THE JORDAN ALGEBRAS OF RIEMANN, WEYL AND CURVATURE COMPATIBLE TENSORS

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Abstract. Given the Riemann, or the Weyl, or a generalized curvature tensor K, a symmetric tensor b_{ij} is named 'compatible' with the curvature tensor if $b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0$. Amongst showing known and new properties, we prove that they form a special Jordan algebra, i.e. the symmetrized product of K-compatible tensors is K-compatible.

1. Introduction

Let (M, g) be a $n-$dimensional Riemannian or pseudo-Riemannian manifold, and K_{jklm} a generalized curvature tensor (the Riemann, the Weyl, or any tensor with the algebraic properties of the Riemann tensor). In ref.[15] we introduced this concept: a symmetric tensor b_{ij} is $K-$compatible if

$$b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0. \tag{1}$$

We name (K, b) a *compatible pair*. The motivation was the following theorem [15]: if b_{ij} is $K-$compatible with eigenvectors X, Y, Z and eigenvalues x, y, z with $z \neq x, y$, then:

$$K_{ijlm} X^i Y^j Z^m = 0. \tag{2}$$

It extends a result by Derdziński and Shen [7] who proved the same for the Riemann tensor, with the hypothesis that b_{ij} is a Codazzi tensor, $\nabla_i b_{jk} = \nabla_j b_{ik}$. Despite the increased generality, the replacement of the Codazzi condition with the algebraic condition (1), enabled a far simpler proof of the new theorem.

Equation (1) with Riemann's tensor originally appeared in a paper by Roter, on conformally symmetric spaces ([21] lemma 1). Riemann and Weyl compatible tensors were studied in refs. [16, 18, 10].

Examples of Riemann compatible tensors are the Codazzi tensors [15], the Ricci tensors of Robertson-Walker or perfect-fluid generalized Robertson-Walker space-times [20], the second fundamental form and the Ricci tensor of a hypersurface embedded in a (pseudo)Riemannian manifold [18], the Ricci tensors of 'weakly Z-symmetric' manifolds ($\nabla_i Z_{jk} = A_i Z_{jk} + B_j Z_{ik} + D_k Z_{ij}$ with $Z_{ij} = R_{ij} + \varphi g_{ij}$, $A_k - B_k$ closed 1-form) [17] that include 'weakly Ricci-symmetric' ones ($\varphi = 0$) [24] and others (see [4, 3]), or 'pseudosymmetric manifolds' [8] ($[\nabla_i, \nabla_j] R_{klmp} = L Q_{klmpij}$, where $L \neq -1/3$ is a scalar function and Q is the Tachibana tensor built with the Riemann and Ricci tensors).

A Riemann compatible tensor is also Weyl compatible, but not the opposite. The Ricci tensors of Gödel ([11], th.2), or pseudo-Z symmetric space times [19] are Weyl compatible.

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In sections 2 and 3 we review Riemann and Weyl compatible tensors, with some new results and examples, and their relation with known identities by Lovelock. Then, in sections 4, 5 and 6, we investigate the algebraic properties of generalized curvature tensors and $K-$compatible tensors. The main result is that the latter form a *special Jordan algebra*, i.e. the set of $K-$compatible tensors is closed for the symmetrized product.

2. Riemann compatible tensors

A symmetric tensor is Riemann compatible if:

$$b_i{}^m R_{jklm} + b_j{}^m R_{kilm} + b_k{}^m R_{ijlm} = 0. \tag{3}$$

The relation may be written $b_{(i}{}^m R_{jk)lm} = 0$, where (ijk) denotes the sum on cyclic permutations of the indices. Contraction with the metric tensor g^{jl} gives $R_{km} b_i^m - b_k{}^m R_{mi} = 0$ i.e. b commutes with the Ricci tensor. Contraction with b^{jl} gives $b_i{}^m R_{jklm} b^{jl} + b_k{}^m R_{ijlm} b^{jl} = 0$ i.e. b commutes with the symmetric tensor $\hat{R}_{jm} = R_{jklm} b^{kl}$.

