

THE MAIN COMPONENT OF A REDUCIBLE HILBERT CURVE OF CONIC FIBRATIONS

MARIA LUCIA FANIA AND ANTONIO LANTERI

ABSTRACT. The study of reducible Hilbert curves of conic fibrations over a smooth surface is carried on in this paper and the question of when the main component is itself the Hilbert curve of some \mathbb{Q} -polarized surface is dealt with. Special attention is paid to the polynomial defining the canonical equation of the Hilbert curve.

1. Introduction

The notion of Hilbert curve of a polarized manifold was introduced in [3] and various properties of such curves have been widely investigated for several classes of special varieties (e.g. see [8], [9], [10]). For conic fibrations over a smooth surface, the Hilbert curve has been studied in [5], with special attention to its reducibility. Here we analyze a further aspect always related to reducibility.

Let (X, L) be a geometric conic fibration over a smooth surface S and let Γ be its Hilbert curve. Then Γ is an affine plane cubic. Suppose that $\Gamma = \ell + \gamma$ for some line ℓ . When the conic γ is irreducible we can call it unambiguously the *main component* of Γ . A natural question arising from [3, Problem 6.6] is whether γ itself can be regarded as the Hilbert curve of some polarized (or more generally a \mathbb{Q} -polarized) surface (Y, \mathcal{L}) , somehow related to (X, L) .

It should be emphasized that, in general, there is no polarized surface (Y, \mathcal{L}) admitting γ as Hilbert curve, see Example 3.3: there (X, L) is a geometric conic fibration over \mathbb{P}^2 such that the residual conic γ of Γ is the Hilbert curve of no polarized surface; however γ is the Hilbert curve of a \mathbb{Q} -polarized surface (Y, \mathcal{L}) with Y a rational surface distinct from \mathbb{P}^2 , the base of the conic fibration. This example, as well as [8, Remark 4.1], but referred to another kind of special varieties, seems to indicate that the appropriate range for (Y, \mathcal{L}) to address

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our problem is in fact that of \mathbb{Q} -polarized smooth surfaces. Moreover, since our expectation is that (Y, \mathcal{L}) , if any, is somehow related to (X, L) , the first natural question is whether we can take $Y = S$, the base surface of our conic fibration. This question however has a negative answer as an example deriving from Remark 2.1 shows.

So when γ is the Hilbert curve of some \mathbb{Q} -polarized surface (Y, \mathcal{L}) , the next obvious question is: how far is S from Y ? Several examples discussed in Section 3, in particular Example 3.4, seem to suggest that there is no a priori relation between Y and S , in general. Nevertheless, it looks natural to wonder if it can be $Y = S$, at least when we confine ourselves to some specific framework. Moreover, restricting to such a framework, one could further investigate a possible relationship between the \mathbb{Q} -ample line bundle \mathcal{L} and a vector bundle on S giving rise to a scroll, in which (X, L) fits naturally as a divisor of relative degree 2.

As to the general problem we are dealing with, many examples produced in Section 3 illustrate various possibilities. Moreover, they show a gradualness of the difficulties we may encounter in dealing with this problem, depending on the requests we place on the pair (Y, \mathcal{L}) . In Section 4, we address our problem focussing on the special case in which (X, L) is a geometric conic fibration over S with no singular fibers. Here the Bogomolov semistability of the vector bundle defining X comes into the picture.

In Section 5, with a slight change of perspective, we move our attention from the Hilbert curve Γ of a conic fibration (X, L) to the polynomial $p_{(X, L)}$ defining the canonical equation of Γ . In particular, letting f denote the general fiber of X , we show that if the equality $p_{(X, L)} = p_{(f, L_f)} p_{(S, \mathcal{L})}$ holds for some ample \mathbb{Q} -line bundle \mathcal{L} on S , then X is a bundle and either $\mathcal{L} = \frac{1}{3}c_1(\mathcal{E})$ or $(3\mathcal{L} - c_1(\mathcal{E}))^2 < 0$, where $\mathcal{E} = \pi_*L$ and $\pi : X \rightarrow S$ is the bundle projection. This relies on results established in Section 4.

Finally, in Section 6, confining to geometric conic fibrations over \mathbb{P}^2 , we investigate under what conditions Γ contains a given line as a component. This leads to Proposition 6.1, which generalizes [5, Proposition 5.4].

2. Preliminaries

Varieties considered in this paper are defined over the field \mathbb{C} of complex numbers. We use the standard notation and terminology from algebraic geometry. According to a standard abuse, tensor products of line bundles are denoted additively. A polarized manifold (X, L) consists of a smooth projective variety X and an ample line bundle L on X . For the notion and the general properties of the Hilbert curve associated to a polarized manifold we refer to [3], see also [8]. Here we just recall some basic facts.

Let (X, L) be a polarized manifold of dimension $n \geq 1$ and let K_X be the canonical bundle of X . For any line bundle D on X the Riemann-Roch theorem provides an expression for the Euler-Poincaré characteristic $\chi(D)$ in terms of D

and of the Chern classes of X . Let $p(x, y)$ denote the complexified polynomial of $\chi(D)$, when we set $D = xK_X + yL$, with x, y complex numbers, namely $p(x, y) = \chi(xK_X + yL)$. The Hilbert curve of (X, L) is the complex affine plane curve Γ of degree n defined by $p(x, y) = 0$. Of course Γ depends only on the class of L in $N(X)$, the numerical equivalence class group of X . Moreover, due to Serre duality, Γ is invariant under the involution $D \mapsto K_X - D$ acting on $N(X)$, hence it is symmetric with respect to $\frac{1}{2}K_X$. To make this symmetry more evident, it is convenient to represent Γ in terms of the affine coordinates $(u = x - \frac{1}{2}, v = y)$ rather than (x, y) . So, rewriting our divisor as $D = \frac{1}{2}K_X + \Delta$, where $\Delta = uK_X + vL$, Γ can be represented with respect to these coordinates by $p(\frac{1}{2} + u, v) = 0$. We will refer to this equation as the *canonical equation* of Γ .

