

# Higgs boson production through $b\bar{b}$ annihilation at threshold in N<sup>3</sup>LO QCD

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**ABSTRACT:** We present threshold enhanced N<sup>3</sup>LO QCD corrections to inclusive Higgs production through bottom anti-bottom annihilation at hadron colliders using threshold resummed cross section. The resummed cross section is obtained using factorization properties and Sudakov resummation of the inclusive cross section. We use the recent results on threshold N<sup>3</sup>LO corrections in QCD for Drell-Yan production and three loop QCD corrections to Higgs form factor with bottom anti-bottom quark to achieve this task. This is the first step towards the evaluation of complete N<sup>3</sup>LO result. We have numerically demonstrated the importance of such corrections at the LHC.

**KEYWORDS:** QCD Phenomenology, NLO Computations

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**1 Introduction**

The discovery of the Higgs boson by ATLAS [1] and CMS [2] collaborations of the Large Hadron Collider (LHC) at CERN has not only shed the light on the dynamics behind the electroweak symmetry breaking but also put the Standard Model (SM) of particle physics on a firmer ground. In the SM, the elementary particles such as quarks, leptons and gauge bosons,  $Z, W^\pm$  acquire their masses through spontaneous symmetry breaking (SSB). The Higgs mechanism provides the framework for SSB. The SM predicts the existence of a Higgs boson whose mass is a parameter of the model. The recent discovery of the Higgs boson provides a valuable information on this, namely on its mass which is about 125.5 GeV. The searches for the Higgs boson have been going on for several decades in various experiments. Earlier experiments such as LEP [3] and Tevatron [4] played an important role in the discovery by the LHC collaborations through narrowing down its possible mass range. LEP excluded Higgs boson of mass below 114.4 GeV and their precision electroweak measurements [5] hinted the mass less than 152 GeV at 95% confidence level (CL), while Tevatron excluded Higgs boson of mass in the range 162 – 166 GeV at 95% CL.

Higgs bosons are produced dominantly at the LHC via gluon gluon fusion through top quark loop, while the sub-dominant ones are vector boson fusion, associated production of the Higgs boson with vector bosons, with top anti-top pairs and also in bottom anti-bottom annihilation. The inclusive productions of the Higgs boson in gluon gluon [6–14], vector boson fusion processes [15] and associated production with vector bosons [16, 17] are known to next to next to leading order (NNLO) accuracy in QCD. The Higgs production in bottom anti-bottom annihilation is also known to NNLO accuracy in the variable flavour scheme (VFS) [18–23], while it is known to NLO in the fixed flavour scheme (FFS) [24–29]. In the minimal super symmetric standard model (MSSM), the coupling of bottom quarks to Higgs becomes large in the large  $\tan\beta$  region, where  $\tan\beta$  is the ratio of vacuum expectation values of up and down type Higgs fields. This can enhance contributions from bottom anti-bottom annihilation subprocesses.

While the theoretical predictions of NNLO [6–14] and next to next to leading log (NNLL) [30] QCD corrections and of two loop electroweak effects [31–36] played an important role in the Higgs discovery, the theoretical uncertainties resulting from factorization and renormalization scales are not fully under control. Hence, the efforts to go beyond

NNLO are going on intensively. Some of the ingredients to obtain N<sup>3</sup>LO QCD corrections are already available. For example, quark and gluon form factors [37–41], the mass factorization kernels [42, 43] and the renormalization constant [44] for the effective operator describing the coupling of the Higgs boson with the SM fields in the infinite top quark mass limit up to three loop level in dimensional regularization are known for some time. In addition, NNLO soft contributions are known [45] to all orders in  $\epsilon$  for both DY and the Higgs productions using dimensional regularization with space time dimension being  $d = 4 + \epsilon$ . They were used to obtain the partial N<sup>3</sup>LO threshold effects [46–50] to Drell-Yan production of di-leptons and inclusive productions of the Higgs boson through gluon gluon fusion and in bottom anti-bottom annihilation. Threshold contribution to the inclusive production cross section is expanded in terms of  $\delta(1 - z)$  and  $\mathcal{D}_i(z)$  where

$$\mathcal{D}_i(z) = \left( \frac{\ln^i(1 - z)}{1 - z} \right)_+ \tag{1.1}$$

with the scaling parameter  $z = m_H^2/\hat{s}$  for the Higgs boson and  $z = m_{l+l-}^2/\hat{s}$  for DY. Here  $m_H$ ,  $m_{l+l-}$  and  $\hat{s}$  are mass of the Higgs boson, invariant mass of the di-leptons and center of mass energy of the partonic reaction responsible for production mechanism respectively. The missing  $\delta(1 - z)$  terms for the complete N<sup>3</sup>LO threshold contributions to the Higgs boson production through gluon gluon fusion are now available due to the seminal work by Anastasiou et al. [51] where the relevant soft contributions were obtained from the real radiations at N<sup>3</sup>LO level. The resummation of threshold effects [52, 53] to infra-red safe observables resulting from their factorization properties as well as Sudakov resummation of soft gluons provides an elegant framework to obtain threshold enhanced contributions to inclusive and semi inclusive observables order by order in perturbation theory. In [54], using this framework, we exploited the universal structure of the soft radiations to obtain the corresponding soft gluon contributions to DY production, which led to the evaluation of missing  $\delta(1 - z)$  part of the N<sup>3</sup>LO threshold corrections. In [55], relevant one loop double real emissions from light-like Wilson lines were computed to obtain threshold corrections to the Higgs boson as well as Drell-Yan productions upto N<sup>3</sup>LO level providing an independent approach. In [56] the universality of soft gluon contributions near threshold and the results of [51] were used to obtain general expression of the hard-virtual coefficient which contributes to N<sup>3</sup>LO threshold as well as threshold resummation at next-to-next-to-next-to-leading-logarithmic (N<sup>3</sup>LL) accuracy for the production cross section of a colourless heavy particle at hadron colliders. For the Higgs boson production through  $b\bar{b}$  annihilation, till date, only partial N<sup>3</sup>LO threshold corrections are known [49, 50, 57] where again the framework of threshold resummation was used. In both [49, 50] and [57], it was not possible to determine the  $\delta(1 - z)$  at N<sup>3</sup>LO due to the lack of information on three loop finite part of bottom anti-bottom Higgs form factor in QCD and the soft gluon radiation at N<sup>3</sup>LO level. In [57], subleading corrections were also obtained through the method of Mellin moments. The recent results on the Higgs form factor with bottom anti-bottom by Gehrmann and Kara [58] and on the universal soft distribution obtained for the Drell-Yan production [54] can now be used to obtain  $\delta(1 - z)$  part of the threshold N<sup>3</sup>LO contribution. For the soft gluon radiations in the  $b\bar{b}$  annihilation, the results from [54] can

be used as they do not depend on the flavour of the incoming quark states. We have set bottom quark mass to be zero throughout except in the Yukawa coupling.

In the next section, we present the details of the threshold resummation and in section 3 we present our results for threshold N<sup>3</sup>LO QCD contributions to the Higgs boson production through  $b\bar{b}$  annihilation at hadron colliders and their numerical impact. The numerical impact of threshold enhanced N<sup>3</sup>LO contributions is demonstrated for the LHC energy  $\sqrt{s} = 14$  TeV by studying the stability of the perturbation theory under factorization and renormalization scales. Finally we give a brief summary of our findings.

