TOPOLOGICAL COMPARISON THEOREMS
FOR BREDON MOTIVIC COHOMOLOGY

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ABSTRACT. We prove equivariant versions of the Beilinson–Lichtenbaum conjecture for Bredon motivic cohomology of smooth complex and real varieties with an action of the group of order 2. This identifies equivariant motivic and topological invariants in a large range of degrees.

CONTENTS

1. INTRODUCTION

A major achievement of motivic homotopy theory is the proof of the Bloch–Kato conjecture relating Milnor K-theory to Galois cohomology [Voe03a], [Voe11], and consequently to the solution of the Beilinson–Lichtenbaum conjecture by the results in [SV00, Section 7]. With finite mod-n coefficients and X being a smooth scheme of finite type over a field k of characteristic coprime to n, one form of the Beilinson–Lichtenbaum conjecture asserts that the comparison map between motivic and étale cohomology

$$(1.1) \qquad H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}/n) \to H^p_{\mathrm{et}}(X, \mu_n^{\otimes q})$$

is an isomorphism when $p \leq q$ [SV00, Conjecture 6.8] and [VSF00, Chapter 1]. This "étale descent" property identifies a large range of motivic cohomology groups, aka higher Chow groups, with more computable étale cohomology groups. For

a smooth complex variety X, the étale cohomology groups in (1.1) agree with the singular cohomology groups of its corresponding analytic space $X(\mathbb{C})$. Thus the Beilinson–Lichtenbaum conjecture provides a powerful link between algebro-geometric and topological invariants. This can further be enhanced to prove the Quillen–Lichtenbaum conjectures comparing the algebraic and Hermitian K-theories of X with their analytic or étale counterparts; see, e.g., [Sus95, Theorem 4.7], [Voe03a, Theorem 7.10], [BKSØ15, Theorem 5.1].

A framework for invariants of smooth varieties equipped with a group scheme action has recently been organized in the subject of equivariant motivic homotopy theory [Del09], [HKO11], [HKØ15], [Her13], [Hoy15], [CJ14]. Of particular interest are actions by the group C_2 of order 2 governing the examples of Hermitian K-theory and motivic Real cobordism [HKO11]. Bredon motivic cohomology introduced in [HVØ15, Section 5] is an equivariant generalization of motivic cohomology for finite group actions. This theory is now amenable to a homotopical analysis on account of the equivariant cancellation theorem shown in [HVØ15, Theorem 9.7].

We employ this setup to prove an equivariant Beilinson–Lichtenbaum comparison theorem for smooth complex (and real) varieties. Before stating our main theorem, we review the players involved. If M is a topological space with C_2-action and A is an abelian group, the Bredon cohomology groups $H_{C_2}^{a+p\sigma}(M, \underline{A})$ are an equivariant analogue of singular cohomology groups. Here, a, p are integers and σ stands for the sign representation; i.e., topological Bredon cohomology is graded by virtual representations of C_2. Now if X is a smooth complex C_2-scheme of finite type, the Bredon motivic cohomology groups $H_{C_2}^{a+p\sigma, b+q\sigma}(X, \underline{A})$ have a grading by a 4-tuple of integers. This 4-tuple is a pair of virtual representations $V = a + p\sigma$ and $W = b + q\sigma$ which are respectively the "cohomological degree" and the "weight" of the grading.

In the Appendix we construct a natural comparison map between Bredon motivic cohomology and its equivariant topological counterpart,

$$(1.2) \qquad H_{C_2}^{a+p\sigma, b+q\sigma}(X, \underline{A}) \to H_{C_2}^{a+p\sigma}(X(\mathbb{C}), \underline{A}).$$

The map in (1.2) is induced by a Betti realization functor for the stable C_2-equivariant motivic homotopy category. In fact, the groups on the left-hand side of (1.2) are represented by a C_2-equivariant Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$, whose Betti realization agrees with the C_2-equivariant Eilenberg–MacLane spectrum for the constant Mackey functor $\underline{A}$, see Definition 3.2, Section A.6, and Theorem A.29.

A virtual C_2-representation $U = a + p\sigma$ has the dimension given by $\dim(U) = a + p$ and $\dim(U^{C_2}) = a$. The Beilinson–Lichtenbaum comparison theorem says that the comparison map (1.1) is an isomorphism (resp., injection) for cohomological degrees less than or equal to the weight (resp., weight plus 1). Our main result, which appears as Theorem 6.8, asserts that the condition of isomorphism depends on the dimensions of the representations and their fixed points. A precise statement is the following.

Theorem 1.3. *Let X be a smooth complex scheme of finite type with C_2-action. Write $V = a + p\sigma$ and $W = b + q\sigma$. Then the comparison map (1.2) is an isomorphism for $a + p \le b + q$ and $a \le \min\{b, b - q\}$ and an injection when $a + p \le b + q + 1$, and $a \le \min\{b, b - q\} + 1$ for any finite abelian group A.*

The condition $a \leq b - q$ cannot be improved, as we can see from Remark 5.6. From this theorem it also results that in the above range of degrees it follows that $H_{C_2}^{a+p\sigma,b+q\sigma}(X, \underline{A})$ is a finite abelian group.

We also establish an equivariant version of the Beilinson–Lichtenbaum comparison theorem in the case of smooth real C_2-schemes of finite type; see Theorem 7.10. In this case, if X is a finite-type C_2-scheme over $\mathbb{R}$, the space of complex points $X(\mathbb{C})$ has a $(C_2 \times C_2)$-action. One copy of C_2 acts algebraically through the action on X, and the other acts via complex conjugation. Thus Theorem 7.10 is a comparison result which relates C_2-equivariant Bredon motivic cohomology of X with $(C_2 \times C_2)$-equivariant topological Bredon cohomology of $X(\mathbb{C})$.

We expect that over other fields, there is an analogous comparison between Bredon motivic cohomology and an appropriate equivariant étale analogue of the topological Bredon cohomology. However, it does not appear that this has been developed yet. Doing so here would lead us too far astray, so we leave this for a future paper.

The nonequivariant Beilinson–Lichtenbaum comparison theorem is a fundamental ingredient in the proof of Theorem 6.8. We leverage this nonequivariant comparison result to an equivariant comparison by means of a motivic version of the isotropy separation cofiber sequence together with some computations in Borel motivic cohomology. We emphasize that these arguments rely both on the representability of Bredon motivic cohomology in the stable equivariant motivic homotopy category as well as that it can be computed as sheaf hypercohomology. That the representable theory coincides with sheaf hypercohomology is basically a recasting of the homotopy invariance and equivariant cancellation theorems, established in [HVØ15], and is proved in Theorem 3.4.

The equivariant Beilinson–Lichtenbaum comparison theorem is a first important step toward understanding the Bredon motivic cohomology ring $H_{C_2}^{\star,\star}(k, \underline{\mathbb{Z}/2})$ of a field k. As in (1.2), the gradings are sums of C_2-representations. These invariants are fundamental for understanding key features of C_2-equivariant motivic homotopy theory; e.g., $H_{C_2}^{\star,\star}(k, \underline{\mathbb{Z}/2})$ forms part of the largely unknown C_2-equivariant motivic Steenrod algebra of cohomology operations. Another fundamental aspect of Bredon motivic cohomology is that the zeroth slice of the C_2-equivariant motivic sphere spectrum turns out to be the highly structured C_2-equivariant Bredon motivic cohomology spectrum $\mathbf{M}\underline{\mathbb{Z}}$ introduced in Definition 3.2; see [HØ16].

This paper is structured as follows. Sections 2–7 are devoted to the proof of our main result for the comparison map (1.2). The proof rests on techniques from equivariant motivic homotopy theory. Roughly speaking, taking complex or real points induces a Betti realization functor which in turn gives rise to the comparison map for Bredon motivic cohomology. The details of these constructions are found in the Appendix. In Section 2 we develop the homotopical techniques that we need in the sequel; in particular, the motivic isotropy separation cofiber sequence plays a central role in our approach. The computational core of the paper lies in Sections 3–7. First, in Section 3 we establish that Bredon motivic cohomology is represented by $\mathbf{M}\underline{\mathbb{Z}}$ and show that it affords Thom classes for a certain class of equivariant vector bundles as well as Gysin sequences. In Section 4 we compare Bredon motivic cohomology with Edidin–Graham's equivariant higher Chow groups. In Section 5 we show that the generalized "geometric" Borel motivic cohomology ring $H_{C_2}^{\star,\star}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ is periodic with period $(2\sigma - 2, \sigma - 1)$. These preliminary results

are used to prove the complex and real comparison theorems for Bredon motivic cohomology in Sections 6 and 7, respectively.

Notation and conventions. Throughout k is a perfect field of characteristic $\mathrm{char}(k) \neq 2$. Unless said otherwise, a scheme is always assumed to be separated. For a finite group G, let $G\mathrm{Sch}/k$ denote the category of separated schemes of finite type over k with left G-actions and equivariant maps. Similarly, $G\mathrm{Sm}/k$ is the category of smooth schemes of finite type over k with left G-actions and equivariant maps. We use the term k-*variety* synonymous with separated, finite-type scheme over k.

We write $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ for the affine scheme associated to a vector space V over k, and $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee))$ for the associated projective scheme.

The construction of the stable G-equivariant motivic homotopy category $\mathrm{SH}_G(k)$ is recalled in the Appendix. We write $[-,-]_G$ for maps in $\mathrm{SH}_G(k)$. We distinguish four sphere objects in the C_2-equivariant motivic homotopy category; see Section 2.3. These are denoted S^1, S^σ, S_t^1, and S_t^σ. The sphere S^1 is the usual simplicial sphere, and S^σ the simplicial sign representation sphere. The algebro-geometric sphere S_t^1 is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with the trivial action, and S_t^σ is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with the inversion action, $x \mapsto x^{-1}$. We write $V = a + p\sigma$ for the C_2-representation which is the sum of a-copies of the trivial representation and p-copies of the sign representation, and we define

$$(1.4) \qquad S^{a+p\sigma, b+q\sigma} := S^{a-b} \wedge S^{(p-q)\sigma} \wedge S_t^b \wedge S_t^{q\sigma}.$$

We adopt the convention that $*$ refers to an integer grading of homotopy or cohomology groups, while $\star$ refers to grading by representations.

2. BACKGROUND

Equivariant motivic homotopy theory was introduced by Voevodsky [Del09] as a tool for understanding symmetric products and motivic Eilenberg–MacLane spaces. Stable equivariant motivic homotopy category was introduced by Hu, Kriz, and Ormsby [HKO11] as part of their study of the homotopy limit problem for Hermitian K-theory of fields. In this section we recall definitions and basic results about equivariant motivic homotopy theory. Technical details and a fairly complete, self-contained discussion are given in the Appendix.

2.1. **Equivariant Nisnevich topology.** The equivariant Nisnevich topology was introduced by Voevodsky [Del09, Section 3]. See [HVØ15, Section 3] and [HKØ15, Section 2] for more details concerning the equivariant Nisnevich topology.

Definition 2.1. An *equivariant distinguished square*

$$(2.2) \qquad \begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \overset{i}{\hookrightarrow} & X \end{array}$$

is a cartesian square in $G\mathrm{Sch}/k$ such that p is étale, $i : U \subseteq X$ is an open embedding, and p induces an isomorphism of reduced schemes $(Y \setminus W)_{\mathrm{red}} \cong (X \setminus U)_{\mathrm{red}}$. An elementary Nisnevich cover is the cover $\{U \to X, Y \to X\}$ associated to an equivariant distinguished square. The *equivariant Nisnevich topology* on $G\mathrm{Sm}/k$ is the smallest Grothendieck topology containing the elementary Nisnevich covers.

Recall that the set-theoretic stabilizer S_x of a point $x \in X$ is the subgroup $S_x := \{g \in G \mid gx = x\}$. By [HVØ15, Proposition 3.5] an equivariant étale map $f : Y \to X$ is an equivariant Nisnevich cover if and only if for every $x \in X$ there is $y \in Y$ such that f induces an isomorphism $k(y) \cong k(x)$ on residue fields, as well as an isomorphism $S_x \cong S_y$ on set-theoretic stabilizers.

Remark 2.3. Using this characterization of Nisnevich covers, we see that every smooth G-scheme is locally affine. Every point $x \in X$ has an S_x-invariant affine neighborhood (take any affine neighborhood and consider the intersection of the translates by elements of S_x). Let U_x be such a neighborhood. Then $G \times^{S_x} U_x \to X$, where for a subgroup $H \subseteq G$ and H-scheme Z we write $G \times^H Z$ for $(G \times Z)/H$, is an equivariant Nisnevich neighborhood of $G \cdot x$ in the sense of [HKØ15, Section 2]. The collection $\{G \times^{S_x} U_x \to X\}$ is an "infinite" equivariant Nisnevich cover of X by smooth affine G-schemes admitting a finite subcover by [HKØ15, Remark 2.18].

2.2. **Motivic G-spaces and spectra.** A *motivic G-space* over k is a presheaf of simplicial sets on GSm/k. We write $GSpc(k)$ and $GSpc_{\bullet}(k)$, respectively, for the categories of motivic G-spaces and pointed motivic G-spaces over k. The unstable equivariant motivic homotopy category is constructed following a pattern familiar from ordinary motivic homotopy theory; technical details are found in the Appendix. In brief, it is the homotopy category of a model structure which is constructed so that the following two relations hold.

(i) Any equivariant distinguished square (2.2) is a homotopy cocartesian square.

(ii) The projection $X \times \mathbb{A}^1 \to X$ is an equivariant motivic weak equivalence for any X in GSm/k.

These relations have nonobvious consequences. For example, by the Whitehead theorem a map inducing isomorphisms on equivariant Nisnevich sheaves of homotopy groups is an equivariant motivic weak equivalence, see [MV99, Section 3.2, Proposition 2.14]. Moreover, every G-equivariant vector bundle is an equivariant motivic weak equivalence [HKØ15, Proposition 4.10].

Let V be a representation of G, e.g., the regular representation $\rho_G = k[G]$. The associated *motivic representation sphere* is defined to be the pointed motivic G-space
$$T^V := \mathbb{P}(V \oplus 1)/\mathbb{P}(V).$$
For an integer $n \geq 0$ we use the smash product in $GSpc_{\bullet}(k)$ to define
$$T^{nV} := (T^V)^{\wedge n}.$$

Consider the equivariant distinguished square

$$
\begin{array}{ccc}
\mathbb{A}(V) \setminus \{0\} & \longrightarrow & \mathbb{A}(V) \\
\downarrow & & \downarrow \\
\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1) & \longrightarrow & \mathbb{P}(V \oplus 1).
\end{array}
$$

It is in particular a homotopy cocartesian square. The inclusion of $\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1)$ into $\mathbb{P}(V \oplus 1)$ is equivariantly $\mathbb{A}^1$-homotopic to the inclusion $\mathbb{P}(V) \subseteq \mathbb{P}(V \oplus 1)$; the requisite deformation is given by $([x_0 : \cdots : x_{n+1}], t) \mapsto [x_0 : \cdots : x_n : tx_{n+1}]$. We conclude that there is an equivariant motivic weak equivalence
$$T^V \simeq \mathbb{A}(V)/\mathbb{A}(V) \setminus \{0\}.$$

Also note that $T^{nV} \simeq T^{V^{\oplus n}}$ and more generally $T^V \wedge T^W \simeq T^{V \oplus W}$.

The stable equivariant motivic homotopy category $\mathrm{SH}_G(k)$ is the stabilization of $G\mathrm{Spc}_\bullet(k)$ with respect to the sphere T^{ρ_G}. We use symmetric T^{ρ_G}-spectra as a model for $\mathrm{SH}_G(k)$; see Section A.4. It is a tensor triangulated category with unit the sphere spectrum $\mathbf{1} = \Sigma^\infty_{T^{\rho_G}} S^0$. Here, $S^0 = \mathrm{Spec}(k)_+$ is the unit for the smash product in $G\mathrm{Spc}_\bullet(k)$. If X is an unbased motivic G-space, e.g., a smooth G-scheme, we have an associated based motivic G-space X_+, by adding a disjoint basepoint, and an associated suspension spectrum $\Sigma^\infty_{T^{\rho_G}} X_+$. When no confusion should arise, we sometimes simply write X for $\Sigma^\infty_{T^{\rho_G}} X_+$ and S^0 for the sphere spectrum.

2.3. C_2-equivariant spheres.
When $G = C_2$, there are only two representations, the trivial representation and the sign representation which we write as σ. It is convenient to introduce the following sphere objects. The *sign Tate sphere* S^1_t is the pointed C_2-scheme $(\mathbb{G}_m, 1)$ where $\mathbb{G}_m$ is equipped with the action $x \mapsto x^{-1}$. The *simplicial sign representation sphere* S^σ is defined to be the unreduced suspension of C_2; i.e., it is the homotopy cofiber of $C_{2+} \to S^0$. We have as well the usual simplicial sphere S^1 and the Tate sphere S^1_t which is the pointed scheme $(\mathbb{G}_m, 1)$ considered with trivial action. As observed in [HKO11, Section 4.1], there is an equivariant motivic weak equivalence $S^\sigma \wedge S^\sigma_t \simeq T^\sigma$.

The indexing convention

$$S^{a+p\sigma, b+q\sigma} := S^{a-b} \wedge S^{(p-q)\sigma} \wedge S^b_t \wedge S^{q\sigma}_t$$

in (1.4) is a mixture between the convention standardly used in motivic homotopy theory and that used in classical equivariant homotopy theory. The translation between the convention of indexing here and the one in [HKO11, Section 4.1] is given by

$$S^{a+p\sigma, b+q\sigma} = S^{(a-b)+(p-q)\gamma+b\alpha+q\gamma\alpha}.$$

The convention we use in this paper has the feature that the effect of the complex Betti realization functor (constructed in Section A.5) is the first entry of the index,

$$\mathrm{Re}_\mathbb{C}(S^{a+p\sigma, b+q\sigma}) = S^{a+p\sigma}.$$

A real Betti realization (taking value in $\mathrm{SH}_{C_2 \times \Sigma_2}$) is constructed in Section A.7. In this case,

$$\mathrm{Re}_{\mathbb{C}, \Sigma_2}(S^{a+p\sigma, b+q\sigma}) = S^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon},$$

where σ is the sign representation corresponding to the factor C_2 and ϵ is the sign representation corresponding to Σ_2.

The following two fundamental homotopy cofiber sequences of pointed motivic C_2-spaces are useful for computations:

(2.4) $$C_{2+} \to S^0 \to S^{\sigma, 0}$$

and

(2.5) $$(\mathbb{A}(n\sigma) \setminus \{0\})_+ \to S^0 \to S^{2n\sigma, n\sigma}.$$

Here, the first maps in these sequences are induced, respectively, by the projections $C_2 \to \mathrm{Spec}(k)$ and $\mathbb{A}(n\sigma) \setminus \{0\} \to \mathrm{Spec}(k)$.

2.4. **Motivic isotropy separation.** The isotropy separation cofiber sequence is a fundamental tool for analyzing equivariant homotopy types in classical equivariant homotopy theory. Our proof of the main theorems 6.8 and 7.10 makes use of an appropriate motivic version. We shall restrict our attention to the group C_2. See [GH16] for a general discussion of motivic isotropy separation.

Recall that the classical topological isotropy separation cofiber sequence is

$$\mathrm{E}_\bullet C_{2+} \to S^0 \to \widetilde{\mathrm{E}}_\bullet C_2,$$

where the last term is defined by this sequence. A check of fixed points shows that a model for $\widetilde{\mathrm{E}}_\bullet C_2$ is given by

$$(2.6) \qquad \widetilde{\mathrm{E}}_\bullet C_2 \simeq \operatorname*{colim}_n S^{n\sigma}.$$

The *geometric classifying space* $\mathbf{B}C_2$ for C_2 is defined as the quotient of

$$\mathbf{E}C_2 := \operatorname*{colim}_n \mathbb{A}(n\sigma) \setminus \{0\}$$

by the free C_2-action. This space plays an important role in nonequivariant motivic homotopy theory because it is a geometric model for the étale classifying space [MV99, Section 4.2]. A similar interpretation of the geometric classifying space is also true in equivariant motivic homotopy theory as well; see [GH16].

The *motivic isotropy separation cofiber sequence* is the cofiber sequence

$$(2.7) \qquad \mathbf{E}C_{2+} \to S^0 \to \widetilde{\mathbf{E}}C_2,$$

where the space $\widetilde{\mathbf{E}}C_2$ is defined by this cofiber sequence. Because of the definition of the geometric classifying space, we have an equivariant motivic equivalence

$$(2.8) \qquad \widetilde{\mathbf{E}}C_2 \simeq \operatorname*{colim}_n S^{2n\sigma, n\sigma}.$$

Proposition 2.9. *The maps $S^0 \to S^{2\sigma,\sigma}$ and $S^0 \to S^{\sigma,0}$ induce equivariant motivic equivalences $\widetilde{\mathbf{E}}C_2 \simeq S^{2\sigma,\sigma} \wedge \widetilde{\mathbf{E}}C_2$ and $\widetilde{\mathbf{E}}C_2 \simeq S^{\sigma,0} \wedge \widetilde{\mathbf{E}}C_2$.*

Proof. The first equivalence follows from (2.8) together with the fact that the cyclic permutation on $T^{3\sigma}$ is the identity by Lemma 2.10. Using Lemma A.9 and that $\mathbf{E}C_{2,+}$ is nonequivariantly contractible [MV99, Proposition 4.2.3], we have equivariant motivic equivalences

$$\mathbf{E}C_{2+} \wedge C_{2+} \simeq (\mathbf{E}C_{2+})^e \wedge C_{2+} \simeq C_{2+}.$$

It follows that $(C_2)_+ \wedge \widetilde{\mathbf{E}}C_2 \simeq *$. Thus the second equivalence follows from the cofiber sequence (2.4). $\qquad\square$

Lemma 2.10. *Let W be a representation. Let $\gamma \in \Sigma_k$ be an even permutation. Then the induced map on T^{kW} is the identity map in $\mathrm{H}_{G,\bullet}(k)$.*

Proof. Since $W^{\oplus k} = 1^{\oplus k} \otimes W$, any automorphism of $1^{\oplus k}$ induces an equivariant automorphism of $W^{\oplus k}$, and thus a pairing $\mathrm{GL}_k \times \mathbb{A}(W^{\oplus k}) \to \mathbb{A}(W^{\oplus k})$ in $G\mathrm{Sm}_k$ (where GL_k has trivial action). This induces in turn a pairing $(\mathrm{GL}_k)_+ \wedge T^{kW} \to T^{kW}$ in $G\mathrm{Spc}_\bullet(S)$. An even permutation γ is the product of row multiplication and addition of elementary matrices. We therefore have a map $\mathbb{A}^1 \to \mathrm{GL}_k$ connecting the identity and γ. This implies that the map on T^{kW} induced by an even permutation is equivariantly $\mathbb{A}^1$-homotopic to the identity. $\qquad\square$

3. Bredon motivic cohomology

In [HVØ15, Section 5] we introduced a Bredon motivic cohomology theory on $G\mathrm{Sm}/k$. Here, we define an equivariant motivic spectrum which is a representing object for the Bredon motivic cohomology groups of [HVØ15], and we record some of its basic properties. To keep the exposition streamlined, we restrict our discussion to the group C_2.

3.1. **Motivic complexes.** For X and Y smooth k-schemes of finite type we write $\mathrm{Cor}_k(X, Y)$ for the group of finite correspondences [SV00, Section 3.1]. If X and Y have an action by C_2, then C_2 acts on $\mathrm{Cor}_k(X, Y)$ as well. The category of equivariant correspondences $C_2\,\mathrm{Cor}_k$ has the same objects as $C_2\mathrm{Sm}/k$ and maps $\mathrm{Cor}_k(X, Y)^{C_2}$. A *presheaf with equivariant transfers* on $C_2\mathrm{Sm}/k$ is an additive presheaf on $C_2\,\mathrm{Cor}_k$.

If Y is a smooth C_2-scheme over k, write $\mathbb{Z}_{\mathrm{tr},C_2}(Y)$ for the free presheaf with equivariant transfers,

$$\mathbb{Z}_{\mathrm{tr},C_2}(Y)(X) := \mathrm{Cor}_k(X, Y)^{C_2}.$$

More generally, if A is an abelian group, we write $A_{\mathrm{tr},C_2}(Y) = \mathbb{Z}_{\mathrm{tr},C_2}(Y) \otimes A$. It is useful to extend the definition of $A_{\mathrm{tr},C_2}(-)$ to quotients of G-schemes. If $\mathcal{X} = \mathrm{colim}_i\, X_i$, where X_i are smooth C_2-schemes over k and the colimit is in the category of presheaves, then we define $A_{\mathrm{tr},C_2}(\mathcal{X}) := \mathrm{colim}_i\, A_{\mathrm{tr},C_2}(X_i)$, where the colimit is computed in the category of presheaves of abelian groups.

Defining $\mathbb{Z}_{\mathrm{tr},C_2}(X) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr},C_2}(Y) := \mathbb{Z}_{\mathrm{tr},C_2}(X \times Y)$ determines a symmetric monoidal product $\otimes^{\mathrm{tr}}$ on the category of presheaves with equivariant transfers.

