Anomaly Non-renormalization in Interacting Weyl Semimetals

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Abstract: Weyl semimetals are 3D condensed matter systems characterized by a degenerate Fermi surface, consisting of a pair of 'Weyl nodes'. Correspondingly, in the infrared limit, these systems behave effectively as Weyl fermions in $3 + 1$ dimensions. We consider a class of interacting 3D lattice models for Weyl semimetals and prove that the quadratic response of the quasi-particle flow between the Weyl nodes is universal, that is, independent of the interaction strength and form. Universality is the counterpart of the Adler–Bardeen non-renormalization property of the chiral anomaly for the infrared emergent description, which is proved here in the presence of a lattice and at a non-perturbative level. Our proof relies on constructive bounds for the Euclidean ground state correlations combined with lattice Ward Identities, and it is valid arbitrarily close to the critical point where the Weyl points merge and the relativistic description breaks down.

1. Introduction

The behaviour of relativistic fermionic particles is described by the Dirac equation and by its interacting extensions, such as Quantum ElectroDynamics (QED) or other standard Quantum Field Theory (QFT) models, whose spectacular and often counter-intuitive predictions have been tested for decades in high energy experiments. On the other hand, the electrons present in ordinary matter, like the conduction electrons in metals, which live at much lower energy scales, are described by the many-body Schrödinger equation, which breaks Lorentz invariance explicitly.

By scale separation, the Dirac high energy physics is not expected to play directly any role in the behaviour of conduction electrons. Nevertheless, there are cases in which the interaction of the conduction electrons with the underlying lattice can produce an effective description in terms of Dirac particles, with a velocity of propagation much smaller than the speed of light. Such emergent QFT description is valid in great generality for one dimensional metals, see [48,58,67] (see also [55] for a recent review), but is also

possible in higher dimensional systems: notable examples were proposed in [37,68] and in [63]. The discovery of graphene [44,64] and of topological insulators [38] provides a condensed matter realization of systems of 2D Dirac fermions. In connection with their emergent QFT description, the transport coefficients of these systems are characterized by remarkable universality properties: examples include the optical conductivity of graphene [60] and the Hall conductance [11,66]. A mathematical proof of universality in the presence of interactions has been established in [6,12,13,26,28–31,39,57]. More recently, 3D Dirac systems have been experimentally realized [16,46,47,62], following the theoretical predictions of [63] and [24,59,65], see [9,38] for reviews; these semimetals are dubbed 'Dirac' or 'Weyl' semimetals, depending on whether the Fermi points coincide or are at distinct locations in the Brillouin zone. Such experimental discoveries open the way to the observation of the counterpart of one of the most interesting phenomena characterizing 3D massless Dirac fermions: the Adler-Bell-Jackiw (ABJ) axial anomaly [2,14], which corresponds to the breaking of the classical chiral symmetry by quantum mechanical radiative corrections and results in the non-conservation of the axial current. Nielsen and Ninomiya [63] predicted that this quantum anomaly should have a realization, or 'simulation', in the quadratic response of the quasi-particle flow between the Weyl nodes to an external electromagnetic field, see also [18,69]; in particular, this quasi-particle flow is expected to be proportional to $E \cdot B$, with coefficient given by the value of the axial anomaly in QED$_4$. The experimental observation of the 'chiral anomaly' in Weyl semimetals has been reported in [41,43,45,71,72], see also [15] for an observation in the context of superfluid ^{3}He. Other applications of the chiral anomaly in condensed matter physics have been discussed in [23], in the context of a gauge theory of phases of matter, see also [22] for new proposed applications and [21] for a recent review.

The theoretical analysis in [63] neglects the effects of many-body interaction. What happens to the quadratic response of the quasi-particle flow in the presence of electron interactions, which are unavoidable in real systems? One of the distinctive features of the axial anomaly in QFT is the Adler–Bardeen (AB) *non-renormalization property* [3], which says that all the radiative corrections cancel out at all orders in the fine structure constant, so that the axial anomaly equals the value of the chiral triangle graph. However, the AB argument cannot be applied to Weyl semimetals: the analysis in [3] requires cancellations between classes of Feynman graphs based on Lorentz and chiral symmetries, which are broken explicitly in any lattice model of interacting electrons. Strangely enough, the issue of universality has not been addressed theoretically until now, despite the huge literature on the effects of interactions[1] [10,20,35,36,49,51,70]. In this paper, we consider a paradigmatic class of lattice models for 3D Weyl semimetals with short-range interactions, generalizing a widely used 'standard' model for these materials, see [9]. For this class of models,

we prove that the quadratic response of the quasi-particle flow between the Weyl points, which is the counterpart of the chiral anomaly in the emergent QFT description of the system, is universal, that is, independent of the interaction strength and form.

Therefore, this quantity can be added to the limited list of universal transport coefficients in condensed matter, like the Hall conductivity and graphene's optical conductiv-

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[1] An even more extensive and rapidly growing literature is available on the effects of disorder, which we cannot fully account for here, see [4,61] for a couple of representative results, see also [7,54] for recent rigorous results. Even in that context, the issue of universality of the 'chiral anomaly' for Weyl semi-metals has not been satisfactorily discussed yet.

ity, and is the only known example of universal quantum transport coefficient in three dimensions. This quadratic response has been measured [41,43,45,71,72] in different Weyl semimetal, such as Na_3Bi, TaAs, GdPtBi, $ZrTe_5$, and the proportionality between the quasi-particle flow between the Weyl points and $E \cdot B$ has been experimentally verified. The interaction does not appear to qualitatively modify such a response, in agreement with our result, even though a precise measurement of the proportionality coefficient remains to be performed. Remarkably, *our result is valid arbitrarily close to the critical point where the Weyl nodes merge* and the natural parameter measuring the strength of the interaction, that is, the coupling strength divided by the quasi-particle velocity, diverges: this is important for applications to real 3D Weyl semimetals, where typically the distance between the Weyl nodes is very small, as compared to the size of the Brillouin zone; see Sect. 2 for further discussion of this point.

