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LINEAR SCHRÖDINGER EQUATION WITH AN ALMOST PERIODIC POTENTIAL*

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Abstract. We study the reducibility of a linear Schrodinger equation subject to a small unbounded almost periodic perturbation which is analytic in time and space. Under appropriate assumptions on the smallness, analyticity, and on the frequency of the almost periodic perturbation, we prove that such an equation is reducible to constant coefficients via an analytic almost periodic change of variables. This implies control of both Sobolev and analytic norms for the solution of the corresponding Schrödinger equation for all times.

Key words. linear Schrödinger equations, almost periodic potentials, KAM-reducibility

AMS subject classifications. 35B15, 35K55

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1. Introduction.

The problem of control of Sobolev norms for linear Schrödinger operators on a torus with smooth time dependent potential has been studied by various authors. Groundbreaking results were proved by Bourgain in [12] in the case of quasi-periodic bounded potentials with a Diophantine frequency and then in [13] for general time dependent potentials. The main result was an upper bound on the growth in time of the Sobolev norm, logarithmic in time in [12] and polynomial in [13]. Such results were generalized to unbounded potentials in [17], [34], [35], [8], [5], [36], [37], [9], [25].

The main feature of such results is that they are very general, require little or no conditions on the time dependence of the potential, and can often also be applied in nonperturbative settings. At this level of generality such results are in fact optimal, as shown in [13]. See also [33], [27] for examples of growth.

A parallel point of view is to study the reducibility of Schrödinger equations with small quasi-periodic potentials by requiring stronger nonresonance conditions on the frequency; see [18]. We recall that a first order linear differential equation $\dot{u} = L(t)u$ is said to be reducible if there exists a (uniformly bounded) time dependent linear operator which conjugates it to an equation $\dot{v} = Dv$, where D is time independent and diagonal (or block diagonal). Thus one gets a uniform control in time of the Sobolev norms to the price of restricting to small quasi-periodic potentials with rather involuted nonresonance conditions on the frequency. We remark that reducibility is a key argument in KAM for nonlinear PDEs. This is a strong motivation for studying reducibility for linear PDEs. Conversely, many KAM results can be adapted to the reducibility setting.

As can be expected, the (block) diagonalization algorithm relies on lower bounds

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on the difference of distinct eigenvalues (the spectral gaps) as well as on a strong control on their possible multiplicity. Indeed, the first results were for bounded potentials in the case of Dirichlet boundary conditions on $[0, \pi]$, where the eigenvalues are simple (see, for instance, [30], [38], [39], [32], [31]). The last ten years have seen considerable progress in this field, particularly in the case of unbounded potentials. The first results were in [28] in the case of periodic potentials and [1], [2] for the quasi-periodic case. Regarding Schrödinger equations we mention [24], [21], [3], [4]. Note that all of the preceding papers deal with Sobolev stability; generalizing to the analytic case, especially in the case of unbounded potentials of order two and in the context of a nonlinear KAM scheme, is not straightforward. A strategy was discussed in [15], [23]. While the literature on reducibility of quasi-periodic potentials is quite extensive in the case of one space dimension, the case of higher dimensional manifolds is still largely open. We mention [20], [26], [19] and finally [6], [22], [37], [16], [7] for an unbounded potential.

Common features of the reduction algorithms are (1) they are perturbative, (2) they require complicated nonresonance conditions depending on the potential, and (3) they strongly depend on the number of frequencies.

In the present paper we study the reducibility of Schrödinger equations on the circle with a small *unbounded almost periodic* potential of the form

$$\begin{aligned} &\partial_t u = \mathrm{i}\Big(\partial_x^2 + \varepsilon P(t)\Big)u\,, \\ &P(t) := V_2(x,t)\partial_x^2 + V_1(x,t)\partial_x + V_0(x,t)\,, \quad x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}),\ t \in \mathbb{R}\,. \end{aligned} \tag{1.1}$$

Here V_0, V_1, V_2 are analytic (in an appropriate sense) almost periodic functions of time with frequency ω which is an infinite dimensional Diophantine vector in $\ell^\infty(\mathbb{N}, \mathbb{R})$ (see Definitions 1.1 and 1.3). For small ε we prove a reducibility result under the assumption that for any $t \in \mathbb{R}$, the operator $P(t)$ is L^2 self-adjoint and that ω belongs to some (explicit but convoluted) Cantor set of asymptotically full measure.

Of course the difficulty of such a result is strongly related to the regularity of the almost periodic potential. Indeed, by definition, an almost periodic function is the limit of quasi-periodic ones with an increasing number of frequencies. If the limit $P_n \to P$ is reached sufficiently quickly, the most direct strategy is to reduce iteratively the approximations of (1.1) with *quasi-periodic* potentials by considering at each step n the operator P_n as a small perturbation of the one of the previous step. This procedure in fact works if one considers sufficiently smoothing and regular potentials but becomes very delicate in the case of unbounded potentials.

Good comparisons are [40], which studies a smoothing nonlinear Schrödinger equation with external parameters and proves existence of almost periodic solutions with superexponential decay in the Fourier modes, and [14], which looks at almost periodic solutions for a nonlinear Schrödinger equation with external parameters with subexponential decay in the Fourier modes. In the first paper, the very fast decay implies that at each KAM step, one only needs to construct quasi-periodic solutions (with an increasing number of frequencies), which is a well-known result; the only point is to show that they converge superexponentially to a nontrivial almost periodic solution. In the second paper the author does not rely on quasi-periodic approximations; this requires completely revisiting the KAM scheme but leads to solutions with much less regularity. In this paper we follow the general point of view of [14] (see also [11]), using the same infinite dimensional Diophantine vectors and various technical lemmas (detailed proofs of all the technical lemmas can be found in [10]). The strategy is to generalize the approach of [1] to the context of almost periodically forced

PDEs. This requires developing pseudodifferential calculus in the context of analytic functions on infinite dimensional tori; see page 6 for a more detailed presentation of the novelties.

In order to give the precise statement of our theorems, we introduce some notation and definitions.

We define the parameter space of frequencies as a subset of[1] $\ell^\infty(\mathbb{N},\mathbb{R})$, where we recall that

$$\ell^\infty(\mathbb{N},\mathbb{R}) := \Big\{ \omega = (\omega_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|\omega\|_\infty := \sup_{j\in\mathbb{N}} |\omega_j| < \infty \Big\}\,.$$

More precisely, our set of frequencies is the infinite dimensional cube

$$\mathtt{R}_0 := \Big[1\,,\,2\Big]^{\mathbb{N}}\,. \tag{1.2}$$

We endow the space of parameters $\mathtt{R}_0$ with the ℓ^∞ metric; namely we set

$$d_\infty(\omega_1,\omega_2) := \|\omega_1-\omega_2\|_\infty \quad \forall \omega_1,\omega_2\in\mathtt{R}_0\,. \tag{1.3}$$

Furthermore, we endow $\mathtt{R}_0$ with the probability measure $\mathbb{P}$ induced by the product measure of the infinite dimensional cube $\mathtt{R}_0$.

We now define the set of *Diophantine* frequencies. The following definition is a slight generalization of the one given by Bourgain in [14]. Here and in the following we denote $\langle j\rangle := \max\{1,|j|\}$.

DEFINITION 1.1. *Given $\gamma\in(0,1)$, $\mu>1$, we denote by $\mathtt{D}_{\gamma,\mu}$ the set of Diophantine frequencies*

$$\mathtt{D}_{\gamma,\mu} := \left\{ \omega\in\mathtt{R}_0 \,:\, |\omega\cdot\ell| > \gamma\prod_{j\in\mathbb{N}}\frac{1}{(1+|\ell_j|^\mu\langle j\rangle^\mu)} \quad \forall \ell\in\mathbb{Z}^{\mathbb{N}} : 0<\sum_{j\in\mathbb{N}}|\ell_j|<\infty \right\}. \tag{1.4}$$

In the following we shall fix $\mu=2$ and denote $\mathtt{D}_\gamma := \mathtt{D}_{\gamma,2}$.

For all $\mu>1$, Diophantine frequencies are *typical* in the set $\mathtt{R}_0$ in the sense of the following measure estimate, proved in [14] (see also [10]).

LEMMA 1.2. *For $\mu>1$ there exists a positive constant $C(\mu)>0$ such that*

$$\mathbb{P}\big(\mathtt{R}_0\setminus\mathtt{D}_{\gamma,\mu}\big) \le C(\mu)\gamma\,.$$

For $\eta>0$, we define the set of infinite integer vectors with *finite support*

$$\mathbb{Z}^\infty_* := \Big\{ \ell\in\mathbb{Z}^{\mathbb{N}} : |\ell|_\eta := \sum_{j\in\mathbb{N}} j^\eta|\ell_j| < \infty \Big\}. \tag{1.5}$$

Note that $\ell_j\neq0$ only for finitely many indices $j\in\mathbb{N}$.

DEFINITION 1.3. *Given $\omega\in\mathtt{D}_\gamma$ and a Banach space $(X,\|\cdot\|_X)$, we say that $F(t):\mathbb{R}\to X$ is almost periodic in time with frequency ω and analytic in the strip $\sigma>0$ if we write it in totally convergent Fourier series*

$$F(t) = \sum_{\ell\in\mathbb{Z}^\infty_*}\widehat{F}(\ell)e^{\mathrm{i}\ell\cdot\omega t} \quad \textit{such that} \quad \widehat{F}(\ell)\in X \;\; \forall\ell\in\mathbb{Z}^\infty_*$$
$$\textit{and} \quad \sum_{\ell\in\mathbb{Z}^\infty_*}\|\widehat{F}(\ell)\|_X e^{\sigma|\ell|_\eta} < \infty.$$

[1]Here and in the following $\mathbb{N}$ does not contain $\{0\}$.

We shall be particularly interested in almost periodic functions where $X = \mathcal{H}(\mathbb{T}_\sigma)$,

$$\mathcal{H}(\mathbb{T}_\sigma) := \Big\{ u = \sum_{n\in\mathbb{Z}} \hat u_n e^{inx}\,,\ \hat u_j \in \mathbb{C}: \quad \|u\|_{\mathcal{H}(\mathbb{T}_\sigma)} := \sum_{n\in\mathbb{Z}} |\hat u_n| e^{\sigma|n|} < \infty \Big\},$$

is the space of analytic functions $\mathbb{T}_\sigma \to \mathbb{C}$, where $\mathbb{T}_\sigma := \{\varphi \in \mathbb{C} : \mathrm{Re}(\varphi) \in \mathbb{T}, \quad |\mathrm{Im}(\varphi)| \le \sigma\}$ is the thickened torus.

Now we are ready to precisely state our main result. We make the following assumptions.

- **(H1)** The functions V_0, V_1, V_2 are almost periodic and analytic, in the sense of Definition 1.3, for $\overline{\sigma} > 0$ and $X = \mathcal{H}(\mathbb{T}_{\overline{\sigma}})$.
- **(H2)** We assume that

$$(1.6)\qquad \begin{aligned} &V_2(x,t) = \overline{V_2(x,t)} \quad \forall (x,t) \in \mathbb{T}\times\mathbb{R}\,,\\ &V_1(x,t) = 2\overline{\partial_x V_2(x,t)} - \overline{V_1(x,t)} \quad \forall (x,t) \in \mathbb{T}\times\mathbb{R}\,,\\ &V_0(t,x) = \overline{V_0(x,t)} - \overline{\partial_x V_1(x,t)} + \overline{\partial_{xx} V_2(x,t)} \quad \forall (x,t) \in \mathbb{T}\times\mathbb{R}\,. \end{aligned}$$

  This implies that the operator $P(t)$ in (1.1) is L^2 self-adjoint for $t \in \mathbb{R}$. Here and in the following we denote by $\mathcal{B}(E,F)$ the space of bounded linear operators from E to F. If $E = F$, we write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$.

Theorem 1.4 (Reducibility). *Let $\overline{\sigma} > 0$ and assume the hypotheses* **(H1)** *and* **(H2)**. *Then there exists a subset $\Omega_\varepsilon \subset \mathtt{R}_0 = [1,2]^{\mathbb{N}}$ satisfying*

$$(1.7)\qquad \lim_{\varepsilon\to 0} \mathbb{P}(\Omega_\varepsilon) = 1$$

such that the following holds. For any $\omega \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $0 < \sigma < \sigma' \le \overline{\sigma}/4$, $\rho > 0$ there exists $\delta = \delta(\sigma, \sigma') \in (0,1)$ such that if $\varepsilon \le \delta$, then there exists a unitary (in $L^2(\mathbb{T})$) operator $W_\infty(t) \equiv W_\infty(t;\omega)$ such that the following hold:

1. *$W_\infty(t), W_\infty(t)^{-1}$ are almost periodic and analytic maps on the strip $\overline{\sigma}/4$ into $X = \mathcal{B}(\mathcal{H}(\mathbb{T}_{\sigma'}), \mathcal{H}(\mathbb{T}_\sigma))$.*
2. *$u(\cdot,t)$ is a solution of the Schrödinger equation (1.1) if and only if $v(\cdot,t) = W_\infty(t)^{-1}[u(\cdot,t)]$ is a solution of the time independent equation*

$$(1.8)\qquad \partial_t v = \mathrm{i}\mathcal{D}_\infty v,$$

   where $\mathcal{D}_\infty$ is a linear, self-adjoint, time independent, 2×2 block-diagonal operator[2] of order two such that the commutator $[\mathcal{D}_\infty, \partial_{xx}] = 0$.
3. *For any $s \ge 0$, the maps $\mathbb{R} \to \mathcal{B}(H^s(\mathbb{T}))$, $t \mapsto W_\infty(t)^{\pm 1}$ are bounded.*

From the theorem stated above, we can deduce the following corollaries.

Corollary 1.5 (Asymptotics of the eigenvalues). *The spectrum of the operator $\mathcal{D}_\infty$ is given by*

$$(1.9)\qquad \begin{aligned} &\mathrm{spec}(\mathcal{D}_\infty) = \{\mu_0(\omega)\} \cup \{\mu_j^{(+)}(\omega), \mu_j^{(-)}(\omega)\}_{j\in\mathbb{N}_0} \subset \mathbb{R}\,,\\ &\mu_j^\sigma(\omega) = \lambda_2 j^2 + \sigma\lambda_1 j + \lambda_0(\omega) + \sigma\frac{\lambda_{-1}(\omega)}{j} + \frac{r_j^\sigma}{j^2}\,,\ j > 0, \end{aligned}$$

where $\lambda_2 - 1$, $\lambda_1 \sim \varepsilon$ do not depend on ω, while $\lambda_0, \lambda_{-1}, r_j^\sigma$ are Lipschitz with respect to ω and of order ε. Finally μ_0 is Lipschitz with respect to ω and of order ε.

[2] We recall that an operator L on a vector space V is $d \times d$ block diagonal if there exists a decomposition of $V = \overline{\oplus V_j}$ such that L maps each V_j in itself and all the V_j have dimension at most d.

For compactness of notation we set $\mu_0^{(+)} = \mu_0^{(-)} = \mu_0$.

COROLLARY 1.6 (Characterization of the Cantor set). *The Cantor set Ω_ε, given in Theorem* 1.4*, is defined explicitly in terms of the spectrum of the block diagonal operator $\mathcal{D}_\infty$. More precisely, it is equal to the set $\Omega_\infty(\gamma)$, $\gamma = \varepsilon^a$, for some $a \in (0,1)$, where*

(1.10)

$$\Omega_\infty(\gamma) := \Big\{ \omega \in \mathtt{D}_\gamma : |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| \geq \frac{2\gamma}{\mathtt{d}(\ell)} \quad \forall (\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0,$$
$$j \neq j', \quad \sigma, \sigma' \in \{+,-\}, \quad |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_j^{(\sigma')}| \geq \frac{2\gamma}{\mathtt{d}(\ell)\langle j \rangle^2}$$
$$\forall (\ell, j) \in (\mathbb{Z}^\infty_* \setminus \{0\}) \times \mathbb{N}_0, \quad \sigma, \sigma' \in \{+,-\} \Big\},$$

where

$$\mathtt{d}(\ell) := \prod_{n \in \mathbb{N}} (1 + |\ell_n|^4 \langle n \rangle^4) \quad \forall \ell \in \mathbb{Z}^\infty_* .$$

COROLLARY 1.7 (Dynamical consequences). *Under the same assumptions of Theorem* 1.4*, the following hold.*

- Analytic stability. *For any $0 < \sigma < \overline{\sigma}/4$, $\rho > 0$, $u_0 \in \mathcal{H}(\mathbb{T}_{\overline{\sigma}})$, the unique solution of the equation* (1.1) *with initial datum $u(x,0) = u_0(x)$ satisfies the estimate $\|u(\cdot,t)\|_{\mathcal{H}(\mathbb{T}_\sigma)} \lesssim_{\sigma,\overline{\sigma}} \|u_0\|_{\mathcal{H}(\mathbb{T}_{\overline{\sigma}})}$ uniformly with respect to $t \in \mathbb{R}$.*
- Sobolev stability. *For any $s \geq 0$, $u_0 \in H^s(\mathbb{T})$, the unique solution of the equation* (1.1) *with initial datum $u(x,0) = u_0(x)$ satisfies the estimate $\|u(\cdot,t)\|_{H^s(\mathbb{T})} \lesssim_s \|u_0\|_{H^s(\mathbb{T})}$ uniformly with respect to $t \in \mathbb{R}$.*

Remark 1.8. By Theorem 1.4, items (1) and (3), one gets boundedness properties of the maps $W_\infty(t)^{\pm 1}$ both on analytic and Sobolev spaces. This is the reason why, in Corollary 1.7, we get a stability result for both analytic and Sobolev initial data; see section 7.

Strategy of the proof. The overall strategy of the proof is the one proposed in [1] and consists of two main steps: a *regularization procedure* and a *KAM reduction scheme*. The aim of the first step is to conjugate (1.1) to a simpler dynamical system where the vector field is space and time independent up to a sufficiently smoothing remainder. Here one uses the fact that the linear operator in (1.1) has a pseudo-differential structure.

In the second step one completes the reduction by applying a KAM scheme, which relies on the fact that the eigenvalues are at most double, with a quantitative control on the differences.

In order to explain the main difficulties that must be overcome in order to deal with almost periodic potentials, let us describe the strategy in more detail.

It is convenient to think of almost periodic in time functions as restrictions of functions on an infinite dimensional torus. To this purpose we define analytic functions of infinitely many angles as the class of totally convergent Fourier series with a prescribed (and very strong) decay on the Fourier coefficients. One may verify that in fact this definition coincides with the set of holomorphic functions on the *thickened torus*

$$\mathbb{T}^\infty_\sigma := \{\varphi = (\varphi_j)_{j \in \mathbb{N}}\,, \quad \varphi_j \in \mathbb{C} \,:\, \mathrm{Re}(\varphi_j) \in \mathbb{T}\,,\ |\mathrm{Im}(\varphi_j)| \leq \sigma \langle j \rangle^\eta \}\,,$$

namely analytic functions such that the *radius of analyticity* of each angle increases as $j \to \infty$. This is quite a strong condition, but it is not at all clear to us whether it may be weakened, even in apparently harmless ways like requiring $|\mathrm{Im}(\varphi_j)| \leq \sigma \log(1+\langle j \rangle)^p$ with $p \gg 1$.

In a nutshell the main novelties are the following: in the regularization step, we need a normal form procedure which uses (1) operators induced by diffeomorphisms of infinite dimensional tori and (2) pseudodifferential operators with symbols depending analytically on $\varphi \in \mathbb{T}^\infty$. This basically requires developing a pseudodifferential calculus for "classical" symbols $a(\varphi, x, \xi)$, with $\varphi \in \mathbb{T}^\infty_\sigma$. On the other hand, in the KAM reducibility scheme the main difficulty is the presence of extremely small divisors, which have to be controlled by shrinking the radius of analyticity appropriately. Let us give a more detailed description of our approach.

In the *regularization* procedure the first step is to reparameterize the x variable ($x \rightsquigarrow x + \beta(x, \omega t)$) in order to remove the space dependence in the leading order term V_2 of (1.1). This induces an invertible linear operator which acts on the dynamical system removing the x dependence from V_2. Here the time behaves as a parameter, so no condition on the time dependence of the potential is needed. Note, however, that this change of variables *mixes* time and space. Namely, if we start with a potential which is analytic in time but only Sobolev in space, after the change of variables it will have finite regularity both in time and in space. For this reason, since we need to preserve analyticity in time throughout our procedure, we require that our potentials are analytic also in space.

In the second step one reparameterizes the variables $\varphi \in \mathbb{T}^\infty_\sigma$ so as to remove the angle dependence in V_2. Here there are various nontrivial points to discuss, both in order to guarantee that the change of variables is well defined and "invertible" and in order to describe the action on analytic functions.

Indeed, even in the much simpler case of a finite number of angles, the regularization procedure is usually performed on C^∞ potentials, and working in the analytic class requires some extra care (see also [23]).

In dealing with infinitely many angles, one uses the fact that ω is Diophantine in the sense of (1.4) as well as the fact that the potentials are analytic with growing radius of analyticity as $j \to \infty$ (see formula (1.11) and the comments following it).

The remaining steps in the regularization procedure do not introduce further problems with respect to the first two steps. As is typical in this kind of result, one could further push the regularization procedure up to an arbitrarily smoothing remainder. We have chosen to regularize our problem up to order -2 because this is the "minimal action" required in order to complete the successive KAM iterative procedure.

An interesting point is that all the regularization steps apart from the first three do not mix the regularity of time and space so that one could work with potentials that are only analytic in time. A simple consequence is that if in (1.1) we assume that V_2 and V_1 are constant in time, then we can require that V_0 has only finite regularity in space (but is still analytic in time).

Since we work with a perturbation which is a differential operator whose coefficients are analytic in both time and space, we cannot apply as a black box the regularization procedure in [5], [35], which is based on Egorov-type theorems and is developed for general pseudo-differential perturbations of class $\mathcal{C}^\infty$. Indeed, developing a general Egorov-type theorem in analytic class does not appear to be a straightforward question (actually the quantitative estimates that we need might not hold true in a general setting).

Therefore, we perform the regularization procedure, in the class of analytic functions, with quantitative estimates; see subsection 3.1 and section 4. The main feature which we exploit is that our perturbation P is a classical pseudodifferential operator (i.e., it admits an expansion in homogeneous symbols of decreasing order).

We remark that in the regularization procedure, one could impose much weaker analyticity conditions. One sees that in fact the only condition needed here is that there exists $\rho > 0$ such that

$$\sup_{\ell \in \mathbb{Z}^\infty_*} \prod_{i \in \mathbb{N}} (1 + \langle i \rangle^2 \ell_i^2) e^{-\rho \sum_j \langle j \rangle^\eta |\ell_j|} < \infty \,. \tag{1.11}$$

If we choose different radii of analyticity, such as

$$\widehat{\mathbb{T}}^\infty_\rho := \{ \varphi = (\varphi_j)_{j \in \mathbb{N}} \,, \quad \varphi_j \in \mathbb{C} \,:\, \mathrm{Re}(\varphi_j) \in \mathbb{T} \,,\ |\mathrm{Im}(\varphi_j)| \leq \rho F(j) \} \,,\ F(j) \geq 1 \,,$$

condition (1.11) becomes

$$\sup_{\ell \in \mathbb{Z}^\infty_*} \prod_{i \in \mathbb{N}} (1 + \langle i \rangle^2 \ell_i^2) e^{-\rho \sum_j |\ell_j| F(j)} < \infty \,,$$

and one can construct many examples where this holds.

In the KAM scheme most difficulties come from quantitative issues, particularly measure estimates. At a purely formal level our scheme is essentially classical. At each step one considers a linear operator of the form $\mathcal{D} + \mathcal{P}(\varphi)$, where $\mathcal{P}$ is very small while $\mathcal{D}$ is time independent and block-diagonal with blocks of dimension at most two. First we introduce an "ultraviolet cut-off" operator, so that $\Pi_N \mathcal{P}$ depends on finitely many angles (depending on N), while the remainder $(\mathrm{Id} - \Pi_N)\mathcal{P}$ is very small.

Then one applies a linear change of variables $e^{\mathcal{F}(\varphi)}$ where $\mathcal{F}$ solves the homological equation

$$-\omega \cdot \partial_\varphi \mathcal{F} + [\mathrm{i}\mathcal{D}, \mathcal{F}] + \Pi_N \mathcal{P} = [\widehat{\mathcal{P}}(0)] \,,$$

where $[\widehat{\mathcal{P}}(0)]$ is the time-independent and block-diagonal part of P.

Direct computations show that (at least at a purely formal level) this change of variables conjugates $\mathcal{D} + \mathcal{P}(\varphi)$ to an operator of the form $\mathcal{D}_+ + \mathcal{P}_+(\varphi)$, where $\mathcal{P}_+(\varphi) \ll \mathcal{P}(\varphi)$. In order to ensure that a solution to the homological equation exists and in order to give quantitative estimates, one restricts ω to a set where the spectrum of the operator

$$L(\varphi) \mapsto -\omega \cdot \partial_\varphi L(\varphi) + [\mathrm{i}\mathcal{D}, L(\varphi)] \tag{1.12}$$

is appropriately bounded from below. Iterating this *KAM step* infinitely many times, one reduces the operator $\mathcal{D} + \mathcal{P}(\varphi)$ for all ω in some implicitly defined set where the condition (1.12) holds throughout the procedure.

The difficult part is to verify that the Melnikov conditions (1.10) are such that

1. the Cantor set $\Omega_\infty(\gamma)$ has positive measure;
2. for all $\omega \in \Omega_\infty(\gamma)$ (1.12) holds at each KAM step with a quantitative control in the solution of the homological equation;
3. the iterative scheme converges.

Here one needs not only for (1.11) to hold for all $\rho > 0$ but also for the supremum in (1.11) to not diverge too badly when $\rho \to 0$. It is here that the special choice of analyticity comes into play, and it is not clear to us if it can be weakened in any significant way.

The paper is organized as follows. In section 2 we state the properties of the analytic functions on the infinite dimensional torus that we need in our proofs. In section 3, we provide some definitions and quantitative estimates for the class of linear operators that we deal with. In particular, in sections 4 and 5 we define the norms that we use and their corresponding properties. In section 4 we show that our equation can be reduced to another one whose vector field is a two-smoothing perturbation of a diagonal one. This is enough to perform the KAM reducibility scheme of section 5. In section 6 we provide the measure estimate of the nonresonant set of parameters $\Omega_\infty(\gamma)$ (see (1.10)), and in section 7 we conclude the proofs of Theorem 1.4 and Corollary 1.7. Finally, in Appendices A and B we collect some technical proofs of some lemmas that we use throughout our proofs.

2. Analytic functions on an infinite dimensional torus. As is habitual in the theory of quasi-periodic functions, we shall study almost periodic functions in the context of analytic functions on an infinite dimensional torus. To this purpose, for $\eta, \sigma > 0$, we define the *thickened* infinite dimensional torus $\mathbb{T}^\infty_\sigma$ as

$$\varphi = (\varphi_j)_{j\in\mathbb{N}}\,, \quad \varphi_j \in \mathbb{C}\,:\, \mathrm{Re}(\varphi_j) \in \mathbb{T}\,,\ |\mathrm{Im}(\varphi_j)| \le \sigma\langle j\rangle^\eta\,.$$

Given a Banach space $(X, \|\cdot\|_X)$ we consider the space $\mathcal{F}$ of pointwise absolutely convergent formal Fourier series $\mathbb{T}^\infty_\sigma \to X$,

$$u(\varphi) = \sum_{\ell\in\mathbb{Z}^\infty_*} \widehat{u}(\ell) e^{\mathrm{i}\ell\cdot\varphi}\,, \quad \widehat{u}(\ell) \in X, \tag{2.1}$$

and define the analytic functions as follows.

Definition 2.1. *Given a Banach space $(X, \|\cdot\|_X)$ and $\sigma > 0$, we define the space of analytic functions $\mathbb{T}^\infty_\sigma \to X$ as the subspace*

$$\mathcal{H}(\mathbb{T}^\infty_\sigma, X) := \Big\{ u(\varphi) = \sum_{\ell\in\mathbb{Z}^\infty_*} \widehat{u}(\ell) e^{\mathrm{i}\ell\cdot\varphi} \in \mathcal{F} \quad : \quad \|u\|_\sigma := \sum_{\ell\in\mathbb{Z}^\infty_*} e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X < \infty \Big\}.$$

In the case $\mathcal{H}(\mathbb{T}^\infty_\sigma, \mathbb{C})$, we shall use the shortened notation $\mathcal{H}(\mathbb{T}^\infty_\sigma)$.