According to [5], by *geometric conic fibration* over a smooth surface S we mean a polarized threefold (X, L) such that X is a conic fibration over S in the classical sense and, at the same time, (X, L) is an adjunction theoretic conic fibration over S . So, X is endowed with a fibration $\pi : X \rightarrow S$ such that $(f, L_f) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ for its general fiber f , and $K_X + L = \pi^*H$ for some ample line bundle H on S . We recall some facts from [5, Section 2]. First of all, $\mathcal{E} := \pi_*L$ is a vector bundle of rank 3 on S and X is embedded fiberwise in $P := \mathbb{P}(\mathcal{E})$, as a member of the linear system $|2\xi + \tilde{\pi}^*\mathcal{B}|$, where ξ is the tautological line bundle of \mathcal{E} on P , $\tilde{\pi} : P \rightarrow S$ is the bundle projection extending π , and \mathcal{B} is some line bundle on S ; moreover, $L = \xi_X$ (ξ restricted to X). We have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$. Furthermore, $H = K_S + c_1(\mathcal{E}) + \mathcal{B}$ ([5, (4)]). The discriminant curve of (X, L) , that is the divisor \mathcal{D} on S parameterizing the singular fibers of π , satisfies $\mathcal{D} = 2c_1(\mathcal{E}) + 3\mathcal{B}$ ([5, (5)]). We say that (X, L) is a smooth conic fibration or that X is a bundle, to mean that the conic fibration has no singular fibers, in which case $\mathcal{D} = 0$ and therefore $H = K_S + \frac{1}{3}c_1(\mathcal{E})$. For instance, suppose that $X = \mathbb{P}(\mathcal{F})$ with \mathcal{F} a rank two vector bundle on S (which certainly happens if S is a simply connected surface with $p_g = 0$) and let $\pi : X \rightarrow S$ be the bundle projection. Then $L = 2\xi_{\mathcal{F}} + \pi^*\mathcal{M}$, where $\xi_{\mathcal{F}}$ denotes the tautological line bundle of \mathcal{F} , \mathcal{M} some line bundle on S and the threefold X is a bundle over S via π and what we said before applies. In particular, $\mathcal{E} = \pi_*L = \pi_*(2\xi_{\mathcal{F}} + \pi^*\mathcal{M}) = S^2(\mathcal{F}) \otimes \mathcal{M}$.

For polarized threefolds (X, L) which are conic fibrations over a smooth surface S , the canonical equation of the Hilbert curve is explicitly described in [5, Proposition 4.1].

Here, in view of our frequent use throughout the paper, it is useful to recall that the canonical equation of the Hilbert curve of a \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) is:

$$(1) \quad \begin{aligned} p_{(Y, \mathcal{L})}(\tfrac{1}{2} + u, v) &= \tfrac{1}{2}(K_Y^2 u^2 + 2K_Y \cdot \mathcal{L}uv + \mathcal{L}^2 v^2 + 2\chi(\mathcal{O}_Y) - \tfrac{1}{4}K_Y^2) \\ &= 0, \end{aligned}$$

see [9, p. 141].

So, up to the factor $\frac{1}{2}$, the matrix associated to this conic is

$$(2) \quad M = \begin{bmatrix} K_Y^2 & K_Y \cdot \mathcal{L} & 0 \\ K_Y \cdot \mathcal{L} & \mathcal{L}^2 & 0 \\ 0 & 0 & 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2 \end{bmatrix}.$$

Remark 2.1. If $(Y, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$ for some $r \in \mathbb{Q}$, then the matrix M , up to the factor $\frac{1}{2}$, is

$$M = \begin{bmatrix} 9 & -3r & 0 \\ -3r & r^2 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}.$$

Note that M has rank 2. Therefore an irreducible conic γ can never be the Hilbert curve of the \mathbb{Q} -polarized surface $(Y, \mathcal{L}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r))$.

Now let (X, L) be a smooth conic fibration over \mathbb{P}^2 . Then, according to [5, Theorem 5.2], its Hilbert curve has the form $\Gamma = \ell + \gamma$, where ℓ is the line of equation $u - v = 0$. Moreover, the conic γ is irreducible provided that $K_X^3 + 48\chi(\mathcal{O}_X) \neq 0$, which happens if (X, L) is general. Then, according to Remark 2.1, there is no \mathbb{Q} -polarization on \mathbb{P}^2 for which γ is the Hilbert curve. This shows that, even enlarging the framework to \mathbb{Q} -polarized surfaces, if γ is the Hilbert curve of (Y, \mathcal{L}) , in general we cannot expect that $Y = S$, the base surface of X .

The notion of Bogomolov semistability for a vector bundle over a surface, see [4], as well as the expressions of the Chern classes of the t -th symmetric power of a rank two vector bundle over a surface, is needed, hence we give it for the reader's convenience.

Definition. Let \mathcal{E} be a vector bundle of rank r over a smooth surface S and let c_i denote the i -th Chern class $c_i(\mathcal{E})$ of \mathcal{E} . The vector bundle \mathcal{E} is Bogomolov semistable, also B -semistable, if and only if $(r-1)c_1^2 - 2rc_2 \leq 0$. When equality holds, \mathcal{E} is said to be properly B -semistable.

Let \mathcal{F} be a vector bundle of rank 2 over a smooth surface S and for every positive integer t let $S^t(\mathcal{F})$ be the t -th symmetric power of \mathcal{F} . Then

$$c_1(S^t(\mathcal{F})) = \frac{1}{2}t(t+1)c_1(\mathcal{F})$$

and

$$c_2(S^t(\mathcal{F})) = \frac{1}{24}t(t^2 - 1)(3t + 2)c_1(\mathcal{F})^2 + \frac{1}{6}t(t+1)(t+2)c_2(\mathcal{F}).$$

By applying the splitting principle and using [7, Exercise 5.16 c), p. 127] one can compute the Chern classes $c_i(S^t(\mathcal{F}))$. In particular, for the vector bundle $\mathcal{E} = S^t(\mathcal{F}) \otimes \mathcal{M}$, where $\mathcal{M} \in \text{Pic}(S)$, we have

$$(3) \quad c_1 = c_1(S^t(\mathcal{F}) \otimes \mathcal{M}) = (t+1)\mathcal{M} + \frac{1}{2}t(t+1)c_1(\mathcal{F}),$$

$$(4) \quad c_2 = c_2(S^t(\mathcal{F}) \otimes \mathcal{M}) = \frac{1}{2}t(t+1)\mathcal{M}^2 + \frac{1}{2}t^2(t+1)c_1(\mathcal{F}) \cdot \mathcal{M}$$

$$+ \frac{1}{24}t(t^2 - 1)(3t + 2)c_1(\mathcal{F})^2 + \frac{1}{6}t(t + 1)(t + 2)c_2(\mathcal{F}).$$

Proposition 2.2. *Let \mathcal{F} be a rank 2 vector bundle over a smooth surface S and let $\mathcal{E} = S^t(\mathcal{F}) \otimes \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}(S)$. Then \mathcal{E} is B -semistable if and only if \mathcal{F} is B -semistable. Moreover \mathcal{E} is properly B -semistable if and only if \mathcal{F} is so.*

