

Comment on “Breakdown of the tensor component in the Skyrme energy density functional”

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In a recent paper [Phys. Rev. C 101, 014305 (2020)], Dong and Shang claim that the Skyrme original tensor interaction is invalid. Their conclusion is based on the misconception that the Fourier transform of tensor interaction is difficult or even impossible, so that the Skyrme-type tensor interaction was introduced in an unreasonable way. We disagree on their claim. In this note, we show that one can easily get the Skyrme force in momentum space by Fourier transformation if one starts from a general central, spin-orbit or tensor interaction with a radial dependence.

I. INTRODUCTION

In a recent paper published in Phys. Rev. C (Ref. [1], hereafter referred to as Dong and Shang), the authors claim in the abstract that “the Skyrme original tensor interaction [...] is invalid.” One has to remind that, although the original idea by Skyrme dates back to long time ago and the Skyrme tensor force has been written out in Refs. [2, 3], there have been many studies on how to implement it and fit its parameters since then, even in recent years. A first series of papers appeared in the 1970s, when the first Skyrme forces had been proposed as a practical tool for Hartree-Fock (HF) and Random Phase Approximation (RPA). A second series, much more numerous, has appeared in the years since 2000. In 2014, some of us published a review paper on the tensor force within mean-field and Density Functional Theory (DFT) approaches to nuclear structure [4]. In that review, we quoted ≈ 30 -40 papers where the Skyrme tensor is implemented and studied, authored by different colleagues. The conclusion of that review paper is that evidences for a strong neutron-proton tensor force exist, even in mean-field and DFT studies, while the role of the tensor force between equal particles (neutron-neutron or proton-proton) is less well established albeit not completely ruled out. Therefore, if the tensor force proposed by Skyrme were “invalid”, this would impact on a lot of published works and conclusions drawn so far. Accordingly, it would be appropriate to have a strong argument regarding this “invalidity”.

The argument against the Skyrme tensor force should

be found in Sec. II of the paper. In this Section, one can read: “The Skyrme original tensor force was introduced in an unreasonable way, because the tensor-force operator S_{12} in momentum space but with an r -dependent strength, i.e., $f_T(r)S_{12}(\mathbf{k})$, is applied as a starting point.” We remind, for the reader’s convenience, that

$$S_{12}(\mathbf{k}) = (\sigma_1 \cdot \mathbf{k})(\sigma_2 \cdot \mathbf{k}) - k^2 \frac{\sigma_1 \cdot \sigma_2}{3}. \quad (1)$$

We cannot see any logic behind the latter sentence in the work by Dong and Shang. In fact, it does not have any formal ground. In physics, nothing forbids an interaction to be at the same time position-dependent and momentum-dependent. Bethe [5] was one of the first to advocate that this ought to be the case if one wishes to introduce an effective potential for finite nuclei. Skyrme echoed this at the beginning of his first paper on this topic, and it is useful at this stage to quote literally the sentence from Ref. [6]: “in the case of a finite system the effective potential must depend upon both momenta and coordinate.”

In fact, this latter sentence does not simply lie as a ground for the Skyrme tensor force, but rather it is at the basis of the whole philosophy behind Skyrme forces and Skyrme functionals [6, 7]. Several authors have proposed Skyrme-type forces with terms that have both momentum- and density-dependence. If the Skyrme tensor force is unreasonable because of the reason advocated by Dong and Shang, the whole Skyrme force will be unreasonable. If this were the case, this would disgrace not only the results of a few hundred papers in which Skyrme tensor terms are introduced, but also some more $\approx 10^3$ papers in which Skyrme forces are used, including the paper by Dong and Shang themselves. As we said, we see no reason to rule out a force because it looks like a momentum-dependent operator times a function of the relative coordinate r .

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It should also be noticed that any comparison with a realistic potential used in Brückner-HF calculations, or similar cases, is immaterial in this context. We are not discussing any *realistic* potential but only the case of an *effective* potential to be used in HF or DFT. Skyrme himself, in Ref. [2], employed the word “pseudopotential” to distinguish clearly his approach from any one based on a fundamental interaction. In modern language, we would say that we deal with an effective field theory in which the coupling terms depend on $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ times derivatives. An analogous case is that of the pionless EFT.

In the paper by Dong and Shang, it is also claimed that the Skyrme-type tensor interaction is introduced in an unreasonable way since the Fourier transform of such tensor interaction is difficult or even impossible. We disagree on this claim based on the procedure that we discuss explicitly in the next Section. In the next Section we will show that if we start from a general central, spin-orbit or tensor interaction, with a radial dependence such that the range is very short, the Fourier transform produces the Skyrme force in momentum space. This is a further, more detailed and mathematically rigorous, way to show that the arguments in the paper by Dong and Shang are invalid.