Remark 2.2. We have chosen to work with an infinite torus $\mathbb{T}^\infty_\sigma$ whose angles are φ_j with $j \in \mathbb{N}$ which in our notation does not contain 0. Of course it would be completely equivalent to working on $\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma$ with angles θ_j with $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

To this purpose one just needs to define $\widehat{\mathbb{Z}}^\infty_* := \{k \in \mathbb{Z}^{\mathbb{N}_0} \,:\, |k|_\eta := \sum_{i\in\mathbb{N}_0} \langle i\rangle^\eta |k_i| < \infty\} = \mathbb{Z} \times \mathbb{Z}^\infty_*$ and consider Fourier series

$$u = \sum_{k\in\widehat{\mathbb{Z}}^\infty_*} \widehat{u}(k) e^{\mathrm{i}k\cdot\theta} \quad \text{such that} \sum_{k\in\widehat{\mathbb{Z}}^\infty_*} |\widehat{u}(k)| e^{\sigma|k|_\eta} < \infty.$$

This notation is useful when working with the space $\mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{H}(\mathbb{T}_\sigma))$, which can thus be identified with $\mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma, \mathbb{C}) \equiv \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$. Indeed, $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{H}(\mathbb{T}_\sigma))$ means

$$u = \sum_{\ell\in\mathbb{Z}^\infty_*} \widehat{u}(\ell, x) e^{\mathrm{i}\ell\cdot\varphi} = \sum_{(\ell,n)\in\mathbb{Z}^\infty_*\times\mathbb{Z}} \widehat{u}_n(\ell) e^{\mathrm{i}\ell\cdot\varphi + \mathrm{i}nx} = \sum_{k\in\widehat{\mathbb{Z}}^\infty_*} \widehat{u}(k) e^{\mathrm{i}k\cdot\theta},$$

where $\theta = (x, \varphi) \in \mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma$ and $k = (n, \ell)$.

With this definition, an almost periodic function as in Definition 1.3 is the restriction of a function in $\mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ to $\varphi = \omega t$. Given $\mathcal{F} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ we define $f(t) = \mathcal{F}(\omega t)$. Note that the condition $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ implies that the series in (2.1) is totally convergent for $\varphi \in \mathbb{T}^\infty_\sigma$.

2.1. Reformulation of the reducibility problem. In order to prove Theorem 1.4, we consider analytic φ-dependent families of linear operators $\mathcal{R} : \mathbb{T}^\infty_\sigma \to \mathcal{B}(L^2_0(\mathbb{T}_x))$, $\varphi \mapsto \mathcal{R}(\varphi)$. Recall that the definition of $\mathcal{B}(E)$, for any Banach space E, is given above Theorem 1.4. Given a frequency vector $\omega \in \mathtt{R}_0$ and two operators $\mathcal{L}, \Phi : \mathbb{T}^\infty_\sigma \to \mathcal{B}(L^2_x)$, under the change of coordinates $u = \Phi(\omega t)v$, the dynamical system

$$\partial_t u = \mathcal{L}(\omega t)u$$

transforms into

$$\partial_t v = \mathcal{L}_+(\omega t)u, \quad \mathcal{L}_+(\varphi) \equiv (\Phi_{\omega *})\mathcal{L}(\varphi) := \Phi(\varphi)^{-1}\mathcal{L}(\varphi)\Phi(\varphi) - \Phi(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi(\varphi)\,, \tag{2.2}$$

where[3]

$$\omega \cdot \partial_\varphi \Phi := \sum_{\ell \in \mathbb{Z}^\infty_*} \mathrm{i}(\ell \cdot \omega)\widehat{\Phi}(\ell) e^{\mathrm{i}\ell\cdot\varphi}\,. \tag{2.3}$$

A direct calculation shows that if $\mathcal{L}(\omega t)$ is skew-self-adjoint and $\Phi(\omega t)$ is unitary, then $\mathcal{L}_+(\omega t)$ is skew-self-adjoint too.

In conclusion, our goal is to prove the existence of maps

$$\mathcal{W}, \mathcal{W}^{-1} \in \mathcal{H}(\mathbb{T}^\infty_{\bar\sigma/4}, \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma), \mathcal{H}(\mathbb{T}_{\sigma'}))),$$

such that $W(t) = \mathcal{W}(\omega t)$ and $W(t) = \mathcal{W}^{-1}(\omega t)$, which solve the reduction equation:

$$\mathcal{W}(\varphi)^{-1}\mathrm{i}(\partial_x^2 + \varepsilon\mathcal{P}(\varphi))\mathcal{W}(\varphi) - \mathcal{W}(\varphi)^{-1}\omega\cdot\partial_\varphi\mathcal{W}(\varphi) = \mathrm{i}\mathcal{D}_\infty\,, \tag{2.4}$$

where the operator $\mathcal{P}(\varphi) \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma), \mathcal{H}(\mathbb{T}_{\sigma'})))$ is of the form

$$\mathcal{P}(\varphi) = \mathcal{V}_2(x,\varphi)\partial_x^2 + \mathcal{V}_1(x,\varphi)\partial_x + \mathcal{V}_0(x,\varphi)$$

with $\mathcal{V}_i \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{H}(\mathbb{T}_\sigma))$ and is such that $P(t) = \mathcal{P}(\omega t)$. Note that for $\varphi \in \mathbb{T}^\infty$, $(\partial_x^2 + \varepsilon\mathcal{P}(\varphi))$ is self-adjoint; hence $\mathcal{W}(\varphi)$ is unitary.

We remark that solving (2.4) is equivalent to diagonalizing the linear operator

$$\mathrm{i}\omega\cdot\partial_\varphi + \partial_x^2 + \varepsilon\mathcal{P} \in \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma, \mathbb{C}), \mathcal{H}(\mathbb{T}^\infty_{\sigma'} \times \mathbb{T}_{\sigma'}, \mathbb{C}))$$

via a bounded change of variables with the special property that it is *Toeplitz in time*.

2.2. Properties of analytic functions. We now discuss some fundamental properties of the space $\mathcal{H}(\mathbb{T}^\infty_\sigma, X)$; note that all the results hold verbatim for $\mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma, X)$.

For any function $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$, given $N \geq 0$, we define the projector $\Pi_N u$ as

$$\Pi_N u(\varphi) := \sum_{|\ell|_\eta \leq N} \widehat{u}(\ell) e^{\mathrm{i}\ell\cdot\varphi} \quad \text{and} \quad \Pi_N^\perp u := u - \Pi_N u\,. \tag{2.5}$$

The following lemma holds.

LEMMA 2.3. *Let $\sigma, \rho > 0$, $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$. Then the following holds:*

$$\|\Pi_N^\perp u\|_\sigma \leq e^{-\rho N}\|u\|_{\sigma+\rho}\,.$$

[3]If we set $F(t) = \Phi(\omega t)$, since the series expansion for $t \in \mathbb{R}$ is totally convergent, we have clearly $\partial_t F(t) = \omega \cdot \partial_\varphi \Phi(\omega t)$.

Proof. One has

$$\|\Pi_N^\perp u\|_\sigma = \sum_{|\ell|_\eta > N} e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X \le e^{-\rho N} \sum_{\ell\in\mathbb{Z}_*^\infty} e^{(\sigma+\rho)|\ell|_\eta}\|\widehat{u}(\ell)\|_X,$$

and the lemma follows. ∎

LEMMA 2.4. *Let $\sigma > 0$, $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$. Then $\|u\|_{L^\infty(\mathbb{T}^\infty_\sigma, X)} \le \|u\|_\sigma$.*

Proof. For any $\varphi \in \mathbb{T}^\infty_\sigma$, one has

$$\|u(\varphi)\|_X \le \sum_{\ell\in\mathbb{Z}_*^\infty} \|\widehat{u}(\ell)\|_X e^{\sigma|\ell|_\eta} = \|u\|_\sigma\,. \qquad ∎$$

LEMMA 2.5. *Assume that X is a Banach algebra and $u, v \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$. Then $uv \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ and $\|uv\|_\sigma \le \|u\|_\sigma\|v\|_\sigma$.*

Proof. One has

$$u(\varphi)v(\varphi) = \sum_{\ell,k\in\mathbb{Z}_*^\infty} \widehat{u}(\ell-k)\widehat{v}(k)e^{\mathrm{i}\ell\cdot\varphi},$$

and therefore, one obtains that

$$\|uv\|_\sigma \le \sum_{\ell,k\in\mathbb{Z}_*^\infty} e^{\sigma|\ell|_\eta}\|\widehat{u}(\ell-k)\|_X\|\widehat{v}(k)\|_X\,.$$

Using the triangular inequality $|\ell|_\eta \le |\ell-k|_\eta + |k|_\eta$, one gets $e^{\sigma|\ell|_\eta} \le e^{\sigma|\ell-k|_\eta}e^{\sigma|k|_\eta}$, implying that

$$\|uv\|_\sigma \le \sum_{\ell,k\in\mathbb{Z}_*^\infty} e^{\sigma|\ell-k|_\eta}\|\widehat{u}(\ell-k)\|_X e^{\sigma|k|_\eta}\|\widehat{v}(k)\|_X \le \|u\|_\sigma\|v\|_\sigma\,. \qquad ∎$$

LEMMA 2.6. *Let $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$. Then*

$$\int_{\mathbb{T}^\infty} u(\varphi)\,d\varphi := \lim_{N\to+\infty} \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} u(\varphi)d\varphi_1\dots d\varphi_N = \widehat{u}(0)\,. \tag{2.6}$$

Moreover, for any $\ell \in \mathbb{Z}_*^\infty \setminus \{0\}$,

$$\widehat{u}(\ell) = \int_{\mathbb{T}^\infty} u(\varphi)e^{-\mathrm{i}\ell\cdot\varphi}\,d\varphi = \lim_{N\to\infty}\frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} u(\varphi)e^{-\mathrm{i}\ell\cdot\varphi}\,. \tag{2.7}$$

Proof. Let $\ell \in \mathbb{Z}_*^\infty \setminus \{0\}$, and let $N^\eta \le |\ell|_\eta$. Then surely $\ell_j = 0$ for all $j > N$; thus

$$e^{\mathrm{i}\ell\cdot\varphi} = e^{\mathrm{i}\ell_1\varphi_1}\dots e^{\mathrm{i}\ell_N\varphi_N},$$

implying that

$$\frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} e^{\mathrm{i}\ell\cdot\varphi}\,d\varphi_1\dots d\varphi_N = 0\,.$$

Hence

$$\begin{aligned}\frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} u(\varphi)d\varphi_1\dots d\varphi_N &= \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N}\Big(\widehat{u}(0) + \sum_{0<|\ell|_\eta\le N^\eta}\widehat{u}(\ell)e^{\mathrm{i}\ell\cdot\varphi}\Big)d\varphi_1\dots d\varphi_N \\ &\quad + \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N}\sum_{|\ell|_\eta>N^\eta}\widehat{u}(\ell)e^{\mathrm{i}\ell\cdot\varphi}d\varphi_1\dots d\varphi_N \\ &= \widehat{u}(0) + \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N}\sum_{|\ell|_\eta>N^\eta}\widehat{u}(\ell)e^{\mathrm{i}\ell\cdot\varphi}d\varphi_1\dots d\varphi_N\,.\end{aligned}$$

Since $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$, the tail of the series $\sum_{|\ell|_\eta > N^\eta}$ goes to zero as $N \to \infty$. This proves (2.6).

Now let $\ell \in \mathbb{Z}^\infty_* \setminus \{0\}$. Then we set

$$u_\ell(\varphi) := u(\varphi) e^{-\mathrm{i}\ell\cdot\varphi} = \sum_{k \in \mathbb{Z}^\infty_*} \widehat{u}(k) e^{\mathrm{i}(k-\ell)\cdot\varphi} = \sum_{h \in \mathbb{Z}^\infty_*} \widehat{u}(h+\ell) e^{\mathrm{i}h\cdot\varphi}\,.$$

By applying the claim (2.6) to the function u_ℓ and observing that $\widehat{u}_\ell(0) = \widehat{u}(\ell)$, the equality (2.7) follows. ◻

Given two Banach spaces X and Y, for any $k \in \mathbb{N}$, we define the space $\mathcal{M}_k(X, Y)$ of the k-linear and continuous forms endowed by the norm

$$\|M\|_{\mathcal{M}_k(X,Y)} := \sup_{\|u_1\|_X, \ldots, \|u_k\|_X \le 1} \|M[u_1, \ldots, u_k]\|_Y \quad \forall M \in \mathcal{M}_k(X, Y)\,. \tag{2.8}$$

To shorten notation, we denote $\ell^\infty := \ell^\infty(\mathbb{N}, \mathbb{C})$; moreover, for $k \in \mathbb{N}$, we write $\mathcal{M}_k$ instead of $\mathcal{M}_k(\ell^\infty, X)$ where X is an arbitrary Banach space.

Let us now discuss the differentiability of functions. We define for $\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k \in \ell^\infty$

$$d^k_\varphi u[\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k] := \sum_{\ell \in \mathbb{Z}^\infty_*} \mathrm{i}^k \prod_{j=1}^k (\ell \cdot \hat{\varphi}_j) \widehat{u}(\ell) e^{\mathrm{i}\ell\cdot\varphi}\,. \tag{2.9}$$

Note that if $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$ for any $\rho > 0$, then the series in (2.9) is totally convergent on $\mathbb{T}^\infty_\sigma$.

Lemma 2.7 (Cauchy estimates). *Let $\sigma, \rho > 0$ and $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$. Then for any $k \in \mathbb{N}$, the kth differential $d^k_\varphi u$ satisfies the estimate*

$$\|d^k_\varphi u\|_{\mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{M}_k)} \lesssim_k \rho^{-k} \|u\|_{\sigma+\rho}\,.$$

Proof. For any $k \in \mathbb{N}$, $\varphi \in \mathbb{T}^\infty_\sigma$, $\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k \in \ell^\infty$, $\|\widehat{\varphi}_j\|_\infty \le 1$ for any $j = 1, \ldots, k$, one has by duality $|\ell \cdot \widehat{\varphi}| \le \|\ell\|_1 \|\widehat{\varphi}\|_\infty \le |\ell|_\eta \|\widehat{\varphi}\|_\infty$, and substituting in (2.9) one gets

$$\|d^k_\varphi u(\varphi)[\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k]\|_\sigma \le \sum_{\ell \in \mathbb{Z}^\infty_*} |\ell|^k_\eta e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X \le \sup_{\ell \in \mathbb{Z}^\infty_*} \Big(|\ell|^k_\eta e^{-\rho|\ell|_\eta}\Big) \|u\|_{\sigma+\rho}\,.$$

A straightforward calculation shows that

$$\sup_{\ell \in \mathbb{Z}^\infty_*} |\ell|^k_\eta e^{-\rho|\ell|_\eta} \le \sup_{x \ge 0} x^k e^{-\rho x} = k^k \rho^{-k} e^{-k} \lesssim_k \rho^{-k},$$

which implies the claimed estimate. ◻

Remark 2.8. Note that if we endow the torus $\mathbb{T}^\infty_\sigma$ with the ℓ^∞ metric, namely given two angles $\varphi_1 = (\varphi_{1,j})_{j \in \mathbb{N}} \in \mathbb{T}^\infty_\sigma$ and $\varphi_2 = (\varphi_{2,j})_{j \in \mathbb{N}} \in \mathbb{T}^\infty_\sigma$, we define

$$d_\infty(\varphi_1, \varphi_2) := \sup\nolimits_{j \in \mathbb{N}} \Big(|\mathrm{Re}(\varphi_{1,j} - \varphi_{2,j})|_{\mathrm{mod}\, 2\pi} + |\mathrm{Im}(\varphi_{1,j}) - \mathrm{Im}(\varphi_{2,j})| \Big)\,, \tag{2.10}$$

then (2.9) is the kth differential in the usual sense. Moreover, the tangent space to $\mathbb{T}^\infty_\sigma$ is $\ell^\infty(\mathbb{C})$.

Given a frequency vector $\omega \in \mathtt{R}_0$ and $u \in \mathcal{H}^\sigma(X)$, we define $\omega \cdot \partial_\varphi u$ as in (2.3):

$$\omega \cdot \partial_\varphi u(\varphi) := \sum_{\ell \in \mathbb{Z}^\infty_*} \mathrm{i}(\omega \cdot \ell)\widehat{u}(\ell) e^{\mathrm{i}\ell \cdot \varphi} = du(\varphi)[\omega] \,. \tag{2.11}$$

If we set $f(t) = u(\omega t)$, since the series expansion for $t \in \mathbb{R}$ is totally convergent, we have clearly $\partial_t f(t) = \omega \cdot \partial_\varphi u(\omega t)$.

The following lemma holds.

LEMMA 2.9. *Let $\sigma, \rho > 0$, $u \in \mathcal{H}^{\sigma+\rho}(X)$, $\omega \in \mathtt{R}_0$. Then*

$$\|\omega \cdot \partial_\varphi u\|_\sigma \lesssim \rho^{-1}\|u\|_{\sigma+\rho} \,.$$

Proof. The lemma follows by the formula (2.11) and by applying Lemma 2.7 in a straightforward way. ∎

Parameter dependence. Let Y be a Banach space and $\gamma \in (0,1)$. If $f : \Omega \to Y$, $\Omega \subseteq \mathtt{R}_0 := [1,2]^{\mathbb{N}}$, is a Lipschitz function, we define

$$\begin{aligned} &\|f\|_Y^{\sup} := \sup_{\omega \in \Omega} \|f(\omega)\|_Y, \quad \|f\|_Y^{\mathrm{lip}} := \sup_{\substack{\omega_1,\omega_2 \in \Omega \\ \omega_1 \neq \omega_2}} \frac{\|f(\omega_1) - f(\omega_2)\|_Y}{\|\omega_1 - \omega_2\|_\infty}, \\ &\|f\|_Y^{\mathrm{Lip}(\gamma,\Omega)} := \|f\|_Y^{\sup} + \gamma \|f\|_Y^{\mathrm{lip}} \,. \end{aligned} \tag{2.12}$$

If $Y = \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$, we simply write $\|\cdot\|_\sigma^{\sup}$, $\|\cdot\|_\sigma^{\mathrm{lip}}$, $\|\cdot\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}$. If Y is a finite dimensional space, we write $\|\cdot\|^{\sup}$, $\|\cdot\|^{\mathrm{lip}}$, $\|\cdot\|^{\mathrm{Lip}(\gamma,\Omega)}$.

The following result follows directly.

LEMMA 2.10. *In Lemmas* 2.3, 2.5, 2.7, *and* 2.9, *if $u(\cdot;\omega)$ is Lipschitz with respect to $\omega \in \Omega \subseteq \mathtt{R}_0$, the same estimates hold verbatim replacing $\|\cdot\|_\sigma$ by $\|\cdot\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}$.*

As is typical in KAM reduction schemes, a fundamental tool in reducibility is to solve the "homological equation," i.e., to invert the operator $\omega \cdot \partial_\varphi$.

LEMMA 2.11 (Homological equation). *Let $\sigma, \rho > 0$, $f \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$, $\omega \in \mathtt{D}_{\gamma,\mu}$ (see* (1.4)*), with $\widehat{f}(0) = 0$. Then there exists a unique solution $u := (\omega \cdot \partial_\varphi)^{-1} f \in \mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ of the equation*

$$\omega \cdot \partial_\varphi u = f$$

satisfying the estimates

$$\|u\|_\sigma \lesssim \exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}} \ln\Big(\frac{\tau}{\rho}\Big)\Big)\|f\|_{\sigma+\rho} \tag{2.13}$$

for some constant $\tau = \tau(\eta,\mu) > 0$. If $f(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$ is Lipschitz with respect to $\omega \in \Omega \subseteq \mathtt{D}_\gamma$, then

$$\|u\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}} \ln\Big(\frac{\tau}{\rho}\Big)\Big)\|f\|_{\sigma+\rho}^{\mathrm{Lip}(\gamma,\Omega)}$$

for some constant $\tau(\eta,\mu) > 0$ (eventually larger than the one in (2.13)*).*

Proof. Since $\omega \in \mathtt{D}_\gamma$, the solution u of the equation $\omega \cdot \partial_\varphi u = f$ is given by

$$u(\varphi) = (\omega \cdot \partial_\varphi)^{-1} f(\varphi) = \sum_{\ell \in \mathbb{Z}^\infty_* \setminus \{0\}} \frac{\widehat{f}(\ell)}{\mathrm{i}\omega \cdot \ell} e^{\mathrm{i}\ell \cdot \varphi} \,.$$

Hence, using that $\omega \in \mathtt{D}_{\gamma,\mu}$, we obtain

$$\begin{aligned}\|u\|_\sigma &\le \gamma^{-1} \sum_{\ell\in\mathbb{Z}^\infty_*\setminus\{0\}} \prod_i (1+\langle i\rangle^\mu |\ell_i|^\mu)\|\widehat f(\ell)\|_X e^{\sigma|\ell|_\eta} \\ &\le \gamma^{-1} \sup_{\ell\in\mathbb{Z}^\infty_*} \Big(e^{-\rho|\ell|_\eta} \prod_i (1+\langle i\rangle^\mu |\ell_i|^\mu)\Big)\|f\|_{\sigma+\rho},\end{aligned}$$

and the claimed estimate follows by applying Lemma B.1(i). ◻

We conclude this section by discussing how the definition of $\mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ (or equivalently $\mathcal{H}(\mathbb{T}^\infty_\sigma \times \mathbb{T}_\sigma, X)$) depends on the coordinates on $\mathbb{T}^\infty_\sigma$.

DEFINITION 2.12. *Recall that $\ell^\infty := \ell^\infty(\mathbb{N},\mathbb{C})$. We say that a function $a \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho})$ is real on real if $a(\varphi) \in \mathbb{R}$ for all $\varphi \in \mathbb{T}^\infty$. Similarly, $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$ is real on real if $\alpha_j(\varphi) \in \mathbb{R}$ for all $\varphi \in \mathbb{T}^\infty, j \in \mathbb{N}$.*

PROPOSITION 2.13 (Torus diffeomorphism). *Let $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$ be real on real. Then there exists $\varepsilon = \varepsilon(\rho)$ such that if $\|\alpha\|_{\sigma+\rho} \le \varepsilon$, then the map $\varphi \mapsto \varphi + \alpha(\varphi)$ is an invertible diffeomorphism of the infinite dimensional torus $\mathbb{T}^\infty_\sigma$ (with respect to the ℓ^∞-topology) and its inverse is given by the map $\vartheta \mapsto \vartheta + \widetilde\alpha(\vartheta)$, where $\widetilde\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, \ell^\infty)$ is real on real and satisfies the estimate $\|\widetilde\alpha\|_{\sigma+\frac{\rho}{2}} \lesssim \|\alpha\|_{\sigma+\rho}$. Furthermore, if $\alpha(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$ is Lipschitz with respect to $\omega \in \Omega \subseteq \mathtt{R}_0$, then $\|\widetilde\alpha\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\frac{\rho}{2}} \lesssim \|\alpha\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho}$.*

COROLLARY 2.14. *Given $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$ as in Proposition 2.13, the operators*

$$\begin{aligned}&\Phi_\alpha : \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X) \to \mathcal{H}(\mathbb{T}^\infty_\sigma, X), \quad u(\varphi) \mapsto u(\varphi+\alpha(\varphi)),\\ &\Phi_{\tilde\alpha} : \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, X) \to \mathcal{H}(\mathbb{T}^\infty_\sigma, X), \quad u(\vartheta) \mapsto u(\vartheta+\tilde\alpha(\vartheta))\end{aligned} \tag{2.14}$$

are bounded and satisfy

$$\|\Phi_\alpha\|_{\mathcal{B}(\mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho},X),\mathcal{H}(\mathbb{T}^\infty_\sigma,X))}, \|\Phi_{\tilde\alpha}\|_{\mathcal{B}(\mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho},X),\mathcal{H}(\mathbb{T}^\infty_\sigma,X))} \le 1,$$

and for any $\varphi \in \mathbb{T}^\infty_\sigma$, $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X), v \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, X)$ one has

$$\Phi_{\tilde\alpha} \circ \Phi_\alpha u(\varphi) = u(\varphi), \quad \Phi_\alpha \circ \Phi_{\tilde\alpha} v(\varphi) = u(\varphi).$$

In order to prove our result, we shall proceed in steps, proving a series of technical lemmas.

LEMMA 2.15. *For $\sigma, \rho > 0$, let $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X)$ and $\alpha \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \ell^\infty)$ with $\|\alpha\|_\sigma \le \rho$. Then the function $f(\varphi) := u(\varphi+\alpha(\varphi))$ belongs to the space $\mathcal{H}(\mathbb{T}^\infty_\sigma, X)$ and $\|f\|_\sigma \le \|u\|_{\sigma+\rho}$. As a consequence, the linear operator*

$$\Phi_\alpha : \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, X) \to \mathcal{H}(\mathbb{T}^\infty_\sigma, X), \quad u(\varphi) \mapsto u(\varphi+\alpha(\varphi))$$

is bounded and satisfies $\|\Phi_\alpha\|_{\mathcal{B}(\mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho},X),\mathcal{H}(\mathbb{T}^\infty_\sigma,X))} \le 1$.

Proof. One has that

$$f(\varphi) = \sum_{\ell\in\mathbb{Z}^\infty_*} \widehat u(\ell) e^{\mathrm{i}\ell\cdot\varphi} e^{\mathrm{i}\ell\cdot\alpha(\varphi)}. \tag{2.15}$$

Moreover, for any $\ell \in \mathbb{Z}^\infty_*$, one has

$$\begin{aligned}e^{\mathrm{i}\ell\cdot\alpha(\varphi)} &= \sum_{n\in\mathbb{N}} \frac{\mathrm{i}^n}{n!}(\ell\cdot\alpha(\varphi))^n \\ &= \sum_{n\in\mathbb{N}} \sum_{\ell_1,\dots,\ell_n\in\mathbb{Z}^\infty_*} \frac{\mathrm{i}^n}{n!}(\ell\cdot\widehat\alpha(\ell_1))\dots(\ell\cdot\widehat\alpha(\ell_n)) e^{\mathrm{i}(\ell_1+\dots+\ell_n)\cdot\varphi}.\end{aligned} \tag{2.16}$$

By the formulae (2.15) and (2.16) one then gets that

$$
\begin{aligned}
f(\varphi) &= \sum_{k\in\mathbb{Z}^\infty_*} \widehat{f}(k) e^{\mathrm{i}k\cdot\varphi}, \\
\widehat{f}(k) &:= \sum_{n\in\mathbb{N}} \frac{\mathrm{i}^n}{n!} \sum_{\ell+\ell_1+\ldots+\ell_n=k} (\ell\cdot\widehat{\alpha}(\ell_1))\ldots(\ell\cdot\widehat{\alpha}(\ell_n))\widehat{u}(\ell)\,.
\end{aligned}
\tag{2.17}
$$

Using that for $k = \ell + \ell_1 + \cdots + \ell_n$ one has that $e^{\sigma|k|_\eta} \le e^{\sigma|\ell|_\eta} e^{\sigma|\ell_1|_\eta} \ldots e^{\sigma|\ell_n|_\eta}$ and $|(\ell\cdot\widehat{\alpha}(\ell_i))| \le \|\ell\|_1 \|\widehat{\alpha}(\ell_i)\|_\infty$, one gets that

(2.18)

$$
\begin{aligned}
\|f\|_\sigma &= \sum_{k\in\mathbb{Z}^\infty_*} e^{\sigma|k|_\eta} \|\widehat{f}(k)\|_X \\
&\le \sum_{n\in\mathbb{N}} \frac{1}{n!} \sum_{\ell,\ell_1,\ldots,\ell_n\in\mathbb{Z}^\infty_*} (\|\ell\|_1)^n e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X e^{\sigma|\ell_1|_\eta} \|\widehat{\alpha}(\ell_1)\|_\infty \ldots e^{\sigma|\ell_n|_\eta} \|\widehat{\alpha}(\ell_n)\|_\infty \\
&\overset{\|\ell\|_1\le|\ell|_\eta}{\le} \sum_{\ell\in\mathbb{Z}^\infty_*} e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X \sum_{n\in\mathbb{N}} \frac{|\ell|^n_\eta}{n!} \prod_{j=0}^n \sum_{\ell_j\in\mathbb{Z}^\infty_*} e^{\sigma|\ell_j|_\eta} \|\widehat{\alpha}(\ell_j)\|_\infty \\
&\le \sum_{\ell\in\mathbb{Z}^\infty_*} e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X \sum_{n\in\mathbb{N}} \frac{|\ell|^n_\eta \|\alpha\|^n_\sigma}{n!} \\
&\le \sum_{\ell\in\mathbb{Z}^\infty_*} e^{\sigma|\ell|_\eta} \|\widehat{u}(\ell)\|_X \exp\Big(|\ell|_\eta \|\alpha\|_\sigma\Big) \\
&\overset{\|\alpha\|_\sigma\le\rho}{\le} \sum_{\ell\in\mathbb{Z}^\infty_*} e^{(\sigma+\rho)|\ell|_\eta} \|\widehat{u}(\ell)\|_X = \|u\|_{\sigma+\rho}.
\end{aligned}
$$

∎

For $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$ we now consider the map

$$
\Psi_\alpha(u)(\varphi) := -\alpha(\varphi + u(\varphi)),
\tag{2.19}
$$

which, by Lemma 2.15 (with $\sigma \rightsquigarrow \sigma + \frac{\rho}{2}$ and $\rho \rightsquigarrow \frac{\rho}{2}$) is well defined, and $\mathcal{B}_{\sigma+\frac{\rho}{2}}(0,R) \to \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, \ell^\infty)$, where

$$
u \in \mathcal{B}_\sigma(0,R) := \Big\{ u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \ell^\infty) : \|u\|_\sigma \le R \Big\}
$$

provided $R < \frac{\rho}{2}$.