In this paper, we focus our attention on the quadratic response coefficient of the 'chiral current' to an external electromagnetic field directly, without discussing the linear response coefficient, which is not expected to have any remarkable universality feature. We stress that in the experiments the 'chiral anomaly' is detected by measuring the peak of quasi-particle flow between the Weyl nodes as the angle between the external electric and magnetic fields E and B is varied, see e.g. [71]: such a procedure has the effect of automatically subtracting the linear response, which is uninteresting for the effect under investigation.

The paper is organized as follows: in Sect. 2, we introduce a class of lattice Weyl semimetals, we define the notions of lattice current and lattice 'chiral current', and state our main results, summarized in Theorem 2.1. In Sect. 3 we present the proof: first we describe the general strategy and the core argument (Sect. 3.1); next we discuss the technical details: the Grassmann representation of the generating function of correlations and its multiscale Renormalization Group (RG) computation are presented in Sects. 3.2 and 3.3; the RG flow of the effective couplings is studied in Sect. 3.4; the regularity properties of the Euclidean correlations is discussed in Sect. 3.5, and the relativistic contribution to the quadratic response of the chiral current to the electromagnetic field is computed in Sect. 3.6. A few additional technical aspects of the proof are deferred to the appendices: in Appendix A we prove a number of key symmetry properties of the kernels of the effective potentials produced by the RG iterations; in Appendix B we discuss a few technical aspects of the tree expansion for the effective potentials; in Appendix C we compute the quadratic response coefficient via time-dependent perturbation theory and prove a Wick rotation theorem that allows us to express it in terms of Euclidean correlations.

2. Weyl Semimetals and Response Functions

2.1. A basic non-interacting model for Weyl semimetals. We start by describing a paradigmatic non-interacting lattice model for Weyl semimetal, see [9, Section II.B.1], with broken time-reversal symmetry, for which inversion symmetry is preserved. This example motivates the class of models which our analysis applies to. It describes an electron system on a simple cubic lattice subject to an external magnetic field and with the following features: the system is magnetically ordered, so that the bands have no spin degeneracy; moreover, every lattice site comes with two internal degrees of freedom, $r = 1, 2$, playing the role of pseudo-spin labels, corresponding to, *e.g.*, orbital degrees of freedom; finally, these orbitals (say, s, p orbitals) have opposite parity. i.e., they behave differently under inversion. The model is defined in terms of the following Bloch Hamiltonian (see

next section for details):

$$\hat{H}^0(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ \beta^*(k) & -\alpha(k) \end{pmatrix}, \tag{2.1}$$

where

$$\alpha(k) = 2 + \zeta - \cos k_1 - \cos k_2 - \cos k_3, \qquad \beta(k) = t_1 \sin k_1 - i t_2 \sin k_2 \tag{2.2}$$

and ζ, t_1, t_2 are three real parameters. We assume that $t_1, t_2 > 0$ and $-1 < \zeta < 1$, so that $\hat{H}^0(k)$ is singular only at two Weyl nodes, or Fermi points, $k = p_F^{\pm}$, with $p_F^{\pm} = (0, 0, \pm \arccos \zeta)$. We shall think of t_1, t_2 as being fixed once and for all, while ζ is tunable: it controls the distance among the Weyl nodes, which tend to merge in the limit as $|\zeta| \to 1^-$. Note that the relative distance between the two nodes vanishes like $\sqrt{1 - |\zeta|}$ as $|\zeta| \to 1^-$.

Remark. Having a parameter tuning the distance between Weyl nodes is extremely natural: in real systems, magnetic fields in orthogonal directions are used to induce and control a 'topological' transitions from, e.g., a Dirac semimetal to a magnetic semiconductor, or to a Weyl semimetal; once a Weyl semimetal phase is entered, the intensity of the magnetic fields is used to tune the location of the Weyl nodes. See [9, Fig.2] and the corresponding discussion; see also [20]. In the limit as the Weyl nodes tend to merge, the valence and conduction bands generically tend to touch quadratically at the doubly-degenerate node. Experimentally, the Weyl nodes may be quite close to each other, compared to the size of the Brillouin zone. The fact that the Weyl points can be arbitrarily close is a peculiarity of Weyl semimetals, which distinguish them from other Dirac materials, like graphene.

In the vicinity of the Weyl nodes, $k \simeq p_F^{\omega}$, the Bloch Hamiltonian can be linearized as:

$$\hat{H}^0(k) = t_1 \sigma_1 k_1 + t_2 \sigma_2 k_2 + \omega \sqrt{1 - \zeta^2} \sigma_3 (k_3 - p_{F,3}^{\omega}) + O(|k - p_F^{\omega}|^2), \tag{2.3}$$

with σ_i the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The parameters $t_1, t_2, \omega \sqrt{1 - \zeta^2}$ in front of the three terms in the right side of (2.3) play the role of 'Fermi velocities' in the three coordinate directions. Note that, as $|\zeta| \to 1$, the third component of the Fermi velocity (the one along which the Weyl nodes are located), $v_{3,\omega}^0 := \omega \sqrt{1 - \zeta^2}$, tends to zero at the *same speed* as $p_F^+ - p_F^-$. This is no accident, it is a generic feature in the limit as two Weyl nodes merge.

