisor* on X if for every $x \in D$, there exists an étale morphism $U \rightarrow X$ with x in its image such that $D \times_X U$ a simple normal crossing divisor on U .

Definition 2.12. Let k be a field and X a reduced quasi-projective k -scheme. Let $\{X_1, \dots, X_n\}$ be the set of irreducible components of X . We shall say that X is a *normal crossing variety* of dimension $d \geq 0$ if for every nonempty subset $J \subset [1, n]$, the scheme theoretic intersection $X_J := \bigcap_{i \in J} X_i$ is a smooth k -scheme which is either empty or of pure dimension $d + 1 - |J|$.

2.6. The double construction. Let k be any field. Let X be an integral regular quasi-projective scheme of dimension d over k and let $D \subset X$ be an effective Cartier divisor. Recall from [4, § 2.1] that the double (or join) of X along D is a quasi-projective scheme $S(X, D) = X \amalg_D X$ so that

$$(2.6) \quad \begin{array}{ccc} D & \xrightarrow{\iota} & X \\ \iota \downarrow & & \downarrow \iota_+ \\ X & \xrightarrow{\iota_-} & S(X, D) \end{array}$$

is a co-Cartesian square in \mathbf{Sch}_k . In particular, the identity map of X induces a finite flat map $\Delta_X: S(X, D) \rightarrow X$ such that $\Delta_X \circ \iota_{\pm} = \text{id}_X$ and $\pi = \iota_+ \amalg \iota_-: X \amalg X \rightarrow S(X, D)$ is the normalization map. We let $X_{\pm} = \iota_{\pm}(X) \subset S(X, D)$ denote the two irreducible components of $S(X, D)$. We use S_X as a shorthand for $S(X, D)$ when the divisor D is understood. S_X is a reduced quasi-projective scheme whose singular locus is $D \subset S_X$. It is projective (resp. affine) whenever X is so. It follows from [35, Lem. 2.2] that (2.6) is a bi-Cartesian square.

We have a pullback isomorphism $\mathcal{Z}_0(S_X, D) \xrightarrow{(\iota_+^*, \iota_-^*)} \mathcal{Z}_0(X_+, D) \oplus \mathcal{Z}_0(X_-, D)$, as well as pushforward inclusion maps $p_{\pm, *}: \mathcal{Z}_0(X, D) \rightarrow \mathcal{Z}_0(S_X, D)$ such that $\iota_+^* \circ p_{+, *} = \text{id}$ and $\iota_+^* \circ p_{-, *} = 0$. The main result of [4] connects the lci Chow group of zero cycles on S_X with Fulton's Chow group of X and the Kerz-Saito Chow group with modulus $\text{CH}_0(X|D)$ as follows.

If k is perfect², there are push-forward maps $p_{\pm, *}: \text{CH}_0(X|D) \rightarrow \text{CH}_0^{BK}(S_X)$ and pull-back maps $\iota_{\pm}^*: \text{CH}_0^{BK}(S_X) \rightarrow \text{CH}_0(X)$, where the latter are split by $\Delta_X^*: \text{CH}_0(X) \rightarrow \text{CH}_0^{BK}(S_X)$. By [4, Thm. 7.1], there is a split short exact sequence

$$(2.7) \quad 0 \rightarrow \text{CH}_0(X|D) \xrightarrow{p_{+, *}} \text{CH}_0^{BK}(S_X) \xrightarrow{\iota_+^*} \text{CH}_0(X) \rightarrow 0$$

and a similar split exact sequence corresponding to the maps $(p_{-, *}, \iota_+^*)$.

²The perfectness condition has been removed recently in [25, Thm. 1.1].

3. CHOW GROUP VS. MOTIVIC COHOMOLOGY

The goal of this section is to prove that the composition of the map (2.2) with the Levine-Wiebel Chow group is an isomorphism for S_X . In combination with (2.7), this will be the key step in the proof of Theorem 1.1. We begin by recalling Serre's Albanese varieties and Albanese maps and state Theorem 3.2 which we shall use in the next section.

3.1. The semi-abelian Albanese variety for the double. We fix an algebraically closed field k of any characteristic. We fix a pair (X, D) consisting of an integral smooth projective k -scheme X of dimension $d \geq 1$ and an effective Cartier divisor D on it. We write S_X for the double $S(X, D)$ of § 2.6 and $U = X \setminus D$.

Let $\pi: \bar{S}_X = X_+ \sqcup X_- \rightarrow S_X$ denote again the normalization map. $NS(\bar{S}_X)$ denote the Neron-Severi group of \bar{S}_X . Recall that this is the quotient of the class group of \bar{S}_X by the subgroup of Weil divisors which are algebraically equivalent to zero. Let $\text{Div}(S_X)$ denote the free abelian group of Weil divisors on S_X . Let $\Pi_1(S_X)$ denote the subgroup of $\text{Div}(\bar{S}_X)$ generated by the Weil divisors which are supported on $\pi^{-1}(D)$. There is a canonical map $\iota_{S_X}: \Pi_1(S_X) \rightarrow NS(\bar{S}_X)$. Let $\Pi_{\text{Serre}}(S_X)$ denote the kernel of this map and let $\Pi(S_X)$ be the kernel of the canonical map

$$(3.1) \quad \Pi_1(S_X) \xrightarrow{(\iota_{S_X}, \pi_*)} NS(\bar{S}_X) \oplus \text{Div}(S_X).$$

Let $J^d(S_X)$ and $J_{\text{Serre}}^d(S_X)$ denote the Cartier duals of the 1-motives

$$[\Pi(S_X) \rightarrow \text{Pic}^0(\bar{S}_X)], \text{ and } [\Pi_{\text{Serre}}(S_X) \rightarrow \text{Pic}^0(\bar{S}_X)],$$

respectively. There is a surjective group homomorphism

$$\rho_{S_X}: \text{CH}_0^{LW}(S_X)_{\text{deg } 0} \twoheadrightarrow J^d(S_X)$$

which is universal among regular homomorphisms from $\text{CH}_0^{LW}(S_X)_{\text{deg } 0}$ to semi-abelian varieties. $J^d(S_X)$ is called the *semi-abelian Albanese variety* of S_X . By [4, Prop. 9.7], there is a factorization of ρ_X :

$$(3.2) \quad \text{CH}_0^{LW}(S_X)_{\text{deg } 0} \rightarrow \text{CH}_0^{BK}(X)_{\text{deg } 0} \xrightarrow{\tilde{\rho}_{S_X}} J^d(S_X)$$

and $J^d(S_X)$ is in fact the universal regular semi-abelian variety quotient of $\text{CH}_0^{BK}(S_X)_{\text{deg } 0}$.

One knows from [51] that $J_{\text{Serre}}^d(S_X)$ is Serre's Albanese variety of $(S_X)_{\text{reg}} = U \sqcup U$, the universal object in the category of morphisms from $U \sqcup U$ to semi-abelian varieties. We let $J^d(U)$ denote Serre's Albanese variety of U so that $J_{\text{Serre}}^d(S_X) = J^d(U) \times J^d(U)$. Using the above constructions, one checks that there is an exact sequence of morphisms of semi-abelian varieties

$$(3.3) \quad 0 \rightarrow T \rightarrow J_{\text{Serre}}^d(S_X) \rightarrow J^d(S_X) \rightarrow 0,$$

where T is an algebraic torus of rank bounded by the number of components of D .

We also have a commutative diagram

$$(3.4) \quad \begin{array}{ccc} \text{CH}^0(D) & \longrightarrow & NS(X) \\ \downarrow \iota_{-,*} & & \downarrow \iota_{-,*} \\ \Pi_1(S_X) & \longrightarrow & NS(X_+ \sqcup X_-). \end{array}$$

If we let $\Pi_{\text{Serre}}(X|D) = \text{Ker}(\text{CH}^0(D) \rightarrow \text{NS}(X))$, we obtain a commutative diagram of 1-motives

$$(3.5) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ [0 \rightarrow \text{Pic}^0(X)] & \xrightarrow{\alpha_X} & [\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)] \\ \downarrow \iota_{-,*} & & \downarrow \iota_{-,*} \\ [\Pi(S_X) \rightarrow \text{Pic}^0(\overline{S}_X)] & \xrightarrow{\beta_X} & [\Pi_{\text{Serre}}(S_X) \rightarrow \text{Pic}^0(\overline{S}_X)] \\ \downarrow \iota_+^* & & \downarrow \iota_+^* \\ [\Pi(S_X) \rightarrow \text{Pic}^0(X)] & \xrightarrow{\gamma_X} & [\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)] \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

in which the two columns are exact.

We now consider the commutative diagram

$$(3.6) \quad \begin{array}{ccc} \text{CH}^0(D) & \longrightarrow & \text{NS}(X) \\ \tilde{\Delta}_X^* \downarrow & & \downarrow \tilde{\Delta}_X^* \\ \Pi_1(S_X) & \longrightarrow & \text{NS}(\overline{S}_X), \end{array}$$

where we let $\tilde{\Delta}_X^*(a) = (a, -a)$. Note that $\Pi_1(S_X) = \text{CH}^0(D) \oplus \text{CH}^0(D)$ and $\text{NS}(\overline{S}_X) = \text{NS}(X) \oplus \text{NS}(X)$. It is clear then that $\tilde{\Delta}_X^*$ factors through $\text{CH}^0(D) \rightarrow \text{Ker}(\Pi_1(S_X) \xrightarrow{\pi^*} \text{Div}(S_X))$. We thus get a morphism of 1-motives $\tilde{\Delta}_X^* : [\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)] \rightarrow [\Pi(S_X) \rightarrow \text{Pic}^0(\overline{S}_X)]$. Composing this with ι_+^* in the left column of (3.5), we get a map $[\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)] \rightarrow [\Pi(S_X) \rightarrow \text{Pic}^0(X)]$.

Lemma 3.1. *The morphism of 1-motives*

$$\iota_+^* \circ \tilde{\Delta}_X^* : [\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)] \rightarrow [\Pi(S_X) \rightarrow \text{Pic}^0(X)]$$

is an isomorphism.

Proof. To prove the lemma, we first note that at the level of the abelian varieties, this map of 1-motives is given by $a \mapsto (a, -a) \mapsto a$, so it is actually identity. Thus, we only need to show that the map $\tilde{\Delta}_X^* : \Pi_{\text{Serre}}(X|D) \rightarrow \Pi(S_X)$ is an isomorphism.

To prove this, let $\Pi_2(S_X) = \text{Ker}(\Pi_1(S_X) \xrightarrow{\pi^*} \text{Div}(S_X))$. We then have the commutative diagram

$$(3.7) \quad \begin{array}{ccccc} \Pi_{\text{Serre}}(X|D) & \longrightarrow & \text{CH}^0(D) & \xrightarrow{cl_X} & \text{NS}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \Pi(S_X) & \longrightarrow & \Pi_2(S_X) & \xrightarrow{cl_{S_X}} & \text{NS}(\overline{S}_X), \end{array}$$

where the vertical arrows are $\tilde{\Delta}_X^*$. Since $D \sqcup D \xrightarrow{\pi} D$ is just the collapse map, it is immediate from various definitions that any element of $\Pi_2(S_X)$ must be of the form $\tilde{\Delta}_X^*(a)$ for some $a \in \text{CH}^0(D)$. Furthermore, we have that $\iota_+^* \circ \tilde{\Delta}_X^*(a)$ is identity. It

follows that the middle vertical arrow in (3.7) is an isomorphism. The right vertical arrow is injective and is split by ι_+^* .

If we let $\overline{NS(X)}$ and $\overline{NS(S_X)}$ be the images of cl_X and cl_{S_X} , respectively, we therefore get a commutative diagram of short exact sequences

$$(3.8) \quad \begin{array}{ccccccc} 0 & \rightarrow & \Pi_{\text{Serre}}(X|D) & \rightarrow & \text{CH}^0(D) & \xrightarrow{cl_X} & \overline{NS(X)} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Pi(S_X) & \longrightarrow & \Pi_2(S_X) & \xrightarrow{cl_{S_X}} & \overline{NS(S_X)} \rightarrow 0, \end{array}$$

where the middle and the right vertical arrows are isomorphisms. It follows that the left vertical arrow is an isomorphism too. This proves the lemma. \square

3.2. Albanese homomorphisms for 0-cycles. We continue with the setup of § 3.1.

We let $J^d(X|D) := \text{Ker}(J^d(S_X) \xrightarrow{\iota_-^*} J^d(X))$, where the map ι_-^* is surjective and is split by $\Delta_X^* : J^d(X) \rightarrow J^d(S_X)$. Using [4, Prop. 9.7, § 11.1], one shows that $J^d(X|D)$ is the universal regular semi-abelian variety quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$. Furthermore, there is a commutative diagram of split short exact sequences

$$(3.9) \quad \begin{array}{ccccccc} 0 & \rightarrow & \text{CH}_0(X|D)_{\text{deg } 0} & \xrightarrow{p_{+,*}} & \text{CH}_0^{BK}(S_X)_{\text{deg } 0} & \xrightarrow{\iota_-^*} & \text{CH}_0(X)_{\text{deg } 0} \rightarrow 0 \\ & & \downarrow \rho_{X|D} & & \downarrow \tilde{\rho}_{S_X} & & \downarrow \rho_X \\ 0 & \longrightarrow & J^d(X|D) & \xrightarrow{p_{+,*}} & J^d(S_X) & \xrightarrow{\iota_-^*} & J^d(X) \longrightarrow 0. \end{array}$$

Next, recall from [52, §3] that the semi-abelian Albanese variety $J^d(U)$ of U is the universal regular quotient of $H_0^{sus}(U)_{\text{deg } 0}$. We write ρ_U for the induced surjection $\rho_U : H_0^{sus}(U)_{\text{deg } 0} \twoheadrightarrow J^d(U)$. Composing ρ_U with the canonical map $\lambda_{X|D}$ of (2.5), we get a (surjective) morphism

$$\text{CH}_0(X|D)_{\text{deg } 0} \rightarrow J^d(U),$$

and the universality of $J^d(X|D)$ as a regular quotient of $\text{CH}_0(X|D)_{\text{deg } 0}$ induces then a canonical map $\lambda_{X|D}^{alb} : J^d(X|D) \rightarrow J^d(U)$. Using (3.5) and Lemma 3.1, we easily obtain the following description of various semi-abelian Albanese varieties.