Proof. Because the rank of \mathcal{E} is $t + 1$, one needs to compute $tc_1^2 - 2(t + 1)c_2$. From (3) and (4) it follows that

$$tc_1^2 - 2(t + 1)c_2 = \frac{1}{12}t(t + 1)^2(t + 2)[c_1(\mathcal{F})^2 - 4c_2(\mathcal{F})].$$

Hence our claim follows. □

3. Examples

Let (X, L) be a geometric conic fibration over a smooth surface. Assume that its Hilbert curve Γ is reducible and of the form $\Gamma = \ell + \gamma$, where γ is a conic. An obvious question is whether γ itself is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) . Because in [5] we produced examples of geometric conic fibrations over a smooth surface having a reducible Hilbert curve, it is natural to see what would be the answer to our question in these cases. Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ be the Segre–Hirzebruch surface of invariant e . We use the standard notation as in [7, Ch. V].

Example 3.1. Let (X, L) be as in [5, Remark 5.4]. Here (X, L) is a geometric smooth conic fibration over \mathbb{F}_e , $X = \mathbb{P}(\mathcal{L}^{\oplus 2})$, where $\mathcal{L} = aC_0 + bf$ with $a > 0$, $b = ae + k$ for some integer $k \geq 1$, see [7, Ch. V, Corollary 2.18], and its Hilbert curve Γ has the canonical equation

$$(5) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = 2(v - u)(2u - 3av) \left[2u - 3\left(\frac{ae}{2} + k\right)v\right] = 0.$$

Let $\ell' : 2u - 3av = 0$ and $\ell'' : 2u - 3\left(\frac{ae}{2} + k\right)v = 0$ be the lines residual to $\ell_0 : u - v = 0$. They coincide if and only if $\frac{ae}{2} + k = a$, that is $a(e - 2) + 2k = 0$ and being a and k positive it follows that either $e = 0$ and $a = k$, or $e = 1$ and $a = 2k$.

If $e = 0$ and $a = k$, then

$$(6) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = 2(-u + v)(2u - 3kv)^2 = 0.$$

In this case the conic $\gamma : 4u^2 - 12kuv + 9k^2v^2 = 0$ is a double line, which, according to (1), is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) if and only if there exists a non zero rational number ρ such that

$$(7) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right) = \rho(4, -6k, 9k^2, 0).$$

This gives four relations. The last one combined with the first gives $\chi(\mathcal{O}_Y) = \frac{\rho}{2}$. From the third relation it follows that $\rho > 0$, thus $K_Y^2 = 4\rho > 0$ and $K_Y \cdot \mathcal{L} = -3k\rho < 0$, since $k \geq 1$. Hence Y is ruled, in view of the Enriques ruledness criterion, and being $K_Y^2 > 0$ it follows that Y is rational and thus $\chi(\mathcal{O}_Y) = 1$. Hence $\rho = 2$, $K_Y^2 = 8$ and thus $Y = \mathbb{F}_n$ for some integer $n \geq 0$. Let $\pi : \mathbb{F}_n \rightarrow \mathbb{P}^1$ be the bundle projection and use again f and C_0 to denote a fiber of π and the section (a section if $n = 0$) with $C_0^2 = -n$. Then $\mathcal{L} = \alpha C_0 + \beta f$, with $\alpha > 0$, $\beta \geq \alpha n + 1$, being \mathcal{L} ample. Because $\mathcal{L}^2 = 18k^2$ and $K_Y \cdot \mathcal{L} = -12k$ we get the following equations $18k^2 = -\alpha^2 n + 2\alpha\beta$ and $-12k = \alpha n - 2\alpha - 2\beta$. From the latter equation we get that $\beta = \frac{1}{2}(\alpha n - 2\alpha + 12k)$ and from the former one $(3k - \alpha)^2 = 0$, hence $\alpha = 3k$. Thus $\mathcal{L} = \alpha C_0 + \beta f = 3kC_0 + \frac{3k}{2}(n + 2)f$. Now since $\beta > \alpha n$ it follows that $3k(n - 2) < 0$ and thus either $n = 0$, or $n = 1$. Therefore the \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) whose Hilbert curve is $\gamma : (2u - 3kv)^2 = 0$ is either $(\mathbb{F}_0, 3k(C_0 + f))$ or $(\mathbb{F}_1, 3k(C_0 + \frac{3}{2}f))$.

Note that this case falls into [5, Example 5.3] with $a_1 = a_2 = 2s$.

If $e = 1$ and $a = 2k$, then

$$(8) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = 8(-u + v)(u - 3kv)^2 = 0.$$

In this case the conic $\gamma : u^2 - 6kuv + 9k^2v^2 = 0$ is a double line which is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) if and only if there exists a non zero rational number ρ such that

$$(9) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right) = \rho\left(1, -3k, 9k^2, 0\right).$$

Reasoning as in the previous case we get $\rho = 8$ and then we see that the \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) whose Hilbert curve is $\gamma : (u - 3kv)^2 = 0$ is either $(\mathbb{F}_0, 6k(C_0 + f))$ or $(\mathbb{F}_1, 3k(2C_0 + 3f))$.

Example 3.2. Let (X, L) be as in [5, Example 7.1]. Then (X, L) is a geometric smooth conic fibration over \mathbb{P}^2 whose Hilbert curve Γ has the canonical equation

$$(10) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = (v - u)\left[3u - (m + 1)v\right]^2 - \frac{1}{4} = 0 \quad (m \geq 3).$$

In this case $\gamma : 9u^2 - 6(m + 1)uv + (m + 1)^2v^2 - \frac{1}{4} = 0$. Thus the conic γ is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) if and only if there exists a non zero rational number ρ such that

$$(11) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right) = \rho\left(9, -3(m + 1), (m + 1)^2, -\frac{1}{4}\right).$$

We get four relations again. The last one combined with the first gives $\chi(\mathcal{O}_Y) = \rho$. From the third relation it follows that $\rho > 0$ and thus $K_Y^2 = 9\rho > 0$ and $K_Y \cdot \mathcal{L} = -3(m + 1)\rho < 0$, since $m \geq 3$. Hence Y is ruled and being $K_Y^2 > 0$ it follows that Y is rational and thus $\chi(\mathcal{O}_Y) = 1$. Hence $\rho = 1$, $K_Y^2 = 9$ and thus $Y = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(m + 1)$, $m \geq 3$.