In order to provide an asymptotic description of the system that holds uniformly in the limit as the Weyl nodes merge, it is convenient to explicitly extract the quadratic part of the Bloch Hamiltonian in the third coordinate direction. In our case:

$$\begin{aligned} \hat{H}^0(k) = t_1 \sigma_1 k_1 + t_2 \sigma_2 k_2 + \sigma_3 \big[v_{3,\omega}^0 (k_3 - p_{F,3}^{\omega}) + \tfrac{\zeta}{2} (k_3 - p_{F,3}^{\omega})^2 \big] \\ + O(k_1^2, k_2^2, |k_3 - p_{F,3}^{\omega}|^3). \end{aligned} \tag{2.5}$$

Note that the prefactor of the quadratic term that we extracted, equal to $\zeta/2$, tends to a non-zero constant in the limit as the Weyl nodes tend to merge, $|\zeta| \to 1^-$. This is again no accident, it is another generic feature in the limit as two Weyl nodes merge.

2.2. A class of non-interacting models for Weyl semimetals. Motivated by the model introduced in the previous section, we now consider a class of non-interacting lattice models, including (2.1) as a special case. We will use adapted coordinates, such that, as in the model above, the Weyl nodes are located along the third coordinate direction.

Let Λ_L be a cubic lattice of side L, with periodic boundary conditions; that is $\Lambda_L = \mathbb{Z}_L^3$, with $\mathbb{Z}_L = \mathbb{Z}/L\mathbb{Z}$ the integers modulo L. The coordinates of the particle on the lattice are specified by (x, r), with $x \in \Lambda_L$ the position and $r = 1, 2$ the internal degree of freedom, called 'color', or 'orbital'. Thus, the single-particle Hilbert space is $\mathfrak{h}_L = \mathbb{C}^{L^3} \otimes \mathbb{C}^2$. We denote by H^0 the single-particle Hamiltonian on $\mathfrak{h}_L$, which we assume to be Hermitian. We consider finite-ranged, translation invariant systems, such that $H^0(x, y) \equiv H^0(y-x)$. We always assume that L is larger than the range of H^0. We also assume that, after identifying $y - x$ with its image in $\mathbb{Z}^3$, $H^0(y - x)$ is independent of L (i.e., the kernel of H^0 is independent of L, up to the periodicity condition). Due to periodicity and translation invariance, we can represent the single-particle Hamiltonian as:

$$H^0(x, y) = \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} e^{-ik \cdot (x-y)} \hat{H}^0(k), \qquad (2.6)$$

where $\mathcal{B}_L$ is the finite-volume Brillouin zone, defined as:

$$\mathcal{B}_L = \frac{2\pi}{L} \mathbb{Z}_L^3, \qquad (2.7)$$

and $\hat{H}^0(k)$ is the Bloch Hamiltonian, a two-by-two matrix parametrized by $k \in \mathcal{B}_L$. Under the above assumptions, the Bloch Hamiltonian is an L-independent function, Hermitian and analytic in $k \in \mathcal{B}_\infty \equiv \mathbb{T}^3$. We also allow for $\hat{H}^0$ to depend smoothly on an additional paremeter $\zeta \in [\zeta_0, \zeta_1) \equiv I$, which can be used to tune the location of the Weyl points, see below.

We assume the following *symmetry properties*, valid for all $\zeta \in I$, which are all satisfied by the model in Eq. (2.1). The first two symmetries are the most physically significant: they characterize the class of Weyl semimetals under consideration. The other two are not crucial, but simplify the structure of the linearized Hamiltonian around the Fermi points.

(i) *Broken time-reversal symmetry.* $\hat{H}^0(k) \neq [\hat{H}^0(-k)]^*$, where $*$ indicates complex conjugation.[2]
(ii) *Inversion symmetry.*[3] $\hat{H}^0(k) = \sigma_3 \hat{H}^0(-k)\sigma_3$.
(iii) *Reflection about a horizontal plane.* $\hat{H}^0(k) = \hat{H}^0\big((k_1, k_2, -k_3)\big)$
(iv) *Reflection about a vertical plane and color exchange.* $\hat{H}^0(k) = -\sigma_1 \hat{H}^0\big((-k_1, k_2, k_3)\big)\sigma_1$.

In addition to these symmetries, we assume the following properties on the energy bands of the Bloch Hamiltonian.

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[2] We denote by z^*, rather than by $\bar{z}$, the complex conjugate of $z \in \mathbb{C}$. Given a matrix $A \in \mathbb{C}^{n \times n}$, we denote by $A^\dagger$ its Hermitian conjugate, and by A^* its complex conjugate, that is, the matrix whose elements satisfy $(A^*)_{ij} = (A_{ij})^*$.

[3] As mentioned in the previous section, we are assuming that the two orbitals $r = 1$ and $r = 2$ transform differently under inversion, that is, one is even and one is odd under $x \to -x$, which explains the presence of the third Pauli matrix σ_3.

Weyl points. We write the Bloch Hamiltonian in the form

$$\hat{H}^0(k) = \sigma_1 a(k) + \sigma_2 b(k) + \sigma_3 c(k) + d(k). \tag{2.8}$$