Theorem 3.2. *The canonical map $\text{CH}_0(X|D) \rightarrow H_0^{sus}(U)$ induces a commutative diagram*

$$(3.10) \quad \begin{array}{ccc} \text{CH}_0(X|D)_{\text{deg } 0} & \xrightarrow{\lambda_{X|D}} & H_0^{sus}(U)_{\text{deg } 0} \\ \rho_{X|D} \downarrow & & \downarrow \rho_U \\ J^d(X|D) & \xrightarrow{\lambda_{X|D}^{alb}} & J^d(U) \end{array}$$

such that the following hold.

- (1) $\lambda_{X|D}^{alb}$ is an isomorphism of semi-abelian varieties.
- (2) $J^d(X|D)$ is the Cartier dual of the 1-motive $[\Pi(S_X) \rightarrow \text{Pic}^0(X)]$.
- (3) $J^d(U)$ is the Cartier dual of the 1-motive $[\Pi_{\text{Serre}}(X|D) \rightarrow \text{Pic}^0(X)]$.

(4) *There is a commutative diagram of short exact sequences of semi-abelian varieties*

$$(3.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J^d(U) & \xrightarrow{\iota_{+,*}} & J^d(U) \times J^d(U) & \xrightarrow{\iota_{-}^*} & J^d(U) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J^d(X|D) & \xrightarrow{p_{+,*}} & J^d(S_X) & \xrightarrow{\iota_{-}^*} & J^d(X) \longrightarrow 0, \end{array}$$

where the vertical arrows are surjective.

3.3. Chow group vs. motivic cohomology of the double. We let k be an algebraically closed field of characteristic $p > 0$. We let (X, D) be a pair consisting of an integral smooth projective k -scheme X of dimension $d \geq 1$ and a reduced effective Cartier divisor D on it. Under this setup, we shall identify the Levine-Weibel Chow group of S_X to its motivic cohomology. We shall use the following consequence of [35, Thm. 6.5] which uses our assumption on D in an essential way. But note that it does not use our assumption on $\text{char}(k)$.

Lemma 3.3. ([35, Thm. 6.5]) *The canonical map $\delta_{S_X}: \text{CH}_0^{LW}(S_X) \rightarrow \text{CH}_0^{BK}(S_X)$ is an isomorphism.*

Lemma 3.4. *The map $\lambda_{S_X}: \text{CH}_0^{BK}(S_X)_\Lambda \rightarrow H^{2d}(S_X, \Lambda(d))$ is surjective and the map $\lambda_{S_X}: \text{CH}_0^{BK}(S_X)_\Lambda\{p'\} \rightarrow H^{2d}(S_X, \Lambda(d))\{p'\}$ is an isomorphism.*

Proof. In this proof, we shall assume all abelian groups to be tensored with Λ . By construction, there is a commutative diagram (see [37, (8.13)])

$$(3.12) \quad \begin{array}{ccc} \mathcal{Z}_0(S_X, D) & \xlongequal{\quad} & \mathcal{Z}_0(S_X, D) \\ \downarrow & & \downarrow \searrow \\ \mathcal{Z}_0(S_X, D) & & H_0^{sus}(S_X \setminus D) \\ \downarrow & \swarrow & \downarrow \\ \text{CH}_0^{BK}(S_X) & \xrightarrow{\lambda_{S_X}} & H^{2d}(S_X, \Lambda(d)), \end{array}$$

where the map $H_0^{sus}(S_X \setminus D) \rightarrow H^{2d}(S_X, \Lambda(d))$ is induced by the localization sequence (2.4) associated to the open embedding $S_X \setminus D \rightarrow S_X$. It follows from [37, Thm. 5.1] that this map is surjective. This in particular implies that the motivic cohomology group $H^{2d}(S_X, \Lambda(d))$ is generated by the free abelian group on the set of closed points in $S_X \setminus D$. But this is equivalent to λ_{S_X} being surjective.

To show the isomorphism between the primed torsion subgroups (i.e., on the prime-to- p -torsion part), we consider the commutative diagram of split short exact sequences (note that the lines stay exact even after applying $(-)\{p'\}$, since (2.7) is split)

$$(3.13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & (\text{CH}_0(X|D))\{p'\} & \longrightarrow & (\text{CH}_0^{BK}(S_X))\{p'\} & \longrightarrow & (\text{CH}_0(X))\{p'\} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (H_0^{sus}(X \setminus D))\{p'\} & \longrightarrow & (H^{2d,d}(S_X))\{p'\} & \longrightarrow & (H^{2d,d}(X))\{p'\} \longrightarrow 0. \end{array}$$

The right vertical arrow is an isomorphism as X is smooth. It suffices therefore to show that the left vertical arrow is an isomorphism. For this, we consider the commutative diagram (3.10). It follows from Theorem 3.2(1) that the lower horizontal arrow

in (3.10) is an isomorphism. The left (resp. right) vertical arrow is an isomorphism on the primed torsion subgroups by [4, Thm. 11.5] (resp. [52, Thm. 1.1]). Our assertion now follows. \square

We can now state and prove the main result of § 3.

Theorem 3.5. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective k -scheme of pure dimension $d \geq 1$ and $D \subset X$ a reduced effective Cartier divisor. Then the composition*

$$\lambda_{S_X}^{LW} : \mathrm{CH}_0^{LW}(S_X)_\Lambda \rightarrow \mathrm{CH}_0^{BK}(S_X)_\Lambda \rightarrow H^{2d}(S_X, \Lambda(d))$$

is an isomorphism.

Proof. We can assume X is integral. The map $\lambda_{S_X}^{LW}$ is surjective by Lemmas 3.3 and 3.4. To show it is injective, we first assume $\Lambda = \mathbb{Z}[\frac{1}{p}]$.

It follows from [37, Thm. 5.9, Lem. 8.1] that there is a commutative diagram

$$(3.14) \quad \begin{array}{ccc} \mathrm{CH}_0^{LW}(S_X)_\Lambda & \longrightarrow & K_0(S_X)_\Lambda \\ \lambda_{S_X}^{LW} \downarrow & & \downarrow \\ H^{2d,d}(S_X)_\Lambda & \longrightarrow & KH_0(S_X)_\Lambda, \end{array}$$

where $KH_*(S_X)$ is Weibel's homotopy K -theory. We remark here that even as [37, Lem. 8.1] is proven over the field of complex numbers, its proof holds over any algebraically closed field.

The right vertical arrow in (3.14) is an isomorphism by [57] (see also [54, Exer. 9.11]) as $p > 0$. The kernel of the top horizontal arrow is a torsion group of exponent $(d-1)!$ by [41, Cor. 2.7]. It follows that the kernel of $\lambda_{S_X}^{LW}$ is torsion of exponent $(d-1)!$. We therefore have a short exact sequence

$$(3.15) \quad 0 \rightarrow \mathrm{Ker}(\lambda_{S_X}^{LW})_\Lambda \rightarrow \mathrm{CH}_0^{LW}(S_X)_\Lambda \rightarrow H^{2d,d}(S_X)_\Lambda \rightarrow 0,$$

where the group on the left is torsion of exponent $(d-1)!$. On the other hand, since k is algebraically closed and S_X is projective, one knows that the degree zero part of $\mathrm{CH}_0^{LW}(S_X)_\Lambda$ is divisible (see the last part in the proof of Theorem 6.4). Since $\mathrm{Ker}(\lambda_{S_X}^{LW})$ lies in this subgroup, it follows from Lemma 3.4 that $\mathrm{Ker}(\lambda_{S_X}^{LW})_\Lambda$ is also divisible. The result now follows since a divisible torsion group of bounded exponent must be zero.

We are now left with proving the injectivity of $\lambda_{S_X}^{LW}$ when k admits resolution of singularities (even if $\mathrm{char}(k) > 0$) and $\Lambda = \mathbb{Z}$. In this case, the above argument shows that $\lambda_{S_X}^{LW}$ is surjective whose kernel is a p -primary torsion group. We now consider the commutative diagram

$$(3.16) \quad \begin{array}{ccc} \mathrm{CH}_0^{LW}(S_X)\{p\} & \xrightarrow{\pi^*} & \mathrm{CH}_0(\overline{S}_X)\{p\} \\ \rho_{S_X} \downarrow & & \downarrow \rho_{\overline{S}_X} \\ J^d(S_X)\{p\} & \xrightarrow{\pi^*} & J^d(\overline{S}_X)\{p\}. \end{array}$$

It follows from [35, Thm. 6.4] that the vertical arrows are isomorphisms. On the other hand, we know that $J^d(S_X)$ is a semi-abelian variety whose abelian variety quotient is $J^d(\overline{S}_X)$. Since an algebraic torus over k has no p -primary torsion (note that $\mathrm{char}(k) = p > 0$), it follows that the lower horizontal arrow in (3.16) is an isomorphism. In particular, the upper horizontal arrow is also an isomorphism.

In the final step, we consider the commutative diagram

$$(3.17) \quad \begin{array}{ccc} \mathrm{CH}_0^{LW}(S_X)\{p\} & \xrightarrow{\lambda_{S_X}^{LW}} & H^{2d,d}(S_X)\{p\} \\ \pi^* \downarrow & & \downarrow \pi^* \\ \mathrm{CH}_0(\overline{S}_X)\{p\} & \xrightarrow{\lambda_{\overline{S}_X}} & H^{2d,d}(\overline{S}_X)\{p\}. \end{array}$$

Since \overline{S}_X is smooth, $\lambda_{\overline{S}_X}$ is an isomorphism. We have shown above that the left vertical arrow in (3.17) is an isomorphism. It follows that $\lambda_{S_X}^{LW}$ is injective on the p -primary torsion subgroups. We conclude that $\mathrm{Ker}(\lambda_{S_X}^{LW})$ is zero. This completes the proof. \square

We end this section with the following question. Using the diagrams (3.4) and (3.6), one shows that there is a 1-motive $[\Lambda_{\mathrm{Serre}}(X|D) \rightarrow \mathrm{Pic}^0(\overline{S}_X)]$ and a map of 1-motives $[\Lambda_{\mathrm{Serre}}(X|D) \rightarrow \mathrm{Pic}^0(\overline{S}_X)] \rightarrow [\Lambda(S_X) \rightarrow \mathrm{Pic}^0(\overline{S}_X)]$. Let $J_{\mathrm{cdh}}^d(S_X)$ denote the Cartier dual of $[\Lambda_{\mathrm{Serre}}(X|D) \rightarrow \mathrm{Pic}^0(\overline{S}_X)]$.

Question 3.6. Is there a surjective homomorphism $\rho_{S_X}^{\mathrm{cdh}} : H^{2d,d}(S_X)_{\mathrm{deg} 0} \twoheadrightarrow J_{\mathrm{cdh}}^d(S_X)$ and a commutative diagram

$$(3.18) \quad \begin{array}{ccc} \mathrm{CH}_0^{LW}(S_X)_{\mathrm{deg} 0} & \xrightarrow{\lambda_{S_X}} & H^{2d,d}(S_X)_{\mathrm{deg} 0} \\ \rho_{S_X} \downarrow & & \downarrow \rho_{S_X}^{\mathrm{cdh}} \\ J^d(S_X) & \longrightarrow & J_{\mathrm{cdh}}^d(S_X) \end{array}$$

such that the vertical arrows are the universal regular quotient maps?

4. PROOFS OF THEOREMS 1.1 AND 1.2

In this section, we shall prove Theorems 1.1 and 1.2. We shall prove the two parts of Theorem 1.1 in separate subsections for the sake of clarity of exposition. We begin with the proof of its part (1).

4.1. Positive characteristic case. Let k be an algebraically closed field of characteristic $p > 0$. Let (X, D) be a pair consisting of an integral smooth projective k -scheme X of dimension $d \geq 1$ and an effective (but not necessarily reduced) Cartier divisor D on it. We let $U = X \setminus D$. We need a lemma and a proposition.

Lemma 4.1. *Let $E \subset X$ be an effective Cartier divisor which is contained in D and whose support coincides with that of D . Then the kernel of the canonical surjection $\mathrm{CH}_0(X|D) \twoheadrightarrow \mathrm{CH}_0(X|E)$ is a p -primary torsion group.*

Proof. We shall prove the lemma by induction on d . If $d \leq 2$, the cycle class map $\mathrm{CH}_0(X|D) \rightarrow H_{\mathrm{zar}}^d(X, \mathcal{K}_{d,(X,D)}^M)$ is an isomorphism by [4, Thm. 1.8], where $\mathcal{K}_{n,(X,D)}^M = \mathrm{Ker}(\mathcal{K}_{n,X}^M \twoheadrightarrow \mathcal{K}_{n,D}^M)$ is the Zariski sheaf of relative Milnor K -groups. It suffices therefore to show that the kernel of the canonical surjection $H_{\mathrm{zar}}^d(X, \mathcal{K}_{d,(X,D)}^M) \twoheadrightarrow H_{\mathrm{zar}}^d(X, \mathcal{K}_{d,(X,E)}^M)$ is a p -primary torsion group. Using the long exact cohomology sequence associated to the exact sequence of Zariski sheaves

$$0 \rightarrow \mathcal{K}_{d,(X,D)}^M \rightarrow \mathcal{K}_{d,(X,E)}^M \rightarrow \mathcal{K}_{d,(D,E)}^M \rightarrow 0,$$

our assertion is implied by the claim that $H_{\mathrm{zar}}^{d-1}(D, \mathcal{K}_{d,(D,E)}^M)$ is a p -primary torsion group (in fact, of bounded exponent). But this is well known (e.g., see [24, Lem. 2.6]).

We now assume $d \geq 3$. Let $\nu: C \rightarrow X$ be a finite morphism from a regular integral projective curve whose image is not contained in D and let $f \in \text{Ker}(\mathcal{O}_{C,E'}^\times \rightarrow \mathcal{O}_{E'}^\times)$, where $E' = \nu^{-1}(E)$. We need to show that $\nu_*(\text{div}(f)) \in \text{CH}_0(X|D)$ is killed by a power of p .

We can get a factorization $C \xrightarrow{\nu'} \mathbb{P}_X^n \xrightarrow{\pi} X$ of ν , where ν' is a closed immersion and π is the canonical projection. It is clear that $\nu'_*(\text{div}(f))$ dies in $\text{CH}_0(\mathbb{P}_X^n|\mathbb{P}_E^n)$ and $\nu_*(\text{div}(f)) = \pi_*(\nu'_*(\text{div}(f)))$. Using the push-forward map on the Chow groups with modulus, it suffices therefore to show that $\nu'_*(\text{div}(f))$ is killed by some power of p in $\text{CH}_0(\mathbb{P}_X^n|\mathbb{P}_D^n)$. We can thus assume that $\nu: C \rightarrow X$ is a closed immersion.