## 2 Threshold resummation

The interaction of bottom quarks and the Higgs boson is given by the action

$$S_I^b = -\frac{\lambda}{\sqrt{2}} \int d^4x \phi(x) \bar{\psi}_b(x) \psi_b(x) \quad (2.1)$$

where  $\psi_b(x)$  denotes the bottom quark field and  $\phi(x)$  the scalar field.  $\lambda$  is the Yukawa coupling given by  $\sqrt{2}m_b/\nu$ , with the bottom quark mass  $m_b$  and the vacuum expectation value  $\nu \approx 246$  GeV. In MSSM, for the pseudo-scalar Higgs boson, we need to replace  $\lambda\phi(x)\bar{\psi}_b(x)\psi_b(x)$  by  $\tilde{\lambda}\tilde{\phi}(x)\bar{\psi}_b(x)\gamma_5\psi_b(x)$  in the above equation. The MSSM couplings are given by

$$\tilde{\lambda} = \begin{cases} -\frac{\sqrt{2}m_b \sin \alpha}{\nu \cos \beta}, & \tilde{\phi} = h, \\ \frac{\sqrt{2}m_b \cos \alpha}{\nu \cos \beta}, & \tilde{\phi} = H, \\ \frac{\sqrt{2}m_b \tan \beta}{\nu}, & \tilde{\phi} = A \end{cases}$$

respectively. The angle  $\alpha$  measures the mixing of weak and mass eigenstates of neutral Higgs bosons. We use VFS scheme throughout, hence except in the Yukawa coupling,  $m_b$  is taken to be zero like other light quarks in the theory.

The inclusive Higgs boson production through bottom anti-bottom annihilation can be computed using

$$\sigma^b(s, q^2) = \sigma_{b\bar{b}}^{(0)}(\mu_R^2) \sum_{ac} \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} f_a(x_1, \mu_F^2) f_c(x_2, \mu_F^2) \Delta_{ac}^b(\hat{s}, q^2, \mu_F^2, \mu_R^2), \quad (2.2)$$

where  $f_a(x_1, \mu_F^2)$  and  $f_c(x_2, \mu_F^2)$  are parton distribution functions with momentum fractions  $x_1$  and  $x_2$  respectively.  $\mu_F$  is the factorization scale and  $\hat{s} = x_1 x_2 s$  where  $s$  ( $\hat{s}$ ) is the square of hadronic (partonic) center of mass energy. The born cross section is given by

$$\sigma_{b\bar{b}}^{(0)} = \frac{\pi \lambda^2(\mu_R^2)}{12m_H^2}. \quad (2.3)$$

The born normalized partonic subprocesses after mass factorization are denoted by  $\Delta_{ac}^b$  where initial state partons are  $a$  and  $c$ . Here,  $q^2 = m_H^2$ ,  $\mu_F$  results from mass factorization and the renormalization scale  $\mu_R$  is due to UV renormalization.  $\Delta_{ac}^b$  can be decomposed into two parts denoted by  $\Delta_b^{sv}$  and  $\Delta_{ac}^{b,R}$ .

$$\Delta_{ac}^b(z, q^2, \mu_R^2, \mu_F^2) = \Delta_b^{sv}(z, q^2, \mu_R^2, \mu_F^2) + \Delta_{ac}^{b,R}(z, q^2, \mu_R^2, \mu_F^2) \quad (2.4)$$

where  $\Delta_b^{sv}$  contains only distributions such as  $\delta(1-z)$  and  $\mathcal{D}_i$ , often called threshold contributions and the second term contains regular terms in  $z$ . Following [49, 50], the threshold resummed cross section at the partonic level in  $d = 4 + \epsilon$  dimensions is given by

$$\Delta_b^{sv}(z, q^2, \mu_R^2, \mu_F^2) = \mathcal{C} \exp\left(\Psi^b(z, q^2, \mu_R^2, \mu_F^2, \epsilon)\right)\Big|_{\epsilon=0} \quad (2.5)$$

where the scaling variable  $z = q^2/\hat{s}$  and  $\Psi^b(z, q^2, \mu_R^2, \mu_F^2, \epsilon)$  is a finite distribution. The above resummed expression follows from factorization properties of the inclusive cross section and Sudakov resummation of soft gluons in the QCD amplitudes. The symbol  $\mathcal{C}$  implies convolution with the following expansion

$$\mathcal{C}e^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}f(z) \otimes f(z) + \dots \quad (2.6)$$

Here  $\otimes$  means Mellin convolution and  $f(z)$  is a distribution of the kind  $\delta(1-z)$  and  $\mathcal{D}_i$ . We drop all the regular terms in  $z$  in the evaluation of threshold contribution  $\Delta_b^{sv}$ .

In  $d = 4 + \epsilon$  dimensions, the distribution  $\Psi^b$  receives contributions from UV renormalization constant  $Z_b$  for Yukawa coupling  $\lambda$ , the Higgs form factor  $\hat{F}^b$  from bottom anti-bottom, soft gluon distribution  $\Phi^b$  from the real radiations in partonic subprocesses and the mass factorization kernels  $\Gamma_{bb}$  that remove the collinear singularities from the initial state bottom quark states. That is

$$\Psi^b(z, q^2, \mu_R^2, \mu_F^2, \epsilon) = \left(\ln\left[Z^b(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)\right]^2 + \ln\left|\hat{F}^b(\hat{a}_s, Q^2, \mu^2, \epsilon)\right|^2\right)\delta(1-z) \quad (2.7)$$

$$+ 2\Phi^b(\hat{a}_s, q^2, \mu^2, z, \epsilon) - 2\mathcal{C} \ln \Gamma_{bb}(\hat{a}_s, \mu^2, \mu_F^2, z, \epsilon). \quad (2.8)$$

While the individual contributions are divergent, the sum is a finite distribution. The scale  $\mu$  is introduced to define the dimensionless coupling constant  $\hat{a}_s = \hat{g}_s^2/16\pi^2$  in dimensional regularization and  $Q^2 = -q^2$ . The renormalized strong coupling constant  $a_s(\mu_R^2)$  is related to bare  $\hat{a}_s$  through strong coupling constant renormalization  $Z(\mu_R^2)$ ,

$$\hat{a}_s = \left(\frac{\mu}{\mu_R}\right)^\epsilon Z(\mu_R^2)S_\epsilon^{-1}a_s(\mu_R^2), \quad S_\epsilon = \exp\left((\gamma_E - \ln 4\pi)\frac{\epsilon}{2}\right). \quad (2.9)$$

where  $\gamma_E$  is Euler-Mascheroni constant. In  $4 + \epsilon$  dimensions,  $Z(\mu_R^2)$  can be expressed in terms of the coefficients  $\beta_i$  of the  $\beta$  function of the strong coupling RG equation. Solving the RGE, we obtain, up to three loop level

$$Z(\mu_R^2) = 1 + a_s(\mu_R^2)\frac{2\beta_0}{\epsilon} + a_s^2(\mu_R^2)\left(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\right) + a_s^3(\mu_R^2)\left(\frac{8\beta_0^3}{\epsilon^3} + \frac{14\beta_0\beta_1}{3\epsilon^2} + \frac{2\beta_2}{3\epsilon}\right). \quad (2.10)$$

The coefficients  $\beta_0$ ,  $\beta_1$  and  $\beta_2$  are given in [59] as

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f,$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4T_F n_f C_F - \frac{20}{3}T_F n_f C_A,$$

$$\begin{aligned} \beta_2 = & \frac{2857}{54}C_A^3 - \frac{1415}{27}C_A^2T_F n_f + \frac{158}{27}C_A T_F^2 n_f^2 \\ & + \frac{44}{9}C_F T_F^2 n_f^2 - \frac{205}{9}C_F C_A T_F n_f + 2C_F^2 T_F n_f \end{aligned} \quad (2.11)$$

where the  $SU(N)$  QCD color factors are given by

$$C_A = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_F = \frac{1}{2} \quad (2.12)$$

and  $n_f$  is the number of active flavours.