Example 3.3. Let (X, L) be as in [5, Example 7.2, Type 1], that is (X, L) is a geometric conic fibration over \mathbb{P}^2 , embedded by $|L|$ in \mathbb{P}^5 and of degree 9. The Hilbert curve Γ of (X, L) has the canonical equation

$$(12) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = (2u - 3v)(4u^2 - 4v^2 - 9) = 0.$$

In this case the conic γ has equation $4u^2 - 4v^2 - 9 = 0$ and it is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) if and only if there exists a non zero rational number ρ such that

$$(13) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right) = \rho(4, 0, -4, -9).$$

The third relation says that $\rho < 0$ hence $K_Y^2 = 4\rho < 0$. Moreover from the last relation it follows that $2\chi(\mathcal{O}_Y) = \frac{1}{4}K_Y^2 - 9\rho = -8\rho$, that is $\chi(\mathcal{O}_Y) = -4\rho > 0$.

Since $K_Y \neq 0$, from the second relation we see that Y is ruled, by the Enriques ruledness criterion, hence Y is rational because $\chi(\mathcal{O}_Y) > 0$. Thus $\chi(\mathcal{O}_Y) = 1$. So, $\rho = -1/4$, and then $K_Y^2 = -1$. This says that Y is obtained from \mathbb{P}^2 via a sequence of 10 blowing-ups. Moreover, $\mathcal{L}^2 = -4\rho = 1$ and then $\mathcal{L}^2 + K_Y \cdot \mathcal{L} = 1$, which is impossible by the genus formula if $\mathcal{L} \in \text{Pic}(Y)$. This gives no contradiction however if \mathcal{L} is only a \mathbb{Q} -line bundle. So let \mathcal{L} be a \mathbb{Q} -line bundle, hence there exists an integer $m \geq 2$ such that $A = m\mathcal{L} \in \text{Pic}(Y)$. Let p be the arithmetic genus of A . From the genus formula we get that $2(p - 1) = A^2 = m^2$, hence m^2 is even and a multiple of four. Thus $p = 2r^2 + 1$ and $m = 2r$ for some positive integer r . So the possible values of p are $p = 3, 9, 19, 33, \dots$

If $r = 1$, then $m = 2$ and $p = 3$. Note that the polarized surface (Y, A) is such that

$$K_Y^2 = -1, K_Y \cdot A = 0, A^2 = 4, \chi(\mathcal{O}_Y) = 1.$$

Going through the list of polarized surfaces (Y, A) of sectional genus three we see [11, Theorem (VIII) (2)] that our Y is the blow up of \mathbb{F}_e at 9 points lying on distinct fibers via a birational morphism $\eta : Y \rightarrow \mathbb{F}_e$, $e \leq 2$ and $A = \eta^*(4C_0 + (2e + 5)f) - [2E_1 + \dots + 2E_9]$, the E_i 's standing for the exceptional divisors. Hence the \mathbb{Q} -polarized smooth surface $(Y, \mathcal{L}) = (Y, \frac{1}{2}A)$ has Hilbert curve $\gamma : 4u^2 - 4v^2 - 9 = 0$.

Example 3.4. Let (X, L) be as in [5, Example 7.2, Type 2], that is (X, L) is a geometric conic fibration over a smooth quartic surface $\Sigma \subset \mathbb{P}^3$, embedded by $|L|$ in \mathbb{P}^5 and of degree 12. Its Hilbert curve Γ has the canonical equation

$$(14) \quad p_{(X,L)}\left(\frac{1}{2} + u, v\right) = \frac{1}{2}(u - v)(4u^2 - 8uv - 4v^2 - 9) = 0.$$

In this case the conic γ has equation $4u^2 - 8uv - 4v^2 - 9 = 0$ and it is the Hilbert curve of some \mathbb{Q} -polarized smooth surface (Y, \mathcal{L}) if and only if there exists a non zero rational number ρ such that

$$(15) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right) = \rho(4, -4, -4, -9).$$

The third relation says that $\rho < 0$ hence $K_Y^2 = 4\rho < 0$. From the last relation it follows that $\chi(\mathcal{O}_Y) = -4\rho > 0$. By the genus formula it follows that $g(\mathcal{L}) = 1 - 4\rho$. Since \mathcal{L} is a \mathbb{Q} -line bundle, this simply says that $g(\mathcal{L})$ is a rational number > 1 , in view of the above inequality.

Let, for instance, $\rho = -1/4$, then

$$(16) \quad \left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, \chi(\mathcal{O}_Y), g(\mathcal{L}) \right) = \left(-1, 1, 1, 1, 2 \right).$$

We like to remark that there are at least two examples of smooth polarized surfaces (Y, \mathcal{L}) of sectional genus two having the conic $\gamma : 4u^2 - 8uv - 4v^2 - 9 = 0$ as Hilbert curve. They have Kodaira dimension $\kappa(Y) = 0$ and $\kappa(Y) = -\infty$, respectively.

Going through [2] we see that if (S, A) is an Enriques surface polarized by an ample line bundle of genus two as in [2, Theorem 2.7, (2.7.1) b)], $\eta : Y = \text{Bl}_p(S) \rightarrow S$ is the blow up at a general point p of S and $\mathcal{L} = \eta^*A - \eta^{-1}(p)$, then the polarized surface (Y, \mathcal{L}) satisfies (16). The description of the pair (S, A) is as follows: let $\pi : \tilde{S} \rightarrow S$ be the universal cover of S ; then \tilde{S} is a $K3$ surface, which is endowed with a double cover $\varphi : \tilde{S} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with branch locus of bidegree $(4, 4)$; take $A \equiv \pi(\varphi^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1)))$ (see also [1, p. 238]). Such surface Y has $\kappa(Y) = 0$.

On the other hand, going through [6, Theorem 15.2] we see that the polarized surface (Y, \mathcal{L}) appearing in case 8) has the conic $\gamma : 4u^2 - 8uv - 4v^2 - 9 = 0$ as Hilbert curve. Precisely, the polarized surface (Y, \mathcal{L}) is so obtained: there are a del Pezzo surface (S'', L'') with $L''^2 = 1$ and two points p_1, p_2 on S'' such that Y is the blow up of S'' at these two points and $\mathcal{L} = 3\eta^*L'' - 2E_1 - 2E_2$, where $\eta : Y \rightarrow S''$ is the blow up, each E_i is a (-1) -curve. Clearly $\mathcal{L}^2 = 1$. See [12, Theorem 5.1(b)] for the existence of such polarized pair (Y, \mathcal{L}) . Of course, $\kappa(Y) = -\infty$.