We now fix a closed embedding $X \hookrightarrow \mathbb{P}_k^N$. We can use [2, Thm. 7] to find a hypersurface $H \subset \mathbb{P}_k^N$ containing C and not containing X such that the scheme theoretic intersection $X \cap H$ has the property that it is smooth and contains no irreducible component of D . By [19, Cor. 3.13], we can assume that $X \cap H$ is integral. We let $Y = X \cap H$, $D_Y = D \cap H$ and $E_Y = E \cap H$.

Let $C \xrightarrow{\nu'} Y \xrightarrow{\iota} X$ be the factorization of ν . It follows from the choice of Y that $\nu'_*(\text{div}(f))$ dies in $\text{CH}_0(Y|E_Y)$. This implies by induction that $\nu'_*(\text{div}(f)) \in \text{CH}_0(Y|D_Y)$ is killed by a power of p . It follows that $\nu_*(\text{div}(f)) = \iota_*(\nu'_*(\text{div}(f)))$ is killed by a power of p in $\text{CH}_0(X|D)$. This concludes the proof. \square

Proposition 4.2. *The kernel of the map $\lambda_{X|D}: \text{CH}_0(X|D) \rightarrow H_0^{sus}(U)$ is a p -primary torsion group. Furthermore, the map $\lambda_{X|D}: \text{CH}_0(X|D)\{\ell\} \rightarrow H_0^{sus}(U)\{\ell\}$ is an isomorphism for any prime $\ell \neq p$.*

Proof. By Lemma 4.1, we can assume that D is reduced. Now, the second part of the proposition is already shown in the proof of Lemma 3.4 (see the argument after (3.13)). To prove the first part, we let S_X be the double of X along D and consider the commutative diagram (see (3.13) again)

$$(4.1) \quad \begin{array}{ccc} \text{CH}_0(X|D)[\frac{1}{p}] & \rightarrow & \text{CH}_0^{BK}(S_X)[\frac{1}{p}] \\ \lambda_{X|D} \downarrow & & \downarrow \gamma_{S_X} \\ H_0^{sus}(U)[\frac{1}{p}] & \longrightarrow & H^{2d,d}(S_X)[\frac{1}{p}]. \end{array}$$

The horizontal arrows are split injective, and the right vertical arrow is an isomorphism by Theorem 3.5. It follows that the left vertical arrow is also an isomorphism. This implies our assertion. \square

We can now prove Theorem 1.1(1). We restate it for convenience.

Theorem 4.3. *Let X be a smooth projective scheme over an algebraically closed field k of characteristic $p > 0$. Let $D \subset X$ be a reduced effective Cartier divisor with complement U . Then the canonical map*

$$\lambda_{X|D}: \text{CH}_0(X|D) \rightarrow H_0^{sus}(U)$$

is an isomorphism.

Proof. We can assume X to be integral. By Proposition 4.2, we only need to show that $\lambda_{X|D}$ induces an isomorphism between the torsion subgroups. For this, we use the commutative diagram (3.10). It follows from Theorem 3.2(1) that the bottom horizontal arrow in this diagram is an isomorphism. The left (resp. right) vertical arrow is an isomorphism on the torsion subgroups by [35, Thm. 1.2] (resp. [20, Thm. 1.7]). This implies that the same is true for the top horizontal arrow. \square

4.2. Characteristic zero case. Let k be any field (not necessarily algebraically closed) of characteristic zero. We let Ω_k be the module of absolute Kähler differentials of k . The second part of Theorem 1.1 is an easy consequence of the following general result.

Theorem 4.4. *Let $\overline{C} \subset \mathbb{P}_k^2$ be the cuspidal curve given by the homogeneous equation $x_0x_1^2 - x_2^3 = 0$. Then $\mathrm{CH}_0(\mathbb{P}_k^2|\overline{C}) \cong \Omega_k \oplus \mathbb{Z}$ and $H_0^{sus}(\mathbb{P}_k^2 \setminus \overline{C}) \cong \mathbb{Z}$.*

Proof. We shall freely use the duality results from [55]. We drop the subscript k from the notations of \mathbb{P}_k^n and \mathbb{A}_k^n .

Let us first prove that $H_0^{sus}(\mathbb{P}_k^2 \setminus \overline{C}) \cong \mathbb{Z}$. Let $C_\infty = (0 : 1 : 0)$ be the point at infinity and $P = (1 : 0 : 0)$ the singular point of \overline{C} . We write $C = \overline{C} \setminus C_\infty = (\mathbb{P}_k^2 \setminus H_\infty) \cap \overline{C}$, where $H_\infty \cong \mathbb{P}_k^1$ is the line $\{X_0 = 0\}$. We claim that there is a split exact sequence

$$(4.2) \quad 0 \rightarrow k^\times \rightarrow H_c^4(\mathbb{A}^2 \setminus C, \mathbb{Z}(2)) \rightarrow \mathbb{Z} \rightarrow 0$$

Indeed, letting $U = \mathbb{A}^2 \setminus C$, the localization triangle

$$M^c(C) \rightarrow M^c(\mathbb{A}^2) \rightarrow M^c(U) \rightarrow M^c(C)[1]$$

in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$ gives rise to an exact sequence of cohomology with compact support

$$(4.3) \quad \cdots \rightarrow H_c^3(C, \mathbb{Z}(2)) \rightarrow H_c^4(U, \mathbb{Z}(2)) \rightarrow H_c^4(\mathbb{A}^2, \mathbb{Z}(2)) \rightarrow H_c^4(C, \mathbb{Z}(2)).$$

By duality and homotopy invariance, we have

$$H_c^4(\mathbb{A}^2, \mathbb{Z}(2)) = H_0(\mathbb{A}^2, \mathbb{Z}(0)) = H_0(k, \mathbb{Z}(0)) = \mathbb{Z}.$$

Since k is infinite and U is a rational variety, we can choose $x \in U(k)$. We then get a sequence of maps

$$H^0(k(x), \mathbb{Z}(0)) \rightarrow H_c^4(U, \mathbb{Z}(2)) \rightarrow H_c^4(\mathbb{A}^2, \mathbb{Z}(2)) \rightarrow H^0(k, \mathbb{Z}(0)),$$

where the first arrow is the Gysin homomorphism and the last arrow is the push-forward map between cohomology with compact support induced by the flat morphism $\mathbb{A}^2 \rightarrow \mathrm{Spec}(k)$. Since the composition is known to be an isomorphism, we get a split surjection $H_c^4(U, \mathbb{Z}(2)) \twoheadrightarrow \mathbb{Z}$.

We are left to compute $H_c^3(C, \mathbb{Z}(2))$. Comparing the exact triangles for C and for \mathbb{A}^1 , again we get $H_c^3(C, \mathbb{Z}(2)) = H_c^3(\mathbb{A}^1, \mathbb{Z}(2))$. By duality and homotopy invariance, the latter is isomorphic to $H_{-1}(k, \mathbb{Z}(-1)) = H^1(k, \mathbb{Z}(1)) = k^\times$. The long exact sequence (4.3) then gives

$$H_c^3(\mathbb{A}^2, \mathbb{Z}(2)) \rightarrow k^\times \rightarrow H_c^4(U, \mathbb{Z}(2)) \rightarrow \mathbb{Z} \rightarrow 0.$$

Since $H_c^3(\mathbb{A}^2, \mathbb{Z}(2)) = H_1(\mathbb{A}^2, \mathbb{Z}) = H_1(k, \mathbb{Z}) = 0$ and $H_0(U, \mathbb{Z}(0)) \cong H_c^4(U, \mathbb{Z}(2))$ by duality, we finally get (4.2).

Next, note that $H_\infty \setminus C_\infty = \mathbb{A}^1$. Looking at the triangle in $\mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$:

$$M^c(H_\infty \setminus C_\infty) \rightarrow M^c(\mathbb{P}^2 \setminus \overline{C}) \rightarrow M^c(U) \rightarrow M^c(H_\infty \setminus C_\infty)[1],$$

we get

$$k^\times = H_c^3(\mathbb{A}^1, \mathbb{Z}(2)) \rightarrow H_c^4(U, \mathbb{Z}(2)) \rightarrow H_c^4(\mathbb{P}^2 \setminus \overline{C}, \mathbb{Z}(2)) \rightarrow 0.$$

Since $H_c^4(U, \mathbb{Z}(2)) = k^\times \oplus \mathbb{Z}$ by (4.2), it's clear by construction that $H_c^4(\mathbb{P}^2 \setminus \overline{C}, \mathbb{Z}(2)) = H_0^{sus}(\mathbb{P}^2 \setminus \overline{C}, \mathbb{Z}(0)) = \mathbb{Z}$.

We now turn to the first statement of the theorem. By [6, Thm. 1.1], we have an identification $\mathrm{CH}_0(\mathbb{P}^2|\overline{C}) \cong H_{\mathrm{Zar}}^2(\mathbb{P}^2, \mathcal{K}_{2,(\mathbb{P}^2, \overline{C})}^M)$. Since the relative K_0 -group $K_0(\mathbb{P}^2, \overline{C})$ decomposes as $H_{\mathrm{Zar}}^2(\mathbb{P}^2, \mathcal{K}_{2,(\mathbb{P}^2, \overline{C})}^M) \oplus \mathrm{Pic}(\mathbb{P}^2, \overline{C})$ and the group $\mathrm{Pic}(\mathbb{P}^2, \overline{C})$ vanishes by direct inspection, it is enough to compute $K_0(\mathbb{P}^2, \overline{C})$.

This can be done explicitly by means of the long localization sequence for K -theory with support. In particular, there is a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow K_0(\mathbb{P}^2, \overline{C}) \rightarrow K_0(\mathbb{A}^2, C) \cong SK_1(C) \rightarrow 0$$

and we are left to show that $SK_1(C) \cong \Omega_k$. This is classical, and can be done by descent along the normalization morphism $\nu: \mathbb{A}^1 \rightarrow C$ using the $K_1(A, B, I)$ computation of [18]. We leave the details to the interested reader. \square

4.3. Kerz-Saito vs. Russell's Chow groups with modulus. There exists another notion different from the one used in this paper (and specifically used in the previous theorem) of Chow group of 0-cycles with modulus. This Chow group was introduced by H. Russell [47] in the study of a higher dimensional analogue of the generalized Jacobian of Rosenlicht-Serre. Russell's Chow group with modulus is defined as the cokernel of the map

$$\text{div}: \bigoplus_{\phi_C: C \rightarrow X} G(C, \phi_C^*(D - D_{\text{red}}) + (\phi_C^*(D))_{\text{red}}) \rightarrow \mathcal{Z}_0(X, D).$$

In particular, in view of Theorem 2.4, it is clear that it agrees with $H_0^{sus}(X \setminus D)$ when D is reduced and X is proper. Theorem 4.4 then exhibits explicitly an example (at least in characteristic zero) for which Russell's Chow group and the Kerz-Saito Chow group do not agree. Several authors have enquired in the past about this comparison. Note that in view of [6, Thm. 1.1], it follows that Russell's Chow group does not satisfy Bloch's formula, even for D reduced. In particular, it may not admit a cycle class map to relative K -theory.

4.4. Finite coefficient case. We now prove Theorem 1.2. We restate it for convenience.

Theorem 4.5. *Let k be a perfect field and X a smooth projective k -scheme. Let $D \subset X$ be an effective Cartier divisor with complement U . Assume that the irreducible components of D_{red} are regular. Then the map*

$$\lambda_{X|D}: \text{CH}_0(X|D)_R \rightarrow H_0^{sus}(U)_R$$

is an isomorphism, where $R = [\frac{1}{p}]$ if $\text{char}(k) = p > 0$ and \mathbb{Z}/m with $m \neq 0$ if $\text{char}(k) = 0$.

Proof. When $\dim(X) = 1$, we can use [46, Thm. 1.3(2)] to assume that D is reduced. In the latter case, the theorem holds integrally as one can directly check. We therefore assume $d \geq 2$. We can also assume X is integral. Since both sides of (2.5) are quotients of the free abelian group $\mathcal{Z}_0(U)$, we need to show that any relation imposed on $\mathcal{Z}_0(X|D)$ to define $H_0^{sus}(U)$ lies in the image of $\mathcal{R}_0(X|D) \rightarrow \mathcal{Z}_0(U)_R$, where $\mathcal{R}_0(X|D) = \text{Ker}(\mathcal{Z}_0(U) \twoheadrightarrow \text{CH}_0(X|D))$.

So let $C \subset X$ be an integral curve not contained in D and let $\nu: C^N \rightarrow C \hookrightarrow X$ be the composite map, where C^N is the normalization of C . Let $f \in k(C^N)^\times$ be a rational function on C^N which is regular in an affine neighborhood of $\nu^{-1}(D)$ such that $f(x) = 1$ for every $x \in \nu^{-1}(D)$. We need to show that $\text{div}(f)$ dies in $\text{CH}_0(X|D)_R$.

Since ν is finite, we can find a factorization

$$(4.4) \quad \begin{array}{ccc} C^N & \xrightarrow{\phi} & \mathbb{P}_X^n \\ & \searrow \nu & \downarrow \pi \\ & & X, \end{array}$$

where ϕ is a closed immersion and π is the canonical projection. Since π is smooth, the irreducible components of $D' := \mathbb{P}_D^n$ are regular. Furthermore, $C^N \not\subset D'$. Since $\pi_*(\operatorname{div}(f)_{C^N}) = \operatorname{div}(f)_C = \operatorname{div}(f)$ under the proper push-forward map $\pi_*: \mathcal{Z}_0(\mathbb{P}_U^n) \rightarrow \mathcal{Z}_0(U)$ and since $\pi_*(\mathcal{R}_0(\mathbb{P}_X^n|D')) \subset \mathcal{R}_0(X|D)$ (see e.g., [36, § 2] or [7, Lem. 2.7]), it suffices to show that $\operatorname{div}(f)_{C^N}$ dies in $\operatorname{CH}_0(\mathbb{P}_X^n|\mathbb{P}_D^n)_R$. Up to replacing X with \mathbb{P}_X^n and C with C^N , we can therefore assume that $C \subset X$ is regular so that ν is a closed immersion of regular schemes.