If W is a finite C_2-set, viewed as a smooth C_2-scheme over k, we have isomorphisms as presheaves on $C_2\mathrm{Sm}/k$, $\mathbb{Z}_{\mathrm{tr},C_2}(W) \cong \mathbb{Z}(W) := \mathbb{Z}\,\mathrm{Hom}_k(-, W)^{C_2}$. Moreover, if F is a presheaf with equivariant transfers, then we also have an isomorphism $F \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr},C_2}(W) \cong F \otimes \mathbb{Z}(W)$ of presheaves on $C_2\mathrm{Sm}/k$.

If F is a presheaf of abelian groups, we write $C_*F(X)$ for the simplicial abelian group $F(X \times \Delta_k^\bullet)$. We use the same notation for the associated cochain complex. Here, $\Delta_k^\bullet$ is the standard cosimplicial object in Sm/k.

Write $D^-(C_2\,\mathrm{Cor}_k)$ for the derived category of bounded above chain complexes of equivariant Nisnevich sheaves with equivariant transfers on $C_2\mathrm{Sm}/k$. According to [HVØ15, Lemma 5.12], the cone $\mathbb{Z}_{\mathrm{top}}(\sigma)$ of the map $\mathbb{Z}_{\mathrm{tr},C_2}(C_2) \to \mathbb{Z}$ is invertible in $(D^-(C_2\,\mathrm{Cor}_k), \otimes^{\mathrm{tr}})$. If $F_\bullet$ is a cochain complex of presheaves with equivariant transfers, write $F_\bullet[\sigma] = F_\bullet \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{top}}(\sigma)$.

Let $V = a + b\sigma$ be a representation of C_2, and let A be an abelian group. Define the *motivic Bredon complex*

$$\underline{A}(V) := a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(T^V))[-2a - 2b\sigma],$$

where a_{Nis} denotes sheafification in the equivariant Nisnevich topology; see Definition 2.1. There are quasi-isomorphisms $a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(T)) \simeq a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(S_t^1))[1]$ and $a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(T^\sigma)) \simeq a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(S_t^\sigma))[\sigma]$; see [HVØ15, p. 328]. It follows that there is a quasi-isomorphism

$$(3.1) \qquad a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(T^V)) \simeq a_{\mathrm{Nis}}(C_*A_{\mathrm{tr},C_2}(S_t^a \wedge S_t^{b\sigma}))[a + b\sigma].$$

Now let A be a commutative ring. We construct a product pairing

$$\underline{A}(V) \otimes \underline{A}(W) \to \underline{A}(V \oplus W).$$

First, we have an associative pairing $A_{\mathrm{tr},C_2}(T^V) \otimes A_{\mathrm{tr},C_2}(T^W) \to A_{\mathrm{tr},C_2}(T^V \wedge T^W)$ of presheaves which is induced by the pairing

$$A_{\mathrm{tr},C_2}(\mathbb{P}(V \oplus 1))(U) \otimes A_{\mathrm{tr},C_2}(\mathbb{P}(W \oplus 1))(U)$$

$$\xrightarrow{\times} A_{\mathrm{tr},C_2}(\mathbb{P}((V \oplus 1) \times \mathbb{P}(W \oplus 1))(U \times U)$$

$$\xrightarrow{\Delta^*} A_{\mathrm{tr},C_2}(\mathbb{P}(V \oplus 1) \times \mathbb{P}(W \oplus 1))(U).$$

Let $A_{*,*}$ be a bisimplicial abelian group. We write $A_{*,*}$ as well for the associated cochain complex. By the Eilenberg–Zilber theorem [GJ99, Theorem IV.2.4] taking totalizations and diagonals yields a natural quasi-isomorphism of chain complexes $\mathrm{Tot}(A_{*,*}) \to \mathrm{diag}(A_{*,*})$. We thus obtain the natural pairing

$$\mathrm{Tot}\left(A_{\mathrm{tr},C_2}(T^V)(U \times \Delta_k^\bullet) \otimes A_{\mathrm{tr},C_2}(T^W)(U \times \Delta_k^\bullet)\right)$$

$$\to \mathrm{diag}\left(A_{\mathrm{tr},C_2}(T^V)(U \times \Delta_k^\bullet) \otimes A_{\mathrm{tr},C_2}(T^W)(U \times \Delta_k^\bullet)\right)$$

$$\to A_{\mathrm{tr},C_2}(T^V \wedge T^W)(U \times \Delta_k^\bullet) \xleftarrow{\cong} A_{\mathrm{tr},C_2}(T^{V \oplus W})(U \times \Delta_k^\bullet).$$

This induces our desired pairing upon sheafification. Now the quasi-isomorphism obtained from the Eilenberg–Zilber theorem is homotopy associative, so the pairing is also homotopy associative.

3.2. **Stable representability.** Let A be an abelian group. Let $\mathcal{F}$ be a presheaf of sets. we may consider $A_{\mathrm{tr},C_2}(\mathcal{F})$ as a presheaf of sets, and therefore as a based motivic C_2-space, the basepoint is 0. We have a canonical map $\gamma : \mathcal{F} \to A_{\mathrm{tr},C_2}(\mathcal{F})$ of motivic C_2-spaces, and there is a pairing $\mu : A_{\mathrm{tr},C_2}(X) \wedge \mathbb{Z}_{\mathrm{tr},C_2}(Y) \to A_{\mathrm{tr},C_2}(X \times Y)$ of motivic C_2-spaces.

Definition 3.2. Let A be an abelian group. The *motivic Bredon cohomology spectrum* $\mathbf{M}\underline{A}$ is defined by letting $\mathbf{M}\underline{A}_n := A_{\mathrm{tr},C_2}(T^{n\rho_{C_2}})$ with structure maps

$$A_{\mathrm{tr},C_2}(T^{n\rho_{C_2}}) \wedge T^{\rho_{C_2}} \xrightarrow{\mathrm{id} \wedge \gamma} A_{\mathrm{tr},C_2}(T^{n\rho_{C_2}}) \wedge \mathbb{Z}_{\mathrm{tr},C_2}(T^{\rho_{C_2}})$$

$$\xrightarrow{\mu} A_{\mathrm{tr},C_2}(T^{(n+1)\rho_{C_2}}).$$

The symmetric group Σ_n acts on $\mathbf{M}\underline{A}_n$ by permuting the factors of $T^{\rho_{C_2}}$. The iterated structure maps $\mathbf{M}\underline{A}_n \wedge T^{k\rho_{C_2}} \to \mathbf{M}\underline{A}_{n+k}$ are $(\Sigma_n \times \Sigma_k)$-equivariant. This means that $\mathbf{M}\underline{A}$ is a symmetric motivic C_2-spectrum; see Definition A.10. Moreover, if A is a commutative ring, there are pairings $\mathbf{M}\underline{A}_n \wedge \mathbf{M}\underline{A}_k \to \mathbf{M}\underline{A}_{n+k}$ which give $\mathbf{M}\underline{A}$ the structure of a commutative ring spectrum (i.e., a commutative monoid in the category of equivariant symmetric motivic spectra $\mathrm{Spt}_{C_2}^\Sigma(k)$).

Finally, we note that the spectrum $\mathbf{M}\underline{A}$ is stably equivalent to $\mathbf{M}\mathbb{Z} \wedge \mathbb{S}A$, where $\mathbb{S}A$ is a Moore spectrum associated to A. This can be seen by noting that the assignment $A \mapsto \mathbf{M}\underline{A}$ defines a functor $\mathrm{Ab} \to \mathrm{Spt}_{C_2}^\Sigma(k)$ which has the following properties.

   (i) If $A = \mathrm{colim}_i A_i$ is a filtered colimit of abelian groups, then $\mathrm{colim}_i \mathbf{M}\underline{A}_i \simeq \mathbf{M}\underline{A}$.
   (ii) If $A = \oplus_j A_j$, then $\vee_j \mathbf{M}\underline{A}_j \simeq \mathbf{M}\underline{A}$.
   (iii) If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of abelian groups, then $\mathbf{M}\underline{A}_1 \to \mathbf{M}\underline{A}_2 \to \mathbf{M}\underline{A}_3$ is a homotopy cofiber sequence of spectra.

Moore spectra satisfy similar properties, so it suffices to check the statement when $A = \mathbb{Z}$, where it is trivial. Of course, an important advantage of the model $\mathbf{M}\underline{A}$ is

that it is a commutative ring spectrum when A is a commutative ring, while this is not necessarily a priori clear for $\mathbf{M}\mathbb{Z} \wedge \mathbb{S}A$.

Definition 3.3. The *Bredon motivic cohomology* of a motivic C_2-spectrum E with coefficients in the abelian group A is defined by

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\mathsf{E}, \underline{A}) := [\mathsf{E}, S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)}.$$

If X is a smooth C_2-scheme, then its unreduced Bredon motivic cohomology is defined via its suspension spectrum by setting

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) := \widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\Sigma^\infty_{T^{\rho C_2}} X_+, \underline{A}).$$

Next we verify that the definition of Bredon motivic cohomology which we have just given agrees with the one given in [HVØ15]. (Note, however, that the indexing we use in the present paper is slightly different than in [HVØ15]). The fact that Bredon motivic cohomology defined in the stable equivariant motivic homotopy category is equal to the hypercohomology groups of the motivic complexes plays a crucial role in our arguments in the later sections. This fact relies on the homotopy invariance and equivariant cancellation theorems for presheaves with equivariant transfers proved in [HVØ15, Theorems 8.12 and 9.7].

If $\mathcal{X} = \mathrm{colim}\, X_i$ is a colimit (in presheaves) of smooth C_2-schemes over k, write $\mathbb{Z}(\mathcal{X}) := \mathrm{colim}_i \mathbb{Z}(X_i)$ and

$$H^a_{C_2\mathrm{Nis}}(\mathcal{X}, \underline{A}(V)[p\sigma]) := \mathrm{Ext}^a_{C_2\mathrm{Nis}}(\mathbb{Z}(\mathcal{X}), \underline{A}(V)[p\sigma]).$$

Theorem 3.4. *Let $V = b + q\sigma$ be a virtual representation of C_2, let $W = c + d\alpha$ be a representation such that $V \oplus W$ is a representation, let X be a smooth C_2-scheme of finite type over k, and let A be an abelian group. Then there is a natural isomorphism*

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) \cong H^a_{C_2\mathrm{Nis}}(T^W \wedge X_+, \underline{A}(W \oplus V)[2c + (2d + p)\sigma]).$$

Proof. We assume that $V = b + q\sigma$ is an actual representation; the more general case of a virtual representation is similar. Let $\mathbf{M}\underline{A} \to \mathbf{M}\underline{A}'$ be a levelwise motivic fibrant replacement, i.e., for each n, $\mathbf{M}\underline{A}'_n$ is motivic fibrant and $\mathbf{M}\underline{A}_n \to \mathbf{M}\underline{A}'_n$ is a motivic weak equivalence (see Section A.4 for details on the motivic model structure). We claim that $\mathbf{M}\underline{A}'$ is already a fibrant motivic C_2-spectrum. Indeed, using Theorem A.6, the map

$$\pi_i(\mathbf{M}\underline{A}'_n)(X) \to \pi_i(\Omega_{T^{\rho C_2}}\mathbf{M}\underline{A}'_{n+1})(X)$$

is naturally identified with the map

$$H^{-i}_{C_2\mathrm{Nis}}(X, \underline{A}(n\rho_{C_2})) \to H^{-i}_{C_2\mathrm{Nis}}(T^{\rho C_2} \wedge X_+, \underline{A}((n+1)\rho_{C_2})).$$

This is the map of the equivariant cancellation theorem [HVØ15, Theorem 9.8], which is an isomorphism for all i. Therefore, $\mathbf{M}\underline{A}'_n \to \Omega_{T^{\rho C_2}}\mathbf{M}\underline{A}'_{n+1}$ is a weak equivalence of motivic C_2-spaces, which implies that $\mathbf{M}\underline{A}'$ is an $\Omega_{T^{\rho C_2}}$-spectrum and so is a fibrant motivic C_2-spectrum; see Section A.4.

Let i, j, k, l be nonnegative integers such that

$$S^{(i+k)+(j+l)\sigma, k+l\sigma} \wedge S^{a+p\sigma, b+q\sigma} \simeq T^{m\rho_{C_2}}$$

for some $m \geq 0$. In particular, $m = i + a - b = j + p - q = k + b = l + q$. We have $H_{C_2}^{a+p\sigma,b+q\sigma}(X, \underline{A}) = [X_+, S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)}$ and

$$
\begin{aligned}
[X_+, & S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^{(i+k)+(j+l)\sigma,k+l\sigma} \wedge X_+, T^{m\rho_{C_2}} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^i \wedge S^{j\sigma} \wedge S_t^k \wedge S_t^{l\sigma} \wedge X_+, T^{m\rho_{C_2}} \wedge \mathbf{M}\underline{A}']_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^i \wedge S^{j\sigma} \wedge S_t^k \wedge S_t^{l\sigma} \wedge X_+, (\mathbf{M}\underline{A}')_m]_{\mathrm{H}_{\bullet,C_2}(k)} \\
&\cong H_{C_2\mathrm{Nis}}^{-i}(S^{j\sigma} \wedge S_t^k \wedge S_t^{l\sigma} \wedge X_+, C_*A_{\mathrm{tr},C_2}(T^m)) \\
&\cong H_{C_2\mathrm{Nis}}^{-i}(S^{j\sigma} \wedge S_t^k \wedge S_t^{l\sigma} \wedge X_+, C_*A_{\mathrm{tr},C_2}(S_t^m \wedge S_t^{m\sigma})[m + m\sigma]) \\
&\cong H_{C_2\mathrm{Nis}}^{0}(X, C_*A_{\mathrm{tr},C_2}(S_t^{m-k} \wedge S_t^{(m-l)\sigma})[(m-i)+(m-j)\sigma]) \\
&\cong H_{C_2\mathrm{Nis}}^{a}(X, \underline{A}(V)[p\sigma]).
\end{aligned}
$$

The first two isomorphisms are immediate. The third follows from [Hov01, Theorem 8.10] and the standard adjunction relating $\mathrm{H}_{\bullet,C_2}(k)$ and $\mathrm{SH}_{C_2}(k)$, the fourth from Theorem A.6, and the remaining isomorphisms from (3.1) and equivariant cancellation [HVØ15, Theorem 9.8]. $\qquad\square$

Remark 3.5. Under the isomorphisms above, the product structures arising from the pairing of spectra $\mathbf{M}\underline{A} \wedge \mathbf{M}\underline{A} \to \mathbf{M}\underline{A}$ agrees with that arising from the pairing of complexes $\underline{A}(V) \otimes \underline{A}(W) \to \underline{A}(V \oplus W)$.

3.3. Basic properties.
We record some of the basic properties of Bredon motivic cohomology.

3.3.1. *Cancellation.* Let $V = s + t\sigma$ be a virtual representation. We defined Bredon motivic cohomology via a representing spectrum in $\mathrm{SH}_{C_2}(k)$. Immediate from this definition we have natural isomorphisms

$$
(3.6) \qquad \widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(T^V \wedge \mathsf{E}, \underline{A}) \cong \widetilde{H}_{C_2}^{a-2s+(p-2t)\sigma, b-s+(q-t)\sigma}(\mathsf{E}, \underline{A}).
$$

3.3.2. *Equivariant transfers.* Recall that we write $D^-(C_2 \mathrm{Cor}_k)$ for the derived category of bounded above chain complexes of equivariant Nisnevich sheaves with equivariant transfers on $C_2\mathrm{Sm}/k$.

Proposition 3.7. *Let X be a smooth C_2-scheme over k, and let $K_\bullet$ be a cochain complex of presheaves with equivariant transfers. Then there is a natural isomorphism $\mathrm{Ext}_{D^-(C_2 \mathrm{Cor}_k)}^{n}(\mathbb{Z}_{\mathrm{tr},C_2}(X), K_\bullet) \cong H_{C_2\mathrm{Nis}}^{n}(X, K_\bullet)$.*

Proof. The argument is as in [Voe00, Proposition 3.1.9], using that smooth C_2-schemes have finite equivariant Nisnevich cohomological dimension [HVØ15, Corollary 3.9] together with [HVØ15, Theorem 4.15]. $\qquad\square$

Corollary 3.8. *If $K_\bullet$ is a cochain complex of sheaves with equivariant transfers then $H_{C_2\mathrm{Nis}}^{n}(-, K_\bullet)$ is a presheaf with equivariant transfers.*

Theorem 3.4 shows that for any a, b, p, q, $H_{C_2}^{a+p\sigma,b+q\sigma}(X, \underline{A})$ is naturally identified with equivariant Nisnevich hypercohomology groups with coefficients in a (bounded above) cochain complex of sheaves with equivariant transfers. It is therefore a presheaf with equivariant transfers.

3.3.3. *Mayer–Vietoris sequences.* Since Bredon motivic cohomology is a representable theory, associated to an equivariant distinguished square (2.2) is a Mayer–Vietoris long exact sequence

$$\cdots \to H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A}) \to H_{C_2}^{a+p\sigma,b+q\sigma}(U,\underline{A}) \oplus H_{C_2}^{a+p\sigma,b+q\sigma}(Y,\underline{A})$$
$$\to H_{C_2}^{a+p\sigma,b+q\sigma}(W,\underline{A}) \to H_{C_2}^{a+1+p\sigma,b+q\sigma}(X,\underline{A}) \to \cdots.$$

3.3.4. *Ring structure.* When A is a commutative ring, $\mathbf{M}\underline{A}$ is a commutative ring spectrum. We thus have a cup product pairing for smooth C_2-varieties

$$H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A}) \times H_{C_2}^{c+s\sigma,d+t\sigma}(X,\underline{A}) \xrightarrow{\cup} H_{C_2}^{(a+c)+(p+s)\sigma,(b+d)+(q+t)\sigma}(X,\underline{A})$$

given by the usual formula

$$x \cup y := \Delta^*(x \boxtimes y),$$

where $\Delta : X \to X \times X$ is the diagonal and $\boxtimes$ is the external product. The cup product makes $H_{C_2}^{*,*}(X,\underline{A})$ into a $\mathbb{Z}^4$-graded ring. Since it is a representable theory, it is a module over $\pi_0^{C_2}(\mathbf{1}) := \mathrm{End}_{\mathrm{SH}_{C_2}(k)}(\mathbf{1})$ and in fact an algebra over this ring.

Consider the following endomorphisms of $\mathbf{1}$ in $\mathrm{SH}_{C_2}(k)$:

$$\epsilon = \Sigma_{S_t^1}^{-2}\left(S_t^1 \wedge S_t^1 \xrightarrow{\mathrm{twist}} S_t^1 \wedge S_t^1\right),$$

(3.9)
$$\epsilon' = \Sigma_{S_t^\sigma}^{-2}\left(S_t^\sigma \wedge S_t^\sigma \xrightarrow{\mathrm{twist}} S_t^\sigma \wedge S_t^\sigma\right),$$

$$u = \Sigma_{S^\sigma}^{-2}\left(S^\sigma \wedge S^\sigma \xrightarrow{\mathrm{twist}} S^\sigma \wedge S^\sigma\right).$$

We also write $\epsilon, \epsilon', u \in H_{C_2}^{0,0}(k,\underline{A})$ for the respective elements in cohomology, induced by these endomorphisms.

Proposition 3.10. *Let* $x \in H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A})$, *and let* $y \in H_{C_2}^{c+s\sigma,d+t\sigma}(X,\underline{A})$. *Then*

$$x \cup y = (-1)^{ac}(u)^{ps}(\epsilon)^{bd}(\epsilon')^{qt}(y \cup x).$$

Proof. This follows by the same argument as [Dug14, Proposition 6.13] (see also [Dug14, Remark 6.14]). The point is to carefully analyze the endomorphism of $S^{\alpha_1+\alpha_2,\beta_1+\beta_2}$ arising from the twist $S^{\alpha_1,\beta_1} \wedge S^{\alpha_2,\beta_2} \to S^{\alpha_2,\beta_2} \wedge S^{\alpha_1,\beta_1}$, where $(\alpha_1,\beta_1) = (a+p\sigma,b+q\sigma)$ and $(\alpha_2,\beta_2) = (c+s\sigma,d+t\sigma)$. □

Remark 3.11. We will see in Proposition 3.24 that $\epsilon = 1$, $\epsilon' = 1$, and $u = -1$ in $H_{C_2}^{\star,\star}(k,\underline{A})$, and therefore $x \cup y = (-1)^{ac+ps}(y \cup x)$.

3.3.5. *Topological realization.* Let $k = \mathbb{C}$. The functor $C_2\mathrm{Sm}/\mathbb{C} \to C_2\mathrm{Top}$, $X \mapsto X(\mathbb{C})$ extends to a functor $\mathrm{Re}_\mathbb{C} : \mathrm{SH}_{C_2}(\mathbb{C}) \to \mathrm{SH}_{C_2}$ from the stable equivariant motivic homotopy category over $\mathbb{C}$ to the classical stable equivariant homotopy category; see Section A.5. For a topological space M with C_2-action we write $H_{C_2}^{a+p\sigma}(M,\underline{A})$ for the topological Bredon cohomology theory, and $\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(\mathcal{X},\underline{A})$ for the reduced Bredon cohomology of a pointed motivic C_2-space $\mathcal{X}$. By Theorem A.29, the Betti realization $\mathrm{Re}_\mathbb{C}(\mathbf{M}\underline{A})$ represents topological Bredon cohomology with $\underline{A}$-coefficients. This yields the comparison functor

$$\mathrm{Re}_\mathbb{C} : H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A}) \to H_{C_2}^{a+p\sigma}(X(\mathbb{C}),\underline{A}).$$

Moreover, this yields a ring homomorphism since $\mathrm{Re}_\mathbb{C} : \mathrm{SH}_{C_2}(\mathbb{C}) \to \mathrm{SH}_{C_2}$ is a symmetric monoidal functor; see Theorem A.14.

3.3.6. *Change of groups.* Consider the adjunction $C_2 \times - : \mathrm{Sm}/k \rightleftarrows C_2\mathrm{Sm}/k : (-)^e$, where X^e is the underlying smooth scheme of a smooth scheme with C_2-action. This extends to an adjunction on pointed motivic spaces, and by Proposition A.11 it stabilizes to yield the adjunction

$$C_{2+} \wedge - : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_{C_2}(k) : (-)^e$$

on stable homotopy categories. The functor $(-)^e$ should be thought of as forgetting the action. Similarly, the adjunction $(-)^{\mathrm{triv}} : \mathrm{Sm}/k \rightleftarrows C_2\mathrm{Sm}/k : (-)^{C_2}$ induces an adjunction on stable homotopy categories

$$(-)^{\mathrm{triv}} : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_{C_2}(k) : (-)^{C_2}.$$

Here, $(-)^{\mathrm{triv}} : \mathrm{Sm}/k \to C_2\mathrm{Sm}/k$ sends a smooth scheme X to the C_2-scheme consisting of X with the trivial action. If E is a motivic spectrum, then $C_{2+} \wedge \mathsf{E}$ agrees with $C_{2+} \wedge (\mathsf{E})^{\mathrm{triv}}$ (the latter being the smash product of two equivariant motivic spectra). Typically we simply write E again for the spectrum $\mathsf{E}^{\mathrm{triv}}$.

Write $H_{\mathcal{M}}^{*,*}(-, A)$ for motivic cohomology theory and $\mathbf{M}A$ for the representing motivic spectrum. There is a natural map

$$(3.12) \qquad \phi : \widetilde{H}_{C_2}^{a+p\sigma, b+q\sigma}(\mathcal{X}, \underline{A}) \to \widetilde{H}_{\mathcal{M}}^{a+p, b+q}(\mathcal{X}^e, A)$$

obtained from $(-)^e : [\mathcal{X}, S^{a+b\sigma, b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \to [\mathcal{X}^e, (S^{a+b\sigma, b+q\sigma} \wedge \mathbf{M}\underline{A})^e]_{\mathrm{SH}(k)}$ and the facts that $(S^{a+b\sigma, b+q\sigma} \wedge \mathbf{M}\underline{A})^e = S^{a+b, b+q} \wedge (\mathbf{M}\underline{A})^e$ and $(\mathbf{M}\underline{A})^e$ is a T^2-spectrum representing motivic cohomology. When $k = \mathbb{C}$, by Proposition A.15 and Theorem A.29 we have a commutative square

$$(3.13) \qquad \begin{array}{ccc} \widetilde{H}_{C_2}^{a+p\sigma, b+q\sigma}(\mathcal{X}, \underline{A}) & \xrightarrow{\phi} & \widetilde{H}_{\mathcal{M}}^{a+p, b+q}(\mathcal{X}^e, A) \\ \downarrow & & \downarrow \\ \widetilde{H}_{C_2}^{a+p\sigma}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X}), \underline{A}) & \xrightarrow{\phi} & \widetilde{H}_{\mathrm{sing}}^{a+p}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e, A). \end{array}$$

Proposition 3.14. *Let $\mathcal{X}$ be a pointed motivic C_2-space. For integers a, b, p, q there is a natural isomorphism*

$$\widetilde{H}_{C_2}^{a+p\sigma, b+q\sigma}(C_{2+} \wedge \mathcal{X}, \underline{A}) \xrightarrow{\cong} \widetilde{H}_{\mathcal{M}}^{a+p, b+q}(\mathcal{X}^e, A).$$

Moreover, when $k = \mathbb{C}$, there is a commutative square

$$\begin{array}{ccc} \widetilde{H}_{C_2}^{a+p\sigma, b+q\sigma}(C_{2+} \wedge \mathcal{X}, \underline{A}) & \xrightarrow{\cong} & \widetilde{H}_{\mathcal{M}}^{a+p, b+q}(\mathcal{X}^e, A) \\ {\scriptstyle \mathrm{Re}_{\mathbb{C}}} \downarrow & & \downarrow {\scriptstyle \mathrm{Re}_{\mathbb{C}}} \\ \widetilde{H}_{C_2}^{a+p\sigma}(C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X}), \underline{A}) & \xrightarrow{\cong} & \widetilde{H}_{\mathrm{sing}}^{a+p}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e, A). \end{array}$$

Proof. By Lemma A.9 the adjoint of the inclusion $\mathcal{X}^e \to (C_{2+} \wedge \mathcal{X})^e = \mathcal{X}^e \coprod \mathcal{X}^e$, as the first summand is an isomorphism $i : C_{2+} \wedge \mathcal{X}^e \cong C_{2+} \wedge \mathcal{X}$ in the equivariant motivic homotopy category. Together with the adjunction isomorphism and the observation that $(\mathbf{M}\underline{A})^e$ represents motivic cohomology, we obtain isomorphisms

$$[C_{2+} \wedge \mathcal{X}, \Sigma^{a+p\sigma, b+q\sigma} \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \cong [C_{2+} \wedge (\mathcal{X})^e, \Sigma^{a+p\sigma, b+q\sigma} \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)}$$
$$\cong [\mathcal{X}^e, \Sigma^{a+p, b+q} \mathbf{M}A]_{\mathrm{SH}(k)},$$

yielding the first part of the proposition.

