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On very effective hermitian K-theory

Alexey Ananyevskiy[1] · Oliver Röndigs[2] · Paul Arne Østvær[3]

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Abstract
We argue that the very effective cover of hermitian K-theory in the sense of motivic homotopy theory is a convenient algebro-geometric generalization of the connective real topological K-theory spectrum. This means the very effective cover acquires the correct Betti realization, its motivic cohomology has the desired structure as a module over the motivic Steenrod algebra, and that its motivic Adams and slice spectral sequences are amenable to calculations.

Keywords Hermitian K-theory · $\mathbf{A}^1$-homotopy theory · Slice filtration

Mathematics Subject Classification 14F42

1 Introduction

Algebraic and hermitian K-theory have been widely studied since the pioneering works on the Grothendieck–Riemann–Roch theorem [3] and on rings with anti-involutions [14]. Both theories are representable in the stable motivic homotopy category $\mathbf{SH}$ of a field of characteristic $\neq 2$, and more generally over regular noetherian base schemes of finite Krull dimension on which 2 is invertible [10,29]. Fundamental properties imply that, with respect to the standard motivic spheres $S^{p,q} := S^{p-q} \wedge \mathbf{G}_{\mathrm{m}}^{q}$ for $p \geq q$, the motivic spectra of algebraic K-theory $\mathbf{KGL}$ and hermitian K-theory $\mathbf{KQ}$ are $(2, 1)$- and $(8, 4)$-periodic, respectively. More precisely, there exist Bott elements in the Grothendieck group $\pi_{2,1}\mathbf{KGL} \cong K_0$ and in the Grothendieck–Witt group $\pi_{8,4}\mathbf{KQ} \cong GW_0$ inducing motivic weak equivalences

✉ Oliver Röndigs
oliver.roendigs@uni-osnabrueck.de

Alexey Ananyevskiy
alseang@gmail.com

Paul Arne Østvær
paularne@math.uio.no

1 St. Petersburg Department, Steklov Mathematical Institute, and Chebyshev Laboratory, St. Petersburg State University, St. Petersburg, Russia

2 Institut für Mathematik, Universität Osnabrück, Osnabrück, Germany

3 Department of Mathematics, University of Oslo, Oslo, Norway

$$S^{2,1} \wedge \mathbf{KGL} \xrightarrow{\cong} \mathbf{KGL} \quad \text{and} \quad S^{8,4} \wedge \mathbf{KQ} \xrightarrow{\cong} \mathbf{KQ}. \tag{1}$$

Because of (1), **KGL** and **KQ** are non-connective and should be thought of as "large" motivic spectra. When using K-theoretic invariants to inform the homotopy sheaves of the sphere **1** in [25], it is convenient to employ smaller "connective" versions of the motivic K-theory spectra. The geometrical meaning of this notion is still not well understood.

One fascinating aspect of motivic homotopy theory is that it offers different notions of "connectivity" based on:

- Voevodsky's slice filtration for the localizing triangulated subcategory of effective motivic spectra $\mathbf{SH}^{\mathbf{eff}} \subset \mathbf{SH}$ generated under homotopy colimits by motivic $\mathbf{P}^1$-suspension spectra of smooth schemes [30, §2].
- Morel's homotopy t-structure [18, §6.2] characterized over perfect fields by the vanishing of homotopy sheaves, and its extension to general base schemes in [11, §2.1].
- Spitzweck–Østvær's very effective slice filtration for the full subcategory $\mathbf{SH}^{\mathbf{veff}} \subset \mathbf{SH}$ generated under homotopy colimits and extensions by motivic $\mathbf{P}^1$-suspension spectra of smooth schemes [28, Definition 5.5].

The essential difference between the effective and the very effective slice filtrations is that the former records slices with respect to the multiplicative group scheme $\mathbf{G}_{\mathrm{m}}$ and the latter with respect to the projective line $\mathbf{P}^1$. By construction $\mathbf{SH}^{\mathbf{veff}}$ is a subcategory of $\mathbf{SH}^{\mathbf{eff}}$ closed under the tensor product, but it is not closed under simplicial desuspension and hence not a triangulated subcategory of **SH**. The sphere **1**, algebraic cobordism **MGL**, and quotients thereof such as motivic Moore spectra and the effective cover of **KGL** are examples of very effective motivic spectra. Both of the slice filtrations interact well with A_∞- and E_∞-structures [7], but only the very effective one maps to (the even part of) the topological Postnikov filtration under Betti realization [7, §3.3]. Over perfect fields $\mathbf{SH}^{\mathbf{veff}}$ is the nonnegative part of the t-structure on $\mathbf{SH}^{\mathbf{eff}}$, and the identification of the very effective slices of **KQ** up to extensions in [2, Theorem 16] makes a strong case for further investigations of the very effective slice filtration.

In this paper we argue that the very effective cover **kq** of hermitian K-theory **KQ** is a convenient algebro-geometric generalization of the connective cover ko of real topological K-theory KO. The question of whether there is a motivic spectrum with similar properties as ko was first addressed in [13] and [9, Conjecture 5.8] over the fields of complex and real numbers, respectively. As we learned from the referee, work by Mike Hill and Kyle Ormsby concerning such a motivic spectrum was presented during a sectional meeting of the AMS in 2014.

Section 2 begins with some preliminary results on **KGL**. We identify the effective and very effective covers of **KGL** over perfect fields of characteristic $\neq 2$, and similarly for **KGL**/2 and base schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. Proposition 11 relates the very effective covers **kq** of **KQ** and **kgl** of **KGL** to the motivic Hopf map $\eta\colon \mathbf{A}^2 \smallsetminus \{0\} \to \mathbf{P}^1$ via the cofiber sequence