Example 2.1. *Codazzi tensors are Riemann compatible.*
Proof: in the identity $[\nabla_i, \nabla_j] b_{kl} = -R_{ijl}{}^m b_{km} - R_{ijk}{}^m b_{ml}$ sum on cyclic permutations of ijk. The first Bianchi identity $R_{(ijk)}{}^m = 0$, gives:

$$[\nabla_i, \nabla_j] b_{kl} + [\nabla_j, \nabla_k] b_{il} + [\nabla_k, \nabla_i] b_{jl} = -(b_i{}^m R_{jklm} + b_j{}^m R_{kilm} + b_k{}^m R_{ijlm}).$$

The left hand side is zero for Codazzi tensors.

Example 2.2. *If $\nabla_j A_k = p_j A_k$, then $A_i A_j$ is Riemann compatible.*
Proof: $A_i[\nabla_j, \nabla_k] A_l = A_i(\nabla_j p_k - \nabla_k p_j) A_l = A_l [\nabla_j, \nabla_k] A_i$. Then $A_i R_{jkl}{}^m A_m = A_l R_{jki}{}^m A_m$; the sum on cyclic permutations of ijk gives zero in r.h.s.

2.1. **Codazzi deviation.** In ref.[16] we introduced the natural concept of *Codazzi deviation* of a symmetric tensor:

$$\mathscr{C}_{jkl} = \nabla_j b_{kl} - \nabla_k b_{jl}. \tag{4}$$

Properties: $\mathscr{C}_{jkl} = -\mathscr{C}_{kjl}$, $\mathscr{C}_{jkl} + \mathscr{C}_{klj} + \mathscr{C}_{ljk} = 0$, and

$$\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} = -(b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m). \tag{5}$$

Once again we read that a Codazzi tensor is Riemann compatible. By eq.(5) the differential condition $\nabla_{(i} \mathscr{C}_{jk)l} = 0$ is equivalent to the algebraic eq.(3).
A Veblen-like identity holds:

$$\begin{aligned} &\nabla_i \mathscr{C}_{jlk} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{lji} + \nabla_l \mathscr{C}_{ikj} \\ &= b_{im} R_{jlk}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{lji}{}^m + b_{lm} R_{ikj}{}^m. \end{aligned} \tag{6}$$

Example 2.3. *For a concircular vector, $\nabla_i X_j = \rho g_{ij}$, the tensor $X_i X_j$ is Riemann compatible.*
Proof: It is $\mathscr{C}_{jkl} = (\nabla_j \rho) g_{kl} - (\nabla_k \rho) g_{jl}$ and $\nabla_i \mathscr{C}_{jkl} = (\nabla_i \nabla_j \rho) g_{kl} - (\nabla_i \nabla_k \rho) g_{jl}$. The cyclic sum in (5) gives zero.
Note: the existence of a concircular time-like vector is necessary and sufficient for a space-time to be generalized Robertson-Walker [6].

Example 2.4 (Lovelock's identities).
1) *The Codazzi deviation of the Ricci tensor is:* $\mathscr{C}_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} = -\nabla^m R_{jklm}$. *Property* (5) *becomes a Lovelock's identity for the Riemann tensor* ([14], p.289)*:*

$$(7) \qquad \nabla_i \nabla^m R_{jklm} + \nabla_j \nabla^m R_{kilm} + \nabla_k \nabla^m R_{ijlm} = -R^m{}_{(i} R_{jk)lm}.$$