These two examples seem to show that there is no relation at all between the base surface of the conic fibration (X, L) having its Hilbert curve Γ reducible and of the form $\Gamma = \ell + \gamma$, with γ a conic, and the possible \mathbb{Q} -polarized smooth surfaces (Y, \mathcal{L}) having γ as Hilbert curve.

We like to point out that further examples can be obtained by taking other values of ρ .

4. The special framework of smooth conic fibrations

We will now focus on the special case in which (X, L) is a smooth conic fibration over a smooth surface S . Here the Bogomolov semistability of the vector bundle defining X comes into the picture.

Let (Y, \mathcal{L}) be a \mathbb{Q} -polarized surface. If $\text{rk}\langle K_Y, \mathcal{L} \rangle = 1$, then we say that (Y, \mathcal{L}) is in the *degenerate case*.

Lemma 4.1. *Let $C \subset \mathbb{A}^2$ be a conic and let \mathcal{J} be the quadratic invariant of C . If C is the Hilbert curve of some \mathbb{Q} -polarized surface (Y, \mathcal{L}) , then $\mathcal{J} \leq 0$ with equality if and only if (Y, \mathcal{L}) is in the degenerate case.*

Proof. Let $K = K_Y$. We know from (2) that $\mathcal{J} = K^2 \mathcal{L}^2 - (K \cdot \mathcal{L})^2$. Let m be a positive integer such that $m\mathcal{L} \in \text{Pic}(Y)$, and let $U = m\mathcal{L}$. The line bundle U is ample and by the Hodge index Theorem it follows that

$$(17) \quad \mathcal{J} = K^2 \mathcal{L}^2 - (K \cdot \mathcal{L})^2 = \frac{1}{m^2} (K^2 U^2 - (K \cdot U)^2) \leq 0,$$

with equality if and only if $\text{rk}\langle K, \mathcal{L} \rangle = \text{rk}\langle K, U \rangle = 1$. □

Let (X, L) be a geometric smooth conic fibration over a smooth surface S . According to the notation in Section 2, $X = \mathbb{P}(\mathcal{F})$ is a bundle. Hence the discriminant divisor $\mathcal{D} = 2c_1(\mathcal{E}) + 3\mathcal{B}$ has to be zero. So $\mathcal{B} = -\frac{2}{3}c_1(\mathcal{E})$ and thus

$$(18) \quad H = K_S + \frac{1}{3}c_1(\mathcal{E}).$$

Let $\delta := \frac{1}{2}[c_1(\mathcal{F})^2 - 4c_2(\mathcal{F})]$ and recall that $\delta \leq 0$ if and only if \mathcal{F} is Bogomolov semistable. Let Γ be the Hilbert curve of (X, L) , whose equation is

$$Cu^3 + Du^2v + Euv^2 + Fv^3 + Au + Bv = 0.$$

Set $c_i = c_i(\mathcal{E})$; according to [5, Proposition 4.1 and Remark 5.1] we have

$$\begin{aligned} C &= \frac{1}{6}K_X^3 = -\frac{1}{6}\left(\frac{2}{3}c_1^2 - 2c_2 + 6K_S^2\right), \\ D &= -\frac{1}{6}(6c_2 - 2c_1^2 - 6K_S^2 + 4K_S \cdot c_1), \\ E &= \frac{1}{6}(6c_2 + 4K_S \cdot c_1 - \frac{8}{3}c_1^2), \\ F &= \frac{1}{6}\left(\frac{4}{3}c_1^2 - 2c_2\right), \\ A = -B &= -\frac{1}{24}(K_X^3 + 48\chi(\mathcal{O}_X)) = \frac{1}{24}\left(\frac{2}{3}c_1^2 - 2c_2 + 6K_S^2 - 48\chi(\mathcal{O}_S)\right). \end{aligned}$$

We know from [5, Theorem 5.2] that Γ contains the line $\ell_0 : u - v = 0$ and thus $\Gamma = \gamma + \ell_0$ where γ is a conic of equation

$$Mu^2 + 2Nuv + Pv^2 + T = 0.$$

So, factoring the equation of Γ in the form

$$(19) \quad (Mu^2 + 2Nuv + Pv^2 + T)(u - v) = 0,$$

where the first factor defines the conic γ , we see that

$$(20) \quad \begin{cases} M = C = \frac{1}{6}K_X^3 = -\frac{1}{6}\left(\frac{2}{3}c_1^2 - 2c_2 + 6K_S^2\right), \\ N = \frac{1}{2}(C + D) = -\frac{1}{12}\left(-\frac{4}{3}c_1^2 + 4c_2 + 4K_S \cdot c_1\right), \\ P = -F = -\frac{1}{6}\left(\frac{4}{3}c_1^2 - 2c_2\right), \\ T = A = \frac{1}{24}\left(\frac{2}{3}c_1^2 - 2c_2 + 6K_S^2 - 48\chi(\mathcal{O}_S)\right). \end{cases}$$

Lemma 4.2. *Let (X, L) be as above and let \mathcal{J} be the quadratic invariant of γ . Then*

$$(21) \quad \mathcal{J} = \frac{2}{3}\delta\left(K_S + \frac{1}{3}c_1\right)^2 + [K_S^2c_1^2 - (K_S \cdot c_1)^2].$$

Proof. It is enough to recall that $\mathcal{J} = MP - N^2$ and an easy computation gives the result. \square

Lemma 4.3. *The polarized surface (S, H) is in the degenerate case if and only if the second addendum in (21) is zero and at least one of K_S^2 and c_1^2 is positive.*

Proof. If $K_S = aH$ for some $a \in \mathbb{Q}$, then from (18) it follows that $c_1 = 3(1 - a)H$. Because H is ample we have that $K_S^2 = a^2H^2 \geq 0$ and $c_1^2 = 9(1 - a)^2H^2 \geq 0$ and equality cannot hold in both inequalities at the same time. Moreover $K_S \cdot c_1 = 3a(1 - a)H^2$, thus the second addendum in (21) is zero. Conversely, if $K_S^2 > 0$ or $c_1^2 > 0$, then by Hodge index Theorem the vanishing of the second addendum in (21) implies that $\text{rk}(K_S, c_1) = 1$ which, by (18), implies that $\text{rk}(K_S, H) = 1$. \square