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \to \mathbf{kgl}. \tag{2}$$

Over the complex numbers, (2) has been obtained independently by Bachmann. The same result holds for η, **KQ**, **KGL**, and base schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ by [23, Theorem 3.4], but it is plainly false for the effective covers of **KQ** and **KGL** by the proof of [24, Corollary 5.1]. By using (2) we identify the Betti realization of **kq** with ko and calculate the mod-2 motivic cohomology $\mathbf{M}\mathbb{Z}/2^{\star}\mathbf{kq}$ as $\mathcal{A}^{\star}//\mathcal{A}^{\star}(1)$; the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^{\star}$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^{\star}$-subalgebra generated by Sq^1 and Sq^2 [12,32]. By dualizing, the mod-2 motivic homology $\mathbf{M}\mathbb{Z}/2_{\star}\mathbf{kq}$ identifies with $\mathcal{A}_{\star}\square_{\mathcal{A}_{\star}(1)}\mathbf{M}\mathbb{Z}/2_{\star}$ as an $\mathcal{A}_\star$-comodule algebra, and by change-of-rings the $\mathbf{M}\mathbb{Z}/2$-based Adams spectral sequence for $\mathbf{kq}$ takes the form

$$\mathrm{Ext}^{*,\star}_{\mathcal{A}_\star(1)}(\mathbf{M}\mathbb{Z}/2_\star, \mathbf{M}\mathbb{Z}/2_\star) \Rightarrow \mathbf{kq}_{\star}{}^{\wedge}_{2,\eta}. \tag{3}$$

As indicated in the notation, the filtered target groups of the spectral sequence (3) are all $(2, \eta)$-completed. The Ext-algebra over $\mathcal{A}_\star(1)$ appearing in (3) is accessible via homological algebra. For explicit calculations with (3) we refer to [9] and [13].

Section 3 is concerned with slice calculations. The negative slices of $\mathbf{kq}$ are evidently zero because $\mathbf{kq}$ is an effective motivic spectrum. Over a perfect field of characteristic $\neq 2$ and $i \geq 0$, we show in Theorem 17 the calculation

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{2n,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+2,2n}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n-2,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4n,2n}\mathbf{M}\mathbb{Z} & q = 2n, \\ \Sigma^{2n+1,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+3,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n+1,2n+1}\mathbf{M}\mathbb{Z}/2 & q = 2n+1. \end{cases} \tag{4}$$

The slices of $\mathbf{kq}$ are considerably "smaller" than those of $\mathbf{KQ}$ [23]. This is a helpful fact which is used in the calculation of the first stable homotopy groups of motivic spheres [25].

An immediate consequence of (4) is the explicit form of the slice spectral sequence given by mod-2 motivic cohomology groups $h^\star$ and integral motivic cohomology groups $H^\star$

$$\pi_{p,w}\mathsf{s}_q\mathbf{kq} = \begin{cases} h^{2n-p,2n-w} \oplus \cdots \oplus h^{4n-2-p,2n-w} \oplus H^{4n-p,2n-w} & q = 2n \\ h^{2n+1-p,2n+1-w} \oplus \cdots \oplus h^{4n+1-p,2n+1-w} & q = 2n+1 \end{cases} \Rightarrow \mathbf{kq}_{p,w}. \tag{5}$$

In Theorem 20 we identify the d_1-differentials in (5) in terms of motivic Steenrod operations. We also calculate the slices and the slice differentials for the η-inversion of $\mathbf{kq}$.

In Sect. 4 we identify the 0-line of $\mathbf{kq}$ with the Milnor–Witt K-theory over fields of characteristic not 2, and determine the associated graded for the groups on the 1-line of $\mathbf{kq}$.

Throughout the paper we employ the following assumptions and notations. In all results

F, S	Perfect field, finite dimensional separated noetherian scheme
$\mathbf{Sm}_S$	Smooth schemes of finite type over S
$S^{s,t}$, $\Omega^{s,t}$, $\Sigma^{s,t}$	Motivic (s,t)-sphere, (s,t)-loop space, (s,t)-suspension
$\mathbf{SH}$, $\mathbf{SH}^{\mathbf{eff}}$	Motivic and effective motivic stable homotopy categories of S
E, $\mathbf{1} = S^{0,0}$	Generic motivic spectrum, the motivic sphere spectrum
Λ, $\mathbf{M}A$	Ring, motivic Eilenberg–MacLane spectra of a Λ-module A
$\mathbf{KGL}$, $\mathbf{KQ}$, $\mathbf{KW}$	Algebraic and hermitian K-theory, Witt-theory
f_q, $\tilde{\mathsf{f}}_q$, s_q	qth effective cover, very effective cover, and slice

concerning $\mathbf{KQ}$ and $\mathbf{kq}$ we assume that 2 is invertible on the base scheme S, as for $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and following [27] we impose the condition that

$$S \text{ is essentially smooth over a Dedekind domain.} \tag{6}$$

Applications will mostly concern perfect fields of characteristic $\neq 2$.

2 Connecting connective K-theories

Definition 1 Following [28, §5] we let $\mathbf{kq} \to \mathbf{KQ}$ denote the very effective cover of the hermitian K-theory spectrum $\mathbf{KQ}$ of quadratic forms [10] and let $\mathbf{kgl} \to \mathbf{KGL}$ denote the very effective cover of the algebraic K-theory spectrum $\mathbf{KGL}$ [29].

Remark 2 Following [2,28], and working over F we can identify the very effective cover $\tilde{\mathsf{f}}_q\mathsf{E}$ of E with $\mathsf{f}_0(\mathsf{E}_{\geq 0})$, the effective cover of the connective cover $\mathsf{E}_{\geq 0}$ of E with respect to the homotopy t-structure on $\mathbf{SH}$ [18].

Lemma 3 *If F admits a complex embedding, the Betti realization of* $\mathbf{kgl}$ *coincides with the connective cover* ku *of the complex topological K-theory spectrum* KU *in the topological stable homotopy category.*

Proof Recall from [28, Proposition 5.12] that $\mathbf{kgl}$ is a homotopy quotient of $\mathbf{MGL}$ under the orientation or Todd genus map, and similarly but easier that ku is a homotopy quotient of $\mathbf{MU}$. The Betti realization functor [21, Appendix A] preserves homotopy colimits, and sends $\mathbf{MGL}$ to $\mathbf{MU}$. □

Lemma 4 *Over F the effective and very effective covers of* $\mathbf{KGL}$ *coincide in* $\mathbf{SH}$.