Using the symmetries (ii), (iii), (iv), we see that: a is odd in k_1, even in k_2 and in k_3; b is odd in k_2, even in k_1 and in k_3; c is even in k_1, in k_2 and in k_3; d is odd in k_1 and in k_2, even in k_3. We let $\varepsilon_\pm(k)$ be the k-dependent eigenvalues of the Bloch Hamiltonian, also known as the 'energy bands':

$$\varepsilon_\pm(k) = d(k) \pm \sqrt{a^2(k) + b^2(k) + c^2(k)}. \tag{2.9}$$

For any $\zeta \in I$, we assume that $\varepsilon_+(k) \geq 0 \geq \varepsilon_-(k)$, with equalities only at *two* points, called the Weyl nodes, denoted by $p_F^\pm$, at which we assume $\varepsilon_\pm(k)$ to vanish *linearly*. Thanks to the symmetries (ii), (iii), (iv), the *pair* of Weyl nodes is invariant under reflections about any of the coordinate axes. Therefore, in order for the nodes to be exactly two, they need to be located along one of the three axes. Moreover, the requirement that $\varepsilon_\pm(k)$ vanish linearly at $p_F^\pm$, combined with the fact that a, b, c, d are all even functions of k_3, implies that the Weyl nodes are located along the third axis:

$$p_F^\omega = (0, 0, p_{F,3}^\omega), \tag{2.10}$$

where $p_{F,3}^+ = -p_{F,3}^- \neq 0$. Of course, $p_F^\pm$ depend smoothly on ζ. We allow for the possibility that the Weyl nodes merge in the limit $\zeta \to \zeta_1^-$. For definiteness, we assume once and for all that

$$\lim_{\zeta \to \zeta_1^-} (p_F^+ - p_F^-) = 0. \tag{2.11}$$

If the limit were different from zero, the model could be treated in the same way as if $\zeta \in [\zeta_0, \zeta_1'] \subset [\zeta_0, \zeta_1)$, which is a sub-case of the problem studied in this paper.[4]

Let us now discuss the structure of the Bloch Hamiltonian in the vicinity of the Weyl nodes. We have $a(p_F^\pm) = b(p_F^\pm) = c(p_F^\pm) = d(p_F^\pm) = 0$. If we Taylor expand the Bloch Hamiltonian around p_F^ω, with $\omega \in \{\pm\}$, at first order in k_1, k_2 and at second order in $k_3 - p_{F,3}^\omega$, recalling the parity properties of a, b, c, d, we obtain the analogue of (2.5), namely

$$\hat{H}^0(k) = \sigma_1(v_1^0 k_1 + a_R(k)) + \sigma_2(v_2^0 k_2 + b_R(k))$$
$$+ \sigma_3 \left[\omega v_3^0 (k_3 - p_{F,3}^\omega) + \frac{b^0}{2}(k_3 - p_{F,3}^\omega)^2 + c_{R,\omega}(k) \right] + d_R(k), \tag{2.12}$$

where v_1^0, v_2^0, v_3^0, b^0 are real variables, smoothly depending on ζ, and $a_R, b_R, c_{R,\omega}, d_R$ are the remainders of the Taylor expansion (note that $d_R = d$). We assume that there exists a constant $c_0 \geq 1$ such that

$$\min\{|v_1^0|, |v_2^0|, |b^0|\} \geq c_0^{-1}, \qquad c_0^{-1} |v_3^0| \leq |p_F^+ - p_F^-| \leq c_0 |v_3^0|, \tag{2.13}$$

uniformly in ζ, for $\zeta \in I$. Thanks to their parity properties of a, b, c, d, the remainder terms satisfy the following bounds in the vicinity of p_F^ω:

$$|a_R(k)| \leq C|k_1|(k_1^2 + k_2^2 + |k_3 - p_{F,3}^\omega|), \quad |b_R(k)| \leq C|k_2|(k_1^2 + k_2^2 + |k_3 - p_{F,3}^\omega|),$$

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[4] In the 'non-singular' case, that is, in the case that the two Weyl points are well separated uniformly in ζ, the infrared Renormalization Group analysis of the interacting model simplifies, as we shall see below. In particular, the regime $h > h_*$, discussed in Sect. 3.3.1, disappears, see also the remark after (3.41).

$$|c_{R,\omega}(k)| \leq C(k_1^2 + k_2^2 + |k_3 - p_{F,3}^\omega|^3), \quad |d_R(k)| \leq C|k_1||k_2|, \tag{2.14}$$

for a constant C that we assume to be independent of ζ. By the 'vicinity of p_F^ω', we mean $|k - p_F^\omega| \leq 2|p_F^+|$. In the complementary region, $\min_\omega |k - p_F^\omega| > 2|p_F^+|$, we have:

$$|a_R(k)| \leq C|k_1||k|^2, \quad |b_R(k)| \leq C|k_2||k|^2,$$
$$|c(k) - c(0) - \tfrac{b^0}{2}k_3^2| \leq C(k_1^2 + k_2^2 + k_3^4), \quad |d_R(k)| \leq C|k_1||k_2|, \tag{2.15}$$

for a constant C that, without loss of generality, can be taken to be the same as in (2.14). Note that, in view of the bounds (2.13), (2.14), (2.15), it is possible to find $c_1 > 0$ such that

$$\min_\omega |k - p_F^\omega| \leq c_1 \quad \Rightarrow \quad |d(k)| \leq \tfrac{1}{2}\sqrt{a^2(k) + b^2(k) + c^2(k)}, \tag{2.16}$$

uniformly in $\zeta \in I$. Moreover, by the smoothness of $\hat{H}^0(k)$ and the fact that $\det \hat{H}^0(k)$ vanishes only at $\pi_F^\pm$, there exists $c_2 > 0$ such that

$$\min_\omega |k - p_F^\omega| \geq c_1 \quad \Rightarrow \quad -\det \hat{H}^0(k) \geq c_2, \tag{2.17}$$

uniformly in $\zeta \in I$. Under these assumptions, we will be able to construct and analyze the ground state of a many-body interacting version of H^0, uniformly in ζ, for $\zeta \in I$. In particular, our analysis will be valid in the limit $\zeta \to \zeta_1^-$ as the Weyl nodes merge.

For other examples of models in this class, in addition to the one discussed in the previous section, see [19,42].

