Global gradient estimates for a general type of nonlinear parabolic equations

Cecilia Cavaterra $^{(1,4)}$

) Serena Dipierro⁽²⁾ Enrico Valdinoci⁽²⁾ Zu $Gao^{(3)}$

July 29, 2021

 Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano Via Saldini 50, I-20133 Milano (Italy)

(2) – Department of Mathematics and Statistics, University of Western Australia 35 Stirling Highway, WA6009 Crawley (Australia)

(3) – Department of Mathematics, School of Science, Wuhan University of Technology 122 Luoshi Road, 430070 Hubei, Wuhan (China)

(4) – Istituto di Matematica Applicata e Tecnologie Informatiche "Enrico Magenes", CNR Via Ferrata 1, 27100 Pavia (Italy)

cecilia.cavaterra@unimi.it, serena.dipierro@uwa.edu.au, gaozu7@whut.edu.cn, enrico.valdinoci@uwa.edu.au

July 29, 2021

Abstract

We provide global gradient estimates for solutions to a general type of nonlinear parabolic equations, possibly in a Riemannian geometry setting.

Our result is new in comparison with the existing ones in the literature, in light of the validity of the estimates in the global domain, and it detects several additional regularity effects due to special parabolic data.

Moreover, our result comprises a large number of nonlinear sources treated by a unified approach, and it recovers many classical results as special cases.

Keywords: Parabolic equations on Riemannian manifolds, Maximum Principle, global gradient estimates.

MSC 2010: 35B09, 35B50, 35K05, 35R01.

1 Introduction

The goal of this paper is to consider a general type of nonlinear parabolic equations, possibly in a Riemannian geometry setting, and to provide new global gradient estimates.

The method that we use relies on the Maximum Principle, as developed by Cheng and Yau in [9] and Hamilton in [17], and on suitable properties of the cut-off function introduced by Li and Yau in [21], which are also the key tool for the classical gradient estimates proved by Souplet and Zhang in [30].

Though several gradient estimate results have been obtained in different cases (see, e.g., [5, 8, 11–13, 18, 19, 22, 23, 33–38]), we provide here a general framework dealing, at once, with various nonlinearities of interest (as a matter of fact, a number of classical and recent results can be re-obtained as special cases of our general approach). Also, we will provide "global" (rather than "local") estimates that take into account the parabolic boundary behavior, thus improving the estimates when the data of the equation are particularly favorable.

We point out that the pointwise gradient estimates for parabolic equations have also a natural counterpart for elliptic equations, see e.g. [4,7,10,15,16,24,26,31], and, in general, pointwise gradient estimates based on Maximum Principles are a classical, yet still very active, topic of investigation.

Now we introduce the mathematical framework in details. Let \mathcal{M} be a Riemannian manifold of dimension $n \ge 2$, with Ricci curvature denoted by $\operatorname{Ric}(\mathcal{M})$. In this article, we will always suppose that the Ricci curvature of \mathcal{M} is bounded from below, namely,

$$\operatorname{Ric}(\mathcal{M}) \ge -k,\tag{1.1}$$

for some $k \in \mathbb{R}$.

As customary, we also use the "positive part" notation

$$k_{+} := \max\{k, 0\}$$

The geodesic ball centered at $x_0 \in \mathcal{M}$ of radius R > 0 will be denoted by $B(x_0, R)$.

Given $x_0 \in \mathcal{M}, R > 0, t_0 \in \mathbb{R}$, and T > 0, we consider a classical parabolic equation of the form

$$u_t = \Delta u + S(x, t, u) \qquad \text{in } Q_{R,T}. \tag{1.2}$$

In this setting u = u(x, t), with $x \in B(x_0, R) \subset \mathcal{M}$ and $t \in [t_0 - T, t_0]$, and we have used the classical notation

$$Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0].$$

We will always suppose that

$$u(x,t) \in (0,M] \qquad \text{for all } (x,t) \in Q_{R,T}, \tag{1.3}$$

for some M > 0.

Also, in (1.2) we denote by S a nonlinear source for the equation, that we suppose to be C^1 in x and u, and continuous in t.

In this setting, we define

$$\gamma := \sup_{(x,t,u)\in Q_{R,T}\times(0,M]} \frac{\left|\nabla S(x,t,u)\right|}{u},\tag{1.4}$$

where ∇ stands for the gradient with respect the components of the space variable x.

Following [30], it is also convenient to consider the auxiliary function

$$v(x,t) := \ln \frac{u(x,t)}{M}.$$
(1.5)

Furthermore, given k as in (1.1), we set

$$\mu := \sup_{(x,t)\in Q_{R,T}} \left(k + \partial_u S(x,t,u) - \frac{S(x,t,u)}{u} + \frac{S(x,t,u)}{u(1-v)} \right)_+.$$
 (1.6)

Our main goal is to establish global gradient bounds for solutions of (1.2). Since these bounds may degenerate near the parabolic boundary (e.g., if the initial or boundary data are not regular enough), we exploit suitable cut-off functions. Specifically, given $\delta \in (0, T)$ and $\rho \in (0, R)$, we consider the functions

$$\begin{aligned}
\mathcal{B}_{1}(x,t) &:= \chi_{B(x_{0},R-\rho)}(x)\chi_{[t_{0}-T,t_{0}-T+\delta)}(t), \\
\mathcal{B}_{2}(x,t) &:= \chi_{B(x_{0},R)\setminus B(x_{0},R-\rho)}(x)\chi_{[t_{0}-T+\delta,t_{0}]}(t), \\
\mathcal{B}_{3}(x,t) &:= \chi_{B(x_{0},R)\setminus B(x_{0},R-\rho)}(x)\chi_{[t_{0}-T,t_{0}-T+\delta)}(t) \\
\end{aligned}$$
and
$$\begin{aligned}
\mathcal{J}(x,t) &:= \chi_{B(x_{0},R-\rho)}(x)\chi_{[t_{0}-T+\delta,t_{0}]}(t).
\end{aligned}$$
(1.7)

We point out that the functions \mathcal{B}_1 , \mathcal{B}_2 and \mathcal{B}_3 are localized in a neighborhood of the parabolic boundary (namely, \mathcal{B}_1 near the time-boundary but in the interior of the spaceboundary, \mathcal{B}_2 near the space-boundary but in the interior of the time-boundary, and \mathcal{B}_3 near the space- and time-boundary). Conversely, the function \mathcal{I} is supported well-inside the domain $Q_{R,T}$, and

$$\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{I} = \chi_{Q_{R,T}},$$

hence the supports of these auxiliary functions can be seen as a partition of the domain under consideration. Furthermore, recalling the notation in (1.4) and (1.6), we define

$$C := \gamma^{1/3} + \sqrt{\mu},$$

$$\mathcal{T} := \frac{1}{\sqrt{\delta}},$$
(1.8)
and
$$S := \frac{1}{\rho} + \frac{1}{\sqrt{\rho(R-\rho)}} + \frac{\sqrt[4]{k_+}}{\sqrt{\rho}}.$$

We notice that \mathcal{C} , \mathcal{T} and \mathcal{S} are constants, depending on the nonlinearity, the geometry of the manifold and the parameters R, ρ and δ .

Moreover, we set

$$\tau_{u} := \sup_{\substack{x \in B(x_{0},R) \\ t \in [t_{0}-T,t_{0}]}} \frac{|\nabla u|}{u(1-v)} (x,t_{0}-T),$$
and
$$\sigma_{u} := \sup_{\substack{x \in \partial B(x_{0},R) \\ t \in [t_{0}-T,t_{0}]}} \frac{|\nabla u|}{u(1-v)} (x,t).$$
(1.9)

We remark that τ_u and σ_u are known-objects, once we know the parabolic boundary data of u (in particular, both τ_u and σ_u are controlled by the supremum of $\frac{|\nabla u|}{u(1-v)}$ over the parabolic boundary of $Q_{R,T}$).

The terms in (1.8) are the building blocks of our chief estimate, since they comprise the different pointwise behavior of the solution, in different regions of the space-time domain. More specifically:

- the term C is a "common" term in all the domain, produced by the nonlinearity S and by the curvature of the ambient manifold,
- the term $\ensuremath{\mathfrak{T}}$ is a localization term due to a cut-off function in the time variable,
- the term S is a localization term due to a cut-off function in the space variable.

Our strategy would then be to choose, in our chief estimate, the "best option" between the boundary datum and the universal smoothing effect produced by the heat equation. To this end, given a constant C > 0 (that will be taken conveniently large in the following Theorem 1.1) we define

$$\beta_{1} := \tau_{u} + \min \left\{ \sigma_{u}, CS \right\},$$

$$\beta_{2} := \sigma_{u} + \min \left\{ \tau_{u}, CT \right\},$$

$$\beta_{3} := \sigma_{u} + \tau_{u},$$
and
$$\iota := \min \left\{ \sigma_{u} + \tau_{u}, \sigma_{u} + CT, \tau_{u} + CS, C(T + S) \right\}.$$

$$(1.10)$$

We point out that the quantities in (1.10) are constants. In this framework, the term β_1 will take care of the region of the domain in the interior of the space variable and near the parabolic boundary in the time variable: such a term takes into account the initial datum in time and, thanks to the spatially interior smoothing effect, takes the "best possible choice" between the spatial boundary datum and the estimate produced by a spatial cut-off.

Similarly, the term β_2 in (1.10) will take care of the region of the domain in the interior of the time variable and near the spatial boundary: such a term takes into account the boundary datum in space and, thanks to the smoothing effect for positive times, takes the "best possible choice" between the initial datum and the estimate produced by a cut-off in the time variable.

The term β_3 in (1.10) deals with the case of proximity to the boundary for both the space and time variables, and clearly reflects the influence of the data on the whole of the parabolic boundary.

Finally, the term ι in (1.10) considers the case of interior points, both in space and time: in this case, one can take the "best possible choice" between the data along the parabolic boundary and the universal smoothing effect of the heat equation in the interior of the domain.

In view of these considerations, the coupling between the auxiliary functions and the coefficients is encoded by the function

$$\mathcal{Z} := \beta_1 \,\mathcal{B}_1 + \beta_2 \,\mathcal{B}_2 + \beta_3 \,\mathcal{B}_3 + \iota \,\mathcal{I}. \tag{1.11}$$

With this notation, the main estimate of this paper goes as follows:

Theorem 1.1. Suppose that u is a solution of equation (1.2) satisfying (1.3).

Then, there exists C > 0, only depending on n, such that, for any $\delta \in (0,T)$ and $\rho \in (0,R)$, we have that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant \left(C\mathcal{C} + \mathcal{Z}(x,t)\right) \left(1 + \ln\frac{M}{u(x,t)}\right) \quad \text{for all } (x,t) \in Q_{R,T}.$$
 (1.12)

We observe that the estimate in (1.12) is dimensionally coherent: indeed, taking the time variable to have the same measure units of the square of the space variable, and S to have the same units of u over the square of the space variables, we obtain that all the terms in (1.12) have the units of the inverse of the space (for this, we recall that the Ricci curvature scales like the inverse of the square of the space variable, see for instance Example 8.5.5 on page 418 and Theorem 8.5.22 on page 427 in [25]).

We also point out that gradient estimates as the one in (1.12) can be also considered as extension of classical gradient estimates inspired by the Bernstein technique, see [1, 1-3, 14, 20, 28, 29, 32].

With respect to this feature, we stress that the quantities τ_u and σ_u introduced in (1.9) are not just natural and convenient objects motivated by the methods relying on the Maximum Principle, but they play an interesting role in *improving the known estimates* for cases in which the parabolic data are particularly nice. More specifically, the classical estimates have the striking feature of being "universal", and thus independent on the data: this property is certainly advantageous when dealing with poor boundary data, since these classical results still ensure suitable gradient bounds in the interior, but they become somehow "suboptimal" when the boundary data are extremely good (these are precisely the pros and cons of an estimate holding true regardless of the specific boundary conditions!). Instead, the estimates that we provide in this paper are flexible enough, on the one hand, to recover several classical results of universal type and, on the other hand, to provide enhanced estimates when the data are extraordinarily nice (this general concept will be quantified, for instance, at the end of Section 3.2).

The rest of this paper is organized as follows. We provide the proof of Theorem 1.1 in Section 2. Then, in Section 3, we give some specific applications of our general result, also showing how it comprises and improves some classical and recent results from the literature.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on a general estimate, given in the forthcoming Lemma 2.1. With that, one will obtain the desired claim in Theorem 1.1 by considering different regions, according to the cut-off functions, and exploiting the Maximum Principle in the interior.

To this aim, we set

$$w := \frac{|\nabla v|^2}{(1-v)^2},$$
(2.1)

and we have:

Lemma 2.1. Let u be as in Theorem 1.1, v be as in (1.5) and w be as in (2.1). Then, in $Q_{R,T}$, it holds

$$\frac{\Delta w - w_t}{2} \ge (1 - v) w^2 + \frac{v \left\langle \nabla w, \nabla v \right\rangle}{1 - v} - \frac{\gamma \left| \nabla v \right|}{(1 - v)^2} - \mu w$$

where γ and μ are as in (1.4) and (1.6).