Proof When $\mathsf{char}(F) = 0$ this is shown in [28, Corollary 5.13] by writing the effective cover of $\mathbf{KGL}$ as a homotopy quotient of $\mathbf{MGL}$ (the latter is very effective over any base scheme [28, Theorem 5.7]). If $\mathsf{char}(F) > 0$ we follow the proof of [2, Theorem 16] where the effective cover $\mathsf{f}_0\mathbf{KGL} \to \mathbf{KGL}$ is shown to be connective. For $t \geq 0$ the presheaf on $\mathbf{Sm}_F$

$$X \mapsto [\Sigma^{s,t} X_+, \mathsf{f}_0\mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s < 2t$, e.g., for $s - t < 0$ (this holds if X is regular, hence over any regular base scheme S). The case $t = 0$ implies by [2, Proposition 4] that $\mathsf{f}_0\mathbf{KGL}$ is connective, and by [2, Lemma 10] that $\mathsf{f}_0\mathbf{KGL}$ is the very effective cover. □

Remark 5 Lemma 4 holds more generally for motivic Landweber exact spectra over a field in the sense of [20].

Over a noetherian scheme S of finite Krull dimension d, the presheaf on $\mathbf{Sm}_S$

$$X \mapsto [\Sigma^{s,t} X_+, \mathbf{KGL}] = KH_{s-2t}(X)$$

is zero for $s - 2t < -d$ by [15], since $\mathbf{KGL}$ represents homotopy K-theory over S [4]. Thus for $t \geq q$, the presheaf

$$X \mapsto [\Sigma^{s,t} X_+, \mathsf{f}_q\mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s - t + d < q$, and $\mathsf{f}_q\mathbf{KGL}$ is q-connected in the sense of [25, Definition 3.16]. If the very effective slice filtration coincides with the combination of the homotopy t-structure and the effective slice filtration over S, then $\mathsf{f}_0\mathbf{KGL}$ is the very effective cover, i.e., the effective and very effective slices of $\mathbf{KGL}$ agree. We can argue differently for $\mathbf{KGL}/2$ when 2 is invertible as follows (this proof can also be adapted to motivic Landweber exact spectra).

Lemma 6 *Over a base scheme S as in (6) on which 2 is invertible, the effective and very effective covers of* $\mathbf{KGL}/2$ *coincide in* $\mathbf{SH}$.

Proof We claim $\mathbf{KGL}/2$ affords the description as a homotopy quotient of $\mathbf{MGL}/2$ for the generators of the Lazard ring $x_i \in \pi_{2i,i}\mathbf{MGL}$. Since $\mathbf{MGL}$ is effective the orientation map for $\mathbf{KGL}$ factors through

$$\phi\colon \mathbf{MGL} \to \mathsf{f}_0\mathbf{KGL}.$$

For $i \geq 2$ we have $\pi_{2i,i}\phi(x_i) = 0$, so that ϕ admits a factorization

$$\mathbf{MGL}/(x_2, x_3, \dots) \to \mathsf{f}_0\mathbf{KGL}.$$

We claim there is a canonically induced motivic weak equivalence

$$\psi\colon \mathbf{MGL}/(2, x_2, x_3, \dots) \xrightarrow{\cong} \mathsf{f}_0\mathbf{KGL}/2.$$

The map ψ yields an isomorphism on slices by [27, Theorem 10.3] and an appropriate adaption of [26, Proposition 5.4]. We show that $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\psi$ is a map between contractible motivic spectra, i.e., ψ is a map between slice complete spectra. For $\mathbf{KGL}/2$ this follows by the argument prior to Lemma 6: By [15] we know $\mathsf{f}_q\mathbf{KGL}$ is q-connected in the sense of [25, Definition 3.16]. Thus $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\mathbf{KGL} \cong *$, and likewise for $\mathsf{f}_0\mathbf{KGL}/2$. The contractibility of $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\mathbf{MGL}/(2, x_2, x_3, \dots)$ follows from the description of the covers $\mathsf{f}_q\mathbf{MGL}$ in the proof of [26, Theorem 4.6]. To conclude for ψ we use that slices detect motivic weak equivalences between slice complete motivic spectra, cf. [11, §8.3]. Recall that $\mathbf{MGL}$ is a very effective motivic spectrum [28, Theorem 5.7]. The lemma follows from the canonically induced motivic weak equivalences in the commutative diagram

$$\begin{array}{ccccc} \tilde{\mathsf{f}}_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & \tilde{\mathsf{f}}_0\mathsf{f}_0\mathbf{KGL}/2 & \xrightarrow{\cong} & \tilde{\mathsf{f}}_0\mathbf{KGL}/2 \\ \cong\downarrow & & \downarrow & & \downarrow \\ \mathsf{f}_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & \mathsf{f}_0\mathsf{f}_0\mathbf{KGL}/2 & \xrightarrow{\cong} & \mathsf{f}_0\mathbf{KGL}/2. \end{array}$$

□

The Bott element $\mathbf{P}^1 \to \mathbf{KGL}$ lifts canonically to a map $\beta\colon \mathbf{P}^1 \to \mathbf{kgl}$ because $\mathbf{P}^1$ is very effective. Let γ denote the canonical composite

$$\mathbf{kgl} \to \mathsf{f}_0(\mathbf{KGL}) \to \mathsf{s}_0\mathbf{KGL}.$$

Proposition 7 *Over F multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl} \xrightarrow{\beta} \mathbf{kgl} \xrightarrow{\gamma} \mathbf{M}\mathbb{Z} \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}.$$

Proof By Lemma 4 we have $\mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \cong \mathbf{kgl}$ and by $(2,1)$-periodicity $\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \cong \Sigma^{-2,-1}\mathbf{kgl}$. Let $\beta'\colon \mathsf{f}_1(\mathbf{KGL}_{\geq 1}) \to \mathsf{f}_0(\mathbf{KGL}_{\geq 0})$ denote the natural map in the very effective slice filtration for $\mathbf{KGL}$ The commutative diagram

$$\begin{array}{ccccccc} \Sigma^{2,1}\mathsf{f}_0(\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & \mathsf{f}_1(\Sigma^{2,1}\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & \mathsf{f}_1((\Sigma^{2,1}\mathbf{KGL})_{\geq 1}) & \xrightarrow{\cong} & \mathsf{f}_1(\mathbf{KGL}_{\geq 1}) \\ \beta\downarrow & & & & & & \downarrow\beta' \\ \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) & & & \xrightarrow{\mathrm{id}} & & & \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \end{array}$$

shows that it corresponds to multiplication with the Bott element on $\mathbf{kgl}$. The cofiber sequence

$$\mathsf{f}_1(\mathbf{KGL}_{\geq 1}) \xrightarrow{\beta'} \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \to \mathbf{M}\mathbb{Z},$$

for the very effective zero slice of **KGL** [2, Lemma 7], which coincides with the usual zero slice $\mathsf{s}_0\mathbf{KGL} \cong \mathbf{M}\mathbb{Z}$ computed in [16], [31], concludes the proof. □