The UV renormalization constant  $Z^b(\hat{a}_s, \mu_R^2, \mu^2, \epsilon)$  for Yukawa coupling  $\lambda$  satisfies

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z^b(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = \sum_{i=1}^{\infty} a_s^i(\mu_R^2) \gamma_{i-1}^b.$$

The above renormalization group equation (RGE) can be solved in  $4 + \epsilon$  dimensions to obtain up to  $\mathcal{O}(a_s^3)$  level:

$$\begin{aligned} Z^b(\hat{a}_s, \mu_R^2, \mu^2, \epsilon) = & 1 + \hat{a}_s \left( \frac{\mu_R^2}{\mu^2} \right)^{\frac{\epsilon}{2}} S_\epsilon \left[ \frac{1}{\epsilon} \left( 2 \gamma_0^b \right) \right] + \hat{a}_s^2 \left( \frac{\mu_R^2}{\mu^2} \right)^\epsilon S_\epsilon^2 \left[ \frac{1}{\epsilon^2} \left( 2 \left( \gamma_0^b \right)^2 - 2 \beta_0 \gamma_0^b \right) \right. \\ & + \frac{1}{\epsilon} \left( \gamma_1^b \right) \left. \right] + \hat{a}_s^3 \left( \frac{\mu_R^2}{\mu^2} \right)^{3\frac{\epsilon}{2}} S_\epsilon^3 \left[ \frac{1}{\epsilon^3} \left( \frac{4}{3} \left( \gamma_0^b \right)^3 - 4 \beta_0 \left( \gamma_0^b \right)^2 + \frac{8}{3} \beta_0^2 \gamma_0^b \right) \right. \\ & \left. + \frac{1}{\epsilon^2} \left( 2 \gamma_0^b \gamma_1^b - \frac{2}{3} \beta_1 \gamma_0^b - \frac{8}{3} \beta_0 \gamma_1^b \right) + \frac{1}{\epsilon} \left( \frac{2}{3} \gamma_2^b \right) \right] \end{aligned} \quad (2.13)$$

where the anomalous dimensions  $\gamma_i^b$  can be obtained from the quark mass anomalous dimensions [60–62]

$$\begin{aligned} \gamma_0^b = & 3C_F, \\ \gamma_1^b = & \frac{3}{2}C_F^2 + \frac{97}{6}C_F C_A - \frac{10}{3}C_F T_F n_f, \\ \gamma_2^b = & \frac{129}{2}C_F^3 - \frac{129}{4}C_F^2 C_A + \frac{11413}{108}C_F C_A^2 + \left( -46 + 48\zeta_3 \right) C_F^2 T_F n_f \\ & + \left( -\frac{556}{27} - 48\zeta_3 \right) C_F C_A T_F n_f - \frac{140}{27}C_F T_F^2 n_f^2. \end{aligned} \quad (2.14)$$

The bare form factor  $\hat{F}^b(\hat{a}_s, Q^2, \mu^2, \epsilon)$  satisfies the following differential equation [63–66]

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}^b = \frac{1}{2} \left[ K^b \left( \hat{a}_s, \frac{\mu_R^2}{\mu^2}, \epsilon \right) + G^b \left( \hat{a}_s, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \epsilon \right) \right] \quad (2.15)$$

where  $K^b$  contains all the poles in  $\epsilon$  and  $G^b$  contains the terms finite in  $\epsilon$ . Renormalization group invariance of  $\hat{F}^b(\hat{a}_s, Q^2, \mu^2, \epsilon)$  gives

$$\mu_R^2 \frac{d}{d\mu_R^2} K^b = -\mu_R^2 \frac{d}{d\mu_R^2} G^b = -\sum_{i=1}^{\infty} a_s^i(\mu_R^2) A_i^q. \quad (2.16)$$

where  $A_i^q$ 's are the cusp anomalous dimensions [42, 43, 67–69] given by

$$\begin{aligned}
 A_1^q &= 4C_F, \\
 A_2^q &= 8C_F C_A \left\{ \frac{67}{18} - \zeta_2 \right\} + 8C_F n_f \left\{ -\frac{5}{9} \right\}, \\
 A_3^q &= 16C_F C_A^2 \left\{ \frac{245}{24} - \frac{67\zeta_2}{9} + \frac{11\zeta_3}{6} + \frac{11\zeta_2^2}{5} \right\} + 16C_F^2 n_f \left\{ -\frac{55}{24} + 2\zeta_3 \right\} \\
 &\quad + 16C_F C_A n_f \left\{ -\frac{209}{108} + \frac{10\zeta_2}{9} - \frac{7\zeta_3}{3} \right\} + 16C_F n_f^2 \left\{ -\frac{1}{27} \right\}. \tag{2.17}
 \end{aligned}$$

Expanding  $\mu_R^2$  independent part of the solution of RG equation for  $G^b$  as

$$G^b(a_s(Q^2), 1, \epsilon) = \sum_{i=1}^{\infty} a_s^i(Q^2) G_i^b(\epsilon), \tag{2.18}$$

one finds that  $G_i^b$  can be decomposed in terms of collinear  $B_i^q$  and soft  $f_i^q$  anomalous dimensions as follows [70–72]

$$G_i^b(\epsilon) = 2(B_i^q - \gamma_i^b) + f_i^q + C_i^b + \sum_{k=1}^{\infty} \epsilon^k g_i^{b,k}. \tag{2.19}$$

The collinear anomalous dimensions  $B_i^q$  [42, 43] are given by

$$\begin{aligned}
 B_1^q &= 3C_F, \\
 B_2^q &= \frac{1}{2} \left( C_F^2 \left\{ 3 - 24\zeta_2 + 48\zeta_3 \right\} + C_A C_F \left\{ \frac{17}{3} + \frac{88}{3}\zeta_2 - 24\zeta_3 \right\} + n_f T_F C_F \left\{ -\frac{4}{3} - \frac{32}{3}\zeta_2 \right\} \right), \\
 B_3^q &= -16C_A^2 C_F \left\{ \frac{1}{8}\zeta_2^2 - \frac{281}{27}\zeta_2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5 + \frac{1657}{576} \right\} + 16C_A C_F^2 \left\{ -\frac{247}{60}\zeta_2^2 + \zeta_2\zeta_3 \right. \\
 &\quad \left. - \frac{205}{24}\zeta_2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5 + \frac{151}{64} \right\} + 16C_A C_F n_f \left\{ \frac{1}{20}\zeta_2^2 - \frac{167}{54}\zeta_2 + \frac{25}{18}\zeta_3 + \frac{5}{4} \right\} \\
 &\quad + 16C_F^3 \left\{ \frac{18}{5}\zeta_2^2 - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{17}{4}\zeta_3 - 15\zeta_5 + \frac{29}{32} \right\} \\
 &\quad - 16C_F^2 n_f \left\{ -\frac{29}{30}\zeta_2^2 - \frac{5}{12}\zeta_2 + \frac{17}{6}\zeta_3 + \frac{23}{16} \right\} - 16C_F n_f^2 \left\{ -\frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 + \frac{17}{144} \right\},
 \end{aligned}$$

the soft anomalous dimensions  $f_i^q$  [70] are

$$\begin{aligned}
 f_1^q &= 0, \\
 f_2^q &= C_A C_F \left\{ -\frac{22}{3}\zeta_2 - 28\zeta_3 + \frac{808}{27} \right\} + C_F n_f T_F \left\{ \frac{8}{3}\zeta_2 - \frac{224}{27} \right\}, \\
 f_3^q &= C_A^2 C_F \left\{ \frac{352}{5}\zeta_2^2 + \frac{176}{3}\zeta_2\zeta_3 - \frac{12650}{81}\zeta_2 - \frac{1316}{3}\zeta_3 + 192\zeta_5 + \frac{136781}{729} \right\} \\
 &\quad + C_A C_F n_f \left\{ -\frac{96}{5}\zeta_2^2 + \frac{2828}{81}\zeta_2 + \frac{728}{27}\zeta_3 - \frac{11842}{729} \right\} \\
 &\quad + C_F^2 n_f \left\{ \frac{32}{5}\zeta_2^2 + 4\zeta_2 + \frac{304}{9}\zeta_3 - \frac{1711}{27} \right\} + C_F n_f^2 \left\{ -\frac{40}{27}\zeta_2 + \frac{112}{27}\zeta_3 - \frac{2080}{729} \right\} \tag{2.20}
 \end{aligned}$$

and the constants  $C_i^b$  are given by

$$C_1^b = 0, \quad C_2^b = -2\beta_0 g_1^{b,1}, \quad C_3^b = -2\beta_1 g_1^{b,1} - 2\beta_0 (g_2^{b,1} + 2\beta_0 g_1^{b,2}). \quad (2.21)$$