Proof. Recalling (1.3) and (1.5), we see that

$$v \leqslant 0, \tag{2.2}$$

and

$$v_t = \frac{u_t}{u}, \qquad \nabla v = \frac{\nabla u}{u} \qquad \text{and} \qquad \Delta v = \frac{u\Delta u - |\nabla u|^2}{u^2}.$$
 (2.3)

As a consequence, from (1.2), we have

$$v_{t} = \frac{\Delta u + S(x, t, u)}{u} = \frac{u\Delta u - |\nabla u|^{2}}{u^{2}} + \frac{|\nabla u|^{2}}{u^{2}} + \frac{S(x, t, u)}{u} = \Delta v + |\nabla v|^{2} + \frac{S(x, t, u)}{u}.$$
(2.4)

Now we observe that

$$\nabla w = \frac{\nabla |\nabla v|^2}{(1-v)^2} + 2\frac{|\nabla v|^2 \nabla v}{(1-v)^3}.$$
(2.5)

Moreover, we have that

div
$$\left(\frac{\nabla |\nabla v|^2}{(1-v)^2}\right) = \frac{\Delta |\nabla v|^2}{(1-v)^2} + \frac{2\langle \nabla |\nabla v|^2, \nabla v\rangle}{(1-v)^3}.$$
 (2.6)

In addition,

div
$$\left(\frac{|\nabla v|^2 \nabla v}{(1-v)^3}\right) = \frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} + \frac{|\nabla v|^2 \Delta v}{(1-v)^3} + \frac{3|\nabla v|^4}{(1-v)^4}.$$

From this, (2.5) and (2.6), we deduce that

$$\begin{split} &\Delta w \\ &= \frac{\Delta |\nabla v|^2}{(1-v)^2} + \frac{2\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} + \frac{2\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} + \frac{2|\nabla v|^2 \Delta v}{(1-v)^3} + \frac{6|\nabla v|^4}{(1-v)^4} \\ &= \frac{\Delta |\nabla v|^2}{(1-v)^2} + \frac{4\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} + \frac{2|\nabla v|^2 \Delta v}{(1-v)^3} + \frac{6|\nabla v|^4}{(1-v)^4}. \end{split}$$
(2.7)

Moreover, using (2.4), we find that

$$\begin{split} w_t &= \frac{2\langle \nabla v, \nabla v_t \rangle}{(1-v)^2} + \frac{2|\nabla v|^2 v_t}{(1-v)^3} \\ &= \frac{2\langle \nabla v, \nabla \Delta v \rangle}{(1-v)^2} + \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^2} + \frac{2\left\langle \nabla v, \nabla \left(\frac{S(x,t,u)}{u}\right)\right\rangle}{(1-v)^2} \\ &+ \frac{2|\nabla v|^2 \Delta v}{(1-v)^3} + \frac{2|\nabla v|^4}{(1-v)^3} + \frac{2|\nabla v|^2 \frac{S(x,t,u)}{u}}{(1-v)^3}. \end{split}$$

This and (2.7), after the cancellation of one term, give that

$$\Delta w - w_t = \frac{\Delta |\nabla v|^2}{(1-v)^2} - \frac{2\langle \nabla v, \nabla \Delta v \rangle}{(1-v)^2} - \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^2} - \frac{2\left\langle \nabla v, \nabla \left(\frac{S(x,t,u)}{u}\right) \right\rangle}{(1-v)^2} + \frac{4\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} - \frac{2|\nabla v|^4}{(1-v)^3} - \frac{2|\nabla v|^2 \frac{S(x,t,u)}{u}}{(1-v)^3} + \frac{6|\nabla v|^4}{(1-v)^4}.$$
(2.8)

Now we recall the Bochner's formula, according to which

$$\Delta\left(\frac{|\nabla v|^2}{2}\right) = \langle \nabla \Delta v, \nabla v \rangle + |D^2 v|^2 + \operatorname{Ric}(\nabla v, \nabla v).$$

This and (1.1) entail that

$$\begin{split} \Delta |\nabla v|^2 - 2 \langle \nabla \Delta v, \nabla v \rangle &= 2 |D^2 v|^2 + 2 \text{Ric}(\nabla v, \nabla v) \\ \geqslant 2 |D^2 v|^2 - 2k |\nabla v|^2. \end{split}$$

Plugging this information in (2.8), we conclude that

$$\begin{split} \Delta w - w_t &\geq \frac{2|D^2 v|^2 - 2k |\nabla v|^2}{(1 - v)^2} - \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1 - v)^2} - \frac{2\left\langle \nabla v, \nabla \left(\frac{S(x, t, u)}{u}\right) \right\rangle}{(1 - v)^2} \\ &+ \frac{4\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1 - v)^3} - \frac{2|\nabla v|^4}{(1 - v)^3} - \frac{2|\nabla v|^2 \frac{S(x, t, u)}{u}}{(1 - v)^3} + \frac{6|\nabla v|^4}{(1 - v)^4}. \end{split}$$
(2.9)

We also remark that

$$\begin{array}{ll} 0 & \leqslant & \left(\frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{2 |\nabla v|^2} + \frac{|\nabla v|^2}{1 - v}\right)^2 \\ & = & \left(\frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{2 |\nabla v|^2}\right)^2 + \frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{1 - v} + \frac{|\nabla v|^4}{(1 - v)^2} \\ & = & \left(\frac{\langle D^2 v \nabla v, \nabla v \rangle}{|\nabla v|^2}\right)^2 + \frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{1 - v} + \frac{|\nabla v|^4}{(1 - v)^2} \\ & \leqslant & |D^2 v|^2 + \frac{\langle \nabla |\nabla v|^2, \nabla v \rangle}{1 - v} + \frac{|\nabla v|^4}{(1 - v)^2}. \end{array}$$

From this and (2.9), one finds that

$$\Delta w - w_t \ge -\frac{2k |\nabla v|^2}{(1-v)^2} - \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^2} - \frac{2\left\langle \nabla v, \nabla \left(\frac{S(x,t,u)}{u}\right) \right\rangle}{(1-v)^2} + \frac{2\langle \nabla |\nabla v|^2, \nabla v \rangle}{(1-v)^3} - \frac{2|\nabla v|^4}{(1-v)^3} - \frac{2|\nabla v|^2 \frac{S(x,t,u)}{u}}{(1-v)^3} + \frac{4|\nabla v|^4}{(1-v)^4}.$$
(2.10)

/

Furthermore, in light of (2.5),

$$\langle \nabla w, \nabla v \rangle = \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^2} + \frac{2|\nabla v|^4}{(1-v)^3},$$

and, as a result,

and
$$\begin{aligned} 2\langle \nabla w, \nabla v \rangle &= \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^2} + \frac{4|\nabla v|^4}{(1-v)^3} \\ \frac{2\langle \nabla w, \nabla v \rangle}{1-v} &= \frac{2\langle \nabla v, \nabla |\nabla v|^2 \rangle}{(1-v)^3} + \frac{4|\nabla v|^4}{(1-v)^4}. \end{aligned}$$

These identities, combined with (2.10), yield that

$$\begin{split} \Delta w - w_t \geqslant &-\frac{2k \, |\nabla v|^2}{(1-v)^2} - \frac{2\left\langle \nabla v, \nabla\left(\frac{S(x,t,u)}{u}\right)\right\rangle}{(1-v)^2} \\ &+ \frac{2|\nabla v|^4}{(1-v)^3} - \frac{2|\nabla v|^2 \frac{S(x,t,u)}{u}}{(1-v)^3} - 2\langle \nabla w, \nabla v \rangle + \frac{2\langle \nabla w, \nabla v \rangle}{1-v}. \end{split}$$

We rewrite this formula as

$$\frac{\Delta w - w_t}{2} \ge -\frac{k |\nabla v|^2}{(1 - v)^2} - \frac{\left\langle \nabla v, \nabla \left(\frac{S(x, t, u)}{u}\right) \right\rangle}{(1 - v)^2} + \frac{|\nabla v|^4}{(1 - v)^3} - \frac{|\nabla v|^2 \frac{S(x, t, u)}{u}}{(1 - v)^3} + \frac{v \left\langle \nabla w, \nabla v \right\rangle}{1 - v}.$$

As a result, recalling (2.1), we conclude that

$$\frac{\Delta w - w_t}{2} \ge -kw - \frac{\left\langle \nabla v, \nabla \left(\frac{S(x,t,u)}{u}\right) \right\rangle}{(1-v)^2} + (1-v)w^2 - \frac{w\,S(x,t,u)}{(1-v)u} + \frac{v\,\left\langle \nabla w, \nabla v \right\rangle}{1-v}.$$
 (2.11)

We also exploit (2.3) to write that $\nabla u = u \nabla v$, and accordingly

$$\nabla\left(\frac{S(x,t,u)}{u}\right) = \frac{\nabla S(x,t,u)}{u} + \frac{\partial_u S(x,t,u)\nabla u}{u} - \frac{S(x,t,u)\nabla u}{u^2}$$
$$= \frac{\nabla S(x,t,u)}{u} + \left(\partial_u S(x,t,u) - \frac{S(x,t,u)}{u}\right)\nabla v.$$

Consequently,

$$\frac{\left\langle \nabla v, \nabla \frac{S(x,t,u)}{u} \right\rangle}{(1-v)^2} = \frac{\left\langle \nabla S(x,t,u), \nabla v \right\rangle}{(1-v)^2 u} + \left(\partial_u S(x,t,u) - \frac{S(x,t,u)}{u} \right) \frac{|\nabla v|^2}{(1-v)^2} \\ = \frac{\left\langle \nabla S(x,t,u), \nabla v \right\rangle}{(1-v)^2 u} + \left(\partial_u S(x,t,u) - \frac{S(x,t,u)}{u} \right) w.$$

This, (1.4), (1.6) and (2.11) lead to

$$\begin{split} \frac{\Delta w - w_t}{2} &\geq (1 - v)w^2 + \frac{v\left\langle \nabla w, \nabla v \right\rangle}{1 - v} - \frac{\left\langle \nabla S(x, t, u), \nabla v \right\rangle}{(1 - v)^2 u} \\ &- \left(k + \partial_u S(x, t, u) - \frac{S(x, t, u)}{u} + \frac{S(x, t, u)}{(1 - v)u}\right)w \\ &\geq (1 - v)w^2 + \frac{v\left\langle \nabla w, \nabla v \right\rangle}{1 - v} - \frac{\left\langle \nabla S(x, t, u), \nabla v \right\rangle}{(1 - v)^2 u} \\ &- \left(k + \partial_u S(x, t, u) - \frac{S(x, t, u)}{u} + \frac{S(x, t, u)}{(1 - v)u}\right)_+ w \\ &\geq (1 - v)w^2 + \frac{v\left\langle \nabla w, \nabla v \right\rangle}{1 - v} - \frac{\gamma \left| \nabla v \right|}{(1 - v)^2} - \mu w, \end{split}$$

as desired.

In the proof of Theorem 1.1, we will exploit suitable cut-off functions. An important property of these auxiliary functions lies in their precise detachment with respect to the zero level set. The details of their construction are given in the following result:

Lemma 2.2. Let $a \in (0,1)$, R > 0 and $\rho \in (0,R)$. Then, there exists a decreasing function $\bar{\psi} \in C^2(\mathbb{R}, [0,1])$ such that

$$\bar{\psi}(r) = 1 \text{ for all } r \in [0, R - \rho], \quad \bar{\psi}(r) = 0 \text{ for all } r \ge R,$$

$$(2.12)$$

and, for every $r \ge 0$,

$$\rho |\bar{\psi}'(r)| + \rho^2 |\bar{\psi}''(r)| \leqslant C (\bar{\psi}(r))^a,$$
(2.13)

for some C > 0, depending only on a.

Proof. We introduce an increasing function $\alpha \in C^2(\mathbb{R}, [0, 1])$ such that:

$$\alpha(t) = t^{\frac{2}{1-a}} \text{ for all } t \in [0, 1/4],$$

$$\alpha(t) = 1 - (1-t)^4 \text{ for all } t \in [3/4, 1],$$

$$\alpha(t) = 0 \text{ for all } t < 0,$$

and
$$\alpha(t) = 1 \text{ for all } t > 1.$$

(2.14)

Let also

$$[0,\infty) \ni r \longmapsto \overline{\psi}(r) := \alpha \left(\frac{R-r}{\rho}\right).$$

We observe that if $r \in [0, R - \rho]$, then $\frac{R-r}{\rho} \in \left[1, \frac{R}{\rho}\right]$, hence $\bar{\psi}(r) = 1$. Similarly, if $r \ge R$, then $\frac{R-r}{\rho} \le 0$ and thus $\bar{\psi}(r) = 0$. These considerations establish (2.12).