2) *The Codazzi deviation of Schouten's tensor*[1] *is* $\mathscr{C}_{jkl} = -\frac{1}{n-3}\nabla^m C_{jklm}$. *Property* (5) *is* $\nabla_{(i}\mathscr{C}_{jk)l} = -(n-3)S^m{}_{(i}R_{jk)lm}$. *The term with the metric tensor in* S_{ij} *does not contribute (Bianchi identity), and one is left with* (see [16])*:*

$$(8) \qquad \nabla_i \nabla^m C_{jklm} + \nabla_j \nabla^m C_{kilm} + \nabla_k \nabla^m C_{ijlm} = -\frac{n-3}{n-2} R^m{}_{(i} R_{jk)lm}.$$

In particular in $n > 3$, *if* $\nabla_m C_{jkl}{}^m = 0$ *(conformally symmetric spaces, Roter* [21]*) the Ricci tensor is Riemann compatible.*

Proposition 2.5. *If* $u_i u_j$ *is Riemann compatible, and* $u^k u_k \neq 0$, *then* u_i *is eigenvector of the Ricci tensor.*

Proof. Since $u_i u_j$ is Riemann compatible, it commutes with the Ricci tensor: $R_{ij} u^j u_k = R_{kj} u^j u_i$. Contraction with u^k gives: $R_{ij} u^j (u_k u^k) = (R_{kj} u^j u^k) u_i = 0$. □

We extrapolate a simple statement from Proposition 5.1 in [10]. A direct proof is possible, by writing (3) for the Ricci tensor in the warping coordinates:

Proposition 2.6. *In a warped spacetime* $ds^2 = \pm dt^2 + a(t)^2 g^*_{\mu\nu} dx^\mu dx^\nu$ *the Ricci tensor is Riemann compatible if and only if the Ricci tensor of the Riemannian submanifold* (M^*, g^*) *is compatible with the Riemann tensor of the submanifold:*

$$R^*_{\mu\sigma} R^*_{\nu\rho\lambda}{}^\sigma + R^*_{\nu\sigma} R^*_{\rho\mu\lambda}{}^\sigma + R^*_{\rho\sigma} R^*_{\mu\nu\lambda}{}^\sigma = 0.$$

2.2. **Geodesic maps.** A map $(M, g) \to (M, \overline{g})$ is *geodesic* if every geodesic line is mapped to a geodesic line. It is necessary and sufficient that there exists a 1-form such that the Christoffel symbols are related by $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta_i{}^k X_j + X_i \delta^k{}_j$ (Levi-Civita, 1896). The relation between the Riemann tensors is

$$\overline{R}_{jkl}{}^m = -\partial_j \overline{\Gamma}^m_{kl} + \partial_k \overline{\Gamma}^m_{jl} - \overline{\Gamma}^d_{kl} \overline{\Gamma}^m_{jd} + \overline{\Gamma}^d_{jl} \overline{\Gamma}^m_{kd} = R_{jkl}{}^m - \delta_k{}^m P_{jl} + \delta_j{}^m P_{kl},$$

where $P_{kl} = \nabla_k X_l - X_k X_l = P_{lk}$. It is: $\overline{R}_{jl} = R_{jl} + (n-1) P_{jl}$.
Geodesic maps preserve the (3,1) projective curvature tensor [22]: $\overline{P}_{jkl}{}^m = P_{jkl}{}^m$, where $P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl})$.

Proposition 2.7 ([16])**.** *If* $b_{ij} = b_{ji}$, *a geodesic map satisfies*

$$(9) \qquad b_{im} \overline{R}_{jkl}{}^m + b_{jm} \overline{R}_{kil}{}^m + b_{km} \overline{R}_{ijl}{}^m = b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m$$

Then, if (R, b) *is a compatible pair, also* $(\overline{R}, b)$ *is.*

[1] Schouten tensor: $S_{ij} = \frac{1}{n-2}\left[R_{ij} - \frac{R}{2(n-1)} g_{ij}\right]$. Properties: $\nabla_k S^k{}_j = \nabla_j S^k{}_k$, $\nabla^m C_{jklm} = (n-3)(\nabla_k S_{jl} - \nabla_j S_{kl})$.