Since  $G_1^b(\epsilon)$  and  $G_2^b(\epsilon)$  are known to all orders in  $\epsilon$  and  $G_3^b(\epsilon)$  is known to  $\mathcal{O}(\epsilon^3)$  [58], the coefficients  $g_i^{b,k}$  for  $i = 1, 2, 3$  can be readily obtained from  $G_i^b(\epsilon)$ . The relevant one loop terms are found to be

$$g_1^{b,1} = C_F(-2 + \zeta_2), \quad g_1^{b,2} = C_F\left(2 - \frac{7}{3}\zeta_3\right), \quad g_1^{b,3} = C_F\left(-2 + \frac{1}{4}\zeta_2 + \frac{47}{80}\zeta_2^2\right), \quad (2.22)$$

the relevant two loop terms [23, 49, 50] are

$$\begin{aligned} g_2^{b,1} &= C_F n_f \left\{ \frac{616}{81} + \frac{10}{9}\zeta_2 - \frac{8}{3}\zeta_3 \right\} + C_F C_A \left\{ -\frac{2122}{81} - \frac{103}{9}\zeta_2 + \frac{88}{5}\zeta_2^2 + \frac{152}{3}\zeta_3 \right\} \\ &\quad + C_F^2 \left\{ 8 + 32\zeta_2 - \frac{88}{5}\zeta_2^2 - 60\zeta_3 \right\}, \\ g_2^{b,2} &= C_F n_f \left\{ \frac{7}{12}\zeta_2^2 - \frac{55}{27}\zeta_2 + \frac{130}{27}\zeta_3 - \frac{3100}{243} \right\} + C_A C_F \left\{ -\frac{365}{24}\zeta_2^2 + \frac{89}{3}\zeta_2\zeta_3 + \frac{1079}{54}\zeta_2 \right. \\ &\quad \left. - \frac{2923}{27}\zeta_3 - 51\zeta_5 + \frac{9142}{243} \right\} + C_F^2 \left\{ \frac{96}{5}\zeta_2^2 - 28\zeta_2\zeta_3 - 44\zeta_2 + 116\zeta_3 + 12\zeta_5 - 24 \right\} \end{aligned}$$

and finally the relevant three loop term [58] is

$$\begin{aligned} g_3^{b,1} &= C_A^2 C_F \left\{ -\frac{6152}{63}\zeta_2^3 + \frac{2738}{9}\zeta_2^2 + \frac{976}{9}\zeta_2\zeta_3 - \frac{342263}{486}\zeta_2 - \frac{1136}{3}\zeta_3^2 + \frac{19582}{9}\zeta_3 \right. \\ &\quad \left. + \frac{1228}{3}\zeta_5 + \frac{4095263}{8748} \right\} + C_A C_F^2 \left\{ -\frac{15448}{105}\zeta_2^3 - \frac{3634}{45}\zeta_2^2 - \frac{2584}{3}\zeta_2\zeta_3 + \frac{13357}{9}\zeta_2 \right. \\ &\quad \left. + 296\zeta_3^2 - \frac{11570}{9}\zeta_3 - \frac{1940}{3}\zeta_5 - \frac{613}{3} \right\} + C_A C_F n_f \left\{ -\frac{1064}{45}\zeta_2^2 + \frac{392}{9}\zeta_2\zeta_3 + \frac{44551}{243}\zeta_2 \right. \\ &\quad \left. - \frac{41552}{81}\zeta_3 - 72\zeta_5 - \frac{6119}{4374} \right\} + C_F^2 n_f \left\{ \frac{772}{45}\zeta_2^2 - \frac{152}{3}\zeta_2\zeta_3 - \frac{3173}{18}\zeta_2 + \frac{15956}{27}\zeta_3 - \frac{368}{3}\zeta_5 \right. \\ &\quad \left. + \frac{32899}{324} \right\} + C_F n_f^2 \left\{ -\frac{40}{9}\zeta_2^2 - \frac{892}{81}\zeta_2 + \frac{320}{81}\zeta_3 - \frac{27352}{2187} \right\} + C_F^3 \left\{ \frac{21584}{105}\zeta_2^3 - \frac{1644}{5}\zeta_2^2 \right. \\ &\quad \left. + 624\zeta_2\zeta_3 - 275\zeta_2 + 48\zeta_3^2 - 2142\zeta_3 + 1272\zeta_5 + 603 \right\}. \quad (2.23) \end{aligned}$$

The mass factorization kernel  $\Gamma_{bb}(z, \mu_F^2, \epsilon)$  removes the collinear singularities resulting from massless partons. It satisfies the following RG equation

$$\mu_F^2 \frac{d}{d\mu_F^2} \Gamma_{bb}(z, \mu_F^2, \epsilon) = \frac{1}{2} \sum_c P_{bc}(z, \mu_F^2) \otimes \Gamma_{cb}(z, \mu_F^2, \epsilon), \quad (2.24)$$

where  $P_{bc}(z, \mu_F^2)$  are Altarelli-Parisi splitting functions. In perturbative QCD, we can expand them as

$$P_{bc}(z, \mu_F^2) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) P_{bc}^{(i-1)}(z). \quad (2.25)$$



The off diagonal splitting functions are regular as  $z \rightarrow 1$ . On the other hand diagonal ones contain  $\delta(1-z)$  and  $\mathcal{D}_0$  as well as regular terms i.e.

$$P_{bb}^{(i)}(z) = 2(B_{i+1}^b \delta(1-z) + A_{i+1}^b \mathcal{D}_0) + P_{reg,bb}^{(i)}(z). \quad (2.26)$$

We find that the regular part of the splitting function,  $P_{reg,bb}^{(i)}$ , does not contribute to threshold corrections.

