# Contributed Discussion

Matteo Iacopini[*,§], Francesco Ravazzolo[†], and Luca Rossini[‡]

We have greatly appreciated the work by Leisen, Villa and Walker, who have proposed a novel method for constructing objective prior distributions which circumvents the need to specify a statistical model and relies solely on a specific proper scoring rule.

Scoring rules are of tantamount importance in practical statistical analysis, since they encode the preferences of the stakeholder (e.g., a policymaker). In fact, since scoring rules provide a simple mean to assess the performance of a set of statistical models with respect to a specific user-defined goal, they have been widely used as a tool for evaluation, comparison and ranking of competing statistical models.

Recently, Iacopini et al. (2020) proposed the asymmetric continuous probabilistic score (ACPS), a new proper scoring rule for evaluating density forecasts according to asymmetric preferences of the stakeholder. Being the ACPS a proper scoring rule, the methodology developed by the Authors can be directly applied to derive a class of objective prior distributions. Moreover, since ACPS has one free parameter specifying the type and degree of asymmetric preferences, the ensuing class of priors would inherit this degree of freedom. Intuitively, its role would be analogous to that of the free constants, $c$ and $u(0)$, in Section 3 of the discussed Article.

To obtain the class of objective priors stemming from the ACPS proper scoring rule, first recall its definition. Let $P : \mathcal{D} \to [0, 1]$ be a cumulative distribution function, with $\mathcal{D} \subseteq \mathbb{R}$, let $y \in \mathcal{D}$, and denote with $c \in (0, 1)$ the asymmetry parameter. Then

$$\begin{aligned} ACPS(P, y; c) &= \int_{-\infty}^{y} (c^2 - P(u)^2) f(P, c) \,\mathrm{d}u + \int_{y}^{+\infty} ((1-c)^2 - (1-P(u))^2) f(P, c) \,\mathrm{d}u \\ &= \int_{\mathbb{R}} \Big[ (c^2 - P(u)^2)\mathbb{I}(u < y) + ((1-c)^2 - (1-P(u))^2)\mathbb{I}(u > y) \Big] f(P, c) \,\mathrm{d}u, \end{aligned}$$

where $f(P, c) = \mathbb{I}(P(u) > c)/(1-c)^2 + \mathbb{I}(P(u) \leq c)/c^2$. Therefore, solving for $P$ the equation $ACPS(P, y; c) = k$, with $k \in \mathbb{R}$, would be equivalent to solve the minimization problem

$$\min_{P} \Big| \int_{\mathbb{R}} L(u, P, P') \,\mathrm{d}u - k \Big|, \tag{1}$$

where $L(u, P, P') = \big[ (c^2 - P(u)^2)\mathbb{I}(u < y) + ((1-c)^2 - (1-P(u))^2)\mathbb{I}(u > y) \big] f(P, c)$. This minimization problem can be reconciled to a standard calculus of variations problem, as follows. First, one can obtain an approximation by substituting $\mathbb{R}$ with a bounded

*Vrije Universiteit Amsterdam, The Netherlands, m.iacopini@vu.nl
†Free University of Bozen, Italy, francesco.ravazzolo@unibz.it
‡Queen Mary University, United Kingdom, l.rossini@qmul.ac.uk
§Corresponding author.

region $(a, b) \subset \mathbb{R}$. Second, the absolute value can be removed by a suitable choice of the constant $k$. Since the equality $ACPS(P, y; c) = k$ can be satisfied for any arbitrary choice of $k \in \mathbb{R}$, and $ACPS(P, y; c) < +\infty$, then there exists $k^* \in \mathbb{R}$ such that $ACPS(P, y; c) - k^* > 0$. Therefore, by choosing $k = k^*$ one gets

$$ACPS(P, y; c) = k^*, \quad \text{where} \quad |ACPS(P, y; c) - k^*| = ACPS(P, y; c) - k^* > 0.$$

Putting all together one gets the following minimization problem

$$\min_P \Big\{ ACPS(P, y; c) - k^* \Big\} \equiv \min_P \Big\{ \int_a^b L(u, P, P') \, \mathrm{d}u - k^* \Big\}. \tag{2}$$

Finally, since $k^*$ is constant, the optimum is $P^* = k^* + P^{**}$, where $P^{**}$ is the solution of the problem

$$\min_P \int_a^b L(u, P, P') \, \mathrm{d}u,$$

for which the corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial P} = 0.$$

Therefore, solving the minimization problem (2) yields a class of cumulative distribution functions, $P^*$, which depends on two free parameters: (i) the asymmetry parameter of the ACPS, $c$, and (ii) the constant of integration, $\bar{c}$.

## References

Iacopini, M., Ravazzolo, F., and Rossini, L. (2020). "Proper scoring rules for evaluating asymmetry in density forecasting." *CAMP BI Working Paper 06/2020*. 1392