Proposition 8 *Over a base scheme S as in (6) on which 2 is invertible, multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl}/2 \xrightarrow{\beta} \mathbf{kgl}/2 \xrightarrow{\gamma} \mathbf{M}\mathbb{Z}/2 \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}/2.$$

Proof This follows from Lemma 6. □

Lemma 9 *If 2 is invertible on a base scheme S as in (6), then the composite*

$$\mathbf{M}\mathbb{Z} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\delta\wedge\mathbf{M}\mathbb{Z}/2} \Sigma^{3,1}\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\gamma\wedge\mathbf{M}\mathbb{Z}/2} \Sigma^{3,1}\mathbf{M}\mathbb{Z} \wedge \mathbf{M}\mathbb{Z}/2$$

is given by multiplication with the first Milnor operation

$$Q_1 = Sq^1Sq^2 + Sq^2Sq^1 \colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{3,1}\mathbf{M}\mathbb{Z}/2.$$

Proof The proof of Lemma 6 shows **KGL** and $\mathsf{f}_0\mathbf{KGL}/2$ are invariant under base change, being homotopy quotients of **MGL**. The same holds for $\mathbf{M}\mathbb{Z}$ and the dual motivic Steenrod algebra $\mathbf{M}\mathbb{Z}/2 \wedge \mathbf{M}\mathbb{Z}/2$ by [27, Section 9, Theorem 10.26]. Hence it suffices to show the statement in the case $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. The inclusion $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{C}$ induces a bijection on the bidegree (3, 1) summand of the motivic Steenrod algebra (which in both cases is given as in [27, Theorem 10.26, Remark 10.27] over the mod two motivic cohomology of the base), because the Picard group of $\mathbb{Z}[\frac{1}{2}]$ vanishes. Hence it remains to prove the statement over $\mathbb{C}$, where it follows by complex realization from Lemma 3 and the corresponding topological result [1, p. 366]. □

Remark 10 Following [13, Theorem 5.4], Lemma 9 shows the mod-2 motivic cohomology $\mathbf{M}\mathbb{Z}/2^{\star}\mathbf{kgl}$ is the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^{\star}$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^{\star}$-subalgebra generated by $\mathsf{Q}_0 = \mathsf{Sq}^1$ and Q_1.

Proposition 11 *Over a field of characteristic $\neq 2$, multiplication with the Hopf map η induces a cofiber sequence*

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\mathrm{f}} \mathbf{kgl} \xrightarrow{\mathrm{h}} \Sigma^{2,1}\mathbf{kq}. \tag{7}$$

Here f *and* h *are functorially induced by the forgetful and hyperbolic maps between algebraic and hermitian K-theory, respectively.*

Proof Consider the fiber F of the naturally induced forgetful map $\mathrm{f}_{\geq 0} \colon \mathbf{KQ}_{\geq 0} \to \mathbf{KGL}_{\geq 0}$. Since f_0 is a triangulated functor, $\mathsf{f}_0(\mathsf{F})$ is the fiber of $\mathrm{f} := \mathsf{f}_0(\mathrm{f}_{\geq 0})$. The composite map

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\mathrm{f}} \mathbf{kgl}$$

is trivial because the first negative algebraic K-group $\pi_{1,1}\mathbf{kgl} = \pi_{1,1}\mathbf{KGL} = K_{-1}$ vanishes over regular schemes. We show there is an induced motivic weak equivalence $\Sigma^{1,1}\mathbf{kq} \to \mathsf{f}_0(\mathsf{F})$ of effective motivic spectra by checking the map of homotopy sheaves $\pi_{s,t}$ is an isomorphism for $t \geq 0$. This follows if (7) induces a long exact sequence of sheaves for $t \geq 0$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_{s,t}\Sigma^{1,1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{\mathrm{f}} & \pi_{s,t}\mathbf{kgl} & \xrightarrow{\mathrm{h}} \cdots \\
& \downarrow \cong & & \downarrow = & & \downarrow = & \\
\cdots \longrightarrow & \pi_{s-1,t-1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{\mathrm{f}} & \pi_{s,t}\mathbf{kgl} & \xrightarrow{\mathrm{h}} \cdots .
\end{array} \tag{8}$$

By construction, (8) is exact for $t \geq 1$ and $s \geq t$ since in the said range it coincides with the long exact sequence

$$\cdots \to \pi_{s,t}\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \pi_{s,t}\mathbf{KQ} \xrightarrow{\mathrm{f}} \pi_{s,t}\mathbf{KGL} \xrightarrow{\mathrm{h}} \cdots$$

induced by the Wood cofiber sequence for η, $\mathbf{KQ}$, and $\mathbf{KGL}$ [23, Theorem 3.4].

If $t \geq 1$ and $s = t$, $\pi_{t,t}(\mathrm{f})\colon \pi_{t,t}\mathbf{kq} \to \pi_{t,t}\mathbf{kgl}$ is surjective since its target is trivial. Thus (8) is exact for $t \geq 1$ and all s; recall that $\pi_{s,t}\mathbf{kq} = \pi_{s,t}\mathbf{kgl} = 0$ for all $s < t$.