The fact that  $\Delta_b^{sv}$  is finite as  $\epsilon \rightarrow 0$  implies that soft distribution function  $\Phi^b(\hat{a}_s, q^2, \mu^2, z, \epsilon)$  also satisfies Sudakov type differential equations [49, 50] namely

$$q^2 \frac{d}{dq^2} \Phi^b = \frac{1}{2} \left[ \overline{K}^b(\hat{a}_s, \frac{\mu_R^2}{\mu^2}, z, \epsilon) + \overline{G}^b(\hat{a}_s, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, z, \epsilon) \right].$$

$\overline{K}^b$  and  $\overline{G}^b$  take the forms similar to those of  $K^b$  and  $G^b$  of the form factors. This guarantees  $\Psi^b$  is finite order by order in perturbation theory as  $\epsilon \rightarrow 0$ . The solution to the above equation is found to be

$$\Phi^b = \sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q^2(1-z)^2}{\mu^2} \right)^{i \frac{\epsilon}{2}} S_\epsilon^i \left( \frac{i\epsilon}{1-z} \right) \hat{\phi}^{b,(i)}(\epsilon) \quad (2.27)$$

where

$$\hat{\phi}^{b,(i)}(\epsilon) = \frac{1}{i\epsilon} [\overline{K}^{b,(i)}(\epsilon) + \overline{G}^{b,(i)}(\epsilon)] \quad (2.28)$$

and the constants  $\overline{K}^b$  can be obtained using

$$\mu_R^2 \frac{d\overline{K}^b}{d\mu_R^2} = -\delta(1-z) \mu_R^2 \frac{dK^b}{d\mu_R^2}. \quad (2.29)$$

This implies that  $\overline{K}^{b,(i)}(\epsilon)$  can be written in terms of  $A_i^q$  and  $\beta_i$ . Defining  $\overline{\mathcal{G}}_i^b(\epsilon)$  through

$$\sum_{i=1}^{\infty} \hat{a}_s^i \left( \frac{q_z^2}{\mu^2} \right)^{i \frac{\epsilon}{2}} S_\epsilon^i \overline{\mathcal{G}}^{b,(i)}(\epsilon) = \sum_{i=1}^{\infty} a_s^i(q_z^2) \overline{\mathcal{G}}_i^b(\epsilon) \quad (2.30)$$

where  $q_z^2 = q^2(1-z)^2$  and using the fact that  $\Delta_b^{sv}$  is finite as  $\epsilon \rightarrow 0$ , we can express  $\overline{\mathcal{G}}_i^b(\epsilon)$  as

$$\begin{aligned} \overline{\mathcal{G}}_1^b(\epsilon) &= -f_1^q + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_1^{b,(k)}, \\ \overline{\mathcal{G}}_2^b(\epsilon) &= -f_2^q - 2\beta_0 \overline{\mathcal{G}}_1^{b,(1)} + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_2^{b,(k)}, \\ \overline{\mathcal{G}}_3^b(\epsilon) &= -f_3^q - 2\beta_1 \overline{\mathcal{G}}_1^{b,(1)} - 2\beta_0 \left( \overline{\mathcal{G}}_2^{b,(1)} + 2\beta_0 \overline{\mathcal{G}}_1^{b,(2)} \right) + \sum_{k=1}^{\infty} \epsilon^k \overline{\mathcal{G}}_3^{b,(k)}. \end{aligned} \quad (2.31)$$

The constants  $\overline{\mathcal{G}}_i^{b,j}$  arise from the soft part of the partonic reactions. Since the soft part does not depend on the hard process,  $\overline{\mathcal{G}}_i^b(\epsilon)$  can be directly obtained from  $\overline{\mathcal{G}}_i^q(\epsilon)$  that contributes to Drell-Yan production [49, 50]

$$\overline{\mathcal{G}}_i^b(\epsilon) = \overline{\mathcal{G}}_i^q(\epsilon). \quad (2.32)$$

We list the relevant ones that contribute up to N<sup>3</sup>LO level. The terms required for one and two loops [49, 50] are

$$\begin{aligned}
\bar{\mathcal{G}}_1^{q,(1)} &= C_F(-3\zeta_2), \\
\bar{\mathcal{G}}_1^{q,(2)} &= C_F\left(\frac{7}{3}\zeta_3\right), \\
\bar{\mathcal{G}}_1^{q,(3)} &= C_F\left(-\frac{3}{16}\zeta_2^2\right), \\
\bar{\mathcal{G}}_2^{q,(1)} &= C_F n_f \left(-\frac{328}{81} + \frac{70}{9}\zeta_2 + \frac{32}{3}\zeta_3\right) + C_A C_F \left(\frac{2428}{81} - \frac{469}{9}\zeta_2 + 4\zeta_2^2 - \frac{176}{3}\zeta_3\right), \\
\bar{\mathcal{G}}_2^{q,(2)} &= C_A C_F \left(\frac{11}{40}\zeta_2^2 - \frac{203}{3}\zeta_2\zeta_3 + \frac{1414}{27}\zeta_2 + \frac{2077}{27}\zeta_3 + 43\zeta_5 - \frac{7288}{243}\right) \\
&\quad + C_F n_f \left(-\frac{1}{20}\zeta_2^2 - \frac{196}{27}\zeta_2 - \frac{310}{27}\zeta_3 + \frac{976}{243}\right)
\end{aligned}$$

and for three loops [54]

$$\begin{aligned}
\bar{\mathcal{G}}_3^{q,(1)} &= C_F \left\{ C_A^2 \left( \frac{152}{63}\zeta_2^3 + \frac{1964}{9}\zeta_2^2 + \frac{11000}{9}\zeta_2\zeta_3 - \frac{765127}{486}\zeta_2 + \frac{536}{3}\zeta_3^2 - \frac{59648}{27}\zeta_3 \right. \right. \\
&\quad \left. \left. - \frac{1430}{3}\zeta_5 + \frac{7135981}{8748} \right) + C_A n_f \left( -\frac{532}{9}\zeta_2^2 - \frac{1208}{9}\zeta_2\zeta_3 + \frac{105059}{243}\zeta_2 + \frac{45956}{81}\zeta_3 \right. \right. \\
&\quad \left. \left. + \frac{148}{3}\zeta_5 - \frac{716509}{4374} \right) + C_F n_f \left( \frac{152}{15}\zeta_2^2 - 88\zeta_2\zeta_3 + \frac{605}{6}\zeta_2 + \frac{2536}{27}\zeta_3 + \frac{112}{3}\zeta_5 \right. \right. \\
&\quad \left. \left. - \frac{42727}{324} \right) + n_f^2 \left( \frac{32}{9}\zeta_2^2 - \frac{1996}{81}\zeta_2 - \frac{2720}{81}\zeta_3 + \frac{11584}{2187} \right) \right\}.
\end{aligned}$$

### 3 Results

Expanding eq. (2.5) in powers of  $a_s(\mu_R^2)$  using eqn (2.6) and performing the convolutions we find

$$\Delta_b^{sv}(z) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) \Delta_b^{sv,(i)}(z, \mu_R^2) \tag{3.1}$$

where  $\Delta_b^{sv,(i)}$  can be expressed in terms of the distributions  $\delta(1-z)$  and  $\mathcal{D}_j$ :

$$\Delta_b^{sv,(i)} = \Delta_b^{sv,(i)}(\mu_R^2)|_{\delta} \delta(1-z) + \sum_{j=0}^{2i-1} \Delta_b^{sv,(i)}(\mu_R^2)|_{\mathcal{D}_j} \mathcal{D}_j. \tag{3.2}$$

The results up to N<sup>3</sup>LO are given below

$$\begin{aligned}
\Delta_b^{sv,(1)} &= \left\{ \delta(1-z) \left( 2\bar{\mathcal{G}}_1^{b,(1)} + 2g_1^{b,1} + 3\zeta_2 A_1^b \right) - 2\mathcal{D}_0 f_1^b + 4\mathcal{D}_1 A_1^b \right\}, \\
\Delta_b^{sv,(2)} &= \left\{ \delta(1-z) \left( \bar{\mathcal{G}}_2^{b,(1)} + 2\bar{\mathcal{G}}_1^{b,(1)2} + g_2^{b,1} + 4g_1^{b,1} \bar{\mathcal{G}}_1^{b,(1)} + 2g_1^{b,12} + \beta_0 \left( 2\bar{\mathcal{G}}_1^{b,(2)} + 2g_1^{b,2} \right) \right. \right. \\
&\quad \left. \left. - 8\zeta_3 f_1^b A_1^b + \zeta_2 \left( 3A_2^b - 2(f_1^b)^2 + 6\bar{\mathcal{G}}_1^{b,(1)} A_1^b + 6g_1^{b,1} A_1^b \right) + \zeta_2 \beta_0 \left( -6\gamma_0^b + 6B_1^b \right) \right. \right.
\end{aligned}$$