We have $\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e \cong \mathrm{Re}_{\mathbb{C}}(\mathcal{X}^e)$ and $C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X}) \cong \mathrm{Re}_{\mathbb{C}}(C_{2+} \wedge \mathcal{X})$ by Proposition A.13. The map $\mathrm{Re}_{\mathbb{C}}(i)$ is an isomorphism $C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e \cong C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X})$. Together with the isomorphism $\mathrm{Re}_{\mathbb{C}}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in Theorem A.29 this implies the second part of the proposition. $\qquad \square$

Proposition 3.15. *Let X be a smooth k-scheme considered as a C_2-scheme with trivial action. For integers m, n the map (3.12) is an isomorphism*

$$\phi : H_{C_2}^{m,n}(X, \underline{A}) \xrightarrow{\cong} H_{\mathcal{M}}^{m,n}(X, A).$$

Proof. The functor $(-)^e : C_2\mathrm{Sm}/k \to \mathrm{Sm}/k$ takes equivariant Nisnevich covers to Nisnevich covers and so induces a morphism of sites $(\mathrm{Sm}/k)_{\mathrm{Nis}} \to (C_2\mathrm{Sm}/k)_{C_2\mathrm{Nis}}$. This induces a map on cohomology $\phi : H_{C_2\mathrm{Nis}}^m(X, \underline{A}(n)) \to H_{\mathrm{Nis}}^m(X, A(n))$. Here, $A(n)$ is the Nisnevich sheafification of $C_*A_{\mathrm{tr}}(T^{\wedge n})[-2n]$; i.e., it is the "usual" weight-n motivic complex. Under the isomorphism of Theorem 3.4 this map ϕ is identified with the map ϕ in the statement of the proposition.

There is also a morphism of sites $t : (C_2\mathrm{Sm}/k)_{C_2\mathrm{Nis}} \to (\mathrm{Sm}/k)_{\mathrm{Nis}}$, induced by $(-)^{\mathrm{triv}} : \mathrm{Sm}/k \to C_2\mathrm{Sm}/k$. The functor t^* is exact, and $t^*A(n) = \underline{A}(n)$; see [HVØ15, Lemma 3.19]. We thus have an induced isomorphism $\psi : H_{\mathrm{Nis}}^m(X, A(n)) \cong H_{C_2\mathrm{Nis}}^m(X, \underline{A}(n))$; this is an inverse to ϕ. $\qquad \square$

Proposition 3.16. *Let X be a smooth quasi-projective C_2-scheme over k. Suppose that X has free action. For integers m, n there are isomorphisms*

$$H_{\mathcal{M}}^{m,n}(X/C_2, A) \xleftarrow[\phi]{\cong} H_{C_2}^{m,n}(X/C_2, \underline{A}) \xrightarrow[\pi^*]{\cong} H_{C_2}^{m,n}(X, \underline{A}),$$

where $\pi : X \to X/C_2$ is the quotient. Moreover, when $k = \mathbb{C}$, there are commutative squares

$$
\begin{array}{ccccc}
H_{\mathcal{M}}^{m,n}(X/C_2, A) & \xleftarrow{\cong}_{\phi} & H_{C_2}^{m,n}(X/C_2, \underline{A}) & \xrightarrow{\cong}_{\pi^*} & H_{C_2}^{m,n}(X, \underline{A}) \\
\downarrow{\mathrm{Re}_{\mathbb{C}}} & & \downarrow{\mathrm{Re}_{\mathbb{C}}} & & \downarrow{\mathrm{Re}_{\mathbb{C}}} \\
H_{\mathrm{sing}}^{m}(X(\mathbb{C})/C_2, A) & \xleftarrow{\cong}_{\phi} & H_{C_2}^{m}(X(\mathbb{C})/C_2, \underline{A}) & \xrightarrow{\cong}_{\pi^*} & H_{C_2}^{m}(X(\mathbb{C}), \underline{A}).
\end{array}
$$

Proof. The first arrow is an isomorphism by Proposition 3.15. By Theorem 3.4 the second map is identified with the map between equivariant Nisnevich hyper-cohomology groups $\pi^* : H_{C_2\mathrm{Nis}}^m(X/C_2, \underline{A}(n)) \to H_{C_2\mathrm{Nis}}^m(X, \underline{A}(n))$. Since X/C_2 has trivial action, we have the isomorphism

$$H_{C_2\mathrm{Nis}}^m(X/C_2, \underline{A}(n)) \cong H_{\mathrm{Nis}}^m(X/C_2, A(n)).$$

Under this identification the isomorphism

$$H_{\mathrm{Nis}}^m(X/C_2, A(n)) \cong H_{C_2\mathrm{Nis}}^m(X, \underline{A}(n))$$

of [HVØ15, Lemma 3.19] is $\pi^* : H_{C_2\mathrm{Nis}}^m(X/C_2, \underline{A}(n)) \to H_{C_2\mathrm{Nis}}^m(X, \underline{A}(n))$.

When $k = \mathbb{C}$, we have that $X(\mathbb{C})/C_2 = (X/C_2)(\mathbb{C})$. The commutativity of the first square is a specialization of (3.13). The commutativity of the second is immediate. $\qquad \square$

We embed $C_2 \subseteq \mathbb{A}(\sigma)$ via $C_2 = \{\pm 1\}$. By the equivariant homotopical purity theorem [HKØ15, Theorem 7.6], [Hoy15, Theorem 3.23] there is an equivariant motivic weak equivalence

$$C_{2+} \wedge T^\sigma \simeq \mathbb{P}(\sigma \oplus 1)/\mathbb{P}(\sigma \oplus 1) \setminus C_2.$$

Since $\mathbb{P}(\sigma \oplus 1) \setminus \mathbb{A}(\sigma) \subseteq \mathbb{P}(\sigma \oplus 1) \setminus C_2$ and $T^\sigma \simeq \mathbb{P}(\sigma \oplus 1)/\mathbb{P}(\sigma \oplus 1) \setminus \mathbb{A}(\sigma)$, we have a map $\tau' : T^\sigma \to C_{2+} \wedge T^\sigma$, and hence a stable map $\tau : S^0 \to C_{2+}$.

Remark 3.17. Presumably topological Bredon cohomology is a presheaf with equivariant transfers (in the sense of [HVØ15, Section 4]), but establishing this would require a lengthy digression. For this reason we use the map τ in the following proposition rather than transfer maps coming from the theory of presheaves with equivariant transfers.

Proposition 3.18. *Let* E *be a motivic* C_2*-spectrum over* $\mathbb{C}$. *Write* $\pi : C_{2+} \wedge \mathsf{E} \to \mathsf{E}$ *for the projection. Then the diagram below commutes and* $\tau^*\pi^* = 2$,

$$\begin{array}{ccccc}
\widetilde{H}_{C_2}^{\star,\star}(\mathsf{E}, \underline{A}) & \xrightarrow{\pi^*} & \widetilde{H}_{C_2}^{\star,\star}(C_{2+} \wedge \mathsf{E}, \underline{A}) & \xrightarrow{\tau^*} & \widetilde{H}_{C_2}^{\star,\star}(\mathsf{E}, \underline{A}) \\
\downarrow{\scriptstyle \mathrm{Re}_\mathbb{C}} & & \downarrow{\scriptstyle \mathrm{Re}_\mathbb{C}} & & \downarrow{\scriptstyle \mathrm{Re}_\mathbb{C}} \\
\widetilde{H}_{C_2}^{\star}(\mathrm{Re}_\mathbb{C}(\mathsf{E}), \underline{A}) & \xrightarrow{\pi^*} & \widetilde{H}_{C_2}^{\star}(C_{2+} \wedge \mathrm{Re}_\mathbb{C}(\mathsf{E}), \underline{A}) & \xrightarrow{\tau^*} & \widetilde{H}_{C_2}^{\star}(\mathrm{Re}_\mathbb{C}(\mathsf{E}), \underline{A}).
\end{array}$$

Proof. The commutativity of the diagram is immediate. It suffices to treat the case $\mathsf{E} = S^0$ and to see that $\tau^*(1) = 2$. The topological realization of τ is the Spanier–Whitehead dual of the projection $C_{2+} \to S^0$. In particular, $\tau^*(1) = 2$ in $H_{C_2}^0(\mathrm{pt.}, \underline{A})$. It remains to show that $\tau^*(1) = 2 \in H_{C_2}^{0,0}(k, \underline{A}) = A$. This follows from the commutative diagram

$$\begin{array}{ccc}
A = \widetilde{H}_{C_2}^{0,0}(C_{2+}, \underline{A}) & \xrightarrow{\tau^*} & \widetilde{H}_{C_2}^{0,0}(S^0, \underline{A}) = A \\
{\scriptstyle \cong}\downarrow{\scriptstyle \phi} & & {\scriptstyle \phi}\downarrow{\scriptstyle \cong} \\
A = \widetilde{H}_{\mathcal{M}}^{0,0}(S^0 \vee S^0, \underline{A}) & \xrightarrow{\tau^*} & \widetilde{H}_{\mathcal{M}}^{0,0}(S^0, \underline{A}) = A
\end{array}$$

and that the bottom arrow sends 1 to 2. $\square$

3.4. **Thom isomorphisms.** Let R be a C_2-equivariant motivic commutative ring spectrum, let X be a smooth C_2-scheme over k, and let $E \to X$ be a C_2-equivariant vector bundle.

Definition 3.19. An R-*Thom class* (or simply *Thom class*, where R is understood) for E is a class $u \in \mathsf{R}_{C_2}^{\star,\star}(\mathrm{Th}(E))$ with the property that for any equivariant map $f : Y \to X$ of smooth C_2-schemes over k, the composition

$$\mathsf{R}_{C_2}^{\star,\star}(Y_+) \xrightarrow{\mathrm{id}\otimes f^* u} \mathsf{R}_{C_2}^{\star,\star}(Y_+) \otimes \mathsf{R}_{C_2}^{\star,\star}(\mathrm{Th}(f^*E)) \xrightarrow{\Delta^*} \mathsf{R}_{C_2}^{\star,\star}(\mathrm{Th}(f^*E))$$

is an isomorphism. Here, $\Delta : \mathrm{Th}(f^*E) \to Y_+ \wedge \mathrm{Th}(f^*E)$ is the Thom diagonal.

Proposition 3.20. *Let* $V = a + b\sigma$. *There are classes* $u_V \in \widetilde{H}_{C_2}^{2a+2b\sigma, a+b\sigma}(T^V, \underline{\mathbb{Z}})$ *such that*

$$\widetilde{H}_{C_2}^{\star,\star}(X_+, \underline{\mathbb{Z}}) \xrightarrow{-\cup(1_X \times u_V)} \widetilde{H}_{C_2}^{\star+2a+2b\sigma, \star+a+b\sigma}(X_+ \wedge T^V, \underline{\mathbb{Z}})$$

is an isomorphism, for any C_2*-variety* X. *Moreover, if* ϕ *is an automorphism of the* C_2*-equivariant vector bundle* $X \times \mathbb{A}(V) \to X$, *then*

$$(3.21) \qquad \phi^*(1_X \times u_V) = 1_X \times u_V.$$

Proof. Let $u_1 \in \widetilde{H}^{2,1}_{C_2}(T, \underline{\mathbb{Z}})$ and $u_\sigma \in \widetilde{H}^{2\sigma,\sigma}_{C_2}(T^\sigma, \underline{\mathbb{Z}})$ be elements corresponding to the unit under the suspension isomorphism (3.6). For a representation $V = a + b\sigma$ define $u_V = (u_1)^a (u_\sigma)^b$. This element satisfies the first condition, and it remains to check that (3.21) holds for any equivariant bundle automorphism ϕ of $X \times \mathbb{A}(V)$. We will proceed by induction on the dimension of V.

We first consider the case $V = \sigma$. Write $\alpha = a + p\sigma$, $\beta = b + q\sigma$, and write

$$H^{\alpha,\beta}_{C_2}(X, \underline{\mathbb{Z}})_{(-\sigma)} = \operatorname{coker}\left(H^{\alpha,\beta}_{C_2}(X \times \mathbb{A}(\sigma), \underline{\mathbb{Z}}) \xrightarrow{i^*} H^{\alpha,\beta}_{C_2}(X \times (\mathbb{A}(\sigma) \setminus \{0\}), \underline{\mathbb{Z}}) \right).$$

Since $H^{\alpha,\beta}_{C_2}(-, \underline{\mathbb{Z}})$ is a presheaf with equivariant transfers (see Section 3.3.2), if X is affine, then by [HVØ15, Proposition 8.3] the map i^* has a retraction, and we have a natural splitting

$$(3.22) \qquad H^{\alpha,\beta}_{C_2}(X \times (\mathbb{A}(\sigma) \setminus \{0\}), \underline{\mathbb{Z}}) = H^{\alpha,\beta}_{C_2}(X \times \mathbb{A}(\sigma), \underline{\mathbb{Z}}) \oplus H^{\alpha,\beta}_{C_2}(X, \underline{\mathbb{Z}})_{(-\sigma)}.$$

Consider the cofiber sequence $(\mathbb{A}(\sigma) \setminus \{0\})_+ \to \mathbb{A}(\sigma)_+ \to T^\sigma$. The induced long exact sequences break into (split) short exact sequences

$$0 \to H^{\alpha,\beta}_{C_2}(\mathbb{A}(\sigma), \underline{\mathbb{Z}}) \xrightarrow{i^*} H^{\alpha,\beta}_{C_2}(\mathbb{A}(\sigma) \setminus \{0\}, \underline{\mathbb{Z}}) \xrightarrow{\delta} \widetilde{H}^{\alpha+1,\beta}_{C_2}(T^\sigma, \underline{\mathbb{Z}}) \to 0.$$

The element u_σ lifts to an element $u'_\sigma \in H^{2\sigma-1,\sigma}_{C_2}(\mathbb{A}(\sigma) \setminus \{0\}, \underline{\mathbb{Z}})$. We choose u'_σ so that under the splitting (3.22), we have $u'_\sigma \in H^{2\sigma-1,\sigma}_{C_2}(\operatorname{Spec}(k), \underline{\mathbb{Z}})_{(-\sigma)}$. Let ϕ be an automorphism of the equivariant vector bundle $X \times \mathbb{A}(\sigma)$ over X. By naturality, to show that $\phi^*(1_X \times u_\sigma) = 1_X \times u_\sigma$ for a smooth affine C_2-variety X, we are reduced to showing

$$(3.23) \qquad\qquad \phi_0^*(1_X \times u'_\sigma) = (1_X \times u'_\sigma) + \beta,$$

where β is some element in $\ker(\delta)$, and ϕ_0 is the restriction of ϕ to $X \times \mathbb{A}(\sigma) \setminus \{0\}$. For any X (not necessarily affine), the group of equivariant linear automorphisms of $X \times \mathbb{A}(\sigma)$ over X is $(\mathcal{O}_X^*)^{C_2}$; i.e., an equivariant automorphism is given by multiplication with an invariant unit. An invariant unit is specified (uniquely) by an equivariant map $X \to \mathbb{G}_m$. By naturality, to verify the relation (3.21) for $1_X \times u_\sigma$ and any automorphism ϕ of $X \times \mathbb{A}(\sigma)$, it suffices to verify it for $X = \mathbb{G}_m$ and ϕ the automorphism of $\mathbb{G}_m \times \mathbb{A}(\sigma) \setminus \{0\} \to \mathbb{G}_m$ given by multiplication with the canonical unit t of $\mathbb{G}_m$. In this case, ϕ_0 is the map $\langle pr_1, \mu \rangle$, where pr_1 is the projection to the first factor and $\mu : \mathbb{G}_m \times (\mathbb{A}(\sigma) \setminus \{0\}) \to \mathbb{A}(\sigma) \setminus \{0\}$ is the multiplication. We have that $\phi_0^*(1_{\mathbb{G}_m} \times u'_\sigma) = 1 \cup \mu^*(u'_\sigma) = \mu^*(u'_\sigma)$.

From the naturality of the decomposition (3.22) we see that μ^* restricts to a map

$$\mu^* : H^{2\sigma-1,\sigma}(k, \underline{\mathbb{Z}})_{(-\sigma)} \to H^{2\sigma-1,\sigma}(\mathbb{G}_m, \underline{\mathbb{Z}})_{(-\sigma)}.$$

Moreover, this map has a splitting induced by $e : \operatorname{Spec}(k) \to \mathbb{G}_m$, the inclusion at $1 \in \mathbb{G}_m$.

It follows that $\mu^*(1_{\mathbb{G}_m} \times u'_\sigma) = 1_{\mathbb{G}_m} \times u'_\sigma + \gamma$, where $\gamma \in \ker(e^*)$. An easy diagram chase shows that $\delta(\ker(e^*)) = 0$, so (3.23) holds. It follows that (3.21) holds for $1_X \times u_\sigma$ for any X and any automorphism of $X \times \mathbb{A}(\sigma)$. A similar (but easier) argument establishes (3.21) for $1_X \times u_1$ as well. This takes care of the case when V has dimension 1.

Now we proceed by induction on the dimension of V. Let $V = a + b\sigma$ be an n-dimensional representation. Let $\underline{\operatorname{Aut}}^{C_2}(V)$ be the presheaf whose value on X is the group of equivariant bundle automorphisms of $X \times \mathbb{A}(V)$. It is represented by a group scheme in $C_2\operatorname{Sm}/k$; in particular, it is an equivariant Nisnevich sheaf. Under

the isomorphism (3.6), the assignment $1_X \times u_V \mapsto \phi^*(1_X \times u_V)$ is an automorphism of $H_{C_2}^{0,0}(X, \underline{\mathbb{Z}}) = H_{C_2\mathrm{Nis}}^0(X, \mathbb{Z})$. This assignment is natural and defines a morphism of sheaves $\underline{\mathrm{Aut}}^{C_2}(V) \to \mathbb{Z}^\times$. To check that the image of this map is $\{1\}$, we may assume that X is a point of the equivariant Nisnevich topology. Recall that for a finite group G, the points of the equivariant Nisnevich topology are of the form $G \times_H \mathrm{Spec}(R)$, where $H \subseteq G$ is a subgroup and R is an essentially smooth Henselian local k-algebra with H-action; see [HVØ15, Theorem 3.14]. For $G = C_2$ there are two possibilities, $X = C_2 \times \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(R)$, where R is a smooth local ring with C_2-action. In the first case the claim follows from Proposition 3.14 and the fact that the u_V are nonequivariant Thom classes. Now we consider the case that $X = \mathrm{Spec}(R)$, where R is a smooth local ring with C_2-action. The equivariant automorphisms of $X \times \mathbb{A}(V)$ are $(\mathrm{Aut}_R(V_R))^{C_2}$. If A is a matrix with entries in R, write A^σ for the matrix obtained by applying the involution σ to its entries. Under the identification $\mathrm{End}_R(V_R) = \mathrm{Mat}_{n \times n}(R)$ the C_2-action is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A^\sigma & -(B^\sigma) \\ -(C^\sigma) & D^\sigma \end{bmatrix}.$$

Here, A is an $(a \times a)$-matrix and B is a $(b \times b)$-matrix, and $a + b = n$. It follows that this matrix is in $\mathrm{Mat}_{n \times n}(R)^{C_2}$ if and only if A and D have coefficients in R^{C_2} and σ acts by -1 on the coefficients of B, C.

Let $x \in R$, and for $i \neq j$ write $E_{ij}(x)$ for the elementary matrix corresponding to adding x times row j to row i; i.e., it has x in position (i, j) and is the same as the identity matrix in all other entries. Then $E_{ij}(x)$ is in $\mathrm{GL}_n(R)^{C_2}$ if

   (i) either $i, j \leq a$ or $a < i, j$ and $x \in R$ is invariant, or
   (ii) $i \leq a$, $j > a$ or $i > a$, $j \leq a$ and $\sigma x = -x$.

The matrix $E_{ij}(x)$ is equivariantly homotopic to the identity via the equivariant $\mathbb{A}^1$-homotopy $t \mapsto E_{ij}(tx)$.

Let T_{ij} be the elementary matrix corresponding to switching the ith and jth rows. If $i, j \leq a$, then T_{ij} is in $\mathrm{Mat}_{n \times n}(R)^{C_2}$. Moreover, in this case there is an algebraic map $\mathbb{A}^1 \to \mathrm{GL}_a(R^{C_2}) \subseteq \mathrm{GL}_n(R)^{C_2}$ joining T_{ij} and the diagonal matrix $\langle -1, 1, \ldots, 1 \rangle$. Thus there is an equivariant $\mathbb{A}^1$-homotopy joining T_{ij} and $\langle -1, 1, \ldots, 1 \rangle$ in $\mathrm{GL}_{a+b}(R)$. Similarly, T_{ij} is equivariantly $\mathbb{A}^1$-homotopic to the identity for $i, j > a$.

Let $M = (m_{ij})$ be an invertible matrix in $\mathrm{Mat}_{n \times n}(R)^{C_2}$. We show that there is an $\mathbb{A}^1$-homotopy

$$M \simeq_{\mathbb{A}^1} \begin{bmatrix} u & 0 \\ 0 & M' \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} M' & 0 \\ 0 & u \end{bmatrix}.$$

Here, $u \in R$ is an invariant unit, and M' is an invertible $((n-1) \times (n-1))$-matrix in $\mathrm{Mat}_{(n-1) \times (n-1)}(R)^{C_2}$. First, we assume the entry m_{11} of M is a unit. In this case, the claim follows by multiplying by the elementary matrices $E_{i1}(-m_{i1}/m_{11})$, $i > 1$ on the left and by $E_{1j}(-m_{1j}/m_{11})$, $j > 1$ on the right. If m_{11} is not a unit but some m_{ij} is a unit for $i, j \leq a$, then by multiplying by T_{1i} on the left and T_{j1} on the right, we are reduced to the previous case. If all entries m_{ij} of A (i.e., $i, j \leq a$) are nonunits, we can repeat the previous arguments for the elements of D if at least one of its entries is a unit. The only remaining case is when all entries of A and of D are in the maximal ideal of R. For $t \in \mathbb{A}^1$ we write M_t for the matrix, which agrees with M in all positions except $(M_t)_{11} = t + m_{11}$. This gives a map $\mathbb{A}^1 \to \mathrm{Mat}_{n \times n}(R)^{C_2}$. Write M_t' for the $(a \times a)$-matrix consisting of $(M_t)_{ij}$

for $i, j \leq a$. The reduction of M_t modulo the maximal ideal of R is

$$\overline{M_t} = \left[\begin{array}{cc} \overline{M'_t} & \overline{B} \\ \hline \overline{C} & 0 \end{array} \right].$$

Since $M = M_0$ is invertible, so is $\overline{M_t}$. It follows that M_t is invertible for all t. Now M_1 is equivariantly $\mathbb{A}^1$-homotopic to M and the previous case applies to M_1, so we are done. $\qquad \square$

Recall from (3.9) that we have elements $\epsilon, \epsilon', u \in H^{0,0}_{C_2}(k, \underline{\mathbb{Z}})$ which determine the commutativity properties of the ring $H^{\star,\star}_{C_2}(X, \underline{\mathbb{Z}})$. These elements are $\epsilon = \Sigma^{-2}_T \tau^*_T (\Sigma^2_T 1)$, $\epsilon' = \Sigma^{-2}_{T^\sigma} \tau^*_{T^\sigma} (\Sigma^2_{T^\sigma} 1)$, and $u = \Sigma^{-2}_{S^\sigma} \tau^*_{S^\sigma} (\Sigma^2_{S^\sigma} 1)$, where $\tau_{\mathsf{E}} : \mathsf{E} \wedge \mathsf{E} \to \mathsf{E} \wedge \mathsf{E}$ is the twist endomorphism.