It remains to consider the case $t = 0$. By [2, Theorem 16] the composite

$$\mathsf{f}_0(\mathbf{KQ}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KQ}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KQ}_{\geq -1})$$

is an equivalence. The canonical map $\mathbf{KQ}_{\geq 0} \to \mathbf{KQ}_{\geq -1}$ is an isomorphism on homotopy sheaves $\pi_{s,t}$ for all $t \geq -1$ and all s. When $s < t-1$ and $s \geq t$ this follows by construction. The case $s = t-1$ holds since $\pi_{t-1,t}\mathbf{KQ} = 0$ for all $t \geq -1$. More precisely, the vanishing for $t \geq 0$ is implied by comparison with Witt theory because $\pi_{t-1,t}\mathbf{KW} = 0$ for all t. The case $t = -1$ follows from the long exact sequence

$$\cdots \to \pi_{0,0}\mathbf{KGL} \xrightarrow{0} \pi_{-2,-1}\mathbf{KQ} \xrightarrow{\eta} \pi_{-1,0}\mathbf{KQ} \xrightarrow{\mathrm{f}} \pi_{-1,0}\mathbf{KGL} = 0,$$

and surjectivity of the rank map $\mathrm{f}\colon \pi_{0,0}\mathbf{kq} \to \pi_{0,0}\mathbf{kgl}$. It follows that there is a canonical motivic weak equivalence

$$\mathsf{f}_0(\mathbf{KQ}_{\geq 0}) \xrightarrow{\cong} \mathsf{f}_{-1}(\mathbf{KQ}_{\geq 0}),$$

which implies exactness of (8) for $t = 0$. □

Lemma 12 *If 2 is invertible on a base scheme S as in (6), then the composite*

$$\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathrm{h}\wedge\mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathrm{f}\wedge\mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2$$

is given by multiplication with $\mathsf{Sq}^2\colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{2,1}\mathbf{M}\mathbb{Z}/2$.

Proof As in the proof of Lemma 9 it suffices to work over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and hence over $\mathbb{C}$. The result follows from Lemma 13 and the corresponding topological statement. □

Lemma 13 *If F admits a complex embedding, the Betti realization of* $\mathbf{kq}$ *coincides with the connective cover* ko *of the real topological K-theory spectrum* KO *in the topological stable homotopy category.*

Proof This follows since the Betti realization sends $\mathbf{KQ}$ to KO, $\mathbf{kgl}$ to ku by Lemma 3, and preserves the Wood cofiber sequence. □

Remark 14 As in [13, Theorem 5.11], Lemma 9 identifies $\mathbf{M}\mathbb{Z}/2^{\star}\mathbf{kq}$ with the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^{\star}$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^{\star}$-subalgebra generated by Sq^1 and Sq^2, and the homotopy of $\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2$ as a comodule over the dual motivic Steenrod algebra recorded by the $\mathbf{M}\mathbb{Z}/2$-based Adams spectral sequence for $\mathbf{kq}$ (3).

Remark 15 If F admits a real embedding, the Betti realization of $\mathbf{kq}$ acquires the structure of a genuine $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant spectrum [8]. Equivariant forms of connective real and complex topological K-theory exist. The survey [5], pointed out to us by John Rognes, emphasizes a particularly well-behaved form. A natural question following Lemma 13 is whether the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant Betti realization of $\mathbf{kq}$ is this $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant connective real topological K-theory spectrum. Dan Isaksen sketched an argument, based on the Adams spectral sequence, that the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant Betti realization of $\mathbf{kq}$ coincides with the form employed in [6], at least after completion at the prime two.

Next we observe that $\mathbf{kgl}$ differs from the cover of algebraic K-theory introduced in [13]. By the cofiber sequence

$$\mathbf{kgl} = \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \to \mathsf{s}_{-1}\mathbf{KGL} = \Sigma^{-2,-1}\mathbf{M}\mathbb{Z},$$

we obtain a long exact sequence and an isomorphism

$$\cdots \to \pi_{0,-1}\mathsf{s}_{-1}\mathbf{KGL} \to \pi_{-1,-1}\mathbf{kgl} \xrightarrow{\cong} \pi_{-1,-1}\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \to \pi_{-1,-1}\mathsf{s}_{-1}\mathbf{KGL} \to \cdots . \tag{9}$$

The outer terms in (9) are trivial. Since $\pi_{-1,-1}\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \cong \pi_{-1,-1}\mathbf{KGL}$ it follows that $\pi_{-1,-1}\mathbf{kgl} \cong K_1(F) \cong F^{\times}$. Over the complex numbers, this calculation distinguishes $\mathbf{kgl}$ from the (2-complete) positive cellular cover of $\mathbf{KGL}$ in [13] because $\pi_{-1,-1}$ of the latter is trivial by construction.

Finally, we remark that $\mathbf{kq}$ does not coincide with the effective cover $\mathsf{f}_0\mathbf{KQ}$ featuring in the solution of the homotopy limit problem for the C_2-action on $\mathbf{kgl}$ in [24].

3 Slice computations

We shall identify the slices of $\mathbf{kq}$ similarly to the slices of $\mathbf{KQ}$ in [23]. The crucial ingredients are the Wood cofiber sequence (7) and the slices of connective algebraic K-theory $\mathbf{kgl}$.

Theorem 16 *Over F the canonical map $\mathbf{kgl} \to \mathbf{KGL}$ induces an isomorphism on all non-negative slices. The negative slices of $\mathbf{kgl}$ are zero.*

Proof Since $\mathbf{kgl} = \mathsf{f}_0\mathbf{KGL}$ by Lemma 4, this follows by construction. □

Identifying the slices of $\mathbf{kq}$ is more involved because $\mathbf{kq} \neq \mathsf{f}_0\mathbf{KQ}$.

Theorem 17 *When char$(F) \neq 2$ the nonnegative slices of $\mathbf{kq}$ are given as*

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{2n,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+2,2n}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n-2,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4n,2n}\mathbf{M}\mathbb{Z} & q = 2n, \\ \Sigma^{2n+1,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+3,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n+1,2n+1}\mathbf{M}\mathbb{Z}/2 & q = 2n+1. \end{cases}$$

The negative slices of $\mathbf{kq}$ are zero. Moreover, the canonical map $\mathbf{kq} \to \mathbf{KQ}$ induces a natural inclusion on slices, and respects the multiplicative structure.