LEMMA 2.16. *Let $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$. Then there exists $\varepsilon = \varepsilon(\rho)$ such that if $\|\alpha\|_{\sigma+\rho} \le \varepsilon$, there exists a unique solution $u \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, \ell^\infty)$ of the fixed point equation $u = \Psi_\alpha(u)$ satisfying the estimate $\|u\|_{\sigma+\frac{\rho}{2}} \le \|\alpha\|_{\sigma+\rho}$. If $\alpha(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \ell^\infty)$, $\omega \in \Omega \subseteq \mathtt{R}_0 = [1,2]^\mathbb{N}$ is Lipschitz, then $\|u\|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma \lesssim \|\alpha\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho}$.*

Proof. To start with we show the following claim.

- *Claim.* There exist $\varepsilon = \varepsilon(\rho)$, $R = R(\rho) > 0$ such that if $\|\alpha\|_{\sigma+\rho} \le \varepsilon$, then the map (2.19) is a contraction on

$$
\mathcal{B}_\sigma(0,R) := \Big\{ u \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \ell^\infty) : \|u\|_\sigma \le R \Big\}\,.
$$

Proof of the claim. By taking $R = R(\rho)$ sufficiently small, by applying Lemma 2.15, one gets that for any $u \in \mathcal{B}_{\sigma+\frac{\rho}{2}}(0, R)$, $\Psi_\alpha(u) \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, \ell^\infty)$ and $\|\Psi_\alpha(u)\|_{\sigma+\frac{\rho}{2}} \leq \|\alpha\|_{\sigma+\rho}$. Then, if $\|\alpha\|_{\sigma+\rho} \leq \varepsilon \leq R$, one has that $\Psi_\alpha : \mathcal{B}_{\sigma+\frac{\rho}{2}}(0, R) \to \mathcal{B}_{\sigma+\frac{\rho}{2}}(0, R)$. Now, given $u_1, u_2 \in \mathcal{B}_{\sigma+\frac{\rho}{2}}(0, R)$, we want to bound $\|\Psi_\alpha(u_1) - \Psi_\alpha(u_2)\|_\sigma$. By the mean value theorem, one has

$$\Psi_\alpha(u_1) - \Psi_\alpha(u_2) = \int_0^1 d_\varphi \alpha\Big(\varphi + t u_1(\varphi) + (1-t) u_2(\varphi)\Big)[u_2 - u_1]\, dt. \tag{2.20}$$

Since $\|u_1\|_{\sigma+\frac{\rho}{2}}, \|u_2\|_{\sigma+\frac{\rho}{2}} \leq R$, by taking $R \leq \frac{\rho}{4}$, by Lemmas 2.7 and 2.15 one has the estimate

$$\begin{aligned} \|\Psi_\alpha(u_1) - \Psi_\alpha(u_2)\|_{\sigma+\frac{\rho}{2}} &\leq \|d_\varphi \alpha\|_{\mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{3\rho}{2}}, \mathcal{M}_1)} \|u_1 - u_2\|_{\sigma+\frac{\rho}{2}} \\ &\lesssim \rho^{-1} \|\alpha\|_{\sigma+\rho} \|u_1 - u_2\|_{\sigma+\frac{\rho}{2}}\,. \end{aligned} \tag{2.21}$$

Hence by taking $\|\alpha\|_{\sigma+\rho} \leq \varepsilon(\rho)$ small enough, one gets that the map Ψ_α is a contraction and by recalling Lemma 2.15 the unique solution of the fixed point equation satisfies $\|u\|_{\sigma+\frac{\rho}{2}} \leq \|\alpha\|_{\sigma+\rho}$. Now assume that $\alpha(\cdot;\omega)$, $\omega \in \Omega$ is Lipschitz with respect to ω. Recalling (2.19) and using the fixed point equation $u = \Psi_\alpha(u)$, one computes for any $\omega_1, \omega_2 \in \Omega$

$$\begin{aligned} \Delta_{\omega_1\omega_2} u(\varphi) &= \alpha(\varphi + u(\varphi;\omega_1);\omega_1) - \alpha(\varphi + u(\varphi;\omega_2);\omega_2) \\ &= \alpha(\varphi + u(\varphi;\omega_1);\omega_1) - \alpha(\varphi + u(\varphi;\omega_1);\omega_2) \\ &\quad + \alpha(\varphi + u(\varphi;\omega_1);\omega_2) - \alpha(\varphi + u(\varphi;\omega_2);\omega_2)\,. \end{aligned}$$

By taking $R = R(\rho)$ small enough, using the mean value theorem, the Cauchy estimates of Lemma 2.7, and the composition Lemma 2.15, one gets

$$\|\Delta_{\omega_1\omega_2} u\|_{\sigma+\frac{\rho}{2}} \leq \|\Delta_{\omega_1\omega_2}\alpha\|_{\sigma+\rho} + C(\rho) \sup_{\omega\in\Omega} \|\alpha(\cdot;\omega)\|_{\sigma+\rho} \|\Delta_{\omega_1\omega_2} u\|_{\sigma+\frac{\rho}{2}}\,.$$

Hence, taking $C(\rho)\sup_{\omega\in\Omega}\|\alpha(\cdot;\omega)\|_{\sigma+\rho} \leq \frac{1}{2}$, one gets $\|\Delta_{\omega_1\omega_2} u\|_{\sigma+\frac{\rho}{2}} \leq 2\|\Delta_{\omega_1\omega_2}\alpha\|_{\sigma+\rho}$, and the claimed Lipschitz estimate follows. ∎

Proof of Proposition 2.13. Clearly the map $\varphi \mapsto \varphi + \alpha(\varphi)$ is invertible by taking $\|\alpha\|_{\sigma+\rho} \leq \varepsilon$ small enough. By applying Lemma 2.16 there exists a unique $\widetilde{\alpha} \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}, \ell^\infty)$ with $\|\widetilde{\alpha}\|_{\sigma+\frac{\rho}{2}} \lesssim \|\alpha\|_{\sigma+\rho}$ satisfying the equation

$$\widetilde{\alpha}(\vartheta) + \alpha(\vartheta + \widetilde{\alpha}(\vartheta)) = 0$$

for $\vartheta \in \mathbb{T}^\infty_{\sigma+\frac{\rho}{2}}$. The same holds exchanging $\vartheta \rightsquigarrow \varphi$ and $\alpha \rightsquigarrow \widetilde{\alpha}$ for $\varphi \in \mathbb{T}^\infty_\sigma$. Hence $\vartheta \mapsto \vartheta + \widetilde{\alpha}(\vartheta)$ is the inverse of $\varphi \mapsto \varphi + \alpha(\varphi)$ and vice versa, and the proof is concluded. ∎

3. Linear operators.

Given a linear operator $\mathcal{R} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, we identify it with its matrix representation $(\mathcal{R}_k^{k'})_{k,k'\in\mathbb{Z}}$ with respect to the exponential basis where

$$\mathcal{R}_k^{k'} := \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{R}[e^{\mathrm{i}k'x}]e^{-\mathrm{i}kx}\, dx.$$

Clearly, given $\mathcal{R}$ as above, the adjoint with respect to the standard hermitian product in $L^2(\mathbb{C})$ is given by

$$(\mathcal{R}^*)_k^{k'} = \overline{\mathcal{R}}_{k'}^{k}\,. \tag{3.1}$$

We may also give a block-matrix decomposition by grouping together the matrix-Fourier indices with the same absolute values. More precisely, we define for any $j \in \mathbb{N}_0$ the space $\mathbf{E}_j$ as

$$\mathbf{E}_0 := \mathrm{span}\{1\}\,, \qquad \mathbf{E}_j := \mathrm{span}\{e^{\mathrm{i}jx}, e^{-\mathrm{i}jx}\} \quad \forall j \in \mathbb{N}, \tag{3.2}$$

and we define the corresponding projection operator π_j as

$$\begin{aligned} &\pi_0 : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\\ &u(x) = \sum_{j\in\mathbb{Z}} \widehat{u}(j)e^{\mathrm{i}jx} \mapsto \pi_0 u(x) := \widehat{u}(0),\\ &\pi_j : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\\ &u(x) = \sum_{j\in\mathbb{Z}} \widehat{u}(j)e^{\mathrm{i}jx} \mapsto \pi_j u(x) := \widehat{u}(j)e^{\mathrm{i}jx} + \widehat{u}(-j)e^{-\mathrm{i}jx}\,, \quad j \in \mathbb{N}\,. \end{aligned} \tag{3.3}$$

The following properties follow directly from (3.2) and (3.3):

$$\begin{aligned} &\pi_j^2 = \pi_j \quad \forall j \in \mathbb{N}_0, \quad \pi_j \pi_{j'} = 0 \quad \forall j \neq j' \in \mathbb{N}_0,\\ &\sum_{j\in\mathbb{N}_0} \pi_j = \mathrm{Id}, \quad L^2(\mathbb{T}) = \oplus_{j\in\mathbb{N}_0} \mathbf{E}_j\,. \end{aligned} \tag{3.4}$$

Hence, any linear operator $\mathcal{R} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ can be written in 2×2 *block*-decomposition

$$\mathcal{R} = \sum_{j,j'\in\mathbb{N}_0} \pi_j \mathcal{R} \pi_{j'}, \tag{3.5}$$

where $j, j' \in \mathbb{N}_0$, and the operator $\pi_j \mathcal{R} \pi_{j'}$ is a linear operator in $\mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$. If $j, j' \in \mathbb{N}$, the operator $\pi_j \mathcal{R} \pi_{j'}$ can be identified with the 2×2 matrix defined by

$$\begin{pmatrix} \mathcal{R}_j^{j'} & \mathcal{R}_j^{-j'} \\ \mathcal{R}_{-j}^{j'} & \mathcal{R}_{-j}^{-j'} \end{pmatrix}. \tag{3.6}$$

The action of any linear operator $M \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$, $j, j' \in \mathbb{N}$, is given by

$$\begin{aligned} &Mu(x) = \sum_{\substack{k=\pm j\\ k'=\pm j'}} M_k^{k'} \widehat{u}(k') e^{\mathrm{i}kx}\\ &\qquad\qquad \forall u \in \mathbf{E}_{j'}\,, \quad u(x) = \widehat{u}(j')e^{\mathrm{i}j'x} + \widehat{u}(-j')e^{-\mathrm{i}j'x}\,. \end{aligned} \tag{3.7}$$

The operator $\pi_0 \mathcal{R} \pi_0 \in \mathcal{B}(\mathbf{E}_0)$ is identified with the multiplication operator by the matrix element $\mathcal{R}_0^0$, and if $j, j' \in \mathbb{N}$, the operators $\pi_j \mathcal{R} \pi_0$, $\pi_0 \mathcal{R} \pi_j$ are identified with the vectors

$$\begin{pmatrix} \mathcal{R}_j^0 \\ \mathcal{R}_{-j}^0 \end{pmatrix} \quad \text{and} \quad \big(\mathcal{R}_0^{j'}, \mathcal{R}_0^{-j'}\big).$$

We denote by $[\mathcal{R}]$ the block-diagonal part of the operator $\mathcal{R}$, namely

$$[\mathcal{R}] := \sum\nolimits_{j\in\mathbb{N}_0} \pi_j \mathcal{R} \pi_j\,. \tag{3.8}$$

If $\pi_j \mathcal{R} \pi_{j'} = 0$ for any $j \neq j'$, we have $\mathcal{R} = [\mathcal{R}]$, and we refer to such operators as 2×2 block-diagonal operators. Note that for any $j, j' \in \mathbb{N}_0$, the adjoint operator $M^* \in \mathcal{B}(\mathbf{E}_j, \mathbf{E}_{j'})$ is thus defined as[4]

$$(M^*)_k^{k'} := \overline{M_{k'}^k}\,. \tag{3.9}$$

We denote by $\mathcal{S}(\mathbf{E}_j)$ the space of self-adjoint matrices in $\mathcal{B}(\mathbf{E}_j)$.

For any $j, j' \in \mathbb{N}_0$, we endow $\mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ with the *Hilbert–Schmidt* norm

$$\|X\|_{\mathtt{HS}} := \sqrt{\operatorname{Tr}(XX^*)} = \Big(\sum_{\substack{|k|=j \\ |k'|=j'}} |X_k^{k'}|^2 \Big)^{\frac{1}{2}}. \tag{3.10}$$

For any $\sigma > 0$, $m \in \mathbb{R}$ we define the class of linear operators of order m (densely defined on $L^2(\mathbb{T})$) $\mathcal{B}^{\sigma,m}$ as

$$\begin{aligned} &\mathcal{B}^{\sigma,m} := \Big\{ \mathcal{R} : L^2(\mathbb{T}) \to L^2(\mathbb{T}) : \|\mathcal{R}\|_{\mathcal{B}^{\sigma,m}} < \infty \Big\}, \quad \text{where} \\ &\|\mathcal{R}\|_{\mathcal{B}^{\sigma,m}} := \sup_{j' \in \mathbb{N}_0} \sum_{j \in \mathbb{N}_0} e^{\sigma|j-j'|} \|\pi_j \mathcal{R} \pi_{j'}\|_{\mathtt{HS}} \langle j' \rangle^{-m}. \end{aligned} \tag{3.11}$$

The following monotonicity properties hold:

$$\|\mathcal{R}\|_{\mathcal{B}^{\sigma,m}} \leq \|\mathcal{R}\|_{\mathcal{B}^{\sigma',m}}, \quad \sigma < \sigma', \quad \|\mathcal{R}\|_{\mathcal{B}^{\sigma,m}} \leq \|\mathcal{R}\|_{\mathcal{B}^{\sigma,m'}}, \quad m' \leq m. \tag{3.12}$$

As a notation, if $m = 0$, we write $\mathcal{B}^\sigma$ instead of $\mathcal{B}^{\sigma,0}$. Note that a direct consequence of the definition is that if $\mathcal{R} \in \mathcal{B}^{\sigma,m}$, then (recall that $D = -\mathrm{i}\partial_x$)

$$\|\mathcal{R}\|_{\mathcal{B}^{\sigma,m}} = \|\mathcal{R}\langle D\rangle^{-m}\|_{\mathcal{B}^\sigma}, \tag{3.13}$$

where, for any $\alpha \in \mathbb{R}$, the diagonal operator $\langle D\rangle^\alpha$ is defined by

$$\langle D\rangle^\alpha u(x) := \sum_{j \in \mathbb{Z}} \langle j\rangle^\alpha \widehat{u}(j) e^{\mathrm{i}jx}.$$

Note that $\mathcal{B}^\sigma$ is contained in the set of bounded linear operators $\mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma))$ as shown in the following.

LEMMA 3.1. *Let $\sigma > 0$ and $\Phi \in \mathcal{B}^\sigma$. Then*

(i) $\|\Phi\|_{\mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma))} \leq \|\Phi\|_{\mathcal{B}^\sigma}$.

(ii) *For any $s \geq 0$, $\|\Phi\|_{\mathcal{B}(H^s(\mathbb{T}))} \lesssim_s \sigma^{-s}\|\Phi\|_{\mathcal{B}^\sigma}$.*

Proof. Proof of (i). Let $\Phi \in \mathcal{B}^\sigma$. According to (3.3) and (3.5), for $u \in \mathcal{H}(\mathbb{T}_\sigma)$, set $\Phi u(x) = \sum_{j,j' \in \mathbb{N}_0} \pi_j \Phi \pi_{j'}[\pi_{j'} u]$. Then, using that for any $j, j' \in \mathbb{N}_0$, $e^{\sigma|j|} \leq e^{\sigma|j-j'|} e^{\sigma|j'|}$, one gets the chain of inequalities

$$\begin{aligned} \|\Phi u\|_\sigma &= \sum_{j \in \mathbb{N}_0} e^{\sigma|j|} \Big\| \sum_{j' \in \mathbb{N}_0} \pi_j \Phi \pi_{j'}[\pi_{j'} u] \Big\|_{L^2} \\ &\leq \sum_{j' \in \mathbb{N}_0} e^{\sigma|j'|} \|\pi_{j'} u\|_{L^2} \Big(\sum_{j \in \mathbb{N}_0} e^{\sigma|j-j'|} \|\pi_j \Phi \pi_{j'}\|_{\mathtt{HS}} \Big) \\ &\leq \sup_{j' \in \mathbb{N}_0} \Big(\sum_{j \in \mathbb{N}_0} e^{\sigma|j-j'|} \|\pi_j \Phi \pi_{j'}\|_{\mathtt{HS}} \Big) \|u\|_\sigma \overset{(3.11)}{\leq} \|\Phi\|_{\mathcal{B}^\sigma} \|u\|_\sigma. \end{aligned}$$

---

[4]If $j, j' \in \mathbb{N}$, $A \in \mathcal{B}(\mathbf{E}_0)$, $B \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_0)$, $C \in \mathcal{B}(\mathbf{E}_0, \mathbf{E}_j)$, then

$$(A^*)_0^0 := \overline{A_0^0}, \quad (B^*)_k^0 = \overline{B_0^k}, \quad k = \pm j', \quad (C^*)_0^k = \overline{C_k^0}, \quad k = \pm j.$$

Proof of (ii). Let $s \geq 0$ and $u \in H^s(\mathbb{T})$. Then, using that for any $j, j' \in \mathbb{N}_0$, $\langle j \rangle \lesssim \langle j' \rangle + \langle j - j' \rangle \lesssim \langle j' \rangle \langle j - j' \rangle$, one gets that

$$\begin{aligned} \|\Phi u\|_{H^s}^2 &= \sum_{j \in \mathbb{N}_0} \langle j \rangle^{2s} \Big\| \sum_{j' \in \mathbb{N}_0} \pi_j \Phi \pi_{j'} [\pi_{j'} u] \Big\|_{L^2}^2 \leq \sum_{j \in \mathbb{N}_0} \Big\| \sum_{j' \in \mathbb{N}_0} \langle j \rangle^s \pi_j \Phi \pi_{j'} [\pi_{j'} u] \Big\|_{L^2}^2 \\ &\lesssim_s \sum_{j \in \mathbb{N}_0} \Big(\sum_{j' \in \mathbb{N}_0} \langle j' \rangle^s \langle j - j' \rangle^s \|\pi_j \Phi \pi_{j'}\|_{\mathtt{HS}} \|\pi_{j'} u\|_{L^2} \Big)^2 . \end{aligned}$$

Moreover, by using the Cauchy–Schwarz inequality, one gets

$$\begin{aligned} \|\Phi u\|_{H^s}^2 &\lesssim_s \sum_{j' \in \mathbb{N}_0} \langle j' \rangle^{2s} \|\pi_{j'} u\|_{L^2}^2 \sum_{j \in \mathbb{N}_0} \langle j - j' \rangle^{2(s+1)} \|\pi_j \Phi \pi_{j'}\|_{\mathtt{HS}}^2 \\ &\overset{(3.11)}{\lesssim_s} \sup_{k \in \mathbb{N}_0} \langle k \rangle^{2(s+1)} e^{-\sigma |k|} \|\Phi\|_{\mathcal{B}^\sigma} \|u\|_{H^s} \lesssim_s \sigma^{-s} \|\Phi\|_{\mathcal{B}^\sigma} \|u\|_{H^s} , \end{aligned}$$

which proves the claimed estimate. ∎

3.1. Toeplitz in time linear operators. We now consider φ-dependent families of linear operators on $L^2(\mathbb{T})$, i.e., absolutely convergent Fourier series $\mathbb{T}^\infty_\sigma \to L^2(\mathbb{T})$.

DEFINITION 3.2. *For $\sigma > 0$, $m \in \mathbb{R}$, we consider $\mathcal{R} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m})$. We define the decay norm*

$$|\mathcal{R}|_{\sigma,m} := \sum_{\ell \in \mathbb{Z}^\infty_*} e^{\sigma |\ell|_\eta} \|\widehat{\mathcal{R}}(\ell)\|_{\mathcal{B}^{\sigma,m}} . \tag{3.14}$$

Moreover, given $\gamma \in (0,1)$ and if $\mathcal{R} = \mathcal{R}(\varphi; \omega)$ depends on the parameter $\omega \in \Omega$, we define

$$\begin{aligned} |\mathcal{R}|_{\sigma,m}^{\mathrm{Lip}(\gamma,\Omega)} &:= \sup_{\omega \in \Omega} |\mathcal{R}(\omega)|_{\sigma,m} + \gamma |\mathcal{R}|_{\sigma,m+2}^{\mathrm{lip}} , \\ |\mathcal{R}|_{\sigma,m+2}^{\mathrm{lip}} &:= \sup_{\substack{\omega_1,\omega_2 \in \Omega \\ \omega_1 \neq \omega_2}} \frac{|\mathcal{R}(\omega_1) - \mathcal{R}(\omega_2)|_{\sigma,m+2}}{\|\omega_1 - \omega_2\|_\infty} . \end{aligned} \tag{3.15}$$

If $m = 0$, we write $|\cdot|_\sigma$ instead of $|\cdot|_{\sigma,m}$. By recalling (3.12), one can easily see that the following properties hold:

$$\begin{aligned} &|\cdot|_{\sigma,m} \leq |\cdot|_{\sigma',m} , \quad |\cdot|_{\sigma,m}^{\mathrm{Lip}(\gamma,\Omega)} \leq |\cdot|_{\sigma',m}^{\mathrm{Lip}(\gamma,\Omega)} \quad \forall \sigma \leq \sigma' , \\ &|\cdot|_{\sigma,m} \leq |\cdot|_{\sigma,m'} , \quad |\cdot|_{\sigma,m}^{\mathrm{Lip}(\gamma,\Omega)} \leq |\cdot|_{\sigma,m'}^{\mathrm{Lip}(\gamma,\Omega)} \quad \forall m' \leq m . \end{aligned} \tag{3.16}$$

DEFINITION 3.3. *We say that $\mathcal{R} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m})$ is self-adjoint (resp., skew-self-adjoint or unitary) if for all $\varphi \in \mathbb{T}^\infty$, the operator $\mathcal{R}(\varphi) \in \mathcal{B}^{\sigma,m}$ is self-adjoint (resp., skew-self-adjoint or unitary).*

We now state some standard properties of linear operators in $\mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m})$.

LEMMA 3.4. *Let $\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma \to \mathbb{C}$, $(x,\varphi) \mapsto a(x,\varphi)$ be in $\mathcal{H}(\mathbb{T}_{\sigma+\rho} \times \mathbb{T}^\infty_{\sigma+\rho})$. Then the multiplication operator $\mathcal{M}_a : u \mapsto au$ satisfies $|\mathcal{M}_a|_\sigma \lesssim \rho^{-1} \|a\|_{\sigma+\rho}$. If $a(x,\varphi;\omega)$, $\omega \in \Omega \subseteq \mathtt{R}_0$ is Lipschitz with respect to ω, then $|\mathcal{M}_a|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \rho^{-1} \|a\|_{\sigma+\rho}^{\mathrm{Lip}(\gamma,\Omega)}$.*

LEMMA 3.5. *Let* $N, \sigma, \rho > 0$, $m, m' \in \mathbb{R}$, $\mathcal{R} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m})$, $\mathcal{Q} \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \mathcal{B}^{\sigma+\rho,m'})$.

(i) *The product operator satisfies* $\mathcal{R}\mathcal{Q} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m+m'})$ *with* $|\mathcal{R}\mathcal{Q}|_{\sigma,m+m'} \lesssim_m \rho^{-|m|}|\mathcal{R}|_{\sigma,m}|\mathcal{Q}|_{\sigma+\rho,m'}$. *If* $\mathcal{R}(\omega), \mathcal{Q}(\omega)$ *depend on a parameter* $\omega \in \Omega \subseteq \mathtt{R}_0$, *then* $|\mathcal{R}\mathcal{Q}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m+m'} \lesssim_m \rho^{-(|m|+2)}|\mathcal{R}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m}|\mathcal{Q}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho,m'}$.

(ii) *The projected operator* $|\Pi_N^\perp \mathcal{R}|_{\sigma,m} \le e^{-\rho N}|\mathcal{R}|_{\sigma+\rho,m}$. *If* $\mathcal{R}(\omega)$ *depends on a parameter* $\omega \in \Omega \subseteq \mathtt{R}_0$, *then the same statement holds by replacing* $|\cdot|_{\sigma,m}$ *with* $|\cdot|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m}$.

(iii) *The mean value* $|[\widehat{\mathcal{R}}(0)]|_{\sigma,m} \le |\mathcal{R}|_{\sigma,m}$. *Moreover, if* $\mathcal{R} = \mathcal{R}(\omega)$ *depends on a parameter* $\omega \in \Omega \subseteq \mathtt{R}_0$, *then the same statement holds by replacing* $|\cdot|_{\sigma,m}$ *with* $|\cdot|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m}$.

Iterating the estimate of Lemma 3.5(i), one has that if $\mathcal{R} \in \mathcal{H}^{\sigma+\rho}(\mathcal{B}^{\sigma+\rho,m})$, then there exists a constant $C_0(m) > 0$ such that for any $N \ge 1$, $\mathcal{R}^N \in \mathcal{H}^\sigma(\mathcal{B}^{\sigma,mN})$ and

$$
\begin{aligned}
&|\mathcal{R}^N|_{\sigma,mN} \le \Big(C_0(m)\rho^{-|m|}|\mathcal{R}|_{\sigma+\rho,m}\Big)^{N-1}|\mathcal{R}|_{\sigma,m}\,,\\
&|\mathcal{R}^N|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,mN} \le \Big(C_0(m)^{N-1}\rho^{-(|m|+2)}|\mathcal{R}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho,m}\Big)^{N-1}|\mathcal{R}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m}\,.
\end{aligned}
\tag{3.17}
$$

Let $m \in \mathbb{Z}$. We recall that the operator ∂_x^m is defined by setting

$$
\partial_x^m[1] = 0, \quad \partial_x^m[e^{\mathrm{i}jx}] = \mathrm{i}^m j^m e^{\mathrm{i}jx}, \quad j \ne 0. \tag{3.18}
$$

Note that this means that $\partial_x^0 = \mathrm{Id} - \pi_0$; see formula (3.3).

LEMMA 3.6. *Let* $\sigma, \rho > 0$, $m, m' \in \mathbb{Z}$, $a \in \mathcal{H}(\mathbb{T}_{\sigma+\rho} \times \mathbb{T}^\infty_{\sigma+\rho})$.

(i) *We have* $\partial_x^m a \partial_x^{m'} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m+m'})$ *and* $|\partial_x^m a \partial_x^{m'}|_{\sigma,m+m'} \lesssim \rho^{-|m|}\|a\|_{\sigma+\rho}$. *If* $a(\cdot;\omega)$, $\omega \in \Omega$ *is Lipschitz with respect to* ω, *then* $|\partial_x^m a \partial_x^{m'}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m+m'} \lesssim \rho^{-|m|}\|a\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho}$.