Since D_{red} is reduced with regular irreducible components, we can apply [48, Prop. A.6] to find a finite sequence of blow-ups $\phi: \tilde{X} \rightarrow X$ over the closed points of $D \cap C$ such that the scheme theoretic inverse image $\tilde{D} := \tilde{X} \times_X D$ satisfies the following.

- (1) \tilde{D}_{red} is reduced and its irreducible components are regular.
- (2) The strict transform \tilde{C} of C is regular and intersects \tilde{D}_{red} only in the regular locus of \tilde{D}_{red} and transversely.

Since ϕ is proper, we have a commutative diagram

$$(4.5) \quad \begin{array}{ccc} \mathcal{Z}_0(U) & \twoheadrightarrow & \operatorname{CH}_0(\tilde{X}|\tilde{D}) \\ \phi_* \downarrow \cong & & \downarrow \phi_* \\ \mathcal{Z}_0(U) & \twoheadrightarrow & \operatorname{CH}_0(X|D), \end{array}$$

where ϕ_* is the push-forward map of the 0-cycles groups. Since $\phi: \tilde{C} \rightarrow C$ is an isomorphism, we see that $f \in k(\tilde{C})^\times$ such that $\phi_*(\operatorname{div}(f)_{\tilde{C}}) = \operatorname{div}(f)_C = \operatorname{div}(f)$. Moreover, f is regular in a neighborhood of $\tilde{D} \cap \tilde{C}$ with $f(x) = 1$ for every $x \in \tilde{D} \cap \tilde{C} \subset \phi^{-1}(D \cap C)$. In particular, $\operatorname{div}(f)_{\tilde{C}} \in \operatorname{Ker}(\mathcal{Z}_0(U) \rightarrow H_0^{\text{sus}}(U))$.

It follows from (4.5) that $\operatorname{div}(f)_C$ will die in $\operatorname{CH}_0(X|D)_R$ if we can show that $\operatorname{div}(f)_{\tilde{C}}$ dies in $\operatorname{CH}_0(\tilde{X}|\tilde{D})_R$. We can therefore assume that our original curve $C \subset X$ has the property that it is regular and it intersects D_{red} only in the regular locus of D_{red} and transversely. But it is immediate from various definitions that in this case, $f \in k(C)^\times$ is regular in a neighborhood of E and $f(x) = 1$ for all $x \in E$ if and only if $f \in \operatorname{Ker}(\mathcal{O}_{C,E}^\times \rightarrow \mathcal{O}_E^\times)$, where $E = C \times_X D_{\text{red}}$. The latter condition implies that $\operatorname{div}(f) \in \mathcal{R}_0(X|D_{\text{red}})$. That is, $\operatorname{div}(f)$ dies in $\operatorname{CH}_0(X|D_{\text{red}})$. By [46, Thm. 1.3(2)], this implies that $\operatorname{div}(f)$ dies in $\operatorname{CH}_0(X|D)_R$. This concludes the proof. \square

5. BLOCH'S TORSION THEOREM FOR SUSLIN HOMOLOGY

The goal of this section is to prove Theorem 1.3. We shall first establish some preliminary results of independent interest.

5.1. A torsion theorem for the double. We fix an algebraically closed field k of characteristic zero and a prime ℓ . Let X be an integral smooth projective surface over k and let $D \subset X$ be a reduced effective Cartier divisor. We let S_X be the double of X along D .

Lemma 5.1. *The canonical map*

$$\lambda_{S_X}: \operatorname{CH}_0^{LW}(S_X)\{\ell\} \rightarrow H^4(S_X, \mathbb{Z}(2))\{\ell\}$$

is an isomorphism.

Proof. By a Lefschetz principle argument, we can assume that $k = \mathbb{C}$. In this case, the Albanese homomorphism $\rho_{S_X}: \text{CH}_0^{LW}(S_X)_{\text{deg } 0} \rightarrow J^2(S_X)$ has a factorization

$$\text{CH}_0^{LW}(S_X)_{\text{deg } 0} \xrightarrow{\lambda_{S_X}} H^4(S_X, \mathbb{Z}(2))_{\text{deg } 0} \xrightarrow{\rho'_{S_X}} J^2(S_X)$$

by [37, Thm. 8.4] such that ρ'_{S_X} is an isomorphism on the torsion subgroups. On the other hand, the main result of [3] asserts that ρ_{S_X} is an isomorphism on the torsion subgroups. This forces λ_{S_X} to be an isomorphism on the torsion subgroups. \square

By [35, Thm. 3.6], there is a canonical isomorphism $H_{\text{ét}}^3(S_X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \xrightarrow{\cong} \text{CH}_0^{LW}(S_X)\{\ell\}$. One has a similar isomorphism for X by Bloch. Let γ_{S_X} be the composition of this isomorphism with λ_{S_X} . We denote the corresponding map for X by γ_X .

Lemma 5.2. *There is a commutative diagram*

$$(5.1) \quad \begin{array}{ccc} H_{\text{ét}}^3(S_X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \xrightarrow{\iota_-^*} & H_{\text{ét}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \\ \gamma_{S_X} \downarrow & & \downarrow \gamma_X \\ H^4(S_X, \mathbb{Z}(2))\{\ell\} & \xrightarrow{\iota_-^*} & H^4(X, \mathbb{Z}(2))\{\ell\} \end{array}$$

such that the vertical arrows are isomorphisms.

Proof. By [35, Thm. 3.6], there is a canonical isomorphism $\gamma_{S_X}: H_{\text{ét}}^3(S_X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \xrightarrow{\cong} \text{CH}_0^{LW}(S_X)\{\ell\}$ (this is true for any complete surface). Moreover, it was shown in [35, Thm. 5.4] (see diagram (5.13) of op. cit.) that γ_{S_X} and γ_X are compatible with ι_-^* . Since λ_{S_X} and λ_X are clearly compatible with ι_-^* , our assertion follows by Lemma 5.1. \square

Lemma 5.3. *Let $U = X \setminus D$. There is a canonical isomorphism*

$$\gamma_U: H_{c, \text{ét}}^3(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \xrightarrow{\cong} H_0^{\text{sus}}(U)\{\ell\}.$$

Proof. We have a commutative diagram of (split) exact sequences

$$(5.2) \quad \begin{array}{ccccccc} 0 \rightarrow & H_{c, \text{ét}}^3(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \xrightarrow{p_{+, *}} & H_{\text{ét}}^3(S_X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \xrightarrow{\iota_-^*} & H_{\text{ét}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \rightarrow 0 \\ & \vdots \downarrow & & \downarrow \gamma_{S_X} & & \downarrow \gamma_X & \\ 0 \longrightarrow & H_0^{\text{sus}}(U)\{\ell\} & \xrightarrow{p_{+, *}} & H^4(S_X, \mathbb{Z}(2))\{\ell\} & \xrightarrow{\iota_-^*} & H^4(X, \mathbb{Z}(2))\{\ell\} & \rightarrow 0. \end{array}$$

Using this diagram, we conclude the proof by Lemma 5.2. \square

5.2. Proof of Theorem 1.3. Let k be an algebraically closed field and U a quasi-projective k -scheme of pure dimension $d \geq 0$. Let $U \hookrightarrow X$ be a dense open immersion such that X is a smooth projective k -scheme. Let ℓ be a prime different from $\text{char}(k)$. We can now prove Theorem 1.3. We restate it for convenience.

Theorem 5.4. *There is an isomorphism*

$$(5.3) \quad \vartheta_U: H_0^{\text{sus}}(U)\{\ell\} \xrightarrow{\cong} H_{c, \text{ét}}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)).$$

Proof. We can assume $d \geq 1$ because the theorem is trivial otherwise. We can assume U to be integral. Let $D \subset X$ be the complement of U with reduced closed subscheme structure. Let $p \geq 1$ be the exponential characteristic of p . We consider several cases.

Assume first that $p > 1$ and $\dim(D) \leq d - 2$. Let $\Lambda = \mathbb{Z}[\frac{1}{p}]$. Since $A\{\ell\} \xrightarrow{\cong} A_\Lambda\{\ell\}$ for any abelian group A , we can tensor both sides of (5.3) with Λ . We then have the localization exact sequence (see (2.4) and the discussion below it)

$$H^{2d-1}(D, \Lambda(d)) \rightarrow H_0^{sus}(U)_\Lambda \rightarrow \mathrm{CH}_0(X)_\Lambda \rightarrow H^{2d}(D, \Lambda(d)).$$

The first and the last terms of this exact sequence vanish by [37, Thm. 5.1], and hence we get $H_0^{sus}(U)_\Lambda \xrightarrow{\cong} \mathrm{CH}_0(X)_\Lambda$. Using the localization sequence for étale cohomology and repeating the same argument, we also get $H_{c,\acute{e}t}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\cong} H_{\acute{e}t}^{2d-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. We therefore reduce to the case when U is smooth and projective, where Bloch's theorem applies.

We next assume that $p > 1$ and $\dim(D) = d - 1$. Let $D' \subset D$ be the union of irreducible components of D which have dimension $d - 1$ and let $U' = X \setminus D'$. We have an exact sequence

$$H_c^{2d-1}(U' \setminus U, \Lambda(d)) \rightarrow H_0^{sus}(U)_\Lambda \rightarrow H_0^{sus}(U')_\Lambda \rightarrow H_c^{2d}(U' \setminus U, \Lambda(d)).$$

The two end terms vanish by [38, Lem. 6.2] because $\dim(U' \setminus U) \leq d - 2$. Using the localization sequence for étale cohomology and repeating the same argument, we get $H_{c,\acute{e}t}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\cong} H_{c,\acute{e}t}^{2d-1}(U', \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$. It suffices therefore to prove the theorem for U' . We can therefore assume that D has pure dimension $d - 1$. Since X is smooth, D is necessarily a Cartier divisor on X . In this case, we can combine Theorem 4.3 and [35, Thm. 1.1] to conclude our result.

We now assume $p = 1$. The proof in the positive characteristic case remains valid in characteristic zero if $d \leq 1$. However, it breaks down when $d \geq 2$ because Theorem 4.3 is no more available with us. We shall adopt a different strategy which involves revisiting the proof of [35, Thm. 1.1].

We have already argued above that D can be assumed to be a Cartier divisor on X . This part of the argument does not involve $\mathrm{char}(k)$. We shall assume that X is integral and prove the theorem by induction on d . When $d = 2$, we can apply Lemma 5.3.

We now assume $d \geq 3$. We fix a closed embedding $X \hookrightarrow \mathbb{P}_k^n$. By [2, Thm. 1], we can find a hypersurface $H \subset \mathbb{P}_k^n$ of sufficiently high degree such that the scheme theoretic intersection $Y = X \cap H$ satisfies the following.

- (1) Y is smooth and irreducible of dimension $d - 1$.
- (2) Y contains no component of D .
- (3) $Y \cap U \neq \emptyset$.

It follows from (1) that Y is integral. We let $V = Y \cap U$ and let $\tau: V \hookrightarrow U$ be the inclusion. We consider the diagram

$$(5.4) \quad \begin{array}{ccc} H_{c,\acute{e}t}^{2d-3}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)) & \xrightarrow{\gamma_V} & H_0^{sus}(V)\{\ell\} \\ \tau_* \downarrow & & \downarrow \tau_* \\ H_{c,\acute{e}t}^{2d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \cdots \cdots \cdots & H_0^{sus}(U)\{\ell\} \end{array}$$

The left vertical arrow is an isomorphism by [20, Prop. 5.1]. Since γ_V exists and is an isomorphism by induction, it remains to show that the right vertical arrow is an isomorphism to finish the proof.

We now look at the commutative diagram

$$(5.5) \quad \begin{array}{ccc} H_0^{sus}(V)\{\ell\} & \xrightarrow{\rho_V} & J^{d-1}(V)\{\ell\} \\ \tau_* \downarrow & & \downarrow \tau_* \\ H_0^{sus}(U)\{\ell\} & \xrightarrow{\rho_U} & J^d(U)\{\ell\}, \end{array}$$

where the horizontal arrows are the Albanese maps of Theorem 3.2. The horizontal arrows are isomorphisms by [52, Thm. 1.1] and the right vertical arrow is an isomorphism by [20, Thm. 7.3]. This forces the left vertical arrow to be an isomorphism. \square

Remark 5.5. One of the referees remarked that a proof of Theorem 5.4 can also be obtained using the Roitman torsion theorem of Spieß and Szamuely [52] together with Poincaré duality. From a different perspective, one could use Theorem 5.4 to show that the torsion theorem of [52] in positive characteristic is a direct consequence of Bloch's theorem together with the analogous result for singular varieties.

6. THE FINITENESS THEOREMS

In this section, we shall derive Theorem 1.5 from some general finiteness results for motivic cohomology groups. Let k be a field. Let $m \geq 1$ be an integer which is invertible in k , and $p \nmid m$ if k is a p -adic field. Set $\Lambda' = \mathbb{Z}/m$. Let X be a smooth projective k -scheme.

If k is algebraically closed, then it is classically known that $\mathrm{CH}_0(X)/m \cong \Lambda'$. Otherwise, the structure of $\mathrm{CH}_0(X)/m$ is not fully understood. If k is finite, then the main result of unramified class field theory in geometric case due to Kato-Saito [28] shows that $\mathrm{CH}_0(X)/m$ is finite. If k is a p -adic field, a conjecture of Colliot-Thélène predicted that $\mathrm{CH}_0(X)/m$ is finite. This conjecture was settled by Saito and Sato in [48] (see also [12, Prop. 9.3]).

Let us now recall that there is an isomorphism $\mathrm{CH}_0(X) \cong H_0^{sus}(X)$. One therefore concludes that $H_0^{sus}(X)/m$ is finite if k is either of the above three types. Suppose now that U is a smooth quasi-projective k -scheme which is not necessarily complete. One can then ask if $H_0^{sus}(X)/m$ is finite. One can similarly ask if $\mathrm{CH}_0(X|D)/m$ is finite if D is an effective Cartier divisor on a smooth projective scheme X over k . To our knowledge, no answer is known about these questions yet. We shall answer these in this section.

6.1. A general finiteness result. In this subsection, we prove a general result about the finiteness of motivic cohomology groups of possibly singular schemes.