Proof Since $\mathbf{kq} = \mathsf{f}_0(\mathbf{KQ}_{\geq 0})$ is (very) effective, its negative slices are zero. Applying the slice functor to (7) yields a cofiber sequence. The natural isomorphism $\mathsf{s}_q \circ \Sigma^{1,1} \cong \Sigma^{1,1} \circ \mathsf{s}_{q-1}$ of [23, Lemma 2.1] shows the forgetful map $\mathrm{f}\colon \mathbf{kq} \to \mathbf{kgl}$ induces an isomorphism on zero slices

$$\mathsf{s}_0\mathbf{kq} \xrightarrow{\cong} \mathsf{s}_0\mathbf{kgl},$$

and likewise for the unit map $\mathbf{1} \to \mathbf{kq}$.

For the 1-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_0\mathbf{kq} = \Sigma^{1,1}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_1\mathbf{kq} \xrightarrow{\mathsf{s}_1\mathrm{f}} \mathsf{s}_1\mathbf{kgl} = \Sigma^{2,1}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_1\mathrm{h}} \Sigma^{2,1}\mathsf{s}_0\mathbf{kq} = \Sigma^{2,1}\mathbf{M}\mathbb{Z}.$$

Here $\mathsf{s}_1\mathrm{h}$ can be identified with an integer $n \in \mathbb{Z}$. Comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [23, §4.3] shows that $n = 2$, so that $\mathsf{s}_1\mathbf{kq} = \Sigma^{1,1}\mathbf{M}\mathbb{Z}/2$.

For the 2-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_1\mathbf{kq} = \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\eta} \mathsf{s}_2\mathbf{kq} \xrightarrow{\mathsf{s}_2\mathrm{f}} \mathsf{s}_2\mathbf{kgl} = \Sigma^{4,2}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_2\mathrm{h}} \Sigma^{2,1}\mathsf{s}_1\mathbf{kq} = \Sigma^{3,2}\mathbf{M}\mathbb{Z}/2.$$

Hence $\mathsf{s}_2\mathrm{h} = 0$, the cofiber sequence splits, and we get $\mathsf{s}_2\mathbf{kq} = \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4,2}\mathbf{M}\mathbb{Z}$. Moreover, $\mathsf{s}_2\mathrm{f}$ is the projection map onto $\Sigma^{4,2}\mathbf{M}\mathbb{Z}$.

For the 3-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_2\mathbf{kq} = \Sigma^{3,3}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{5,3}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_3\mathbf{kq} \xrightarrow{\mathsf{s}_3\mathrm{f}} \mathsf{s}_3\mathbf{kgl} = \Sigma^{6,3}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_3\mathrm{h}} \Sigma^{2,1}\mathsf{s}_2\mathbf{kq}.$$

Here $\mathsf{s}_3\mathrm{h}$ maps trivially to $\Sigma^{4,3}\mathbf{M}\mathbb{Z}/2$, while the component of $\mathsf{s}_3\mathrm{h}$ mapping to $\Sigma^{6,3}\mathbf{M}\mathbb{Z}$ can be identified with an integer $n \in \mathbb{Z}$. We deduce $n = 2$ by comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [23, §4.3]. Hence we obtain $\mathsf{s}_3\mathbf{kq} \cong \Sigma^{3,3}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{5,3}\mathbf{M}\mathbb{Z}/2$.

Iterating these arguments produces the claimed calculation. □

Remark 18 Contrary to the calculation of the slices of **KQ** in [23] there is no "mysterious summand" appearing in Theorem 17 thanks to the connectivity of **kq**. Each slice of **kq** is a finite sum of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. The odd slices of **kq** are cellular of finite type for every F [25, §3.3], and likewise for all the slices when $\mathsf{char}(F) = 0$.

The multiplicative structure on the graded slices $\mathsf{s}_*\mathbf{kq}$ can be identified similarly to $\mathsf{s}_*\mathbf{KQ}$ as in [22, Theorem3.3]. In more details, there is a motivic weak equivalence

$$\mathsf{s}_*\mathbf{kq} \cong \mathbf{M}\mathbb{Z}[\eta, \sqrt{\alpha}]/(2\eta = 0, \eta^2 \xrightarrow{\delta} \sqrt{\alpha})$$

where η has bidegree $(1, 1)$ and $\sqrt{\alpha}$ is a class of bidegree $(4, 2)$ arising from the $(8, 4)$-periodicity operator on **KQ** mentioned in the introduction. The multiplicative structure is not quite that of a polynomial ring; for example, the multiplication $\mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} \to \mathsf{s}_2\mathbf{kq}$ is given as the composition

$$\mathsf{s}_1\mathbf{kq} \wedge_{\mathsf{s}_0\mathbf{kq}} \mathsf{s}_1\mathbf{kq} \cong \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{3,2}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathrm{id}\vee\delta} \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4,2}\mathbf{M}\mathbb{Z}$$

where $\delta\colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{1,0}\mathbf{M}\mathbb{Z}$ is the connecting map. Moreover, the action of the Hopf map η on the slices of **kq** can be read off from the proof of Theorem 17, giving us the next result.

Theorem 19 *When* $\mathsf{char}(F) \neq 2$ *the slices of* $\mathbf{kq}[\frac{1}{\eta}] = \mathbf{KW}_{\geq 0}$ *are given by*

$$\mathsf{s}_q(\mathbf{KW}_{\geq 0}) = \Sigma^{q,q}\Big(\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2,0}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4,0}\mathbf{M}\mathbb{Z}/2 \vee \cdots\Big),$$

and

$$\mathsf{s}_*(\mathbf{KW}_{\geq 0}) \cong \mathbf{M}\mathbb{Z}[\eta^{\pm 1}, \sqrt{\alpha}]/(2\eta = 2\sqrt{\alpha} = 0, \eta^2 \xrightarrow{Sq^1} \sqrt{\alpha}).$$

The canonical map $\mathbf{KW}_{\geq 0} \to \mathbf{KW}$ *induces the natural inclusion on slices, and respects the multiplicative structure.*

As in the case of $\mathsf{s}_*\mathbf{kq}$, the multiplicative structure is not quite polynomial, and because of the occurrence of Sq^1 not $\mathbf{M}\mathbb{Z}/2$-linear. Let $\mathbf{d}_1^{\mathbf{kq}}(q)\colon \mathsf{s}_q\mathbf{kq} \to \Sigma^{1,0}\mathsf{s}_{q+1}\mathbf{kq}$ denote the first slice differential as a map of motivic spectra, and similarly for $\mathbf{KW}_{\geq 0}$. By Theorem 17, $\mathbf{d}_1^{\mathbf{kq}}(q)$ is a map between finite sums of motivic Eilenberg–MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. Thus $\mathbf{d}_1^{\mathbf{kq}}(q)$ can be described via its restriction $\mathbf{d}_1^{\mathbf{kq}}(q, i)$ to the summand corresponding to the unique suspension $\Sigma^{q+i,q}$. We note that $\mathbf{d}_1^{\mathbf{kq}}(q,i)$ splits into at most three nontrivial components.