Now, we prove (2.13). For this, in light of (2.12), it is enough to consider the case in which $r \in [R - \rho, R]$, since otherwise $\bar{\psi}'(r) = \bar{\psi}''(r) = 0$. Now, when $r \in [R - \rho, R]$, we have that $\frac{R-r}{\rho} \in [0, 1]$, and we distinguish two cases:

either
$$\frac{R-r}{\rho} \in \left[0, \frac{1}{4}\right],$$
 (2.15)

or
$$\frac{R-r}{\rho} \in \left(\frac{1}{4}, 1\right]$$
. (2.16)

Suppose first that (2.15) is satisfied. Then, $\bar{\psi}(r) = \alpha \left(\frac{R-r}{\rho}\right) = \left(\frac{R-r}{\rho}\right)^{\frac{2}{1-a}}$, and accordingly

$$\begin{split} \rho|\bar{\psi}'(r)| + \rho^2|\bar{\psi}''(r)| &= \frac{2}{1-a} \left(\frac{R-r}{\rho}\right)^{\frac{1+a}{1-a}} + \frac{2(1+a)}{(1-a)^2} \left(\frac{R-r}{\rho}\right)^{\frac{2a}{1-a}} \\ &= \left[\frac{2}{1-a} \frac{R-r}{\rho} + \frac{2(1+a)}{(1-a)^2}\right] \left(\frac{R-r}{\rho}\right)^{\frac{2a}{1-a}} \\ &\leqslant \left[\frac{1}{2(1-a)} + \frac{2(1+a)}{(1-a)^2}\right] \left(\bar{\psi}(r)\right)^a. \end{split}$$

This proves (2.13) in this case, and we now suppose that (2.16) holds true. In this situation, we exploit the monotonicity of α to see that

$$\bar{\psi}(r) \ge \alpha \left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^{\frac{2}{1-a}}$$

As a consequence,

$$\rho|\bar{\psi}'(r)| + \rho^2|\bar{\psi}''(r)| = \left|\alpha'\left(\frac{R-r}{\rho}\right)\right| + \left|\alpha''\left(\frac{R-r}{\rho}\right)\right|$$
$$\leqslant 2\|\alpha\|_{C^2(\mathbb{R})} \leqslant 24^{\frac{2a}{1-a}} \|\alpha\|_{C^2(\mathbb{R})} \left(\bar{\psi}(r)\right)^a.$$

This ends the proof of (2.13), as desired.

As a simple variant of Lemma 2.2, we also provide the details of an auxiliary cut-off function in the time variable:

Lemma 2.3. Let $t_0 \in \mathbb{R}$ and T > 0. Let $a \in (0, 1)$ and $\delta \in (0, T)$. Then, there exists an increasing function $\phi \in C^2(\mathbb{R}, [0, 1])$ such that

$$\phi(t) = 0 \text{ for all } t \leq t_0 - T, \text{ and } \phi(t) = 1 \text{ for all } t \geq t_0 - T + \delta, \qquad (2.17)$$

and, for every $t \in \mathbb{R}$,

$$\delta|\phi'(t)| \leqslant C(\phi(t))^{\frac{1+a}{2}},\tag{2.18}$$

for some C > 0, depending only on a.

Proof. Let α be the function in (2.14) and define

$$\phi(t) := \alpha\left(\frac{t-t_0+T}{\delta}\right).$$

Then, if $t \leq t_0 - T$ we have that $\frac{t-t_0+T}{\delta} \leq 0$ and thus $\phi(t) = 0$. Similarly, if $t \geq t_0 - T + \delta$ then $\frac{t-t_0+T}{\delta} \geq 1$ and therefore $\phi(t) = 1$. This proves (2.17).

To check (2.18), we can suppose that $t \in [t_0 - T, t_0 - T + \delta]$ (otherwise $\phi'(t)$ and the claim is obviously true). We distinguish two cases,

either
$$t \in \left[t_0 - T, t_0 - T + \frac{\delta}{4}\right]$$
 (2.19)

or
$$t \in \left[t_0 - T + \frac{\delta}{4}, t_0 - T + \delta\right].$$
 (2.20)

If (2.19) holds true, we have that

$$\phi(t) = \left(\frac{t - t_0 + T}{\delta}\right)^{\frac{2}{1 - a}},$$

and therefore

$$|\phi'(t)| = \frac{2}{(1-a)\delta} \left(\frac{t-t_0+T}{\delta}\right)^{\frac{1+a}{1-a}} = \frac{2}{(1-a)\delta} (\phi(t))^{\frac{1+a}{2}},$$

giving (2.18) in this case.

If instead (2.20) holds true, we use the monotonicity of α to write that, for every $t \in [t_0 - T + \frac{\delta}{4}, t_0 - T + \delta],$

$$\phi(t) \ge \alpha\left(\frac{1}{4}\right) = \frac{1}{4^{\frac{2}{1-a}}},$$

and consequently

$$|\phi'(t)| = \frac{1}{\delta} \left| \alpha' \left(\frac{t - t_0 + T}{\delta} \right) \right| \leq \frac{\|\alpha\|_{C^1(\mathbb{R})}}{\delta} \leq \frac{4^{\frac{1 + a}{1 - a}} \|\alpha\|_{C^1(\mathbb{R})}}{\delta} \left(\phi(t) \right)^{\frac{1 + a}{2}},$$

so that the proof of (2.18) is concluded.

To complete the proof of Theorem 1.1, we now distinguish four regimes, according to the cut-off functions in (1.7). The estimates in each of these regimes will be dealt with in the forthcoming Lemmata 2.4, 2.5, 2.6 and 2.7. To this end, it is also useful to point out the identity (valid for all smooth and positive functions ψ),

$$\frac{\Delta(w\psi) - (w\psi)_t}{2} - \frac{\langle \nabla(w\psi), \nabla\psi \rangle}{\psi} = \frac{(\Delta w - w_t)\psi}{2} + \frac{(\Delta \psi - \psi_t)w}{2} - \frac{w|\nabla\psi|^2}{\psi}.$$
 (2.21)

Hence, subtracting $\frac{v \langle \nabla v, \nabla(w\psi) \rangle}{1-v}$ to both sides of (2.21),

$$\begin{aligned} \frac{\Delta(w\psi) - (w\psi)_t}{2} - \left\langle \nabla(w\psi), \frac{\nabla\psi}{\psi} + \frac{v\,\nabla v}{1-v} \right\rangle \\ = \frac{(\Delta w - w_t)\,\psi}{2} + \frac{(\Delta\psi - \psi_t)\,w}{2} - \frac{w\,|\nabla\psi|^2}{\psi} - \frac{v\,\langle\nabla v, \nabla(w\psi)\rangle}{1-v} \end{aligned}$$

Whence it follows from Lemma 2.1 that

$$\frac{\Delta(w\psi) - (w\psi)_t}{2} - \left\langle \nabla(w\psi), \frac{\nabla\psi}{\psi} + \frac{v\nabla v}{1 - v} \right\rangle$$

$$\geq (1 - v) w^2 \psi + \frac{v\psi \left\langle \nabla w, \nabla v \right\rangle}{1 - v} - \frac{\gamma\psi \left| \nabla v \right|}{(1 - v)^2} - \mu w\psi$$

$$+ \frac{(\Delta\psi - \psi_t) w}{2} - \frac{w \left| \nabla \psi \right|^2}{\psi} - \frac{v \left\langle \nabla v, \nabla(w\psi) \right\rangle}{1 - v}.$$
(2.22)

One can also notice that

$$\frac{v\psi\left\langle \nabla w, \nabla v \right\rangle}{1-v} - \frac{v\left\langle \nabla v, \nabla (w\psi) \right\rangle}{1-v} = -\frac{vw\left\langle \nabla \psi, \nabla v \right\rangle}{1-v},$$

and thus rewrite (2.22) in the form

$$\frac{\Delta(w\psi) - (w\psi)_t}{2} - \left\langle \nabla(w\psi), \frac{\nabla\psi}{\psi} + \frac{v\,\nabla v}{1-v} \right\rangle$$

$$\geq (1-v)\,w^2\psi - \frac{\gamma\psi\,|\nabla v|}{(1-v)^2} - \mu w\psi$$

$$+ \frac{(\Delta\psi - \psi_t)\,w}{2} - \frac{w\,|\nabla\psi|^2}{\psi} - \frac{vw\,\langle\nabla\psi,\nabla v\rangle}{1-v}.$$
(2.23)

In addition, from (2.1) and Young's inequality with exponents 4 and 4/3,

$$\frac{\gamma\psi |\nabla v|}{(1-v)^2} = \frac{\gamma\psi\sqrt{w}}{1-v} = \sqrt[4]{1-v}\sqrt{w}\sqrt[4]{\psi}\frac{\gamma\psi^{\frac{3}{4}}}{(1-v)^{\frac{5}{4}}} \\ \leqslant \frac{1}{4}(1-v)w^2\psi + \frac{C\gamma^{4/3}\psi}{(1-v)^{5/3}},$$
(2.24)

for some C > 0.

Similarly, the use of (2.1) and of the Young's inequality with exponents 4/3 and 4 gives that, in the support of ψ ,

$$\left| \frac{vw \langle \nabla \psi, \nabla v \rangle}{1 - v} \right| \leq \frac{|v| \, w \, |\nabla \psi| \, |\nabla v|}{1 - v} = |v| \, w^{3/2} \, |\nabla \psi|
= \left[\left(\frac{2}{3} \right)^{3/4} \, (1 - v)^{3/4} \, w^{3/2} \, \psi^{3/4} \right] \left[\left(\frac{3}{2} \right)^{3/4} \, \frac{|v| \, |\nabla \psi|}{(1 - v)^{3/4} \, \psi^{3/4}} \right] \qquad (2.25)
\leq \frac{1}{4} \, (1 - v) \, w^2 \, \psi + \frac{C \, |v|^4 \, |\nabla \psi|^4}{(1 - v)^3 \, \psi^3},$$

up to renaming C > 0.

In light of (2.24) and (2.25), we deduce from (2.23)

$$\frac{\Delta(w\psi) - (w\psi)_t}{2} - \left\langle \nabla(w\psi), \frac{\nabla\psi}{\psi} + \frac{v\,\nabla v}{1-v} \right\rangle$$

$$\geq \frac{(1-v)\,w^2\psi}{4} - \frac{C\gamma^{4/3}\,\psi}{(1-v)^{5/3}} - \mu w\psi$$

$$+ \frac{(\Delta\psi - \psi_t)\,w}{2} - \frac{w\,|\nabla\psi|^2}{\psi} - \frac{C\,|v|^4\,|\nabla\psi|^4}{(1-v)^3\,\psi^3}.$$
(2.26)

Besides, by the Cauchy-Schwarz inequality,

$$\mu w \psi = \left(\sqrt{1-v} \ w \ \sqrt{\psi}\right) \ \left(\frac{\mu \sqrt{\psi}}{\sqrt{1-v}}\right) \leqslant \frac{(1-v)w^2\psi}{8} + \frac{C\mu^2\psi}{1-v},\tag{2.27}$$

up to renaming C, which, combined with (2.26), proves that

$$\frac{\Delta(w\psi) - (w\psi)_{t}}{2} - \left\langle \nabla(w\psi), \frac{\nabla\psi}{\psi} + \frac{v\,\nabla v}{1-v} \right\rangle \\
\geqslant \frac{(1-v)\,w^{2}\psi}{8} - \frac{C\gamma^{4/3}\,\psi}{(1-v)^{5/3}} - \frac{C\mu^{2}\psi}{1-v} \\
+ \frac{(\Delta\psi - \psi_{t})\,w}{2} - \frac{w\,|\nabla\psi|^{2}}{\psi} - \frac{C\,|v|^{4}\,|\nabla\psi|^{4}}{(1-v)^{3}\,\psi^{3}}.$$
(2.28)

We will use (2.28) as a pivotal inequality in the forthcoming computations. We have:

Lemma 2.4. In the setting of Theorem 1.1, in $B(x_0, R - \rho) \times [t_0 - T, t_0]$ it holds

$$w \leqslant \left[\tau_u^2 + C\left(\gamma^{2/3} + \mu + \frac{1}{\rho^2} + \frac{1}{\rho(R-\rho)} + \frac{\sqrt{k_+}}{\rho}\right)\right],$$
(2.29)

for some C > 0.

Proof. Let $a \in (0, 1)$, to be conveniently chosen in what follows. For every $x \in B(x_0, R)$, we define

$$\psi(x) := \bar{\psi}(d(x, x_0)),$$
 (2.30)

where $d(\cdot, \cdot)$ represents the geodesic distance and $\bar{\psi}$ is the function introduced in Lemma 2.2. By (1.1), we know that

$$\Delta d(x, x_0) \leq \frac{n-1}{d(x, x_0)} + \sqrt{(n-1)k_+},$$

see e.g. [5, formula (2.1)] and the references therein.