Lemma 6.1. *Let k be either a finite, or a p -adic or an algebraically closed field. Let X be a smooth projective k -scheme of dimension $d \geq 0$. Then $H^{2d+i}(X, \Lambda'(d+i))$ is finite for every $i \geq 0$.*

Proof. If X is integral of dimension d , then $H^{2a-b}(X, \mathbb{Z}(b)) \cong \mathrm{CH}^a(X, b) = 0$ for $a > d + b$. This allows us to assume that X has pure dimension d . Subsequently, we can assume that X is integral. Using the above discussion of $i = 0$ case, we can assume that $i \geq 1$.

We now note that over any field k , there is a canonical isomorphism $H^{2d+i}(X, \Lambda'(d+i)) \cong \mathrm{CH}^{d+i}(X, i)/m$. Furthermore, Bloch's spectral sequence $H_{\mathrm{zar}}^p(X, \mathcal{CH}^q(X, i)) \Rightarrow \mathrm{CH}^{p+q}(X, i)$ and the isomorphisms $\mathrm{CH}^p(\mathcal{O}_{X,x}, p) \cong H^p(\mathcal{O}_{X,x}, \mathbb{Z}(p)) \cong K_p^M(\mathcal{O}_{X,x})$

(the latter is Gabber-Kerz improved Milnor K -theory, [31]) together yield an exact sequence

$$(6.1) \quad \bigoplus_{y \in X_{(1)}} K_{i+1}^M(k(y)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} K_i^M(k(x)) \rightarrow \mathrm{CH}^{d+i}(X, i) \rightarrow 0.$$

We thus get an exact sequence

$$(6.2) \quad \bigoplus_{y \in X_{(1)}} K_{i+1}^M(k(y))/m \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} K_i^M(k(x))/m \rightarrow \mathrm{CH}^{d+i}(X, i)/m \rightarrow 0.$$

If k is algebraically closed, then $K_i^M(k)$ is divisible for $i \geq 1$ and hence we conclude that $\mathrm{CH}^{d+i}(X, i)/m = 0$ for $i \geq 1$. This proves the lemma for k algebraically closed. For k finite, the lemma follows from [1, Thm. 1.2, 1.3].

Suppose now that k is a p -adic field with residue characteristic p . We first assume $i = 1$. In this case, we can apply Temkin's strengthening of the alteration theorems of de Jong and Gabber (see [53, Thm. 1.2.5]) to find a projective and generically finite morphism $f: X' \rightarrow X$ of degree p^n such that X' is a smooth projective scheme over k which admits a strict semi-stable reduction over k . Since the composition $\mathrm{CH}^{d+1}(X, 1) \xrightarrow{f^*} \mathrm{CH}^{d+1}(X', 1) \xrightarrow{f_*} \mathrm{CH}^{d+1}(X, 1)$ is multiplication by p^n , it follows that $\mathrm{CH}^{d+1}(X, 1)/m \hookrightarrow \mathrm{CH}^{d+1}(X, 1)/m$. We can therefore assume that X admits a strict semi-stable reduction over k . In this case, the finiteness of $\mathrm{CH}^{d+1}(X, 1)/m$ follows from [13, Thm. 3.1.4].

We now assume $i \geq 2$. Let $\mathcal{H}_{\acute{e}t}^p(q)$ be the Zariski sheaf on X associated to the presheaf $U \mapsto H_{\acute{e}t}^p(U, \Lambda'(q))$. Let $X_{(i)}$ be the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = i$. We consider the commutative diagram with exact rows:

$$(6.3) \quad \begin{array}{ccccc} \bigoplus_{y \in X_{(1)}} K_{i+1}^M(k(y))/m & \xrightarrow{\partial} & \bigoplus_{x \in X_{(0)}} K_i^M(k(x))/m & \longrightarrow & \mathrm{CH}^{d+i}(X, i)/m \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{y \in X_{(1)}} H_{\acute{e}t}^{i+1}(k(y), \Lambda'(i+1)) & \xrightarrow{\partial} & \bigoplus_{x \in X_{(0)}} H_{\acute{e}t}^i(k(x), \Lambda'(i)) & \rightarrow & H_{\mathrm{zar}}^d(X, \mathcal{H}_{\acute{e}t}^{d+i}(d+i)) \rightarrow 0, \end{array}$$

where the bottom row is part of the Bloch-Ogus sequence for the étale cohomology sheaves and the first two vertical arrows are the Norm-residue homomorphisms. The two vertical arrows from the left are isomorphisms by the solution of the Bloch-Kato conjecture due to Rost and Voevodsky. It follows that the right vertical arrow is also an isomorphism. It suffices therefore to show that $H_{\mathrm{zar}}^d(X, \mathcal{H}_{\acute{e}t}^{d+i}(d+i))$ is finite.

We now consider the Leray spectral sequence

$$E_2^{p,q} = H_{\mathrm{zar}}^p(X, \mathcal{H}_{\acute{e}t}^q(d+i)) \Rightarrow H_{\acute{e}t}^{p+q}(X, \Lambda'(d+i)).$$

Since the étale cohomological dimension of any affine in X is bounded by $d+2$, one easily checks that $E_2^{d,d+i} = E_{\infty}^{d,d+i}$ and there is an isomorphism $E_{\infty}^{d,d+i} \xrightarrow{\cong} H_{\acute{e}t}^{2d+i}(X, \Lambda'(d+i))$. But the latter group is finite by the proper base change theorem. This concludes the proof. \square

Proposition 6.2. *Let k be as in Lemma 6.1 and let X be any projective k -scheme of dimension $d \geq 0$ over k . Then $H^{2d+i}(X, \Lambda'(d+i))$ is finite for every $i \geq 0$.*

Proof. Let q denote the exponential characteristic of k . Since the localization map $H^{2d+i}(X, \Lambda'(d+i)) \rightarrow H^{2d+i}(X, \Lambda'(d+i))[\frac{1}{q}]$ is injective, it suffices to show that the

latter group is finite. We can assume that X is reduced. If X is smooth, we are done by Lemma 6.1.

If X is not smooth, we argue as follows. By Temkin's strengthening of the alteration theorems of de Jong and Gabber (see [53, Thm. 1.2.5]), there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \rightarrow X$, which is generically finite and étale of degree q^n , $n \geq 1$. Then by a theorem of Raynaud-Gruson [22, Thm. 5.2.2], there exists a blow-up $g : X' \rightarrow X$ with center Z such that the following diagram commutes, where h' is finite, flat and surjective of degree q^n and $g' : W' \rightarrow W$ is the blow-up of W with center $h^{-1}(Z)$:

$$(6.4) \quad \begin{array}{ccc} W' & \xrightarrow{h'} & X' \\ g' \downarrow & & \downarrow g \\ W & \xrightarrow{h} & X. \end{array}$$

Thus we have a cdh -cover $\{X' \sqcup Z \rightarrow X\}$ of X , such that $\dim_k(Z) < \dim_k(X)$ and $\dim_k(E) < \dim_k(X)$, where we set $E = X' \times_X Z$. By cdh -excision of [30, Prop. 5.3.4] (see also Thm. 2.3.14 of the same reference), the following sequence is exact:

$$H^{2d+i-1}(E, \Lambda'(d+i))\left[\frac{1}{q}\right] \rightarrow H^{2d+i}(X, \Lambda'(d+i))\left[\frac{1}{q}\right] \rightarrow H^{2d+i}(X', \Lambda'(d+i))\left[\frac{1}{q}\right] \oplus H^{2d+i}(Z, \Lambda'(d+i))\left[\frac{1}{q}\right].$$

By induction on dimension, this reduces to showing that $H^{2d+i}(X', \Lambda'(d+i))\left[\frac{1}{q}\right]$ is finite.

If we let $Z' = h^{-1}(Z)$ and $E' = W' \times_W Z'$, then the same reasoning as above shows that there is an exact sequence

$$H^{2d+i}(W, \Lambda'(d+i))\left[\frac{1}{q}\right] \rightarrow H^{2d+i}(W', \Lambda'(d+i))\left[\frac{1}{q}\right] \rightarrow \frac{H^{2d+i}(E', \Lambda'(d+i))\left[\frac{1}{q}\right]}{H^{2d+i}(Z', \Lambda'(d+i))\left[\frac{1}{q}\right]}.$$

The left term is finite since W is smooth and projective, and the right term is finite by induction on the dimension. It follows that the middle term is finite too. It suffices therefore to show that the map $h'^* : H^{2d+i}(X', \Lambda'(d+i))\left[\frac{1}{q}\right] \rightarrow H^{2d+i}(W', \Lambda'(d+i))\left[\frac{1}{q}\right]$ is injective. But this follows from [9, Prop. 6.3] because h' is finite and flat (in particular, surjective) of degree q^n . This completes the proof. \square

6.2. Proofs of parts (1) and (2) of Theorem 1.5. The following result generalizes the finiteness theorems of Kato-Saito [28] and Saito-Sato [48] to non-complete schemes. The question whether such a finiteness result would hold originated out of a discussion between Colliot-Thélène and the authors. In the two results below, k is either a finite, or a p -adic or an algebraically closed field.

Theorem 6.3. *Let U be a smooth quasi-projective k -scheme of pure dimension $d \geq 0$. Then $H_0^{sus}(U)/m$ is finite.*

Proof. We can assume that U is integral. Let p be the exponential characteristic of k . It suffices to show that $(H_0^{sus}(U)/m)\left[\frac{1}{p}\right]$ is finite. We have seen in § 2.3 that $(H_0^{sus}(U)/m)\left[\frac{1}{p}\right]$ agrees with $(H_c^{2d}(U, \mathbb{Z}(d))/m)\left[\frac{1}{p}\right]$. The theorem is thus equivalent to the statement that $(H_c^{2d}(U, \mathbb{Z}(d))/m)\left[\frac{1}{p}\right]$ is finite. By the universal coefficient theorem, we have an exact sequence

$$(6.5) \quad 0 \rightarrow (H_c^{2d}(U, \mathbb{Z}(d))/m)\left[\frac{1}{p}\right] \rightarrow H_c^{2d}(U, \Lambda'(d))\left[\frac{1}{p}\right] \rightarrow {}_m H_c^{2d+1}(U, \mathbb{Z}(d))\left[\frac{1}{p}\right].$$

We choose an open embedding $U \subset X$ such that X is an integral projective scheme of dimension d over k . We let $D = X \setminus U$ with reduced closed subscheme structure. We then have an exact sequence

$$H^{2d}(D, \mathbb{Z}(d))\left[\frac{1}{p}\right] \rightarrow H_c^{2d+1}(U, \mathbb{Z}(d))\left[\frac{1}{p}\right] \rightarrow H^{2d+1}(X, \mathbb{Z}(d))\left[\frac{1}{p}\right].$$

The two end terms are zero by [37, Thm. 5.1]. It follows that so is the middle term. Using (6.5), we are reduced to showing that $H_c^{2d}(U, \Lambda'(d))\left[\frac{1}{p}\right]$ is finite. But this follows from the exact sequence

$$H^{2d-1}(D, \Lambda'(d))\left[\frac{1}{p}\right] \rightarrow H_c^{2d}(U, \Lambda'(d))\left[\frac{1}{p}\right] \rightarrow H^{2d}(X, \Lambda'(d))\left[\frac{1}{p}\right]$$

and Proposition 6.2. \square

The second main result of this section is the following generalization of the finiteness theorems of Kato-Saito [28] and Saito-Sato [48] to Chow groups with modulus.

Theorem 6.4. *Let X be a smooth projective k -scheme and $D \subset X$ an effective Cartier divisor. Then $\mathrm{CH}_0(X|D)/m$ is finite.*

Proof. If k is finite, the theorem is already known by [24, Thms. 1.3, 4.8]. Next, suppose that k is a p -adic field. We can then find a proper birational morphism $f: X' \rightarrow X$ from a smooth projective scheme X' such that $f^{-1}(D)_{\mathrm{red}}$ is a strict normal crossing divisor on X' and f is an isomorphism over the complement of D . It is clear from the definition of the Chow group of 0-cycles with modulus that the pull-back map $f^*: \mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0(X'|D')$ exists and is an isomorphism, where $D' = f^*(D)$. We can therefore assume that D_{red} is a simple normal crossing divisor. We are now done by Theorems 1.2 and 6.3.

We now assume that k is algebraically closed. Using (2.7), we only need to show that $\mathrm{CH}_0^{BK}(S_X)_{\mathrm{deg} 0}$ is m -divisible. Since every 0-cycle of degree zero on S_X lies on a complete intersection reduced curve $C \subset S_X$, it suffices to show that $\mathrm{Pic}^0(C)$ is m -divisible. But this is classical. \square

6.3. Proof of Theorem 1.5(3). We shall now complete the proof of Theorem 1.5 by proving its part (3). If k is algebraically closed, the last part of the proof of Theorem 6.4 shows that $\mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0}$ is m -divisible. If k is finite, part (3) of Theorem 1.5 follows from [20, Thm. 1.3]. The remaining case follows from the following general result of independent interest.

Let k be any field and let $m \neq 0$ be an integer. Let X be an integral projective k -scheme which is regular is codimension one. It is shown in [20, Lem. 8.3] that the identity map of $\mathcal{Z}_0(U)$ defines a surjection $\theta_X: \mathrm{CH}_0^{LW}(X) \twoheadrightarrow H_0^{sus}(U)$ if we let $U = X_{\mathrm{reg}}$.

Theorem 6.5. *Assume that $\mathrm{char}(k) = 0$. Then the map*

$$\theta_X/m: \mathrm{CH}_0^{LW}(X)/m \rightarrow H_0^{sus}(U)/m$$

is an isomorphism.

Proof. There is nothing to prove if $d \leq 1$ and we can thus assume that $d \geq 2$. We shall prove the statement by induction on d . We first consider the starting case $d = 2$.

Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities of X such that the reduced exceptional divisor $E \subset \tilde{X}$ has strict normal crossings. For any integer $n \geq 1$, let $nE \hookrightarrow \tilde{X}$ denote the infinitesimal thickening of E in \tilde{X} of order n .