Let $\tau \in h^{0,1} \cong \mu_2(F)$ and $\rho \in h^{1,1} \cong F^\times/2$ denote the classes represented by $-1 \in F$; $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group of F in degree p and weight q. There are canonical maps $\mathrm{pr}\colon \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$ and $\partial\colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{1,0}\mathbf{M}\mathbb{Z}$.

Theorem 20 *When char$(F) \neq 2$ the $\mathbf{d}_1$-differential in the slice spectral sequence for* **kq** *is given by*

$$\mathbf{d}_1^{\mathbf{kq}}(q,i) = \begin{cases} (Sq^3Sq^1, Sq^2, 0) & q-1 > i \equiv 0 \bmod 4 \\ (Sq^3Sq^1, Sq^2+\rho Sq^1, \tau) & q-1 > i \equiv 2 \bmod 4 \end{cases}$$

$$\mathbf{d}_1^{\mathbf{kq}}(q,q) = \begin{cases} (0, Sq^2 \circ \mathrm{pr}, 0) & q \equiv 0 \bmod 4 \\ (0, Sq^2 \circ \mathrm{pr}, \tau \circ \mathrm{pr}) & q \equiv 2 \bmod 4 \end{cases}$$

$$\mathbf{d}_1^{\mathbf{kq}}(q,q-1) = \begin{cases} (\partial Sq^2Sq^1, Sq^2, 0) & q \equiv 1 \bmod 4 \\ (\partial Sq^2Sq^1, Sq^2+\rho Sq^1, \tau) & q \equiv 3 \bmod 4. \end{cases}$$

Proof Use Theorem 17 and the identification of $\mathbf{d}_1^{\mathbf{KQ}}$ for **KQ** in [23, Theorem 5.5]. □

Theorem 21 *When char$(F) \neq 2$ the $\mathbf{d}_1$-differential in the slice spectral sequence for* $\mathbf{KW}_{\geq 0}$ *is given by*

$$\mathbf{d}_1^{\mathbf{KW}_{\geq 0}}(q,i) = \begin{cases} (Sq^3Sq^1, Sq^2, 0) & i \equiv 0 \bmod 4 \\ (Sq^3Sq^1, Sq^2+\rho Sq^1, \tau) & i \equiv 2 \bmod 4. \end{cases}$$

Proof This follows from Theorem 19 and the identification of $\mathbf{d}_1^{\mathbf{KW}}$ for **KW** recorded in [23, Theorem 5.3]. □

Following [22, §4] we calculate the first slice differentials for **kq** and $\mathbf{KW}_{\geq 0}$ in terms of the multiplicative generators for their slices.

We note that $\mathsf{d}_1^{\mathbf{kq}}(\sqrt{\alpha}^m\eta^n)$ is given by

$$\begin{cases} \tau\sqrt{\alpha}^{m-1}\eta^{n+3} + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m \equiv 1(2), n > 1 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m \equiv 0(2), n > 1 \\ \tau\sqrt{\alpha}^{m-1}\eta^4 + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^2 + \delta\mathsf{Sq}^2\mathsf{Sq}^1\sqrt{\alpha}^{m+1} & m \equiv 1(2), n = 1 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^2 + \delta\mathsf{Sq}^2\mathsf{Sq}^1\sqrt{\alpha}^{m+1} & m \equiv 0(2), n = 1 \\ \tau\,\mathrm{pr}\sqrt{\alpha}^{m-1}\eta^3 + \mathsf{Sq}^2\mathrm{pr}\sqrt{\alpha}^m\eta & m \equiv 1(2), n = 0 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta & m \equiv 0(2), n = 0, \end{cases} \tag{10}$$

while $\mathsf{d}_1^{\mathbf{KW}_{\geq 0}}(\sqrt{\alpha}^m\eta^n)$ is given by

$$\begin{cases} \tau\sqrt{\alpha}^{m-1}\eta^{n+3} + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m \equiv 1(2) \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m \equiv 0(2). \end{cases} \tag{11}$$

Remark 22 The corresponding formula for $\mathsf{d}_1^{\mathbf{KQ}}(\sqrt{\alpha}^m\eta^n)$ in [22, §4] contains a typo when $m \equiv 1(2)$, $n = 0$. We thank Bert Guillou for pointing this out to us.

Remark 23 Bachmann [2] determined the very effective slices of $\mathbf{KQ}$ and hence of $\mathbf{kq}$ up to extensions. Additional work is needed to identify the corresponding first very effective slice differentials. A first step is to calculate the endomorphisms of the very effective zero slice of $\mathbf{KQ}$. The very effective slices of $\mathbf{KW}_{\geq 0}$ were determined up to extensions in [2, Lemma 6].

4 Homotopy computations

First we identify the target of the slice spectral sequences for the sphere and very effective hermitian K-theory.

Theorem 24 *Over a field F of characteristic $\neq 2$ there are conditionally convergent slice spectral sequences*

$$\pi_\star s_* \mathbf{1} \Longrightarrow \pi_\star \mathbf{1}^\wedge_\eta, \tag{12}$$

and

$$\pi_\star s_* \mathbf{kq} \Longrightarrow \pi_\star \mathbf{kq}^\wedge_\eta. \tag{13}$$

Proof Here (12) is shown in [25, §3]. The only issue in (13) is to identify the quotient of $\mathbf{kq}$ by η with a slice complete spectrum [24, §4]. This follows directly from Lemma 4, Proposition 11 and [24, Lemma 3.11]. □

To formulate our identification of the 0-line of $\mathbf{kq}$ we recall the definition of Milnor–Witt K-theory $K_*^{MW}(F)$ in [19]. It is the quotient of the free associative integrally graded ring on the set of symbols $[F^\times] := \{[u] \mid u \in F^\times\}$ in degree 1 and η in degree -1 by the homogeneous ideal enforcing the relations

(1) $[uv] = [u] + [v] + \eta[u][v]$ (η-twisted logarithm),
(2) $[u][v] = 0$ for $u + v = 1$ (Steinberg relation),
(3) $[u]\eta = \eta[u]$ (commutativity), and
(4) $(2 + [-1]\eta)\eta = 0$ (hyperbolic relation).