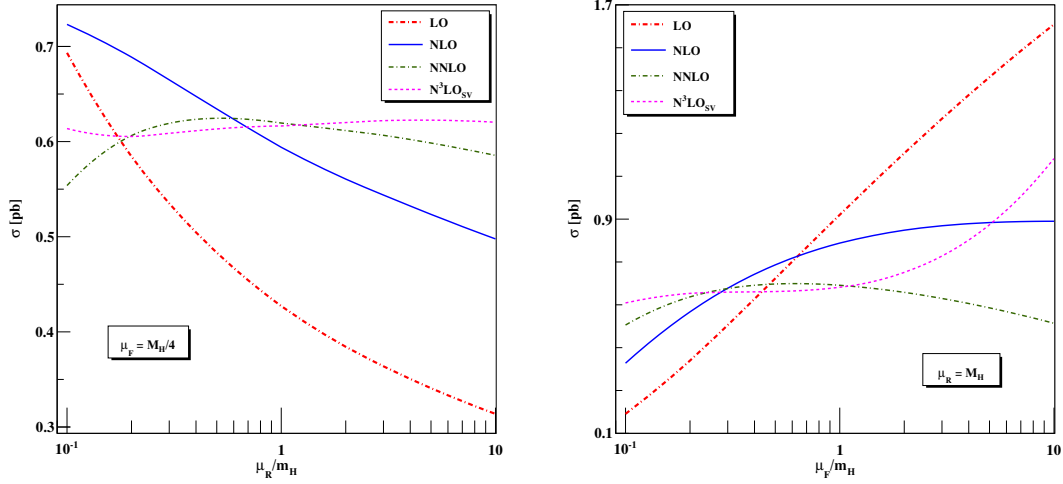
$$\begin{aligned}
 & + 3f_1^b) + \frac{37}{10}\zeta_2^2(A_1^b)^2) + \mathcal{D}_0(-4\beta_0\bar{\mathcal{G}}_1^{b,(1)} + 16\zeta_3(A_1^b)^2 + 2\zeta_2 f_1^b A_1^b - 2f_2^b \\
 & - 4\bar{\mathcal{G}}_1^{b,(1)} f_1^b - 4g_1^{b,1} f_1^b) + \mathcal{D}_1(4\beta_0 f_1^b - 4\zeta_2(A_1^b)^2 + 4A_2^b + 4(f_1^b)^2 + 8\bar{\mathcal{G}}_1^{b,(1)} A_1^b \\
 & + 8g_1^{b,1} A_1^b) + \mathcal{D}_2(-4\beta_0 A_1^b - 12f_1^b A_1^b) + 8\mathcal{D}_3(A_1^b)^2 \Big\}, \\
 \Delta_b^{sv,(3)} = & \delta(1-z) \Big\{ \frac{2}{3}\bar{\mathcal{G}}_3^{b,(1)} + 2\bar{\mathcal{G}}_1^{b,(1)}\bar{\mathcal{G}}_2^{b,(1)} + \frac{4}{3}\bar{\mathcal{G}}_1^{b,(1)3} + \frac{2}{3}g_3^{b,1} + 2g_2^{b,1}\bar{\mathcal{G}}_1^{b,(1)} + 2g_1^{b,1}\bar{\mathcal{G}}_2^{b,(1)} \\
 & + 4g_1^{b,1}\bar{\mathcal{G}}_1^{b,(1)2} + 2g_1^{b,1}g_2^{b,1} + 4g_1^{b,12}\bar{\mathcal{G}}_1^{b,(1)} + \frac{4}{3}g_1^{b,13} + \frac{4}{3}\beta_1(\bar{\mathcal{G}}_1^{b,(2)} + g_1^{b,2}) + 4\beta_0\left(\frac{1}{3}\bar{\mathcal{G}}_2^{b,(2)} \right. \\
 & \left. + \bar{\mathcal{G}}_1^{b,(1)}\bar{\mathcal{G}}_1^{b,(2)} + \frac{1}{3}g_2^{b,2} + g_1^{b,2}\bar{\mathcal{G}}_1^{b,(1)} + g_1^{b,1}\bar{\mathcal{G}}_1^{b,(2)} + g_1^{b,1}g_1^{b,2}\right) + \frac{8}{3}\beta_0^2(\bar{\mathcal{G}}_1^{b,(3)} + g_1^{b,3}) \\
 & - 32\zeta_5(3f_1^b + 2\beta_0)(A_1^b)^2 - 8\zeta_3\left(f_2^b A_1^b + f_1^b A_2^b + \frac{1}{3}(f_1^b)^3 + 2\bar{\mathcal{G}}_1^{b,(1)} f_1^b A_1^b \right. \\
 & \left. + 2g_1^{b,1} f_1^b A_1^b\right) - 8\zeta_3\beta_0\left((f_1^b)^2 + 2\bar{\mathcal{G}}_1^{b,(1)} A_1^b\right) + \frac{160}{3}\zeta_3^2(A_1^b)^3 + \zeta_2(3A_3^b - 4f_1^b f_2^b \\
 & + 3\bar{\mathcal{G}}_2^{b,(1)} A_1^b + 6\bar{\mathcal{G}}_1^{b,(1)} A_2^b - 4\bar{\mathcal{G}}_1^{b,(1)}(f_1^b)^2 + 6\bar{\mathcal{G}}_1^{b,(1)2} A_1^b + 3g_2^{b,1} A_1^b + 6g_1^{b,1} A_2^b \\
 & - 4g_1^{b,1}(f_1^b)^2 + 12g_1^{b,1}\bar{\mathcal{G}}_1^{b,(1)} A_1^b + 6g_1^{b,12} A_1^b) - 3\zeta_2\beta_1(2\gamma_0^b - 2B_1^b - f_1^b) \\
 & + 6\zeta_2\beta_0\left(-2\gamma_1^b + 2B_2^b + f_2^b + \bar{\mathcal{G}}_1^{b,(2)} A_1^b - 2\bar{\mathcal{G}}_1^{b,(1)}\gamma_0^b + 2\bar{\mathcal{G}}_1^{b,(1)} B_1^b - \frac{1}{3}\bar{\mathcal{G}}_1^{b,(1)} f_1^b + g_1^{b,2} A_1^b \right. \\
 & \left. - 2g_1^{b,1}\gamma_0^b + 2g_1^{b,1} B_1^b + g_1^{b,1} f_1^b\right) - 12\zeta_2\beta_0^2 g_1^{b,1} + 40\zeta_2\zeta_3 f_1^b(A_1^b)^2 + 32\zeta_2\zeta_3\beta_0(A_1^b)^2 \\
 & + \frac{37}{5}\zeta_2^2\left(A_1^b A_2^b - \frac{38}{37}(f_1^b)^2 A_1^b + \bar{\mathcal{G}}_1^{b,(1)}(A_1^b)^2 + g_1^{b,1}(A_1^b)^2\right) + \zeta_2^2\beta_0(18A_1^b B_1^b \\
 & - 18A_1^b\gamma_0^b + f_1^b A_1^b) - 3\zeta_2^2\beta_0^2 A_1^b - \frac{283}{42}\zeta_3^3(A_1^b)^3 \Big\} + \mathcal{D}_0\left\{-4\beta_1\bar{\mathcal{G}}_1^{b,(1)} - 4\beta_0(\bar{\mathcal{G}}_2^{b,(1)} \right. \\
 & \left. + \bar{\mathcal{G}}_1^{b,(2)} f_1^b + 2\bar{\mathcal{G}}_1^{b,(1)2} + g_1^{b,2} f_1^b + 2g_1^{b,1}\bar{\mathcal{G}}_1^{b,(1)}) - 8\beta_0^2\bar{\mathcal{G}}_1^{b,(2)} + 192\zeta_5(A_1^b)^3 + 32\zeta_3(A_1^b A_2^b \right. \\
 & \left. + (f_1^b)^2 A_1^b + \bar{\mathcal{G}}_1^{b,(1)}(A_1^b)^2 + g_1^{b,1}(A_1^b)^2\right) + 48\zeta_3\beta_0 f_1^b A_1^b + 2\zeta_2(f_2^b A_1^b + f_1^b A_2^b \\
 & + 2(f_1^b)^3 + 2\bar{\mathcal{G}}_1^{b,(1)} f_1^b A_1^b + 2g_1^{b,1} f_1^b A_1^b) + 2\zeta_2\beta_0(6f_1^b\gamma_0^b - 6f_1^b B_1^b + (f_1^b)^2 \\
 & + 2\bar{\mathcal{G}}_1^{b,(1)} A_1^b) - 80\zeta_2\zeta_3(A_1^b)^3 + 23\zeta_2^2 f_1^b(A_1^b)^2 + 16\zeta_2^2\beta_0(A_1^b)^2 - 2(f_3^b + \bar{\mathcal{G}}_2^{b,(1)} f_1^b \\
 & + 2\bar{\mathcal{G}}_1^{b,(1)} f_2^b + 2\bar{\mathcal{G}}_1^{b,(1)2} f_1^b + g_2^{b,1} f_1^b + 2g_1^{b,1} f_2^b + 4g_1^{b,1}\bar{\mathcal{G}}_1^{b,(1)} f_1^b + 2g_1^{b,12} f_1^b) \Big\} + \mathcal{D}_1\left\{4\beta_1 f_1^b \right. \\
 & \left. + 8\beta_0(f_2^b + \bar{\mathcal{G}}_1^{b,(2)} A_1^b + 3\bar{\mathcal{G}}_1^{b,(1)} f_1^b + g_1^{b,2} A_1^b + g_1^{b,1} f_1^b) + 16\beta_0^2\bar{\mathcal{G}}_1^{b,(1)} - 160\zeta_3 f_1^b(A_1^b)^2 \right. \\
 & \left. - 96\zeta_3\beta_0(A_1^b)^2 - 8\zeta_2\left(A_1^b A_2^b + \frac{7}{2}(f_1^b)^2 A_1^b + \bar{\mathcal{G}}_1^{b,(1)}(A_1^b)^2 + g_1^{b,1}(A_1^b)^2\right) \right. \\
 & \left. - 24\zeta_2\beta_0(A_1^b\gamma_0^b - A_1^b B_1^b + f_1^b A_1^b) - 46\zeta_2^2(A_1^b)^3 + 4(A_3^b + 2f_1^b f_2^b + \bar{\mathcal{G}}_2^{b,(1)} A_1^b \right.
 \end{aligned}$$