It is clear from the definitions of $\mathrm{CH}_0^{LW}(X)$, $\mathrm{CH}_0(\tilde{X}|D)$ and $H_0^{sus}(U)$ that the identity map of $\mathcal{Z}_0(U)$ defines, by the pull-back via π^* , the canonical surjective maps

$$(6.6) \quad \mathrm{CH}_0^{LW}(X) \xrightarrow{\pi^*} \mathrm{CH}_0(\tilde{X}|nE) \twoheadrightarrow \mathrm{CH}_0(\tilde{X}|E) \twoheadrightarrow H_0^{sus}(U)$$

for every integer $n \geq 1$ such that the composite map is θ_X . The first arrow from the left is an isomorphism for all $n \gg 1$ by [20, Thm. 1.5], and the third arrow is an isomorphism modulo m by Theorem 1.2. We thus have to show that the kernel of the surjection $\mathrm{CH}_0(\tilde{X}|nE) \rightarrow \mathrm{CH}_0(\tilde{X}|E)$ is divisible. By [6, Thm. 8.2], this is equivalent to showing that $H_{\mathrm{Zar}}^1(E, \mathcal{K}_{2,(nE,E)})$ is divisible. But this is an immediate consequence of the well known Goodwillie's theorem for the relative K -theory of nilpotent ideals, which implies that $\mathcal{K}_{2,(mE,E)}$ is a sheaf of \mathbb{Q} -vector spaces on E .

We now assume $d \geq 3$. By [20, Cor. 2.5], we can assume that X is normal. We let $J^d(X)$ denote the Albanese variety of X (which coincides with the classical Albanese variety of any desingularization of X) and consider the commutative diagram (see Theorem 3.2)

$$(6.7) \quad \begin{array}{ccc} \mathrm{CH}_0^{LW}(X)_{\mathrm{deg} 0} & \xrightarrow{\theta_X} & H_0^{sus}(U)_{\mathrm{deg} 0} \\ \rho_X \downarrow & & \downarrow \rho_U \\ J^d(X) & \xrightarrow{\theta'_X} & J^d(U). \end{array}$$

The vertical arrows are isomorphisms on the torsion subgroups by [40, Thm. 1] and [52, Thm. 1.1]. Since a surjective morphism of semi-abelian varieties induces surjection on m -torsion subgroups for every $m \geq 1$, it follows that the same happens for θ_X . This implies that $\mathrm{Ker}(\theta_X) \rightarrow \mathrm{Ker}(\theta_X/m)$ is surjective. Hence, it suffices to show that the map $\mathrm{Ker}(\theta_X) \rightarrow \mathrm{CH}_0^{LW}(X)/m$ is zero.

Let $\nu: C \rightarrow X$ be a finite morphism from a regular integral projective curve whose image is not contained in X_{sing} and let $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^{\times} \rightarrow \mathcal{O}_E^{\times})$, where $E = \nu^{-1}(X_{\mathrm{sing}})$ with the reduced closed subscheme structure. It suffices to show that $\nu_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^{LW}(X)/m$. Equivalently, there is a 0-cycle $\alpha \in \mathcal{Z}_0(U)$ such that $\nu_*(\mathrm{div}(f)) - m\alpha \in \mathcal{R}_0^{LW}(X)$.

We can get a factorization $C \xrightarrow{\nu'} \mathbb{P}_X^r \xrightarrow{\pi} X$ of ν , where ν' is a closed immersion and π is the canonical projection. Since the singular locus of \mathbb{P}_X^r coincides with $\mathbb{P}_{X_{\mathrm{sing}}}^r$, it is clear that $\nu'_*(\mathrm{div}(f)) \in \mathcal{R}_0^S(\mathbb{P}_U^r)$ and $\nu_*(\mathrm{div}(f)) = \pi_*(\nu'_*(\mathrm{div}(f)))$. Using [20, Cor. 2.6], it suffices therefore to show that $\nu'_*(\mathrm{div}(f)) - m\alpha' \in \mathcal{R}_0^{LW}(\mathbb{P}_X^r)$ for some $\alpha' \in \mathcal{Z}_0(\mathbb{P}_U^r)$. We can thus assume that $\nu: C \rightarrow X$ is a closed immersion.

We now fix a closed embedding $X \hookrightarrow \mathbb{P}_k^N$. Since U and C are smooth and $d \geq 3$, we can use [2, Thm. 7] or [19, Thm. 3.9, Cor. 3.13] to find a hypersurface $H \subset \mathbb{P}_k^N$ containing C and not containing X such that the scheme theoretic intersection $X \cap H$ has the property that it contains no irreducible component of X_{sing} and $H \cap U$ is smooth and integral.

We let $Y = H \cap U$ and $W = H \cap X$. Since Y is a Cartier divisor on X , we must have $Y_{\mathrm{sing}} = X_{\mathrm{sing}} \cap Y$. Furthermore, $\dim(Y_{\mathrm{sing}}) = \dim(Y \cap X_{\mathrm{sing}}) \leq \dim(X_{\mathrm{sing}}) - 1 \leq d - 3 = \dim(Y) - 2$. In particular, Y is regular in codimension one. Let $C \xrightarrow{\nu'} Y \xrightarrow{\iota} X$ be the factorization of ν . It follows from the choice of Y that $\nu'_*(\mathrm{div}(f)) \in \mathcal{R}_0^S(W)$. It follows by induction on d that $\nu'_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^{LW}(Y)/m$. The push-forward map $\iota_*: \mathcal{Z}_0(W) \rightarrow \mathcal{Z}_0(U)$ and [20, Cor. 2.6] together imply that $\nu_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^{LW}(X)/m$. This concludes the proof. \square

7. CHOW GROUPS OF NORMAL CROSSING SURFACES

We are now left with proving Theorem 1.6, the last of the main results listed in § 1. In order to do this, we need to use an induction procedure on the dimension of normal crossing varieties. In this section, we shall prove some results for surfaces which will allow this induction.

7.1. Chow group with modulus to Levine-Weibel Chow group. The goal of this subsection is to prove the $d = 2$ case of Theorem 8.3. Let k be a field and X a normal crossing variety of dimension $d \geq 1$ over k (see Definition 2.12). We shall use the following notations in this section.

Let $\iota: Y \hookrightarrow X$ be the inclusion of an irreducible component of X and let $E \subset Y$ be the scheme theoretic intersection $\overline{X \setminus Y} \cap Y$. Then Y is a regular and E is a simple normal crossing divisor on Y . Let $D \subset X$ denote the singular locus of X . Let $\iota': Y' = \overline{X \setminus Y} \hookrightarrow X$ be the inclusion of the union of other components of X . Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals on X defining Y' and let $\mathcal{J} \subset \mathcal{O}_Y$ be the sheaf of ideals on Y defining E .

We have the following canonical inclusion of the groups of 0-cycles:

$$(7.1) \quad \iota_*: \mathcal{Z}_0(Y, E) \hookrightarrow \mathcal{Z}_0(X, D).$$

We wish to show that this map preserves the subgroups of rational equivalences when $d \leq 2$. This will be generalized to higher dimensions in the next section. We first deal with the case $d = 1$.

Lemma 7.1. ι_* descends to a homomorphism $\iota_*: \text{CH}_0(Y|E) \rightarrow \text{CH}_0^{LW}(X)$ when $d = 1$.

Proof. Let $f \in \text{Ker}(\mathcal{O}_{Y,E}^\times \rightarrow \mathcal{O}_E^\times)$. The scheme X being a normal crossing variety means that the sequence

$$(7.2) \quad 0 \rightarrow \mathcal{O}_{X,E} \rightarrow \mathcal{O}_{Y,E} \times \mathcal{O}_{Y',E} \rightarrow \mathcal{O}_E \rightarrow 0$$

is exact. If we let $f' = (f, 1) \in \mathcal{O}_{Y,E}^\times \times \mathcal{O}_{Y',E}^\times$, then the modulus condition for f implies that $f' \in \mathcal{O}_{X,E}^\times$. Moreover, we have $\text{div}(f') = \text{div}(f)$ in $\mathcal{Z}_0(X, D)$. \square

In the rest of § 7.1, we shall assume $d = 2$. For any closed immersion of schemes $W_2 \subset W_1$ of k -schemes, we let $\mathcal{K}_{i,(W_1,W_2)}$ denote the Zariski sheaf on W_1 associated to the presheaf $U \mapsto K_i(U, U \cap W_2)$ of relative Quillen K -theory.

Lemma 7.2. The canonical map $\iota^*: \mathcal{I} \rightarrow \iota_*(\mathcal{J})$ is an isomorphism.

Proof. Let A be a reduced affine k -algebra of dimension $d \geq 1$ such that $\text{Spec}(A)$ is a normal crossing variety. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ be the set of minimal primes of A . We write $I = \bigcap_{i=2}^r \mathfrak{p}_i$ and $B = A/\mathfrak{p}_1$. Our problem reduces to showing that the homomorphism $I \rightarrow IB$ is bijective. This homomorphism is clearly surjective. It is also injective because the kernel of the composite map $I \rightarrow IB \rightarrow B$ is $\mathfrak{p}_1 \cap I = \bigcap_{i=1}^r \mathfrak{p}_i = (0)$. \square

Lemma 7.3. The restriction map $\iota^*: H^2(X, \mathcal{K}_{2,(X,Y')}) \rightarrow H^2(Y, \mathcal{K}_{2,(Y,E)})$ is an isomorphism.

Proof. We have a commutative diagram of Zariski sheaves on X with exact rows:

$$(7.3) \quad \begin{array}{ccccccc} \mathcal{K}_{2,(X,Y')} & \longrightarrow & \mathcal{K}_{2,X} & \longrightarrow & \iota'_*(\mathcal{K}_{2,Y'}) & \longrightarrow & 0 \\ \iota^* \downarrow & & \downarrow \iota^* & & \downarrow \iota^* & & \\ \mathcal{K}_{2,(Y,E)} & \longrightarrow & \mathcal{K}_{2,Y} & \longrightarrow & \iota'_*(\mathcal{K}_{2,E}) & \longrightarrow & 0. \end{array}$$

The canonical map $\iota^*: \mathcal{I} \rightarrow \iota_*(\mathcal{J})$ is an isomorphism by Lemma 7.2. It follows from [33] (see also [18, Thm. 0.2]) that the map $\mathcal{K}_{2,(X,Y')} \rightarrow \iota_*(\mathcal{K}_{2,(Y,E)})$ is surjective. It is also clear that the kernel of this map is supported on D . In particular, the map $H^2(X, \mathcal{K}_{2,(X,Y')}) \rightarrow H^2(X, \iota_*(\mathcal{K}_{2,(Y,E)})) \simeq H^2(Y, \mathcal{K}_{2,(Y,E)})$ is an isomorphism. \square

For any reduced quasi-projective surface Z over k , let

$$cyc_Z: \mathrm{CH}_0^{LW}(Z) \rightarrow F^2 K_0(Z) \subset K_0(Z)$$

be the K -theoretic cycle class map (e.g., see [4, Lem. 3.12]). As we have already observed, it factors through $H_{\mathrm{Zar}}^2(Z, \mathcal{K}_{2,Z})$, and we get canonical maps

$$(7.4) \quad \mathrm{CH}_0^{LW}(Z) \rightarrow \mathrm{CH}_0^{BK}(Z) \rightarrow H_{\mathrm{Zar}}^2(Z, \mathcal{K}_{2,Z}) \rightarrow H_{\mathrm{Nis}}^2(Z, \mathcal{K}_{2,Z}).$$

The second arrow is an isomorphism by [6, Thm. 8.1] (or by [39] when k is algebraically closed), the third arrow is an isomorphism by [29, Thm. 2.5] (the cited result assumes X_{reg} to be smooth but [23, § 3.5] explains that this is unnecessary). The first arrow is an isomorphism when k is infinite by [6, Cor. 7.8].

Lemma 7.4. *The map ι_* of (7.1) descends to a homomorphism*

$$\iota_*: \mathrm{CH}_0(Y|E) \rightarrow \mathrm{CH}_0^{BK}(X).$$

Proof. Since $d = 2$ and $H^i(X, \iota'_*(\mathcal{K}_{2,Y'})) \xrightarrow{\simeq} H^i(Y', \mathcal{K}_{2,Y'})$ for $i \geq 0$, we get from (7.3), a commutative diagram of Zariski cohomology groups:

$$(7.5) \quad \begin{array}{ccccccc} H^2(X, \mathcal{K}_{2,(X,Y')}) & \longrightarrow & H^2(X, \mathcal{K}_{2,X}) & \longrightarrow & H^2(Y', \mathcal{K}_{2,Y'}) & \longrightarrow & 0 \\ \downarrow \iota^* \simeq & & \downarrow \iota^* & & \downarrow \iota^* & & \\ H^2(Y, \mathcal{K}_{2,(Y,E)}) & \longrightarrow & H^2(Y, \mathcal{K}_{2,Y}) & \longrightarrow & H^2(E, \mathcal{K}_{2,E}) & \longrightarrow & 0, \end{array}$$

where the two rows are exact.

For a closed point $x \in Y \setminus E = X \setminus Y' \subset X_{\mathrm{reg}}$, we have a commutative diagram

$$(7.6) \quad \begin{array}{ccccc} \mathbb{Z} \xrightarrow{\simeq} H_{\{x\}}^2(X, \mathcal{K}_{2,(X,Y')}) & \longrightarrow & H^2(X, \mathcal{K}_{2,(X,Y')}) & & \\ \downarrow \simeq & \searrow \simeq & \downarrow & \searrow & \\ H_{\{x\}}^2(X, \mathcal{K}_{2,X}) & \longrightarrow & H^2(X, \mathcal{K}_{2,X}) & & \\ \downarrow \simeq & \downarrow \simeq & \downarrow & \searrow & \\ H_{\{x\}}^2(Y, \mathcal{K}_{2,(Y,E)}) & \longrightarrow & H^2(Y, \mathcal{K}_{2,(Y,E)}) & & \\ \downarrow \simeq & \downarrow \simeq & \downarrow & \searrow & \downarrow \iota^* \\ H_{\{x\}}^2(Y, \mathcal{K}_{2,Y}) & \longrightarrow & H^2(Y, \mathcal{K}_{2,Y}) & & \end{array}$$

Since the composite $\mathbb{Z} \xrightarrow{\simeq} H_{\{x\}}^2(X, \mathcal{K}_{2,X}) \hookrightarrow K_0(X)$ is the cycle class of $x \in X_{\mathrm{reg}}$, it follows from Lemma 7.3, (7.5) and (7.6) that the inclusion $Y \setminus E = X \setminus Y' \subset X_{\mathrm{reg}}$

induces a commutative diagram of cycle class maps

$$(7.7) \quad \begin{array}{ccc} \mathcal{Z}_0(Y, E) & \xrightarrow{cyc_{Y|E}} & H^2(Y, \mathcal{K}_{2,(Y,E)}) \\ \downarrow \iota_* & & \downarrow \phi_Y \\ \mathcal{Z}_0(X, D) & \xrightarrow{cyc_X} & H_{\text{Zar}}^2(X, \mathcal{K}_{2,X}) \hookrightarrow K_0(X), \end{array}$$

where $\phi_Y = (\iota^*)^{-1}: H^2(Y, \mathcal{K}_{2,(Y,E)}) \rightarrow H^2(X, \mathcal{K}_{2,(X,Y')})$.