2.3. The interacting model. Let us now include the many-body interaction. We describe the system in the grand-canonical setting, in second quantization. The fermionic Fock space is defined as $\mathcal{F}_L = \bigoplus_{N \geq 0} \mathfrak{h}_L^{\wedge N}$, with $\wedge$ the antisymmetric tensor product. For all $(x, r) \in \Lambda_L \times \{1, 2\}$, we consider fermionic creation operators $a_{x,r}^+ : \mathfrak{h}_L^{\wedge N} \to \mathfrak{h}_L^{\wedge N+1}$ and annihilation operators $a_{x,r}^- : \mathfrak{h}_L^{\wedge N} \to \mathfrak{h}_L^{\wedge N-1}$ verifying the canonical anticommutation relations: $\{a_{x,r}^+, a_{y,r'}^-\} = \delta_{x,y}\delta_{r,r'}$, $\{a_{x,r}^+, a_{y,r'}^+\} = \{a_{x,r}^-, a_{y,r'}^-\} = 0$. The fermionic operators are compatible with the periodic boundary conditions of the model. Their Fourier transforms are defined, for $k \in \mathcal{B}_L$, as:

$$\hat{a}_{k,r}^\pm = \sum_{x \in \Lambda_L} e^{\mp ik \cdot x} a_{x,r}^\pm \iff a_{x,r}^\pm = \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} e^{\pm ik \cdot x} \hat{a}_{k,r}^\pm. \tag{2.18}$$

The many-body Hamiltonian is:

$$\mathcal{H}_L = \sum_{x,y \in \Lambda_L} \sum_{r,r' \in \{1,2\}} a_{x,r}^+ H_{r,r'}^0(x, y) a_{y,r'}^- \tag{2.19}$$

$$+ \lambda \sum_{x,y \in \Lambda_L} \sum_{r,r' \in \{1,2\}} (\rho_{x,r} - \tfrac{1}{2}) w_{r,r'}(x, y)(\rho_{y,r'} - \tfrac{1}{2}) - \nu \mathcal{N}_L$$

where $\rho_{x,r} = a_{x,r}^+ a_{x,r}^-$ is the density operator and $\mathcal{N}_L = \sum_{x \in \Lambda_L}(\rho_{x,1} - \rho_{x,2})$ is the staggered number operator. The first term in the right side of (2.19) is the hopping term, defined in terms of an H^0 satisfying the properties listed in the previous section. The second term is the many-body interaction, defined in terms of a $w_{r,r'}(x, y)$, which we

assume to be even, short-ranged, translational invariant, $w_{r,r'}(x, y) = w_{r,r'}(y - x)$, and periodic over Λ_L. As for $H^0(x, y)$, we assume that $w(x, y)$ is L-independent, up to the periodicity condition. Also, we suppose that the interaction is invariant under the reflections *(iv)* and *(v)* above: that is, we require that $w_{r,r'}(x)$ is invariant under $x \to (x_1, x_2, -x_3)$ and under[5] $x \to (-x_1, x_2, x_3)$. The interaction strength λ will be assumed to be small, compared with the bandwidth $\max_k \|H^0(k)\|$, uniformly in the system size and in the choice of the parameter ζ. The $-1/2$ appearing in the factors $(\rho_{x,r} - \frac{1}{2})$ and $(\rho_{y,r'} - \frac{1}{2})$ correspond to a specific, convenient, choice of the chemical potential. Finally, the third term in the right side of (2.19) is a staggered chemical potential, with $\nu \equiv \nu(\lambda)$ being a free parameter, such that $\nu(0) = 0$, to be chosen in such a way that the Fermi points of the interacting theory do not move as the interaction varies: they will coincide with those of the noninteracting theory, $p_F^\pm$, in a sense to be made precise later.

The Gibbs state of the model is:

$$\langle \cdot \rangle_{\beta,L} = \mathrm{Tr}_{\mathcal{F}_L} \cdot \rho_{\beta,L}, \qquad \rho_{\beta,L} = \frac{e^{-\beta \mathcal{H}_L}}{\mathcal{Z}_{\beta,L}}, \qquad \mathcal{Z}_{\beta,L} = \mathrm{Tr}_{\mathcal{F}_L} e^{-\beta \mathcal{H}_L}. \qquad (2.20)$$

Given a local observable $\mathcal{O}_x$, even in the fermionic fields, we denote by $\mathcal{O}_\mathbf{x} := e^{x_0 \mathcal{H}_L} \mathcal{O}_x e^{-x_0 \mathcal{H}_L}$ its imaginary-time evolution, for $\mathbf{x} = (x_0, x)$ and $x_0 \in [0, \beta)$. We denote by $\hat{O}_\mathbf{p}$ its space-time Fourier transform,

$$\hat{O}_\mathbf{p} = \int_0^\beta dx_0 \sum_{x \in \Lambda_L} e^{-i\mathbf{k} \cdot \mathbf{x}} O_\mathbf{x}, \qquad (2.21)$$

with $\mathbf{p} = (p_0, p) \in \mathcal{M}_\beta^\mathrm{b} \times \mathcal{B}_L$ and where $\mathcal{M}_\beta^\mathrm{b} = \frac{2\pi}{\beta} \mathbb{Z}$ is the set of bosonic Matsubara frequencies. If, instead, $\mathcal{O}_x$ is an operator which is odd in the fermionic fields, such as a_x^-, similar definitions and formulas hold; however, in the definition of Fourier transform, the set of bosonic Matsubara frequencies is replaced by the set of fermionic Matsubara frequencies, $\mathcal{M}_\beta^\mathrm{f} = \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$. Given $\mathbf{k} \in \mathcal{M}_\beta^\mathrm{f} \times \mathcal{B}_L$, we denote the space-time Fourier transforms of the fermionic operators as:

$$\hat{a}_{\mathbf{k},r}^\pm = \int_0^\beta dx_0 \sum_{x \in \Lambda_L} e^{\mp i\mathbf{k} \cdot \mathbf{x}} \hat{a}_{\mathbf{x},r}^\pm. \qquad (2.22)$$