3. Weyl compatible tensors

A symmetric tensor is Weyl compatible if:

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = 0. \tag{10}$$

This identity holds for any symmetric tensor [16]:

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{11}$$
$$+\tfrac{1}{n-2}\left[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)\right].$$

A simple consequence is obtained in dimension $n = 3$, where the Weyl tensor is zero (see [9], in less simple manner):

Proposition 3.1. *In $n = 3$ a Ricci tensor is Riemann compatible.*

If b_{ij} is Riemann compatible, then it commutes with the Ricci tensor. As a result, the identity shows that b_{ij} is also Weyl compatible. Therefore, Riemann compatibility is a stronger condition than Weyl compatibility. The identity (11) can be rewritten in terms of the Codazzi deviation:

$$\begin{aligned} b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m &= \nabla_i\mathscr{D}_{jkl} + \nabla_j\mathscr{D}_{kil} + \nabla_k\mathscr{D}_{ijl} \\ &\quad -\tfrac{1}{n-2}\nabla^m(\mathscr{C}_{ijm}g_{kl} + \mathscr{C}_{jkm}g_{il} + \mathscr{C}_{kim}g_{jl}). \end{aligned} \tag{12}$$

where $\mathscr{D}_{jkl} = \mathscr{C}_{jkl} - \frac{1}{n-2}(\mathscr{C}_{jm}{}^m g_{kl} - \mathscr{C}_{km}{}^m g_{jl})$.

Example 3.2. *If a vector field is torqued* [5]*, i.e. $\nabla_i\tau_j = \rho g_{ij} + \alpha_i\tau_j$ with $\alpha_k\tau^k = 0$, then $\tau_i\tau_j$ is Weyl compatible.*
Proof: one evaluates $\mathscr{C}_{jkl} = -\rho(\tau_j g_{kl} - \tau_k g_{jl})$ and $\mathscr{D}_{jkl} = -\frac{1}{n-2}\mathscr{C}_{jkl}$. It turns out that the r.h.s. of (12) *is zero.*
Note: the existence of a torqued time-like vector is necessary and sufficient for a space-time to be twisted [5].

Proposition 3.3 (see remark 4.2 of [12])**.** *In a space-time of dimension $n = 4$, if u_iu_j is Weyl compatible and time-like unit ($u^ku_k = -1$) then the Weyl tensor is wholly determined by the electric tensor $E_{kl} = C_{jklm}u^ju^m$:*

$$\begin{aligned} C_{abcd} = {}& 2(u_au_dE_{bc} - u_au_cE_{bd} + u_bu_cE_{ad} - u_bu_dE_{ac}) \\ &+ g_{ad}E_{bc} - g_{ac}E_{bd} + g_{bc}E_{ad} - g_{bd}E_{ac} \end{aligned} \tag{13}$$

Proof. In $n = 4$ the following Lovelock's identity holds ([14], ex 4.9 page 128):

$$\begin{aligned} 0 = {}& g_{ar}C_{bcst} + g_{br}C_{cast} + g_{cr}C_{abst} + g_{at}C_{bcrs} + g_{bt}C_{cars} + g_{ct}C_{abrs} \\ &+ g_{as}C_{bctr} + g_{bs}C_{catr} + g_{cs}C_{abtr} \end{aligned}$$

The contraction with u^au^r gives

$$\begin{aligned} 0 = {}& - C_{bcst} + u_bu^rC_{crst} + u_cu^rC_{rbst} + u_tu^rC_{bcrs} + g_{bt}u^au^rC_{cars} + g_{ct}u^au^rC_{abrs} \\ &+ u_su^rC_{bctr} + g_{bs}u^au^rC_{catr} + g_{cs}u^au^rC_{abtr} \\ = {}& - C_{bcst} + u^r(u_bC_{stcr} + u_cC_{rbst} + u_tC_{cbsr} + u_sC_{bctr}) \\ &+ g_{bt}E_{cs} - g_{ct}E_{bs} - g_{bs}E_{ct} + g_{cs}E_{bt} \end{aligned}$$