(ii) *For any* $N \in \mathbb{N}$,

$$
\partial_x^m a \partial_x^{m'} = \sum_{i=0}^{N-1} c_{i,m}(\partial_x^i a)\partial_x^{m+m'-i} + \mathcal{R}_N(a), \tag{3.19}
$$

where the remainder $\mathcal{R}_N(a)$ *satisfies the estimate*

$$
|\mathcal{R}_N(a)|_{\sigma,m+m'-N} \lesssim_{m,N} \rho^{-(2N+|m|+1)}\|a\|_{\sigma+\rho}\,. \tag{3.20}
$$

Moreover, one has $c_{0,m} = 1$, $c_{1,m} = m$. *If* $a(\cdot;\omega)$, $\omega \in \Omega$ *is Lipschitz with respect to* ω, *then*

$$
|\mathcal{R}_N(a)|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m+m'-N} \lesssim_{m,N} \rho^{-(2N+|m|+1)}\|a\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho}\,. \tag{3.21}
$$

(iii) *Let* $b(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_{\sigma+\rho} \times \mathbb{T}^\infty_{\sigma+\rho})$, $\omega \in \Omega$, *and set* $\mathcal{A} = a\partial_x^m$, $\mathcal{B} := b\partial_x^{m'}$. *Then* $\mathcal{A}\mathcal{B} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,m+m'})$ *satisfies, for any* $N \ge 1$, *the expansion*

$$
\mathcal{A}\mathcal{B} = ab\partial_x^{m+m'} + mab_x\partial_x^{m+m'-1} + \sum_{i=2}^{N-1} c_{i,m}a(\partial_x^i b)\partial_x^{m+m'-i} + \mathcal{R}_N(a,b)\,, \tag{3.22}
$$

where $c_{m,i} \in \mathbb{R}$ *for any* $i = 2, \dots, N-1$, *and the remainder* $\mathcal{R}_N(a,b)$ *satisfies the estimate*

$$
|\mathcal{R}_N(a,b)|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,m+m'-N} \lesssim_{m,m',N} \rho^{-\kappa}\|a\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho}\|b\|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho} \tag{3.23}
$$

for some constant $\kappa = \kappa(m, m', N) > 0$. As a consequence, for any $N \geq 1$, the commutator $[\mathcal{A}, \mathcal{B}]$ admits the expansion

$$[\mathcal{A}, \mathcal{B}] = (mab_x - m'a_x b)\partial_x^{m+m'-1} + \sum_{i=2}^{N-1} \big(c_{m,i} a(\partial_x^i b) - c_{m',i}(\partial_x^i a) b\big)\partial_x^{m+m'-i} + \mathcal{R}_N(a, b) - \mathcal{R}_N(b, a)\,.$$

Proof. Proof of (i). The proof follows by Lemmas 3.4 and 3.5 and using that for any $p \in \mathbb{Z}$, $\sigma > 0$, $|\partial_x^p|_{\sigma,p} = |\partial_x^p|_{\sigma,p}^{\mathrm{Lip}(\gamma,\Omega)} \leq 1$.

Proof of (ii). Let $\mathcal{R} := \partial_x^m a \partial_x^{m'}$. Then $\mathcal{R}(\varphi) = \sum_{\ell \in \mathbb{Z}_*^\infty} \widehat{\mathcal{R}}(\ell) e^{\mathrm{i}\ell\cdot\varphi}$, where, for any $\ell \in \mathbb{Z}_*^\infty$, the operator $\widehat{\mathcal{R}}(\ell)$ admits the matrix representation $(\widehat{\mathcal{R}}_j^{j'}(\ell))_{j,j' \in \mathbb{Z}}$,

$$\widehat{\mathcal{R}}_j^{j'}(\ell) = \mathrm{i}^{m+m'} j^m \widehat{a}(\ell, j - j') j'^{m'} \quad \forall j, j' \in \mathbb{Z} \setminus \{0\}. \tag{3.24}$$

We write the Taylor expansion

$$j^m = j'^m + m j'^{m-1}(j - j') + \sum_{k=2}^{N-1} c_{m,k} j'^{m-k}(j - j')^k + r_N(j, j'), \tag{3.25}$$

where the remainder $r_N(j, j')$ is given by

$$r_N(j, j') := c_{N,m} \int_0^1 (1-\tau)^{N-1}(j' + \tau(j - j'))^{m-N}\, d\tau (j - j')^N\,. \tag{3.26}$$

By using the Petree inequality, one has that

$$\frac{(j' + \tau(j - j'))^{m-N}}{j'^{m-N}} \lesssim_{m,N} \langle j - j' \rangle^{N+|m|}\,.$$

This latter inequality implies that

$$|r_N(j, j')| \lesssim_{m,N} \langle j' \rangle^{m-N} \langle j - j' \rangle^{2N+|m|}\,. \tag{3.27}$$

Using the expansion (3.24), we get that the operator $\mathcal{R}$ can be expanded as

$$\mathcal{R}(\varphi) = a\partial_x^{m+m'} + m(\partial_x a)\partial_x^{m+m'-1} + \sum_{i=2}^{N-1} c_{m,i}(\partial_x^i a)\partial_x^{m+m'-i} + \mathcal{R}_N(\varphi),$$

where the operator $\mathcal{R}_N(\varphi) = \sum_{\ell \in \mathbb{Z}_*^\infty} \widehat{\mathcal{R}}_N(\ell) e^{\mathrm{i}\ell\cdot\varphi}$, and for any $\ell \in \mathbb{Z}_*^\infty$, the operator $\widehat{\mathcal{R}}_N(\ell)$ admits the matrix representation

$$(\widehat{\mathcal{R}}_N(\ell))_j^{j'} := \mathrm{i}^{m+m'} \widehat{a}(\ell, j - j') r_N(j, j') j'^{m'}, \quad j, j' \in \mathbb{Z} \setminus \{0\}\,. \tag{3.28}$$

By (3.27), using that $\widehat{a}(\ell, \cdot) \in \mathcal{H}(\mathbb{T}_{\sigma+\rho})$, one gets the estimate

$$|\widehat{\mathcal{R}}_N(\ell)_j^{j'}| \lesssim \langle j - j' \rangle^{2N+|m|} e^{-(\sigma+\rho)|j-j'|} \langle j' \rangle^{m+m'-N} \|\widehat{a}(\ell, \cdot)\|_{\sigma+\rho}\,. \tag{3.29}$$

Furthermore, using that

$$\langle j - j' \rangle^{2N+|m|} e^{-\frac{\rho}{2}|j-j'|} \lesssim_{N,m} \rho^{-(2N+|m|)},$$

one gets the estimate

$$|\widehat{\mathcal{R}}_N(\ell)_j^{j'}| \lesssim \rho^{-(2N+|m|)} e^{-(\sigma+\frac{\rho}{2})|j-j'|} \langle j' \rangle^{m+m'-N} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho} . \tag{3.30}$$

Now if $j, j' \in \mathbb{N}_0$, using that for any $\delta > 0$, $e^{-\delta|j+j'|} \leq e^{-\delta|j-j'|}$, the latter estimate implies also the estimate on the 2×2 block $\pi_j \widehat{\mathcal{R}}_N(\ell)\pi_{j'}$ of the form

$$\|\pi_j \widehat{\mathcal{R}}_N(\ell)\pi_{j'}\| \lesssim_{m,N} \rho^{-(2N+|m|)} e^{-(\sigma+\frac{\rho}{2})|j-j'|} \langle j' \rangle^{m+m'-N} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho} \quad \forall j, j' \in \mathbb{N}_0 . \tag{3.31}$$

Then, for any $j' \in \mathbb{N}_0$, one has that

$$\begin{aligned}\sum_{j \in \mathbb{N}_0} e^{\sigma|j-j'|} \|\pi_j \widehat{\mathcal{R}}_N(\ell)\pi_{j'}\| \langle j' \rangle^{N-(m+m')} &\lesssim_{m,N} \rho^{-(2N+|m|)} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho} \sum_{j \in \mathbb{N}_0} e^{-\frac{\rho}{2}|j-j'|} \\ &\lesssim_{m,N} \rho^{-(2N+|m|+1)} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho},\end{aligned}$$

which implies that

$$\|\widehat{\mathcal{R}}_N(\ell)\|_{\mathcal{B}^{\sigma, m+m'-N}} \lesssim_{m,N} \rho^{-(2N+|m|+1)} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho} .$$

By using this latter estimate one gets that

$$|\mathcal{R}_N|_{\sigma, m+m'-N} \lesssim_{m,N} \rho^{-(2N+|m|+1)} \sum_{\ell \in \mathbb{Z}^\infty_*} e^{\sigma|\ell|_\eta} \|\widehat{a}(\ell,\cdot)\|_{\sigma+\rho} \lesssim_{m,N} \rho^{-(2N+|m|+1)} \|a\|_{\sigma+\rho},$$

which is exactly the claimed estimate (3.20).

If a depends on the parameter $\omega \in \Omega \subseteq \mathtt{R}_0$, given $\omega_1, \omega_2 \in \Omega$, one expands the operator $\partial_x^m(\Delta_{\omega_1\omega_2} a)\partial_x^{m'}$ as in (3.19), where a is replaced by $\Delta_{\omega_1\omega_2} a$, and the remainder $\mathcal{R}_N(\Delta_{\omega_1\omega_2} a)$ is estimated in terms of $\Delta_{\omega_1\omega_2} a$. The Lipschitz estimate then follows.

Proof of (iii). The claimed expansion (3.22) follows by a repeated application of item (i). The estimates of the remainder $\mathcal{R}_N(a,b)$ follow by using the estimates of items (i) and (ii) and by using the composition Lemma 3.5. The expansion of the commutator follows easily by expanding $\mathcal{AB}$ and $\mathcal{BA}$. ∎

LEMMA 3.7 (Exponential map). *Let $\sigma > 0$, $\rho \in (0,1)$, $m \geq 0$, and $\mathcal{R}(\omega) \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho}, \mathcal{B}^{\sigma+\rho,-m})$, $\omega \in \Omega \subseteq \mathtt{R}_0$, and assume that*

$$\rho^{-2} |\mathcal{R}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho} \leq \delta \tag{3.32}$$

for some $\delta \in (0,1)$ small enough. Then, for any $N \geq 1$, the map $\Phi_N := \exp(\mathcal{R}) - \sum_{n=0}^{N-1} \frac{\mathcal{R}^n}{n!} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-Nm})$ with

$$|\Phi_N|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-Nm} \lesssim \Big(C_0 \rho^{-(|m|+2)} |\mathcal{R}|_{\sigma+\rho,-m} \Big)^N . \tag{3.33}$$

As a consequence, $\exp(\mathcal{R}) \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^\sigma)$ *and*

$$|\exp(\mathcal{R})|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma \leq 1 + C\rho^{-(|m|+2)} |\mathcal{R}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma+\rho,-m} \tag{3.34}$$

for some constant $C > 0$.

4. Normal form.

As we said in the introduction, we want to conjugate to constant coefficients the Schrödinger equation $\partial_t u = \mathcal{L}(\omega t)u$, where

$$\mathcal{L}(\varphi) := \mathrm{i}(1 + \varepsilon \mathcal{V}_2(x,\varphi))\partial_{xx} + \varepsilon \mathrm{i} \mathcal{V}_1(x,\varphi)\partial_x + \varepsilon \mathrm{i} \mathcal{V}_0(x,\varphi) .$$

We assume that the functions $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{H}(\mathbb{T}^\infty_{\bar\sigma} \times \mathbb{T}_{\bar\sigma})$ for some $\bar\sigma > 0$ satisfy the condition (1.6), so that $\mathcal{L}(\varphi)$ is an L^2 skew-self-adjoint linear operator.

4.1. Normalization of the x-dependence of the highest order term. We consider an operator induced by an analytic diffeomorphism of the torus

$$(x, \varphi) \mapsto (x + \beta(x, \varphi), \varphi),$$

where β is a real on real analytic function on the infinite dimensional torus that will be determined later. We make the ansatz that

$$\beta \in \mathcal{H}(\mathbb{T}_{\sigma_1} \times \mathbb{T}^\infty_{\sigma_1}), \quad \|\beta\|_\sigma \lesssim_{\sigma_1, \bar\sigma} \delta \quad \forall 0 < \sigma_1 < \bar\sigma \,. \tag{4.1}$$

By Proposition 2.13, for any $0 < \sigma_1 < \bar\sigma$ there exists $\delta_0(\sigma_1, \bar\sigma)$ such that for any $\delta \leq \delta_0$, the map $(x, \varphi) \mapsto (x + \beta(x, \varphi), \varphi)$ is invertible, with inverse given by $(y, \varphi) \mapsto (y + \widetilde\beta(y, \varphi), \varphi)$, and

$$\widetilde\beta \in \mathcal{H}(\mathbb{T}_{\sigma_2} \times \mathbb{T}^\infty_{\sigma_2}), \quad \|\widetilde\beta\|_{\sigma_2} \lesssim_{\sigma_1, \sigma_2} \|\beta\|_{\sigma_1} \quad \forall \sigma_2 < \sigma_1 < \bar\sigma \,. \tag{4.2}$$

We now define the operator

$$\Phi^{(1)}(\varphi)[u] := \sqrt{1 + \beta_x(x, \varphi)}\, u(x + \beta(x, \varphi)) \,. \tag{4.3}$$

A direct calculation shows that this map is unitary and, if β is appropriately small, invertible with inverse given by

$$\Phi^{(1)}(\varphi)^{-1}[u] := \sqrt{1 + \widetilde\beta_y(y, \varphi)}\, u(y + \widetilde\beta(y, \varphi)) \tag{4.4}$$

for $\varphi \in \mathbb{T}^\infty_\sigma$ with $\sigma < \sigma_2$. Note that one has the relation

$$1 + \widetilde\beta_y(y, \varphi) = \frac{1}{1 + \beta_x(y + \widetilde\beta(y, \varphi), \varphi)}, \quad 1 + \beta_x(x, \varphi) = \frac{1}{1 + \widetilde\beta_y(x + \beta(x, \varphi), \varphi)} \,. \tag{4.5}$$

The following lemma holds.

LEMMA 4.1. *For any $\sigma < \sigma' < \bar\sigma$, there exists $\delta \equiv \delta(\sigma, \sigma', \bar\sigma) \in (0, 1)$ such that if $\varepsilon \in (0, \delta)$, the following hold. Define*

$$\begin{aligned} m_2(\varphi) &:= \Big(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{dx}{\sqrt{1 + \varepsilon \mathcal{V}_2(x, \varphi)}}\, dx \Big)^{-2}, \\ \beta(x, \varphi) &:= \partial_x^{-1} \Big[\frac{\sqrt{m_2(\varphi)}}{\sqrt{1 + \varepsilon \mathcal{V}_2(x, \varphi)}} - 1 \Big] . \end{aligned} \tag{4.6}$$

(i) *The map $\mathbb{T}^\infty_\sigma \to \mathcal{B}(\mathcal{H}(\mathbb{T}_{\sigma'}), \mathcal{H}(\mathbb{T}_\sigma))$, $\varphi \mapsto \Phi^{(1)}(\varphi)^{\pm 1}$ is bounded.*

(ii) *For any $s \geq 0$, the map $\mathbb{T}^\infty \to \mathcal{B}(H^s(\mathbb{T}))$, $\varphi \mapsto \Phi^{(1)}(\varphi)^{\pm 1}$ is bounded.*

(iii) *$\Phi^{(1)}(\varphi)$ transforms the operator $\mathcal{L}(\varphi)$ into*

$$\mathcal{L}^{(1)}(\varphi) := (\Phi^{(1)}_{\omega *})\mathcal{L}(\varphi) = \mathrm{i} m_2(\varphi)\partial_x^2 + a_1(x, \varphi)\partial_x + a_0(x, \varphi), \tag{4.7}$$

where the functions $m_2 \in \mathcal{H}(\mathbb{T}^\infty_\sigma), \beta, \widetilde\beta, a_1, a_0 \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$ are independent of the parameter ω and satisfy the estimates

$$\|m_2 - 1\|_\sigma, \|\beta\|_\sigma, \|\widetilde\beta\|_\sigma\,, \|a_1\|_\sigma, \|a_0\|_\sigma \lesssim_{\sigma, \bar\sigma} \varepsilon \,. \tag{4.8}$$

Finally, $\mathcal{L}^{(1)}$ is skew-self-adjoint, and hence $m_2(\varphi), a_1(x, \varphi)$ are real on real while $a_0 = -\bar a_0 + \partial_x a_1$.

Proof. The proof of item (i) follows by (4.3) and (4.4), by using the estimates on $\beta, \widetilde{\beta}$ (4.8), and by applying Lemmas 2.5 and 2.15.

To prove the item (ii), we argue as follows. Since β and $\widetilde{\beta}$ are analytic, then for any $\varphi \in \mathbb{T}^\infty$ one has $\beta(\varphi, \cdot), \widetilde{\beta}(\varphi, \cdot) \in \mathcal{C}^\infty(\mathbb{T})$ and

$$\sup_{\varphi \in \mathbb{T}^\infty} \|\beta(\varphi, \cdot)\|_{\mathcal{C}^s(\mathbb{T})}\,, \ \sup_{\varphi \in \mathbb{T}^\infty} \|\widetilde{\beta}(\varphi, \cdot)\|_{\mathcal{C}^s(\mathbb{T})} < \infty$$

for any $s \geq 0$. A direct calculation then shows that $\sup_{\varphi \in \mathbb{T}^\infty} \|\Phi(\varphi)\|_{\mathcal{B}(H^s(\mathbb{T}))} \leq C(\sup_{\varphi \in \mathbb{T}^\infty} \|\beta(\varphi, \cdot)\|_{\mathcal{C}^s(\mathbb{T})})$ and

$$\sup_{\varphi \in \mathbb{T}^\infty} \|\Phi(\varphi)^{-1}\|_{\mathcal{B}(H^s(\mathbb{T}))} \leq C\Big(\sup_{\varphi \in \mathbb{T}^\infty} \|\widetilde{\beta}(\varphi, \cdot)\|_{\mathcal{C}^s(\mathbb{T})}\Big),$$

and the result follows.

In order to prove (iii), we remark that the map $\Phi^{(1)}(\varphi)$ satisfies the following conjugation rules:

$$\begin{aligned} &\Phi^{(1)}(\varphi)^{-1} \circ a(x, \varphi) \circ \Phi^{(1)}(\varphi) = a(y + \widetilde{\beta}(y, \varphi), \varphi)\,, \\ &\Phi^{(1)}(\varphi)^{-1} \circ \partial_x \circ \Phi^{(1)}(\varphi) = (1 + \beta_x(y + \widetilde{\beta}(y, \varphi), \varphi))\partial_y \\ &\qquad\qquad + \frac{1}{2}(1 + \widetilde{\beta}_y(y, \varphi))\beta_{xx}(y + \widetilde{\beta}(y, \varphi), \varphi)\,, \\ &\Phi^{(1)}(\varphi)^{-1} \omega \cdot \partial_\varphi \Phi^{(1)}(\varphi) = \omega \cdot \partial_\varphi \beta(y + \widetilde{\beta}(y, \varphi), \varphi)\partial_y \\ &\qquad\qquad + \frac{1}{2}(1 + \widetilde{\beta}_y(y, \varphi))\omega \cdot \partial_\varphi \beta_x(y + \widetilde{\beta}(y, \varphi), \varphi)\,. \end{aligned} \tag{4.9}$$

Then, recalling (2.2), the transformed operator is

$$\begin{aligned} &\mathcal{L}^{(1)}(\varphi) = \mathrm{i} a_2(y, \varphi)\partial_y^2 + a_1(y, \varphi)\partial_y + a_0(y, \varphi)\,, \\ &\quad a_2 := \Big((1 + \varepsilon \mathcal{V}_2)(1 + \beta_x)^2\Big)_{x = y + \widetilde{\beta}(y, \varphi)}\,, \\ &\quad a_1 := \Big(2\mathrm{i}(1 + \varepsilon \mathcal{V}_2)\beta_{xx} + \varepsilon \mathrm{i} \mathcal{V}_1(1 + \beta_x) - \omega \cdot \partial_\varphi \beta\Big)_{x = y + \widetilde{\beta}(y, \varphi)}\,, \\ &\quad a_0 := \mathrm{i}\sqrt{1 + \widetilde{\beta}_y}\Big((1 + \varepsilon \mathcal{V}_2)\partial_{xx}\sqrt{1 + \beta_x}\Big)\Big|_{x = y + \widetilde{\beta}(y, \varphi)} \\ &\qquad + \frac{1}{2}\mathrm{i}(1 + \widetilde{\beta}_y)\Big(\varepsilon \mathcal{V}_1 \beta_{xx} + \omega \cdot \partial_\varphi \beta_x\Big)\Big|_{x = y + \widetilde{\beta}(y, \varphi)} + \varepsilon \mathcal{V}_0(y, \varphi + \widetilde{\beta}(y, \varphi))\,. \end{aligned} \tag{4.10}$$

By the definitions of the functions $\beta(x, \varphi)$ and $m_2(\varphi)$ given in (4.6) one gets

$$a_2(x, \varphi) = m_2(\varphi), \quad \text{namely} \quad (1 + \varepsilon \mathcal{V}_2)(1 + \beta_x)^2 = m_2(\varphi); \tag{4.11}$$

hence the operator $\mathcal{L}^{(1)}(\varphi)$ in (4.10) takes the form (4.7). Since $\Phi^{(1)}$ is unitary, by construction $\mathcal{L}^{(1)}$ is skew-self-adjoint.

Since $\mathcal{V}_2 \in \mathcal{H}^{\bar{\sigma}}_{x,\varphi}$, by applying Lemma A.4 (applied to the analytic function $f(u) = \frac{1}{\sqrt{1+u}}$, $|u| \leq \frac{1}{2}$), and by (4.6), one gets that for ε small enough, $\beta \in \mathcal{H}(\mathbb{T}_{\sigma_1} \times \mathbb{T}^\infty_{\sigma_1})$, $m_2 \in \mathcal{H}(\mathbb{T}^\infty_{\sigma_1})$ for any $0 < \sigma_1 < \bar{\sigma}$. Using the mean value theorem, one gets the estimate $\|\beta\|_{\sigma_1}, \|m_2 - 1\|_{\sigma_1} \lesssim_{\sigma_1, \bar{\sigma}} \varepsilon$. The ansatz (4.1) is then proved. The ansatz (4.2) follows by Proposition 2.13. Finally, by applying Lemmas 2.7, 2.15, and A.4, and using that $\mathcal{V}_2, \mathcal{V}_1, \mathcal{V}_0 \in \mathcal{H}(\mathbb{T}_{\bar{\sigma}} \times \mathbb{T}^\infty_{\bar{\sigma}})$, one deduces the claimed properties on the functions a_0 and a_1. ∎

4.2. Reduction to constant coefficients of the highest order term. Our next purpose is to eliminate the φ-dependence from the highest order coefficient $m_2(\varphi)\partial_{xx}$ of the operator $\mathcal{L}^{(1)}(\varphi)$ in (4.7). To achieve this we conjugate the equation $\partial_t u = \mathrm{i}\mathcal{L}^{(1)}(\omega t)u$ by means of a reparameterization of time $t \mapsto t + \alpha(\omega t)$, where α is a suitable analytic function which has to be determined. More precisely, we consider the change of variables

$$u(t,x) = \Phi^{(2)} v(t,x) := v(x, t + \alpha(\omega t)), \quad (x,t) \in \mathbb{T} \times \mathbb{R} . \tag{4.12}$$

We assume that $\alpha(\varphi)$ is real on real and satisfies the ansatz

$$\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma_1}), \quad \|\alpha\|_{\sigma_1} \lesssim_{\sigma_1, \bar\sigma} \delta \quad \forall 0 < \sigma_1 < \bar\sigma . \tag{4.13}$$

By applying Proposition 2.13, for any $\sigma_2 < \bar\sigma$ there exists $\delta_0 = \delta_0(\sigma_2, \sigma_1, \bar\sigma)$ small enough such that if $\delta \le \delta_0$, the map $\varphi \mapsto \varphi + \omega\alpha(\varphi)$ is invertible with inverse given by $\vartheta \mapsto \vartheta + \omega\widetilde\alpha(\vartheta)$ and

$$\widetilde\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma_2}), \quad \|\widetilde\alpha\|_{\sigma_2} \lesssim_{\sigma_1,\sigma_2} \|\alpha\|_{\sigma_1} \quad \forall \sigma_2 < \sigma_1 < \bar\sigma . \tag{4.14}$$

The inverse of the map $\Phi^{(2)}$ in (4.12) is then given by

$$(\Phi^{(2)})^{-1} u(x,\tau) := u(x, \tau + \widetilde\alpha(\omega\tau)) . \tag{4.15}$$

Remark 4.2. If $u(x)$ is a function independent of φ, then $(\Phi^{(2)})^{\pm 1} u = u$.

The following lemma holds.

LEMMA 4.3. *Let $\omega \in \mathtt{D}_\gamma$. For any $\sigma < \bar\sigma$ there exists $\delta(\sigma,\bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \le \delta$, then, setting*

$$\lambda_2 := \widehat{m}_2(0) = \int_{\mathbb{T}^\infty} m_2(\varphi)\, d\varphi , \quad \alpha := (\omega \cdot \partial_\varphi)^{-1} \Big[\frac{m_2}{\lambda_2} - 1 \Big] , \tag{4.16}$$

$\Phi^{(2)}$ transforms the operator $\mathcal{L}^{(1)}(\varphi)$ into

$$\mathcal{L}^{(2)}(\vartheta) = \mathrm{i}\lambda_2 \partial_x^2 + b_1(\vartheta,x)\partial_x + b_0(\vartheta,x) . \tag{4.17}$$

The constant $\lambda_2 \in \mathbb{R}$ is independent of ω. For all $\omega \in \mathtt{D}_\gamma$ the functions $\alpha(\cdot;\omega), \widetilde\alpha(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_\sigma), b_1(\cdot;\omega), \mathrm{i}b_0(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$ are well defined and real on real. Furthermore, for any $\Omega \subseteq \mathtt{D}_\gamma$ the following estimates hold:

$$|\lambda_2 - 1|, \|b_0\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}, \|b_1\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \varepsilon , \quad \|\alpha\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} , \|\widetilde\alpha\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \varepsilon\gamma^{-1} .$$

Proof. A direct calculation shows that formula (2.2) reads

$$\mathcal{L}^{(2)}(\vartheta) := (\Phi^{(2)}_{\omega *})\mathcal{L}^{(1)}(\varphi) = \frac{1}{\rho(\vartheta)} \mathcal{L}^{(1)}(\vartheta + \omega\widetilde\alpha(\vartheta)) , \quad \rho(\vartheta) := 1 + \omega \cdot \partial_\varphi \alpha(\vartheta + \omega\widetilde\alpha(\vartheta)) . \tag{4.18}$$

Note that, since $\mathcal{L}^{(1)}(\omega t)$ is skew-self-adjoint, then also $\mathcal{L}^{(2)}(\omega t)$ is skew-self-adjoint. By (4.18), one has

$$\begin{aligned} \mathcal{L}^{(2)}(\vartheta) &= \mathrm{i}b_2(\vartheta)\partial_x^2 + b_1(\vartheta,x)\partial_x + b_0(\vartheta,x) , \\ b_2(\vartheta) &:= \Big[\frac{m_2}{1 + \omega\cdot\partial_\varphi\alpha} \Big]\Big|_{\varphi = \vartheta + \omega\widetilde\alpha(\vartheta)} , \\ b_1(\vartheta,x) &:= \Big[\frac{a_1}{1 + \omega\cdot\partial_\varphi\alpha} \Big]\Big|_{\varphi = \vartheta + \omega\widetilde\alpha(\vartheta)} , \\ b_0(\vartheta,x) &:= \Big[\frac{a_0}{1 + \omega\cdot\partial_\varphi\alpha} \Big]\Big|_{\varphi = \vartheta + \omega\widetilde\alpha(\vartheta)} . \end{aligned} \tag{4.19}$$

By the definitions of $\alpha(\varphi)$ and $\lambda_2 \in \mathbb{R}$ given in (4.16), one obtains that

$$b_2(\vartheta) = \lambda_2, \quad \text{namely} \quad \frac{m_2(\varphi)}{1 + \omega \cdot \partial_\varphi \alpha(\varphi)} = \lambda_2, \tag{4.20}$$

and therefore the linear operator $\mathcal{L}^{(2)}(\varphi)$ defined in (4.19) takes the form given in (4.17). Note that the function $m_2(\varphi)$ defined in (4.6) is independent of ω and therefore also λ_2 does not depend on ω. By applying Lemma 4.1, by (4.16) and by Lemma 2.11 and Proposition 2.13, one gets that $|\lambda_2 - 1| \lesssim \varepsilon$ and that for any $0 < \sigma < \bar\sigma$, for $\varepsilon\gamma^{-1} \leq \delta$, for some $\delta = \delta(\sigma, \bar\sigma)$ small enough, $\alpha, \widetilde\alpha \in \mathcal{H}(\mathbb{T}^\infty_\sigma)$ with $\|\alpha\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}, \|\widetilde\alpha\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_{\sigma,\bar\sigma} \varepsilon\gamma^{-1}$. Finally, recalling (4.19), using the properties on a_0 and a_1 stated in Lemma 4.1 and by applying Lemma A.4 (with $f(u) = \frac{1}{1+u}$, $|u| \leq \frac{1}{2}$) and Lemmas 2.9 and 2.15, we can deduce the claimed properties on b_0 and b_1. ∎

4.3. Elimination of the x-dependence from the first order term. The next aim is to eliminate the dependence on x from the first order term in (4.17). To this aim, we conjugate the vector field $\mathcal{L}^{(2)}(\varphi)$ by means of a multiplication operator

$$\Phi^{(3)}(\varphi) : u \mapsto e^{\mathrm{i}p(x,\varphi)} u, \tag{4.21}$$

where p is an analytic real on real function which has to be determined. The following lemma holds.