Later, we shall be interested in the Schwinger correlation functions of the model, defined, for $x_{0,i} \in [0, \beta)$, as:

$$\langle \mathbf{T} a_{\mathbf{x}_1,r_1}^{\varepsilon_1} \cdots a_{\mathbf{x}_n,r_n}^{\varepsilon_n} \rangle_{\beta,L}, \qquad (2.23)$$

with $\mathbf{T}$ the fermionic time-ordering, ordering the operators in decreasing imaginary-time order, see, e.g., [30, Eq. (3.3)]. We extend (2.23) anti-periodically to all imaginary times, $x_{0,i} \in \mathbb{R}$. Of particular interest is the two-point Schwinger function,

$$S_{2;r_1,r_2}^{\beta,L}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{T} a_{\mathbf{x},r_1}^- a_{\mathbf{y},r_2}^+ \rangle_{\beta,L},$$

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[5] Since we assumed that $w_{rr'}$ is even, these two reflection symmetries also imply the invariance of $w_{r,r'}$ under $x \to (x_1, -x_2, x_3)$.

together with its thermodynamic and zero temperature limit[6]

$$S_{2;r_1,r_2}(\mathbf{x}, \mathbf{y}) = \lim_{\beta,L\to\infty} \langle \mathbf{T} a_{\mathbf{x},r_1}^- a_{\mathbf{y},r_2}^+ \rangle_{\beta,L}, \tag{2.24}$$

which is translationally invariant, whenever it exists. We also denote by $\hat{S}_{2;r_1,r_2}(\mathbf{p})$ its Fourier transform.

In the absence of interaction, $\lambda = \nu = 0$, the two-point function (thought of as a 2×2 matrix of elements $S_{2;r_1,r_2}$) can be written, for $x_0 - y_0 \notin \beta\mathbb{Z}$, as:

$$S_2^{\beta,L}(\mathbf{x}; \mathbf{y})|_{\lambda=0} = \frac{1}{\beta L^3} \sum_{k_0 \in \mathcal{M}_\beta^f} \sum_{k \in \mathcal{B}_L} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-ik_0 + \hat{H}^0(k)} \equiv g_{\beta,L}(\mathbf{x}, \mathbf{y}). \tag{2.25}$$

We refer to $g_{\beta,L}$ as the free propagator of the model, we set $g(\cdot) = \lim_{\beta,L\to\infty} g_{\beta,L}(\cdot)$, and we denote by $\hat{g}$ its Fourier transform. Let $\mathbf{p}_F^\omega := (0, p_F^\omega)$. In the absence of interactions, as $\beta, L \to \infty$, the Fourier transform of the free propagator reads, for $\mathbf{k} = \mathbf{k}' + \mathbf{p}_F^\omega$ and $\mathbf{k}'$ small,

$$\hat{g}(\mathbf{k}' + \mathbf{p}_F^\omega) = \begin{pmatrix} -ik_0 + \omega v_3^0 k_3' & v_1^0 k_1' - iv_2^0 k_2' \\ v_1^0 k_1' + iv_2^0 k_2' & -ik_0 - \omega v_3^0 k_3' \end{pmatrix}^{-1} (1 + R_\omega^0(\mathbf{k}')), \tag{2.26}$$

with $\|R_\omega^0(\mathbf{k}')\| \le C|\mathbf{k}'|$. We see that $g_\omega(\mathbf{k}') := g(\mathbf{k}'+\mathbf{p}_F^\omega)$ agrees at leading order with the propagator of chiral relativistic fermions, with chirality $\omega = \pm$, and anisotropic velocities. We will prove that, for λ small, by fixing $\nu = \nu(\lambda)$ appropriately, the singularities of the Fourier transform of the interacting propagator are located at the same points, $\mathbf{p}_F^\pm$, as the non-interacting one; moreover, at those points, it has the same singularity structure:

$$\hat{S}_2(\mathbf{k}' + \mathbf{p}_F^\omega) = \frac{1}{Z} \begin{pmatrix} -ik_0 + \omega v_3 k_3' & v_1 k_1' - iv_2 k_2' \\ v_1 k_1' + iv_2 k_2' & -ik_0 - \omega v_3 k_3' \end{pmatrix}^{-1} (1 + R_\omega(\mathbf{k}')), \tag{2.27}$$

with $v_j = v_j^0(1 + O(\lambda))$, $j = 1, 2, 3$, the interacting Fermi velocities, $Z = 1 + O(\lambda)$ the wave function renormalization and, for any $\theta \in (0, 1)$, the remainder satisfies $\|R_\omega(\mathbf{k}')\| = O(|\mathbf{k}'|^\theta)$, non-uniformly in θ as $\theta \to 1^-$ (and, possibly, non-uniformly in the distance between the Weyl nodes).

2.4. Coupling to an external gauge field. Our analysis will focus on the transport properties of the model, after introducing an external electromagnetic field. The coupling of the model with an external vector potential is defined via the Peierls' substitution. This means that, both in the Hamiltonian and in the physical observables, any product of fermionic operators $a_{x,r}^+ a_{y,r'}^-$ is replaced by:

$$a_{x,r}^+ a_{y,r'}^- \longrightarrow a_{x,r}^+ a_{y,r'}^- e^{i\int_{x\to y} A\cdot d\ell}, \tag{2.28}$$

where $A_x \in \mathbb{R}^3$, with $x \in Q_L := (\mathbb{R}/L\mathbb{Z})^3$, and $\int_{x\to y} A \cdot d\ell$ denotes the line integral:

$$\int_0^1 A_{x+s(y-x)} \cdot (y - x)\, ds. \tag{2.29}$$

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[6] Whenever we write or refer to the limit $\beta, L \to \infty$, we mean $L \to \infty$ first, then $\beta \to \infty$.