This and (2.13) entail that

$$|\nabla\psi(x)| = |\bar{\psi}'(d(x,x_0))\nabla d(x,x_0)| \leq \frac{C(\psi(x))^a}{\rho}$$

and $-\Delta\psi(x) = -\bar{\psi}'(d(x,x_0))\Delta d(x,x_0) - \bar{\psi}''(d(x,x_0))|\nabla d(x,x_0)|^2$ (2.31)
 $\leq \frac{C(\psi(x))^a}{\rho} \left(\frac{n-1}{d(x,x_0)} + \sqrt{(n-1)k_+}\right) + \frac{C(\psi(x))^a}{\rho^2}.$

Now, we consider $\tilde{w} := w\psi$ and, in the support of ψ , we can exploit (2.28) and find that

$$\frac{\Delta \tilde{w} - \tilde{w}_t}{2} - \left\langle \nabla \tilde{w}, \frac{\nabla \psi}{\psi} + \frac{v \nabla v}{1 - v} \right\rangle$$

$$\geq \frac{(1 - v) w^2 \psi}{8} - \frac{C \gamma^{4/3} \psi}{(1 - v)^{5/3}} - \frac{C \mu^2 \psi}{1 - v}$$

$$+ \frac{\Delta \psi w}{2} - \frac{w |\nabla \psi|^2}{\psi} - \frac{C |v|^4 |\nabla \psi|^4}{(1 - v)^3 \psi^3}.$$
(2.32)

We take (x_1, t_1) in the closure of $Q_{R,T}$ realizing the maximum of \tilde{w} . Since $\tilde{w}(x, t) = 0$ if $x \in \partial B(x_0, R)$, necessarily x_1 is an interior point of $B(x_0, R)$. Consequently $\nabla \tilde{w}(x_1, t_1) = 0$ and $\Delta \tilde{w}(x_1, t_1) \leq 0$. Hence, inserting this information into (2.32), we obtain that

$$0 \ge \frac{\tilde{w}_t}{2} + \frac{(1-v)w^2\psi}{8} - \frac{C\gamma^{4/3}\psi}{(1-v)^{5/3}} - \frac{C\mu^2\psi}{1-v} + \frac{\Delta\psi w}{2} - \frac{w|\nabla\psi|^2}{\psi} - \frac{C|v|^4|\nabla\psi|^4}{(1-v)^3\psi^3}\Big|_{(x,t)=(x_1,t_1)}.$$
(2.33)

We now distinguish two cases,

either
$$t_1 = t_0 - T$$
, (2.34)

or
$$t_1 \in (t_0 - T, t_0].$$
 (2.35)

Suppose first that (2.34) holds true. Then, for every $(x, t) \in Q_{R,T}$,

$$\begin{split} \tilde{w}(x,t) &\leqslant \tilde{w}(x_1,t_0-T) \\ &\leqslant \sup_{x \in B(x_0,R)} \tilde{w}(x,t_0-T) \\ &\leqslant \sup_{x \in B(x_0,R)} w(x,t_0-T) \\ &= \sup_{x \in B(x_0,R)} \left(\frac{|\nabla u|^2}{u^2(1-v)^2}\right) (x,t_0-T) \\ &\leqslant \tau_u^2, \end{split}$$

thanks to (1.9) and (2.1). In particular, for all $(x,t) \in B(x_0, R-\rho) \times [t_0 - T, t_0]$,

$$w(x,t) = \tilde{w}(x,t) \leqslant \tau_u^2,$$

and this proves (2.29) in this case.

Hence, to complete the proof of (2.29), we now consider the case in which (2.35) is satisfied. Then, $\tilde{w}_t(x_1, t_1) \ge 0$, and consequently (2.33) entails that

$$0 \ge \frac{(1-v)w^{2}\psi}{8} - \frac{C\gamma^{4/3}\psi}{(1-v)^{5/3}} - \frac{C\mu^{2}\psi}{1-v} + \frac{\Delta\psi w}{2} - \frac{w|\nabla\psi|^{2}}{\psi} - \frac{C|v|^{4}|\nabla\psi|^{4}}{(1-v)^{3}\psi^{3}}\Big|_{(x,t)=(x_{1},t_{1})}.$$
(2.36)

Our goal is now to estimate the terms in (2.36) that contain powers of w strictly less than 2, in order to "reabsorb" them into the quadratic term. To this end, exploiting (2.31)inside (2.36), and renaming C > 0 (possibly depending on a), we find that

$$\frac{1}{8} (1-v) w^2 \psi \bigg|_{(x,t)=(x_1,t_1)}
\leq \frac{C\gamma^{4/3} \psi}{(1-v)^{5/3}} + \frac{C\mu^2 \psi}{1-v} - \frac{\Delta \psi w}{2} + \frac{Cw\psi^{2a-1}}{\rho^2} + \frac{C |v|^4 \psi^{4a-3}}{(1-v)^3 \rho^4} \bigg|_{(x,t)=(x_1,t_1)}.$$
(2.37)

Also, from (2.31),

$$-\frac{\Delta\psi\,w}{2} \leqslant \frac{Cw}{2} \left[\frac{\psi^a}{\rho}\left(\frac{n-1}{d} + \sqrt{(n-1)k_+}\right) + \frac{\psi^a}{\rho^2}\right]$$
$$= \frac{C\sqrt{1-v}\,w\,\sqrt{\psi}}{2} \left[\rho\left(\frac{n-1}{d} + \sqrt{(n-1)k_+}\right) + 1\right]\frac{\psi^{a-\frac{1}{2}}}{\sqrt{1-v}\,\rho^2}$$

where $d := d(x, x_0)$. Then, by the Cauchy-Schwarz inequality,

$$-\frac{\Delta\psi\,w}{2} \leqslant \frac{(1-v)w^2\psi}{16} + C\left[\rho\left(\frac{n-1}{d} + \sqrt{(n-1)k_+}\right) + 1\right]^2 \frac{\psi^{2a-1}}{(1-v)\rho^4},\tag{2.38}$$

up to renaming C.

We also remark that when $x \in B(x_0, R - \rho)$, then $d = d(x, x_0) \in [0, R - \rho)$, and thus $\frac{R-d}{\rho} > 1$, which gives that

$$\psi(x) = \overline{\psi}(d) = \alpha\left(\frac{R-d}{\rho}\right) = 1.$$

In particular,

$$\Delta \psi(x) = 0 \qquad \text{for all } x \in B(x_0, R - \rho). \tag{2.39}$$

Now we claim that

$$-\frac{\Delta\psi\,w}{2} \leqslant \frac{(1-v)w^2\psi}{16} + \frac{C\psi^{2a-1}}{(1-v)\rho^2(R-\rho)^2} + \frac{Ck_+\psi^{2a-1}}{(1-v)\rho^2} + \frac{C\psi^{2a-1}}{(1-v)\rho^4}, \tag{2.40}$$

up to renaming C. To prove this, we first observe that in $B(x_0, R-\rho)$ the claim is obvious, thanks to (2.39). Hence, we can focus on the complement of $B(x_0, R-\rho)$, where

$$d \geqslant R - \rho. \tag{2.41}$$

Then, we can exploit (2.38), combined with (2.41), and obtain (2.40), as desired.

Thus, we insert (2.40) into (2.37) and we obtain that

$$\frac{1}{16} (1-v) w^2 \psi \bigg|_{(x,t)=(x_1,t_1)} \leqslant \frac{C\gamma^{4/3} \psi}{(1-v)^{5/3}} + \frac{C\mu^2 \psi}{1-v} + \frac{Cw\psi^{2a-1}}{\rho^2} + \frac{C|v|^4 \psi^{4a-3}}{(1-v)^3 \rho^4} + \frac{C\psi^{2a-1}}{(1-v)\rho^2 (R-\rho)^2} + \frac{Ck_+\psi^{2a-1}}{(1-v)\rho^2} + \frac{C\psi^{2a-1}}{(1-v)\rho^4} \bigg|_{(x,t)=(x_1,t_1)} .$$
(2.42)

Furthermore, using the Cauchy-Schwarz inequality,

$$\frac{Cw\psi^{2a-1}}{\rho^2} = \sqrt{1-v} \ w \ \sqrt{\psi} \ \frac{C\psi^{2a-\frac{3}{2}}}{\sqrt{1-v} \ \rho^2} \leqslant \frac{1}{32}(1-v)w^2\psi + \frac{C\psi^{4a-3}}{(1-v)\rho^4}, \tag{2.43}$$

up to renaming C, which together with (2.42) entails that at the point $(x, t) = (x_1, t_1)$

$$\frac{1}{32} (1-v) w^2 \psi \leqslant \frac{C\gamma^{4/3} \psi}{(1-v)^{5/3}} + \frac{C\mu^2 \psi}{1-v} + \frac{C\psi^{4a-3}}{(1-v)\rho^4} + \frac{C|v|^4 \psi^{4a-3}}{(1-v)^3 \rho^4} + \frac{C\psi^{2a-1}}{(1-v)\rho^2 (R-\rho)^2} + \frac{Ck_+ \psi^{2a-1}}{(1-v)\rho^2} + \frac{C\psi^{2a-1}}{(1-v)\rho^4}.$$
(2.44)

Now, up to renaming C, and recalling the maximizing property of (x_1, t_1) , we can rewrite (2.44) in case (2.35) in the form

$$\sup_{Q_{R,T}} w^2 \psi^2 \leqslant \frac{C\gamma^{4/3} \psi^2}{(1-v)^{8/3}} + \frac{C\mu^2 \psi^2}{(1-v)^2} + \frac{C\psi^{4a-2}}{(1-v)^2 \rho^4} + \frac{C|v|^4 \psi^{4a-2}}{(1-v)^4 \rho^4} + \frac{C\psi^{2a}}{(1-v)^2 \rho^2 (R-\rho)^2} + \frac{Ck_+ \psi^{2a}}{(1-v)^2 \rho^2} + \frac{C\psi^{2a}}{(1-v)^2 \rho^4} \bigg|_{(x,t)=(x_1,t_1)}.$$

$$(2.45)$$

Now we choose a := 1/2, and we recall that $0 \le \psi \le 1$ and that $\psi = 1$ in $B(x_0, R - \rho)$.

In this way, in case (2.35) we deduce from (2.45) that

$$\sup_{B(x_0,R-\rho)\times(t_0-T,t_0]} w^2 = \sup_{B(x_0,R-\rho)\times(t_0-T,t_0]} w^2 \psi^2$$

$$\leqslant \frac{C\gamma^{4/3}}{(1-v)^{8/3}} + \frac{C\mu^2}{(1-v)^2} + \frac{C}{(1-v)^2\rho^4}$$

$$+ \frac{C|v|^4}{(1-v)^4\rho^4} + \frac{C}{(1-v)^2\rho^2(R-\rho)^2} + \frac{Ck_+}{(1-v)^2\rho^2}\Big|_{(x,t)=(x_1,t_1)}.$$
(2.46)

Up to renaming C, we can also rewrite (2.46) as

$$\sup_{B(x_0,R-\rho)\times(t_0-T,t_0]} w^2 \leqslant \frac{C\gamma^{4/3}}{(1-v)^{8/3}} + \frac{C\mu^2}{(1-v)^2} + \frac{C(1+v^4)}{(1-v)^4\rho^4} + \frac{C}{(1-v)^2\rho^2(R-\rho)^2} + \frac{Ck_+}{(1-v)^2\rho^2} \bigg|_{(x,t)=(x_1,t_1)}$$

That is, recalling (2.2),

$$\sup_{B(x_0,R-\rho)\times(t_0-T,t_0]} w^2 \leqslant C\gamma^{4/3} + C\mu^2 + \frac{C}{\rho^4} + \frac{C}{\rho^2(R-\rho)^2} + \frac{Ck_+}{\rho^2},$$

which establishes (2.29), as desired.

Lemma 2.5. In the setting of Theorem 1.1, in $B(x_0, R) \times [t_0 - T + \delta, t_0]$ it holds

$$w \leqslant \left[\sigma_u^2 + C\left(\gamma^{2/3} + \mu + \frac{1}{\delta}\right)\right],\tag{2.47}$$

for some C > 0.

Proof. We take ϕ as in Lemma 2.3 (say, with a := 1/2), and we define $\tilde{w}(x,t) := w(x,t)\phi(t)$. Then, in light of (2.28),

$$\frac{\Delta \tilde{w} - \tilde{w}_t}{2} - \left\langle \nabla \tilde{w}, \frac{v \, \nabla v}{1 - v} \right\rangle \ge \frac{(1 - v) \, w^2 \phi}{8} - \frac{C \gamma^{4/3} \, \psi}{(1 - v)^{5/3}} - \frac{C \mu^2 \psi}{1 - v} - \frac{\phi_t \, w}{2}. \tag{2.48}$$

Suppose that the maximum of \tilde{w} in the closure of $Q_{R,T}$ is reached at (x_1, t_1) . Since $\tilde{w} = 0$ when $t = t_0 - T$, we know that $t_1 \in (t_0 - T, t_0]$. As a result,

$$\tilde{w}_t(x_1, t_1) \ge 0. \tag{2.49}$$

We then distinguish two cases,

either $x_1 \in \partial B(x_0, R),$ (2.50)

or
$$x_1 \in B(x_0, R)$$
. (2.51)

If (2.50) holds true, then, in $Q_{R,T}$,

$$\begin{split} \tilde{w} \leqslant \tilde{w}(x_{1}, t_{1}) \leqslant \sup_{\substack{x \in \partial B(x_{0}, R) \\ t \in [t_{0} - T, t_{0}]}} \tilde{w}(x, t) \\ \leqslant \sup_{\substack{x \in \partial B(x_{0}, R) \\ t \in [t_{0} - T, t_{0}]}} w(x, t) \\ = \sup_{\substack{x \in \partial B(x_{0}, R) \\ t \in [t_{0} - T + \delta/2, t_{0}]}} \Big| \frac{|\nabla u|^{2}}{u^{2}(1 - v)^{2}} \Big| (x, t) \\ = \sigma_{u}^{2}, \end{split}$$

due to (1.9) and (2.1). In particular, since $\phi = 1$ for any $t \ge t_0 - T + \delta$, we have that

$$w = \tilde{w} \leqslant \sigma_u^2$$

in $B(x_0, R) \times [t_0 - T + \delta, t_0]$.