This gives the Weyl tensor in terms of its single and double contractions with u^i. If u_iu_j is Weyl compatible, the single contraction is: $C_{jklr}u^r = u_kE_{jl} - u_jE_{kl}$, and the result is obtained. For an extension to $n > 4$ see [12]. □

3.1. **Conformal maps.** A map $(M,g) \to (M,\hat g)$ is *conformal* if $\hat g_{kl} = e^{2\sigma} g_{kl}$. The Christoffel symbols transform according to: $\hat\Gamma_{ij}^m = \Gamma_{ij}^m + \delta^m{}_i X_j + X_i \delta^m{}_j - g_{ij} X^m$, where $X_i = \nabla_i \sigma$. A conformal map leaves the Weyl tensor (3,1) unchanged: $\hat C_{jkl}{}^m = C_{jkl}{}^m$. Therefore, Weyl compatibility is an invariant property of conformal maps.

4. K-compatible tensors

Riemann and Weyl compatibility extend to $K-$compatibility, where K is a generalised curvature tensor (GCT), i.e. a tensor with the algebraic properties of the Riemann tensor under permutation of indices [13]:

$$K_{jklm} = -K_{kjlm} = -K_{jkml}, \tag{14}$$

$$K_{jklm} + K_{kljm} + K_{ljkm} = 0, \tag{15}$$

$$K_{jklm} = K_{lmjk}. \tag{16}$$

In analogy with the Riemann tensor, one shows that (14) and (15) imply the symmetry (16), and the identity $K_{j(klm)} = 0$. The tensor $K_{jl} = K_{jml}{}^m$ is symmetric.

A symmetric tensor b_{ij} is $K-$compatible if:

$$b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0 \tag{17}$$

and (K,b) is a compatible pair. The property can be written $b^m{}_{(i} K_{jk)lm} = 0$.
The metric tensor is $K-$compatible, by the Bianchi property (15). The tensors b_{ij} and K_{ij} commute: $b_i{}^m K_{mk} - K_{im} b^m{}_k = 0$ (contract (17) with g^{jl} and use symmetry).
Examples of $K-$compatible tensors were obtained by Shaikh et al. starting from specific metrics (see for example [23, 1]). Bourguignon proved that if b_{ij} is a Codazzi tensor then $\mathring{\mathrm{R}}_{jklm} = R_{jkrs} b^r{}_l b^s{}_m$ is a GCT, [2]. We prove a more general statement:

Proposition 4.1. *If a_{ij} and b_{ij} are $K-$compatible, then $\mathring{K}_{jklm} = K_{jkrs}(a^r{}_l b^s{}_m + b^r{}_l a^s{}_m)$ is a* GCT.

Proof. The properties (14) and (16) are obvious; the Bianchi property (15) completes the proof: $\mathring{\mathrm{K}}_{(jkl)m} = a^r{}_{(l} K_{jk)rs}\, b^s{}_m + b^r{}_{(l} K_{jk)rs}\, a^s{}_m = 0$ because each term is zero being a or b $K-$compatible. □

4.1. **Properties of $K-$compatible tensors.** A linear combination of K-compatible tensors obvioulsy is K-compatible. Now we prove:

Theorem 4.2. *If a and b are $K-$compatible, then $\frac{1}{2}(ab+ba)$ is $K-$compatible.*