Proposition 3.24. *In $H^{0,0}_{C_2}(k, \underline{\mathbb{Z}})$, $\epsilon = 1$, $\epsilon' = 1$, and $u = -1$. In particular, if $x \in H^{a+p\sigma, b+q\sigma}_{C_2}(X, \underline{\mathbb{Z}})$ and $y \in H^{c+s\sigma, d+t\sigma}_{C_2}(X, \underline{\mathbb{Z}})$, then*

$$x \cup y = (-1)^{ac+ps}(y \cup x).$$

Proof. We have that $\widetilde{H}^{4,2}_{C_2}(T^2, \underline{\mathbb{Z}})$ is a free $H^{0,0}_{C_2}(k, \underline{\mathbb{Z}})$-module with basis u_2. The map τ_T is induced by the twist automorphism of the vector bundle $\mathbb{A}^2 \to \mathrm{Spec}(k)$, so by Proposition 3.20 $\tau^*_T(u_2) = u_2$. In particular, $\tau^*_T = \mathrm{id}$, and it follows that $\epsilon = \Sigma^{-2}_T \tau^*_T (\Sigma^2_T 1) = 1$, as claimed. The argument for ϵ' is entirely similar.

Now we compute u. Recall from Section 3.1 the complex $\mathbb{Z}_{\mathrm{top}}(\sigma)$ of presheaves with transfers. Under the identification of Theorem 3.4 the element u corresponds to the twist isomorphism $\mathbb{Z}_{\mathrm{top}}(\sigma) \otimes^{\mathrm{tr}, \mathbb{L}} \mathbb{Z}_{\mathrm{top}}(\sigma) \to \mathbb{Z}_{\mathrm{top}}(\sigma) \otimes^{\mathrm{tr}, \mathbb{L}} \mathbb{Z}_{\mathrm{top}}(\sigma)$ in $D^-(C_2 \, \mathrm{Cor}_k)$. The twist isomorphism of complexes $C \otimes D \to D \otimes C$ is given componentwise by $(-1)^{pq}$ times the twist $C^p \otimes D^q \to D^q \otimes C^p$. The complex $\mathbb{Z}_{\mathrm{top}}(\sigma) \otimes^{\mathrm{tr}, \mathbb{L}} \mathbb{Z}_{\mathrm{top}}(\sigma)$ is isomorphic to

$$\mathbb{Z}_{\mathrm{tr}, C_2}(C_2 \times C_2) \xrightarrow{\langle q_2, -q_1 \rangle} \mathbb{Z}_{\mathrm{tr}, C_2}(C_2) \oplus \mathbb{Z}_{\mathrm{tr}, C_2}(C_2) \xrightarrow{p_* \oplus p_*} \mathbb{Z},$$

where $\mathbb{Z}$ is in degree 0, $p : C_2 \to \mathrm{Spec}(k)$ is the projection, and $q_i : C_2 \times C_2 \to C_2$ is the projection to the ith factor. Under this isomorphism, the twist isomorphism is given, respectively, in degrees -2, -1, and 0, by $-\tau_*$, τ, and id.

A chain homotopy between the twist map and $-\mathrm{id}$ is given by $\{s_i\}$, $s_i = 0$, $i \neq 0, -1$, and $s_0 = \langle 0, p^t \rangle$, $s_{-1} = \Delta'_* \oplus \Delta'_*$. Here, $\Delta' : C_2 \to C_2 \times C_2$ is given by $e \mapsto \sigma \times e$, $\sigma \mapsto e \times \sigma$, and p^t is the transpose of p. Indeed, it is quickly checked that $s_{-1} \langle q_2, -q_1 \rangle = \Delta'_* q_2 - \Delta'_* q_1 = -\tau_* + \mathrm{id}$ and $(p_* \oplus p_*) s_0 = 2 \cdot \mathrm{id}$. For the remaining chain homotopy relation, we have that $\langle q_2, -q_1 \rangle s_1 + s_0(p_* \oplus p_*)$ is equal to

$$\begin{pmatrix} q_2 \Delta'_* & q_2 \Delta'_* \\ p^t p_* - q_1 \Delta'_* & p^t p_* - q_1 \Delta'_* \end{pmatrix} = \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ (\mathrm{id} + m_*) - m_* & (\mathrm{id} + m_*) - m_* \end{pmatrix} = \tau + \mathrm{id},$$

where $m : C_2 \to C_2$ is the nontrivial involution. $\qquad \square$

Definition 3.25. Let V be a C_2-representation, and let $E \to X$ be a C_2-equivariant vector bundle. Say that E is *type V* if every point $x \in X$ is contained in an invariant open neighborhood $U \subseteq X$ such that $E|_U$ is isomorphic, as a C_2-equivariant vector bundle, to the product bundle $U \times \mathbb{A}(V) \to U$.

Theorem 3.26. *Let X be a smooth C_2-scheme over k, let $V = a + b\sigma$, and let $E \to X$ be a C_2-equivariant vector bundle of type V. Then there are Thom classes*

$$\mathrm{th}(E) \in \widetilde{H}^{2a+2b\sigma, a+b\sigma}_{C_2}(\mathrm{Th}(\mathrm{E}), \underline{A}).$$

Proof. By assumption there is a cover $X = U_1 \cup \cdots \cup U_n$ by open invariant subschemes such that $E|_{U_i} \cong U_i \times \mathbb{A}(V)$. Proceeding by induction on n, we can use the Mayer–Vietoris long exact sequence to patch the elements $1_{U_i} \times u_V$ constructed in the previous proposition. The condition of (3.21) guarantees that they patch together. $\qquad\square$

In the following, A denotes an abelian group.

Corollary 3.27. *Let $i : Z \hookrightarrow X$ be a closed immersion of smooth C_2-schemes over k, with open complement $j : U \hookrightarrow X$ and normal bundle $\mathcal{N}_i$. Suppose that $Z = \coprod Z_r$, with each Z_r invariant, and that $\mathcal{N}_i|_{Z_r}$ is of type $a_r + b_r\sigma$. Then there is a Gysin long exact sequence*

$$\cdots \to \oplus_r H_{C_2}^{\star-2\alpha_r, \star-\alpha_r}(Z_r, \underline{A}) \to H_{C_2}^{\star,\star}(X, \underline{A}) \xrightarrow{j^*} H_{C_2}^{\star,\star}(U, \underline{A}) \to \cdots.$$

Proof. By equivariant homotopical purity [HKØ15, Theorem 7.6], we have a cofiber sequence of motivic C_2-spaces

(3.28) $$U \to X \to \mathrm{Th}(\mathcal{N}_i).$$

This induces a long exact sequence

$$\cdots \to \widetilde{H}_{C_2}^{\star,\star}(\mathrm{Th}(\mathcal{N}_i), \underline{A}) \to H_{C_2}^{\star,\star}(X, \underline{A}) \xrightarrow{j^*} H_{C_2}^{\star,\star}(U, \underline{A}) \to \widetilde{H}_{C_2}^{\star+1,\star}(\mathrm{Th}(\mathcal{N}_i), \underline{A}) \to \cdots.$$

Note that $\mathrm{Th}(\mathcal{N}_i) = \vee_r \mathrm{Th}(\mathcal{N}_i|_{Z_r})$. Applying the previous theorem to each $\mathrm{Th}(\mathcal{N}_i|_{Z_\alpha})$ identifies the long exact sequence induced by (3.28) with the desired Gysin sequence. $\qquad\square$

Remark 3.29. An important case is the following. Let X be a smooth C_2-scheme, and let Z be a connected component of the fixed point subscheme $X^{C_2} \subseteq X$. Then the fibers of the normal bundle of $Z \subseteq X$ are of type $\mathrm{codim}_X(Z)\sigma$. Indeed, we have $(T_z X)^{C_2} = T_z(X^{C_2}) = T_z(Z)$ for any $z \in Z$ by, e.g., [HVØ15, Lemma 8.10].

When $k = \mathbb{C}$, the same construction as in the proof of Theorem 3.26 applies to topological Bredon cohomology. Moreover, this construction is compatible with the Betti realization functor $\mathrm{Re}_\mathbb{C} : \widetilde{H}_{C_2}^{\star,\star}(\mathrm{Th}(E), \underline{A}) \to \widetilde{H}_{C_2}^{\star}(\mathrm{Th}(E)(\mathbb{C}), \underline{A})$.

Proposition 3.30. *Let $k = \mathbb{C}$, and let $E \to X$ be a C_2-equivariant vector bundle, and suppose that $\mathrm{th}(E)$ is a Thom class. Then $\mathrm{Re}_\mathbb{C}(\mathrm{th}(E)) \in \widetilde{H}_{C_2}^{\star}(\mathrm{Th}(E)(\mathbb{C}), \underline{A})$ is a Thom class.*

Proof. To show that $\mathrm{Re}_\mathbb{C}(\mathrm{th}(E))$ is a Thom class, it suffices to show that $i^*(\mathrm{th}(E))$ is a generator of the free $H^*(C_2/H, \underline{A})$-module $\widetilde{H}^*(\mathrm{Th}(i^*E), \underline{A})$, where $H \subseteq C_2$ is a subgroup and $i : C_2/H \to X(\mathbb{C})$ is an equivariant map [May96, XVI.9]. Write $V = a + b\sigma$, where $i^*E = C_2/H \times \mathbb{A}(V)$. Then, $i^*(\mathrm{th}(E)) = a(\Sigma_{TV} 1)$ in $\widetilde{H}_{C_2}^{2a+2b\sigma, a+b\sigma}(Th(i^*E), \underline{A})$ for some $a \in H_{C_2}^{0,0}(\mathrm{Spec}(\mathbb{C}), \underline{A}) = A$. It follows that $i^*\mathrm{Re}_\mathbb{C}(\mathrm{th}(E)) = a(\Sigma_{SV(\mathbb{C})} 1)$, which is a generator of $\widetilde{H}_{C_2}^{2a+2b\sigma}(\mathrm{Th}(i^*E(\mathbb{C})), \underline{A})$, so $\mathrm{Re}_\mathbb{C}(\mathrm{th}(E))$ is a Thom class. $\qquad\square$

4. Bredon cohomology and equivariant higher Chow groups

In [HVØ15, Theorem 5.19] we constructed a natural comparison map between the Bredon motivic cohomology groups and Edidin–Graham's equivariant higher Chow groups. In this section we elaborate on the comparison between these two

constructions of equivariant motivic cohomology. Throughout, A denotes an abelian group.

Proposition 4.1. *Let X be a smooth quasi-projective C_2-scheme over k. There is a natural isomorphism*

$$CH^b_{C_2}(X, 2b - a, A) \cong H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}).$$

Proof. By definition $CH^b_{C_2}(X, 2b - a, A) = CH^b(X \times_{C_2} (\mathbb{A}(n\sigma) \setminus \{0\}), 2b - a, A)$ for n sufficiently large; see [EG98, pp. 599 and 605]. In particular, the value of this latter group is constant for $n \gg 0$. Write $U_n = \mathbb{A}(n\sigma) \setminus \{0\}$. Using the isomorphism between higher Chow groups and motivic cohomology [Voe02, Corollary 2] together with Proposition 3.16, we obtain the natural isomorphisms

$$CH^b_{C_2}(X, 2b - a, A) \cong \lim_n H^{a,b}_{\mathcal{M}}(X \times_{C_2} U_n, A)$$
$$\cong \lim_n H^{a,b}_{C_2}(X \times U_n, \underline{A})$$
$$\cong H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}).$$

For the last isomorphism we have used the Milnor exact sequence

$$0 \to \lim_n{}^1 H^{a,b}_{C_2}(X \times U_n) \to H^{a,b}_{C_2}(X \times \mathbf{E}C_2) \to \lim_n H^{a,b}_{C_2}(X \times U_n) \to 0,$$

and the fact that the $\lim^1$-term vanishes. $\square$

The groups $H^{\star,\star}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ define the Borel motivic cohomology of X. In light of the identification above, we view $H^{\star,\star}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ as a generalized version of equivariant higher Chow groups (in which the grading is by representations instead of just integers). By the motivic isotropy separation cofiber sequence (2.7) the projection map $\mathbf{E}C_{2+} \to S^0$ induces the comparison map between the Bredon motivic cohomology and the Borel motivic cohomology theories. Their difference is measured by $\widetilde{\mathbf{E}}C_2$.

Lemma 4.2. *Let X be a smooth, quasi-projective C_2-scheme. Suppose that* (i) X *has free action,* (ii) $b < 0$, *or* (iii) $a \leq 1$ *and that A is finite. Then*

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) = 0.$$

Proof. By Proposition 2.9 we have isomorphisms

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}^{a,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}).$$

First, we consider the case in which X has free action. Then X/G is smooth, and applying Proposition 3.16, the map $H^{a,b}_{C_2}(X, \underline{A}) \to H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ becomes identified with the natural isomorphism

$$H^{a,b}_{\mathcal{M}}(X/G, A) \xrightarrow{\cong} H^{a,b}_{\mathcal{M}}(X \times^G \mathbf{E}C_2, A)$$

for all a, b. This establishes the vanishing in case (i).

Now consider the case in which X has trivial action. In this case, we have

$$H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}) = H^{a,b}_{\mathcal{M}}(X \times \mathbf{B}C_2, A).$$

The projection map $X \times \mathbf{B}C_2 \to X$ affords a section, and the long exact sequence associated to the motivic isotropy sequence breaks up into short exact sequences

$$0 \to H^{a,b}_{C_2}(X, \underline{A}) \to H^{a,b}_{\mathcal{M}}(X \times \mathbf{B}C_2, A) \to \widetilde{H}^{a+1,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to 0.$$

The two left groups vanish whenever $b < 0$, and so does the third, which establishes case (ii). The space $\mathbf{B}C_2$ is the complement of the zero section of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^\infty$; see, e.g., [Voe03b, Lemma 6.4]. It thus fits into a cofiber sequence

$$\mathbf{B}C_2 \to \mathbb{P}^\infty \to \mathrm{Th}(\mathcal{O}(-2))$$

of motivic spaces. Now we suppose that A is finite. We have, as a consequence of the Bloch–Kato conjectures, that $H_{\mathcal{M}}^{a,b}(X, A) = 0$ for $a < 0$. Combined with the sequence in motivic cohomology resulting from the above cofiber sequence, the Thom isomorphism, and the projective bundle theorem, we find an isomorphism $H_{\mathcal{M}}^{a,b}(X, A) \to H_{\mathcal{M}}^{a,b}(X \times \mathbf{B}C_2, A)$ for $a \le 0$ and A finite.

Now consider an arbitrary smooth C_2-scheme X. The fixed points $X^{C_2} \subseteq X$ are smooth and its open complement $U = X \setminus X^{C_2}$ has free action. The previous paragraphs applied to X^{C_2} and U together with the Gysin sequence of Corollary 3.27 and Remark 3.29 yield the result. □

Theorem 4.3. *Let X be a smooth quasi-projective C_2-scheme over k. Then the natural map*

$$H_{C_2}^{a+p\sigma, b+q\sigma}(X, \underline{A}) \to H_{C_2}^{a+p\sigma, b+q\sigma}(X \times \mathbf{E}C_2, \underline{A})$$

is an isomorphism if (i) *X has free action,* (ii) *$b < 0$, or* (iii) *A is finite and $a \le 0$. In case* (iii) *the map is injective if $a = 1$.*

Proof. This follows immediately from the previous lemma and the motivic isotropy separation cofiber sequence (2.7). □

Remark 4.4. Combining the identification of Proposition 4.1 and the periodicity of Corollary 5.5, we find that there is a natural map

$$H_{C_2}^{a+p\sigma, b+q\sigma}(X, \underline{\mathbb{Z}/2}) \to H_{\mathcal{M}}^{a+p, b+q}(X \times^{C_2} \mathbf{E}C_2, \mathbb{Z}/2)$$
$$\cong CH^{b+q}(X, 2(b+q) - a - p, \mathbb{Z}/2)$$

which is an isomorphism if (i) X has free action, (ii) $b < 0$, or (iii) $a \le 0$. Moreover, in case (iii) the map is an injection for $a = 1$.

5. Periodicity and Borel motivic cohomology

In this section we show that the ring $H_{C_2}^{\star,\star}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ is periodic with period $(2\sigma - 2, \sigma - 1)$. It follows that the groups $H_{C_2}^{\star,\star}(X \times \mathbf{E}C_2, \underline{\mathbb{Z}/2})$ are also periodic. These form a generalized geometric Borel motivic cohomology theory. The integer graded portion of these groups is isomorphic to equivariant higher Chow groups; see Section 4. This periodicity of the generalized Borel motivic cohomology will play an important role in our comparison theorem between motivic and topological Bredon cohomology.

We remind the reader that our convention is that $(*,*)$ stands for an integer bigrading and $(\star, \star)$ stands for the bigrading determined by representations; see (1.4). We begin with the cohomology of C_2.

Lemma 5.1. *Let A be an abelian group. There is an $H_{\mathcal{M}}^{*,*}(k, A)$-algebra isomorphism*

$$H_{C_2}^{\star,\star}(C_2, \underline{A}) \cong H_{\mathcal{M}}^{*,*}(k, A)[s^{\pm 1}, t^{\pm 1}],$$

where $s \in H_{C_2}^{\sigma-1, 0}(C_2, \underline{A})$ and $t \in H_{C_2}^{\sigma-1, \sigma-1}(C_2, \underline{A})$.

Proof. Write $(\alpha, \beta) = (a + p\sigma, b + q\sigma)$ and $|a + p\sigma| = a + p$. We have isomorphisms

$$H^{\star,\star}_{C_2}(C_2, \underline{A}) \xrightarrow{\cong} \widetilde{H}^{\star+\alpha,\star+\beta}_{C_2}(S^{\alpha,\beta} \wedge C_{2+}, \underline{A}) \xrightarrow{\cong} \widetilde{H}^{\star+\alpha,\star+\beta}_{C_2}(S^{|\alpha|,|\beta|} \wedge C_{2+}, \underline{A})$$
$$\xleftarrow{\cong} H^{\star+\alpha-|\alpha|,\star+\beta-|\beta|}_{C_2}(C_2, \underline{A});$$

the first and last isomorphisms are instances of the suspension isomorphism, and the middle isomorphism follows from Lemma A.9. This is an isomorphism of free one-dimensional $H^{\star,\star}_{C_2}(C_2, \underline{A})$-modules and thus is given by multiplication with an invertible element $x_{\alpha,\beta} \in H^{\alpha-|\alpha|,\beta-|\beta|}_{C_2}(C_2, \underline{A})$. Taking $s = x_{\sigma-1,0}$ and $t = x_{\sigma-1,\sigma-1}$, we get an $H^{*,*}_{\mathcal{M}}(k, A)$-algebra map

$$H^{*,*}_{\mathcal{M}}(k, A)[s^{\pm 1}, t^{\pm 1}] \to H^{\star,\star}_{C_2}(C_2, \underline{A}),$$

which is an isomorphism. $\square$

Lemma 5.2. *The map $C_{2+} \to S^0$ induces an isomorphism*

$$H^{2\sigma-2,\sigma-1}_{C_2}(k, \underline{\mathbb{Z}/2}) \xrightarrow{\cong} H^{2\sigma-2,\sigma-1}_{C_2}(C_2, \underline{\mathbb{Z}/2}).$$

Proof. By Theorem 3.4 this map is identified with the map

$$H^0_{C_2\mathrm{Nis}}(T, \underline{\mathbb{Z}/2}(\sigma)[2\sigma]) \to H^0_{C_2\mathrm{Nis}}(C_{2+} \wedge T, \underline{\mathbb{Z}/2}(\sigma)[2\sigma]).$$

We have that $\underline{\mathbb{Z}/2}(\sigma)[2\sigma] = C_*\mathbb{Z}_{\mathrm{tr},C_2}(T^\sigma) \otimes^{\mathbb{L}} \mathbb{Z}/2$ and, by [HVØ15, Proposition 5.14],

$$C_*\mathbb{Z}_{\mathrm{tr},C_2}(T^\sigma) \simeq \mathrm{cone}((\mathcal{O}^*)^{C_2} \oplus \mathbb{Z} \to \mathbb{Z}).$$

We now see that

$$H^i_{C_2\mathrm{Nis}}(T, C_*\mathbb{Z}_{\mathrm{tr},C_2}(T^\sigma)) \xrightarrow{\cong} H^i_{C_2\mathrm{Nis}}(C_{2+} \wedge T, C_*\mathbb{Z}_{\mathrm{tr},C_2}(T^\sigma))$$

is an isomorphism for all i (for example, by applying [HVØ15, Lemma 3.19]). $\square$

Lemma 5.3. *Let A be an abelian group. There are isomorphisms*

$$H^{0,0}_{C_2}(\mathbf{E}C_2, \underline{A}) \cong \lim_n H^{0,0}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{A}) \cong A.$$

Proof. We have $H^{0,0}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{A}) \cong H^{0,0}_{\mathcal{M}}((\mathbb{A}(n\sigma) \setminus \{0\})/C_2, A) = A$ by Proposition 3.16, and the maps

$$H^{0,0}_{\mathcal{M}}((\mathbb{A}(n\sigma) \setminus \{0\})/C_2, A) \to H^{0,0}_{\mathcal{M}}((\mathbb{A}((n+1)\sigma) \setminus \{0\})/C_2, A)$$

are isomorphisms. $\square$

Write $v = st \in H^{2\sigma-2,\sigma-1}_{C_2}(k, \underline{\mathbb{Z}/2})$ for the element obtained from Lemmas 5.1 and 5.2. We also write v for the corresponding image in $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$.

Theorem 5.4. *The element $v \in H^{2(\sigma-1),\sigma-1}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ is invertible.*

Proof. Consider the equivariant embedding $i_n : C_2 \subseteq \mathbb{A}(n\sigma)\setminus\{0\}$ given by including at $\{\pm 1\}$. We show that i_n induces an isomorphism

$$i_n^* : H^{2-2\sigma,1-\sigma}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2}) \xrightarrow{\cong} H^{2-2\sigma,1-\sigma}_{C_2}(C_2, \underline{\mathbb{Z}/2}).$$

This will imply the theorem as follows. For each n, the elements v^{-1} in $H^{\star,\star}_{C_2}(C_2, \underline{\mathbb{Z}/2})$ lift uniquely to elements u_n in $H^{\star,\star}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2})$. The uniqueness of these lifts implies that $\{u_n\}$ determine elements (u_n) in $\lim_n H^{\star,\star}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2})$. These in turn lift to an element $\overline{u}$ in $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$. We now find

that $v \cup \overline{u} \in H_{C_2}^{0,0}(\mathbf{E}C_2, \underline{\mathbb{Z}/2}) = \mathbb{Z}/2$ must be equal to 1 since it maps to $1 \in H_{C_2}^{0,0}(C_2, \underline{\mathbb{Z}/2}) = \mathbb{Z}/2$.

For typographical simplicity we will suppress the coefficients $\mathbb{Z}/2$ of the cohomology groups from the notation. We will also write $U_n := \mathbb{A}(n\sigma) \setminus \{0\}$. Consider the comparison of exact sequences induced by (2.5),

$$
\begin{array}{ccccccc}
H_{C_2}^{1,1}(U_n) & \longrightarrow & H_{C_2}^{1,1}(U_1 \times U_n) & \longrightarrow & \widetilde{H}_{C_2}^{2,1}(T^\sigma \wedge U_{n+}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{C_2}^{1,1}(C_2) & \longrightarrow & H_{C_2}^{1,1}(U_1 \times C_2) & \longrightarrow & \widetilde{H}_{C_2}^{2,1}(T^\sigma \wedge C_{2+}) & \longrightarrow & 0.
\end{array}
$$

The quotient $U_1 \times_{C_2} U_n$ is the complement of the zero section of the line bundle $L := \mathbb{A}(\sigma) \times_{C_2} U_n$ on U_n/C_2. By Proposition 3.16 the left-hand square of the above diagram is identified with the left-hand square of the commutative diagram

$$
\begin{array}{ccccccc}
H_{\mathcal{M}}^{1,1}(U_n/C_2) & \longrightarrow & H_{\mathcal{M}}^{1,1}(U_1 \times_{C_2} U_n) & \longrightarrow & \widetilde{H}_{\mathcal{M}}^{2,1}(\mathrm{Th}(L)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{\mathcal{M}}^{1,1}(\mathrm{Spec}(k)) & \longrightarrow & H_{\mathcal{M}}^{1,1}(\mathbb{A}^1 \setminus \{0\}) & \longrightarrow & \widetilde{H}_{\mathcal{M}}^{2,1}(T) & \longrightarrow & 0.
\end{array}
$$

The map $\widetilde{H}_{\mathcal{M}}^{2,1}(\mathrm{Th}(L)) \to \widetilde{H}_{\mathcal{M}}^{2,1}(T)$ sends the Thom class of L to a generator, so this map is an isomorphism. In particular, $\widetilde{H}_{C_2}^{2,1}(T^\sigma \wedge U_{n+}) \to \widetilde{H}_{C_2}^{2,1}(T^\sigma \wedge C_{2+})$ is an isomorphism, as desired. $\square$

Corollary 5.5. *Multiplication by* $v^{-q} \in H_{C_2}^{2q(1-\sigma),q(1-\sigma)}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ *induces a natural isomorphism*

$$\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(\mathsf{E} \wedge (\mathbf{E}C_2)_+, \underline{\mathbb{Z}/2}) \cong \widetilde{H}_{C_2}^{(a+2q)+(p-2q)\sigma,b+q}(\mathsf{E} \wedge (\mathbf{E}C_2)_+, \underline{\mathbb{Z}/2})$$

of $H_{C_2}^{\star,\star}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$-modules for any motivic C_2-spectrum E.