$$\begin{aligned}
 & + 2\bar{\mathcal{G}}_1^{b,(1)} A_2^b + 2\bar{\mathcal{G}}_1^{b,(1)} (f_1^b)^2 + 2\bar{\mathcal{G}}_1^{b,(1)2} A_1^b + g_2^{b,1} A_1^b + 2g_1^{b,1} A_2^b + 2g_1^{b,1} (f_1^b)^2 \\
 & + 4g_1^{b,1} \bar{\mathcal{G}}_1^{b,(1)} A_1^b + 2g_1^{b,1,2} A_1^b \Big\} + \mathcal{D}_2 \Big\{ -4\beta_1 A_1^b - 8\beta_0 \left( A_2^b + \frac{3}{2} (f_1^b)^2 + 4\bar{\mathcal{G}}_1^{b,(1)} A_1^b \right. \\
 & + g_1^{b,1} A_1^b \Big) - 8\beta_0^2 f_1^b + 160\zeta_3 (A_1^b)^3 + 60\zeta_2 f_1^b (A_1^b)^2 + 36\zeta_2 \beta_0 (A_1^b)^2 - 12 \left( f_2^b A_1^b \right. \\
 & + f_1^b A_2^b + \frac{1}{3} (f_1^b)^3 + 2\bar{\mathcal{G}}_1^{b,(1)} f_1^b A_1^b + 2g_1^{b,1} f_1^b A_1^b \Big) \Big\} + \mathcal{D}_3 \Big\{ \frac{80}{3} \beta_0 f_1^b A_1^b + \frac{16}{3} \beta_0^2 A_1^b \\
 & - 40\zeta_2 (A_1^b)^3 + 16 \left( A_1^b A_2^b + (f_1^b)^2 A_1^b + \bar{\mathcal{G}}_1^{b,(1)} (A_1^b)^2 + g_1^{b,1} (A_1^b)^2 \right) \Big\} \\
 & + \mathcal{D}_4 \Big\{ -\frac{40}{3} \beta_0 (A_1^b)^2 - 20 f_1^b (A_1^b)^2 \Big\} + 8\mathcal{D}_5 (A_1^b)^3. \tag{3.3}
 \end{aligned}$$

In the above equation  $A_i^b = A_i^q$ ,  $B_i^b = B_i^q$ ,  $f_i^b = f_i^q$  and we have set  $\mu_R^2 = \mu_F^2 = q^2$ .

The finite  $\Delta_b^{sv,(i)}(Q^2)$  depend on the anomalous dimensions  $A_i^b$ ,  $B_i^b$ ,  $f_i^b$  and  $\gamma_i^b$ , the  $\beta$  function coefficients  $\beta_i$  and  $\epsilon$  expansion coefficients of  $G^b(\epsilon)$ ,  $g_j^{b,i}$ 's and of the corresponding  $\bar{\mathcal{G}}^b(\epsilon)$ ,  $\bar{\mathcal{G}}_j^{b,i}$ 's. The results for the  $\Delta_b^{sv,(1)}$  and  $\Delta_b^{sv,(2)}$  agree with those from the exact NLO and NNLO results [23]. At N<sup>3</sup>LO level, only  $\Delta_b^{sv,(3)}|_{\mathcal{D}_i}$ 's were known [49, 50] as the terms  $g_2^{b,2}, g_3^{b,1}$  of the form factor and  $\bar{\mathcal{G}}_2^{q,(2)}, \bar{\mathcal{G}}_3^{q,(1)}$  needed for  $\Delta_b^{sv,(3)}|_\delta$  were not available. The recent results for  $g_2^{b,2}$  and  $g_3^{b,1}$  from [58],  $\bar{\mathcal{G}}_2^{q,2}$  from [45] and the N<sup>3</sup>LO  $\bar{\mathcal{G}}_3^{q,(1)}$  from [54] can be used to obtain the missing  $\delta(1-z)$  part namely  $\Delta_b^{sv,(3)}|_\delta$ . This completes the evaluation of full threshold N<sup>3</sup>LO contributions in QCD for the Higgs boson production through bottom anti-bottom annihilation at hadron colliders. Below we present our results up to N<sup>3</sup>LO level after substituting  $A_i, B_i, f_i, \gamma_i$  and  $\beta_i$  terms as well as the constants  $g_j^{b,i}$  from the form factors and  $\bar{\mathcal{G}}_j^{b,(i)}$  from the soft distributions function in eq. (3.3).