II. FOURIER TRANSFORM OF THE TERMS OF THE SKYRME FORCE

Let us write the Fourier transform of an interaction $V(r)$ as

$$V(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}} V(r) d\mathbf{r}, \quad (2)$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer (\mathbf{k} and \mathbf{k}' are the initial and final relative momenta). For a central interaction $V_C(r)$, the integral in (2) can be performed by using the multipole expansion of a plane wave

$$e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \sum_{\lambda\mu} i^\lambda j_\lambda(qr) Y_{\lambda\mu}(\hat{q}) Y_{\lambda\mu}^*(\hat{r}), \quad (3)$$

where $j_\lambda(qr)$ is the spherical Bessel function of rank λ , and $Y_{\lambda\mu}(q)$ is a spherical harmonic. Thus, the integral for a central interaction becomes

$$V(\mathbf{q}) \propto \int j_0(qr) V_C(r) r^2 dr. \quad (4)$$

The constant and the lowest-order momentum-dependent terms of the Skyrme interaction are obtained by means of the Taylor expansion $j_0(qr) = 1 - (qr)^2/2 + \dots$. The constant term of this expansion provides the t_0 term in the Skyrme force. The second term is written as

$$V(\mathbf{q}) \propto \mathbf{q}^2 \int V_C(r) r^4 dr, \quad (5)$$

with

$$\mathbf{q}^2 = \mathbf{k}^2 + \mathbf{k}'^2 - 2\mathbf{k} \cdot \mathbf{k}'. \quad (6)$$

The first two terms in Eq. (6) provide the t_1 term of the Skyrme force, while the third term gives the t_2 term. These two terms mimic the finite range in the effective two-body interaction. The radial integrals in Eqs. (4) and (5) can be easily performed for any commonly adopted Yukawa-type or Gaussian-type finite-range interaction.

The spin-orbit interaction V_{LS} ,

$$V_{LS} = f_{LS}(r)(\sigma_1 + \sigma_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \times (\mathbf{p}_1 - \mathbf{p}_2), \quad (7)$$

can be also expanded, after being Fourier-transformed through Eq. (2), in the momentum operator \mathbf{q} . The radial dependence of $V_{LS}(r)$ is expressed by the spherical harmonics $Y_{1\mu}$, so that the expansion of Eq. (3) produces the spherical Bessel function $j_1(qr)$. We do not repeat these steps here as they can be found in Ref. [7].

The tensor interaction in the coordinate space is expressed as

$$V_T(r) = f_T(r) S_{12}(\mathbf{r}), \quad (8)$$

where

$$S_{12}(\mathbf{r}) = (\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r}) - r^2 \frac{\sigma_1 \cdot \sigma_2}{3}. \quad (9)$$

The tensor operator of (9) is rewritten using the spherical harmonic $Y_{2\mu}$ and becomes

$$S_{12}(\mathbf{r}) = \sqrt{\frac{8\pi}{3}} [(\sigma_1 \cdot \sigma_2)^{(2)} \times r^2 Y_2(\hat{r})]^{(0)}. \quad (10)$$

The Fourier transform can be evaluated as

$$V_T(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}} V_T(r) d\mathbf{r} \propto \int r^4 dr j_2(qr) [(\sigma_1 \cdot \sigma_2)^{(2)} \times Y_2(\hat{q})]^{(0)}. \quad (11)$$

The spherical Bessel function $j_2(qr)$ is proportional to q^2 , as $j_2(qr) \sim (qr)^2/5!!$ in the lowest order of expansion, and $V_T(q)$ becomes

$$V_T(\mathbf{q}) \simeq -\frac{4\pi}{15} \sqrt{\frac{8\pi}{3}} [(\sigma_1 \cdot \sigma_2)^{(2)} \times q^2 Y_2(\hat{q})]^{(0)} \int r^6 f_T(r) dr. \quad (12)$$

In this way, we obtain the tensor interactions in momentum space in the form

$$S_{12}(\mathbf{q}) = \sqrt{\frac{8\pi}{3}} [(\sigma_1 \cdot \sigma_2)^{(2)} \times q^2 Y_2(\hat{q})]^{(0)} \quad (13)$$

$$= (\sigma_1 \cdot \mathbf{q})(\sigma_2 \cdot \mathbf{q}) - \mathbf{q}^2 \frac{\sigma_1 \cdot \sigma_2}{3} \quad (14)$$

$$= \{[(\sigma_1 \cdot \mathbf{k}')(\sigma_2 \cdot \mathbf{k}') - \frac{1}{3}(\sigma_1 \cdot \sigma_2) \mathbf{k}'^2]$$

$$+ [(\sigma_1 \cdot \mathbf{k})(\sigma_2 \cdot \mathbf{k}) - \frac{1}{3}(\sigma_1 \cdot \sigma_2) \mathbf{k}^2]\}$$

$$- \{(\sigma_1 \cdot \mathbf{k}')(\sigma_2 \cdot \mathbf{k}) + (\sigma_2 \cdot \mathbf{k}')(\sigma_1 \cdot \mathbf{k})$$

$$- \frac{2}{3} [(\sigma_1 \cdot \sigma_2) \mathbf{k}' \cdot \mathbf{k}]\}. \quad (15)$$

In the above expression, the operator $\mathbf{k} = (\nabla_1 - \nabla_2)/2i$ acts on the right and $\mathbf{k}' = -(\nabla'_1 - \nabla'_2)/2i$ acts on the left. The first two terms in Eq. (15) are the so-called triplet-even tensor term (T-term in Skyrme), whereas

the third and fourth terms correspond to a triplet-odd term (U-term). In this way we can validate the Skyrme type tensor interaction as the lowest-order expansion of a finite-range tensor force.

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