Proof. Let $c_{ij} = a_i{}^k b_{kj} + b_i{}^k a_{kj}$. Then:

$$\begin{aligned}
c^m{}_{(i} K_{jk)rm} =\ & a_i{}^s b_s{}^m K_{jkrm} + a_j{}^s b_s{}^m K_{kirm} + a_k{}^s b_s{}^m K_{ijrm} + a \leftrightarrows b \\
=& -a_i{}^s (b_j{}^m K_{ksrm} + b_k{}^m K_{sjrm}) - a_j{}^s (b_k{}^m K_{isrm} + b_i{}^m K_{skrm}) \\
& - a_k{}^s (b_i{}^m K_{jsrm} + b_j{}^m K_{sirm}) + a \leftrightarrows b \\
=& -(a_i{}^s b_j{}^m - a_j{}^s b_i{}^m) K_{ksrm} - (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m) K_{isrm} \\
& - (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m) K_{jsrm} + a \leftrightarrows b \\
=& -(a_i{}^s b_j{}^m - a_j{}^s b_i{}^m)(K_{ksrm} - K_{kmrs}) - (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m)(K_{isrm} - K_{imrs}) \\
& - (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m)(K_{jsrm} - K_{jmrs})
\end{aligned}$$

$$= (a_i{}^s b_j{}^m - a_j{}^s b_i{}^m)K_{krsm} + (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m)K_{irsm} + (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m)K_{jrsm}$$
$$= (a_i{}^s b_j{}^m + b_i{}^s a_j{}^m)K_{krsm} + (a_j{}^s b_k{}^m + b_j{}^s a_k{}^m)K_{irsm} + (a_k{}^s b_i{}^m + b_k{}^s a_i{}^m)K_{jrsm}$$
$$= \mathring{K}_{krij} + \mathring{K}_{irjk} + \mathring{K}_{jrki} = \mathring{K}_{(kri)j} = 0$$

because $\mathring{K}$ is a GCT by Prop.4.1. □

Therefore, the linear space of $K-$compatible tensors is a special Jordan algebra. In particular, the powers of b are $K-$compatible (powers $n, n+1, ...$ are linear combinations of lower powers by Cayley-Hamilton theorem). In particular (with an exchange of indices) the tensor $(b^2)_j{}^s (b^2)_k{}^r K_{rslm}$ is a GCT. This enables the simple proof of the theorem in [15], so short that we reproduce it:

Theorem 4.3 (Extended Derdziński-Shen theorem). *Let b_{ij} be K-compatible, X^i, Y^i, Z^i be eigenvectors of $b_i{}^m$ with eigenvalues x, y, z. If $x \neq z$ and $y \neq z$ then:*

$$K_{ijkl} X^i Y^j Z^k = 0. \tag{18}$$

Proof. Consider the identities $g^m{}_{(i}K_{jk)lm} = 0$, $b^m{}_{(i}K_{jk)lm} = 0$, $(b^2)^m{}_{(i}K_{jk)lm} = 0$ and contract them with $X^i Y^j Z^k$. The three algebraic relations are put in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \begin{bmatrix} K_{jkli} X^i Y^j Z^k \\ K_{kilj} X^i Y^j Z^k \\ K_{ijlk} X^i Y^j Z^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the matrix is $(x-y)(x-z)(z-y)$. If the eigenvalues are all different then $K_{ijkl} X^i Y^j Z^k = 0$ (with contraction of any three indices). If $x = y \neq z$, the reduced system of equations still implies $K_{ijkl} X^i Y^j Z^k = 0$. □

Proposition 4.4. *If b is K-compatible and invertible, then b^{-1} is K-compatible:*

$$(b^{-1})^j{}_{(s}K_{rl)kj} = 0 \tag{19}$$

Proof. Multiply (17) by $(b^{-1})^i{}_r (b^{-1})^j{}_s$ and obtain the identity: $(b^{-1})^j{}_s K_{jklr} + (b^{-1})^i{}_r K_{kils} + (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} = 0$. Rewrite it as:

$$(b^{-1})^j{}_{(s}K_{rl)kj} - (b^{-1})^j{}_l K_{srkj} + (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} = 0$$