Remark 5.6. This $(2\sigma - 2, \sigma - 1)$-periodicity does not decompose into a $(\sigma - 1, 0)$- and a $(\sigma - 1, \sigma - 1)$-periodicity (contrary to an erroneous claim in a previous version of this paper). We also note that the failure of $(\sigma - 1, \sigma - 1)$-periodicity shows that the condition $a \le b - q$ in Proposition 6.4 is unavoidable.

To see the failure of $(\sigma - 1, 0)$-periodicity, consider the exact sequence, where $U_n = \mathbb{A}(n\sigma) \setminus 0$,

$$H_{C_2}^{0,0}(U_n, \underline{\mathbb{Z}/2}) \to H_{C_2}^{0,0}(C_2 \times U_n, \underline{\mathbb{Z}/2}) \to \widetilde{H}_{C_2}^{1,0}(S^\sigma \wedge U_{n+}, \underline{\mathbb{Z}/2}) \to H_{C_2}^{1,0}(U_n, \underline{\mathbb{Z}/2}).$$

Since $H_{C_2}^{1,0}(U_n, \underline{\mathbb{Z}/2}) \cong H_{\mathcal{M}}^{1,0}(U_n/C_2, \underline{\mathbb{Z}/2}) = 0$ and the left-hand map is an isomorphism, we have $H_{C_2}^{1-\sigma,0}(U_n, \underline{\mathbb{Z}/2}) = 0$. Therefore, $H_{C_2}^{1-\sigma,0}(\mathbf{E}C_2, \underline{\mathbb{Z}/2}) = 0$; in particular, $H_{C_2}^{\star,\star}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ is not $(\sigma - 1, 0)$-periodic. Consequently it is also not $(\sigma - 1, \sigma - 1)$-periodic.

We note as well that Proposition 2.9 immediately implies that the cohomology of any $X_+ \wedge \widetilde{\mathbf{E}}C_2$ is periodic in the following sense.

Proposition 5.7. *Let X be a smooth C_2-scheme over k, and let A be a commutative ring. For all integers a,b,p,q there are $H_{C_2}^{\star,\star}(X, \underline{A})$-module isomorphisms*

$$\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}_{C_2}^{a+(p-1)\sigma,b+q\sigma}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A})$$

and

$$\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}_{C_2}^{a+(p-2)\sigma,b+(q-1)\sigma}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}).$$

6. Comparing motivic and topological Bredon cohomology over $\mathbb{C}$

Let X be a smooth variety over a field. The Beilinson–Lichtenbaum conjecture [SV00, Conjecture 6.8] is the assertion that the map

$$(6.1) \qquad\qquad H_{\mathcal{M}}^{p,q}(X, \mathbb{Z}/n) \to H_{\text{et}}^p(X, \mu_n^{\otimes q})$$

is an isomorphism when $p \leq q$ and is an injection when $p = q + 1$. The validity of this conjecture is a consequence of the Milnor and Bloch–Kato conjectures [Voe03a, Voe11] together with [SV00, Theorem 7.4]. Now if X is a complex variety, this can be rephrased using singular cohomology; topological realization

$$(6.2) \quad H_{\mathcal{M}}^{p,q}(X, \mathbb{Z}/n) \to H_{\text{sing}}^p(X(\mathbb{C}), \mathbb{Z}/n) \ \text{ is } \begin{cases} \text{an isomorphism} & \text{if } p \leq q, \\ \text{a monomorphism} & \text{if } p = q + 1. \end{cases}$$

In this section we establish a C_2-equivariant generalization of the Beilinson–Lichtenbaum conjecture for smooth complex varieties X with involution. We begin with a consideration of the Borel part of the Bredon cohomologies.

Lemma 6.3. *Let U be a smooth quasi-projective complex C_2-variety with free action. Then*

$$\text{Re}_{\mathbb{C}} : H_{C_2}^{m+i\sigma,n}(U, \underline{\mathbb{Z}/2}) \to H_{C_2}^{m+i\sigma}(U(\mathbb{C}), \underline{\mathbb{Z}/2})$$

is an isomorphism if $m \leq n$ and $m + i \leq n$. It is a monomorphism if $m \leq n+1$ and $m + i \leq n + 1$.

Proof. When $i = 0$, the result holds by (6.2) together with Proposition 3.16. The result follows in general using induction on i and considering the comparison of exact sequences obtained from (2.4),

$$\longrightarrow H_{C_2}^{m+j\sigma,n}(U) \longrightarrow H_{C_2}^{m+(j+1)\sigma,n}(U) \longrightarrow H_{\mathcal{M}}^{m+j+1,n}(U) \longrightarrow H_{C_2}^{m+1+j\sigma,n}(U) \longrightarrow$$

$$\longrightarrow H_{C_2}^{m+j\sigma}(U(\mathbb{C})) \longrightarrow H_{C_2}^{m+(j+1)\sigma}(U(\mathbb{C})) \longrightarrow H_{\text{sing}}^{m+j+1}(U(\mathbb{C})) \longrightarrow H_{C_2}^{m+1+j\sigma}(U(\mathbb{C})) \longrightarrow .$$

$\square$

Proposition 6.4. *Let X be a smooth complex C_2-variety, and let A be a finite abelian group. Then*

$$\text{Re}_{\mathbb{C}} : H_{C_2}^{a+p\sigma,b+q\sigma}(X \times \mathbf{E}C_2, \underline{A}) \to H_{C_2}^{a+p\sigma}(X(\mathbb{C}) \times \mathbf{E}C_2(\mathbb{C}), \underline{A})$$

is

(i) *an isomorphism if $a \leq b - q$ and $a + p \leq b + q$,*
(ii) *a monomorphism if $a \leq b - q + 1$ and $a + p \leq b + q + 1$.*

Proof. We first assume that X is quasi-projective. It suffices to assume that $A = \mathbb{Z}/\ell^i$ where ℓ is a prime. Suppose that 2 is invertible in A. Consider the commutative

diagram coming from Propositions 3.14 and 3.18, where the coefficient group A has been suppressed,

$$
\begin{array}{ccc}
H_{C_2}^{a+p\sigma,b+q\sigma}(X \times \mathbf{E}C_2) \xrightarrow{\ \pi^*\ } & H_{\mathcal{M}}^{a+p,b+q}(X \times \mathbf{E}C_2) \xrightarrow{\ \tau^*\ } & H_{C_2}^{a+p\sigma,b+q\sigma}(X \times \mathbf{E}C_2) \\
\downarrow{\scriptstyle\mathrm{Re}_{\mathbb{C}}} & \downarrow{\scriptstyle\mathrm{Re}_{\mathbb{C}}} & \downarrow{\scriptstyle\mathrm{Re}_{\mathbb{C}}} \\
H_{C_2}^{a+p\sigma}((X \times \mathbf{E}C_2)(\mathbb{C})) \xrightarrow{\ \pi^*\ } & H_{\mathrm{sing}}^{a+p}((X \times \mathbf{E}C_2)(\mathbb{C})) \xrightarrow{\ \tau^*\ } & H_{C_2}^{a+p\sigma}((X \times \mathbf{E}C_2)(\mathbb{C})).
\end{array}
$$

The horizontal compositions are multiplied by 2 and so are isomorphisms. By the Beilinson–Lichtenbaum conjecture (6.2), the middle map is an isomorphism for $a + p \leq b + q$ and an injection for $a + p = b + q + 1$. This implies that the same is thus true of the outer vertical arrows.

It remains to consider the case $A = \mathbb{Z}/2^i$. By comparing the exact sequences arising from the short exact sequence $0 \to \mathbb{Z}/2^{i-1} \to \mathbb{Z}/2^i \to \mathbb{Z}/2 \to 0$ and induction, we are reduced to the case $A = \mathbb{Z}/2$.

We now assume that $A = \mathbb{Z}/2$ (and continue suppressing the coefficients as needed). Using the periodicity from Corollary 5.5, we may replace $(a + p\sigma, b + q\sigma)$ with $((a + 2q) + (p - 2q)\sigma, b + q)$, and we write $(m + i\sigma, b)$ for this new bidegree. The hypothesis on $(a + p\sigma, b + q\sigma)$ implies that $(m + i\sigma, b)$ satisfies the hypothesis of the previous lemma. The result now follows in this case from the previous lemma together with the comparison of Milnor exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_n^1 H_{C_2}^{m-1+i\sigma,b}(U_n) & \longrightarrow & H_{C_2}^{m+i\sigma,b}(X \times \mathbf{E}C_2) & \longrightarrow & \lim_n H_{C_2}^{m+i\sigma,b}(U_n) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \lim_n^1 H_{C_2}^{m-1+i\sigma}(U_n(\mathbb{C})) & \to & H_{C_2}^{m+i\sigma}((X \times \mathbf{E}C_2)(\mathbb{C})) & \to & \lim_n H_{C_2}^{m+i\sigma}(U_n(\mathbb{C})) & \to & 0,
\end{array}
$$

where $U_n := X \times (\mathbb{A}(n\sigma) \setminus \{0\})$, since U_n is a smooth quasi-projective variety with free action.

To deduce the proposition for a general smooth X from the quasi-projective case, we use the fact that X is locally affine in the equivariant Nisnevich topology; see Remark 2.3. There are several ways to turn this observation into a formal argument; we proceed directly as follows. Suppose that we have a cartesian square of smooth C_2-complex varieties

$$
\begin{array}{ccc}
W & \hookrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle\phi} \\
U & \hookrightarrow & X,
\end{array}
$$

where Y is quasi-projective, ϕ is equivariant étale, U is an invariant open, and the restriction $\phi|_{Y \setminus W}$ has an equivariant section. Then if the proposition is true for U, it is also true for X. Indeed, this square leads to a distinguished equivariant Nisnevich square via standard techniques,

$$
\begin{array}{ccc}
W' & \hookrightarrow & Y' \\
\downarrow & & \downarrow \\
U & \hookrightarrow & X,
\end{array}
$$

where Y' is open in Y; in particular, it is quasi-projective. Comparing the resulting Mayer–Vietoris long exact sequences then shows that under these assumptions, the proposition holds for X.

Now we let $A \subseteq X$ be a dense invariant affine open and Y any quasi-projective equivariant Nisnevich cover of X. Let $\emptyset = Z_{n+1} \subseteq Z_n \subseteq \cdots Z_1 \subseteq Z_0 := X \setminus A$ be an equivariant splitting sequence for $Y|_{Z_0}$. Set $X_i = X \setminus Z_i$ and $Y_i = Y|_{X_i}$. The cartesian square

$$
\begin{array}{ccc}
Y_i & \hookrightarrow & Y_{i+1} \\
\downarrow & & \downarrow \\
X_i & \hookrightarrow & X_{i+1}
\end{array}
$$

satisfies the conditions of the previous paragraph, so proceeding by induction, we find that the proposition holds for each X_i. $\qquad\square$

Next we consider the isotropic part of the Bredon cohomologies.

Proposition 6.5. *Let X be a smooth complex C_2-variety and A a finite abelian group. Then*

$$
\mathrm{Re}_{\mathbb{C}} : \widetilde{H}_{C_2}^{a+p\sigma, b+q\sigma}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to \widetilde{H}_{C_2}^{a+p\sigma}(X(\mathbb{C})_+ \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C}), \underline{A})
$$

is an isomorphism if $a \leq b$ and an injection if $a = b+1$.

Proof. First, we consider the special case when X has trivial action. By the periodicities supplied by Proposition 5.7 and the corresponding ones in topological Bredon cohomology, we may assume that $p = q = 0$.

Since X has trivial action, the map $H_{C_2}^{a,b}(X, \underline{A}) \to H_{C_2}^a(X(\mathbb{C}), \underline{A})$ is naturally isomorphic to the map $H_{\mathcal{M}}^{a,b}(X, A) \to H_{\mathrm{sing}}^a(X(\mathbb{C}), A)$, by Proposition 3.15. In particular, by the Beilinson–Lichtenbaum conjecture, it is an isomorphism if $a \leq b$, and an injection if $a = b+1$. The map $H_{C_2}^{a,b}(X \times \mathbf{E}C_2, \underline{A}) \to H_{C_2}^a(X(\mathbb{C}) \times \mathbf{E}C_2(\mathbb{C}), \underline{A})$ is an isomorphism for $a \leq b$ and an injection for $a = b+1$, by the previous proposition. The proposition thus follows for X with trivial action by comparing the long exact sequences associated to the motivic isotropy cofiber sequence

$$
(6.6) \qquad X_+ \wedge \mathbf{E}C_{2+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2.
$$

We now treat the general case. The fixed point scheme X^{C_2} is smooth, so by the equivariant homotopical purity theorem [HKØ15, Theorem 7.6], we have a cofiber sequence

$$
X \setminus X^{C_2} \to X \to \mathrm{Th}(\mathcal{N}),
$$

where $\mathcal{N}$ is the normal bundle of the inclusion $X^{C_2} \subseteq X$. By Lemma 4.2 the map $X \to \mathrm{Th}(\mathcal{N})$ induces an isomorphism

$$
(6.7) \qquad \widetilde{H}_{C_2}^{\star;\star}(\mathrm{Th}(\mathcal{N}) \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \xrightarrow{\cong} \widetilde{H}_{C_2}^{\star;\star}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}),
$$

and it is similar for the topological Bredon cohomology. Note that X^{C_2} is a disjoint union $X^{C_2} = \coprod_r Z_r$ of connected smooth varieties with trivial action. We may apply Theorem 3.26 and Proposition 3.30 together with Remark 3.29 to each $\mathcal{N}|_{Z_r}$

in order to obtain the commutative square

$$\begin{array}{ccc}
\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(\mathrm{Th}(\mathcal{N}) \wedge \widetilde{\mathbf{E}}C_2) & \xrightarrow{\cong} & \bigoplus^r \widetilde{H}_{C_2}^{a+(p-2p_r)\sigma,\,b+(q-p_r)\sigma}(Z_{r\,+} \wedge \widetilde{\mathbf{E}}C_2) \\
\downarrow & & \downarrow \\
\widetilde{H}_{C_2}^{a+p\sigma}(\mathrm{Th}(\mathcal{N}(\mathbb{C})) \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C})) & \xrightarrow{\cong} & \bigoplus^r \widetilde{H}_{C_2}^{a+(p-2p_r)\sigma}(Z_r(\mathbb{C})_+ \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C})),
\end{array}$$

where $p_r = \mathrm{codim}_X(Z_r)$. The proposition holds for the right-hand map, and thus it holds for the left-hand map as well. Applying the isomorphism (6.7) yields the conclusion of the proposition. $\qquad\square$

Combining the previous two results now implies our equivariant generalization of the Beilinson–Lichtenbaum conjectures over $\mathbb{C}$.

Theorem 6.8. *Let X be a smooth complex C_2-variety, and let A be a finite abelian group. The comparison map*

$$\mathrm{Re}_{\mathbb{C}} : H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A}) \to H_{C_2}^{a+p\sigma}(X(\mathbb{C}),\underline{A})$$

is

(i) *an isomorphism if both $a + p \le b + q$ and $a \le \min\{b - q, b\}$,*
(ii) *an injection if both $a + p \le b + q + 1$ and $a \le \min\{b - q, b\} + 1$.*

Proof. This follows by comparing the long exact sequences induced by the motivic isotropy cofiber sequence $X_+ \wedge \mathbf{E}C_{2\,+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2$ together with Propositions 6.4 and 6.5. $\qquad\square$

Notice that in the case $p = q = 0$ and X complex variety with trivial $(\mathbb{Z}/2)$-action, the above theorem reduces to the usual Beilinson–Lichtenbaum conjecture for complex varieties (which we actually used in the proof). (See also (6.2).)

7. Comparing motivic and topological Bredon cohomology over $\mathbb{R}$

Let X be a smooth real variety and write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The space $X(\mathbb{C})$ has an Σ_2-action. This extends to the topological realization functor $\mathrm{Re}_{\mathbb{C},\Sigma_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}_{\Sigma_2}$; see [HO14, Proposition 4.8]. We have $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) = \mathrm{H}\underline{A}$ [HO14, Theorem 4.17], and thus a comparison map relating motivic cohomology and Bredon cohomology. By [HV12, Corollary 5.11] the Beilinson–Lichtenbaum conjecture (6.1) for real varieties can be reinterpreted as the statement that the map

$$(7.1) \quad H_{\mathcal{M}}^{a,b}(X,A) \to H_{\Sigma_2}^{a-b+b\sigma}(X(\mathbb{C}),\underline{A}) \text{ is } \begin{cases} \text{an isomorphism} & \text{if } a \le b, \\ \text{a monomorphism} & \text{if } a = b + 1. \end{cases}$$

We now consider a smooth real variety X with a C_2-action. The space of complex points $X(\mathbb{C})$ has two involutions, one coming from complex conjugation and the other coming from action on the variety X. To avoid confusing these actions, we write C_2 for the group acting algebraically on the real variety X, while we write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the Galois group, and its action is by complex conjugation. Thus $X(\mathbb{C})$ has a $(C_2 \times \Sigma_2)$-action. This extends to the topological realization functor (see Theorem A.17),

$$\mathrm{Re}_{\mathbb{C},\,\Sigma_2} : \mathrm{SH}_{C_2}(\mathbb{R}) \to \mathrm{SH}_{C_2 \times \Sigma_2}.$$

Write τ_1 (resp., τ_2) for the nontrivial element of C_2 (resp., of Σ). Write σ for the $(C_2 \times \Sigma_2)$-representation which is defined by letting τ_1 act by -1 and τ_2 by the

identity. Write ϵ for the representation which is defined by letting τ_1 act by the identity and τ_2 by -1. The four representations of the Klein group $C_2 \times \Sigma_2$ are 1, σ, ϵ, and $\sigma \otimes \epsilon$.

The effect of the topological realization functor on spheres is as follows.

Lemma 7.2. *We have* $\mathrm{Re}_{\mathbb{C},\Sigma_2}(S^1) \simeq S^1$, $\mathrm{Re}_{\mathbb{C},\Sigma_2}(S^\sigma) \simeq S^\sigma$, $\mathrm{Re}_{\mathbb{C},\Sigma_2}(S_t^1) \simeq S^\epsilon$, *and* $\mathrm{Re}_{\mathbb{C},\Sigma_2}(S_t^\sigma) \simeq S^{\sigma \otimes \epsilon}$. *Thus*

$$\mathrm{Re}_{\mathbb{C},\Sigma_2}(S^{a+p\sigma,b+q\sigma}) \simeq S^{(a-b)+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}.$$

Proof. The first relation is obvious. The second follows from the cofiber sequence $C_{2+} \to S^0 \to S^\sigma$. We have that $S_t^1(\mathbb{C})$ and $S_t^\sigma(\mathbb{C})$ are equivariantly homotopic to the unit circle in $\mathbb{C}^*$. In the first case $\tau_1(z) = z$ and $\tau_2(z) = \overline{z}$, and in the second case $\tau_1(z) = 1/z$ and $\tau_2(z) = \overline{z}$. The displayed equalities follow immediately from these formulae. □

Since $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in $\mathrm{SH}_{C_2\times\Sigma_2}$ (see Theorem A.29), there exists a comparison map

$$(7.3) \qquad \mathrm{Re}_{\mathbb{C},\Sigma_2} : H_{C_2}^{a+p\sigma,b+q\sigma}(X,\underline{A}) \to H_{C_2\times\Sigma_2}^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}(X(\mathbb{C}),\underline{A}).$$

Observe that $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbb{A}(\sigma)) = \sigma + \sigma \otimes \epsilon$. Applying $\mathrm{Re}_{\mathbb{C},\Sigma_2}$ to the motivic isotropy cofiber sequence (2.7) yields the cofiber sequence of $(C_2 \times \Sigma_2)$-spaces,

$$\mathrm{E}_{\Sigma_2}C_{2+} \to S^0 \to \widetilde{\mathrm{E}}_{\Sigma_2}C_2.$$

Here, $\mathrm{E}_{\Sigma_2}C_2$ is the Σ_2-equivariant universal C_2-space; see [May96, VII.1] or [HHR09, Definition B.108]. We have $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 = \mathrm{colim}_n\, S^{n\sigma\otimes\epsilon} \wedge S^{n\sigma}$. For any representation V, the cyclic permutation on S^{3V} is equivariantly homotopic to the identity (e.g., an argument as in Lemma 2.10 works topologically), so we conclude the following.

Proposition 7.4. *The unit maps* $S^0 \to S^\sigma$, $S^0 \to S^{\sigma\otimes\epsilon}$ *induce* $(C_2 \times \Sigma_2)$-*equivariant homotopy equivalences* $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 \simeq \widetilde{\mathrm{E}}_{\Sigma_2}C_2 \wedge S^\sigma$ *and* $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 \simeq \widetilde{\mathrm{E}}_{\Sigma_2}C_2 \wedge S^{\sigma\otimes\epsilon}$. *In particular,* $\widetilde{H}_{C_2\times\Sigma_2}^\star(\mathsf{E} \wedge \widetilde{\mathrm{E}}_{\Sigma_2}C_2,\underline{A})$ *is* σ-*periodic as well as* $\sigma \otimes \epsilon$-*periodic for any* $(C_2 \times \Sigma_2)$-*equivariant spectrum* E.

Since (7.3) is a ring map, Theorem 5.4 implies that there is an invertible element $v \in H_{C_2\times\Sigma_2}^\star(\mathrm{E}_{\Sigma_2}C_2,\underline{\mathbb{Z}/2})$; the degree of v is $\sigma - 1 + \sigma \otimes \epsilon - \epsilon$. This immediately implies the following.

Proposition 7.5. *Multiplication by* $v^{-q} \in H_{C_2\times\Sigma_2}^{q-q\sigma+q\epsilon-q\sigma\otimes\epsilon}(\mathrm{E}_{\Sigma_2}C_2,\underline{\mathbb{Z}/2})$ *induces a natural isomorphism*

$$\widetilde{H}_{C_2\times\Sigma_2}^{a+p\sigma+c\epsilon+q\sigma\otimes\epsilon}(\mathsf{E} \wedge (\mathrm{E}_{\Sigma_2}C_2)_+,\underline{\mathbb{Z}/2}) \cong \widetilde{H}_{C_2\times\Sigma_2}^{(a+q)+(p-q)\sigma+(c+q)\epsilon}(\mathsf{E} \wedge (\mathrm{E}_{\Sigma_2}C_2)_+,\underline{\mathbb{Z}/2})$$

of $H_{C_2\times\Sigma_2}^\star(\mathrm{E}_{C_2}\Sigma_2,\underline{\mathbb{Z}/2})$-*modules for any* $(C_2 \times \Sigma_2)$-*spectrum* E.

We proceed, as in the previous section, toward our equivariant version of the Beilinson–Lichtenbaum conjecture over the reals by establishing the following case.

Lemma 7.6. *Let* U *be a smooth quasi-projective real* C_2-*variety with free action. Then*

$$\mathrm{Re}_{\mathbb{C}} : H_{C_2}^{m+i\sigma,n}(U,\underline{\mathbb{Z}/2}) \to H_{C_2\times\Sigma_2}^{m-n+i\sigma+n\epsilon}(U(\mathbb{C}),\underline{\mathbb{Z}/2}),$$

is an isomorphism if $m \leq n$ *and* $m+i \leq n$. *It is a monomorphism if* $m \leq n+1$ *and* $m+i \leq n+1$.

Proof. If Y is a $(C_2 \times \Sigma_2)$-space, then we have $H^{s+t\epsilon}_{C_2 \times \Sigma_2}(Y, \underline{A}) \cong H^{s+t\sigma}_{\Sigma_2}(Y/C_2, \underline{A})$. Moreover, if $Y = X(\mathbb{C})$, where X is a real variety with free C_2-action, arguing as in Proposition 3.16, we have a commutative square

$$
\begin{array}{ccc}
H^{m,n}_{C_2}(X, \underline{A}) & \xrightarrow{\cong} & H^{m,n}_{\mathcal{M}}(X/C_2, A) \\
{\scriptstyle \mathrm{Re}_{\mathbb{C}, \Sigma_2}} \downarrow & & \downarrow {\scriptstyle \mathrm{Re}_{\mathbb{C}, \Sigma_2}} \\
H^{m-n+n\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C}), \underline{A}) & \xrightarrow{\cong} & H^{m-n+n\sigma}_{\Sigma_2}(X(\mathbb{C})/C_2, \underline{A}).
\end{array}
$$

When $i = 0$, the result thus holds by (7.1). The result follows in general using induction and considering the comparison of exact sequences obtained from (2.4),

$$
\begin{array}{ccccccc}
\longrightarrow & H^{m+(j-1)\sigma,n}_{C_2}(U) & \longrightarrow & H^{m+j\sigma,n}_{C_2}(U) & \longrightarrow & H^{m+j,n}_{\mathcal{M}}(U) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\rightarrow & H^{m-n+n\epsilon+(j-1)\sigma}_{C_2 \times \Sigma_2}(U(\mathbb{C})) & \rightarrow & H^{m-n+n\epsilon+j\sigma}_{C_2 \times \Sigma_2}(U(\mathbb{C})) & \rightarrow & H^{m+j-n+n\sigma}_{\Sigma_2}(U(\mathbb{C})) & \rightarrow .
\end{array}
$$

$\square$

Proposition 7.7. *Let X be a smooth real variety with C_2-action, and let A be a finite abelian group. Then the map*

$$
H^{a+p\sigma,b+q\sigma}_{C_2}(X \times \mathbf{E}C_2, \underline{A}) \to H^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C}) \times \mathrm{E}_{\Sigma_2} C_2, \underline{A})
$$

is

    (i) *an isomorphism if $a \le b - q$ and $a + p \le b + q$,*
    (ii) *a monomorphism if $a \le b - q + 1$ and $a + p \le b + q + 1$.*

Proof. It suffices to consider the case of a quasi-projective X by the same argument as in the second half of Proposition 6.4. It also suffices to consider the case $A = \mathbb{Z}/2$ by the same argument as in Proposition 6.4, using an obvious variant of Proposition 3.18.