It follows from [6, Thm. 3.6] that cyc_X factors through the quotient $\text{CH}_0^{BK}(X)$ and it follows from [34, Thm. 1.2, Lem. 2.1] that the map $cyc_{Y|E}$ factors through the quotient $\text{CH}_0(Y|E)$. We thus get a commutative diagram with solid arrows

$$(7.8) \quad \begin{array}{ccc} \mathcal{Z}_0(Y, E) & \twoheadrightarrow & \text{CH}_0(Y|E) \xrightarrow{cyc_{Y|E}} H_{\text{Zar}}^2(Y, \mathcal{K}_{2,(Y,E)}) \\ \downarrow \iota_* & & \downarrow \phi_Y \\ \mathcal{Z}_0(X, D) & \twoheadrightarrow & \text{CH}_0^{BK}(X) \xrightarrow{cyc_X} H_{\text{Zar}}^2(X, \mathcal{K}_{2,X}) \hookrightarrow K_0(X). \end{array}$$

Now, the map $cyc_X: \text{CH}_0^{BK}(X) \rightarrow H_{\text{Zar}}^2(X, \mathcal{K}_{2,X})$ is an isomorphism as shown above, and the map $\text{CH}_0(Y|E) \rightarrow H_{\text{Zar}}^2(Y, \mathcal{K}_{2,(Y,E)})$ is an isomorphism by [6, Thm. 8.2]. It follows that the composite map $\iota_*: \mathcal{Z}_0(Y, E) \hookrightarrow \mathcal{Z}_0(X, D) \twoheadrightarrow \text{CH}_0^{BK}(X)$ factors through $\text{CH}_0(Y|E)$. This proves the lemma. \square

7.2. 0-cycles on normal crossing surfaces. In this subsection, we shall prove the $d = 2$ case of Theorem 8.4. We continue with the setup of § 7.1. Let $\{X_1, \dots, X_r\}$ be the set of irreducible components of X . Let $C \subset X$ be a reduced curve. The following definition is borrowed from [12].

Definition 7.5. We shall say that C is a *snc subcurve* in X if its scheme theoretic intersection with each irreducible component of X is either empty or smooth and integral of dimension one, its intersection with each $X_i \cap X_j$ ($i \neq j$) is either empty or reduced and 0-dimensional, and its intersection with each $X_i \cap X_j \cap X_\ell$ ($i \neq j \neq \ell \neq i$) is empty.

Let X be as above and let $C \subset X$ be a snc subcurve. The following is easy to verify.

Lemma 7.6. *The following hold for C .*

- (1) *The inclusion $C \hookrightarrow X$ is a regular embedding.*
- (2) *The embedding dimension of C is at most two.*
- (3) *C is regular and integral if and only if it is contained in one component X_i of X and $C \cap X_j = \emptyset$ for every component $X_j \neq X_i$.*

Let X be a normal crossing variety of dimension $d \geq 2$ over k . Let $D \subset X$ denote the singular locus of X . We let $\mathcal{R}_0^{\text{snc}}(X, D)$ denote the subgroup of $\mathcal{Z}_0(X, D)$ generated by $\text{div}(f)$, where f is a rational function on a snc subcurve $C \subset X$ such that f is regular and invertible along $C \cap D$. We let

$$(7.9) \quad \text{CH}_0^{\text{snc}}(X) = \frac{\mathcal{Z}_0(X, D)}{\mathcal{R}_0^{\text{snc}}(X, D)}.$$

It follows from Lemma 7.6 that a snc subcurve on X is a Cartier curve relative to X_{sing} . This implies that there are canonical surjections $\text{CH}_0^{\text{snc}}(X) \twoheadrightarrow \text{CH}_0^{LW}(X) \twoheadrightarrow \text{CH}_0^{BK}(X)$. We will now prove the following result. This shows that the Levine-Weibel Chow group of a normal crossing surface can be described using a simpler notion of rational equivalence. This will be used in the proof of Theorem 8.4.

Proposition 7.7. *Assume that k is infinite and $d = 2$, then the maps*

$$\mathrm{CH}_0^{\mathrm{snc}}(X) \rightarrow \mathrm{CH}_0^{\mathrm{LW}}(X) \rightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)$$

are isomorphisms.

Proof. We have remarked previously that the second arrow is an isomorphism. We shall show that the first arrow is injective using the ideas of [4, Lem. 5.5], that in turn is inspired by [41, Lem. 1.4]. We fix a locally closed embedding $X \hookrightarrow \mathbb{P}_k^n$. Let $C \subset X$ be a reduced Cartier curve relative to D and let $f \in k(C)^\times$ be such that it is regular and invertible along $C \cap D$. We need to show that $\mathrm{div}(f) \in \mathcal{R}_0^{\mathrm{snc}}(X, D)$.

As in the proof of [41, Lem. 1.4], we can reduce to the case when there is a very ample line bundle \mathcal{L} on X and a section $s \in H^0(X, \mathcal{L})$ such that $C = (t_0)$. Since \mathcal{L} is very ample, the Bertini theorem (see the proof of [41, Lem. 1.3] and [27, Chap. 6], and also [19] for the case where k is not assumed to be perfect) now allow us to choose another Cartier divisor C_∞ in the linear system $H^0(X, \mathcal{L})$ such that:

- (1) $C_\infty \subset X$ is a snc subcurve.
- (2) $C_\infty \cap C \cap D = \emptyset$.
- (3) C_∞ contains no component of C .

Denote by t_∞ the section of \mathcal{L} with $C_\infty = (t_\infty)$. We extend the function f on C to a function h on $C_\infty \cup C$ by setting $h = f$ on C and $h = 1$ on C_∞ . Notice that h is meromorphic on $C_\infty \cup C$ and regular invertible in a neighborhood of $(C_\infty \cup C) \cap D$ by (2). Let S denote the finite set of points $S = \{\text{poles of } f\} \cup (C \cap C_\infty)$. Note that $S \cap D = \emptyset$.

We now choose a very ample line bundle $\mathcal{M} = j^*(\mathcal{O}_{\mathbb{P}_k^r}(1))$ on X , corresponding to an embedding $j: X \hookrightarrow \mathbb{P}_k^r$ for $r \gg 0$ of X as a locally closed subscheme. Since k is infinite, there is a dense open subset V_X of the dual projective space $(\mathbb{P}_k^r)^\vee$ such that for $L \in V_X$, the scheme theoretic intersection $L \cdot X = L \times_{\mathbb{P}_k^r} X$ satisfies the following list of properties:

- a) $L \cdot X$ is a snc subcurve in X ,
- b) $L \cdot X \cap (C \cup C_\infty) \cap D = \emptyset$,
- c) $L \cdot X \supset S$,
- d) $L \cdot X$ intersects $C_\infty \cup C$ in a finite set of points.

The hyperplane L corresponds to a section l of the linear system $H^0(\mathbb{P}_k^r, \mathcal{O}_{\mathbb{P}_k^r}(1))$. Write s_∞ for the global section of \mathcal{L} that is the restriction of l to X . Then $L \cdot X = (s_\infty)$. Note that we can assume that s_∞ does not have poles on $(C \cup C_\infty) \cap D$. Let \overline{X} be the closure of X in \mathbb{P}_k^r and let \mathcal{I} be the ideal sheaf of $\overline{X} \setminus X$ in \mathbb{P}_k^r . We can find a section s'_∞ of the sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}_k^r}(m)$ for some $m \gg 0$, which restricts to a section s_∞ on X satisfying the properties (a) - (d) on $X = \overline{X} \setminus V(\mathcal{I})$. This implies in particular that $X \setminus (s_\infty) = \overline{X} \setminus (s'_\infty)$ is affine. Thus, up to taking a further Veronese embedding of \mathbb{P}_k^r (replacing s_∞ with s'_∞) we can assume that (a) - (d) as above as well as the following hold:

- e) $X \setminus (s_\infty) = X \setminus (L \cdot X)$ is affine.

Consider again the function h on $C \cup C_\infty$. By our choice of S , h is regular on $(C \cup C_\infty) \setminus S$. By (e) above, h extends to a regular function H on the affine open $U = X \setminus (s_\infty)$. Since H is a meromorphic function on X which has poles only along (s_∞) , it follows that for $N \gg 0$ the section $H s_\infty^N$ is an element of $H^0(U, \mathcal{L}^N)$ which extends to a section s_0 of \mathcal{L}^N on all of X . Since h is regular and invertible at each point of $C \cup C_\infty \cap D$ and since s_∞ does not have zeros or poles on $C \cup C_\infty \cap D$, it follows that $(s_0) \cap (C \cup C_\infty) \cap D = \emptyset$ and (using (d) above) that (s_0) does not contain any

component of $C \cup C_\infty$. Note that up to replacing s_0 by $s_0 s_\infty^i$, we are free to choose N as large as needed.

Write $\mathcal{I}_{C \cup C_\infty}$ for the ideal sheaf of $C \cup C_\infty$ in X . We can then find sections s_1, \dots, s_m of $H^0(X, \mathcal{L}^N \otimes \mathcal{I}_{C \cup C_\infty})$ such that the rational map $\phi: X \rightarrow \mathbb{P}_k^{m-1}$ that they define is a locally closed immersion on $X \setminus (C \cup C_\infty)$. In particular, there exists an affine open neighbourhood U_x of every $x \in X \setminus (C \cup C_\infty)$ where at least one of the s_i is not identically zero and where the k -algebra $k[s_1/s_i, \dots, s_m/s_i]$ generated by $s_1/s_i, \dots, s_m/s_i$ coincides with the coordinate ring of U_x . But then, the same must be true for the algebra $k[s_0/s_1, s_1/s_i, \dots, s_m/s_i]$ obtained by adding the element s_0/s_i . Hence, the rational map $\psi: X \rightarrow \mathbb{P}_k^m$ given by the sections (s_0, s_1, \dots, s_m) of $H^0(X, \mathcal{L}^N)$ is also a locally closed immersion on $X \setminus (C \cup C_\infty)$, and since the base locus of the linear system associated to (s_0, s_1, \dots, s_m) is $(s_0) \cap (C \cup C_\infty)$, it is in fact a morphism away from $(s_0) \cap (C \cup C_\infty)$.

In particular, ψ is birational (hence separable) to its image so that the linear system $V = (s_0, \dots, s_m)$ is not composite with a pencil. Since X is a simple normal crossing divisor, the classical Theorem of Bertini (see, for example, [59, Thm. I.6.3] and [27, Chap. 6]) shows that a general divisor E in V is a snc subcurve in X . We can therefore assume that there is a global section s'_0 of $H^0(X, \mathcal{L}^N)$ of the form $s'_0 = s_0 + \alpha$ with $\alpha \in (s_1, \dots, s_m)$, that satisfies the following properties:

- a') (s'_0) is a snc subcurve in X .
- b') $(s'_0) \cap (C \cup C_\infty) \cap D = \emptyset$.
- c') $C_\infty \cup C$ contains no component of (s'_0) .

We then have

$$\frac{s'_0}{s_\infty^N} = \frac{H s_\infty^N + (\alpha s_\infty^{-N}) s_\infty^N}{s_\infty^N} = H + \alpha s_\infty^{-N} = H', \text{ (say).}$$

Since α vanishes along $C \cup C_\infty$ and s_∞ does not vanish identically on $U \cap (C \cup C_\infty)$ by (d), it follows that $H'|_{(C \cup C_\infty) \cap U} = H|_{(C \cup C_\infty) \cap U} = h|_U$. In other words, we have $s'_0/s_\infty^N = h$ as rational functions on $C \cup C_\infty$. We can now compute:

$$\begin{aligned} \nu_*(\operatorname{div}(f)) &= (s'_0) \cdot C - N(s_\infty) \cdot C \\ 0 = \operatorname{div}(1) &= (s'_0) \cdot C_\infty - N(s_\infty) \cdot C_\infty. \end{aligned}$$

We therefore have

$$\begin{aligned} \nu_*(\operatorname{div}(f)) &= (s'_0) \cdot (C - C_\infty) - N(s_\infty)(C - C_\infty) \\ &= (s'_0) \cdot (\operatorname{div}(t_0/t_\infty)) - N(s_\infty) \cdot (\operatorname{div}(t_0/t_\infty)) \\ &= \nu_{s'_0,*}(\operatorname{div}(f')) - N \nu_{s_\infty,*}(\operatorname{div}(f'')), \end{aligned}$$

where $f' = (t_0/t_\infty)|_{(s'_0)} \in \mathcal{O}_{(s'_0), D \cap (s'_0)}^\times$ (by (b')) and $f'' = (t_0/t_\infty)|_{(s_\infty)} \in \mathcal{O}_{(s_\infty), D \cap (s_\infty)}^\times$ (by (b)). Since (s'_0) and (s_∞) are snc subcurves on X , we are done. \square

8. PROOF OF THEOREM 1.6

In this section, we study groups of 0-cycles on higher dimensional normal crossing varieties and prove Theorem 1.6. We begin with a moving lemma from [4] and [25].

8.1. A moving lemma. Let k be a perfect field and X an integral smooth projective k -scheme with an effective Cartier divisor $D \subset X$. We shall use the following condition in some results of this section.

(8.1)

The canonical map $\mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0^{BK}(S_X)$ is an isomorphism.

The question as to when this condition is satisfied is a subtle one. We summarize here some known facts about it.

Lemma 8.1. *The condition (8.1) holds in the following cases.*

- (1) X is affine and k is algebraically closed.
- (2) $\mathrm{char}(k) = 0$ and X is projective.
- (3) $\dim(X) \leq 2$ and k is infinite.
- (4) k is algebraically closed and D is reduced.

Proof. Items (1) and (2) are respectively, the items (2) and (3) of [4, Thm. 3.17]. The item (2) is [6, Cor. 7.8] and (4) follows from [35, Thm. 6.6]. \square

Let $A \subset D$ be a closed subscheme of dimension at most $\dim(X) - 2$. Let $\mathcal{R}_0^{\mathrm{mod}}(X|D) \subset \mathcal{Z}_0(X, D)$ be the subgroup generated by $\mathrm{div}(f)$, where $C \subset X$ is an integral curve and $f \in k(C)^\times$ is such that the following hold.