This proves (2.47) in this case, and we now suppose that (2.51) holds true. Then, we have that $\Delta \tilde{w}(x_1, t_1) \leq 0$ and $\nabla \tilde{w}(x_1, t_1) = 0$. These observations and (2.49), together with (2.48), yield that

$$0 \ge \frac{(1-v)w^2\phi}{8} - \frac{C\gamma^{4/3}\psi}{(1-v)^{5/3}} - \frac{C\mu^2\psi}{1-v} - \frac{\phi_t w}{2}\Big|_{(x,t)=(x_1,t_1)}.$$
(2.52)

Moreover, by (2.18) and the Cauchy-Schwarz inequality,

$$\frac{\phi_t w}{2} \leqslant \frac{C \phi^{\frac{1+a}{2}} w}{2\delta} = \frac{\sqrt{1-v} w \sqrt{\phi}}{4} \frac{2C\phi^{\frac{a}{2}}}{\delta\sqrt{1-v}} \leqslant \frac{(1-v)w^2\phi}{16} + \frac{C\phi^a}{\delta^2(1-v)}, \quad (2.53)$$

possibly renaming constants. Plugging this into (2.52), we conclude that

$$(1-v) w^2 \phi \bigg|_{(x,t)=(x_1,t_1)} \leq \frac{C\gamma^{4/3} \phi}{(1-v)^{5/3}} + \frac{C\mu^2 \phi}{1-v} + \frac{C\phi^a}{\delta^2(1-v)} \bigg|_{(x,t)=(x_1,t_1)}$$

That is

$$w^2 \phi \bigg|_{(x,t)=(x_1,t_1)} \leqslant \frac{C\gamma^{4/3} \phi}{(1-v)^{8/3}} + \frac{C\mu^2 \phi}{(1-v)^2} + \frac{C\phi^a}{\delta^2 (1-v)^2} \bigg|_{(x,t)=(x_1,t_1)}$$

Now, since $0 \leq \phi \leq 1$ and $\phi = 1$ for any $t \geq t_0 - T + \delta$, this implies that

$$\sup_{B(x_0,R)\times[t_0-T+\delta,t_0]} w^2 = \sup_{B(x_0,R)\times[t_0-T+\delta,t_0]} w^2 \phi^2 \leqslant \frac{C\gamma^{4/3}}{(1-v)^{8/3}} + \frac{C\mu^2}{(1-v)^2} + \frac{C}{\delta^2(1-v)^2} \bigg|_{(x,t)=(x_1,t_1)}$$

As a consequence, recalling also (2.2), we obtain (2.47), as desired.

ı.

Lemma 2.6. In the setting of Theorem 1.1, in $B(x_0, R) \times [t_0 - T, t_0]$ it holds

$$w \leqslant \left[\sigma_u^2 + \tau_u^2 + C\left(\gamma^{2/3} + \mu\right)\right],\tag{2.54}$$

for some C > 0.

Proof. We suppose that the maximum of w in the closure of $Q_{R,T}$ is reached at the point (x_1, t_1) . We distinguish three possibilities:

either $x_1 \in B(x_0, R)$ and $t_1 \in (t_0 - T, t_0],$ (2.55)

or
$$x_1 \in B(x_0, R)$$
 and $t_1 = t_0 - T$, (2.56)

or
$$x_1 \in \partial B(x_0, R)$$
 and $t_1 \in [t_0 - T, t_0].$ (2.57)

Suppose first that (2.55) holds true. Then, we have that $\Delta w(x_1, t_1) \leq 0$, $\nabla w(x_1, t_1) = 0$ and $w_t(x_1, t_1) \geq 0$. Therefore, in light of Lemma 2.1, we obtain that

$$0 \ge (1-v) w^2 - \frac{\gamma |\nabla v|}{(1-v)^2} - \mu w \bigg|_{(x,t)=(x_1,t_1)}.$$
(2.58)

We insert (2.24) and (2.27) (used here with $\psi := 1$) into (2.58) to see that

$$\frac{1}{2} (1-v) w^2 \bigg|_{(x,t)=(x_1,t_1)} \leqslant \frac{C\gamma^{4/3}}{(1-v)^{5/3}} + \frac{C\mu^2}{1-v} \bigg|_{(x,t)=(x_1,t_1)}.$$

Hence, using the maximality of (x_1, t_1) and recalling (2.2),

$$\sup_{B(x_0,R)\times[t_0-T,t_0]} w^2 \leqslant \frac{C\gamma^{4/3}}{(1-v)^{5/3}} + \frac{C\mu^2}{1-v} \bigg|_{(x,t)=(x_1,t_1)} \leqslant C\gamma^{4/3} + C\mu^2.$$

This proves (2.54) in this case. Hence we now assume that (2.56) is satisfied. Then, in $Q_{R,T}$,

$$w \leqslant w(x_1, t_0 - T) = \frac{|\nabla v(x_1, t_0 - T)|^2}{(1 - v(x_1, t_0 - T))^2}$$

= $\frac{|\nabla u(x_1, t_0 - T)|^2}{(u(x_1, t_0 - T))^2(1 - v(x_1, t_0 - T))^2} \leqslant \tau_u^2,$ (2.59)

thanks to (1.9), (2.1) and (2.3).

This establishes (2.54) in this case, and thus we now suppose that (2.57) is satisfied. In this case, the computation in (2.59) gives that, in $Q_{R,T}$,

$$w \leqslant w(x_1, t_1) = \frac{|\nabla u(x_1, t_1)|^2}{(u(x_1, t_1))^2 (1 - v(x_1, t_1))^2} \leqslant \sigma_u^2,$$

whence the proof of (2.54) is complete.

Lemma 2.7. In the setting of Theorem 1.1, in $B(x_0, R - \rho) \times [t_0 - T + \delta, t_0]$ it holds

$$w \leqslant C \left(\gamma^{2/3} + \mu + \frac{1}{\rho(R-\rho)} + \frac{\sqrt{k_+}}{\rho} + \frac{1}{\delta} + \frac{1}{\rho^2} \right),$$
(2.60)

for some C > 0.

Proof. Let $a \in (0, 1)$ to be conveniently chosen in what follows. Let also ψ be as in (2.30) and ϕ be as in Lemma 2.3. We define $\Phi(x, t) := \psi(x)\phi(t)$ and $\tilde{w} := w\Phi$. Suppose that the maximum of \tilde{w} in the closure of $Q_{R,T}$ is reached at some point (x_1, t_1) . Since Φ vanishes along the parabolic boundary, we know that $x_1 \in B(x_0, R)$ and $t_1 \in (t_0 - T, t_0]$. As a consequence,

$$\Delta \tilde{w}(x_1, t_1) \leqslant 0, \qquad \nabla \tilde{w}(x_1, t_1) = 0 \qquad \text{and} \qquad \tilde{w}_t(x_1, t_1) \ge 0. \tag{2.61}$$

Combining this information with (2.28), we obtain that

$$0 \ge \frac{(1-v)w^{2}\Phi}{8} - \frac{C\gamma^{4/3}\psi}{(1-v)^{5/3}} - \frac{C\mu^{2}\psi}{1-v} + \frac{(\Delta\Phi - \Phi_{t})w}{2} - \frac{w|\nabla\Phi|^{2}}{\Phi} - \frac{C|v|^{4}|\nabla\psi|^{4}}{(1-v)^{3}\psi^{3}}\Big|_{(x,t)=(x_{1},t_{1})}.$$
(2.62)

Now, from (2.38),

$$-\frac{\Delta\Phi\,w}{2} = -\frac{\phi\Delta\psi\,w}{2} \leqslant \frac{(1-v)w^2\Phi}{32} + C\left[\rho\,\left(\frac{n-1}{d} + \sqrt{(n-1)k_+}\right) + 1\right]^2 \frac{\psi^{2a-1}\phi}{(1-v)\rho^4},\tag{2.63}$$

and, from (2.53),

$$\frac{\Phi_t w}{2} = \frac{\psi \phi_t w}{2} \leqslant \frac{(1-v)w^2 \Phi}{32} + \frac{C\phi^a \psi}{\delta^2 (1-v)}.$$
(2.64)

We plug these items of information into (2.62), and we find that

$$\begin{split} & \left. \frac{1}{16} (1-v) \, w^2 \Phi \right|_{(x,t)=(x_1,t_1)} \\ & \leqslant \frac{C \gamma^{4/3} \Phi}{(1-v)^{5/3}} + \frac{C \mu^2 \Phi}{1-v} + C \left[\rho \, \left(\frac{n-1}{d} + \sqrt{(n-1)k_+} \right) + 1 \right]^2 \frac{\psi^{2a-1} \phi}{(1-v)\rho^4} \\ & + \frac{C \phi^a \psi}{\delta^2 (1-v)} + \frac{w \, |\nabla \Phi|^2}{\Phi} + \frac{C \, |v|^4 \, |\nabla \psi|^4}{(1-v)^3 \, \psi^3} \right|_{(x,t)=(x_1,t_1)}. \end{split}$$

This and (2.43) entail that

$$\begin{split} & \left. \frac{1}{32} (1-v) \, w^2 \Phi \right|_{(x,t)=(x_1,t_1)} \\ & \leqslant \frac{C \gamma^{4/3} \Phi}{(1-v)^{5/3}} + \frac{C \mu^2 \Phi}{1-v} + C \left[\rho \, \left(\frac{n-1}{d} + \sqrt{(n-1)k_+} \right) + 1 \right]^2 \frac{\psi^{2a-1} \phi}{(1-v)\rho^4} \\ & + \frac{C \phi^a \psi}{\delta^2 (1-v)} + \frac{C \psi^{4a-3} \phi}{(1-v)\rho^4} + \frac{C \, |v|^4 \, |\nabla \psi|^4}{(1-v)^3 \, \psi^3} \right|_{(x,t)=(x_1,t_1)}. \end{split}$$

Using this and (2.31) (and adjusting the constants) we conclude that

$$\begin{split} & (1-v) \, w^2 \Phi \bigg|_{(x,t)=(x_1,t_1)} \\ & \leqslant \frac{C \gamma^{4/3} \Phi}{(1-v)^{5/3}} + \frac{C \mu^2 \Phi}{1-v} + C \left[\rho \, \left(\frac{n-1}{d} + \sqrt{(n-1)k_+} \right) + 1 \right]^2 \frac{\psi^{2a-1} \phi}{(1-v)\rho^4} \\ & + \frac{C \phi^a \psi}{\delta^2 (1-v)} + \frac{C \psi^{4a-3} \phi}{(1-v)\rho^4} + \frac{C |v|^4 \, \psi^{4a-3} \phi}{(1-v)^3 \, \rho^4} \bigg|_{(x,t)=(x_1,t_1)}. \end{split}$$

Therefore, choosing a := 3/4, we see that, for every $x \in B(x_0, R-\rho)$ and $t \in [t_0 - T + \delta, t_0]$,

$$\begin{split} w^2(x,t) &= w^2(x,t) \Phi^2(x,t) = \tilde{w}^2(x,t) \leqslant \tilde{w}^2(x_1,t_1) = w^2(x_1,t_1) \Phi^2(x_1,t_1) \\ &\leqslant \frac{C\gamma^{4/3}}{(1-v)^{8/3}} + \frac{C\mu^2}{(1-v)^2} + \left[\rho \left(\frac{n-1}{d} + \sqrt{(n-1)k_+} \right) + 1 \right]^2 \frac{C}{(1-v)^2 \rho^4} \\ &+ \frac{C}{\delta^2(1-v)^2} + \frac{C}{(1-v)^2 \rho^4} + \frac{C|v|^4}{(1-v)^4 \rho^4} \bigg|_{(x,t)=(x_1,t_1)}, \end{split}$$

that, recalling (2.2), yields the desired estimate in (2.60).

Now we use the notation

$$\widetilde{\mathbb{C}} := \gamma^{2/3} + \mu,$$

$$\widetilde{\mathbb{T}} := \frac{1}{\delta},$$
and
$$\widetilde{\mathbb{S}} := \frac{1}{\rho^2} + \frac{1}{\rho(R-\rho)} + \frac{\sqrt{k_+}}{\rho}.$$
(2.65)

In this setting, the term $\widetilde{\mathbb{C}}$ in (2.65) denotes a common quantity for our main estimates, while the terms $\widetilde{\mathfrak{T}}$ and $\widetilde{\mathfrak{S}}$ play the role of parabolic boundary terms in time and in space respectively. As a matter of fact, by combining Lemmata 2.4, 2.5, 2.6 and 2.7 we find that

Corollary 2.8. In the setting of Theorem 1.1, the function w can be estimated by

$$C\widetilde{\mathfrak{C}} + (\tau_u^2 + C\widetilde{\mathfrak{S}}) \quad in \ B(x_0, R - \rho) \times [t_0 - T, t_0],$$

$$C\widetilde{\mathfrak{C}} + (\sigma_u^2 + C\widetilde{\mathfrak{T}}) \quad in \ B(x_0, R) \times [t_0 - T + \delta, t_0],$$

$$C\widetilde{\mathfrak{C}} + (\sigma_u^2 + \tau_u^2) \quad in \ B(x_0, R) \times [t_0 - T, t_0],$$

$$C\widetilde{\mathfrak{C}} + C\left(\widetilde{\mathfrak{S}} + \widetilde{\mathfrak{T}}\right) \quad in \ B(x_0, R - \rho) \times [t_0 - T + \delta, t_0].$$

for some C > 0.