Using Corollary 5.5 and Proposition 7.5, we can replace $(a + p\sigma, b + q\sigma)$ by $(a + 2q + (p-2q)\sigma, b+q)$ and $a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon$ with $(a-b-q)+(p-2q)\sigma+(b+q)\epsilon$. For simplicity we reindex, replacing $a + 2q + (p - q)\sigma$ with $m + i\sigma$ and $b + q$ with b. The hypothesis on $(a + p\sigma, b + q\sigma)$ implies that $(m + i\sigma, b)$ satisfies the hypothesis of Lemma 7.6.

Considering the comparison of Milnor exact sequences, as in Proposition 6.4, we see from Lemma 7.6 that

$$
(7.8) \qquad\qquad H^{m+i\sigma,b}_{C_2}(U_n, \mathbb{Z}/2) \to H^{m-b+i\sigma+b\epsilon}_{C_2 \times C_2}(U_n(\mathbb{C}), \mathbb{Z}/2)
$$

is an isomorphism for $m + i \le b$ and $m \le b$ and a monomorphism for $m + i = b + 1$ and $m \le b + 1$. Here, $U_n := X \times (\mathbb{A}(n\sigma) \setminus \{0\})$. Replacing back m with $a + 2q$ and i for $p - 2q$ and b for $b + q$, we get the result. $\square$

We also have the following.

Proposition 7.9. *Let X be a smooth real variety with C_2-action, and let A be a finite abelian group. The map*

$$
\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to \widetilde{H}^{a-b+b\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C})_+ \wedge \widetilde{\mathrm{E}}_{\Sigma_2} C_2, \underline{A})
$$

is an isomorphism for $a \le b$ and a monomorphism for $a = b + 1$.

Proof. In the case of a free C_2-action on X the above map is an isomorphism for all of the indexes because the groups are 0. Consider the comparison of long exact sequences in cohomology associated to the motivic isotropy cofiber sequence

$$X_+ \wedge \mathbf{E}C_2 \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2.$$

Suppose that X has trivial action. Using Propositions 5.7 and 7.4, we may assume that $p = 0$ and $q = 0$. Consider the map

$$\widetilde{H}^{a,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to \widetilde{H}^{a-b+b\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C})_+ \wedge \widetilde{\mathbf{E}}_{\Sigma_2}C_2, \underline{A}).$$

Because X has a trivial action, we have that the middle map in the long exact sequence given by the motivic isotropy cofiber sequence is identified with

$$H^{a,b}_{\mathcal{M}}(X, \underline{A}) \to H^{a-b,b}_{C_2}(X(\mathbb{C}), \underline{A}).$$

This is an isomorphism if $a \le b$ and a monomorphism if $a = b + 1$ by the Beilinson–Lichtenbaum conjecture (7.1).

The other significant map in the diagram is

$$\widetilde{H}^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}) \to \widetilde{H}^{a-b+b\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C}) \times E_{\Sigma_2}C_2, \underline{A}),$$

which according to Proposition 7.7 is an isomorphism for any $a \le b$ and a monomorphism for any $a = b + 1$. This confirms the isomorphism in the statement when the action is trivial.

The isomorphism in the general case follows as in Proposition 6.5 by considering the equivariant cofiber sequence for the real variety X,

$$X \setminus X^{C_2} \to X \to \mathrm{Th}(\mathcal{N}),$$

where $\mathcal{N}$ is the normal bundle of the inclusion $X^{C_2} \subset X$. $\square$

Theorem 7.10. *Let X be a smooth real variety with C_2-action, and let A be an abelian group. Then the map*

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) \to H^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}_{C_2 \times \Sigma_2}(X(\mathbb{C}), \underline{A})$$

is

(i) *an isomorphism if both $a + p \le b + q$ and $a \le \min\{b - q, b\}$,*
(ii) *an injection if both $a + p \le b + q + 1$ and $a \le \min\{b - q, b\} + 1$.*

Proof. This follows by comparing the long exact sequences induced by the motivic isotropy cofiber sequence $X_+ \wedge \mathbf{E}C_{2+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2$ together with Propositions 7.7 and 7.9. $\square$

When $p = q = 0$ and X is a smooth real variety with trivial C_2-action, the above theorem reduces to the version of the Beilinson–Lichtenbaum conjecture for a real variety (see (7.1)) established in [HV12]. (Of course, this was used to prove the theorem.)

APPENDIX A. EQUIVARIANT MOTIVIC HOMOTOPY THEORY

Unstable equivariant motivic homotopy theory was first considered by Voevodsky [Del09]. A stable version was considered by Hu, Kriz, and Ormsby [HKO11] as part of their work on the completion problem in Hermitian K-theory. General foundations and model structures are constructed in [HKØ15], and representability results for equivariant algebraic K-theory are also established. A general framework for stable equivariant motivic homotopy theory emphasizing the six functor

formalism is introduced in [Hoy15]. Alternate versions of a homotopy theory for smooth G-schemes are studied in [Her13], [CJ14]; however, theories of interest, such as equivariant algebraic K-theory, are not representable in the homotopy categories constructed there.

The main results in this paper rely on a Betti or topological realization functor, which is constructed in this appendix, relating equivariant motivic homotopy and classical equivariant homotopy theory. We begin by giving a brief but self-contained construction of a model for the unstable and stable equivariant motivic homotopy categories, which is geared toward the construction of the Betti realization functor. We also record the details of the construction of several comparison functors between equivariant and nonequivariant homotopy categories in this setting. Finally, in the last sections of this appendix we verify that the topological realization of the Bredon motivic cohomology spectrum is the topological Bredon cohomology spectrum.

A.1. **Unstable equivariant motivic homotopy theory.** To keep exposition streamlined, we restrict attention to the case of a finite group G over a field k. Furthermore we always assume that the order of G is invertible in k. A *motivic G-space* over k is defined to be a presheaf of simplicial sets on GSm/k. We write $GSpc(k)$ and $GSpc_\bullet(k)$, respectively, for the categories of motivic G-spaces and pointed motivic G-spaces over k. We are primarily interested in the stable equivariant motivic homotopy category, so we only treat the unstable model structure for pointed motivic G-spaces. The category $GSpc_\bullet(k)$ is a symmetric monoidal category via the pointwise smash product $(F \wedge G)(X) := F(X) \wedge G(X)$ of motivic G-spaces.

We make use of an equivariant version of the "closed flasque model structure" introduced in [PPR09], a variation on the flasque model structure of [Isa05]. This model structure is particularly well suited for topological realization.

Let $\mathcal{Z} = \{Z_i \hookrightarrow X\}$ be a finite collection of closed immersions ($\emptyset \to X$ is allowed) in GSm/k. Define

$$\bigcup \mathcal{Z} := \mathrm{coeq}(\coprod_{r,r'} Z_r \times_X Z_{r'} \rightrightarrows \coprod_r Z_r),$$

where the coequalizer is computed in $Spc(k)$ (i.e., $\bigcup Z$ is the categorical union of the $Z_r \subseteq X$). Write $i_{\mathcal{Z}} : \bigcup \mathcal{Z} \to X$ for the resulting monomorphism.

The pushout product $i \square j$ of two maps $i : A \to X$ and $j : B \to Y$ is defined to be the map $A \wedge Y \coprod_{A \wedge B} X \wedge B \to X \wedge Y$. Define two sets of maps:

(1) I^c is the set of maps of the form $(i_{\mathcal{Z}})_+ \square g_+$, where $\mathcal{Z}$ is a finite set of closed equivariant immersions in GSm/k and $g : \partial \Delta^n \to \Delta^n$, $n \geq 0$ is the standard inclusion.

(2) J^c is the set of maps of the form $(i_{\mathcal{Z}})_+ \square g_+$, where $\mathcal{Z}$ is a finite set of closed equivariant immersions in GSm/k and $g : \Lambda^{n,k} \to \Delta^n$, $n \geq 1$, $0 \leq i \leq n$ is the standard inclusion.

Definition A.1. Let $f : F \to G$ be a map of pointed motivic G-spaces.

(1) Say that f is a *schemewise weak equivalence* provided that $f : F(U) \to G(U)$ is a weak equivalence of simplicial sets for all U in GSm/k.

(2) Say that f is a *closed flasque fibration* if it has the right lifting property with respect to J^c.

(3) Say that f is a *closed flasque cofibration* if it has the left lifting property with respect to acyclic closed schemewise fibrations.

The schemewise weak equivalences, closed flasque cofibrations, and closed flasque fibrations define the *global closed flasque model structure* on $G\mathrm{Spc}_\bullet(k)$; see, e.g., [HKØ15, Theorem 3.8].

Proposition A.2. *The global closed flasque model structure is a simplicial, proper, cellular, monoidal model structure on $G\mathrm{Spc}_\bullet(k)$. The sets I^c and J^c are, respectively, generating cofibrations and acyclic cofibrations.*

For a distinguished equivariant Nisnevich square Q as in (2.2), write P_Q for the pushout of $A \leftarrow B \rightarrow Y$ in $G\mathrm{Spc}(k)$.

Definition A.3.

(1) The *closed flasque (equivariant Nisnevich) local model structure* on $G\mathrm{Spc}_\bullet(k)$ is the left Bousfield localization of the global closed flasque model structure at the set of maps $\{(P_Q)_+ \rightarrow X_+\}$, where Q ranges over the set of distinguished equivariant Nisnevich squares. The associated homotopy category is denoted $\mathrm{H}_{G,\bullet}^{\mathrm{Nis}}(k)$.

(2) The *closed flasque motivic model structure* on $G\mathrm{Spc}_\bullet(k)$ is the left Bousfield localization of the closed flasque local model structure at the set of projections $(X \times \mathbb{A}^1)_+ \rightarrow X_+$ for all X in $G\mathrm{Sm}/k$. For brevity we refer to the weak equivalences and fibrations of this model structure as *motivic weak equivalences* and *motivic fibrations*. The associated homotopy category is the unstable equivariant motivic homotopy category and is denoted $\mathrm{H}_{G,\bullet}(k)$.

Theorem A.4. *The closed flasque local and closed flasque motivic model structures are simplicial, proper, cellular, monoidal model structures on $G\mathrm{Spc}_\bullet(k)$. The identity functor from the projective model structures to the closed flasque model structures is a left Quillen equivalence. In particular, the homotopy category $\mathrm{H}_{G,\bullet}(k)$ coincides with the one defined in [HKØ15] (and hence also with the one defined by Voevodsky in [Del09]).*

Proof. The global closed flasque model structure is left proper (in fact, proper) as well as cellular, so by [Hir03, Theorem 4.1.1] the left Bousfield localization of this model structure at a set of maps exists (and is again left proper and cellular). This implies that the above model structures exist and are simplicial, left proper, and cellular. Right properness follows from the fact that the local projective and motivic model structures are proper; see [HKØ15, Theorem 4.3].

Every projective cofibration is a closed flasque cofibration, and the weak equivalences in the global model structures coincide so that the identity is a Quillen equivalence between the global model structures, and hence a Quillen equivalence on the localized model structures.

To show that these are symmetric monoidal model structures, we need to check that if f and g are cofibrations, then the pushout product of $f \,\square\, g$ is also a cofibration, and that it is a weak equivalence if either f or g is an acyclic cofibration. It suffices to assume that f and g are generating cofibrations. Note that taking smash products preserves motivic weak equivalences since every object is injective cofibrant. The pushout product of maps of the form $(i_\mathscr{Z})_+ \rightarrow X_+$ are again of the same form. The pushout product axiom in simplicial sets therefore implies that

the pushout product of closed flasque cofibrations is again a cofibration. If one of f or g is a weak equivalence, so is the pushout product because the smash product preserves equivariant motivic equivalences. □

Remark A.5. It follows from the definitions that F is motivic fibrant on GSm/k if and only if the following three conditions hold.

(1) F is fibrant in the global closed flasque model structure.
(2) F is *equivariant Nisnevich excisive*, i.e., for any distinguished square in GSm/k the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{array}$$

is homotopy cartesian.
(3) F is $\mathbb{A}^1$-*invariant*; i.e., $F(X \times \mathbb{A}^1) \to F(X)$ is a weak equivalence for all X.

A.2. **Hypercohomology and motivic homotopy.** Let $\mathcal{F}$ be a presheaf of simplicial abelian groups on GSm/k. Write $\mathcal{NF}$ for the associated presheaf of normalized cochain complexes. Forgetting the group structure, we view $\mathcal{F}$ as a pointed motivic G-space, with 0 as basepoint. There is a natural isomorphism (see, e.g., [MV99, Proposition 2.1.26]) of homotopy classes of maps in $\mathrm{H}_{G,\bullet}^{\mathrm{Nis}}(k)$ and sheaf cohomology groups

$$[S^n \wedge X_+, \mathcal{F}]_{\mathrm{H}_{G,\bullet}^{\mathrm{Nis}}(k)} \cong H_{G\mathrm{Nis}}^{-n}(X, (\mathcal{NF})_{G\mathrm{Nis}}).$$

Moreover, if $\mathcal{F}$ is equivariant Nisnevich excisive, then both of these groups agree with the homotopy group $\pi_n\mathcal{F}(X)$.

In [HVØ15] we introduced an equivariant generalization of Voevodsky's machinery of presheaves with transfers. A presheaf with equivariant transfers is an additive presheaf on the category $G\mathrm{Cor}_k$, which has the same objects as GSm/k and whose maps are given by $\mathrm{Cor}_k(X,Y)^G$. From the viewpoint of motivic homotopy theory, a fundamental feature of the transfer structure is that it allows one to construct a "small" motivic fibrant replacement.

Write $\mathcal{L}_{G\mathrm{Nis}}$ for a fibrant replacement functor in the equivariant Nisnevich local model structure of Theorem A.4. If F is a presheaf of abelian groups, recall that $C_*F(X)$ is the simplicial abelian group $F(X \times \Delta_k^\bullet)$. By the exponent $\exp(G)$ of a group G we mean the least common multiple of the orders of elements of the group.

Theorem A.6. *Suppose that G is an abelian group, $|G|$ is coprime to $\mathrm{char}(k)$, and k contains a primitive $\exp(G)$th-root of unity. Let $\mathcal{F}$ be a presheaf with equivariant transfers. Then $\mathcal{F} \to \mathcal{L}_{G\mathrm{Nis}}C_*\mathcal{F}$ is a motivic fibrant replacement functor. In particular, we have a natural isomorphism*

$$[S^n \wedge X_+, \mathcal{F}]_{\mathrm{H}_{G,\bullet}(k)} \cong H_{G\mathrm{Nis}}^{-n}(X, \mathcal{N}C_*\mathcal{F}).$$

Proof. By definition $\mathcal{L}C_*\mathcal{F}$ is closed flasque fibrant and equivariant Nisnevich excisive. It remains to see that $\mathcal{L}_{G\mathrm{Nis}}C_*\mathcal{F}$ is $\mathbb{A}^1$-invariant. We have natural isomorphisms

$$\pi_n\mathcal{L}_{G\mathrm{Nis}}C_*\mathcal{F}(X) \cong [S^n \wedge X_+, C_*\mathcal{F}]_{\mathrm{H}_{G,\bullet}^{\mathrm{Nis}}(k)} \cong H_{G\mathrm{Nis}}^{-n}(X, \mathcal{N}C_*\mathcal{F}).$$

The $\mathbb{A}^1$-invariance of this presheaf follows from [HVØ15, Theorem 8.12] together with an application of the spectral sequence

$$H^p_{G\mathrm{Nis}}(X, \mathcal{H}^q) \Longrightarrow H^{p+q}_{G\mathrm{Nis}}(X, \mathcal{N}C_*\mathcal{F}),$$

where $\mathcal{H}^q$ is the sheafification of the presheaf $U \mapsto H^q\mathcal{N}C_*\mathcal{F}(U)$. $\qquad\square$

A.3. Adjunctions of motivic spaces.

Next we record some motivic analogues of familiar adjunctions in topology relating G-spaces and ordinary spaces. We begin with the adjunctions

$$i := (-)^{\mathrm{triv}} : \mathrm{Sm}/k \rightleftarrows G\mathrm{Sm}/k : (-)^G =: \phi,$$
$$\epsilon := G \times - : \mathrm{Sm}/k \rightleftarrows G\mathrm{Sm}/k : (-)^e =: \rho.$$

Here, $(-)^e$ simply forgets the action, the underlying scheme of X^{triv} is X and it is considered a G-scheme with trivial action, and X^G is the fixed point scheme, which is smooth since $|G|$ is invertible in k; see, e.g., [Edi92, Proposition 3.4]). We obtain several adjoint pairs of functors on based motivic spaces,

$$i^* : \mathrm{Spc}_\bullet(k) \rightleftarrows G\mathrm{Spc}_\bullet(k) : i_*,$$
$$i_* = \phi^* : G\mathrm{Spc}_\bullet(k) \rightleftarrows \mathrm{Spc}_\bullet(k) : \phi_*,$$
$$\epsilon^* : \mathrm{Spc}_\bullet(k) \rightleftarrows G\mathrm{Spc}_\bullet(k) : \epsilon_*,$$
$$\epsilon_* = \rho^* : G\mathrm{Spc}_\bullet(k) \rightleftarrows \mathrm{Spc}_\bullet(k) : \rho_*.$$

Proposition A.7. *Each of the pairs (i^*, i_*), (ϕ^*, ϕ_*), (ϵ^*, ϵ_*), and (ρ^*, ρ_*) are Quillen adjoints on the closed flasque motivic model structures.*

Proof. Each of the functors i, ϕ, ϵ, and ρ commute with fiber products and preserve closed immersions. It follows that i^*, ϕ^*, ϵ^*, and ρ^* preserve generating cofibrations and generating acyclic cofibrations for the global closed flasque model structure. These are thus left Quillen adjoints on the global closed flasque model structure.

Both i and ϵ send distinguished Nisnevich squares to distinguished equivariant Nisnevich squares. The functor ρ sends distinguished equivariant Nisnevich squares to distinguished Nisnevich squares, and by [Her13, Corollary 3.2.6] ϕ does as well. Moreover, i, ϕ, ϵ, and ρ all send maps of the form $X \times \mathbb{A}^1 \to X$ to maps of the same form. It follows by the universal property of Bousfield localization that the functors i^*, ϕ^*, ϵ^*, and ρ^* are also left Quillen functors on the closed flasque motivic model structure. $\qquad\square$

We may now define motivic analogues of the classical change of group functors.

Definition A.8. Define the

(1) *trivial action* functor by $(-)^{\mathrm{triv}} := \mathbb{L}i^*$,
(2) *G-fixed points* functor by $(-)^G := \mathbb{R}i_*$,
(3) *induced motivic G-space* functor by $G_+ \wedge - := \mathbb{L}\epsilon^*$,
(4) *coinduced motivic G-space* functor by $F(G_+, -) := \mathbb{R}\rho_*$,
(5) *underlying motivic space* functor by $(-)^e := \mathbb{R}\epsilon_*$.

Note that $i_* = \phi^*$ and $\epsilon_* = \rho^*$ are both Quillen left and Quillen right adjoints. We thus have natural motivic equivalences $\mathbb{R}i_* = (-)^G \simeq \mathbb{L}i_*$ and $\mathbb{R}\epsilon_* = (-)^e \simeq \mathbb{L}\epsilon_*$.

In summary, we have adjunctions

$$(-)^{\mathrm{triv}} : \mathrm{H}_{\bullet}(k) \rightleftarrows \mathrm{H}_{G,\bullet}(k) : (-)^G,$$
$$G_+ \wedge - : \mathrm{H}_{\bullet}(k) \rightleftarrows \mathrm{H}_{G,\bullet}(k) : (-)^e,$$
$$(-)^e : \mathrm{H}_{G,\bullet}(k) \rightleftarrows \mathrm{H}_{\bullet}(k) : F(G_+, -).$$

We note that $(G_+ \wedge X)^e \simeq \coprod_{|G|} X$. Indeed, we have $\epsilon_* \epsilon^*(X) = \coprod_{|G|} X$ since this formula holds for smooth schemes and both sides commute with colimits. If X^{cof} is a cofibrant replacement for X, then we have the weak equivalences $(G_+ \wedge X)^e = \mathbb{R}\epsilon_* \mathbb{L}\epsilon^*(X) \simeq \mathbb{L}\epsilon_* \mathbb{L}\epsilon^*(X) \simeq \epsilon_* \epsilon^*(X^{\mathrm{cof}}) = \coprod_{|G|} X^{\mathrm{cof}}$, which establishes the claim. The inclusion $X^e \to (G_+ \wedge X)^e$ at the summand corresponding to $e \in G$ induces the map $i : G_+ \wedge X^e \to G_+ \wedge X$.

Lemma A.9. *The map* $i : G_+ \wedge X^e \xrightarrow{\cong} G_+ \wedge X$ *is an isomorphism in* $\mathrm{H}_{G,\bullet}(k)$.

Proof. We have that $G_+ \wedge (-)^e \cong \mathbb{L}\epsilon^* \mathbb{L}\epsilon_*(-) = \mathbb{L}\epsilon^* \mathbb{L}\rho^*(-)$. Note that these commute with homotopy colimits, as does $G_+ \wedge -$. It thus suffices to assume that X is representable. If Y is a G-scheme, then the map $G \times Y^e \to G \times Y$ given by $(g, y) \mapsto (g, gy)$ yields the desired equivariant isomorphism. $\square$

A.4. Stable equivariant motivic homotopy theory.

Stable model structures on motivic spectra are constructed in [Jar00], yielding the stable motivic homotopy category. A stable equivariant motivic homotopy category was first constructed in [HKO11]. In this paper we will work with model structures for equivariant motivic spectra by using Hovey's machinery [Hov01, Section 8].

Let V be a representation. The associated motivic representation sphere is the quotient $T^V := \mathbb{P}(V \oplus 1)/\mathbb{P}(V)$, where the quotient is taken in the category of presheaves. It is naturally an object of $G\mathrm{Spc}_{\bullet}(k)$. We also write $T^{nV} := (T^V)^{\wedge n}$. Since $\mathbb{P}(V) \to \mathbb{P}(V \oplus 1)$ is a closed flasque cofibration, T^V is a closed flasque cofibrant motivic G-space. We write ρ_G for the regular representation.

Let K be a pointed motivic G-space. We write $\mathbf{hom}(K, -)$ for the right adjoint of $K \wedge -$. In particular, $\Sigma_{T^{\rho_G}} F = T^{\rho_G} \wedge F$ and $\Omega_{T^{\rho_G}} F = \mathbf{hom}(T^{\rho_G}, F)$.

Definition A.10. A symmetric K-spectrum is a sequence $E = (E_0, E_1, \ldots)$ consisting of objects E_n of $G\mathrm{Spc}_{\bullet}(k)$ together with the following data:

(i) a Σ_n-action on E_n,
(ii) Σ_n-equivariant maps $\sigma_n : E_n \wedge K \to E_{n+1}$,

where the structure maps σ_n are required to satisfy the condition that the composites $E_n \wedge K^{\wedge p} \to E_{n+p}$ are $(\Sigma_n \times \Sigma_p)$-equivariant for all $n, p \geq 0$.

A map $E \to F$ of symmetric K-spectra is a collection of Σ_n-equivariant maps $E_n \to F_n$ which are compatible with the structure maps. Write $G\mathrm{Spt}_K^{\Sigma}(k)$ for the category of symmetric K-spectra in $G\mathrm{Spc}_{\bullet}(k)$. Now suppose that K is a closed flasque cofibrant based motivic G-space. The category $G\mathrm{Spc}_{\bullet}(k)$ equipped with the closed flasque motivic model structure is a left proper, cellular, simplicial, symmetric monoidal model category, so we can use [Hov01, Definition 8.7] to define a stable model structure on the category of symmetric K-spectra. It is again a symmetric monoidal model category by [Hov01, Theorem 8.11]. If K' is another closed flasque cofibrant based motivic G-space, we write $G\mathrm{Spt}_{K,K'}^{\Sigma}(k)$ for the category of symmetric (K, K')-bispectra (i.e., symmetric K'-spectra in $G\mathrm{Spt}_K^{\Sigma}(k)$). This is again equipped with the stable model structure of [Hov01, Section 8].