Proposition 4.4. *Let (X, L) be a geometric smooth conic fibration over a smooth surface S , with $X = \mathbb{P}(\mathcal{F})$ and such that (S, H) is in the degenerate case. If the Hilbert curve of (X, L) , $\Gamma = \gamma + \ell_0$, is such that γ is the Hilbert curve of some \mathbb{Q} -polarized surface, then \mathcal{F} is B -semistable. In particular if γ is the Hilbert curve of (Y, \mathcal{L}) , then (Y, \mathcal{L}) is in the degenerate case if and only if \mathcal{F} is properly B -semistable.*

Proof. By Lemma 4.3 and taking into account (21) and (18) it follows that the quadratic invariant of γ is $\mathcal{J} = \frac{2}{3}\delta H^2$. We know that $\mathcal{J} \leq 0$ by Lemma 4.1. Since $H^2 > 0$, H being ample, this implies $\delta \leq 0$. In particular $\mathcal{J} = 0$ if and only if $\delta = 0$. The last claim follows from Lemma 4.1. \square

Let (X, L) be a geometric smooth conic fibration over a smooth surface S . Then the Hilbert curve $\Gamma = \gamma + \ell_0$ is described by (19). Recalling (1) we thus see that γ is the Hilbert curve $\Gamma_{(Y, \mathcal{L})}$ of the \mathbb{Q} -polarized surface (Y, \mathcal{L}) if and only if there exists a non-zero rational number ρ such that

$$(M, N, P, T) = \rho\left(K_Y^2, K_Y \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2\right).$$

Taking into account the expressions of M, N, P, T provided by (20) and replacing ρ with $\rho' = -6\rho$, this leads to the following four equations:

$$(22) \quad \frac{2}{3}c_1^2 - 2c_2 + 6K_S^2 = \rho' K_Y^2,$$

$$(23) \quad -\frac{2}{3}c_1^2 + 2c_2 + 2K_S \cdot c_1 = \rho' K_Y \cdot \mathcal{L},$$

$$(24) \quad \frac{4}{3}c_1^2 - 2c_2 = \rho' \mathcal{L}^2,$$

$$(25) \quad -\frac{1}{6}c_1^2 + \frac{1}{2}c_2 - \frac{3}{2}K_S^2 + 12\chi(\mathcal{O}_S) = \rho'(2\chi(\mathcal{O}_Y) - \frac{1}{4}K_Y^2).$$

Combining (22) with (25) we get $\chi(\mathcal{O}_Y) = \frac{6}{\rho'}\chi(\mathcal{O}_S)$. Hence, if $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S)$, then

$$\text{either } \chi(\mathcal{O}_S) = 0, \text{ or } \rho' = 6.$$

In particular, if $\chi(\mathcal{O}_S) \neq 0$ and we require that Y is birational to the surface S itself (e.g., even $Y = S$), this gives $\rho' = 6$. Thus, if $\chi(\mathcal{O}_S) \neq 0$, (22) implies that $c_1^2 = 3c_2$, i.e., that \mathcal{E} is properly B -semistable. On the other hand, if we suppose that \mathcal{E} is properly B -semistable, then (22) and (23) simplify considerably, giving

$$6K_S^2 = \rho' K_Y^2$$

and

$$2K_S \cdot c_1 = \rho' K_Y \cdot \mathcal{L},$$

respectively. Moreover (24) becomes

$$2c_2 (= \frac{2}{3}c_1^2) = \rho' \mathcal{L}^2.$$

If S is not ruled, $\chi(\mathcal{O}_S) \neq 0$ and Y is birational to S , then the fact that $\rho' = 6$ implies $K_Y^2 = K_S^2$, hence Y and S are isomorphic if they are minimal (by the Enriques theorem on minimal models), while they dominate the common minimal model via birational morphisms factoring via the same number of blowing-ups, if they are not minimal.

This suggests to continue exploring what happens if we require, in particular, that Y is the base surface S itself of the smooth conic fibration (X, L) , with the further assumption that \mathcal{E} is properly B -semistable. In this situation it looks plausible that $\mathcal{L} = \frac{1}{3} \det \mathcal{E}$ (which, when it is ample, corresponds to the average polarization induced on S by the rank 3 vector bundle \mathcal{E}).

In fact, we have:

Proposition 4.5. *Let $Y = S$ and suppose that $\chi(\mathcal{O}_S) \neq 0$. If $\gamma = \Gamma_{(S, \mathcal{L})}$, then either $\mathcal{L} = \frac{1}{3}c_1$, or $(c_1 - 3\mathcal{L})^2 < 0$.*

Proof. Since $Y = S$, by what we said before we have $\rho' = 6$. Moreover, $c_1^2 = 3c_2 = 9\mathcal{L}^2$. Thus

$$(c_1 - 3\mathcal{L}) \cdot (c_1 + 3\mathcal{L}) = 0.$$

On the other hand by (23)

$$(c_1 - 3\mathcal{L}) \cdot K_S = 0.$$

Summing up the former equation with the triple of the latter we get

$$3(K_S + \frac{1}{3}c_1 + \mathcal{L}) \cdot (c_1 - 3\mathcal{L}) = 0.$$

Then the assertion follows from the Hodge index theorem, recalling that $H = K_S + \frac{1}{3}c_1$ is ample. \square

We will now give examples of pairs (S, \mathcal{L}) as in Proposition 4.5 where \mathcal{L} is such that either $\mathcal{L} = \frac{1}{3}c_1$, or $(c_1 - 3\mathcal{L})^2 < 0$.

The following, which is also a generalization of [5, Example 5.3], provides an example with $\mathcal{L} = \frac{1}{3}c_1$.

Example 4.6. Let S be any surface and consider an ample line bundle $\mathcal{A} \in \text{Pic}(S)$ such that $K_S + 3\mathcal{A}$ is ample. Let $\mathcal{F} = \mathcal{A}^{\oplus 2}$ and set $X = \mathbb{P}(\mathcal{F})$. Then $X \cong S \times \mathbb{P}^1$ is a trivial \mathbb{P}^1 -bundle over S via the first projection $p : X \rightarrow S$. Let $L = 2\xi_{\mathcal{F}} + p^*\mathcal{A}$, where $\xi_{\mathcal{F}}$ is the tautological line bundle of \mathcal{F} . Then (X, L) is a geometric smooth conic fibration, in view of the ampleness of $K_S + 3\mathcal{A}$. Letting $\mathcal{E} := p_*L = S^2(\mathcal{F}) \otimes \mathcal{A}$, we have that X is contained in $P := \mathbb{P}(\mathcal{E})$, as a divisor of relative degree 2, and $L = (\xi_{\mathcal{E}})_X$, the tautological line bundle of \mathcal{E} , restricted to X . Since $c_1(\mathcal{F}) = 2\mathcal{A}$ and $c_2(\mathcal{F}) = \mathcal{A}^2$, from (3), (4) it follows that

$$c_1 = 3(c_1(\mathcal{F}) + \mathcal{A}) = 9\mathcal{A}$$

and

$$c_2 = 3\mathcal{A}^2 + 6c_1(\mathcal{F}) \cdot \mathcal{A} + 2c_1(\mathcal{F})^2 + 4c_2(\mathcal{F}) = 27\mathcal{A}^2.$$