LEMMA 4.4. *Let $\omega \in \mathtt{D}_\gamma$. For any $0 < \sigma < \bar\sigma$ there exists $\delta(\sigma, \bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \leq \delta$, the following holds. Define*

$$m_1(\varphi) := \frac{1}{2\pi} \int_{\mathbb{T}} b_1(x,\varphi)\, dx\,, \quad p(x,\varphi) := \frac{\partial_x^{-1}[b_1(x,\varphi) - m_1(\varphi)]}{2\lambda_2}\,. \tag{4.22}$$

(i) *The map $\mathbb{T}^\infty_\sigma \to \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma))$, $\varphi \mapsto \Phi^{(3)}(\varphi)^{\pm 1}$ is bounded.*
(ii) *For any $s \geq 0$, the map $\mathbb{T}^\infty \to \mathcal{B}(H^s(\mathbb{T}))$, $\varphi \mapsto \Phi^{(3)}(\varphi)^{\pm 1}$ is bounded.*
(iii) *The operator $\Phi^{(3)}(\varphi)$ transforms $\mathcal{L}^{(2)}(\varphi)$ into*

$$\mathcal{L}^{(3)}(\varphi) = \mathrm{i}\lambda_2 \partial_{xx} + m_1(\varphi)\partial_x + c_0(x,\varphi), \tag{4.23}$$

where the functions $p(\cdot;\omega), \mathrm{i}c_0(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$, $m_1(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_\sigma)$ are real on real, are well defined for $\omega \in \mathtt{D}_\gamma$, and satisfy for $\Omega \subseteq \mathtt{D}_\gamma$ the estimates

$$\|p\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}, \|c_0\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)}, \|m_1\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_{\sigma,\bar\sigma} \varepsilon\,. \tag{4.24}$$

Proof. Item (i) follows by (4.21), by Lemmas 2.5 and A.4, and by the estimates (4.24) on p, which are a straightforward computation.

(ii) Since p is analytic, then $p(\varphi,\cdot) \in \mathcal{C}^\infty(\mathbb{T})$ for any $\varphi \in \mathbb{T}^\infty$ and

$$M(s) := \sup_{\varphi \in \mathbb{T}^\infty} \|p(\varphi,\cdot)\|_{\mathcal{C}^s(\mathbb{T})} < \infty$$

for any $s \geq 0$. A direct calculation shows that

$$\sup_{\varphi \in \mathbb{T}^\infty} \|\Phi^{(3)}(\varphi)^{\pm 1}\|_{\mathcal{B}(H^s(\mathbb{T}))} \lesssim_s \sup_{\varphi \in \mathbb{T}^\infty} \|\exp(\mathrm{i}p)\|_{\mathcal{C}^s(\mathbb{T})} \lesssim_s \exp(M(s)).$$

The latter estimate proves item (ii).

(iii) A direct calculation shows that

(4.25)
$$\begin{aligned}\mathcal{L}^{(3)}(\varphi) &:= (\Phi^{(3)}_{\omega*})\mathcal{L}^{(2)}(\varphi) = \Phi^{(3)}(\varphi)^{-1}\mathcal{L}^{(2)}(\varphi)\Phi^{(3)}(\varphi) - \Phi^{(3)}(\varphi)^{-1}\omega\cdot\partial_\varphi\Phi^{(3)}(\varphi)\\ &= \mathrm{i}\lambda_2\partial_{xx} + c_1(x,\varphi)\partial_x + c_0(x,\varphi),\end{aligned}$$

where

$$\begin{aligned} c_0 &:= -\mathrm{i}\lambda_2 p_x^2 - \lambda_2 p_{xx} + \mathrm{i}b_1 p_x - \mathrm{i}\omega\cdot\partial_\varphi p + b_0\,,\\ c_1 &:= -2\lambda_2 p_x + b_1\,. \end{aligned} \tag{4.26}$$

The definitions of p and m_1 given in (4.22) allow us to solve the equation

$$-\,2\lambda_2 p_x(x,\varphi) + b_1(x,\varphi) = m_1(\varphi)\,. \tag{4.27}$$

Therefore, the operator $\mathcal{L}^{(3)}(\varphi)$ in (4.25) takes the form (4.23).

Note that the skew-self-adjoint structure guarantees that $\mathrm{i}m_1(\varphi)$ is a real function (meaning that it is real on real). The claimed properties on the functions p and m_1 follow by (4.22) and by applying Lemma 4.3. The claimed properties on the function c_0 defined in (4.26) follow by Lemma 4.3 and by applying Lemmas 2.7 and 2.9. ∎

4.4. Reduction to constant coefficients of the first order term. In order to reduce the first order term in (4.23) to constant coefficients, we consider the transformation

$$\Phi^{(4)}(\varphi) : u(x) \mapsto u(x+q(\varphi)), \tag{4.28}$$

where q is an analytic function on $\mathbb{T}^\infty_\sigma$ to be determined. Clearly, the inverse of $\Phi^{(4)}(\varphi)$ is given by

$$\Phi^{(4)}(\varphi)^{-1} : u(x) \mapsto u(x-q(\varphi))\,.$$

LEMMA 4.5. *Let $\omega \in \mathtt{D}_\gamma$. For any $\sigma < \bar\sigma$ there exists $\delta(\sigma,\bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \le \delta$, the following holds. Define*

$$\lambda_1 := \int_{\mathbb{T}^\infty} m_1(\varphi)\,d\varphi = \widehat{m}_1(0), \quad q(\varphi) := (\omega\cdot\partial_\varphi)^{-1}[m_1(\varphi) - \lambda_1]\,. \tag{4.29}$$

(i) *The map $\mathbb{T}^\infty_\sigma \to \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma))$, $\varphi \mapsto \Phi^{(4)}(\varphi)^{\pm1}$ is bounded.*

(ii) *For any $s \ge 0$, the map $\mathbb{T}^\infty \to \mathcal{B}(H^s(\mathbb{T}))$, $\varphi \mapsto \Phi^{(4)}(\varphi)^{\pm1}$ is bounded.*

(iii) *The map $\Phi^{(4)}(\varphi)$ transforms the operator $\mathcal{L}^{(3)}(\varphi)$ into*

$$\mathcal{L}^{(4)}(\varphi) = \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x + d_0(x,\varphi), \tag{4.30}$$

where the constant $\lambda_1 \in \mathbb{R}$ does not depend on ω and $q(\cdot;\omega) \in \mathcal{H}(\mathbb{T}^\infty_\sigma), \mathrm{i}d_0(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$ are real on real functions defined for $\omega \in \mathtt{D}_\gamma$. Furthermore, the following bounds hold for any $\Omega \subseteq \mathtt{D}_\gamma$:

$$\|q\|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma\,,\ \|d_0\|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma \lesssim_{\sigma,\bar\sigma} \varepsilon\,, \quad |\lambda_1| \lesssim \varepsilon\,. \tag{4.31}$$

Proof. Items (i)–(ii) follow similarly to the corresponding items of Lemma 4.1, by using the estimate (4.31) on the function $q(\varphi)$, which is a direct computation.

(iii) A direct calculation shows that

$$\begin{aligned} &\mathcal{L}^{(4)}(\varphi) := (\Phi^{(4)}_{\omega *})\mathcal{L}^{(3)}(\varphi) = \mathrm{i}\lambda_2 \partial_{xx} + \big(- \omega \cdot \partial_\varphi q(\varphi) + m_1(\varphi)\big)\partial_x + d_0(x, \varphi)\,, \\ &d_0(x,\varphi) := c_0(x, \varphi - q(\varphi))\,. \end{aligned} \tag{4.32}$$

By (4.29), we solve the equation

$$- \omega \cdot \partial_\varphi q(\varphi) + m_1(\varphi) = \lambda_1\,. \tag{4.33}$$

Then, the operator $\mathcal{L}^{(4)}$ defined in (4.32) takes the form given in (4.30). We now show that λ_1 is independent of ω. By (4.22) and (4.29), one has that

$$\lambda_1 = \frac{1}{2\pi} \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} b_1(\vartheta, x)\, dx\, d\vartheta,$$

where by (4.19) and using the properties (A.12), one has that

$$\begin{aligned} b_1(\vartheta, x) &= \Big[\frac{a_1}{1 + \omega \cdot \partial_\varphi \alpha}\Big]\Big|_{\varphi = \vartheta + \omega \widetilde{\alpha}(\vartheta)} \\ &= a_1(\vartheta + \omega \widetilde{\alpha}(\vartheta), x)\Big(1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta)\Big)\,. \end{aligned}$$

By expanding $a_1(x,\varphi)$ in Fourier series, i.e., $a_1(x, \varphi) = \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^\infty_*} \widehat{a}_1(\ell, j) e^{\mathrm{i}\ell\cdot\varphi} e^{\mathrm{i}jx}$, one has that

$$\begin{aligned} \lambda_1 &= \frac{1}{2\pi} \int_{\mathbb{T}^\infty} \int_{\mathbb{T}} b_1(\vartheta, x)\, dx\, d\vartheta \\ &= \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}^\infty_*} \widehat{a}_1(\ell, j) \int_{\mathbb{T}} e^{\mathrm{i}jx}\, dx \int_{\mathbb{T}^\infty} e^{\mathrm{i}\ell\cdot(\vartheta + \omega\widetilde{\alpha}(\vartheta))}\Big(1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta)\Big)\, d\vartheta \\ &= \sum_{\ell \in \mathbb{Z}^\infty_*} \widehat{a}_1(\ell, 0) \int_{\mathbb{T}^\infty} e^{\mathrm{i}\ell\cdot(\vartheta + \omega\widetilde{\alpha}(\vartheta))}\Big(1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta)\Big)\, d\vartheta \\ &\overset{Lemma\ \mathrm{A.3}}{=} \widehat{a}_1(0,0) = \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{T}^\infty} a_1(x, \varphi)\, d\varphi\, dx\,. \end{aligned}$$

By Lemma 4.1, the function a_1 does not depend on ω and therefore also λ_1 is independent of ω.

The estimates on λ_1, q, d_0 given in (4.29) and (4.32) follow by applying Lemmas 2.11, 2.15, and 4.4(ii). ∎

4.5. Elimination of the x-dependence from the zeroth order term. In order to eliminate the x-dependence from the zeroth order term in the operator $\mathcal{L}^{(4)}(\varphi)$ in (4.32), we conjugate, using (2.2), by a transformation

$$\Phi^{(5)}(\varphi) := \exp(\mathcal{V}(\varphi)), \quad \text{where} \quad \mathcal{V}(\varphi) := \frac{1}{2}\big(v(x,\varphi) \circ \partial_x^{-1} + \partial_x^{-1} \circ v(x, \varphi)\big), \tag{4.34}$$

where $v(x,\varphi)$ is a real on real function to be determined. Note that for real values of the angle $\varphi \in \mathbb{T}^\infty$, one has that $\mathcal{V}(\varphi) = -\mathcal{V}(\varphi)^*$, implying that $\Phi^{(5)}(\varphi)$ is a unitary operator.

LEMMA 4.6. *Let $\omega \in \mathtt{D}_\gamma$. For any $0 < \sigma < \bar\sigma$ there exists $\delta(\sigma, \bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \le \delta$, the following holds. Define*

$$v := \frac{1}{2\mathrm{i}\lambda_2} \partial_x^{-1}\Big(\langle d_0 \rangle_x - d_0\Big)\,. \tag{4.35}$$

(i) *The map* $\mathbb{T}^\infty_\sigma \to \mathcal{B}\big(\mathcal{H}(\mathbb{T}_\sigma)\big)$, $\varphi \mapsto \Phi^{(5)}(\varphi)^{\pm 1}$ *is bounded.*

(ii) *For any* $s \geq 0$, *the map* $\mathbb{T}^\infty \to \mathcal{B}\big(H^s(\mathbb{T})\big)$, $\varphi \mapsto \Phi^{(5)}(\varphi)^{\pm 1}$ *is bounded.*

(iii) *The map* $\Phi^{(5)}(\varphi)$ *transforms the operator* $\mathcal{L}^{(4)}(\varphi)$ *into*

$$\mathcal{L}^{(5)}(\varphi) := (\Phi^{(5)}_{\omega *})\mathcal{L}^{(4)}(\varphi) = \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x + \langle d_0\rangle_x(\varphi) + e_{-1}(x,\varphi)\partial_x^{-1} + \mathcal{R}^{(5)}(\varphi), \tag{4.36}$$

and the functions $v(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$ *and the operator* $\mathcal{R}^{(5)}(\omega) \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big)$ *defined for* $\omega \in \mathtt{D}_\gamma$ *satisfy the estimates*

$$\|v\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)},\ \|e_{-1}\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)},\ |\mathcal{R}^{(5)}|_{\sigma,-2}^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_{\sigma,\bar\sigma} \varepsilon\,. \tag{4.37}$$

Proof. By (4.35), using the estimates on d_0 given in Lemma 4.5, one gets that v satisfies the estimate (4.37). By Lemma 3.6, one has that the operator $\mathcal{V}(\varphi)$ admits an expansion of the form

$$\mathcal{V}(\varphi) = v(x,\varphi)\partial_x^{-1} - \frac{1}{2}v_x(x,\varphi)\partial_x^{-2} + c_{-3}v_{xx}\partial_x^{-3} + \mathcal{R}_{\mathcal{V}}(\varphi), \tag{4.38}$$

where $c_{-3} \in \mathbb{R}$ is a constant and for any $0 < \sigma < \bar\sigma$, $\mathcal{R}_{\mathcal{V}} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-4}\big)$ and

$$|\mathcal{V}|_{\sigma,-1}^{\mathrm{Lip}(\gamma,\Omega)},\ |\mathcal{R}_{\mathcal{V}}|_{\sigma,-4}^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_{\sigma,\bar\sigma} \varepsilon\,. \tag{4.39}$$

By (4.34) and (4.39) and Lemma 3.6(i) and the estimate (3.34), there exists $\delta = \delta(\sigma,\overline{\sigma}) \in (0,1)$ such that if $\varepsilon\gamma^{-1} \leq \delta$, then $|(\Phi^{(5)})^{\pm 1}|_\sigma \lesssim_{\sigma,\overline{\sigma}} 1$. Items (i)–(ii) then follow by applying Lemmas 2.4 and 3.1.

(iii) A direct calculation shows that

$$\begin{aligned}\mathcal{L}^{(5)}(\varphi) &:= (\Phi^{(5)}_{\omega *})\mathcal{L}^{(4)}(\varphi) = \Phi^{(5)}(\varphi)^{-1}\mathcal{L}^{(4)}(\varphi)\Phi^{(5)}(\varphi) - \Phi^{(5)}(\varphi)^{-1}\omega\cdot\partial_\varphi\Phi^{(5)}(\varphi)\\ &= \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x + d_0(x,\varphi) + [\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{V}(\varphi)]\\ &\quad - \Phi^{(5)}(\varphi)^{-1}\omega\cdot\partial_\varphi\Phi^{(5)}(\varphi) + \mathcal{R}^{(I)}(\varphi),\end{aligned} \tag{4.40}$$

where the remainder $\mathcal{R}^{(I)}(\varphi)$ is given by

$$\begin{aligned}\mathcal{R}^{(I)}(\varphi) &:= \int_0^1 (1-t)\exp(-\tau\mathcal{V}(\varphi))\,[[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{V}(\varphi)], \mathcal{V}(\varphi)]\exp(\tau\mathcal{V}(\varphi))\,d\tau\\ &\quad + \int_0^1 e^{-\tau\mathcal{V}(\varphi)}[d_0, \mathcal{V}(\varphi)]e^{\tau\mathcal{V}(\varphi)}\,d\tau\,.\end{aligned} \tag{4.41}$$

By recalling (4.38) and (4.39) and by applying Lemma 3.6, using that $\partial_x^0 = \mathrm{Id} - \pi_0$ and $\lambda_2 = 1 + O(\varepsilon)$, $\lambda_1 = O(\varepsilon)$, one obtains that

$$[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{A}(\varphi)] = 2\mathrm{i}\lambda_2 v_x(x,\varphi) + a_v^{(-1)}(x,\varphi)\partial_x^{-1} + \mathcal{R}^{(II)}(\varphi),$$

where for any $0 < \sigma < \bar\sigma$, $a_v^{(1)} \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$, $\mathcal{R}^{(II)} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big)$ with

$$\|a_v^{(-1)}\|_\sigma^{\mathrm{Lip}(\gamma,\Omega)},\ |\mathcal{R}^{(II)}|_{\sigma,-2}^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_\sigma \varepsilon \tag{4.42}$$

and

$$\begin{aligned}&[[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{V}], \mathcal{V}] \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2})\,,\\ &\Big|[[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{V}], \mathcal{V}]\Big|_{\sigma,-2}^{\mathrm{Lip}(\gamma,\Omega)} \lesssim_{\sigma,\bar\sigma} \varepsilon\,.\end{aligned} \tag{4.43}$$

Moreover, using the estimate on d_0 provided in Lemma 4.5 and by applying again Lemma 3.6, one gets that

$$[d_0, \mathcal{V}] \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}), \quad |[d_0, \mathcal{V}]|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar\sigma} \varepsilon \,. \tag{4.44}$$

By applying Lemma 3.5, using Lemma 3.7 and the estimate (4.39) to bound $\exp(\pm\tau\mathcal{V}(\varphi))$ and by applying the estimates (4.43) and (4.44), one obtains that

$$\mathcal{R}^{(I)} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big), \quad |\mathcal{R}^{(I)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar\sigma} \varepsilon \,. \tag{4.45}$$

Moreover, recalling the definition of the operator $\Phi^{(5)}$ given in (4.34), using (4.38) and (4.39) and by applying Lemmas 3.6 and 3.7, one obtains that

$$\begin{aligned} &-\Phi^{(5)}(\varphi)^{-1}\omega\cdot\partial_\varphi\Phi^{(5)}(\varphi) = -\omega\cdot\partial_\varphi v(x,\varphi)\partial_x^{-1} + \mathcal{R}^{(III)}(\varphi), \\ &\mathcal{R}^{(III)}(\varphi) \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big), \quad |\mathcal{R}^{(III)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar\sigma} \varepsilon \quad \forall 0<\sigma<\bar\sigma, \end{aligned} \tag{4.46}$$

and therefore by (4.40) one gets

$$\begin{aligned} &\mathcal{L}^{(5)}(\varphi) = \lambda_2\partial_{xx} + \lambda_1\partial_x + d_0 + 2\lambda_2 v_x + e_{-1}(x,\varphi)\partial_x^{-1} + \mathcal{R}^{(5)}(\varphi)\,, \\ &e_{-1}(x,\varphi) := a_v^{(-1)}(x,\varphi) - \omega\cdot\partial_\varphi v(x,\varphi)\,, \\ &\mathcal{R}^{(5)}(\varphi) := \mathcal{R}^{(I)}(\varphi) + \mathcal{R}^{(II)}(\varphi) + \mathcal{R}^{(III)}(\varphi)\,. \end{aligned} \tag{4.47}$$

The claimed statement then follows since $d_0 + 2\mathrm{i}\lambda_2 v_x = \langle d_0\rangle_x$ (see (4.35)), by the estimate (4.37) on v, the estimate (4.42) on $a_v^{(-1)}$, and the estimates (4.42), (4.45), and (4.46) on $\mathcal{R}^{(I)}, \mathcal{R}^{(II)}, \mathcal{R}^{(III)}$. ∎

4.6. Elimination of the x dependence from the order -1. In order to eliminate the x dependence from the term of order -1 in the operator $\mathcal{L}^{(5)}$ given in (4.36), we conjugate such an operator by means of a transformation

$$\Phi^{(6)}(\varphi) := \exp(\mathcal{G}(\varphi)), \quad \text{where} \quad \mathcal{G}(\varphi) := \frac{\mathrm{i}}{2}\big(g(x,\varphi)\circ\partial_x^{-2} + \partial_x^{-2}\circ g(x,\varphi)\big) \tag{4.48}$$

and $g(x,\varphi)$ is a real on real function to be determined. Note that for real values of the angle $\varphi \in \mathbb{T}^\infty$, one has that $\mathcal{G}(\varphi) = -\mathcal{G}(\varphi)^*$, implying that $\Phi^{(6)}(\varphi)$ is unitary.

LEMMA 4.7. *Let $\omega \in \mathtt{D}_\gamma$. For any $\sigma < \bar\sigma$ there exists $\delta(\sigma,\bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \le \delta$, the following holds. Define*

$$g(x,\varphi) := \frac{1}{2\lambda_2}\partial_x^{-1}\big[e_{-1}(x,\varphi) - \langle e_{-1}\rangle_x(\varphi)\big]\,. \tag{4.49}$$

(i) *The map $\mathbb{T}^\infty_\sigma \to \mathcal{B}\big(\mathcal{H}(\mathbb{T}_\sigma)\big)$, $\varphi \mapsto \Phi^{(6)}(\varphi)^{\pm1}$ is bounded.*
(ii) *For any $s \ge 0$, the map $\mathbb{T}^\infty \to \mathcal{B}\big(H^s(\mathbb{T})\big)$, $\varphi \mapsto \Phi^{(6)}(\varphi)^{\pm1}$ is bounded.*
(iii) *The map $\Phi^{(6)}(\varphi)$ transforms the operator $\mathcal{L}^{(5)}(\varphi)$ into*

$$\mathcal{L}^{(6)}(\varphi) = (\Phi^{(6)}_{\omega *})\mathcal{L}^{(5)}(\varphi) = \lambda_2\partial_{xx} + \lambda_1\partial_x + \langle d_0\rangle_x(\varphi) + \langle e_{-1}\rangle_x(\varphi)\partial_x^{-1} + \mathcal{R}^{(6)}(\varphi), \tag{4.50}$$

where the function $g(\cdot;\omega) \in \mathcal{H}(\mathbb{T}_\sigma \times \mathbb{T}^\infty_\sigma)$ is real on real and the operator $\mathcal{R}^{(6)}(\omega) \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big)$ is skew-self-adjoint. Moreover, they are defined $\omega \in \mathtt{D}_\gamma$ and satisfy for all $\Omega \subseteq \mathtt{D}_\gamma$ the estimates

$$\|g\|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma\,,\ |\mathcal{R}^{(6)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar\sigma} \varepsilon\,. \tag{4.51}$$

Proof. By (4.49), using the estimates on e_{-1} given in Lemma 4.6, one gets that g satisfies the estimate (4.51). By Lemma 3.6 and by the estimate on g one has that for any $0 < \sigma < \bar{\sigma}$,

$$\mathcal{G} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big)\,, \quad |\mathcal{G}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_\sigma \varepsilon\,. \tag{4.52}$$

The above estimate and Lemma 3.7, using that $\omega \cdot \partial_\varphi \Phi^{(6)} = \omega \cdot \partial_\varphi (\Phi^{(6)} - \mathrm{Id})$, imply that for any $0 < \sigma < \bar{\sigma}$

$$\sup_{\tau \in [0,1]} |\exp(\pm\tau\mathcal{G})|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma} \lesssim_{\sigma,\bar{\sigma}} 1, \quad |\omega \cdot \partial_\varphi(\Phi^{(6)})|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar{\sigma}} \varepsilon\,. \tag{4.53}$$

Items (i)–(ii) follow by the estimate (4.53) and by applying Lemmas 2.4 and 3.1.

(iii) A direct calculation shows that

$$\begin{aligned} \mathcal{L}^{(6)}(\varphi) &:= (\Phi^{(6)}_{\omega *})\mathcal{L}^{(5)}(\varphi) = \Phi^{(6)}(\varphi)^{-1}\mathcal{L}^{(5)}(\varphi)\Phi^{(6)}(\varphi) - \Phi^{(6)}(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi^{(6)}(\varphi) \\ &= \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x + \langle d_0\rangle(\varphi) + e_{-1}(x,\varphi)\partial_x^{-1} + [\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{G}(\varphi)] \\ &\quad + \mathcal{R}^{(I)}(\varphi), \end{aligned} \tag{4.54}$$

where the remainder $\mathcal{R}(\varphi)$ is given by

$$\begin{aligned} \mathcal{R}^{(I)}(\varphi) &:= \int_0^1 (1-t)\exp(-\tau\mathcal{G}(\varphi))\,[[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{G}(\varphi)], \mathcal{G}(\varphi)]\,\exp(\tau\mathcal{G}(\varphi))\,d\tau \\ &\quad + \int_0^1 e^{-\tau\mathcal{G}(\varphi)}\Big([\langle d_0\rangle_x + e_{-1}\partial_x^{-1}, \mathcal{G}(\varphi)]\Big)e^{\tau\mathcal{G}(\varphi)}\,d\tau - \Phi^{(6)}(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi^{(6)}(\varphi)\,. \end{aligned} \tag{4.55}$$

By recalling the estimate of Lemma 4.5 on d_0, the estimate of Lemma 4.6 on e_{-1}, and the estimate (4.52) on $\mathcal{G}$, and by applying Lemmas 3.5 and 3.6 and using that $\lambda_2 = 1 + O(\varepsilon)$ and $\lambda_1 = O(\varepsilon)$, one obtains that for any $0 < \sigma < \bar{\sigma}$

$$\begin{aligned} &[[\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{G}(\varphi)], \mathcal{G}(\varphi)]\,,\ [\langle d_0\rangle_x + e_{-1}\partial_x^{-1}, \mathcal{G}(\varphi)] \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big)\,, \\ &\Big|[[\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{G}(\varphi)], \mathcal{G}(\varphi)]\Big|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2}\,,\ \Big|[\langle d_0\rangle_x + e_{-1}\partial_x^{-1}, \mathcal{G}(\varphi)]\Big|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar{\sigma}} \varepsilon\,. \end{aligned} \tag{4.56}$$

Therefore, the estimates (4.53) and (4.56) and Lemma 3.5 imply that the remainder $\mathcal{R}^{(I)}$ defined in (4.55) satisfies

$$\mathcal{R}^{(I)} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big), \quad |\mathcal{R}^{(I)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar{\sigma}} \varepsilon \quad \forall 0 < \sigma < \bar{\sigma}\,. \tag{4.57}$$

Recalling the definition of $\mathcal{G}$, using the estimate (4.51) on g, by applying Lemma 3.6, using that $\lambda_2 = 1 + O(\varepsilon)$, $\lambda_1 = O(\varepsilon)$, one gets that

$$[\mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x, \mathcal{G}(\varphi)] = -2\lambda_2 g_x \partial_x^{-1} + \mathcal{R}^{(II)}(\varphi), \tag{4.58}$$

where, for any $0 < \sigma < \bar{\sigma}$,

$$\mathcal{R}^{(II)} \in \mathcal{H}\big(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2}\big), \quad |\mathcal{R}^{(II)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar{\sigma}} \varepsilon\,. \tag{4.59}$$

Therefore, by (4.54), one gets

$$\begin{aligned} &\mathcal{L}^{(6)}(\varphi) = \lambda_2\partial_{xx} + \lambda_1\partial_x + \langle d_0\rangle_x + \big(-2\lambda_2 g_x + e_{-1}\big)\partial_x^{-1} + \mathcal{R}^{(6)}(\varphi)\,, \\ &\mathcal{R}^{(6)}(\varphi) := \mathcal{R}^{(I)}(\varphi) + \mathcal{R}^{(II)}(\varphi)\,. \end{aligned} \tag{4.60}$$

The claimed statement then follows since $e_{-1} - 2\lambda_2 g_x = \langle e_{-1}\rangle_x$ (see (4.49)) and by recalling (4.57) and (4.59). ∎

4.7. Reduction to constant coefficients up to order −2. In the last step of our regularization procedure, we eliminate the φ-dependence from the term $\langle d_0 \rangle_x(\varphi) + \langle e_{-1} \rangle(\varphi)\partial_x^{-1}$. To achieve this purpose, we consider the map

$$\Phi^{(7)}(\varphi) := \exp(\mathcal{F}(\varphi)), \quad \mathcal{F}(\varphi) := \mathrm{diag}_{j \in \mathbb{Z}} f_j(\varphi), \tag{4.61}$$

where for any $j \in \mathbb{Z}$, f_j are analytic functions to be determined which are purely imaginary for any real value of the angle φ. We prove the following lemma.