We use the following convention for the Fourier transform of A: if $p \in \frac{2\pi}{L}\mathbb{Z}^3$,

$$\hat{A}_p = \int_{Q_L} A_x e^{ip\cdot x} dx \quad \Longleftrightarrow \quad A_x = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} \hat{A}_p e^{-ip\cdot x}.$$

Notice that the many-body interaction is not affected by the presence of the gauge field, due to the fact that the density operator is gauge invariant.

Let us denote by $\mathcal{H}_L^0(A)$ the free Hamiltonian ($\lambda = \nu = 0$) in the presence of the gauge field. The charge current operator is defined as:

$$\hat{J}_{j,p}(A) := -L^3 \frac{\partial \mathcal{H}_L^0(A)}{\partial \hat{A}_{j,p}}. \tag{2.30}$$

We also set $\hat{J}_{j,p} \equiv \hat{J}_{j,p}(0)$. A straightforward computation gives, letting $\eta_p(\delta) := \frac{1-e^{-ip\cdot\delta}}{ip\cdot\delta} = \int_0^1 ds\, e^{-isp\cdot\delta}$,

$$\hat{J}_{j,p} = \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \Big[-i \sum_{\delta \in \Lambda_L} e^{-ik\cdot\delta} \delta_j H^0(\delta) \eta_p(\delta) \Big] \hat{a}_k^- \equiv \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \hat{\mathfrak{J}}_j(k,p) \hat{a}_k^-,$$
$$\tag{2.31}$$

where $\hat{\mathfrak{J}}_j(k,p)$ should be understood as a 2×2 matrix, and a_k^+ (resp. a_k^-) as a row (resp. column) vector of components $a_{k,1}^+, a_{k,2}^+$ (resp. $a_{k,1}^-, a_{k,2}^-$). Note that, in the thermodynamic limit, the kernel $\hat{\mathfrak{J}}_j(k,p)$ can be written more explicitly as

$$\hat{\mathfrak{J}}_j(k,p) = \int_0^1 ds\, \partial_{k_j} \hat{H}^0(k+sp). \tag{2.32}$$

In particular, recalling (2.12), the kernel of the current at $k = p_F^\omega$ and $p = 0$ reads:

$$\hat{\mathfrak{J}}_j(p_F^\omega, 0) = \begin{cases} v_j^0 \sigma_j & \text{if } j = 1, 2, \\ \omega v_3^0 \sigma_3 & \text{if } j = 3. \end{cases} \tag{2.33}$$

Equivalently, in real-space,

$$J_{j,z} = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} e^{ip\cdot z} \hat{J}_{j,p} = \sum_{x,y \in \Lambda_L} a_x^+ \mathfrak{J}_j(y - x, z - x) a_y^-, \tag{2.34}$$

where $\mathfrak{J}_j(y, z) = L^{-6} \sum_{k \in \mathcal{B}_L} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} \hat{\mathfrak{J}}_j(k,p) e^{ik\cdot y} e^{ip\cdot z}$. In the thermodynamic limit,

$$\mathfrak{J}_j(y, z) = -iy_j H^0(y) \int_0^1 ds\, \delta(z - sy), \tag{2.35}$$

where $\delta(z - sy)$ is a Dirac delta. Note that the current satisfies a lattice continuity equation. In fact:

$$p \cdot \hat{J}_p = \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \sum_{\delta \in \Lambda_L} e^{-ik\cdot\delta} H^0(\delta)(e^{-ip\cdot\delta} - 1)\hat{a}_k^-$$

$$= \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ (\hat{H}^0(k+p) - \hat{H}^0(k)) \hat{a}_k^-, \tag{2.36}$$

which can be rewritten as

$$\partial_{x_0} \hat{\rho}_{(x_0,p)} = p \cdot \hat{J}_{(x_0,p)}. \tag{2.37}$$

Here $\hat{\rho}_{(x_0,p)} = e^{x_0 \mathcal{H}_L} \hat{\rho}_p e^{-x_0 \mathcal{H}_L}$, with $\hat{\rho}_p = L^{-3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \hat{a}_k^-$. Similarly, $\hat{J}_{(x_0,p)} = e^{x_0 \mathcal{H}_L} \hat{J}_p e^{-x_0 \mathcal{H}_L}$. We collect the density and the components of the lattice current to form a Euclidean 4-current $(-i \hat{\rho}_p, \hat{J}_p)$, whose components are denoted by $\hat{J}_{\mu,p}$, $\mu \in \{0, 1, 2, 3\}$:

$$\hat{J}_{\mu,p} = \frac{1}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \hat{\mathfrak{J}}_\mu(k, p) \hat{a}_k^-, \tag{2.38}$$

where $\hat{\mathfrak{J}}_0 = -i \mathbb{1}_2$ and $\hat{\mathfrak{J}}_j$, $j = 1, 2, 3$, are given by (2.31).