Hence, considering the more convenient term in any common domain, we deduce from Corollary 2.8 that:

Corollary 2.9. In the setting of Theorem 1.1, at any point in $Q_{R,T}$, we have that

$$w \leqslant C\widetilde{\mathcal{C}} + \left[\min\left\{\sigma_{u}^{2} + \tau_{u}^{2}, \sigma_{u}^{2} + C\widetilde{\mathcal{T}}, \tau_{u}^{2} + C\widetilde{\mathcal{S}}, C(\widetilde{\mathcal{T}} + \widetilde{\mathcal{S}})\right\} \chi_{B(x_{0}, R-\rho) \times [t_{0}-T+\delta, t_{0}]} + \left(\sigma_{u}^{2} + \min\left\{\tau_{u}^{2}, C\widetilde{\mathcal{T}}\right\}\right) \chi_{(B(x_{0}, R) \setminus B(x_{0}, R-\rho)) \times [t_{0}-T+\delta, t_{0}]} + \left(\tau_{u}^{2} + \min\left\{\sigma_{u}^{2}, C\widetilde{\mathcal{S}}\right\}\right) \chi_{B(x_{0}, R-\rho) \times [t_{0}-T, t_{0}-T+\delta]} + \left(\sigma_{u}^{2} + \tau_{u}^{2}\right) \chi_{(B(x_{0}, R) \setminus B(x_{0}, R-\rho)) \times [t_{0}-T, t_{0}-T+\delta]}\right],$$

$$(2.66)$$

for some C > 0.

Completion of the proof of Theorem 1.1. In light of (2.1) and (2.3),

$$w = \frac{|\nabla v|^2}{(1-v)^2} = \frac{|\nabla u|^2}{u^2(1-v)^2}.$$

This and (2.66) give that

$$\begin{aligned} \frac{|\nabla u|^2}{u^2(1-v)^2} &\leqslant C\widetilde{\mathfrak{C}} + \Big[\min\left\{\sigma_u^2 + \tau_u^2, \, \sigma_u^2 + C\widetilde{\mathfrak{I}}, \, \tau_u^2 + C\widetilde{\mathfrak{S}}, \, C(\widetilde{\mathfrak{I}} + \widetilde{\mathfrak{S}})\right\} \chi_{B(x_0,R-\rho)\times[t_0-T+\delta,t_0]} \\ &+ \left(\sigma_u^2 + \min\left\{\tau_u^2, \, C\widetilde{\mathfrak{I}}\right\}\right) \chi_{(B(x_0,R)\setminus B(x_0,R-\rho))\times[t_0-T+\delta]} \\ &+ \left(\tau_u^2 + \min\left\{\sigma_u^2, \, C\widetilde{\mathfrak{S}}\right\}\right) \chi_{B(x_0,R-\rho)\times[t_0-T,t_0-T+\delta]} \\ &+ \left(\sigma_u^2 + \tau_u^2\right) \chi_{(B(x_0,R)\setminus B(x_0,R-\rho))\times[t_0-T,t_0-T+\delta]}\Big].\end{aligned}$$

Taking the square root and recalling (1.8), we obtain (1.12), as desired.

3 Applications of Theorem 1.1

In this part, we will show several applications of Theorem 1.1 also by comparing our general approach with some of the existing results in specific situations.

3.1 The heat equation in \mathbb{R}^n

As a special case, one obtains from Theorem 1.1 a global estimate for the gradient of solutions of the heat equation in Euclidean balls. We state this byproduct of Theorem 1.1 in detail for the sake of clarity:

Corollary 3.1. Let $B(x_0, R) \subset \mathbb{R}^n$ be the n-dimensional Euclidean ball. Let $M > 0, t_0 \in \mathbb{R}$ and T > 0. Let also $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$ and suppose that $u : Q_{R,T} \to (0, M]$ is a solution of

$$u_t = \Delta u \qquad in \ Q_{R,T}.$$

Then, for any $\delta \in (0,T)$ and $\rho \in (0,R)$, there exists C > 0, only depending on n, such that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant \mathcal{Z}^{(0)}(x,t) \left(1 + \ln \frac{M}{u(x,t)}\right) \qquad \text{for all } (x,t) \in Q_{R,T}, \tag{3.1}$$

where

$$\mathcal{Z}^{(0)} := \beta_1^{(0)} \, \mathcal{B}_1 + \beta_2 \, \mathcal{B}_2 + \beta_3 \, \mathcal{B}_3 + \iota^{(0)} \, \mathcal{I},$$

with

$$\beta_1^{(0)} := \tau_u + \min \{ \sigma_u, C S^{(0)} \}$$

and $\iota^{(0)} := \min \{ \sigma_u + \tau_u, \sigma_u + C \mathfrak{T}, \tau_u + C S^{(0)}, C (\mathfrak{T} + S^{(0)}) \}$

being

$$S^{(0)} := \frac{1}{\rho} + \frac{1}{\sqrt{\rho(R-\rho)}}.$$

We recall that \mathcal{T} is as in (1.8), β_2 and β_3 are as in (1.10), τ_u and σ_u are as in (1.9), and \mathcal{B}_1 , \mathcal{B}_2 , \mathcal{B}_3 and \mathcal{I} are as in (1.7).

Proof. Corollary 3.1 follows directly from Theorem 1.1 by recalling (1.7) and (1.10), since here k = 0 and S(x, t, u) = 0.

We remark that the logarithmic function in (3.1) arises naturally in the context of heat equation: for instance, one can consider the Gauß Kernel

$$u_G(x,t) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

and observe that, if $x \in B((1, 0, \dots, 0), 1/2)$ and $t \in (1, 2)$,

$$\frac{|\nabla u_G(x,t)|}{u_G(x,t)} = \frac{|x|}{2t} = -\frac{2}{|x|} \ln\left((4\pi t)^{n/2} u_G(x,t)\right) \simeq \ln\frac{1}{u_G(x,t)}.$$

3.2 The heat equation on manifolds

A small variation of Corollary 3.1 provides the following result:

Corollary 3.2. Let \mathcal{M} be a Riemannian manifold of dimension $n \ge 2$, with Ricci curvature bounded from below by some $k \in \mathbb{R}$. Let $B(x_0, R)$ be a geodesic ball in \mathcal{M} and $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$. Let M > 0, $t_0 \in \mathbb{R}$ and T > 0. Let $u : Q_{R,T} \to (0, M]$ be a solution of

$$u_t = \Delta u$$
 in $Q_{R,T}$.

Then, for any $\delta \in (0,T)$ and $\rho \in (0,R)$, there exists C > 0, only depending on n, such that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant \left(C\sqrt{k_+} + \mathcal{Z}(x,t)\right) \left(1 + \ln\frac{M}{u(x,t)}\right) \qquad \text{for all } (x,t) \in Q_{R,T}, \quad (3.2)$$

where \mathcal{Z} is as in (1.11).

Proof. We can exploit Theorem 1.1 by suitably modifying (1.7) and (1.10), since here S(x, t, u) = 0, and accordingly $\mu = k_+$ and $\mathcal{C} = \sqrt{k_+}$.

As particular cases of Corollary 3.2, one can re-obtain several classical local estimates for the heat equation. We provide one classical application to show the comprehensive nature of the results provided in this paper.

Theorem (Theorem 1.1 in [30]). Let \mathcal{M} be a Riemannian manifold of dimension $n \ge 2$ with Ricci curvature bounded from below by -k, for some $k \ge 0$. Suppose that u is any positive solution to the heat equation in $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$. Suppose also that $u \le M$ in $Q_{R,T}$. Then there exists a dimensional constant C such that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right) \left(1 + \ln\frac{M}{u(x,t)}\right)$$
(3.3)

for each $(x,t) \in Q_{R/2,T/2}$.

Proof. We exploit Corollary 3.2 with $\rho := R/2$ and $\delta := T/2$. In this way, in view of (1.7)

$$Q_{R/2,T/2} = B(x_0, R/2) \times [t_0 - T/2, t_0] = B(x_0, R - \rho) \times [t_0 - T + \delta, t_0]$$

$$\subseteq \{\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = 0\}.$$
(3.4)

Then, from (1.9) and (3.2), for all $(x,t) \in Q_{R/2,T/2}$,

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq \left(C\sqrt{k} + \mathcal{Z}(x,t)\right) \left(1 + \ln\frac{M}{u(x,t)}\right) \\
\leq \left(C\sqrt{k} + \iota\right) \left(1 + \ln\frac{M}{u(x,t)}\right) \\
= \left(C\sqrt{k} + \min\left\{\sigma_u + \tau_u, \sigma_u + C\mathfrak{T}, \tau_u + C\mathfrak{S}, C(\mathfrak{T} + \mathfrak{S})\right\}\right) \\
\cdot \left(1 + \ln\frac{M}{u(x,t)}\right).$$
(3.5)

In particular, by (1.8),

$$\frac{|\nabla u|}{u} \leqslant C \left(\sqrt{k} + \Im + \Im\right) \left(1 + \ln\frac{M}{u}\right) \leqslant C \left(\sqrt{k} + \frac{1}{\sqrt{T}} + \frac{1}{R} + \frac{\sqrt[4]{k}}{\sqrt{R}}\right) \left(1 + \ln\frac{M}{u}\right).$$

Accordingly, since

$$\frac{2\sqrt[4]{k}}{\sqrt{R}} \leqslant \sqrt{k} + \frac{1}{R}$$

we see that

$$\frac{|\nabla u|}{u} \leqslant C \left(\sqrt{k} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right) \left(1 + \ln\frac{M}{u}\right),$$

which implies (3.3), as desired.

It is interesting to remark that the classical dependence on the Ricci curvature in (3.3) is optimal, as shown by the following explicit example. Let $\lambda > 0$ and D_{λ} be the unit disk in the plane with the Poincaré metrics

$$g_{ij} = \frac{4\lambda^2 \,\delta_{ij}}{(1-|x|^2)^2}.\tag{3.6}$$

We recall that in this setting the notion of harmonicity is conformally invariant, hence¹

each harmonic function in the Euclidean unit disk is also harmonic in the hyperbolic metric of this Poincaré disk, (3.7)

which in turn has negative Ricci curvature of order $-\frac{1}{\lambda^2}$. Thus, we take $u(x,t) := x_1 + 2$, we observe that the supremum of u in the disk is equal to 3 and the infimum to 1, and that u is a Euclidean, hence hyperbolic, harmonic function. In particular, u is a global solution of the heat equation (hence we can consider the case $|x| \to +1$, which

¹For completeness we will provide a proof of (3.7) in the appendix.

means $R \to +\infty$ in the hyperbolic disk, and $T \to +\infty$ in $Q_{R,T}$). Now, recalling (see page 20 in [27]) that $\nabla u = g^{ij} u_i \partial_j$, we find that, in $\{|x| < 1/2\}$,

$$|\nabla u|^2 = g_{kj}g^{ij}g^{mk}u_iu_m = g^{mi}u_iu_m = \frac{(1-|x|^2)^2}{4\lambda^2}u_iu_i = \frac{(1-|x|^2)^2}{4\lambda^2} \ge \frac{9}{64\lambda^2}.$$

As a result, in $\{|x| < 1/2\},\$

$$\frac{|\nabla u|}{u} \geqslant \frac{1}{8\lambda},$$

and the latter quantity is of the order of the square root of minus the Ricci curvature of D_{λ} , thus showing the optimality of (3.3) with respect to the Ricci curvature (even in the case of harmonic functions, i.e. stationary solutions of the heat equation).

One of the remarkable aspects of the estimates in [30] is their "universality" with respect to the parabolic boundary data. On the other hand, the approach that we proposed with cut-off functions improve the classical estimates when the parabolic boundary data are "exceptionally good". For instance, suppose that the function $\frac{|\nabla u|}{u(1-v)}$ is controlled by a small ϵ along the parabolic boundary (hence, in view of (1.9), both τ_u and σ_u are bounded by ϵ): then one can deduce from (3.5) that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C\left(\sqrt{k} + \epsilon\right) \left(1 + \ln\frac{M}{u(x,t)}\right),\tag{3.8}$$

which is an improvement of (3.3) when ϵ is small enough. Interestingly, this improvement also occurs in the Euclidean case, for the classical heat equation, in which k = 0.

Let us emphasize the fact that cases of this type take place even in very simple and explicit examples: for instance, given $\epsilon \in (0, 1)$, one can consider the Euclidean case in which

$$u(x,t) = 10 + \epsilon e^{x_1 + t}.$$

In this situation, we have that, if |x| < 1 and $t \in [0, 1]$,

$$\begin{aligned} \Delta u &= u_{11} = \epsilon \, e^{x_1 + t} = u_t, \\ u &\leq 10 + e^2 \epsilon \leq 10 + e^2 < 19, \\ u &\geq 10 - e^2 \epsilon \geq 10 - e^2 > 1, \\ |\nabla u| &= \epsilon \, e^{x_1 + t} \leqslant e^2 \epsilon, \\ 1 - v &= 1 + \ln \frac{M}{u} \in [1, 1 + \ln 19] \end{aligned}$$

Notice in particular that u solves the heat equation in $Q_{1,1}$ and the classical estimate in (3.3) (used here with k = 0, $R = t_0 = T = 1$) entails that

$$\frac{|\nabla u|}{u} \leqslant C \qquad \text{in } Q_{1/2,1/2}.$$

Instead, one can deduce from (3.8) that

$$\frac{|\nabla u|}{u} \leqslant C\epsilon \qquad \text{in } Q_{1/2,1/2},$$

thus clarifying how the techniques developed in this paper lead to an enhancement of the classical estimates even when we are interested only in interior estimates, since they are capable of "trading universality for boundary information" (namely, they can possibly take additional advantage of nice boundary data whenever this information can lead to improved estimates with respect to the classical, universal ones).