LEMMA 4.8. *Let $\omega \in \mathtt{D}_\gamma$. For any $0 < \sigma < \bar\sigma$ there exists $\delta(\sigma, \bar\sigma) > 0$ such that if $\varepsilon\gamma^{-1} \le \delta$, the following holds. Define*

$$\begin{aligned} &\lambda_0 := \frac{1}{\mathrm{i}} \langle d_0 \rangle_{x,\varphi}, \quad \lambda_{-1} := \langle e_{-1} \rangle_{x,\varphi}\,, \\ &\mathcal{F}(\varphi) := (\omega \cdot \partial_\varphi)^{-1}[\langle d_0 \rangle_x - \mathrm{i}\lambda_0] + (\omega \cdot \partial_\varphi)^{-1}[e_{-1} - \lambda_{-1}]\partial_x^{-1}\,. \end{aligned} \tag{4.62}$$

(i) *The map $\mathbb{T}^\infty_\sigma \to \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma))$, $\varphi \mapsto \Phi^{(7)}(\varphi)^{\pm 1}$ is bounded.*
(ii) *For any $s \ge 0$, the map $\mathbb{T}^\infty \to \mathcal{B}(H^s(\mathbb{T}))$, $\varphi \mapsto \Phi^{(7)}(\varphi)^{\pm 1}$ is bounded.*
(iii) *The map $\Phi^{(7)}(\varphi)$ transforms the operator $\mathcal{L}^{(6)}(\varphi)$ into*

$$\mathcal{L}^{(7)}(\varphi) := (\Phi^{(7)}_{\omega *})\mathcal{L}^{(6)}(\varphi) = \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x + \mathrm{i}\lambda_0 + \lambda_{-1}\partial_x^{-1} + \mathcal{R}^{(7)}(\varphi), \tag{4.63}$$

where $\lambda_0, \lambda_{-1} \in \mathbb{R}$ and the operator $\mathcal{R}^{(7)} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^{\sigma,-2})$ satisfy the estimates

$$|\lambda_0|^{\mathrm{Lip}(\gamma,\Omega)}, |\lambda_{-1}|^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \varepsilon\,, \ |\mathcal{R}^{(7)}|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma,-2} \lesssim_{\sigma,\bar\sigma} \varepsilon\,. \tag{4.64}$$

Proof. Since the operator $\mathcal{F}(\varphi)$ is a diagonal operator, one has that $[\mathcal{F}(\varphi), \partial_x^k] = 0$ for any $k \in \mathbb{Z}$, and a direct calculation shows that

$$\Phi^{(7)}(\varphi)^{-1}\omega \cdot \partial_\varphi \Phi^{(7)}(\varphi) = \omega \cdot \partial_\varphi \mathcal{F}(\varphi)\,. \tag{4.65}$$

Therefore, by (4.62), we solve the homological equation

$$-\,\omega \cdot \partial_\varphi \mathcal{F}(\varphi) + \langle d_0 \rangle_x + \langle e_{-1} \rangle_x \partial_x^{-1} = \mathrm{i}\lambda_0 + \lambda_{-1}\partial_x^{-1}\,. \tag{4.66}$$

By the estimates (4.31) on d_0 and (4.37) on e_{-1} one gets that

$$|\lambda_0|^{\mathrm{Lip}(\gamma,\Omega)}, \ |\lambda_{-1}|^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \varepsilon$$

and by applying Lemmas 2.11 and 3.6 one obtains that for any $0 < \sigma < \bar\sigma$,

$$\mathcal{F} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^\sigma), \quad |\mathcal{F}|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma \lesssim_{\sigma,\bar\sigma} \varepsilon\gamma^{-1}\,. \tag{4.67}$$

The latter estimate and Lemma 3.7 imply that

$$(\Phi^{(7)})^{\pm 1} \in \mathcal{H}(\mathbb{T}^\infty_\sigma, \mathcal{B}^\sigma), \quad |(\Phi^{(7)})^{\pm 1}|^{\mathrm{Lip}(\gamma,\Omega)}_\sigma \le 1 + C(\sigma,\bar\sigma)\varepsilon\gamma^{-1} \tag{4.68}$$

for some constant $C(\sigma, \bar\sigma) > 0$. Hence, one obtains that

$$\begin{aligned} \mathcal{L}^{(7)}(\varphi) = (\Phi^{(7)}_{\omega *})\mathcal{L}^{(6)}(\varphi) &= \mathrm{i}\lambda_2\partial_{xx} + \lambda_1\partial_x - \omega \cdot \partial_\varphi \mathcal{F}(\varphi) \\ &\quad + \langle d_0 \rangle_x + \langle e_{-1} \rangle_x \partial_x^{-1} + \mathcal{R}^{(7)}(\varphi)\,, \\ \mathcal{R}^{(7)}(\varphi) := \Phi^{(7)}(\varphi)^{-1}&\mathcal{R}^{(6)}(\varphi)\Phi^{(7)}(\varphi)\,. \end{aligned} \tag{4.69}$$

The estimate (4.64) on the operator $\mathcal{R}^{(7)}$, defined in (4.69), follows by the composition Lemma 3.5, by the estimate (4.51) on $\mathcal{R}^{(6)}$, and by the estimate (4.68) on $(\Phi^{(7)})^{\pm 1}$. ∎

5. The KAM reducibility scheme. In this section we carry out the reducibility of the equation $\partial_t u = \mathcal{L}_0(\omega t)u$, where the operator $\mathcal{L}_0 \equiv \mathcal{L}^{(7)}$ is as given in Lemma 4.8. We fix

$$\sigma_0 := \frac{\bar{\sigma}}{2}\,. \tag{5.1}$$

The operator $\mathcal{L}_0(\varphi) \equiv \mathcal{L}_0(\varphi;\omega)$ defined for $\omega \in \mathtt{D}_\gamma$ has the form

$$\mathcal{L}_0(\varphi) = \mathrm{i}\mathcal{D}_0 + \mathcal{P}_0(\varphi), \tag{5.2}$$

where for all $\Omega \in \mathtt{D}_\gamma$

$$\begin{aligned}
&\mathcal{D}_0 := \lambda_2\partial_{xx} + \frac{1}{\mathrm{i}}\lambda_1\partial_x + \lambda_0 + \frac{1}{\mathrm{i}}\lambda_{-1}\partial_x^{-1}\,,\\
&\lambda_2,\lambda_1,\lambda_0,\lambda_{-1} \in \mathbb{R}, \quad |\lambda_2 - 1|, |\lambda_1|, |\lambda_0|^{\mathrm{Lip}(\gamma,\Omega)}, |\lambda_{-1}|^{\mathrm{Lip}(\gamma,\Omega)} \lesssim \varepsilon\,,\\
&|\mathcal{P}_0|^{\mathrm{Lip}(\gamma,\Omega)}_{\sigma_0,-2} \lesssim_{\sigma_0} \varepsilon\,.
\end{aligned} \tag{5.3}$$

Note that, as we pointed out in the previous section, the real constants λ_2, λ_1 do not depend on the parameter ω. The linear operator $\mathcal{D}_0$ is a 2×2 block diagonal operator $\mathcal{D}_0 = \mathrm{diag}_{j\in\mathbb{N}_0}\mathcal{D}_0(j)$ where, for any $j \in \mathbb{N}_0$, the 2×2 block $\mathcal{D}_0(j)$ is given by

$$\begin{aligned}
&\mathcal{D}_0(j) := \begin{pmatrix} \mu_j^{(0)} & 0 \\ 0 & \mu_{-j}^{(0)} \end{pmatrix},\\
&\mu_j^{(0)} := -\lambda_2 j^2 + \lambda_1 j + \lambda_0 - \lambda_{-1} j^{-1}\,, \quad \mu_{-j}^{(0)} := -\lambda_2 j^2 - \lambda_1 j + \lambda_0 + \lambda_{-1} j^{-1}\,.
\end{aligned} \tag{5.4}$$

In order to state our reducibility theorem, we fix some other constants. For $n \geq 1$, we set

$$\chi \in (1,2), \quad \sigma_n = \sigma_0\Big(1 - \frac{1}{4\pi}\sum_{j=1}^{n}\frac{1}{j^2}\Big), \quad N_n = \langle n\rangle^3\chi^n N_0, \tag{5.5}$$

and to shorten notation, we set

$$\mathtt{d}(\ell) := \prod_{n\in\mathbb{N}}(1 + |\ell_n|^4\langle n\rangle^4) \quad \forall \ell \in \mathbb{Z}^\infty_*\,. \tag{5.6}$$

THEOREM 5.1 (Reducibility). *Let $\gamma \in (0,1)$. Then there exists $\delta \in (0,1)$ small enough such that if $\varepsilon\gamma^{-1} \leq \delta$, for any $n \geq 0$, the following holds.*

$(\mathbf{S1})_\mathbf{n}$ *There exists a linear skew-self-adjoint vector field*

$$\mathcal{L}_n(\varphi) = \mathrm{i}\mathcal{D}_n + \mathcal{P}_n(\varphi), \tag{5.7}$$

where $\mathcal{D}_n$ is a 2×2 self-adjoint block diagonal operator $\mathcal{D}_n = \mathrm{diag}_{j\in\mathbb{N}_0}\mathcal{D}_n(j)$, $\mathcal{P}_n \in \mathcal{H}\big(\mathbb{T}^\infty_{\sigma_n}, \mathcal{B}^{\sigma_n,-2}\big)$ is skew-self-adjoint, and moreover both are defined for $\omega \in \Omega_n(\gamma)$, where $\Omega_0(\gamma) := \mathtt{D}_\gamma$ and for any $n \geq 1$

(5.8)

$$\begin{aligned}
\Omega_n(\gamma) := \Big\{&\omega \in \Omega_{n-1}(\gamma) : \|\mathcal{O}_{n-1}(\ell,j,j')^{-1}\|_{\mathrm{Op}} \leq \frac{\mathtt{d}(\ell)}{\gamma} \quad \forall (\ell,j,j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0,\\
&j \neq j' \quad \textit{and} \quad \|\mathcal{O}_{n-1}(\ell,j,j)^{-1}\|_{\mathrm{Op}} \leq \frac{\mathtt{d}(\ell)j^2}{\gamma}\\
&\forall (\ell,j) \in (\mathbb{Z}^\infty_* \setminus \{0\}) \times \mathbb{N}_0, \quad |\ell|_\eta \leq N_{n-1}\Big\}.
\end{aligned}$$

For any $(\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0$, *the operators* $\mathcal{O}_{n-1}(\ell, j, j') : \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ *are defined by*

$$\mathcal{O}_{n-1}(\ell, j, j') := \omega \cdot \ell \,\mathrm{Id} + M_L(\mathcal{D}_{n-1}(j)) - M_R(\mathcal{D}_{n-1}(j')) \,. \tag{5.9}$$

For any $j \in \mathbb{N}_0$,

$$\|\mathcal{D}_n(j) - \mathcal{D}_0(j)\|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\mathtt{HS}} \lesssim \varepsilon \tag{5.10}$$

and

$$|\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2} \le C_* \varepsilon e^{-\chi^n} \tag{5.11}$$

for some constant $C_* > 0$.

For $n \ge 1$, *there exists a map* $\Phi_n(\varphi) := \exp(\mathcal{F}_n(\varphi))$, *where*

$$\mathcal{F}_n \in \mathcal{H}\Big(\mathbb{T}^\infty_{\frac{\sigma_{n-1}+\sigma_n}{2}}, \mathcal{B}^{\frac{\sigma_{n-1}+\sigma_n}{2}}\Big)$$

is skew-self-adjoint and defined for $\omega \in \Omega_n(\gamma)$, *which satisfies*

$$\mathcal{L}_n(\varphi) = (\Phi_n)_{\omega *} \mathcal{L}_{n-1}(\varphi) \,. \tag{5.12}$$

The operator $\mathcal{F}_n$ *satisfies the estimate*

$$|\mathcal{F}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\frac{\sigma_{n-1}+\sigma_n}{2}} \lesssim \varepsilon \gamma^{-1} e^{-\frac{\chi^{n-1}}{2}} \,. \tag{5.13}$$

$(\mathbf{S2})_{\mathbf{n}}$ *For any* $j \in \mathbb{N}_0$ *there exists a Lipschitz extension of the function* $\mathcal{D}_n(j; \cdot) : \Omega_n(\gamma) \to \mathcal{S}(\mathbf{E}_j)$ *to the set* $\mathtt{D}_\gamma$, *denoted by* $\widetilde{\mathcal{D}}_n(j; \cdot) : \mathtt{D}_\gamma \to \mathcal{S}(\mathbf{E}_j)$, *that, for any* $n \ge 1$, *satisfies the estimate*

$$\begin{aligned} &\sup_{\omega \in \mathtt{D}_\gamma} \|\widetilde{\mathcal{D}}_n(j;\omega) - \widetilde{\mathcal{D}}_{n-1}(j;\omega)\|_{\mathtt{HS}} \lesssim \langle j \rangle^{-2} \varepsilon e^{-\chi^{n-1}} \,, \\ &\|\widetilde{\mathcal{D}}_n(j) - \widetilde{\mathcal{D}}_{n-1}(j)\|^{\mathrm{lip}}_{\mathtt{HS}} \lesssim \varepsilon \gamma^{-1} e^{-\chi^{n-1}} \,. \end{aligned} \tag{5.14}$$

5.1. Proof of Theorem 5.1. *Proof of* $(\mathbf{Si})_0$, $i = 1, 2$. The claims hold by recalling the properties of the operator $\mathcal{L}_0$ listed in (5.2)–(5.4).

$(\mathbf{S2})_0$ holds, since the constants λ_2 and λ_1 are independent of ω and λ_0, λ_{-1} are already defined on $\mathtt{D}_\gamma$.

5.1.1. The reducibility step. *Proof of* $(\mathbf{S1})_{n+1}$. We now describe the inductive step, showing how to define a symplectic transformation $\Phi_{n+1} := \exp(\mathcal{F}_{n+1})$ so that the transformed vector field $\mathcal{L}_{n+1}(\varphi) = (\Phi_{n+1})_{\omega *} \mathcal{L}_n(\varphi)$ has the desired properties. We write Π_n instead of Π_{N_n} to denote the projector on the Fourier modes $|\ell|_\eta \le N_n$, where N_n is defined as in (5.5). A direct calculation shows that

$$\begin{aligned} \mathcal{L}_{n+1}(\varphi) &= (\Phi_{n+1})_{\omega *} \mathcal{L}_n(\varphi) = \Phi_{n+1}(\varphi)^{-1} \mathcal{L}_n(\varphi) \Phi_{n+1}(\varphi) - \Phi_{n+1}(\varphi)^{-1} \omega \cdot \partial_\varphi \Phi_{n+1}(\varphi) \\ &= \mathrm{i}\mathcal{D}_n - \omega \cdot \partial_\varphi \mathcal{F}_{n+1} + [\mathrm{i}\mathcal{D}_n, \mathcal{F}_{n+1}] + \Pi_n \mathcal{P}_n + \Pi_n^\perp \mathcal{P}_n \\ &\quad + \int_0^1 (1-\tau) e^{-\tau \mathcal{F}_{n+1}} [[\mathrm{i}\mathcal{D}_n, \mathcal{F}_{n+1}], \mathcal{F}_{n+1}] e^{\tau \mathcal{F}_{n+1}} \, d\tau \\ &+ \int_0^1 e^{-\tau \mathcal{F}_{n+1}} [\mathcal{P}_n, \mathcal{F}_{n+1}] e^{\tau \mathcal{F}_{n+1}} \, d\tau - \int_0^1 (1-\tau) e^{-\tau \mathcal{F}_{n+1}} [\omega \cdot \partial_\varphi \mathcal{F}_{n+1}, \mathcal{F}_{n+1}] e^{\tau \mathcal{F}_{n+1}} \, d\tau \,. \end{aligned} \tag{5.15}$$

Our next aim is to solve the homological equation

$$- \omega \cdot \partial_\varphi \mathcal{F}_{n+1} + [\mathrm{i}\mathcal{D}_n, \mathcal{F}_{n+1}] + \Pi_n \mathcal{P}_n = [\widehat{\mathcal{P}}_n(0)], \tag{5.16}$$

where the diagonal part of the operator $\widehat{\mathcal{P}}_n(0)$ is defined according to (3.8).

LEMMA 5.2. *For all $\omega \in \Omega_{n+1}(\gamma)$ (see (5.8)), there exists a unique solution $F_{n+1} \in \mathcal{H}(\mathbb{T}^\infty_{\sigma_n-\rho}, \mathcal{B}^{\sigma_n-\rho})$ with $\rho > 0$, $\sigma_n - \rho > 0$, of the homological equation (5.16) satisfying the bound*

$$|\mathcal{F}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\sigma_n-\rho} \lesssim \gamma^{-1} \exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}} \ln\Big(\frac{\tau}{\rho}\Big)\Big) |\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2} \tag{5.17}$$

for some appropriate constant $\tau > 1$.

Proof. In order to simplify notation in this proof, we drop the index n and we write $+$ instead of $n+1$. Passing to the 2×2 block representation of operators and taking the Fourier transform with respect to φ, one gets that (5.16) is equivalent to

$$\begin{aligned} &\mathrm{i}\Big(- \omega \cdot \ell \pi_j \widehat{\mathcal{F}}_+(\ell) \pi_{j'} + \mathcal{D}(j) \pi_j \widehat{\mathcal{F}}_+(\ell) \pi_{j'} - \pi_j \widehat{\mathcal{F}}_+(\ell) \pi_{j'} \mathcal{D}(j')\Big) + \pi_j \widehat{\mathcal{P}}(\ell) \pi_{j'} = 0 \\ &\forall (\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0, \quad (\ell, j, j') \neq (0, j, j), \quad |\ell|_\eta \le N, \\ &\quad \text{and} \quad \pi_j \widehat{\mathcal{F}}_+(0) \pi_j = 0 \quad \forall j \in \mathbb{N}_0 \,. \end{aligned} \tag{5.18}$$

According to the definition given in (5.9), for any $\omega \in \Omega_+(\gamma) \equiv \Omega_{n+1}(\gamma)$, since the operator

$$\mathcal{O}(\ell, j, j') := \omega \cdot \ell \,\mathrm{Id} - M_L(\mathcal{D}(j)) + M_R(\mathcal{D}(j')) \tag{5.19}$$

is invertible, one defines $\mathcal{F}_+$ as

$$\pi_j \widehat{\mathcal{F}}_+(\ell) \pi_{j'} := \begin{cases} -\mathrm{i}\mathcal{O}(\ell, j, j')^{-1} \pi_j \widehat{\mathcal{P}}(\ell) \pi_{j'} & \forall (\ell, j, j') \neq (0, j, j), \\ 0 \quad \forall (\ell, j, j') = (0, j, j)\,. \end{cases} \tag{5.20}$$

For any $(\ell, j, j') \neq (0, j, j)$, $j \neq j'$, $|\ell| \le N$, one obtains that

$$\|\pi_j \widehat{\mathcal{F}}_+(\ell) \pi_{j'}\|_{\mathtt{HS}} \le \frac{\mathtt{d}(\ell)}{\gamma} \|\pi_j \widehat{\mathcal{P}}(\ell) \pi_{j'}\|_{\mathtt{HS}}, \tag{5.21}$$

and for $\ell \neq 0$, $|\ell|_\eta \le N$,

$$\|\pi_j \widehat{\mathcal{F}}_+(\ell) \pi_j\|_{\mathtt{HS}} \le \frac{\mathtt{d}(\ell) \langle j \rangle^2}{\gamma} \|\pi_j \widehat{\mathcal{P}}(\ell) \pi_j\|_{\mathtt{HS}}\,. \tag{5.22}$$

Let $\sigma \equiv \sigma_n$. By recalling (3.11), the estimates (5.21) and (5.22) imply that for any $\ell \in \mathbb{Z}^\infty$, $|\ell|_\eta \le N$

$$\|\widehat{\mathcal{F}}_+(\ell)\|_{\mathcal{B}^{\sigma-\rho}} \le \mathtt{d}(\ell) \gamma^{-1} \|\widehat{\mathcal{P}}(\ell)\|_{\mathcal{B}^{\sigma,-2}}\,. \tag{5.23}$$

Hence in view of the (3.14), one obtains that

$$\begin{aligned} |\mathcal{F}_+|_{\sigma-\rho} &\le \gamma^{-1} \sum_{\ell \in \mathbb{Z}^\infty_*} \mathtt{d}(\ell) e^{(\sigma-\rho)|\ell|_\eta} \|\widehat{\mathcal{P}}(\ell)\|_{\mathcal{B}^{\sigma,-2}} \le \gamma^{-1} \Big(\sup_{\ell \in \mathbb{Z}^\infty_*} \mathtt{d}(\ell) e^{-\rho|\ell|_\eta}\Big) |\mathcal{P}|_{\sigma,-2} \\ &\overset{\textit{Lemma B.1}}{\le} \gamma^{-1} \exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}} \ln\Big(\frac{\tau}{\rho}\Big)\Big) |\mathcal{P}|_{\sigma,-2}\,. \end{aligned} \tag{5.24}$$

Now we show the Lipschitz estimate. Let $\omega_1, \omega_2 \in \Omega_+(\gamma)$. Then for any $(\ell, j, j') \neq (0, j, j')$, $|\ell|_\eta \leq N$,

$$\begin{aligned} \Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{F}}_+(\ell)\pi_{j'}\big) &= -\mathrm{i}\mathcal{O}(\ell, j, j'; \omega_1)^{-1}\Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{P}}(\ell)\pi_{j'}\big) \\ &\quad + \mathrm{i}\mathcal{O}(\ell, j, j'; \omega_1)^{-1}\big(\Delta_{\omega_1\omega_2}\mathcal{O}(\ell, j, j')\big)\mathcal{O}(\ell, j, j'; \omega_2)^{-1}\pi_j \widehat{\mathcal{P}}(\ell; \omega_2)\pi_{j'}\,. \end{aligned} \tag{5.25}$$

By (5.3), (5.4), (5.10), and (A.7), one obtains that

$$\begin{aligned} \|\Delta_{\omega_1\omega_2}\mathcal{O}(\ell, j, j')\|_{\mathrm{Op}} &\leq \|\omega_1 - \omega_2\|_\infty |\ell|_\eta + 2\sup_{j\in\mathbb{N}_0}\|\Delta_{\omega_1\omega_2}\mathcal{D}(j)\|_{\mathtt{HS}} \\ &\lesssim (1 + |\ell|_\eta)\|\omega_1 - \omega_2\|_\infty\,. \end{aligned} \tag{5.26}$$

Hence since $\omega_1, \omega_2 \in \Omega_+(\gamma)$, the formula (5.25) and the estimate (5.26) imply that for any $\ell \in \mathbb{Z}^\infty_*$, $j \neq j'$, $|\ell|_\eta \leq N$,

$$\begin{aligned} \|\Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{F}}_+(\ell)\pi_{j'}\big)\|_{\mathtt{HS}} &\lesssim \frac{\mathtt{d}(\ell)^2}{\gamma^2}(1 + |\ell|_\eta)\|\pi_j \widehat{\mathcal{P}}(\ell; \omega_2)\pi_{j'}\|_{\mathtt{HS}} \\ &\quad + \frac{\mathtt{d}(\ell)}{\gamma}\|\Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{P}}(\ell)\pi_{j'}\big)\|_{\mathtt{HS}}, \end{aligned} \tag{5.27}$$

and for any $\ell \in \mathbb{Z}^\infty_* \setminus \{0\}$, $j \in \mathbb{N}_0$, $|\ell|_\eta \leq N$,

$$\begin{aligned} \|\Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{F}}_+(\ell)\pi_j\big)\|_{\mathtt{HS}} &\lesssim \frac{\mathtt{d}(\ell)^2\langle j\rangle^4}{\gamma^2}(1 + |\ell|_\eta)\|\pi_j \widehat{\mathcal{P}}(\ell; \omega_2)\pi_j\|_{\mathtt{HS}}\|\omega_1 - \omega_2\|_\infty \\ &\quad + \frac{\mathtt{d}(\ell)\langle j\rangle^2}{\gamma}\|\Delta_{\omega_1\omega_2}\big(\pi_j \widehat{\mathcal{P}}(\ell)\pi_j\big)\|_{\mathtt{HS}}\,. \end{aligned} \tag{5.28}$$

Recalling (3.11) and using the estimates (5.27) and (5.28), one obtains that

$$\begin{aligned} \|\Delta_{\omega_1\omega_2}\widehat{\mathcal{F}}_+(\ell)\|_{\mathcal{B}^{\sigma-\rho,2}} &\lesssim \frac{\mathtt{d}(\ell)^2}{\gamma^2}(1 + |\ell|_\eta)\|\widehat{\mathcal{P}}(\ell; \omega_2)\|_{\mathcal{B}^{\sigma,-2}}\|\omega_1 - \omega_2\|_\infty \\ &\quad + \frac{\mathtt{d}(\ell)}{\gamma}\|\Delta_{\omega_1\omega_2}\widehat{\mathcal{P}}(\ell)\|_{\mathcal{B}^\sigma}\,. \end{aligned} \tag{5.29}$$

Hence, recalling (3.14), one gets

$$\begin{aligned} &|\Delta_{\omega_1\omega_2}\mathcal{F}_+|_{\sigma-\rho,2} \lesssim \gamma^{-2}\Big(\sup_{\ell\in\mathbb{Z}^\infty_*}\mathtt{d}(\ell)^2 e^{-\rho|\ell|_\eta}(1 + |\ell|_\eta)\Big)\|\omega_1 - \omega_2\|_\infty \sup_{\omega\in\Omega}|\mathcal{P}(\omega)|_{\sigma,-2} \\ &\quad + \gamma^{-1}\Big(\sup_{\ell\in\mathbb{Z}^\infty_*}\mathtt{d}(\ell)e^{-\rho|\ell|_\eta}\Big)|\Delta_{\omega_1\omega_2}\mathcal{P}|_\sigma \\ &\overset{Lemma\ B.1}{\lesssim} \gamma^{-2}\exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\rho}\Big)\Big)\Big(\|\omega_1 - \omega_2\|_\infty \sup_{\omega\in\Omega}|\mathcal{P}(\omega)|_{\sigma,-2} + \gamma|\Delta_{\omega_1\omega_2}\mathcal{P}|_\sigma\Big) \end{aligned} \tag{5.30}$$

for some $\tau > 0$. The bounds (5.24) and (5.30), together with (3.15), imply the claimed bound. ∎

By the formula (5.15) and using that the operator $\mathcal{F}_{n+1}$ solves the homological equation (5.16), one obtains that

$$
\begin{aligned}
&\mathcal{L}_{n+1}(\varphi) := \mathrm{i}\mathcal{D}_{n+1} + \mathcal{P}_{n+1}(\varphi)\,,\\
&\quad \mathcal{D}_{n+1} := \mathcal{D}_n + \mathcal{Z}_n, \quad \mathcal{Z}_n := \frac{1}{\mathrm{i}}[\widehat{\mathcal{P}}_n(0)]\,,\\
&\quad \mathcal{P}_{n+1} := \Pi_n^\perp \mathcal{P}_n + \int_0^1 (1-\tau) e^{-\tau \mathcal{F}_{n+1}}[[\widehat{\mathcal{P}}_n(0)] - \Pi_n \mathcal{P}_n, \mathcal{F}_{n+1}] e^{\tau \mathcal{F}_{n+1}}\, d\tau\\
&\qquad\qquad + \int_0^1 e^{-\tau \mathcal{F}_{n+1}} [\mathcal{P}_n, \mathcal{F}_{n+1}] e^{\tau \mathcal{F}_{n+1}}\, d\tau\,.
\end{aligned} \tag{5.31}
$$

The new block-diagonal part $\mathcal{D}_{n+1}$. Since by the inductive hypothesis the operator $\mathcal{P}_n(\varphi)$ is skew-self-adjoint, the 2×2 block-diagonal operator $[\widehat{\mathcal{P}}_n(0)] = \mathrm{diag}_{j\in\mathbb{N}_0} \pi_j \widehat{\mathcal{P}}_n(0)\pi_j$ is also skew-self-adjoint; therefore, the 2×2 block-diagonal operator $\mathcal{Z}_n := \frac{1}{\mathrm{i}}[\widehat{\mathcal{P}}_n(0)]$ is self-adjoint. Hence, using the induction hypothesis, one gets that $\mathcal{D}_{n+1}$ is a 2×2 self-adjoint block-diagonal operator. We then set

$$
\mathcal{D}_{n+1}(j) := \pi_j \mathcal{D}_{n+1} \pi_j := \mathcal{D}_n(j) + \mathcal{Z}_n(j), \quad \mathcal{Z}_n(j) := \pi_j \mathcal{Z}_n \pi_j \quad \forall j \in \mathbb{N}_0\,. \tag{5.32}
$$

By the inductive estimate (5.11), one gets that for any $\sigma \le \sigma_n$

$$
|\mathcal{Z}_n|_{\sigma,-2}^{\mathrm{Lip}(\gamma,\Omega_n)} = |\mathcal{D}_{n+1} - \mathcal{D}_n|_{\sigma,-2}^{\mathrm{Lip}(\gamma,\Omega_n)} \le |\mathcal{P}_n|_{\sigma_n,-2}^{\mathrm{Lip}(\gamma,\Omega_n)} \lesssim \varepsilon e^{-\chi^n}\,. \tag{5.33}
$$

The latter estimate implies that

$$
\begin{aligned}
&\sup_{\omega\in\Omega_n(\gamma)} \|\mathcal{Z}_n(j;\omega)\|_{\mathtt{HS}} \lesssim \varepsilon e^{-\chi^n} \langle j\rangle^{-2}\,,\\
&\sup_{\substack{\omega_1,\omega_2\in\Omega_n(\gamma)\\ \omega_1\neq\omega_2}} \frac{\|\mathcal{Z}_n(j;\omega_1) - \mathcal{Z}_n(j;\omega_2)\|_{\mathtt{HS}}}{\|\omega_1-\omega_2\|_\infty} \lesssim \varepsilon \gamma^{-1} e^{-\chi^n}\,,
\end{aligned} \tag{5.34}
$$

uniformly with respect to $j \in \mathbb{N}_0$. The estimate (5.9) at the step $n+1$ then follows by applying (5.33), using a telescoping argument.