The last two terms cancel, as shown by the chain:
$(b^{-1})^j{}_l K_{srkj} = (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} \Leftrightarrow K_{srkb} b^r{}_a = b^i{}_b (b^{-1})^j{}_s b^m{}_k K_{ajlm}$
$\Leftrightarrow b^s{}_c K_{srkb} b^r{}_a = b^l{}_b b^m{}_k K_{aclm} \Leftrightarrow \mathring{K}_{kbca} = \mathring{K}_{acbk}$, which is true as $\mathring{K}$ is a GCT. □

We prove a Veblen-like identity:

Proposition 4.5. *If b_{ij} is K-compatible then:*

$$b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_k{}^m K_{iljm} - b_l{}^m K_{jkim} = 0. \tag{20}$$

Proof. $0 = b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = b_i{}^m K_{jklm} - b_j{}^m (K_{ilkm} + K_{lkim}) + b_k{}^m K_{ijlm} = b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_l{}^m K_{kjim} + b_k{}^m K_{jlim} + b_k{}^m K_{ijlm}$
$= b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_l{}^m K_{kjim} - b_k{}^m K_{lijm}$. □

4.2. **More on generalised curvature tensors.** A linear combination of GCTs is a GCT. Given two compatible pairs (K, a) and (K, b) a new GCT tensor is obtained in Prop.4.1. In particular, if $a_{ij} = g_{ij}$ (the metric tensor) the following K' is a GCT:

$$K'_{jklm} = K_{jkrs}({\delta^r}_l {b^s}_m + {b^r}_l {\delta^s}_m) = K_{jkls}{b^s}_m - K_{jkms}{b^s}_l \tag{21}$$

Proposition 4.6. *If b is K-compatible, then b is K'-compatible.*

Proof. The tensor $K'_{jklm} = K_{jklr}{b^r}_m - K_{jkmr}{b^r}_l$ is a GCT. Let us evaluate: ${b^m}_i K'_{jklm} = {b^m}_i K_{jklr}{b^r}_m - {b^m}_i K_{jkmr}{b^r}_l = {(b^2)^r}_i K_{jklr} - \mathring{K}_{jkim}$. Both tensors vanish if the cyclic sum (ijk) is taken. □

Proposition 4.7. *(K, b) is a compatible pair for any symmetric tensor b if and only if*

$$K_{ijlm} = \frac{K}{n(n-1)}(g_{il}g_{jm} - g_{im}g_{jl}) \tag{22}$$

where K is a scalar field.

Proof. The symmetry of the tensor is made explicit by writing $b_{ij} = \frac{1}{2}b^{rs}(g_{ir}g_{js} + g_{is}g_{jr})$. The compatibility relation must hold for any b^{rs}, then:

$$0 = g_{ir}K_{jkls} + g_{jr}K_{kils} + g_{kr}K_{ijls} + g_{is}K_{jklr} + g_{js}K_{kilr} + g_{ks}K_{ijlr}.$$

Contraction with g^{ks} gives $(n-1)K_{ijlr} = g_{jr}K_{il} - g_{ir}K_{jl}$; contraction with g^{il} gives $K_{jr} = \frac{1}{n}g_{jr}{K^i}_i$ and (22) follows. The reverse, i.e. (22) implies (17), is shown by direct check. □

A pseudo-Riemannian manifold of dimension $n > 2$ is an *Einstein manifold* if $R_{ij} = \frac{1}{n}Rg_{ij}$ where R is the scalar curvature. Since $\nabla_i {R^i}_j = \frac{1}{2}\nabla_j R$, the scalar curvature is constant. A manifold is a *constant curvature manifold* if the Riemann tensor has the form (22). Such manifolds are Einstein manifolds.

Corollary 4.8. *A manifold is a constant curvature manifold if and only if ${b_i}^m R_{jklm} + {b_j}^m R_{kilm} + {b_k}^m R_{ijlm} = 0$ for all symmetric tensors.*

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