Interestingly, this improvement also highlights the persisting effect of "exceptionally good" parabolic boundary data on the interior behavior of the solutions, and in general it captures the boundary behavior of the solutions.

3.3 General nonlinearities

In this part, we will focus on the nonlinear parabolic equation, by comparing our results with the previous literature. A first improvement of our general result in Theorem 1.1 is that it captures the global (and not only the local behavior) of the solution. Furthermore, the result in Theorem 1.1 deals with several nonlinearities "at the same time" and from it we can re-obtain easily, as particular cases, a number of different results that are scattered in several works of the existing literature.

As a matter of fact, Theorem 1.1 here is new even when considered as an interior estimate. We state this particular case explicitly as follows:

Corollary 3.3. Suppose that u is a solution of equation (1.2) satisfying (1.3). Then, there exists C > 0, only depending on n, such that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant \left(C\mathcal{C} + \iota_{\star}\right) \left(1 + \ln\frac{M}{u(x,t)}\right) \qquad \text{for all } (x,t) \in Q_{R/2,T/2}$$

Here, we used the notation in (1.8) for \mathfrak{C} , and

$$\iota_{\star} := \min\left\{\sigma_u + \tau_u, \, \sigma_u + C\mathfrak{T}_{\star}, \, \tau_u + C\mathfrak{S}_{\star}, \, C(\mathfrak{T}_{\star} + \mathfrak{S}_{\star})\right\},\tag{3.9}$$

where

$$\mathfrak{T}_{\star} := \frac{1}{\sqrt{T}} \qquad and \qquad \mathfrak{S}_{\star} := \frac{1}{R} + \frac{\sqrt[4]{k_+}}{\sqrt{R}}. \tag{3.10}$$

Proof. The claim follows from (1.12), (1.10) and (3.4). Notice in particular that, in this setting, we have $\rho := R/2$ and $\delta := T/2$, which give in (1.8) that $\mathfrak{T} \leq C\mathfrak{T}_{\star}$ and $\mathfrak{S} \leq \mathfrak{S}_{\star}$. \Box

As a special case of Corollary 3.3, one obtains the following new uniform interior estimate:

Corollary 3.4. Suppose that u is a solution of equation (1.2) satisfying (1.3). Then, there exists C > 0, only depending on n, such that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C \Big(\mathcal{C} + \mathcal{T}_{\star} + \mathcal{S}_{\star} \Big) \left(1 + \ln \frac{M}{u(x,t)} \right) \qquad \text{for all } (x,t) \in Q_{R/2,T/2}.$$

Here, we used the notation in (1.8) for \mathfrak{C} , and the one in (3.10) for \mathfrak{T}_{\star} and \mathfrak{S}_{\star} .

Proof. By (3.9), we know in particular that $\iota_{\star} \leq C(\mathfrak{T}_{\star} + \mathfrak{S}_{\star})$, hence the desired result follows from Corollary 3.3.

In this way, we re-obtain many results in the literature as particular cases. We list a few of them for the sake of completeness. We start with a result related to the thin film equation.

Theorem (Theorem 1.1 in [22]). Let α , $\lambda \in \mathbb{R}$. Let \mathcal{M} be a Riemannian manifold of dimension $n \ge 2$ with Ricci curvature bounded from below by -k, where k is a non-negative constant. Suppose that u is a positive solution to

$$u_t = \Delta u + \lambda u^{\alpha}$$

in $Q_{R,T} := B(x_0, R) \times [0, T]$. Let $M := \sup_{Q_{R,T}} u$ and $m := \inf_{Q_{R,T}} u$. Then² in $Q_{R/2,T/2}$, we have

• if $\lambda < 0$ and $\alpha \in (-\infty, 1]$,

$$\frac{|\nabla u|^2}{u^2} \leqslant C \left(k + \frac{1}{R^2} + \frac{1}{T} + \lambda(\alpha - 1)m^{\alpha - 1}\right) \left(1 + \ln\frac{M}{u}\right)^2; \quad (3.11)$$

²We think that there are some typos in Theorem 1.1 in [22], since the claim "Then in $Q_{R,T}$ " should read "in $Q_{R/2,T/2}$ ". The necessity of reducing the domain in [22] comes from formula (2.11) there. In addition, it seems there could be some constants missing in formula (1.5), and also in formula (1.6) when $\lambda > 0$ and $\alpha < 0$ of [22], since one term has a negative sign (these constants should probably have appeared in formulas (2.20) and (2.26) in [22] and the proof should take care of the delicate situation in which the maximal point there occurs on small values of the cut-off function). To avoid confusion, we do not include the unclear formulas in our version of the main theorem of [22]. On the other hand, it seems to us that the cases $\lambda = 0$, $\alpha = 0$ and $\alpha = 1$, which were in principle omitted in the original formulation of Theorem 1.1 in [22], can be included without extra effort, hence these cases are explicitly present in the formulation given here. • if $\lambda \ge 0$ and $\alpha \in [1, +\infty)$,

$$\frac{|\nabla u|^2}{u^2} \leqslant C \left(k + \frac{1}{R^2} + \frac{1}{T} + \lambda \alpha M^{\alpha - 1}\right) \left(1 + \ln \frac{M}{u}\right)^2;$$
(3.12)

• if $\lambda \ge 0$ and $\alpha \in [0, 1)$,

$$\frac{|\nabla u|^2}{u^2} \leqslant C \left(k + \frac{1}{R^2} + \frac{1}{T} + \lambda \alpha m^{\alpha - 1}\right) \left(1 + \ln \frac{M}{u}\right)^2.$$
(3.13)

Here, the constant C depends only on the dimension n.

Proof. We can use the setting in (1.2) with $t_0 := T$ and $S(x, t, u) := \lambda u^{\alpha}$. With this, recalling (1.4) and (1.6), we see that

$$\gamma = 0$$
 and $\mu = \sup_{Q_{R,T}} \left(k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v} \right)_+.$

Consequently, by (1.8),

$$\mathcal{C} = \sup_{Q_{R,T}} \sqrt{\left(k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v}\right)_{+}}$$

Hence, in light of (3.10),

$$\mathcal{C} + \mathcal{T}_{\star} + \mathcal{S}_{\star} \leqslant \sup_{Q_{R,T}} \sqrt{\left(k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v}\right)_{+}} + \frac{1}{\sqrt{T}} + \frac{1}{R} + \frac{\sqrt[4]{k}}{\sqrt{R}}$$

Then, we can exploit Corollary 3.4 in this setting, which yields that, in $Q_{R/2,T/2}$,

$$\mathcal{E} := \left(1 + \ln \frac{M}{u(x,t)}\right)^{-2} \frac{|\nabla u|^2}{u^2} \leqslant C \left(\mathcal{C} + \mathcal{T}_\star + \mathcal{S}_\star\right)^2$$

$$\leqslant C \left[\sup_{Q_{R,T}} \left(k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v}\right)_+ + \frac{1}{T} + \frac{1}{R^2} + \frac{\sqrt{k}}{R}\right],$$
(3.14)

up to renaming C line after line.

Now, if $\lambda \ge 0$ and $\alpha \in [0, 1)$, recalling (2.2),

$$k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v} \leqslant k + (\alpha - 1)\lambda u^{\alpha - 1} + \lambda u^{\alpha - 1} \leqslant k + \lambda \alpha m^{\alpha - 1}.$$

From this and (3.14), we find that

$$\mathcal{E} \leqslant C \left[\left(k + \lambda \alpha m^{\alpha - 1} \right) + \frac{1}{T} + \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right]$$
$$\leqslant C \left(k + \lambda \alpha m^{\alpha - 1} + \frac{1}{T} + \frac{1}{R^2} \right),$$

from which (3.13) plainly follows.

If instead $\lambda \ge 0$ and $\alpha \in [1, +\infty)$, using that $v \le 0$, we remark that

 $k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v} \leqslant k + (\alpha - 1)\lambda u^{\alpha - 1} + \lambda u^{\alpha - 1} = k + \lambda \alpha u^{\alpha - 1} \leqslant k + \lambda \alpha M^{\alpha - 1}.$

This and (3.14) give that

$$\mathcal{E} \leqslant C \left[\left(k + \lambda \alpha M^{\alpha - 1} \right) + \frac{1}{T} + \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right]$$
$$\leqslant C \left(k + \lambda \alpha M^{\alpha - 1} + \frac{1}{T} + \frac{1}{R^2} \right),$$

which proves (3.12).

Now, we suppose that $\lambda < 0$ and $\alpha \in (-\infty, 1]$. In this case we see that

$$(\alpha - 1)\lambda u^{\alpha - 1} = \frac{|(\alpha - 1)\lambda|}{u^{|1 - \alpha|}} \leqslant \frac{|(\alpha - 1)\lambda|}{m^{|1 - \alpha|}} = (\alpha - 1)\lambda m^{\alpha - 1}.$$

From this, one deduces that

$$k + (\alpha - 1)\lambda u^{\alpha - 1} + \frac{\lambda u^{\alpha - 1}}{1 - v} \leqslant k + (\alpha - 1)\lambda u^{\alpha - 1} \leqslant k + (\alpha - 1)\lambda m^{\alpha - 1}.$$

Therefore, one can use this information and (3.14) to obtain (3.11), as desired.

From our main results a general gradient estimate for solutions of semilinear parabolic equations follows as a byproduct. For concreteness, we point out the following explicit result:

Corollary 3.5. Let $p \in \mathbb{R}$. Let \mathcal{M} be a Riemannian manifold with $\operatorname{Ric}(\mathcal{M}) \ge -k$ for some $k \in \mathbb{R}$. Let u be a positive solution to the semilinear heat equation

$$u_t = \Delta u + u^p \tag{3.15}$$

in $Q_{R,T}$. Assume that $u \leq M$ in $Q_{R,T}$. Then, there exists C > 0 depending on n such that, on $Q_{R/2,T/2}$, there holds

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C\left(\max\left\{\sqrt{k_+}, \sqrt{(k+p\,\vartheta^{p-1})_+}\right\} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right)\left(1+\ln\frac{M}{u(x,t)}\right),$$

where

$$\vartheta := \begin{cases} M & if \ p > 1, \\ 1 & if \ p = 1, \\ \inf_{Q_{R,T}} u & if \ p \in (0, 1), \\ 0 & if \ p = 0, \\ M & if \ p < 0. \end{cases}$$
(3.16)

Proof. We take $p \in \mathbb{R}$ and $S(x, t, u) := u^p$. In this way, the notations in (1.4) and (1.6) yield that

$$\begin{aligned} \gamma &= 0 \\ \text{and} \qquad \mu &= \sup_{(x,t) \in Q_{R,T}} \left(k + p u^{p-1} - u^{p-1} + \frac{u^{p-1}}{1 - v} \right)_+ \\ &= \sup_{(x,t) \in Q_{R,T}} \left(k + \left((p-1) + \frac{1}{1 - v} \right) u^{p-1} \right)_+ \\ &\leqslant \sup_{(x,t) \in Q_{R,T}} \left(k + p u^{p-1} \right)_+ \\ &\leqslant \left(k + p \vartheta^{p-1} \right)_+. \end{aligned}$$

Hence, by (1.8),

$$\mathcal{C} \leqslant \sqrt{(k+p\,\vartheta^{p-1})_+}.$$

Using this, (3.10) and Corollary 3.4, we thereby conclude that, in $Q_{R/2,T/2}$,

$$\left(1 + \ln \frac{M}{u(x,t)}\right)^{-1} \frac{|\nabla u(x,t)|}{u(x,t)} \leqslant C \left(\mathcal{C} + \mathfrak{T}_{\star} + \mathfrak{S}_{\star}\right)$$

$$\leqslant C \left(\sqrt{(k+p\,\vartheta^{p-1})_{+}} + \frac{1}{\sqrt{T}} + \frac{1}{R} + \frac{\sqrt[4]{k_{+}}}{\sqrt{R}}\right).$$

$$(3.17)$$

We also remark that, by the Cauchy-Schwarz inequality,

$$\frac{\sqrt[4]{k_+}}{2\sqrt{R}} \leqslant \sqrt{k_+} + \frac{1}{R}.$$

This and (3.17) give that

$$\left(1 + \ln \frac{M}{u(x,t)}\right)^{-1} \frac{|\nabla u(x,t)|}{u(x,t)} \leq C \left(\sqrt{k_+} + \sqrt{(k+p\,\vartheta^{p-1})_+} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right).$$

ields the desired result.

This yields the desired result.

We remark that Corollary 3.5 contains, as a special case, a recent result obtained in [6] which dealt with the case p > 1 (see in particular Lemma 3.1 in [6]).

Moreover, we think that an interesting treat of our Corollary 3.5 in its general formulation is that the constant C is independent of p: besides its technical relevance, this fact reveals a telling feature of the nonlinear parabolic equations, in the sense that, at a formal level, for large p, given $a \in (0, 1)$, solutions $u = u_p$ of (3.15) with $0 < u \leq 1 - a$ satisfy, on $Q_{R/2,T/2}$,

$$\frac{|\nabla u|}{u} \leqslant C\left(\max\left\{\sqrt{k_{+}}, \sqrt{(k+p(1-a)^{p-1})_{+}}\right\} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right)(1+|\ln u|),$$

which, as $p \to +\infty$, formally boils down to

$$\frac{|\nabla u|}{u} \leqslant C\left(\sqrt{k_+} + \frac{1}{\sqrt{T}} + \frac{1}{R}\right)\left(1 + |\ln u|\right),$$

which recovers the estimate for the heat equation given in (3.3) (we remark that also equation (3.15) reduces to the heat equation as $p \to +\infty$ in this regime, and that the assumption $u \leq 1 - a$ is equivalent to u bounded in the case of the heat equation due to its linear structure). Though we do not address a rigorous treatment of these limit properties as $p \to +\infty$ here, we think that our unified approach to gradient estimates entails a number of interesting connections between structurally different equations which could be worth a further exploration.