Therefore $c_1^2 = 3c_2$, i.e., \mathcal{E} is properly B-semistable. In the present case, the expressions of M, N, P, T provided in (20) show that

$$-(M, N, P, T) = (K_S^2, 3K_S \cdot \mathcal{A}, 9\mathcal{A}^2, 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2).$$

As a consequence, for the conic γ , residual of the line ℓ_0 in the Hilbert curve $\Gamma_{(X,L)}$, we have that

$$\gamma = \Gamma_{(S,\mathcal{L})},$$

where $\mathcal{L} = \frac{1}{3}c_1 (= 3\mathcal{A})$. Note that the quadratic invariant \mathcal{J} of γ is negative, provided that (S, \mathcal{A}) is not as in the degenerate case. Actually, up to a positive numerical factor we have

$$\mathcal{J} = 9K_S^2\mathcal{A}^2 - (3K_S \cdot \mathcal{A})^2,$$

which is negative in view of the Hodge index theorem. As a consequence, the conic γ is reducible if and only if $8\chi(\mathcal{O}_S) = K_S^2$, which does not happen for a general S .

We now give two examples of pairs (S, \mathcal{L}) as in Proposition 4.5 which fall in the case in which \mathcal{L} is so that $(c_1 - 3\mathcal{L})^2 < 0$. In one of them \mathcal{L} is an ample \mathbb{Q} -line bundle, in the other case \mathcal{L} is just an ample line bundle.

Example 4.7. Let $S = \mathbb{F}_1$, let C_0 be the section of self-intersection $C_0^2 = -1$ and let f be a fiber of the bundle projection $\mathbb{F}_1 \rightarrow \mathbb{P}^1$. Over \mathbb{F}_1 we consider the rank two vector bundle $\mathcal{F} = [3C_0 + 4f] \oplus [C_0 + 3f]$. Let $X = \mathbb{P}(\mathcal{F})$ and let $\pi : X \rightarrow \mathbb{F}_1$ be the projection map. Let $L = 2\xi_{\mathcal{F}} + \pi^*\mathcal{A}$, where $\xi_{\mathcal{F}}$ is the tautological line bundle of \mathcal{F} and $\mathcal{A} = C_0 + 2f$. The vector bundle $\mathcal{E} := \pi_*L = S^2(\mathcal{F}) \otimes \mathcal{A}$, from (3), has $c_1 = c_1(\mathcal{E}) = 14C_0 + 27f$ hence (X, L) is a standard smooth conic fibration because $K_{\mathbb{F}_1} + \frac{1}{3}c_1$ is ample. Note that $c_1(\mathcal{F}) = 4C_0 + 7f$ and $c_2(\mathcal{F}) = (3C_0 + 4f) \cdot (C_0 + 3f)$, from which it follows that $c_1(\mathcal{F})^2 - 4c_2(\mathcal{F}) = 0$ that is, \mathcal{F} is properly B -semistable and, by Proposition 2.2, also \mathcal{E} is properly B -semistable. The \mathbb{Q} -line bundle $\mathcal{L} = \frac{13}{2}C_0 + \frac{33}{4}f$ is such that $(c_1(\mathcal{E}) - 3\mathcal{L})^2 = (-\frac{9}{2}C_0 + \frac{9}{4}f)^2 = -\frac{81}{2} < 0$.

Similarly, if over \mathbb{F}_1 we take $\mathcal{F} = [3C_0 + 5f] \oplus [C_0 + 4f]$ and $\mathcal{A} = 4C_0 + 5f$, then the vector bundle $\mathcal{E} = S^2(\mathcal{F}) \otimes \mathcal{A}$ has $c_1 = 24C_0 + 42f$. Moreover the line bundle $\mathcal{L} = 10C_0 + 13f$ is such that $(c_1(\mathcal{E}) - 3\mathcal{L})^2 = (-6C_0 + 3f)^2 = -72 < 0$.

5. A characterization in terms of the canonical equation of the Hilbert curve

Let (X, L) be a geometric conic fibration over a smooth surface S : its general fiber with the polarization induced by L is $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$. By the Riemann–Roch theorem the polynomial defining its canonical equation is

$$p_{(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))}(\frac{1}{2} + u, v) = \chi((\frac{1}{2} + u)K_{\mathbb{P}^1} + v\mathcal{O}_{\mathbb{P}^1}(2)) = 2(v - u).$$

A variant of the problem discussed in this paper is the following. When is

$$(26) \quad p_{(X, L)} = p_{(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))} p_{(S, \mathcal{L})}$$

for some ample line bundle (or \mathbb{Q} -line bundle) \mathcal{L} on S ?

For instance, for any ample line bundle \mathcal{L} on S , the above equality holds for the pair $(X, L) = (S \times \mathbb{P}^1, \mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2))$, in view of [3, 2.5]. But this is not the only possibility. Actually, the answer is provided by:

Theorem 5.1. *Let (X, L) be a geometric conic fibration over a smooth surface S and let $\pi : X \rightarrow S$ be the fibration morphism. If (26) holds for some ample \mathbb{Q} -line bundle \mathcal{L} on S , then X is a bundle over S , the vector bundle $\mathcal{E} = \pi_*L$ is properly B -semistable, and either $\mathcal{L} = \frac{1}{3}c_1$, or $(c_1 - 3\mathcal{L})^2 < 0$, where $c_1 = c_1(\mathcal{E})$.*

Proof. Since $u - v$ divides $p_{(X, L)}$, [5, Theorem 5.2] implies that X is a bundle. Moreover, taking into account the expressions of M, N, P and T in (20) and

recalling (1), our assumption implies

$$(27) \quad \begin{cases} K_S^2 + \frac{1}{3}(\frac{1}{3}c_1^2 - c_2) = K_S^2, \\ \frac{1}{3}(K_S \cdot c_1 + c_2 - \frac{1}{3}c_1^2) = K_S \cdot \mathcal{L}, \\ \frac{1}{3}(\frac{1}{3}c_1^2 + \frac{1}{3}c_1^2 - c_2) = \mathcal{L}^2, \\ 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2 + \frac{1}{12}(c_2 - \frac{1}{3}c_1^2) = 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2. \end{cases}$$

Therefore $c_1^2 = 3c_2$, i.e., \mathcal{E} is properly B-semistable. Moreover, the second and the third equations of the above system show that $K_S \cdot c_1 = K_S \cdot 3\mathcal{L}$ and $c_1^2 = 9\mathcal{L}^2$ and then the last assertion in the statement follows as in the proof of Proposition 4.5. \square

Remark 5.2. We like to point out that Example 3.1, Example 3.2 and Example 4.7 satisfy the relation (26); moreover the first two examples fall in the case in which $\mathcal{L} = \frac{1}{3}c_1$, while the last example falls in the case in which $(c_1 - 3\mathcal{L})^2 < 0$.