The new remainder $\mathcal{P}_{n+1}$. By applying Lemma 3.5(ii), one obtains the estimates

$$
|\Pi_n^\perp \mathcal{P}_n|_{\sigma_{n+1},-2}^{\mathrm{Lip}(\gamma,\Omega_n)} \le e^{-N_n(\sigma_n-\sigma_{n+1})} |\mathcal{P}_n|_{\sigma_n,-2}^{\mathrm{Lip}(\gamma,\Omega_n)}\,. \tag{5.35}
$$

Furthermore, by applying iteratively Lemma 3.5(i),(iii), one obtains that if $\rho > 0$ satisfies $\sigma_{n+1} + 3\rho < \sigma_n$, then

$$
\begin{aligned}
&\Big| e^{-\tau\mathcal{F}_{n+1}} [\mathcal{P}_n, \mathcal{F}_{n+1}] e^{\tau\mathcal{F}_{n+1}} \Big|_{\sigma_{n+1},-2}^{\mathrm{Lip}(\gamma,\Omega_{n+1})}\\
&+ \Big| e^{-\tau\mathcal{F}_{n+1}} [[\widehat{\mathcal{P}}_n(0)] - \Pi_n\mathcal{P}_n, \mathcal{F}_{n+1}] e^{\tau\mathcal{F}_{n+1}} \Big|_{\sigma_{n+1},-2}^{\mathrm{Lip}(\gamma,\Omega_{n+1})}\\
&\quad \lesssim \rho^{-\mathtt{a}} \Big(\sup_{\tau\in[0,1]} |e^{\pm\tau\mathcal{F}_{n+1}}|_{\sigma_{n+1}+3\rho}^{\mathrm{Lip}(\gamma,\Omega_{n+1})} \Big) |\mathcal{P}_n|_{\sigma_n,-2}^{\mathrm{Lip}(\gamma,\Omega_n)} |\mathcal{F}_{n+1}|_{\sigma_{n+1}+2\rho}^{\mathrm{Lip}(\gamma,\Omega_{n+1})}
\end{aligned} \tag{5.36}
$$

for some constant $\mathtt{a} > 0$.

Now we want to use Lemma 3.7 in order to estimate $\sup_{\tau\in[0,1]} |e^{\pm\tau\mathcal{F}_{n+1}}|_{\sigma_{n+1}+3\rho}^{\mathrm{Lip}(\gamma,\Omega_{n+1})}$. We fix $\rho := \frac{\sigma_n - \sigma_{n+1}}{8}$ so that $\sigma_{n+1} + 4\rho = \sigma_{n+1} + \frac{\sigma_n-\sigma_{n+1}}{2} = \frac{\sigma_n+\sigma_{n+1}}{2} < \sigma_n$. With this choice of ρ, by applying Lemma 5.2 and the inductive estimate (5.11) on $\mathcal{P}_n$, one obtains that

$$
\begin{aligned}
|\mathcal{F}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\frac{\sigma_n+\sigma_{n+1}}{2}} &= |\mathcal{F}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\sigma_{n+1}+4\rho} \\
&\lesssim \gamma^{-1}\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)\Big)|\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2} \\
&\lesssim \varepsilon\gamma^{-1}\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)-\chi^n\Big) \\
&\lesssim \varepsilon\gamma^{-1}e^{-\frac{\chi^n}{2}}\,,
\end{aligned}
\tag{5.37}
$$

using that, by (5.5), one has

$$
\sup_{n\in\mathbb{N}}\Big\{\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)-\frac{\chi^n}{2}\Big)\Big\}<\infty\,.
$$

The estimate (5.37) proves the estimate (5.13) at the step $n+1$. Furthermore,

$$
\frac{1}{(\sigma_n-\sigma_{n+1})^2}|\mathcal{F}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\frac{\sigma_n+\sigma_{n+1}}{2}}\le\delta
\tag{5.38}
$$

for some $\delta\in(0,1)$ small enough by taking $\varepsilon\gamma^{-1}$ small enough and using that by (5.5)

$$
\lim_{n\to\infty}\frac{1}{(\sigma_n-\sigma_{n+1})^2}e^{-\frac{\chi^n}{2}}=0\,.
$$

The smallness condition (3.32) of Lemma 3.7 is verified, and therefore we get the estimate

$$
\sup_{\tau\in[0,1]}|e^{\pm\tau\mathcal{F}_{n+1}}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\sigma_n+3\rho}\lesssim 1\,.
\tag{5.39}
$$

The estimates (5.35)–(5.37) and (5.39) (recalling the definition of the remainder $\mathcal{P}_{n+1}$ given in (5.31)) lead to the inductive estimate

$$
\begin{aligned}
|\mathcal{P}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\sigma_{n+1},-2} &\le e^{-N_n(\sigma_n-\sigma_{n+1})}|\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2} \\
&\quad + C\gamma^{-1}\frac{1}{(\sigma_n-\sigma_{n+1})^{\mathfrak{a}}}\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)\Big)(|\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2})^2\,,
\end{aligned}
\tag{5.40}
$$

where $C>0$ is a positive constant and $\mathfrak{a}>0$ is the constant appearing in the estimate (5.36). The latter estimate and the inductive estimate (5.11) on $|\mathcal{P}_n|^{\mathrm{Lip}(\gamma,\Omega_n)}_{\sigma_n,-2}$ together imply that

$$
\begin{aligned}
|\mathcal{P}_{n+1}|^{\mathrm{Lip}(\gamma,\Omega_{n+1})}_{\sigma_{n+1},-2} &\le e^{-N_n(\sigma_n-\sigma_{n+1})}C_*\varepsilon e^{-\chi^n} \\
&\quad + C\gamma^{-1}\frac{1}{(\sigma_n-\sigma_{n+1})^{\mathfrak{a}}}\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)\Big)C_*^2\varepsilon^2e^{-2\chi^n} \\
&\le C_*\varepsilon e^{-\chi^{n+1}}
\end{aligned}
\tag{5.41}
$$

provided

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}}\Big\{\exp\Big(\chi^n(\chi-1)-N_n(\sigma_n-\sigma_{n+1})\Big)\Big\}\le\frac{1}{2}\,, \\
&CC_*\varepsilon\gamma^{-1}\sup_{n\in\mathbb{N}}\Big\{\frac{1}{(\sigma_n-\sigma_{n+1})^{\mathfrak{a}}}\exp\Big(\frac{\tau}{(\sigma_n-\sigma_{n+1})^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\sigma_n-\sigma_{n+1}}\Big)-(2-\chi)\chi^n\Big)\Big\}\le\frac{1}{2}\,.
\end{aligned}
$$

The first condition above holds by recalling (5.5) and by taking $N_0 > 0$ large enough. The second condition above holds by recalling (5.5) and by taking $\varepsilon\gamma^{-1}$ small enough.

Proof of $(\mathbf{S2})_{n+1}$. We shall need the following lemma (which is a simple adaptation of the extension result Lemma M5 in [29]).

LEMMA 5.3. *Let (X, d) be a metric space, and let $\mathcal{H}$ be a Hilbert space of dimension d. Let $E \subseteq X$ be a subset of X and*

$$f : E \to \mathcal{H}$$

be a function satisfying

$$\|f\|_\infty := \sup_{x \in E} \|f(x)\|_{\mathcal{H}} < \infty, \quad \|f\|^{\rm lip} := \sup_{x_1, x_2 \in E, x_1 \neq x_2} \frac{\|f(x_1) - f(x_2)\|_{\mathcal{H}}}{d(x_1, x_2)} < \infty\,.$$

Then there exists a Lipschitz extension $\widetilde{f} : X \to \mathcal{H}$ satisfying

$$\|\widetilde{f}\|_\infty \lesssim_d \|f\|_\infty, \quad \|\widetilde{f}\|^{\rm lip} \lesssim_d \|f\|^{\rm lip}\,.$$

By recalling the estimate (5.34), for any $j \in \mathbb{N}_0$, the function $\Omega_{n+1}(\gamma) \to \mathcal{S}(\mathbf{E}_j)$, $\omega \mapsto \mathcal{Z}_n(j; \omega) = \mathcal{D}_{n+1}(j; \omega) - \mathcal{D}_n(j; \omega)$ is a Lipschitz function. Hence we apply the extension Lemma 5.3 with $E = \Omega_{n+1}(\gamma)$, $(X, d) = (\mathtt{D}_\gamma, d_\infty)$ (recall Definition 1.1 and (1.3)), and $\mathcal{H} = \mathcal{S}(\mathbf{E}_j)$ equipped with the scalar product (A.3). Hence we get a Lipschitz extension $\widetilde{\mathcal{Z}}_n(j; \cdot) : \mathtt{D}_\gamma \to \mathcal{S}(\mathbf{E}_j)$ of $\mathcal{Z}_n(j)$ preserving the sup norm and the Lipschitz seminorm, namely $\sup_{\omega \in \mathtt{D}_\gamma} \|\widetilde{\mathcal{Z}}_n(j; \omega)\|_{\mathtt{HS}} \lesssim \sup_{\omega \in \Omega_{n+1}(\gamma)} \|\mathcal{Z}_n(j; \omega)\|_{\mathtt{HS}}$, $\|\widetilde{\mathcal{Z}}_n(j)\|_{\mathtt{HS}}^{\rm lip} \lesssim \|\mathcal{Z}_n(j)\|_{\mathtt{HS}}^{\rm lip}$. Therefore, using the bounds (5.34) and defining $\widetilde{\mathcal{D}}_{n+1}(j) := \widetilde{\mathcal{D}}_n(j) + \widetilde{\mathcal{Z}}_n(j)$, the claimed statement follows.

5.2. Convergence. Final blocks. By applying Theorem 5.1$(\mathbf{S2})_{\mathbf{n}}$ the sequence of the Lipschitz functions $\widetilde{\mathcal{D}}_n(j; \cdot) : \mathtt{D}_\gamma \to \mathcal{S}(\mathbf{E}_j)$, $j \in \mathbb{N}_0$, is a Cauchy sequence with respect to the norm $\| \cdot \|^{{\rm Lip}(\gamma, \Omega_0)}$, and, therefore, we can define the *final blocks*

$$\mathcal{D}_\infty(j) := \lim_{n \to \infty} \widetilde{\mathcal{D}}_n(j) \quad \forall j \in \mathbb{N}_0\,. \tag{5.42}$$

By using a telescoping argument one obtains that for any $j \in \mathbb{N}_0$, for any $n \in \mathbb{N}$, the following estimates hold:

$$\begin{aligned} &\sup_{\omega \in \mathtt{D}_\gamma} \|\mathcal{D}_\infty(j; \omega) - \widetilde{\mathcal{D}}_n(j; \omega)\|_{\mathtt{HS}} \lesssim \langle j \rangle^{-2} \varepsilon e^{-\chi^n}\,, \\ &\|\mathcal{D}_\infty(j) - \widetilde{\mathcal{D}}_n(j)\|_{\mathtt{HS}}^{\rm lip} \lesssim \varepsilon \gamma^{-1} e^{-\chi^n}\,. \end{aligned} \tag{5.43}$$

Then, recalling the definition of the norm $| \cdot |_{\sigma, m}^{{\rm Lip}(\gamma, \Omega)}$ given in (3.15), if we define the 2×2 block diagonal operators

$$\widetilde{\mathcal{D}}_n := {\rm diag}_{j \in \mathbb{N}_0} \widetilde{\mathcal{D}}_n(j) \quad \forall n \in \mathbb{N}\,, \quad \mathcal{D}_\infty := {\rm diag}_{j \in \mathbb{N}_0} \mathcal{D}_\infty(j), \tag{5.44}$$

one gets that for any $\sigma > 0$, $n \in \mathbb{N}$, and $\Omega \in \mathtt{D}_\gamma$

$$|\mathcal{D}_\infty - \widetilde{\mathcal{D}}_n|_{\sigma, -2}^{{\rm Lip}(\gamma, \Omega)} \lesssim \varepsilon e^{-\chi^n}\,. \tag{5.45}$$

Final Cantor set. For any $\ell \in \mathbb{Z}_*^\infty$, $j, j' \in \mathbb{N}_0$, we define the linear operator $\mathcal{O}_\infty(\ell, j, j') : \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$,

$$\mathcal{O}_\infty(\ell, j, j') := \omega \cdot \ell\, {\rm Id} - M_L(\mathcal{D}_\infty(j)) + M_R(\mathcal{D}_\infty(j')), \tag{5.46}$$

and we define the set

(5.47)
$$\Omega_\infty(\gamma) := \Big\{\omega \in \mathtt{D}_\gamma : \|\mathcal{O}_\infty(\ell, j, j')^{-1}\|_{\rm Op} \le \frac{\mathtt{d}(\ell)}{2\gamma} \quad \forall (\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0,$$
$$j \neq j' \quad \text{and} \quad \|\mathcal{O}_\infty(\ell, j, j)^{-1}\|_{\rm Op} \le \frac{\mathtt{d}(\ell) j^2}{2\gamma} \quad \forall (\ell, j) \in (\mathbb{Z}^\infty_* \setminus \{0\}) \times \mathbb{N}_0 \Big\}.$$

The following lemma holds.

Lemma 5.4. *One has that*
$$\Omega_\infty(\gamma) \subseteq \cap_{n \in \mathbb{N}_0} \Omega_n(\gamma).$$

Proof. We proceed by induction. By definition $\Omega_\infty(\gamma) \subseteq \mathtt{D}_\gamma$. Now assume that $\Omega_\infty(\gamma) \subseteq \Omega_n(\gamma)$ for some $n \ge 0$ and let us show that $\Omega_\infty(\gamma) \subseteq \Omega_{n+1}(\gamma)$. Let $\omega \in \Omega_\infty(\gamma)$. Since by the induction hypothesis $\omega \in \Omega_n(\gamma)$, the 2×2 blocks $\mathcal{D}_n(j; \omega)$, $j \in \mathbb{N}_0$, are well defined and $\mathcal{D}_n(j; \omega) = \widetilde{\mathcal{D}}_n(j; \omega)$ on such a set. By the estimates (5.43), recalling the property (A.7), one obtains that
$$\|M_L\big(\mathcal{D}_\infty(j) - \mathcal{D}_n(j)\big)\|_{\rm Op}\,,\, \|M_R\big(\mathcal{D}_\infty(j) - \mathcal{D}_n(j)\big)\|_{\rm Op} \lesssim \varepsilon \langle j \rangle^{-2} e^{-\chi^n},$$
and using that
$$\mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j') = -M_L(\mathcal{D}_n(j) - \mathcal{D}_\infty(j)) + M_R(\mathcal{D}_n(j') - \mathcal{D}_\infty(j')),$$
the latter estimate implies that for any $\ell \in \mathbb{Z}^\infty_*$, $|\ell|_\eta \le N_n$, $j, j' \in \mathbb{N}_0$, $j \neq j'$
$$\|\mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j')\|_{\rm Op} \lesssim \varepsilon e^{-\chi^n}, \tag{5.48}$$
and for any $\ell \in \mathbb{Z}^\infty_* \setminus \{0\}$, $|\ell|_\eta \le N_n$, $j \in \mathbb{N}_0$
$$\|\mathcal{O}_n(\ell, j, j) - \mathcal{O}_\infty(\ell, j, j)\|_{\rm Op} \lesssim \varepsilon e^{-\chi^n} \langle j \rangle^{-2}. \tag{5.49}$$
Since $\omega \in \Omega_\infty(\gamma) \subseteq \Omega_n(\gamma)$, we can write
$$\begin{aligned}\mathcal{O}_n(\ell, j, j') &= \mathcal{O}_\infty(\ell, j, j') + \mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j') \\ &= \mathcal{O}_\infty(\ell, j, j')\Big(\mathrm{Id} + \mathcal{O}_\infty(\ell, j, j')^{-1}\Big[\mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j')\Big]\Big),\end{aligned}$$
and using the estimates (5.48) and (5.49), we get for any $(\ell, j, j') \neq (0, j, j)$, $|\ell|_\eta \le N_n$, the bound
$$\begin{aligned}\|\mathcal{O}_\infty(\ell, j, j')^{-1}[\mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j')]\|_{\rm Op} &\lesssim \varepsilon\gamma^{-1} e^{-\chi^n} \sup_{|\ell|_\eta \le N_n} \mathtt{d}(\ell) \\ &\overset{\text{Lemma B.2}}{\lesssim} \varepsilon\gamma^{-1} e^{-\chi^n} (1 + N_n)^{C(\eta,\mu) N_n^{\frac{1}{1+\eta}}} \\ &\lesssim \varepsilon\gamma^{-1} \sup_{n \in \mathbb{N}} \exp\Big(-\chi^n + C(\eta) N_n^{\frac{1}{1+\eta}} \ln(1 + N_n)\Big).\end{aligned} \tag{5.50}$$
By the choice of N_n provided in (5.5), one obtains that
$$\sup_{n \in \mathbb{N}} \exp\Big(-\chi^n + C(\eta, \mu) N_n^{\frac{1}{1+\eta}} \ln(1 + N_n)\Big) < \infty,$$
implying that for $\varepsilon\gamma^{-1}$ small enough
$$\|\mathcal{O}_\infty(\ell, j, j')^{-1}[\mathcal{O}_n(\ell, j, j') - \mathcal{O}_\infty(\ell, j, j')]\|_{\rm Op} \le \frac{1}{2}.$$
Hence, by the Neumann series, $\mathcal{O}_n(\ell, j, j')$ is invertible and $\omega \in \Omega_{n+1}(\gamma)$. ∎

KAM transformations. For every $n \geq 1$, we define the transformation Ψ_n as

$$\Psi_n := \Phi_1 \circ \cdots \circ \Phi_n \,, \tag{5.51}$$

where for any $n \geq 1$ the transformation $\Phi_n = \exp(\mathcal{F}_n)$ is constructed as in Theorem 5.1. Note that for any $n \in \mathbb{N}$, the map Ψ_n is invertible, and the inverse is given by

$$\Psi_n^{-1} := \Phi_n^{-1} \circ \cdots \circ \Phi_1^{-1} \,. \tag{5.52}$$

We now show the convergence of the sequence of transformations $(\Psi_n)_{n\in\mathbb{N}}$, in the space $\mathcal{H}\big(\mathbb{T}^\infty_{\frac{\sigma_0}{2}}, \mathcal{B}^{\frac{\sigma_0}{2}}\big)$.

LEMMA 5.5. (i) *The sequence of transformation $(\Psi_n)_{n\in\mathbb{N}}$ converges to an invertible transformation Ψ_∞, for $\omega \in \Omega_\infty(\gamma)$ with respect to the norm $|\cdot|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{2}}$. Furthermore, the following bounds hold:*

$$|\Psi_\infty - \mathrm{Id}|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{2}} \,,\; |\Psi_\infty^{-1} - \mathrm{Id}|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{2}} \lesssim \varepsilon\gamma^{-1} \,.$$

(ii) *For any $0 < \sigma \leq \frac{\sigma_0}{2}$, for any $s \geq 0$, the maps $\mathbb{T}^\infty_\sigma \to \mathcal{B}(\mathcal{H}(\mathbb{T}_\sigma), \mathcal{H}(\mathbb{T}_\sigma))$, $\varphi \mapsto \Psi_\infty(\varphi)^{\pm 1}$ and $\mathbb{T}^\infty \to \mathcal{B}(H^s(\mathbb{T}), H^s(\mathbb{T}))$, $\varphi \mapsto \Psi_\infty(\varphi)^{\pm 1}$ are bounded.*

Proof. Proof of (i). This is a completely standard argument.

Proof of (ii). The claimed statement follows by item (i) and by applying Lemmas 2.4 and 3.1. ∎

Final normal form. We now show the following lemma.

LEMMA 5.6. *For any $\omega \in \Omega_\infty(\gamma)$ and for any $\varphi \in \mathbb{T}^\infty_{\sigma_0/3}$, the operator $\mathcal{L}_0(\varphi;\omega)$ defined in* (5.2) *is conjugated to the 2×2 block diagonal operator $\mathrm{i}\mathcal{D}_\infty$ (see* (5.42) *and* (5.44)*), namely $(\Psi_\infty)_{\omega*}\mathcal{L}_0(\varphi;\omega) = \mathrm{i}\mathcal{D}_\infty(\omega)$.*

Proof. By applying Theorem 5.1, by recalling (5.51) of the maps Ψ_n, $n \in \mathbb{N}$, and by using that by Lemma 5.4, $\Omega_\infty(\gamma) \subseteq \cap_{n\geq 0}\Omega_n(\gamma)$, one gets that for any $n \in \mathbb{N}$

$$\mathrm{i}\mathcal{D}_n(\omega) + \mathcal{P}_n(\varphi;\omega) = \mathcal{L}_n = (\Psi_n)_{\omega*}\mathcal{L}_0(\varphi;\omega) \quad \forall \omega \in \Omega_\infty(\gamma) \,. \tag{5.53}$$

By (5.2) and (5.3) and by Lemmas 2.9 and 5.5, one has

$$\begin{aligned} &|\omega\cdot\partial_\varphi(\Psi_\infty - \Psi_n)|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{2}-\rho} \lesssim \rho^{-1}|\Psi_\infty - \Psi_n|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{2}} \to 0 \\ &\quad \text{as} \quad n\to\infty, \quad \text{and} \quad |\mathcal{L}_0|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\sigma_0,-2} \lesssim 1 \end{aligned} \tag{5.54}$$

for $\rho > 0$ so that $\frac{\sigma_0}{2} - \rho > 0$. Therefore, by recalling the (2.2), by the estimates (5.54), and by applying Lemma 3.5(i), one gets that

$$\lim_{n\to\infty} |(\Psi_n)_{\omega*}\mathcal{L}_0 - (\Psi_\infty)_{\omega*}\mathcal{L}_0|^{\mathrm{Lip}(\gamma,\Omega_\infty(\gamma))}_{\frac{\sigma_0}{3}} = 0 \,. \tag{5.55}$$

By the estimates (5.11), (5.45), and (5.55) and passing to the limit in (5.53) one obtains the claimed statement. ∎

6. Measure estimates. It remains only to estimate the measure of the set $\Omega_\infty(\gamma)$, defined in (5.47). In order to do this, let us start with some preliminary considerations. For any $j \in \mathbb{N}_0$, the 2×2 block $\mathcal{D}_\infty(j;\omega)$, $\omega \in \mathtt{D}_\gamma$, is self-adjoint and depends in a Lipschitz way on the parameter ω. By (5.42) and (5.43) and by recalling (5.3) and (5.4), for any $j \in \mathbb{N}$, we can write that

$$\mathcal{D}_\infty(j) = \lambda_2 j^2 \mathrm{Id} + R_\infty(j;\omega), \tag{6.1}$$

where the self-adjoint 2×2 block $R_\infty(j;\omega)$ satisfies the estimate

$$\sup_{\omega \in \mathtt{D}_\gamma} \|R_\infty(j;\omega)\|_{\mathtt{HS}} \lesssim \varepsilon \langle j \rangle\,, \quad \|R_\infty(j)\|_{\mathtt{HS}}^{\mathrm{lip}} \lesssim \varepsilon\gamma^{-1}\,. \tag{6.2}$$

By applying Lemma A.2, one then obtains that for any $j \in \mathbb{N}$,

$$\mathrm{spec}(\mathcal{D}_\infty(j;\omega)) = \{\mu_j^{(+)}(\omega), \mu_j^{(-)}(\omega)\}, \quad \mathrm{spec}(R_\infty(j;\omega)) = \{r_j^{(+)}(\omega), r_j^{(-)}(\omega)\},$$

where $\mu_j^{(\pm)}$ and $r_j^{(\pm)}$ depend in a Lipschitz way on the parameter $\omega \in \mathtt{D}_\gamma$ and they satisfy

$$\begin{aligned} &\mu_j^{(\pm)}(\omega) = \lambda_2 j^2 + r_j^{(\pm)}(\omega)\,, \\ &|\lambda_2 - 1| \lesssim \varepsilon\,, \quad \sup_{\omega \in \mathtt{D}_\gamma} |r_j^{(\pm)}(\omega)| \lesssim \varepsilon\langle j \rangle, \quad |r_j^{(\pm)}|^{\mathrm{lip}} \lesssim \varepsilon\gamma^{-1}\,. \end{aligned} \tag{6.3}$$

If $j = 0$, one has $|\mu_0|^{\mathrm{Lip}(\gamma,\mathtt{D}_\gamma)} \lesssim \varepsilon$. For compactness of notation we set $\mu_0^{(+)} = \mu_0^{(-)} = \mu_0$. By applying Lemmas A.1 and A.2(ii) one then obtains that the set $\Omega_\infty(\gamma)$ can be written as

$$\begin{aligned} \Omega_\infty(\gamma) = \Big\{ \omega \in \mathtt{D}_\gamma : |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| \geq \frac{2\gamma}{\mathtt{d}(\ell)} \quad \forall (\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0, \\ j \neq j', \quad \sigma, \sigma' \in \{+,-\}, \quad |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_j^{(\sigma')}| \geq \frac{2\gamma}{\mathtt{d}(\ell)\langle j \rangle^2} \\ \forall (\ell, j) \in (\mathbb{Z}^\infty_* \setminus \{0\}) \times \mathbb{N}_0, \quad \sigma, \sigma' \in \{+,-\} \Big\}, \end{aligned} \tag{6.4}$$

where we recall that

$$\mathtt{d}(\ell) := \prod_{n \in \mathbb{N}} (1 + |\ell_n|^4 \langle n \rangle^4) \quad \forall \ell \in \mathbb{Z}^\infty_*\,.$$

In the remaining part of this section we prove the following proposition.

PROPOSITION 6.1. *Assume that* $\mu > 3$. *For* $\varepsilon\gamma^{-1}$ *and* γ *small enough one has that* $\mathbb{P}(\mathtt{R}_0 \setminus \Omega_\infty(\gamma)) \lesssim \gamma$.