In addition, from Corollary 3.5, one re-obtains a recent result motivated by ancient solutions:

Theorem (Lemma³ 4.1 in [5]). Let \mathcal{M} be a Riemannian manifold with $\operatorname{Ric}(\mathcal{M}) \ge -k$ for some $k \in \mathbb{R}$. Let u be a positive solution to the semilinear heat equation

$$u_t = \Delta u + u^2$$

in $Q_{R,T}$. Assume that $u \leq M$ in $Q_{R,T}$. Then, there exists C > 0 depending on n such that, on $Q_{R/2,T/2}$, there holds

$$\frac{|\nabla u|}{u} \leqslant C \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{(2M+k)_+}\right) \left(1 + \ln\frac{M}{u}\right). \tag{3.18}$$

Proof. Recalling (3.16), we see that, when p = 2,

$$\sqrt{k_+} \leqslant \sqrt{\left(k+2M\right)_+} = \sqrt{\left(k+p\,\vartheta^{p-1}\right)_+}.$$

From this and Corollary 3.5, we obtain (3.18).

Acknowledgments

Cecilia Cavaterra has been partially supported by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica). Serena Dipierro and Enrico Valdinoci are members of INdAM and AustMS. Serena Dipierro has been supported by the Australian Research Council DE-CRA DE180100957 "PDEs, free boundaries and applications". Enrico Valdinoci has been

³See also the enhanced version of [5] available on http://cvgmt.sns.it/paper/3135/

supported by the Australian Laureate Fellowship FL190100081 "Minimal surfaces, free boundaries and partial differential equations". Zu Gao has been supported by the Independent Innovation Research Fund of Wuhan University of Technology (No: 2021IVA058).

A Proof of (3.7)

We recall that a metric g_{ij} is said to be conformal (or, more precisely, conformal to the Euclidean metric) if

$$g_{ij} = \varphi \delta_{ij}, \tag{A.19}$$

for some scalar factor φ . For instance, the metric of the Poincaré disk in the plane given in (3.6) is conformal, with factor $\varphi := \frac{4\lambda^2}{(1-|x|^2)^2}$, being $|\cdot|$ the standard Euclidean norm.

The Laplacian operator (or, more precisely, the Laplace-Beltrami operator) possesses an explicit representation with respect to conformal metrics: roughly speaking, since conformal metrics preserve angles, an infinitesimal orthonormal frame is transformed into an infinitesimal orthogonal frame (the length of the vectors possibly being affected by the conformal factor φ), thus the new Laplacian (being computed as sum of second derivatives with respect to an orthonormal frame) remains the same possibly up to a "curvature" term which accounts for the variation of φ (this additional term pops up because the Laplacian is a second order operator). The case of dimension 2 is somewhat special, since this additional term vanishes.

Here are the explicit computations underpinning this heuristic idea. From (A.19), we have that $g^{ij} = \varphi^{-1} \delta^{ij}$ and det $g = \varphi^n$. Hence, in local coordinates, the Laplacian with respect to the conformal metrics in (A.19) is

$$\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{\det g} \ g^{ij} \partial_j \right) = \frac{1}{\varphi^{\frac{n}{2}}} \sum_{i,j=1}^{n} \partial_i \left(\varphi^{\frac{n-2}{2}} \delta^{ij} \partial_j \right) = \frac{1}{\varphi^{\frac{n}{2}}} \sum_{i=1}^{n} \partial_i \left(\varphi^{\frac{n-2}{2}} \partial_i \right)$$
$$= \frac{1}{\varphi^{\frac{n}{2}}} \sum_{i=1}^{n} \left(\frac{n-2}{2} \varphi^{\frac{n-4}{2}} \partial_i \varphi \partial_i + \varphi^{\frac{n-2}{2}} \partial_{ii} \right) = \sum_{i=1}^{n} \left(\frac{n-2}{2} \varphi^2 \partial_i \varphi \partial_i + \frac{1}{\varphi} \partial_{ii} \right).$$

In dimension 2, this boils down to

$$\frac{1}{\varphi} \sum_{i=1}^{n} \partial_{ii},$$

which is a scalar multiple of the Euclidean Laplacian, and therefore (3.7) plainly follows.

References

- [1] Amal Attouchi, Gradient estimate and a Liouville theorem for a p-Laplacian evolution equation with a gradient nonlinearity, Differential Integral Equations **29** (2016), no. 1-2, 137–150. MR3450752
- [2] Xavier Cabré, Serena Dipierro, and Enrico Valdinoci, *The Bernstein technique for integro-differential equations*, arXiv e-prints (2020), available at 2010.00376.
- [3] Luis A. Caffarelli and Xavier Cabré, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR1351007
- [4] Luis Caffarelli, Nicola Garofalo, and Fausto Segàla, A gradient bound for entire solutions of quasilinear equations and its consequences, Comm. Pure Appl. Math. 47 (1994), no. 11, 1457–1473, DOI 10.1002/cpa.3160471103. MR1296785
- [5] Daniele Castorina and Carlo Mantegazza, Ancient solutions of semilinear heat equations on Riemannian manifolds, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 28 (2017), no. 1, 85–101, DOI 10.4171/RLM/753. MR3621772
- [6] _____, Ancient solutions of superlinear heat equations on Riemannian manifolds, Commun. Contemp. Math. (to appear).
- [7] Cecilia Cavaterra, Serena Dipierro, Alberto Farina, Zu Gao, and Enrico Valdinoci, Pointwise gradient bounds for entire solutions of elliptic equations with non-standard growth conditions and general nonlinearities, J. Differential Equations 270 (2021), 435–475, DOI 10.1016/j.jde.2020.08.007. MR4150380
- [8] Qun Chen and Guangwen Zhao, Li-Yau type and Souplet-Zhang type gradient estimates of a parabolic equation for the V-Laplacian, J. Math. Anal. Appl. 463 (2018), no. 2, 744–759, DOI 10.1016/j.jmaa.2018.03.049. MR3785481
- S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975), no. 3, 333–354, DOI 10.1002/cpa.3160280303. MR385749
- [10] Matteo Cozzi, Alberto Farina, and Enrico Valdinoci, Gradient bounds and rigidity results for singular, degenerate, anisotropic partial differential equations, Comm. Math. Phys. 331 (2014), no. 1, 189–214, DOI 10.1007/s00220-014-2107-9. MR3231999
- Serena Dipierro, Zu Gao, and Enrico Valdinoci, Global gradient estimates for nonlinear parabolic operators, ESAIM Control Optim. Calc. Var. 27 (2021), Paper No. 21, 37, DOI 10.1051/cocv/2021016. MR4238774
- [12] Nguyen Thac Dung and Nguyen Ngoc Khanh, Gradient estimates of Hamilton-Souplet-Zhang type for a general heat equation on Riemannian manifolds, Arch. Math. (Basel) 105 (2015), no. 5, 479–490, DOI 10.1007/s00013-015-0828-4. MR3413923

- [13] Nguyen Thac Dung, Nguyen Ngoc Khanh, and Quôc Anh Ngô, Gradient estimates for some f-heat equations driven by Lichnerowicz's equation on complete smooth metric measure spaces, Manuscripta Math. 155 (2018), no. 3-4, 471–501, DOI 10.1007/s00229-017-0946-3. MR3763415
- [14] Ha Tuan Dung and Nguyen Thac Dung, Sharp gradient estimates for a heat equation in Riemannian manifolds, Proc. Amer. Math. Soc. 147 (2019), no. 12, 5329–5338, DOI 10.1090/proc/14645. MR4021092
- [15] Alberto Farina and Enrico Valdinoci, A pointwise gradient estimate in possibly unbounded domains with nonnegative mean curvature, Adv. Math. 225 (2010), no. 5, 2808–2827, DOI 10.1016/j.aim.2010.05.008. MR2680184
- [16] _____, A pointwise gradient bound for elliptic equations on compact manifolds with nonnegative Ricci curvature, Discrete Contin. Dyn. Syst. 30 (2011), no. 4, 1139–1144, DOI 10.3934/dcds.2011.30.1139. MR2812957
- [17] Richard S. Hamilton, A matrix Harnack estimate for the heat equation, Comm. Anal. Geom. 1 (1993), no. 1, 113–126, DOI 10.4310/CAG.1993.v1.n1.a6. MR1230276
- [18] Guangyue Huang and Bingqing Ma, Hamilton's gradient estimates of porous medium and fast diffusion equations, Geom. Dedicata 188 (2017), 1–16, DOI 10.1007/s10711-016-0201-1. MR3639621
- [19] Xinrong Jiang, Gradient estimate for a nonlinear heat equation on Riemannian manifolds, Proc. Amer. Math. Soc. 144 (2016), no. 8, 3635–3642, DOI 10.1090/proc/12995. MR3503732
- [20] O. A. Ladyženskaya, Solution of the first boundary problem in the large for quasi-linear parabolic equations, Trudy Moskov. Mat. Obšč. 7 (1958), 149–177 (Russian). MR0114050
- [21] Peter Li and Shing-Tung Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156 (1986), no. 3-4, 153–201, DOI 10.1007/BF02399203. MR834612
- [22] Bingqing Ma and Fanqi Zeng, Hamilton-Souplet-Zhang's gradient estimates and Liouville theorems for a nonlinear parabolic equation, C. R. Math. Acad. Sci. Paris, Ser. I 356 (2018), no. 5, 550–557, DOI 10.1016/j.crma.2018.04.003. MR3790428
- [23] Li Ma, Lin Zhao, and Song Xianfa, Gradient estimate for the degenerate parabolic equation $u_t = \Delta F(u) + H(u)$ on manifolds, J. Differential Equations 244 (2008), no. 5, 1157–1177, DOI 10.1016/j.jde.2007.08.014. MR2392508
- [24] Luciano Modica, A gradient bound and a Liouville theorem for nonlinear Poisson equations, Comm.
 Pure Appl. Math. 38 (1985), no. 5, 679–684, DOI 10.1002/cpa.3160380515. MR803255
- [25] John Oprea, Differential geometry and its applications, 2nd ed., Classroom Resource Materials Series, Mathematical Association of America, Washington, DC, 2007. MR2327126
- [26] L. E. Payne, Some remarks on maximum principles, J. Analyse Math. 30 (1976), 421–433, DOI 10.1007/BF02786729. MR454338
- [27] Peter Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998. MR1480173

- [28] James Serrin, Gradient estimates for solutions of nonlinear elliptic and parabolic equations, Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), Academic Press, New York, 1971, pp. 565–601. MR0402274
- [29] Boyan Sirakov and Philippe Souplet, *Liouville-type theorems for unbounded solutions of elliptic equations in half-spaces*, arXiv e-prints (2020), available at 2002.07247.
- [30] Philippe Souplet and Qi S. Zhang, Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds, Bull. London Math. Soc. 38 (2006), no. 6, 1045–1053, DOI 10.1112/S0024609306018947. MR2285258
- [31] René P. Sperb, Maximum principles and their applications, Mathematics in Science and Engineering, vol. 157, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR615561
- [32] Eduardo V. Teixeira and José Miguel Urbano, An intrinsic Liouville theorem for degenerate parabolic equations, Arch. Math. (Basel) 102 (2014), no. 5, 483–487, DOI 10.1007/s00013-014-0648-y. MR3254790
- [33] Jiayong Wu, Gradient Estimates for a Nonlinear Diffusion Equation on Complete Manifolds, J. Partial Differ. Equ. 23 (2010), no. 1, 68–79, DOI 10.4208/jpde.v23.n1.4. MR2640448
- [34] _____, Elliptic gradient estimates for a weighted heat equation and applications, Math. Z. 280 (2015), no. 1-2, 451–468, DOI 10.1007/s00209-015-1432-9. MR3343915
- [35] Xiangjin Xu, Gradient estimates for $u_t = \Delta F(u)$ on manifolds and some Liouville-type theorems, J. Differential Equations **252** (2012), no. 2, 1403–1420, DOI 10.1016/j.jde.2011.08.004. MR2853544
- [36] Xiaobao Zhu, Hamilton's gradient estimates and Liouville theorems for fast diffusion equations on noncompact Remannian manifolds, Proc. Amer. Math. Soc. 139 (2011), no. 5, 1637–1644, DOI 10.1090/S0002-9939-2010-10824-9. MR2763753
- [37] _____, Gradient estimates and Liouville theorems for nonlinear parabolic equations on noncompact Riemannian manifolds, Nonlinear Anal. 74 (2011), no. 15, 5141–5146, DOI 10.1016/j.na.2011.05.008. MR2810695
- [38] _____, Hamilton's gradient estimates and Liouville theorems for porous medium equations on noncompact Riemannian manifolds, J. Math. Anal. Appl. 402 (2013), no. 1, 201–206, DOI 10.1016/j.jmaa.2013.01.018. MR3023250