$$\begin{aligned}
 \Delta_b^{sv,(3)} = & \delta(1-z) \Big\{ C_A^2 C_F \left( \frac{13264}{315} \zeta_2^3 + \frac{2528}{27} \zeta_2^2 - \frac{1064}{3} \zeta_2 \zeta_3 - 272 \zeta_2 - \frac{400}{3} \zeta_3^2 - \frac{14212}{81} \zeta_3 \right. \\
 & - 84 \zeta_5 + \frac{68990}{81} \Big) + C_A C_F^2 \left( -\frac{20816}{315} \zeta_2^3 - \frac{62468}{135} \zeta_2^2 + \frac{27872}{9} \zeta_2 \zeta_3 + \frac{22106}{27} \zeta_2 \right. \\
 & + \frac{3280}{3} \zeta_3^2 - \frac{10940}{9} \zeta_3 - \frac{37144}{9} \zeta_5 - \frac{982}{3} \Big) + C_A C_F n_f \left( -\frac{6728}{135} \zeta_2^2 + \frac{208}{3} \zeta_2 \zeta_3 \right. \\
 & + \frac{3368}{81} \zeta_2 + \frac{2552}{81} \zeta_3 - 8 \zeta_5 - \frac{11540}{81} \Big) + C_F^3 \left( -\frac{184736}{315} \zeta_2^3 + \frac{152}{5} \zeta_2^2 - 64 \zeta_2 \zeta_3 \right. \\
 & - \frac{550}{3} \zeta_2 + \frac{10336}{3} \zeta_3^2 - 1188 \zeta_3 + 848 \zeta_5 + \frac{1078}{3} \Big) + C_F^2 n_f \left( \frac{12152}{135} \zeta_2^2 - \frac{5504}{9} \zeta_2 \zeta_3 \right. \\
 & - \frac{2600}{27} \zeta_2 + \frac{4088}{9} \zeta_3 + \frac{5536}{9} \zeta_5 - \frac{70}{9} \Big) + C_F n_f^2 \left( \frac{128}{27} \zeta_2^2 - \frac{32}{81} \zeta_2 - \frac{1120}{81} \zeta_3 + \frac{16}{27} \right) \Big\} \\
 & + \mathcal{D}_0 \Big\{ C_A^2 C_F \left( -\frac{2992}{15} \zeta_2^2 - \frac{352}{3} \zeta_2 \zeta_3 + \frac{98224}{81} \zeta_2 + \frac{40144}{27} \zeta_3 - 384 \zeta_5 - \frac{594058}{729} \right) \\
 & + C_A C_F^2 \left( \frac{1408}{3} \zeta_2^2 - 1472 \zeta_2 \zeta_3 + \frac{6592}{27} \zeta_2 + \frac{32288}{9} \zeta_3 + \frac{6464}{27} \right) + C_A C_F n_f \left( \frac{736}{15} \zeta_2^2 \right.
 \end{aligned}$$



**Figure 1.** Total cross section for the Higgs boson production in  $b\bar{b}$  annihilation at various orders in  $a_s$  as a function of  $\mu_R/m_H$  (left panel) and of  $\mu_F/m_H$  (right panel) at the LHC with  $\sqrt{s} = 14$  TeV.

$$\begin{aligned}
 & -\frac{29392}{81}\zeta_2 - \frac{2480}{9}\zeta_3 + \frac{125252}{729}) + C_F^3(-6144\zeta_2\zeta_3 - 1024\zeta_3 + 12288\zeta_5) \\
 & + C_F^2 n_f \left( -\frac{1472}{15}\zeta_2^2 - \frac{1504}{27}\zeta_2 - \frac{5728}{9}\zeta_3 + \frac{842}{9} \right) + C_F n_f^2 \left( \frac{640}{27}\zeta_2 + \frac{320}{27}\zeta_3 \right. \\
 & \left. - \frac{3712}{729} \right) \Big\} + \mathcal{D}_1 \left\{ C_A^2 C_F \left( \frac{704}{5}\zeta_2^2 - \frac{12032}{9}\zeta_2 - 704\zeta_3 + \frac{124024}{81} \right) \right. \\
 & + C_A C_F^2 \left( \frac{3648}{5}\zeta_2^2 - \frac{20864}{9}\zeta_2 - 5760\zeta_3 - \frac{544}{3} \right) + C_A C_F n_f \left( 384\zeta_2 - \frac{32816}{81} \right) \\
 & + C_F^3 \left( -\frac{14208}{5}\zeta_2^2 + 1024\zeta_2 - 960\zeta_3 + 256 \right) + C_F^2 n_f \left( \frac{3200}{9}\zeta_2 + 1280\zeta_3 - \frac{184}{3} \right) \\
 & \left. + C_F n_f^2 \left( \frac{1600}{81} - \frac{256}{9}\zeta_2 \right) \right\} + \mathcal{D}_2 \left\{ C_A^2 C_F \left( \frac{704}{3}\zeta_2 - \frac{28480}{27} \right) + C_A C_F^2 \left( \frac{11264}{3}\zeta_2 \right. \right. \\
 & \left. + 1344\zeta_3 - \frac{10816}{9} \right) + C_A C_F n_f \left( \frac{9248}{27} - \frac{128}{3}\zeta_2 \right) + C_F^3(10240\zeta_3) + C_F^2 n_f \left( \frac{1696}{9} \right. \\
 & \left. - \frac{2048}{3}\zeta_2 \right) + C_F n_f^2 \left( -\frac{640}{27} \right) \Big\} + \mathcal{D}_3 \left\{ C_A^2 C_F \left( \frac{7744}{27} \right) + C_A C_F^2 \left( \frac{17152}{9} - 512\zeta_2 \right) \right. \\
 & \left. + C_A C_F n_f \left( -\frac{2816}{27} \right) + C_F^3(-3072\zeta_2 - 512) + C_F^2 n_f \left( -\frac{2560}{9} \right) + C_F n_f^2 \left( \frac{256}{27} \right) \right\} \\
 & + \mathcal{D}_4 \left\{ C_F^2 n_f \left( \frac{1280}{9} \right) + C_A C_F^2 \left( -\frac{7040}{9} \right) \right\} + \mathcal{D}_5 \left\{ 512 C_F^3 \right\}. \tag{3.4}
 \end{aligned}$$

The numerical impact of our results can be studied using the exact LO, NLO, NNLO  $\Delta_{ac}^{b,(i)}$ ,  $i = 0, 1, 2$  and the threshold  $N^3\text{LO}$  result  $\Delta_{bb}^{sv,(3)}$ . We have used  $\sqrt{s} = 14$  TeV for the LHC, the  $Z$  boson mass  $M_Z = 91.1876$  GeV and the Higgs boson mass  $m_H = 125.5$  GeV

throughout. The strong coupling constant  $\alpha_s(\mu_R^2)$  is evolved using the 4-loop RG equations with  $\alpha_s^{\text{N}^3\text{LO}}(m_Z) = 0.117$  and for parton density sets we use MSTW 2008NNLO [73]. The Yukawa coupling is evolved using 4 loop RG with  $\lambda(m_b) = \sqrt{2}m_b(m_b)/\nu$  and  $m_b(m_b) = 4.3 \text{ GeV}$ .

The renormalization scale dependence is studied by varying  $\mu_R$  between  $0.1 m_H$  and  $10 m_H$  keeping  $\mu_F = m_H/4$  fixed. For the factorization scale, we have fixed  $\mu_R = m_H$  and varied  $\mu_F$  between  $0.1 m_H$  and  $10 m_H$ . We find that the perturbation theory behaves better if we include more and more higher order terms (see figure 1).

To summarize, we have systematically developed a framework to compute threshold contributions in QCD to the production of the Higgs boson in bottom anti-bottom annihilation subprocesses at the hadron colliders. Factorization of UV, soft and collinear singularities and exponentiation of their sum allow us to obtain threshold corrections order by order in perturbation theory. Using the recently obtained N<sup>3</sup>LO soft distribution function for Drell-Yan production and the three loop Higgs form factor with bottom anti-bottom quarks, we have obtained threshold N<sup>3</sup>LO corrections to the Higgs boson production through bottom anti-bottom annihilation. We have also studied the stability of our result under renormalization and factorization scales.

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