A symmetric K-spectrum $\mathcal{E}$ is fibrant if and only if it is an Ω_K-spectrum, i.e., if E_i is motivic fibrant and $\mathcal{E}_i \to \Omega_K E_{i+1}$ is a motivic weak equivalence for all i. By Theorem 3.4 the Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$ is an $\Omega_{T^{\rho C_2}}$-spectrum.

The adjunctions of Section A.3 stabilize.

Proposition A.11. *The adjoint pairs of Section* A.3 *induce adjoint pairs*

$$(-)^{\mathrm{triv}} : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_G(k) : (-)^G,$$
$$G_+ \wedge - : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_G(k) : (-)^e,$$
$$(-)^e : \mathrm{SH}_G(k) \rightleftarrows \mathrm{SH}(k) : F(G_+, -).$$

The functors $(-)^{\mathrm{triv}}$, *and* $(-)^e$ *are symmetric monoidal and* $(-)^G$ *is lax monoidal.*

Proof. First, we note that we have an equivalence, as tensor triangulated categories

$$\mathrm{SH}_G(k) \simeq \mathrm{Ho}(G\mathrm{Spt}^{\Sigma}_{T, T^{\rho G}}(k)),$$

where the category $G\mathrm{Spt}^{\Sigma}_{T, T^{\rho G}}(k)$ of symmetric $(T, T^{\rho G})$-bispectra is equipped with the stable model structure as above. Indeed, the model category $G\mathrm{Spt}^{\Sigma}_{T, T^{\rho G}}(k)$ is isomorphic to $G\mathrm{Spt}^{\Sigma}_{T^{\rho G}, T}(k)$, and the endofunctor $- \wedge T$ on $G\mathrm{Spt}^{\Sigma}_{T^{\rho G}}(k)$ is a Quillen equivalence since $T \wedge T^{\tilde{\rho}_G} = T^{\rho G}$ (where $\tilde{\rho}_G$ is the reduced regular representation). By [Hov01, Theorem 9.1] the stabilization functor (which is symmetric monoidal) $G\mathrm{Spt}^{\Sigma}_{T^{\rho G}}(k) \to G\mathrm{Spt}^{\Sigma}_{T^{\rho G}, T}(k)$ is therefore a left Quillen equivalence.

The monoidal functor $i^* : \mathrm{Spc}_\bullet(k) \to G\mathrm{Spc}_\bullet(k)$ makes $G\mathrm{Spc}_\bullet(k)$ into a $\mathrm{Spc}_\bullet(k)$-model category in the sense of [Hov99, Definition 4.2.18], and the Quillen functor i^* is a $\mathrm{Spc}_\bullet(k)$-module functor (see [Hov99, Definition 4.1.7]). By [Hov01, Theorem 9.3] the Quillen pair (i^*, i_*) extends to a Quillen pair on spectra. Combined with the stablization adjunction, we have the composition of Quillen functors

$$\mathrm{Spt}^{\Sigma}_T(k) \underset{i_*}{\overset{i^*}{\rightleftarrows}} G\mathrm{Spt}^{\Sigma}_T(k) \underset{\Omega^{\infty}_{T^{\rho G}}}{\overset{\Sigma^{\infty}_{T^{\rho G}}}{\rightleftarrows}} G\mathrm{Spt}^{\Sigma}_{T, T^{\rho G}}(k).$$

The pair $((-)^{\mathrm{triv}}, (-)^G)$ is the induced adjunction on homotopy categories, i.e., $(-)^{\mathrm{triv}} = \mathbb{L}(\Sigma^{\infty}_{T^{\rho G}} \circ i^*)$ and $(-)^G = \mathbb{R}(i_* \circ \Omega^{\infty}_{T^{\rho G}})$.

The other two adjunctions are attained as follows. We use $\mathrm{Spt}^{\Sigma}_{T^{|G|}}(k)$ as our model for $\mathrm{SH}(k)$. Note that for X in $\mathrm{Spc}_\bullet(k)$ and Y in $G\mathrm{Spc}_\bullet(k)$ we have a natural isomorphism $\epsilon^*(X \wedge \rho^*(Y)) \cong \epsilon^*(X) \wedge Y$. Indeed, this holds when X is in $\mathrm{Sm}k$ and Y is in $G\mathrm{Sm}/k$ and both sides commute with colimits. Since $\rho^*(T^{\rho G}) = T^{|G|}$, we have $\epsilon^*(X \wedge T^{|G|}) = \epsilon^*(X) \wedge T^{\rho G}$. We also have $\rho^*(Y \wedge T^{\rho G}) = \rho^*(Y) \wedge T^{|G|}$. By [HO14, Lemma 4.1], the adjunctions (ϵ^*, ϵ_*) and (ρ^*, ρ_*) extend to Quillen adjunctions on stable model structures

$$\epsilon^* : \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k) \rightleftarrows G\mathrm{Spt}^{\Sigma}_{T^{\rho G}}(k) : \epsilon_*,$$
$$\rho^* : G\mathrm{Spt}^{\Sigma}_{T^{\rho G}}(k) \rightleftarrows \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k) : \rho_*.$$

We have $\epsilon_* = \rho^* : G\mathrm{Spc}_\bullet(k) \to \mathrm{Spc}_\bullet(k)$. The prolongations to functors on spectra are levelwise isomorphic, and it is straightforward to verify that they are in fact isomorphic as spectra. In other words, $\epsilon_* = \rho^* : G\mathrm{Spt}^{\Sigma}_{T^{\rho G}}(k) \to \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k)$. In

particular, $\mathbb{R}\epsilon_* \simeq \mathbb{L}\rho^*$. The adjunctions $(G_+ \wedge -, (-)^e)$ and $((-)^e, F(G_+, -))$ on the homotopy categories are thus obtained as $G_+ \wedge - = \mathbb{L}\epsilon^*$, $F(G_+, -) = \mathbb{R}\rho_*$, and $(-)^e = \mathbb{L}\rho^*$.

Since i^* and ρ^* are symmetric monoidal functors, so are $(-)^{\mathrm{triv}}$ and $(-)^e$. Since $(-)^G$ is right adjoint to a symmetric monoidal functor, it is lax monoidal. $\qquad\square$

A.5. **Topological realization over $\mathbb{C}$ (unstable).** If X is a complex variety, we consider $X(\mathbb{C})$ a topological space with the Euclidean topology. If X has a G-action, then $X(\mathbb{C})$ also has a G-action and $X \mapsto X(\mathbb{C})$ defines a functor $G\mathrm{Sm}/\mathbb{C} \to G\mathrm{Top}$. The topological realization functor $\mathrm{Re}_{\mathbb{C}} : G\mathrm{Spc}_\bullet(\mathbb{C}) \to G\mathrm{Top}_\bullet$ is defined by the Kan extension

$$\mathrm{Re}_{\mathbb{C}}(F) = \operatorname*{colim}_{(X \times \Delta^n)_+ \to F} (X(\mathbb{C}) \times \Delta^n_{\mathrm{top}})_+,$$

where Δ^n_{top} is the standard topological n-simplex, considered with trivial action. It has a right adjoint $K \mapsto \mathcal{S}\mathrm{ing}_{\mathbb{C}}(K)$, defined by $\mathcal{S}\mathrm{ing}_{\mathbb{C}}(K)(X) = \underline{\mathrm{Hom}}_G(X(\mathbb{C}), K)$ (where $\underline{\mathrm{Hom}}_G(-, -)$ is the simplicial set of continuous equivariant maps). Equip $G\mathrm{Top}_\bullet$ with the model structure where a map $X \to Y$ is a weak equivalence or a fibration if $X^H \to Y^H$ is a weak equivalence or a fibration for all subgroups of G; see, e.g., [MM02, Theorem III.1.8]. The resulting homotopy category $\mathrm{H}_{G,\bullet}$ is the classical unstable equivariant homotopy category. For the corresponding model structure on G-simplicial sets see, e.g., [DRØ03, Section 9.2].

Proposition A.12. *The adjoint pair*

$$\mathrm{Re}_{\mathbb{C}} : G\mathrm{Spc}_\bullet(\mathbb{C}) \rightleftarrows G\mathrm{Top}_\bullet : \mathcal{S}\mathrm{ing}_{\mathbb{C}}$$

is a Quillen adjunction and $\mathrm{Re}_{\mathbb{C}}$ is a symmetric monoidal functor.

Proof. The argument is similar to that given in [PPR09, Theorem A.23]. First, we show that $\mathrm{Re}_{\mathbb{C}}$ is a left adjoint on the closed flasque model structure. For any finite collection of closed immersions $\{Z_i \to X\}$ in $G\mathrm{Sm}/\mathbb{C}$, $\mathrm{Re}_{\mathbb{C}}((i_{\mathcal{Z}})_+)$ is the inclusion of an equivariant subcomplex, and hence an equivariant cofibration. It follows that $\mathrm{Re}_{\mathbb{C}}$ sends I^c to cofibrations in $G\mathrm{Top}_\bullet$ and J^c to weak homotopy equivalences in $G\mathrm{Top}_\bullet$. This implies that $\mathcal{S}\mathrm{ing}_{\mathbb{C}}$ preserves trivial fibrations, as well as fibrations between fibrant objects. By Dugger's lemma [Dug01, Corollary A2], $\mathrm{Re}_{\mathbb{C}}$ is thus a left Quillen functor on the closed flasque global model structure.

Next we claim that $\mathrm{Re}_{\mathbb{C}}$ sends equivariant distinguished squares to homotopy pushouts. By [Her13, Corollary 2.13] the H-fixed points of an equivariant distinguished square is a distinguished Nisnevich square in Sm/k. A square in $G\mathrm{Top}_\bullet$ is a homotopy pushout if and only if it is so on all fixed points. Since $X^H(\mathbb{C}) = X(\mathbb{C})^H$, it suffices to show that the square obtained by taking complex points of a distinguished Nisnevich square is a homotopy pushout square in topological spaces. This follows from [DI04, Theorem 5.2], establishing the claim. As $\mathbb{A}^1(\mathbb{C})$ is equivariantly contractible, $\mathrm{Re}_{\mathbb{C}}(X_+) \to \mathrm{Re}_{\mathbb{C}}((X \times \mathbb{A}^1)_+)$ is an equivariant homotopy equivalence. It follows from the universal property of Bousfield localization that $\mathrm{Re}_{\mathbb{C}}$ is a left Quillen functor on the closed flasque motivic model structure.

That $\mathrm{Re}_{\mathbb{C}}$ is symmetric monoidal is a simple consequence of the fact that there is a natural equivariant homeomorphism $(X \times Y)(\mathbb{C}) \cong X(\mathbb{C}) \times Y(\mathbb{C})$. $\qquad\square$

Proposition A.13. *The squares of left adjoints commute up to natural isomorphisms,*

$$
\begin{array}{ccc}
\mathrm{H}_\bullet(\mathbb{C}) \xrightarrow{G_+\wedge -} \mathrm{H}_{G,\bullet}(\mathbb{C}) & \mathrm{H}_\bullet(\mathbb{C}) \xrightarrow{(-)^{\mathrm{triv}}} \mathrm{H}_{G,\bullet}(\mathbb{C}) & \mathrm{H}_{G,\bullet}(\mathbb{C}) \xrightarrow{(-)^e} \mathrm{H}_\bullet(\mathbb{C}) \\
\Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} & {\scriptstyle\mathrm{Re}_\mathbb{C}}\Big\downarrow \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} & {\scriptstyle\mathrm{Re}_\mathbb{C}}\Big\downarrow \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} \\
\mathrm{H}_\bullet \xrightarrow{G_+\wedge -} \mathrm{H}_{G,\bullet}, & \mathrm{H}_\bullet \xrightarrow{(-)^{\mathrm{triv}}} \mathrm{H}_{G,\bullet}, & \mathrm{H}_{G,\bullet} \xrightarrow{(-)^e} \mathrm{H}_\bullet.
\end{array}
$$

Proof. Using the notation from Section A.3, we have a natural isomorphism $\mathrm{Re}_\mathbb{C} \circ \epsilon^*(-) \cong G_+ \wedge \mathrm{Re}_\mathbb{C}(-)$ of functors $\mathrm{Spc}_\bullet(\mathbb{C}) \to G\mathrm{Top}$. Indeed, this is clear on representable motivic spaces, and since these functors commute with colimits, this suffices. Similarly, we have natural isomorphisms $\mathrm{Re}_\mathbb{C} \circ i^*(-) \cong (\mathrm{Re}_\mathbb{C}(-))^{\mathrm{triv}}$ and $\mathrm{Re}_\mathbb{C} \circ \rho^*(-) \cong (\mathrm{Re}_\mathbb{C}(-))^e$. These isomorphisms imply the isomorphisms of derived functors on the homotopy categories. $\qquad\square$

A.6. **Topological realization over $\mathbb{C}$ (stable).** Now we turn our attention to a stable realization functor. For any real orthogonal representation V there is a representation sphere $S^V = V_+$, where $(-)_+$ denotes the one-point compactification. Since $\mathbb{C}[G] = \mathbb{R}[G] \oplus \mathbb{R}[G]$ as real representations, we have $\mathrm{Re}_\mathbb{C}(T^{\rho_G}) = S^{2\rho_G}$.

If E is a motivic G-spectrum, define the topological $S^{2\rho_G}$-spectrum $\mathrm{Re}_\mathbb{C}\mathsf{E}$ by $(\mathrm{Re}_\mathbb{C}\mathsf{E})_i = \mathrm{Re}_\mathbb{C}\mathsf{E}_i$ with structure maps

$$
\mathrm{Re}_\mathbb{C}\mathsf{E}_i \wedge S^{2\rho_G} = \mathrm{Re}_\mathbb{C}(\mathsf{E}_i \wedge T^{\rho_G}) \to \mathrm{Re}_\mathbb{C}\mathsf{E}_{i+1}.
$$

The functor $\mathcal{S}\mathrm{ing}_\mathbb{C}$ extends as well to a functor on $S^{2\rho_G}$-spectra. In fact, we obtain an adjoint pair of functors

$$
\mathrm{Re}_\mathbb{C} : \mathrm{Spt}^\Sigma_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^\Sigma_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_\mathbb{C}.
$$

Theorem A.14. *The adjoint pairs*

$$
\mathrm{Re}_\mathbb{C} : \mathrm{Spt}^\Sigma_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^\Sigma_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_\mathbb{C},
$$
$$
\mathrm{Re}_\mathbb{C} : \mathrm{Spt}^\Sigma_{T^{\rho_G}}(\mathbb{C})^{fp} \rightleftarrows \mathrm{Spt}^\Sigma_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_\mathbb{C}
$$

are both Quillen adjoint pairs. Moreover, $\mathrm{Re}_\mathbb{C}$ is symmetric monoidal.

Proof. This is straightforward from Proposition A.12; see [PPR09, Theorem A.45]. $\qquad\square$

Proposition A.15. *The squares of left adjoints commute up to natural isomorphisms,*

$$
\begin{array}{ccc}
\mathrm{SH}(\mathbb{C}) \xrightarrow{G_+\wedge -} \mathrm{SH}_G(\mathbb{C}) & \mathrm{SH}(\mathbb{C}) \xrightarrow{(-)^{\mathrm{triv}}} \mathrm{SH}_G(\mathbb{C}) & \mathrm{SH}_G(\mathbb{C}) \xrightarrow{(-)^e} \mathrm{SH}(\mathbb{C}) \\
\Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} & {\scriptstyle\mathrm{Re}_\mathbb{C}}\Big\downarrow \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} & {\scriptstyle\mathrm{Re}_\mathbb{C}}\Big\downarrow \qquad \Big\downarrow{\scriptstyle\mathrm{Re}_\mathbb{C}} \\
\mathrm{SH} \xrightarrow{G_+\wedge -} \mathrm{SH}_G, & \mathrm{SH} \xrightarrow{(-)^{\mathrm{triv}}} \mathrm{SH}_G, & \mathrm{SH}_G \xrightarrow{(-)^e} \mathrm{SH}.
\end{array}
$$

Proof. This is straightforward using the unstable result in Proposition A.13. $\qquad\square$

A.7. **Topological realization over** $\mathbb{R}$. Write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. If X is a real variety with G-action, then $X(\mathbb{C})$ is a $(G \times \Sigma_2)$-space, where the Σ_2-action is via complex conjugation. We thus have a functor $G\mathrm{Sm}/\mathbb{R} \to (G \times \Sigma_2)\mathrm{Top}$ which induces the topological realization functor $\mathrm{Re}_{\mathbb{C}, \Sigma_2} : G\mathrm{Spc}_\bullet(\mathbb{C}) \to (G \times \Sigma_2)\mathrm{Top}_\bullet$, defined by the Kan extension

$$\mathrm{Re}_{\mathbb{C}, \Sigma_2}(F) = \operatorname*{colim}_{(X \times \Delta^n)_+ \to F} (X(\mathbb{C}) \times \Delta^n_{\mathrm{top}})_+.$$

Its right adjoint is defined by $\mathcal{S}\mathrm{ing}_{\mathbb{C}, \Sigma_2}(K)(X) = \underline{\mathrm{Hom}}_{G \times \Sigma_2}(X(\mathbb{C}), K)$, where K is a $(G \times \Sigma_2)$-space and X is a smooth real G-variety.

Proposition A.16. *The adjoint pair*

$$\mathrm{Re}_{\mathbb{C}, \Sigma_2} : G\mathrm{Spc}_\bullet(\mathbb{R}) \rightleftarrows (G \times \Sigma_2)\mathrm{Top}_\bullet : \mathcal{S}\mathrm{ing}_{\mathbb{C}, \Sigma_2}$$

is a Quillen adjunction, and $\mathrm{Re}_{\mathbb{C}, \Sigma_2}$ is a symmetric monoidal functor.

Proof. The argument is similar to Proposition A.12. $\square$

Now we turn our attention to a stable realization functor. Since $\mathbb{C}[G] = \mathbb{R}[G \times \Sigma_2]$ as real representations, we have $\mathrm{Re}_{\mathbb{C}}(T^{\rho_G}) = S^{\rho_G \times \Sigma_2}$.

If E is a motivic G-spectrum, define the topological $S^{\rho_G \times \Sigma_2}$-spectrum $\mathrm{Re}_{\mathbb{C}, \Sigma_2}\mathsf{E}$ by $(\mathrm{Re}_{\mathbb{C}, \Sigma_2}\mathsf{E})_i = \mathrm{Re}_{\mathbb{C}, \Sigma_2}\mathsf{E}_i$, with structure maps

$$\mathrm{Re}_{\mathbb{C}, \Sigma_2}\mathsf{E}_i \wedge \mathrm{Re}_{\mathbb{C}, \Sigma_2}(S^{\rho_G \times \Sigma_2}) = \mathrm{Re}_{\mathbb{C}, \Sigma_2}(\mathsf{E}_i \wedge T^{\rho_G}) \to \mathrm{Re}_{\mathbb{C}, \Sigma_2}\mathsf{E}_{i+1}.$$

The functor $\mathcal{S}\mathrm{ing}_{\mathbb{C}}$ extends as well to a functor on $S^{\rho_G \times \Sigma_2}$-spectra, adjoint to $\mathrm{Re}_{\mathbb{C}, \Sigma_2}$.

Theorem A.17. *The adjoint pair*

$$\mathrm{Re}_{\mathbb{C}, \Sigma_2} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{\rho_G \times \Sigma_2}}((G \times \Sigma_2)\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C}, \Sigma_2}$$

is a Quillen adjoint pair. Moreover, $\mathrm{Re}_{\mathbb{C}, \Sigma_2}$ is symmetric monoidal.

Proof. This is straightforward from Proposition A.16; see [PPR09, Theorem A.45]. $\square$

A.8. **Symmetric powers.** In order to analyze the topological realization of the Bredon motivic cohomology spectrum, we need to make precise that it is represented by symmetric powers. In this subsection we introduce and analyze symmetric powers for motivic G-spaces. This discussion parallels the treatment of symmetric powers in [Del09], [Voe10a], [Lev14, Appendix]. Throughout this subsection we assume that $\mathrm{char}(k) = 0$.

Write $G\mathrm{Sm}'/k$ for the category of smooth, quasi-projective G-schemes over k, and $G\mathrm{Spc}'(k)$ and $G\mathrm{Spc}'_\bullet(k)$ for the corresponding categories of simplicial presheaves and pointed simplicial presheaves. These can also be given a motivic model structure as above. The inclusion $\phi : G\mathrm{Sm}'/k \subseteq G\mathrm{Sm}/k$ induces an adjoint pair $\phi^* : G\mathrm{Spc}'(k) \rightleftarrows G\mathrm{Spc}(k) : \phi_*$, and it is similar for based motivic spaces.

Lemma A.18. *The adjunction $\phi^* : G\mathrm{Spc}'(k) \rightleftarrows G\mathrm{Spc}(k) : \phi_*$ is a Quillen equivalence on motivic model structures. It is similar for based motivic G-spaces.*

Proof. This follows easily from the fact that smooth G-schemes are locally affine in the equivariant Nisnevich topology; see Remark 2.3. $\square$

Write $G\mathrm{Sch}'/k$ for the category of reduced, quasi-projective G-schemes of finite type over k. Consider the functor $(-)^{\times n} : G\mathrm{Sch}'/k \to (\Sigma_n \times G)\mathrm{Sch}'/k$, which sends a G-scheme X to the n-fold product $X^{\times n}$, where G acts diagonally and Σ_n acts by permuting the factors. The composition of $(-)^{\times n}$ and the Yoneda embedding gives us a functor $G\mathrm{Sch}'/k \to \mathrm{sPre}((\Sigma_n \times G)\mathrm{Sch}'/k)$, and we define $\Pi^{(n)}(-)$ to be its left Kan extension, yielding the functor

$$\Pi^{(n)} : \mathrm{sPre}(G\mathrm{Sch}'/k) \to \mathrm{sPre}((\Sigma_n \times G)\mathrm{Sch}'/k),$$

and it is similar for pointed motivic spaces.

Let $N \trianglelefteq K$ be a normal subgroup of a group K, and write $\Gamma = K/N$ for the quotient group. If X is a quasi-projective K-scheme over k, then a quotient scheme X/N exists and the K-action on X induces a Γ-action on the scheme X/N. We write $q_{K,N} : K\mathrm{Sch}'/k \to \Gamma\mathrm{Sch}/k$ for the quotient functor, i.e., $q_{K,N}(X) = X/N$. The functor $q_{K,N}$ induces an adjoint pair of functors

$$(q_{K,N})^* : \mathrm{sPre}(K\mathrm{Sch}'/k) \rightleftarrows \mathrm{sPre}(\Gamma\mathrm{Sch}/k) : (q_{K,N})_*,$$

and it is similar for based presheaves. For $\mathcal{X}$ in $K\mathrm{Spc}'(k)$ define $\mathcal{X}/N := (q_{K,N})^*(\mathcal{X})$. If $\mathcal{X}$ is represented by a quasi-projective G-scheme X, then $\mathcal{X}/N$ is represented by X/N.

Let $\mathcal{C}$ be a category with finite coproducts, and let F be a presheaf of sets on $\mathcal{C}$. Say that F is *additive*[1] provided that $F(\emptyset) = *$ and $F(X \coprod Y) = F(X) \times F(Y)$ for any $X, Y \in \mathcal{C}$. By [AR01, Theorem 2.6] a presheaf of sets on $\mathcal{C}$ is additive if and only if the opposite of its category of elements is sifted. In particular, an additive presheaf is canonically a sifted colimit of representable presheaves. Recall that filtered colimits and reflexive coequalizers are sifted colimits.

Note that a presheaf on $G\mathrm{Sch}'/k$ is additive if and only if it is a sheaf for the topology generated by covers by G-connected components. The inclusion of the full subcategory of presheaves $i : \mathrm{Pre}_\Sigma(G\mathrm{Sch}'/k) \subseteq \mathrm{Pre}(G\mathrm{Sch}'/k)$ whose objects are additive presheaves has a left adjoint

$$a_\# : \mathrm{Pre}(G\mathrm{Sch}'/k) \to \mathrm{Pre}_\Sigma(G\mathrm{Sch}'/k)$$

given by sheafification (or as a special case of [Voe10b, Lemma 3.8]). Since we do not work in the category $\mathrm{Pre}_\Sigma(G\mathrm{Sch}'/k)$, we will also simply write $a_\#$ again for the composite $ia_\#$.

Definition A.19. Define the *nth symmetric product* $\mathrm{Sym}^n : \mathrm{sPre}(G\mathrm{Sch}'/k) \to \mathrm{sPre}(G\mathrm{Sch}'/k)$ by

$$\mathrm{Sym}^n(\mathcal{X}) := \Pi^{(n)}(a_\# \mathcal{X})/\Sigma_n.$$

The inclusion of categories $i : G\mathrm{Sm}'/k \subseteq G\mathrm{Sch}'/k$ induces an adjoint pair of functors

$$i^* : G\mathrm{Spc}'(k) \rightleftarrows \mathrm{sPre}(G\mathrm{Sch}'/k) : i_*,$$

and it is similar for pointed motivic spaces. Note that i_* preserves colimits, as these are computed sectionwise. If $\mathcal{X}$ is an object of $\mathrm{sPre}(G\mathrm{Sch}'/k)$, when no confusion should arise, we again write $\mathrm{Sym}^n(\mathcal{X})$ for $i_*(\mathrm{Sym}^n(\mathcal{X}))$ in $G\mathrm{Spc}'(k)$, and also for $\phi^* i_*(\mathrm{Sym}^n(\mathcal{X}))$ in $G\mathrm{Spc}(k)$.