- (1) $C \not\subset D$.
- (2) C is regular along D .
- (3) $C \cap A = \emptyset$.
- (4) $f \in \mathrm{Ker}(\mathcal{O}_{C, \nu^*(D)}^\times \rightarrow \mathcal{O}_{\nu^*(D)}^\times)$, where $\nu : C \hookrightarrow X$ is the inclusion.

Let $\mathrm{CH}_0^{\mathrm{mod}}(X|D)$ be the quotient of $\mathcal{Z}_0(X, D)$ by $\mathcal{R}_0^{\mathrm{mod}}(X|D)$. There is an evident surjection $\mathrm{CH}_0^{\mathrm{mod}}(X|D) \rightarrow \mathrm{CH}_0(X|D)$.

Lemma 8.2. *Assume that k is infinite and the condition (8.1) holds. Then the map*

$$\mathrm{CH}_0^{\mathrm{mod}}(X|D) \rightarrow \mathrm{CH}_0(X|D)$$

is an isomorphism.

Proof. This is a direct consequence of [25, Prop. 4.3] (which is [4, Cor. 7.2] when $A = \emptyset$). By [25, Prop. 4.3], there are maps $\tau_X^* : \mathrm{CH}_0^{LW}(S_X) \rightarrow \mathrm{CH}_0^{\mathrm{mod}}(X|D)$ and $p_{+,*} : \mathrm{CH}_0^{\mathrm{mod}}(X|D) \rightarrow \mathrm{CH}_0^{LW}(S_X)$ such that $\tau_X^* \circ p_{+,*}$ is identity. By [4, Thm. 7.1], there are maps $\tau_X^* : \mathrm{CH}_0^{BK}(S_X) \rightarrow \mathrm{CH}_0(X|D)$ and $p_{+,*} : \mathrm{CH}_0(X|D) \rightarrow \mathrm{CH}_0^{BK}(S_X)$ such that $\tau_X^* \circ p_{+,*}$ is identity. In particular, the horizontal arrows in the commutative diagram

$$(8.2) \quad \begin{array}{ccc} \mathrm{CH}_0^{\mathrm{mod}}(X|D) & \xrightarrow{p_{+,*}} & \mathrm{CH}_0^{LW}(S_X) \\ \mathrm{can} \downarrow & & \downarrow \mathrm{can} \\ \mathrm{CH}_0(X|D) & \xrightarrow{p_{+,*}} & \mathrm{CH}_0^{BK}(S_X) \end{array}$$

are (split) injective. The right vertical arrow is an isomorphism under (8.1). The lemma now follows. \square

8.2. Generalization of Lemma 7.4 in higher dimensions. Let k be a perfect field and X a normal crossing variety of dimension $d \geq 1$ over k (see Definition 2.12). We shall now generalize Lemma 7.4 to higher dimensions. We shall use the notations of § 7.1 in the proof.

Recall that for a k -scheme Y and a point $y \in Y$, the embedding dimension of Y at y is $\mathrm{edim}_y(Y) = \dim_{k(y)}(\Omega_{Y/k}^1 \otimes_{\mathcal{O}_Y} k(y))$, where $\Omega_{Y/k}^1$ is the Zariski sheaf of Kähler

differentials on Y . For any integer $e \geq 0$, we let Y_e denote the subscheme of points $y \in Y$ such that $\text{edim}_y(Y) = e$. Since $y \mapsto \text{edim}_y(Y)$ is an upper semi-continuous function on Y (see [26, Example III.12.7.2]), it follows that Y_e is a locally closed subscheme of Y . We let $\text{edim}(Y) = \max_{e \geq 0} \{e + \dim(Y_e)\}$.

Theorem 8.3. *Assume that the condition (8.1) holds (e.g., k is algebraically closed and D is reduced, or k is infinite and $d \leq 2$). Then the map $\iota_*: \mathcal{Z}_0(Y, E) \rightarrow \mathcal{Z}_0(X, D)$ of (7.1) descends to a group homomorphism*

$$\iota_*: \text{CH}_0(Y|E) \rightarrow \text{CH}_0^{LW}(X).$$

Proof. We shall prove the theorem by induction on d . The case $d \leq 2$ follows from Lemmas 7.1 and 7.4 (and in this case the result is unconditional). So we assume $d \geq 3$. Let $C \subset Y$ be an integral curve not contained in E and let $f \in \text{Ker}(\mathcal{O}_{C^N, \nu^*(E)}^\times \rightarrow \mathcal{O}_{\nu^*(E)}^\times)$, where $\nu: C^N \rightarrow X$ is the canonical map. We need to show that $\text{div}(f) \in \mathcal{R}_0^{LW}(X, D)$.

By Lemma 8.2, we can assume that C is regular in a neighborhood of $C \cap D$ so that $\nu: C^N \rightarrow X$ is a closed embedding in a neighborhood of $C \cap D$ and $C \cap D_{\text{sing}} = C \cap E_{\text{sing}} = \emptyset$. Note here that $E \subset Y$ is a simple normal crossing divisor. In particular, it is reduced.

Let T be the set of singular points of C . Since C is regular along D , we must have $T \cap Z = \emptyset$, where we let $Z = \overline{X} \setminus \overline{Y}$. We can find a finite sequence of blow-ups $\pi: X' \rightarrow X$ over the points of T such that the strict transform C' of C is regular and it intersects the exceptional divisor F transversely in the regular locus of F . As $T \cap Z = \emptyset$, the map $\pi^{-1}(Z) \rightarrow Z$ is an isomorphism. In particular, X' is a normal crossing variety of dimension d over k . Let Y' be the strict transform of Y under π and let $E' = F + E$. Then E' is a simple normal crossing divisor on Y' and C' intersects E' transversely in its regular locus. Note also that $D' := X'_{\text{sing}} \cong D$.

We now choose a locally closed embedding $X' \hookrightarrow \mathbb{P}_k^N$. We then have:

- (1) $\text{edim}(C') = 2 < \dim(Y')$.
- (2) $\text{edim}(C' \cap Y_i) + \dim(C \cap Y_i) \leq 1 < \dim(Y_i)$ for every irreducible component $Y_i \neq Y'$ of X' .
- (3) $\text{edim}(C' \cap D_i) + \dim(C' \cap D_i) \leq 1 < \dim(D_i)$ for every irreducible component D_i of D' .
- (4) $C' \cap D'_{\text{sing}} = \emptyset$.

Let $\overline{C'}$ denote the scheme theoretic closure of C' in \mathbb{P}_k^N and let \mathcal{I} be the sheaf of ideals on \mathbb{P}_k^N defining $\overline{C'}$. Let $\{X_1, \dots, X_r\}$ be the set of irreducible components of X with $X_1 = Y$. Then $\{Y', X_2, \dots, X_r\}$ is the set of irreducible components of X' . Let \mathcal{W} be the collection of all subschemes of X' such that $W \subset X'$ lies in \mathcal{W} if and only if it is an irreducible component of X'_J for some $J \subset [1, n]$ (see § 7.1). Note that any such W is integral and smooth.

If k is infinite, we can use (1) - (4) and [19, Thm. 3.12] to find $d \gg 0$ such that a general member H (which is defined over k) of the linear system $|H^0(\mathbb{P}_k^N, \mathcal{I}(d))|$ has the property that $H \cap W$ is irreducible and smooth for every $W \in \mathcal{W}$. In particular, it is integral. If k is finite, we can use [19, Prop. 5.2, Thm. 8.4] to find a hypersurface $H \in |H^0(\mathbb{P}_k^N, \mathcal{I}(d))|(k)$ for all $d \gg 0$ such that $H \cap W$ is irreducible and smooth for every $W \in \mathcal{W}$. In particular, it is integral.

We choose one hypersurface H as above and let $W' = X' \cap H$. Then it is clear that W' is a normal crossing variety of dimension $d - 1$ with irreducible components $\{Y' \cap W', X_2 \cap W', \dots, X_r \cap W'\}$. We let $V' = Y' \cap W'$. By induction, it follows

that $\operatorname{div}(f) \in \mathcal{R}_0^{LW}(W', D' \cap W')$. Since $\psi: W' \hookrightarrow X'$ is a complete intersection closed subscheme such that $W'_{\text{sing}} = X'_{\text{sing}} \cap W' = D' \cap W'$, there is a push-forward map (e.g., see [4, Lem. 3.10]) $\psi_*: \mathcal{Z}_0(W', D' \cap W') \rightarrow \mathcal{Z}_0(X', D')$ such that $\psi_*(\mathcal{R}_0^{LW}(W', D' \cap W')) \subset \mathcal{R}_0^{LW}(X', D')$. It follows that $\operatorname{div}(f) \in \mathcal{R}_0^{LW}(X', D')$.

Finally, as $\pi: X' \rightarrow X$ is a finite sequence of point blow-ups away from $D = X_{\text{sing}}$, there is a push-forward map $\pi_*: \mathcal{Z}_0(X', D') \rightarrow \mathcal{Z}_0(X, D)$ such that $\pi_*(\mathcal{R}_0^{LW}(X', D')) \subset \mathcal{R}_0^{LW}(X, D)$ (see [4, Lem. 3.18]). We conclude that $\operatorname{div}(f) = \operatorname{div}(f)_C = \pi_*(\operatorname{div}(f)_{C'}) \in \mathcal{R}_0^{LW}(X, D)$. This concludes the proof. \square

8.3. Proof of Theorem 1.6. Let k be an infinite perfect field and X a projective normal crossing variety of dimension $d \geq 0$ over k . We write D for X_{sing} .

We shall now prove our final result which identifies the motivic cohomology in the bi-degree $(2d, d)$ of X to its Levine-Weibel Chow group. One consequence of this is that it allows us to deduce that the main result of [5], which proved a restriction isomorphism for relative 0-cycles in a projective and flat family over a Henselian discrete valuation ring, recovers the earlier result of [12] when the family is semi-stable, i.e., the reduced special fiber is a simple normal crossing divisor. This question was then raised by some of the authors of [12]. The second consequence is that it allows one to describe the motivic cohomology of a normal crossing variety in terms of 0-cycles supported on X_{reg} and only one type of relations, unlike in the description given in [12] which uses two types of relations (see below).

We shall need one more ingredient: a new Chow group of 0-cycles introduced in [12, § 2]. Let $\mathcal{R}_0^{EKW}(X, D) \subset \mathcal{Z}_0(X, D)$ be the subgroup generated by $\operatorname{div}(f)$, where $f \in k(C)^\times$ is a rational function on a curve $C \subset X$ such that the pair (C, f) satisfies either of the conditions (1) and (2) below.

- (1) C is an integral curve not contained in D with normalization $\nu: C^N \rightarrow C \hookrightarrow X$ and $f \in \mathcal{O}_{C^N, \nu^*(D)}^\times$ such that $f(x) = 1$ for all $x \in \nu^*(D)$.
- (2) $C \subset X$ is a snc subcurve and $f \in \mathcal{O}_{C, (C \cap D)}^\times$.

We let $\operatorname{CH}_0^{EKW}(X)$ denote the quotient of $\mathcal{Z}_0(X, D)$ by the subgroup $\mathcal{R}_0^{EKW}(X, D)$. This group was defined in [12, § 2], where it was denoted by $\mathcal{C}(X)$. One of the main computational results of [12] is the comparison between the group $\operatorname{CH}_0^{EKW}(X)[\frac{1}{p}]$ and the cdh motivic cohomology in bi-degree $(2d, d)$ of the normal crossing variety \bar{X} , where p is the exponential characteristic of k , see [12, Thm. 7.1]. There is a canonical surjection

$$(8.3) \quad \operatorname{CH}_0^{\text{snc}}(X) \twoheadrightarrow \operatorname{CH}_0^{EKW}(X)$$

since both groups are quotients of $\mathcal{Z}_0(X, D)$.

We now prove Theorem 1.6. We restate it for convenience.

Theorem 8.4. *Assume that the condition (8.1) holds. Then the map $\lambda_X: \operatorname{CH}_0^{LW}(X)_\Lambda \rightarrow H^{2d}(X, \Lambda(d))$ is an isomorphism in the following cases.*

- (1) $\operatorname{char}(k) = p > 0$ and $\Lambda = \mathbb{Z}[\frac{1}{p}]$.
- (2) $m \in k^\times$ and $\Lambda = \mathbb{Z}/m$.

Proof. It is shown in [12, Thm 7.1] that the maps

$$(8.4) \quad \Lambda \xrightarrow{\cong} H^0(k(x), \Lambda(0)) \rightarrow H_c^{2d}(X \setminus D, \Lambda(d)) \rightarrow H^{2d}(X, \Lambda(d))$$

for $x \in X \setminus D$ induce a map $\gamma_X: \operatorname{CH}_0^{EKW}(X)_\Lambda \xrightarrow{\cong} H^{2d}(X, \Lambda(d))$. Moreover, this is an isomorphism. We remark here that in op. cit., Definition 2.11 is used for normal

crossing varieties which is a stronger notion than the one we use. However, the reader can easily check that the proof of [12, Thm 7.1] only uses our weaker notion. It suffices now to show that there is a canonical surjection $\psi_X: \text{CH}_0^{EKW}(X)_\Lambda \rightarrow \text{CH}_0^{LW}(X)_\Lambda$ such that $\gamma_X = \lambda_X \circ \psi_X$.

To show that the identity map of $\mathcal{Z}_0(X, D)_\Lambda$ descends to ψ_X , we let $C \subset X$ be a curve and $f \in k(C)^\times$ a rational function. We first observe that if the pair (C, f) is of type (1) above, then C must be contained in one and only one irreducible component Y of X . Moreover, for this component Y , the intersection $E = D \cap Y$ must be a simple normal crossing divisor on Y . We now conclude easily from Theorems 1.2 and 8.3 that $\psi_X(\text{div}(f)) \in \mathcal{R}_0^{LW}(X, D)_\Lambda$. If (C, f) is of type (2) above, then we get $\psi_X(\text{div}(f)) \in \mathcal{R}_0^{LW}(X, D)_\Lambda$ by Lemma 7.6. We have thus shown that $\psi_X: \text{CH}_0^{EKW}(X)_\Lambda \rightarrow \text{CH}_0^{LW}(X)_\Lambda$ is defined. It is clear from (8.4) that $\gamma_X = \lambda_X \circ \psi_X$. This concludes the proof. \square

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