We note that

$$\mathbb{P}\Big(\mathtt{R}_0 \setminus \Omega_\infty(\gamma)\Big) \leq \mathbb{P}\Big(\mathtt{R}_0 \setminus \mathtt{D}_\gamma\Big) + \mathbb{P}\Big(\mathtt{D}_\gamma \setminus \Omega_\infty(\gamma)\Big)\,. \tag{6.5}$$

In [10], it is proved that

$$\mathbb{P}\Big(\mathtt{R}_0 \setminus \mathtt{D}_\gamma\Big) \lesssim \gamma\,; \tag{6.6}$$

therefore, we need to estimate the set $\mathtt{D}_\gamma \setminus \Omega_\infty(\gamma)$. In order to shorten notation, we define

$$\mathcal{Z}_1 := \Big\{(\ell, j, j') \in \mathbb{Z}^\infty_* \times \mathbb{N}_0 \times \mathbb{N}_0 : j \neq j'\Big\}, \quad \mathcal{Z}_2 := (\mathbb{Z}^\infty_* \setminus \{0\}) \times \mathbb{N}_0\,. \tag{6.7}$$

One has that

$$\mathtt{D}_\gamma \setminus \Omega_\infty(\gamma) = \Big(\bigcup_{(\ell,j,j') \in \mathcal{Z}_1} \mathcal{R}_{\ell j j'}(\gamma) \Big) \bigcup \Big(\bigcup_{(\ell,j) \in \mathcal{Z}_2} \mathcal{Q}_{\ell j}(\gamma) \Big), \tag{6.8}$$

where for any $(\ell, j, j') \in \mathcal{Z}_1$ we define

$$\mathcal{R}_{\ell j j'}(\gamma) := \bigcup_{\sigma, \sigma' \in \{+,-\}} \Big\{ \omega \in \mathtt{D}_\gamma : |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| < \frac{2\gamma}{\mathtt{d}(\ell)} \Big\} \tag{6.9}$$

and for any $(\ell, j) \in \mathcal{Z}_2$ we define

$$\mathcal{Q}_{\ell j}(\gamma) := \bigcup_{\sigma, \sigma' \in \{+,-\}} \Big\{ \omega \in \mathtt{D}_\gamma : |\omega \cdot \ell + \mu_j^{(\sigma)} - \mu_j^{(\sigma')}| < \frac{2\gamma}{\mathtt{d}(\ell) \langle j \rangle^2} \Big\}. \tag{6.10}$$

LEMMA 6.2. (i) *Let* $(\ell, j, j') \in \mathcal{Z}_1$. *If* $\mathcal{R}_{\ell j j'}(\gamma) \neq \emptyset$, *then* $|j^2 - j'^2| \leq C|\ell|_1$ *and* $\mathbb{P}(\mathcal{R}_{\ell j j'}(\gamma)) \lesssim \frac{\gamma}{\mathtt{d}(\ell)}$.

(ii) *Let* $(\ell, j) \in \mathcal{Z}_2$. *If* $\mathcal{Q}_{\ell j}(\gamma) \neq \emptyset$, *then* $\mathbb{P}(\mathcal{Q}_{\ell j}(\gamma)) \lesssim \frac{\gamma}{\langle j \rangle^2 \mathtt{d}(\ell)}$.

Proof. We prove item (i). The proof of item (ii) can be done arguing in a similar fashion. Let $j, j' \in \mathbb{N}_0$, $j \neq j'$, and $\sigma, \sigma' \in \{+,-\}$. By (6.3) one has that for some constant $C > 0$,

$$|\mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| \geq |\lambda_2||j^2 - j'^2| - C\varepsilon(j + j') - C\varepsilon .$$

Using that $\lambda_2 = 1 + O(\varepsilon)$ and that $|j + j'| \leq |j^2 - j'^2|$, one obtains that for ε small enough

$$|\mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| \geq \frac{1}{2}|j^2 - j'^2|, \tag{6.11}$$

implying that $\mathcal{R}_{0jj'}(\gamma) = \emptyset$ for any $j \neq j'$. Hence, if $(\ell, j, j') \in \mathcal{Z}_1$ and $\mathcal{R}_{\ell j j'}(\gamma) \neq \emptyset$, one has that $\ell \neq 0$. Furthermore, if $\omega \in \mathcal{R}_{\ell j j'}(\gamma) \neq \emptyset$, one has that by using (6.11), one obtains that

$$\frac{1}{2}|j^2 - j'^2| \leq |\mu_j^{(\sigma)} - \mu_{j'}^{(\sigma')}| \leq \frac{2\gamma}{\mathtt{d}(\ell)} + |\omega \cdot \ell| \lesssim 1 + \|\omega\|_\infty \|\ell\|_1 \lesssim 1 + \|\ell\|_1 . \tag{6.12}$$

Now let

$$s := \min\{n \in \mathbb{N} : \ell_n \neq 0\}, \quad S := \max\{n \in \mathbb{N} : \ell_n \neq 0\},$$

and $\mathbf{e}^{(s)} = (\mathbf{e}_n^{(s)})_{n \in \mathbb{N}}$ is the vector whose nth component is 0 if $n \neq s$ and 1 if $n = s$. Similarly we define the vector $\mathbf{e}^{(S)}$. Let

$$\psi(t) := (\omega + t\mathbf{e}^{(s)}) \cdot \ell + \mu_j^{\sigma}(\omega + t\mathbf{e}^{(s)}) - \mu_{j'}^{(\sigma')}(\omega + t\mathbf{e}^{(s)}) .$$

By using the estimate (6.3), for $\varepsilon\gamma^{-1}$ small enough, one has that

$$|\psi(t_1) - \psi(t_2)| \geq |t_1 - t_2||\ell_s| - C\varepsilon\gamma^{-1}|t_1 - t_2| \geq \frac{1}{2}|t_1 - t_2| .$$

The latter estimate implies that

$$\Big| \Big\{ t : \omega + t\mathbf{e}^{(s)} \in \mathcal{R}_{\ell j j'}(\gamma), \quad |\psi(t)| < \frac{2\gamma}{\mathtt{d}(\ell)} \Big\} \Big| \lesssim \frac{\gamma}{\mathtt{d}(\ell)} .$$

Since $\mathcal{R}_{\ell j j'}(\gamma)$ is a cylinder with at most $S - s$ components, one obtains the desired bound. ∎

Proof of Proposition 6.1. By recalling (6.8) and by applying Lemma 6.2, one gets the estimate

$$\mathbb{P}\big(\mathtt{D}_\gamma \setminus \Omega_\infty(\gamma)\big) \lesssim \sum_{\substack{(\ell,j,j')\in\mathcal{Z}_1\\ |j^2-j'^2|\le\|\ell\|_1}} \frac{\gamma}{\mathtt{d}(\ell)} + \sum_{(\ell,j)\in\mathcal{Z}_2} \frac{\gamma}{\langle j\rangle^2\mathtt{d}(\ell)}$$

$$\lesssim \gamma\Big(\sum_{\ell\in\mathbb{Z}^\infty_*} \frac{\|\ell\|_1^2}{\mathtt{d}(\ell)} + \sum_{\ell\in\mathbb{Z}^\infty_*}\frac{1}{\mathtt{d}(\ell)}\sum_{j\in\mathbb{N}_0}\frac{1}{\langle j\rangle^2}\Big) \overset{Lemma\ \mathrm{B.3}}{\lesssim} \gamma\,.$$

The claimed statement then follows by recalling (6.5) and (6.6).

7. Proof of Theorem 1.4 and Corollary 1.7. Let $\gamma := \varepsilon^a$, $a\in(0,1)$. Then the smallness condition $\varepsilon\gamma^{-1}\le\delta$ is fulfilled by taking $\varepsilon\in(0,\varepsilon_0)$ with ε_0 small enough. By setting $\Omega_\varepsilon := \Omega_\infty(\gamma)$, Proposition 6.1 implies (1.7). For any $\omega\in\Omega_\varepsilon$, we define

$$\mathcal{W}_\infty(\varphi) := \Phi^{(1)}(\varphi)\circ\Phi^{(2)}\circ\cdots\circ\Phi^{(7)}(\varphi)\circ\Psi_\infty(\varphi) \quad \varphi\in\mathbb{T}^\infty_{\sigma/4}, \tag{7.1}$$

where the maps $\Phi^{(1)},\dots,\Phi^{(7)}$ are as constructed in section 4 and the map Ψ_∞ is as given in Lemma 5.5. The properties (1) and (2) on the maps $\mathcal{W}_\infty(\varphi)^{\pm1}$ stated in Theorem 1.4 are easily deduced from Lemmas 4.1, 4.3 to 4.8, and 5.5(ii) and from Remark 4.2. Furthermore, by the same lemmas and Lemma 5.6 one obtains that $u(t,x)$ is a solution of (1.1) if and only if $v(\cdot,t) := \mathcal{W}_\infty(\omega t)^{-1}u(\cdot,t)$, $\omega\in\Omega_\varepsilon$, solves the time independent equation $\partial_t v = \mathrm{i}\mathcal{D}_\infty v$ where $\mathcal{D}_\infty$ is the 2×2 time independent self-adjoint block-diagonal operator defined in (5.42)–(5.44). The proof of Theorem 1.4 is then concluded.

Proof of Corollary 1.7. Since $\mathcal{D}_\infty$ is a 2×2 block-diagonal self-adjoint operator, the general solution of the equation $\partial_t v = \mathrm{i}\mathcal{D}_\infty v$ can be written as

$$v(x,t) = \sum_{j\in\mathbb{N}_0} e^{\mathrm{i}t\pi_j\mathcal{D}_\infty\pi_j}[\Pi_j v_0]\,.$$

Since $\pi_j\mathcal{D}_\infty\pi_j : \mathbf{E}_j\to\mathbf{E}_j$ is self-adjoint (recall (3.2)), one has that

$$\big\|e^{\mathrm{i}t\pi_j\mathcal{D}_\infty\pi_j}[\pi_j v_0]\big\|_{L^2} = \|\pi_j v_0\|_{L^2} \quad \forall j\in\mathbb{N}_0.$$

This implies that both analytic and Sobolev norms are preserved; namely for any $\sigma>0$, $\|v(\cdot,t)\|_\sigma = \|v_0\|_\sigma$, and for any $s\ge0$, $\|v(\cdot,t)\|_{H^s} = \|v_0\|_{H^s}$. Hence, by using the properties (1) and (2) stated in Theorem 1.4, one obtains that for any $\omega\in\Omega_\varepsilon$, the solution $u(\cdot,t) := \mathcal{W}_\infty(\omega t)v(\cdot,t)$ of (1.1) satisfies the desired bounds both in analytic and Sobolev norms. The proof of the corollary is therefore concluded.

Appendix A. Technical lemmas.

A.1. Linear operators in finite dimension. Given an operator $A\in\mathcal{B}(\mathbf{E}_j)$, we define its trace as

$$\begin{aligned} \mathrm{Tr}(A) &:= A_0^0, \quad A\in\mathcal{B}(\mathbf{E}_0),\\ \mathrm{Tr}(A) &:= A_j^j + A_{-j}^{-j}\,, \quad A\in\mathcal{B}(\mathbf{E}_j), \quad j\in\mathbb{N}\,. \end{aligned} \tag{A.1}$$

It is easy to check that if $A,B\in\mathcal{B}(\mathbf{E}_j)$, then

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)\,. \tag{A.2}$$

For all $j, j' \in \mathbb{N}_0$, the space $\mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ is a Hilbert space[5] equipped by the inner product given for any $X, Y \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ by

$$\langle X, Y\rangle := \mathrm{Tr}(XY^*)\,. \tag{A.3}$$

This scalar product induces the L^2-norm $\|\cdot\|_{\mathtt{HS}}$ defined in (3.10).

Given a linear operator $\mathbf{L} : \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$, we denote by $\|\mathbf{L}\|_{\mathrm{Op}}$ its operator norm when the space $\mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ is equipped by the L^2-norm (3.10), namely

$$\|\mathbf{L}\|_{\mathrm{Op}} := \sup\Big\{\|\mathbf{L}(M)\|_{\mathtt{HS}} : M \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)\,, \quad \|M\|_{\mathtt{HS}} \le 1\Big\}\,. \tag{A.4}$$

For any operator $A \in \mathcal{B}(\mathbf{E}_j)$ we denote by $M_L(A) : \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ the linear operator defined for any $X \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ as

$$M_L(A)X := AX\,. \tag{A.5}$$

Similarly, given an operator $B \in \mathcal{B}(\mathbf{E}_{j'})$, we denote by $M_R(B) : \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ the linear operator defined for any $X \in \mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ as

$$M_R(B)X := XB\,. \tag{A.6}$$

The following elementary estimates hold:

$$\|M_L(A)\|_{\mathrm{Op}} \le \|A\|_{\mathtt{HS}}\,, \quad \|M_R(B)\|_{\mathrm{Op}} \le \|B\|_{\mathtt{HS}}\,. \tag{A.7}$$

We denote by $\mathcal{S}(\mathbf{E}_j)$, the set of the self-adjoint operators form $\mathbf{E}_j$ onto itself, namely

$$\mathcal{S}(\mathbf{E}_j) := \Big\{A \in \mathcal{B}(\mathbf{E}_j) : A = A^*\Big\}\,. \tag{A.8}$$

Furthermore, for any $A \in \mathcal{B}(\mathbf{E}_j)$ denote by $\mathrm{spec}(A)$ the spectrum of A. The following lemma can be proved by using elementary arguments from linear algebra; hence the proof is omitted.

LEMMA A.1. *Let $j, j' \in \mathbb{N}_0$, $A \in \mathcal{S}(\mathbf{E}_j)$, $B \in \mathcal{S}(\mathbf{E}_{j'})$; then the following hold.*

(i) *The operators $M_L(A)$, $M_R(B)$ defined in* (A.5) *and* (A.6) *are self-adjoint operators with respect to the scalar product defined in* (A.3).

(ii) *Let $j, j' \in \mathbb{N}$, $A \in \mathcal{S}(\mathbf{E}_j)$, $B \in \mathcal{S}(\mathbf{E}_{j'})$. The spectrum of the operator $M_L(A) \pm M_R(B)$ satisfies*

$$\mathrm{spec}\Big(M_L(A) \pm M_R(B)\Big) = \Big\{\lambda \pm \mu : \lambda \in \mathrm{spec}(A)\,, \quad \mu \in \mathrm{spec}(B)\Big\}\,.$$

(iii) *Let $j \in \mathbb{N}$, $A \in \mathcal{S}(\mathbf{E}_j)$, and $B \equiv \lambda_0 \in \mathcal{S}(\mathbf{E}_0)$. Then, the spectra of the operators $M_L(A) \pm M_R(\lambda_0) \equiv M_L(A) \pm \lambda_0\mathrm{Id} : \mathcal{B}(\mathbf{E}_0, \mathbf{E}_j) \to \mathcal{B}(\mathbf{E}_0, \mathbf{E}_j)$ and $M_L(\lambda_0) \pm M_R(A) \equiv \lambda_0\mathrm{Id} \pm M_R(A) : \mathcal{B}(\mathbf{E}_j, \mathbf{E}_0) \to \mathcal{B}(\mathbf{E}_j, \mathbf{E}_0)$ satisfy*

$$\mathrm{spec}\Big(M_L(A) \pm \lambda_0\mathrm{Id}\Big) = \mathrm{spec}\Big(\lambda_0\mathrm{Id} \pm M_R(A)\Big) = \Big\{\lambda \pm \lambda_0 : \lambda \in \mathrm{spec}(A)\Big\}\,.$$

We finish this section by recalling some well-known facts concerning linear self-adjoint operators on finite dimensional Hilbert spaces. Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension n equipped by the inner product $(\cdot\,,\cdot)_{\mathcal{H}}$. For any self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$, we order its eigenvalues as

$$\mathrm{spec}(A) := \big\{\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)\big\}\,. \tag{A.9}$$

[5] Actually all the norms on the finite dimensional space $\mathcal{B}(\mathbf{E}_{j'}, \mathbf{E}_j)$ are equivalent.

LEMMA A.2. *Let $\mathcal{H}$ be a Hilbert space of dimension n. Then the following hold.*

(i) *Let $A_1, A_2 : \mathcal{H} \to \mathcal{H}$ be self-adjoint operators. Then their eigenvalues, ordered as in* (A.9)*, satisfy the Lipschitz property*

$$|\lambda_k(A_1) - \lambda_k(A_2)| \leq \|A_1 - A_2\|_{\mathcal{B}(\mathcal{H})} \qquad \forall k = 1, \ldots, n\,.$$

(ii) *Let $A = y\mathrm{Id}_{\mathcal{H}} + B$, where $y \in \mathbb{R}$, $\mathrm{Id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is the identity, and $B : \mathcal{H} \to \mathcal{H}$ is self-adjoint. Then*

$$\lambda_k(A) = y + \lambda_k(B) \qquad \forall k = 1, \ldots, n\,.$$

(iii) *Let $A : \mathcal{H} \to \mathcal{H}$ be self-adjoint and assume that* $\mathrm{spec}(A) \subset \mathbb{R} \setminus \{0\}$*. Then A is invertible and its inverse satisfies*

$$\|A^{-1}\|_{\mathcal{B}(\mathcal{H})} = \frac{1}{\min_{k=1,\ldots,n} |\lambda_k(A)|}\,.$$

A.2. Properties of torus diffeomorphisms. In subsection 4.2, we have considered diffeomorphisms of the form

$$\varphi \mapsto \varphi + \omega\alpha(\varphi), \tag{A.10}$$

where $\alpha \in \mathcal{H}(\mathbb{T}^\infty_{\sigma+\rho})$, $\sigma, \rho > 0$, and $\omega \in \mathtt{D}_\gamma$. By Proposition 2.13, for $\varepsilon = \varepsilon(\rho)$ small enough, if $\|\alpha\|_{\mathcal{H}^{\sigma+\rho}} \leq \varepsilon$, then the diffeomorphism (A.10) is invertible and its inverse has the form

$$\vartheta \mapsto \vartheta + \omega\widetilde{\alpha}(\vartheta), \tag{A.11}$$

where $\widetilde{\alpha} \in \mathcal{H}(\mathbb{T}^\infty_\sigma)$ and $\|\widetilde{\alpha}\|_\sigma \lesssim \|\alpha\|_{\sigma+\rho}$. Note that by (A.10) and (A.11), one can easily deduce the formulae

$$\begin{aligned} 1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta) &= \frac{1}{1 + \omega \cdot \partial_\varphi \alpha(\vartheta + \omega\widetilde{\alpha}(\vartheta))}, \\ 1 + \omega \cdot \partial_\varphi \alpha(\varphi) &= \frac{1}{1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\varphi + \omega\alpha(\varphi))}\,. \end{aligned} \tag{A.12}$$

The following lemma will be used in the reduction procedure of section 4 in order to show that some averages do not depend on the parameter $\omega \in \Omega$.

LEMMA A.3. *The following holds:*

Let $\omega \in \mathtt{D}_\gamma$ be a Diophantine frequency, and let a be a function in $\mathcal{H}(\mathbb{T}^\infty_\sigma)$. Then $\int_{\mathbb{T}^\infty} \omega \cdot \partial_\vartheta a(\vartheta)\, d\vartheta = 0$. As a consequence one has

$$\int_{\mathbb{T}^\infty} \Big(1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta)\Big) d\vartheta = 1 \tag{A.13}$$

and for any $\ell \in \mathbb{Z}^\infty_* \setminus \{0\}$,

$$\int_{\mathbb{T}^\infty} e^{\mathrm{i}\ell\cdot\big(\vartheta + \omega\widetilde{\alpha}(\vartheta)\big)} \Big(1 + \omega \cdot \partial_\vartheta \widetilde{\alpha}(\vartheta)\Big)\, d\vartheta = 0\,. \tag{A.14}$$

Proof. Let $N \in \mathbb{N}$. Then we split

$$\omega \cdot \partial_\vartheta a(\vartheta) = \sum_{\ell \neq 0\,,\, |\ell|_\eta \leq N} \mathrm{i}\omega \cdot \ell \widehat{a}(\ell) e^{\mathrm{i}\ell\cdot\vartheta} + \sum_{|\ell|_\eta > N} \mathrm{i}\omega \cdot \ell \widehat{a}(\ell) e^{\mathrm{i}\ell\cdot\vartheta}\,.$$

Since a is an analytic function, the second term on the right-hand side goes to zero as $N \to +\infty$. Moreover,

$$\int_{\mathbb{T}^N} \sum_{\ell \neq 0\,,\,|\ell|_\eta \le N} \mathrm{i}\omega \cdot \ell \widehat{a}(\ell) e^{\mathrm{i}\ell\cdot\vartheta}\, d\vartheta = \sum_{\ell \neq 0\,,\,|\ell|_\eta \le N} \mathrm{i}\omega \cdot \ell \widehat{a}(\ell) \int_{\mathbb{T}^N} e^{\mathrm{i}\ell\cdot\vartheta}\, d\vartheta = 0\,.$$

Therefore, one deduces that

$$\int_{\mathbb{T}^\infty} a(\vartheta)\, d\vartheta = \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \sum_{|\ell|_\eta > N} \mathrm{i}\omega \cdot \ell \widehat{a}(\ell) e^{\mathrm{i}\ell\cdot\vartheta}\, d\vartheta = 0\,.$$

The equality (A.13) follows immediately by the previous claim. The equality (A.14) follows by observing that since $\ell \in \mathbb{Z}^\infty_* \setminus \{0\}$ and ω is Diophantine, one has that

$$e^{\mathrm{i}\ell\cdot\big(\vartheta+\omega\widetilde{\alpha}(\vartheta)\big)}\Big(1+\omega\cdot\partial_\vartheta\widetilde{\alpha}(\vartheta)\Big) = \frac{1}{\mathrm{i}\omega\cdot\ell}\omega\cdot\partial_\vartheta\Big(e^{\mathrm{i}\ell\cdot\big(\vartheta+\omega\widetilde{\alpha}(\vartheta)\big)}\Big);$$

hence the result follows by applying the first claim. ∎

LEMMA A.4 (Moser composition lemma). *Let $f : B_R(0) \to \mathbb{C}$ be a holomorphic function defined in a neighborhood of the origin $B_R(0)$ of the complex plane $\mathbb{C}$. Then the composition operator $F(u) := f \circ u$ is a well-defined nonlinear map $\mathcal{H}(\mathbb{T}^\infty_\sigma) \to \mathcal{H}(\mathbb{T}^\infty_\sigma)$.*

Proof. Clearly, since $f(z) = \sum_{n\ge 0} a_n z^n$ is analytic, for any $z \in \mathbb{C}$, $|z| < R$, the series $\sum_{n\ge0} |a_n||z|^n$ is convergent. Moreover, let $u \in \mathcal{H}(\mathbb{T}^\infty_\sigma)$ with $\|u\|_\sigma \le r < R$. By applying Lemma 2.5, for any $n \ge 1$, $u^n \in \mathcal{H}(\mathbb{T}^\infty_\sigma)$ and $\|u^n\|_\sigma \le \|u\|^n_\sigma \le r^n$. The series $\sum_{n\ge0} a_n u^n$ is absolutely convergent with respect to $\|\cdot\|_\sigma$. Indeed, one has

$$\Big\|\sum_{n\ge0} a_n u^n\Big\|_\sigma \le \sum_{n\ge0} |a_n| \|u\|^n_\sigma \le \sum_{n\ge0} |a_n| r^n < \infty\,.$$

This implies that $F(u) = \sum_{n\ge0} a_n u^n$ belongs to the space $\mathcal{H}(\mathbb{T}^\infty_\sigma)$, and the proof of the lemma is concluded. ∎

Appendix B. Some estimates of constants.

LEMMA B.1. (i) *Let $\mu_1, \mu_2 > 0$. Then*

$$\sup_{\substack{\ell\in\mathbb{Z}^\infty_* \\ |\ell|_\eta<\infty}} \prod_i (1+\langle i\rangle^{\mu_1}|\ell_i|^{\mu_2}) e^{-\rho|\ell|_\eta} \le \exp\Big(\frac{\tau}{\rho^{\frac{1}{\eta}}}\ln\Big(\frac{\tau}{\rho}\Big)\Big)$$

for some constant $\tau = \tau(\eta, \mu_1, \mu_2) > 0$.

(ii) *Let $\rho > 0$. Then $\sum_{\ell\in\mathbb{Z}^\infty_*} e^{-\rho|\ell|_\eta} \lesssim \exp(\frac{\tau}{\rho^{\frac{1}{\eta}}}\ln(\frac{\tau}{\rho}))$ for some constant $\tau = \tau(\eta) > 0$.*

Proof. Proof of (i). We remark that the left-hand side can be expressed as

$$\exp\Big(\sum_i -\rho\langle i\rangle^\eta|\ell_i| + \ln\big(1+\langle i\rangle^{\mu_1}|\ell_i|^{\mu_2}\big)\Big) =: \exp\Big(\sum_i f_i(|\ell_i|)\Big),$$

where

$$f_i(x) := \ln\big(1+\langle i\rangle^{\mu_1}x^{\mu_2}\big) - \rho\langle i\rangle^\eta x\,. \tag{B.1}$$

The result follows word for word from Lemma 7.2 of [10], where it is proved in the special case $\mu_1 = 2 + q$, $\mu_2 = 2$.

Proof of (ii). By Lemma 4.1 of [10], one has

$$\sum_{\ell \in \mathbb{Z}^\infty_*} \prod_i \frac{1}{1 + \langle i \rangle^2 |\ell_i|^2} \leq C_0 < \infty .$$

Therefore,

$$\sum_{\ell \in \mathbb{Z}^\infty_*} e^{-\rho |\ell|_\eta} = \sum_{\ell \in \mathbb{Z}^\infty_*} \prod_i \frac{1}{1 + \langle i \rangle^2 |\ell_i|^2} e^{-\rho \langle i \rangle^\eta |\ell_i|} \big(1 + \langle i \rangle^2 |\ell_i|^2\big)$$
$$\lesssim \sup_{\ell \in \mathbb{Z}^\infty_*} \Big(\prod_i e^{-\rho \langle i \rangle^\eta |\ell_i|} \big(1 + \langle i \rangle^2 |\ell_i|^2\big) \Big) .$$

The claimed statement then follows by item (i) with $\mu_1 = \mu_2 = 2$. ∎

LEMMA B.2 (Small divisor estimate). *Let $\mu_1, \mu_2 \geq 1$. We have the following estimate for $N \gg 1$:*

$$\sup_{\ell \in \mathbb{Z}^\infty_* \colon |\ell|_\eta < N} \prod_i (1 + \langle i \rangle^{\mu_1} |\ell_i|^{\mu_2}) \leq (1 + N)^{C(\eta, \mu_1, \mu_2) N^{\frac{1}{1+\eta}}} \tag{B.2}$$

for some constant $C(\eta, \mu_1, \mu_2) > 0$.

Proof. For ℓ fixed, let us denote by k the number of nonzero components of ℓ. We claim that $k \lesssim_\eta N^{\frac{1}{1+\eta}}$; indeed,

$$N \geq |\ell|_\eta = \sum_{j=1}^k \langle i_j \rangle^\eta |\ell_{i_j}| \geq \sum_{j=1}^k \langle i_j \rangle^\eta \geq \sum_{j=1}^k j^\eta \simeq_\eta k^{1+\eta},$$

and the claim follows. Now if $\eta \geq 1$, we have $\langle i \rangle |\ell_i| \leq \langle i \rangle^\eta |\ell_i| \leq N$, and setting $\mu := \max\{\mu_1, \mu_2\}$, we have

$$\sup_{\ell \in \mathbb{Z}^\infty_* \colon |\ell|_\eta \leq N} \sum_i \ln(1 + \langle i \rangle^{\mu_1} |\ell_i|^{\mu_2}) \lesssim_\eta N^{\frac{1}{1+\eta}} \ln(1 + N^\mu) \lesssim_{\eta,\mu} N^{\frac{1}{1+\eta}} \ln(1 + N).$$

Otherwise if $\eta \leq 1$, we have $\langle i \rangle |\ell_i| \leq (\langle i \rangle^\eta |\ell_i|)^{\frac{1}{\eta}} \leq N^{\frac{1}{\eta}}$ and again

$$\sup_{\ell \in \mathbb{Z}^\infty_* \colon |\ell|_\eta \leq N} \sum_i \ln(1 + \langle i \rangle^{\mu_1} |\ell_i|^{\mu_2}) \lesssim_\eta N^{\frac{1}{1+\eta}} \ln(1 + N^{\frac{\mu}{\eta}}) \lesssim_{\eta,\mu} N^{\frac{1}{1+\eta}} \ln(1 + N).$$ ∎

LEMMA B.3. *For $\mu_1, \mu_2 > 3$, we have that $\sum_{\ell \in \mathbb{Z}^\infty_*} \frac{\|\ell\|_1^2}{\mathtt{d}(\ell)} < \infty$, where $\mathtt{d}(\ell) := \prod_{i \in \mathbb{N}} (1 + \langle i \rangle^{\mu_1} |\ell_i|^{\mu_2})$.*

Proof. The proof is very similar to that of the measure estimate Lemma 4.1 of [10]. ∎

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