If $\mathcal{X}$ is represented by a scheme X in $G\mathrm{Sch}'/k$, then $\mathrm{Sym}^n(\mathcal{X})$ is represented by the scheme-theoretic symmetric power $\mathrm{Sym}^n(X)$. Moreover, $\mathrm{Sym}^n(-)$ commutes with sifted colimits.

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[1] In [Voe10b] the term *radditive* is used.

If $(\mathcal{X}, x)$ is an object of $\mathrm{sPre}_\bullet(G\mathrm{Sch}'/k)$, then $\mathrm{Sym}^n(\mathcal{X})$ has a canonical basepoint given by $\mathrm{Sym}^n(x)$, and we write $\mathrm{Sym}^n_\bullet(\mathcal{X})$ for the corresponding pointed motivic space in $G\mathrm{Spc}'_\bullet(k)$. If $\mathcal{X}$ is represented by a pointed scheme X in $G\mathrm{Sch}'/k$, then $\mathrm{Sym}^n(\mathcal{X})$ is represented by the scheme-theoretic symmetric power $\mathrm{Sym}^n(X)$. Note that $\mathrm{Sym}^n_\bullet(-)$ commutes with sifted colimits.

Remark A.20. We avoid here any explicit discussion of the derived functors necessary to make this construction $\mathbb{A}^1$-invariant. However, consideration of the symmetric powers of a coproduct shows that any sensible definition cannot both commute with all colimits and compute the correct value of quotient spaces. Some amount of "derivedness" needs to be baked into the definition. This explains the appearance of $a_\#$ in the definition.

We have natural pairings $(X^{\times m})/\Sigma_m \times (X^{\times n})/\Sigma_n \to (X^{\times(m+n)})/\Sigma_{m+n}$ and $(X^{\times m})/\Sigma_m \times (Y^{\times n})/\Sigma_n \to (X \times Y)^{\times mn}/\Sigma_{mn}$, for $X, Y \in G\mathrm{Sch}'/k$. These induce addition and multiplication maps,

$$\mathrm{Sym}^m_\bullet(\mathcal{X}) \times \mathrm{Sym}^n_\bullet(\mathcal{X}) \xrightarrow{\sigma_{m,n}} \mathrm{Sym}^{m+n}_\bullet(\mathcal{X}),$$

$$\mathrm{Sym}^m_\bullet(\mathcal{X}) \wedge \mathrm{Sym}^n_\bullet(\mathcal{Y}) \xrightarrow{\mu_{m,n}} \mathrm{Sym}^{mn}_\bullet(\mathcal{X} \wedge \mathcal{Y}).$$

Stabilization maps $\mathrm{st}_{n+1} := \sigma_{n,1}(-, *) : \mathrm{Sym}^n_\bullet(\mathcal{X}) \to \mathrm{Sym}^{n+1}_\bullet(\mathcal{X})$ are thus obtained by adding the basepoint. Define

$$\mathrm{Sym}^\infty_\bullet(\mathcal{X}) := \underset{n}{\mathrm{colim}}(\mathcal{X} \xrightarrow{\mathrm{st}_2} \mathrm{Sym}^2_\bullet(\mathcal{X}) \xrightarrow{\mathrm{st}_3} \mathrm{Sym}^3_\bullet(\mathcal{X}) \xrightarrow{\mathrm{st}_4} \cdots)$$

Recall that the group of finite correspondences $\mathrm{Cor}_k(Y, X)$ is the free abelian group generated by integral closed subschemes $Z \subseteq Y \times X$ such that Z is finite over Y and dominates an irreducible component of Y. When G acts on X and Y, there is an induced action on $\mathrm{Cor}_k(Y, X)$. The group of equivariant finite correspondences is defined to be $G\,\mathrm{Cor}_k(Y, X) := \mathrm{Cor}_k(Y, X)^G$. We write $\mathbb{Z}_{\mathrm{tr},G}(X)$ for the presheaf of equivariant correspondences, $\mathbb{Z}_{\mathrm{tr},G}(X)(Y) := G\,\mathrm{Cor}_k(Y, X)$. If $A \subseteq X$ is a closed invariant subscheme, we define $\mathbb{Z}_{\mathrm{tr},G}(X/A) := \mathbb{Z}_{\mathrm{tr},G}(X)/\mathbb{Z}_{\mathrm{tr},G}(A)$, the group quotient.

The submonoid $\mathrm{Cor}^{\mathrm{eff}}_k(Y, X)^G$ of effective equivariant correspondences consists of those equivariant correspondences $\sum n_Z Z$ such that all $n_Z \geq 0$. We write $\mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X)$ for the corresponding presheaf of effective equivariant correspondences. Let $A \subseteq X$ be a closed invariant subscheme, and define an equivalence relation $\sim$ on $\mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X)(Y)$ by declaring $Z \sim Z'$ if $Z - Z' \in \mathbb{Z}_{\mathrm{tr},G}(A)(Y)$. Now define the presheaf $\mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X/A) := \mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X)/\sim$. We have that $\mathbb{Z}_{\mathrm{tr},G}(X) = (\mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X))^+$ and $\mathbb{Z}_{\mathrm{tr},G}(X/A) = (\mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(X/A))^+$, where $(-)^+$ denotes group completion.

Consider the subset $L_n(X)(Y) \subseteq \mathrm{Cor}^{\mathrm{eff}}_k(Y, X)$ of effective correspondences of degree n. Write $L^G_n(X)(Y) = (L_n(X)(Y))^G$ for the subset of equivariant correspondences of degree n. Addition of cycles induces a pairing

$$\sigma_{i,j} : L^G_i(X) \times L^G_j(X) \to L^G_{i+j}(X).$$

Now we consider a pointed G-scheme (X, x). The presheaf $L^G_n(X)$ is pointed by $n(Y \times x)$ in $L^G_n(X)(Y)$. Let $A \subseteq X$ be a closed invariant subscheme containing $x \in X$. Adding the basepoint induces stabilization maps

$$\mathrm{st} = \sigma(-, *) : L^G_i(X) \to L^G_{i+1}(X).$$

Define the presheaf $L_n^G(X/A)$ to be the coequalizer (in $G\mathrm{Spc}(k)$)

$$(A.21) \qquad \coprod_{j=0}^{n} L_j^G(A) \times L_{n-j}^G(X) \rightrightarrows L_n^G(X) \to L_n^G(X/A),$$

where the top arrow is induced by the inclusion $A \subseteq X$ and addition of cycles and the bottom map is the composition

$$L_j^G(A) \times L_{n-j}^G(X) \xrightarrow{\mathrm{proj}} L_{n-j}^G(X) \xrightarrow{\mathrm{st}} L_n^G(X).$$

Note that the presheaf $L_n^G(X/A)$ has a canonical basepoint, and that the quotient map $L_n^G(X) \to L_n^G(X/A)$ is a map of based presheaves. The addition map induces an addition map $L_i^G(X/A) \times L_j^G(X/A) \to L_{i+j}^G(X/A)$, and, in particular, we have stabilization maps $\mathrm{st} : L_i^G(X/A) \to L_{i+1}^G(X/A)$ obtained by adding the basepoint.

Proposition A.22. *Let X be a pointed, semi-normal, quasi-projective G-scheme over k, and let $A \subseteq X$ be an invariant closed reduced subscheme containing the basepoint. There are isomorphisms $L_n^G(X) \cong \mathrm{Sym}_\bullet^n(X)$ in $G\mathrm{Spc}_\bullet(k)$ which induce isomorphisms $L_n^G(X/A) \cong \mathrm{Sym}_\bullet^n(X/A)$.*

Proof. There is a natural isomorphism $\psi : \mathrm{Sym}_\bullet^n(X) \to L_n(X)$ by [SV96, Theorem 6.8], which is given as follows. Let $W_n \subseteq \mathrm{Sym}^n(X) \times X$ be the image of the closed subscheme $X^{n-1} \times \Delta_X \subseteq X^{n+1}$ under $\pi_n \times \mathrm{id}_X$, where $\pi_n : X^n \to \mathrm{Sym}^n(X)$ is the quotient. If $f : Y \to \mathrm{Sym}^n(X)$ is a map, with Y smooth, then $\psi(f) = (f \times \mathrm{id}_X)^*(W_n)$ is defined to be the pullback cycle. This is an equivariant isomorphism. Since $\mathrm{Sym}^n(-)$ commutes with sifted colimits, we can compute $\mathrm{Sym}_\bullet^n(X/A)$ as the coequalizer (in $G\mathrm{Spc}(k)$)

$$(A.23) \qquad \mathrm{Sym}^n(A \coprod X) \rightrightarrows \mathrm{Sym}^n(X) \to \mathrm{Sym}_\bullet^n(X/A),$$

where the top arrow is induced by the inclusion $A \subseteq X$ and the identity on X, while the bottom arrow is induced by mapping A to the basepoint and the identity on X. Since $\mathrm{Sym}^n(A \coprod X) \cong \coprod_{j=0}^n \mathrm{Sym}^j(A) \times \mathrm{Sym}^{n-j}(X)$, comparing with (A.21) yields the isomorphisms $L_n^G(X/A) \cong \mathrm{Sym}_\bullet^n(X/A)$. $\qquad\square$

The maps $L_n^G(X) \to \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X)$ induce maps $L_n^G(X/A) \to \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X/A)$ which are compatible with the stabilization $L_n^G(X/A) \to L_{n+1}^G(X/A)$. We thus have an induced map $\mathrm{colim}_n L_n^G(X/A) \to \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X/A)$.

Proposition A.24. *Let (X, x) be a pointed smooth quasi-projective G-scheme over k, and let $A \subseteq X$ be a closed reduced subscheme which contains the basepoint $x \in X$. Then we have isomorphisms in $G\mathrm{Spc}_\bullet(k)$*

$$\mathrm{Sym}_\bullet^\infty(X/A) \xleftarrow{\cong} \mathrm{colim}_n L_n^G(X/A) \xrightarrow{\cong} \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X/A).$$

Proof. The left-hand isomorphism follows from the previous proposition. The map $\mathrm{colim}_n L_n(X_+) \to \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X)$ is immediately seen to be an isomorphism. The map $f : \mathrm{colim}_n L_n^G(X/A) \to \mathbb{Z}_{\mathrm{tr},G}^{\mathrm{eff}}(X/A)$ is surjective, and we check injectivity. Let $[W], [V] \in L_n^G(X/A)(Y)$, and represent them by elements in $L_n^G(X)(Y)$ which we again write as W, V. We can write $W = W_A + W'$ and $V = V_A + V'$ uniquely as a sum of effective cycles, with W_A, V_A supported on $Y \times A$ and no component of the supports of W', V' contained in $Y \times A$. If $f([W]) = f([V])$, then we have that $W' = V'$. It follows from the definition of $L_n^G(X/A)$ that $[W] = [V]$. $\qquad\square$

Proposition A.25. *Let X be a smooth quasi-projective G-scheme over k, and let V be a representation which contains a copy of the trivial representation. Then the map $\mathrm{Sym}^\infty_\bullet(T^V \wedge\ X_+) \to \mathbb{Z}_{\mathrm{tr},G}(T^V \wedge\ X_+)$ is an equivariant motivic weak equivalence.*

Proof. If E is a presheaf of sets, write C_*E as usual for the presheaf of simplicial sets defined by $C_n E(Y) = E(Y \times \Delta^n_k)$. The map $E \to C_*E$ is an $\mathbb{A}^1$-weak equivalence.

Write $F_1 = \mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(T^V \wedge X_+)$ and $F_2 = \mathbb{Z}_{\mathrm{tr},G}(T^V \wedge X_+)$, and let $\phi : F_1 \to F_2$ be the group completion. Under the isomorphism of Proposition A.24, the map in the statement of the proposition is identified with ϕ. It suffices to show that $\phi : C_*F_1 \to C_*F_2$ is a motivic weak equivalence. In fact, we show that if S is a point for the equivariant Nisnevich topology, then $\phi : C_*F_1(S) \to C_*F_2(S)$ is a weak equivalence of simplicial sets. We claim that $\pi_0 C_* F_1(S) = 0$. Granted this, since ϕ is the group completion of a free commutative monoid, by [FM94, Theorem Q4] the map ϕ is a homology equivalence. Since ϕ is a map between simple spaces, this implies that ϕ is in fact a homotopy equivalence.

It remains to see that $\pi_0 C_* F_1(S) = 0$. We have that $S = G \times_H \mathrm{Spec}(R)$, where $H \subseteq G$ is a subgroup and R is a smooth, local Henselian ring with H-action; see [HVØ15, Theorem 3.14]. Write $\rho_H : G\mathrm{Sch}/k \to H\mathrm{Sch}/k$ for the functor which restricts the action. Note that $\rho_H(\mathrm{Sym}^\infty_\bullet(Y)) = \mathrm{Sym}^\infty_\bullet(\rho_H Y)$ for any based quasi-projective G-scheme Y, so for any H-scheme Z,

$$\mathrm{Hom}_{G\mathrm{Sch}/k}(G \times_H Z, \mathrm{Sym}^\infty_\bullet(X)) = \mathrm{Hom}_{H\mathrm{Sch}/k}(Z, \mathrm{Sym}^\infty_\bullet(\rho_H X)).$$

Thus, replacing G with H and S with $\mathrm{Spec}(R)$, we may assume that S is a local Henselian G-scheme.

Write

$$F'_1 = \mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}\left((\mathbb{P}(V \oplus 1)/(\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1)) \wedge X_+ \right).$$

First, note that since $\mathbb{P}(V) \subseteq \mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1)$ has an equivariant section,

$$\pi_0 C_* F_1(S) \subseteq \pi_0 C_* F'_1(S).$$

(In fact, $C_* F_1(S)$ and $C_* F'_1(S)$ are weakly equivalent, but the weaker statement is enough for our purposes here.) Therefore, it suffices to show that $\pi_0 C_* F'_1(S) = 0$. An element of $C_d F'_1(S)$ is represented by an effective invariant cycle $\sum n_i Z_i$ on $\Delta^d_{S \times X} \times \mathbb{P}(V \oplus 1)$ such that Z_i is finite over S and has nontrivial intersection with $\Delta^d_{S \times X} \times \mathbb{P}(1)$.

Let $\mathcal{Z}$ be an element of $C_0 F'_1(S)$ and $\sum n_i Z_i \in \mathbb{Z}^{\mathrm{eff}}_{\mathrm{tr},G}(\mathbb{P}(V \oplus 1) \times X)(S)$ an element representing $\mathcal{Z}$, as in the previous paragraph. Since the Z_i are finite over S, they are local. Since Z_i has nontrivial intersection with $\mathbb{P}(1)_{S \times X}$, the closed points are supported in $\mathbb{P}(1)_{S \times X}$ and Z_i are contained in the open $\mathbb{A}(V)_{S \times X} \subseteq \mathbb{P}(V \oplus 1)_{S \times X}$ (where $\mathbb{P}(1)_{S \times X} = 0_{S \times X}$ in $\mathbb{A}(V)_{S \times X}$). By assumption V contains a copy of the trivial representation, so we can write $V = 1 \oplus V'$. Consider the equivariant map $\phi : \mathbb{A}^1 \times \mathbb{A}(V)_{S \times X} \to \mathbb{A}(V)_{S \times X}$ given by $(t, (x_i), y) \mapsto ((x_i - t), y)$. Let Φ be the cycle on $\mathbb{A}^1 \times \mathbb{A}(V)_{S \times X}$ obtained by pullback of $\mathcal{Z}$ along ϕ. Then $\Phi \in C_1 F'_1(S)$ and has the property that $\Phi|_0 = \mathcal{Z}$, and $\Phi|_1$ is supported in $(\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1))_{S \times X}$ (the closed points lie in $\{1\}_{S \times X} \subseteq \mathbb{A}(V)_{S \times X}$). Thus $\mathcal{Z} = 0$ in $\pi_0 C_* F'_1(S)$. $\square$

A.9. Realization of Eilenberg–MacLane spectra. Our next goal is to show that the topological realization, in both the complex and real cases, of the Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$ is the topological Bredon cohomology spectrum (for the constant Mackey functor $\underline{A}$). The construction of the motivic G-spectrum

$\mathbf{M}\underline{\mathbb{Z}}$ is spelled out in detail in Section 3.2 for the case $G = C_2$, and the construction for a general finite group G is similar. In brief, we set $\mathbf{M}\underline{\mathbb{Z}}_n := \mathbb{Z}_{\mathrm{tr},G}(T^{n\rho_G})$, and structure maps are defined by $\mathbf{M}\underline{\mathbb{Z}}_n \wedge T^{\rho_G} \to \mathbf{M}\underline{\mathbb{Z}}_n \wedge \mathbb{Z}_{\mathrm{tr},G}(T^{\rho_G}) \to \mathbf{M}\underline{\mathbb{Z}}_{n+1}$, with the first map being induced by the inclusion $T^{\rho_G} \to \mathbb{Z}_{\mathrm{tr},G}(T^{\rho_G})$ together with the addition-of-cycles map.

Lemma A.26.

(1) *Let V be a complex representation which contains a trivial summand. For any X in $G\mathrm{Sch}/\mathbb{C}$ the natural map*
$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V \wedge X_+) \to S^{V(\mathbb{C})} \wedge X(\mathbb{C})_+$$
is an equivariant equivalence in $G\mathrm{Top}_{\bullet}$.

(2) *Let V be a real representation which contains a trivial summand. For any X in $G\mathrm{Sch}/\mathbb{R}$ the natural map*
$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\Sigma_2}(T^V \wedge X_+) \to S^{V(\mathbb{C})} \wedge X(\mathbb{C})_+$$
is an equivariant equivalence in $(G \times \Sigma_2)\mathrm{Top}_{\bullet}$.

Proof. We treat the complex case; the real case is similar.

Over a field of characteristic 0 there are equivariant resolutions of singularities; see, e.g., [Kol07, Proposition 3.9.1]. We may thus find a simplicial scheme $X_{\bullet} \to X$ over X such that $X_{\bullet}^H \to X^H$ is a proper *cdh*-hypercover for every subgroup $H \subseteq G$. We have $|X_{\bullet}|^H = |X_{\bullet}^H|$, and since $V^H \neq 0$, it follows from [Voe10c, Theorem 4.2] that the map $(T^V \wedge |X_{\bullet}|_+)^H \to (T^V \wedge X_+)^H$ is a motivic weak equivalence in $\mathrm{Spc}_{\bullet}(\mathbb{C})$. Therefore, the map $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V \wedge |X_{\bullet}|_+) \to S^{V(\mathbb{C})} \wedge X(\mathbb{C})_+$ is an equivariant weak equivalence in $G\mathrm{Top}_{\bullet}$.

Since each X_n is smooth, $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V \wedge |X_{\bullet}|_+) \simeq S^{V(\mathbb{C})} \wedge |X(\mathbb{C})_{\bullet}|_+$. To complete the proof, it remains to show that $S^{V(\mathbb{C})} \wedge |X(\mathbb{C})_{\bullet}|_+ \to S^{V(\mathbb{C})} \wedge X(\mathbb{C})_+$ is an equivariant weak equivalence. It suffices to show that this map is a weak equivalence on all fixed points. Note that $X^H(\mathbb{C}) = X(\mathbb{C})^H$. Applying $(-)^H$ to the map above, we obtain the map $S^{2|V^H|} \wedge |X^H(\mathbb{C})_{\bullet}|_+ \to S^{2|V^H|} \wedge X^H(\mathbb{C})_+$. Since $X^H(\mathbb{C})_{\bullet} \to X^H(\mathbb{C})$ is a proper hypercover, it is a universal cohomological descent hypercover [Del74, 5.3.5]. It follows that $H_{\mathrm{sing}}^*(|X^H(\mathbb{C})_{\bullet}|, A) \to H_{\mathrm{sing}}^*(X^H(\mathbb{C}), A)$ is an isomorphism for all abelian groups A. It follows that $S^{2|V^H|} \wedge |X^H(\mathbb{C})_{\bullet+}| \to S^{2|V^H|} \wedge X^H(\mathbb{C})_+$ is a homology isomorphism. Since $|V^H| \geq 1$, these are simply connected spaces, and thus this homology isomorphism is a weak equivalence. $\square$

Lemma A.27.

(1) *Let W, V be complex representations with V containing a trivial summand, and let X be a smooth quasi-projective complex variety with G-action. Then for $n \geq 0$*
$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}_{\bullet}^n(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}_{\bullet}^n(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$$
is an equivariant weak equivalence in $G\mathrm{Top}_{\bullet}$.

(2) *Let W, V be real representations with V containing a trivial summand, and let X be a smooth quasi-projective real variety with G-action. Then for $n \geq 0$*
$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\Sigma_2}(\Sigma_{T^V}\mathrm{Sym}_{\bullet}^n(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}_{\bullet}^n(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$$
is an equivariant weak equivalence in $(G \times \Sigma_2)\mathrm{Top}_{\bullet}$.

Proof. We treat the complex case; the real case is similar. From (A.23) we see that $\mathbb{L}\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(\Sigma_{T^W}X_+))$ is the homotopy coequalizer of

$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(\Sigma_{T^V}\mathrm{Sym}^n(A\coprod Y)) \rightrightarrows \mathbb{L}\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(\Sigma_{T^V}\mathrm{Sym}^n(Y)).$$

The result now follows from Lemma A.26. $\qquad\square$

Corollary A.28.

(1) *Let X be a smooth quasi-projective complex variety with G-action. For any $n, m \geq 0$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma^\infty_{T^{\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{T^{\rho_G}}X_+)) \to \Sigma^\infty_{S^{2\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{S^{2\rho_G}}X(\mathbb{C})_+)$$

is a stable equivariant equivalence.

(2) *Let X be a smooth quasi-projective real variety with G-action. For any $n, m \geq 0$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\Sigma_2}(\Sigma^\infty_{T^{\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{T^{\rho_G}}X_+)) \to \Sigma^\infty_{S^{2\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{S^{2\rho_G}}X(\mathbb{C})_+)$$

is a stable equivariant equivalence.

Define

$$(\Sigma^\infty_{T^{\rho_G}}X)^{\mathrm{eff}}_{\mathrm{tr}} = (\mathrm{Sym}^\infty_\bullet(X_+), \mathrm{Sym}^\infty_\bullet(\Sigma_{T^{\rho_G}}X_+), \mathrm{Sym}^\infty_\bullet(\Sigma^2_{T^{\rho_G}}X_+), \ldots)$$

with obvious structure maps.

Theorem A.29. *There is an isomorphism $\mathrm{Re}_{\mathbb{C}}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in SH_G, for any abelian group A. Similarly, there is an isomorphism $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in $\mathrm{SH}_{G\times\Sigma_2}$.*

Proof. We treat the complex case. The real case is similar.

Since $\mathbf{M}\underline{A} = \mathbf{M}\underline{\mathbb{Z}} \wedge \mathcal{M}A$ and $\mathrm{H}\underline{A} = \mathrm{H}\underline{\mathbb{Z}} \wedge \mathcal{M}A$, where $\mathcal{M}A$ is a Moore spectrum for A, and $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\mathcal{M}A) = \mathcal{M}A$, it suffices to establish the result for $A = \mathbb{Z}$. The map $(\Sigma^\infty_{T^{\rho_G}}(S^0))^{\mathrm{eff}}_{\mathrm{tr}} \to \mathbf{M}\underline{\mathbb{Z}}$ is a stable equivalence by Proposition A.25.

It follows from [dS03, Proposition 3.7] that the spectrum $\{\mathbb{Z}S^{2n\rho_G}\}_{n\geq 0}$ is a $S^{2\rho_G}$-spectrum model for $\mathrm{H}\underline{\mathbb{Z}}$. It follows from [Dug05, Proposition A.6] that the natural map $(\Sigma^\infty_{S^{2\rho_G}}(S^0))^{\mathrm{eff}}_{\mathrm{tr}} := \{\mathrm{Sym}^\infty_\bullet(\Sigma_{S^{2n\rho_G}}S^0)\} \to \{\mathbb{Z}S^{2n\rho_G}\}_{n\geq 0}$ is an equivariant stable equivalence. It thus suffices to see that $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma^\infty_{T^{\rho_G}}(S^0))^{\mathrm{eff}}_{\mathrm{tr}} \to (\Sigma^\infty_{S^{2\rho_G}}(S^0))^{\mathrm{eff}}_{\mathrm{tr}}$ is a stable equivalence.

We have the natural isomorphism $\mathrm{colim}_n(\Sigma^\infty_{T^{\rho_G}}E_n)[-n] \cong E$ in $\mathrm{SH}_k(G)$, where $D[n]$ is the shifted T^{ρ_G}-spectrum given by $(D[n])_i = D_{i-n}$. Similarly, we have the natural isomorphism $\mathrm{colim}_n(\Sigma^\infty_{S^{2\rho_G}}F_n)[-n] \cong F$ in SH_G. Since $\mathbb{L}\mathrm{Re}_{\mathbb{C}}$ preserves homotopy colimits and shifts, the result follows from Corollary A.28. $\qquad\square$

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