Milnor–Witt K-theory is ε-commutative for $\varepsilon = -(1+[-1]\eta)$. By work of Morel [17] there is an isomorphism with the graded ring of endomorphisms of the sphere

$$K_*^{MW}(F) \cong \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}\mathbf{1}.$$

Moreover, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck–Witt ring of quadratic forms with its standard presentation, inverting η in $K_*^{MW}(F)$ yields the ring of Laurent polynomials $W(F)[\eta^{\pm 1}]$ over the Witt ring, and $K_*^{MW}(F)/\eta = K_*^M(F)$, the Milnor K-theory ring of F.

Theorem 25 *Over a field F of characteristic $\neq 2$ the unit map $\mathbf{1} \to \mathbf{kq}$ induces an isomorphism on* 0*-lines*

$$K_*^{MW}(F) = \bigoplus_{n\in\mathbb{Z}} K_n^{MW}(F) \overset{\cong}{\to} \bigoplus_{n\in\mathbb{Z}} \pi_{-n,-n}\mathbf{kq}. \tag{14}$$

Proof Recall from [25, §5] the short exact sequence

$$0 \to \pi_{n,n}\mathbf{1} \to \pi_{n,n}\mathbf{1}^{\wedge}_{\eta} \oplus \pi_{n,n}\mathbf{1}[\tfrac{1}{\eta}] \to \pi_{n,n}\mathbf{1}^{\wedge}_{\eta}[\tfrac{1}{\eta}] \to 0. \tag{15}$$

Similarly, following [25, §3], the η-arithmetic square

$$\begin{array}{ccc} \mathbf{kq} & \longrightarrow & \mathbf{kq}[\frac{1}{\eta}] \\ \big\downarrow & & \big\downarrow \\ \mathbf{kq}^{\wedge}_{\eta} & \longrightarrow & \mathbf{kq}^{\wedge}_{\eta}[\frac{1}{\eta}] \end{array}$$

for very effective K-theory yields a short exact sequence

$$0 \to \pi_{n,n}\mathbf{kq} \to \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta} \oplus \pi_{n,n}\mathbf{kq}[\tfrac{1}{\eta}] \to \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}[\tfrac{1}{\eta}] \to 0. \tag{16}$$

Here we use the vanishing of $\pi_{n+1,n}\mathbf{kq}^{\wedge}_{\eta}[\frac{1}{\eta}]$ and $\pi_{n-1,n}\mathbf{kq}$. On the terms contributing to the 0-line, the map from (12) to (13) is an isomorphism. Theorem 20 combined with the same computations as in [25, §4] show the said isomorphism persists to the E^{∞}-page. By invoking Theorem 24 we conclude $\pi_{n,n}\mathbf{1}^{\wedge}_{\eta} \stackrel{\cong}{\to} \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}$ and $\pi_{n,n}\mathbf{1}^{\wedge}_{\eta}[\frac{1}{\eta}] \stackrel{\cong}{\to} \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}[\frac{1}{\eta}]$. As noted above, by [17] we have $\pi_{n,n}\mathbf{1}[\frac{1}{\eta}] \stackrel{\cong}{\to} \pi_{n,n}\mathbf{kq}[\frac{1}{\eta}] \cong \pi_{n,n}\mathbf{KW}_{\geq 0} \cong W(F)$. A straightforward comparison between (15) and (16) allows us to deduce (14). □

Remark 26 It was pointed out to us by Bachmann that the results of [2] yield an isomorphism of the zeroth generalized slices $\tilde{s}_0\mathbf{1} \cong \tilde{s}_0\mathbf{KQ}$. This gives another proof for Theorem 25.

We note the isomorphism $\pi_{n+1,n}\mathbf{kq} \stackrel{\cong}{\to} \pi_{n+1,n}\mathbf{kq}^{\wedge}_{\eta}$ follows as in [25, Proposition 5.3]. Thus for the purpose of identifying the 1-line of $\mathbf{kq}$ we may use Theorem 20 and computations as in [25, §4] to deduce:

Proposition 27 *The only nontrivial terms in* (13) *contributing to* $\pi_{n+1,n}\mathbf{kq}$ *are*

$$E^{\infty}_{n+1,q,n}(\mathbf{kq}) = \begin{cases} h^{-n+1,-n+2}/Sq^2(h^{-n-1,-n+1}) & q=2 \\ h^{-n,-n+1}/Sq^2\mathrm{pr}(H^{-n-2,-n}) & q=1 \\ H^{-n-1,-n} & q=0. \end{cases}$$

Here $h^{i,j}$ and $H^{i,j}$ denote the mod-2 and integral motivic cohomology groups of F in degree i and weight j. This determines the 1-line of $\mathbf{kq}$ up to extensions; these are nontrivial in general, as already the classical computation of $K_3(\mathbb{Q})$ implies. When $n > 1$ we read off the vanishing $\pi_{n+1,n}\mathbf{kq} = 0$. The first nontrivial group on the 1-line is $\pi_{2,1}\mathbf{kq} \cong \mu_2(F) \cong \mathbb{Z}/2$. When $n = 0$ we obtain $\pi_{1,0}\mathbf{kq} \cong \pi_{1,0}\mathbf{KQ} \cong F^{\times}/2 \oplus \mu_2(F)$. Furthermore, there is a short exact sequence

$$0 \to h^{2,3}/\mathsf{Sq}^2(h^{0,2}) \to \pi_{0,-1}\mathbf{kq} \to h^{1,2} \to 0. \tag{17}$$

When $n \leq -2$ the group $\pi_{n+1,n}\mathbf{kq}$ surjects onto the integral motivic cohomology group $H^{-n-1,-n}$, with kernel described by Proposition 27.

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