6. A general line as a component of Γ

Let (X, L) be a geometric conic fibration over a smooth surface S and let Γ be the Hilbert curve of (X, L) . According to [5, Proposition 4.1] the canonical equation of Γ has the form

$$(28) \quad Cu^3 + Du^2v + Euv^2 + Fv^3 + Au + Bv = 0$$

for suitable coefficients $A, B, C, D, E, F \in \mathbb{Q}$. In [5, Theorem 5.2] we proved that Γ contains the line $\ell_0 : u - v = 0$ as a component if and only if X is a bundle. What if Γ contains a different line as a component? We will now also investigate this possibility in the simplest case in which $S = \mathbb{P}^2$. Recall that Γ is symmetric with respect to the origin O of the (u, v) -plane, due to Serre duality, hence, if a line ℓ is a component of Γ , then necessarily $O \in \ell$. The following result can be regarded as a generalization of [5, Proposition 5.4], where we considered the line ℓ of equation $2u - 3v = 0$.

Proposition 6.1. *Let (X, L) be a geometric conic fibration over \mathbb{P}^2 and let Γ be its Hilbert curve. Then Γ contains the line $\ell : pu - qv = 0$, with $p \neq q$, if and only if $[p(a + b - 1) - 2q][p(a + b + 1) - 4q] = 0$, where a and b are such that $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(a)$ and $\mathcal{B} = \mathcal{O}_{\mathbb{P}^2}(b)$.*

Proof. Because the base of the geometric conic fibration (X, L) is \mathbb{P}^2 , $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(a)$ and $c_2(\mathcal{E}) = c$ for some $a, c \in \mathbb{Z}$. Thus the coefficients of the degree three terms in (28) become, up to the factor $\frac{1}{6}$:

$$\begin{aligned} C &= -\left(2a^2 - 2c - 18a - 27b + 4ab + 3b^2 + 54\right), \\ D &= -\left(6c + 3ab + 18b - 54\right), \end{aligned}$$

$$E = 6c - 18a - 9b + 6ab + 3b^2,$$

$$F = 2a^2 - 2c + ab.$$

Likewise the coefficients of u and v are, up to the factor $\frac{1}{24}$, respectively

$$A = 2a^2 - 2c - 18a - 27b + 4ab + 3b^2 + 6,$$

$$B = 4a^2 + 2c + 11ab + 6b^2 - 6.$$

According to the discussion in [5, Section 3, Proposition 3.2], Γ is reducible if and only if

$$(29) \quad A^2(AF - BE) + B^2(AD - BC) = 0.$$

Hence the line $\ell : pu - qv = 0$ is a component of Γ if and only if (29) holds with $A = pk$ and $B = -qk$ for some non zero $k \in \mathbb{Z}$ (since A and B are integers). Recalling that $p \neq q$, the last two conditions, combined with the above expressions of A and B , give

$$c = -\frac{1}{2} \frac{1}{p - q} (4a^2p + 2a^2q - 18qa - 27qb + 4qab + 3qb^2 + 11pab + 6pb^2 - 6p + 6q)$$

and

$$k = \frac{3(2a^2 - 6a - 9b + 5ab + 3b^2)}{p - q}.$$

Replacing the value of k in A and B , it thus follows that, up to a factor, (29) can be rewritten as

$$k^3(p - q) \left[p^2(2a^2 - 2c + ab) + pq(7ab + 3b^2 - 18a - 9b + 4c + 2a^2) + q^2(-27b + 54 + 4ab - 2c + 3b^2 - 18a + 2a^2) \right] = 0.$$

Hence, after plugging the value of c , we get

$$(30) \quad 6(p - q)(-2q - p + ap + bp)(-4q + p + ap + bp)k^3 = 0.$$

As we said $p - q \neq 0$ and $k \neq 0$, thus (30) holds if and only if $[p(a + b - 1) - 2q] \cdot [p(a + b + 1) - 4q] = 0$.

If $p(a + b - 1) - 2q = 0$, then $b = -a + \frac{2q}{p} + 1$ and in this case $\Gamma = \ell + \gamma_1$, where the equation of γ_1 is

$$(5p + pa - 6q)u^2 + (-2pa - 6p + 8q)uv + (pa + p - 2q)v^2 + \frac{3}{4}p - \frac{1}{4}pa + \frac{3}{2}q = 0.$$

Note that if $\frac{3}{4}p - \frac{1}{4}pa + \frac{3}{2}q \neq 0$, then γ_1 is irreducible because the determinant of the matrix associated to γ_1 is $-4(\frac{3}{4}p - \frac{1}{4}pa + \frac{3}{2}q)(p - q)^2$ which is different from zero because $p \neq q$.

However, it could be that $\frac{1}{4}(3p - ap + 6q) = 0$ and in this case γ_1 is reducible. This happens if and only if $q = \frac{1}{6}(a - 3)p$.

If $p(a + b + 1) - 4q = 0$, then $b = -a + \frac{4q}{p} - 1$ and in this case $\Gamma = \ell + \gamma_2$, where the equation of γ_2 is

$$(7p + pa - 12q)u^2 + (-2pa - 6p + 16q)uv + (pa - p - 4q)v^2$$

$$-\frac{3}{4}p - \frac{1}{4}pa + 3q = 0. \quad \square$$

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MARIA LUCIA FANIA
 DISIM, UNIVERSITÀ DEGLI STUDI DELL'AQUILA
 VIA VETOIO, LOC. COPPITO I-67100 L'AQUILA, ITALY
Email address: marialucia.fania@univaq.it

ANTONIO LANTERI
 DIPARTIMENTO DI MATEMATICA "F. ENRIQUES"
 UNIVERSITÀ DEGLI STUDI DI MILANO
 VIA C. SALDINI, 50, I-20133 MILANO, ITALY
Email address: antonio.lanteri@unimi.it