2.5. The lattice chiral current.
We now introduce a lattice current for the quasi-particle flow between the Weyl nodes, playing the role of the 'chiral current' for our lattice model:

$$\hat{J}_{\mu,p}^5 = \frac{Z_{\mu,\mathrm{bare}}^5}{L^3} \sum_{k \in \mathcal{B}_L} \hat{a}_{k+p}^+ \hat{\mathfrak{J}}_\mu^5(k, p) \hat{a}_k^-, \tag{2.39}$$

where $p \in \mathcal{B}_L$ and, letting $\sigma_0 := -i \mathbb{1}_2$,

$$\hat{\mathfrak{J}}_\mu^5(k, p) = \begin{cases} \frac{\sin k_3 + \sin(k_3 + p_3)}{2 \sin(p_{F,3}^+)} \sigma_\mu & \text{if } \mu = 0, 1, 2, \\ \sigma_3 & \text{if } \mu = 3. \end{cases} \tag{2.40}$$

Note that the kernel $Z_{\mu,\mathrm{bare}}^5 \hat{\mathfrak{J}}_\mu^5(k, p)$ of the chiral current at $k = p_F^\omega$ and $p = 0$ reads:

$$Z_{\mu,\mathrm{bare}}^5 \hat{\mathfrak{J}}_\mu^5(p_F^\omega, 0) = \begin{cases} \omega Z_{\mu,\mathrm{bare}}^5 \sigma_\mu & \text{if } \mu = 0, 1, 2, \\ Z_{3,\mathrm{bare}}^5 \sigma_3 & \text{if } \mu = 3. \end{cases} \tag{2.41}$$

Comparing with (2.33), we see that at the Weyl nodes the different components of the chiral current behave like those of the total current, up to an additional 'chirality sign' $\omega = \pm$ and different multiplicative factors. This implies that, in the 'infrared regime' of p small and k close to the Weyl nodes, the 'chiral density' $\hat{J}_{0,p}^5$ defined by (2.39) with $\mu = 0$ reduces to the difference between the quasi-particle densities around the Weyl nodes, up to a multiplicative pre-factor; similarly, the spatial components of the chiral current reduce to the difference between the quasi-particle currents around the Weyl nodes. Making a parallel with QED$_4$, this infrared chiral current coincides with the standard QED$_4$ axial current with different velocities.

The multiplicative normalizations $Z_{\mu,\mathrm{bare}}^5$ are fixed as follows. Let us define the vertex functions

$$\hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p}) = \lim_{\beta, L \to \infty} \frac{1}{\beta L^3} \langle \mathbf{T} \hat{J}_{\mu,\mathbf{p}} ; \hat{a}_{\mathbf{k}+\mathbf{p}}^- \hat{a}_{\mathbf{k}}^+ \rangle_{\beta,L}, \tag{2.42}$$

$$\hat{\Gamma}^5_\mu(\mathbf{k}, \mathbf{p}) = \lim_{\beta, L \to \infty} \frac{1}{\beta L^3} \langle \mathbf{T} \hat{J}^5_{\mu, \mathbf{p}} ; \hat{a}^-_{\mathbf{k}+\mathbf{p}} \hat{a}^+_{\mathbf{k}} \rangle_{\beta, L}, \tag{2.43}$$

which are 2×2 matrices in the color indices of the fermionic operators, for each choice of $\mu \in \{0, 1, 2, 3\}$. Let us also introduce their 'amputated' versions,

$$\hat{\gamma}_\mu(\mathbf{k}, \mathbf{p}) = \left[\hat{S}_2(\mathbf{k} + \mathbf{p})\right]^{-1} \hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p}) \left[\hat{S}_2(\mathbf{k})\right]^{-1}, \tag{2.44}$$

$$\hat{\gamma}^5_\mu(\mathbf{k}, \mathbf{p}) = \left[\hat{S}_2(\mathbf{k} + \mathbf{p})\right]^{-1} \hat{\Gamma}^5_\mu(\mathbf{k}, \mathbf{p}) \left[\hat{S}_2(\mathbf{k})\right]^{-1}, \tag{2.45}$$

where we recall that $\hat{S}_2(\mathbf{k})$ is the interacting propagator, see (2.24) and (2.27). Existence of the limits in (2.42)–(2.43), in the sense of footnote 6 above, is part of the main results of this paper, stated in the following section. We will impose that the amputated vertex functions agree in the infrared limit, up to a sign: that is, if $\mathbf{p}^\omega_F$ is the 4-vector $\mathbf{p}^\omega_F = (0, p^\omega_F)$,

$$\lim_{\substack{\mathbf{k} \to \mathbf{p}^\omega_F \\ \mathbf{p} \to \mathbf{0}}} \hat{\gamma}^5_\mu(\mathbf{k}, \mathbf{p}) \left[\hat{\gamma}_\mu(\mathbf{k}, \mathbf{p})\right]^{-1} = \omega \mathbb{1}_2, \tag{2.46}$$

where, for technical convenience, we take the limit in such a way that $\mathbf{k} - \mathbf{p}^\omega_F$, $\mathbf{p}$, $\mathbf{k} + \mathbf{p} - \mathbf{p}^\omega_F$ are all of the same order as they tend to zero. The normalization condition (2.46) can be interpreted as the requirement that, at the level of the amputated vertex correlation functions, for momenta close to a Weyl node, the insertion of the operator $\hat{J}^5_{\mu, \mathbf{p}}$ is equivalent to the insertion of $\omega \hat{J}_{\mu, \mathbf{p}}$, where ω is the chirality of the Weyl node.

Finally, we denote by $J^5_\mu(A)$ the chiral current coupled to the external gauge field, defined again via the Peierls substitution (2.28). We have:

$$J^5_{\mu, z}(A) = Z^5_{\mu, \text{bare}} \sum_{x, y \in \Lambda_L} a^+_x \mathfrak{J}^5_\mu(y - x, z - x) e^{i \int_{x \to y} A \cdot d\ell} a^-_y, \tag{2.47}$$

where $\mathfrak{J}^5_\mu(y - x, z - x)$ is the inverse Fourier transform of $\hat{\mathfrak{J}}^5_\mu(k, p)$.

Remark. 1) The limiting values of $\hat{\gamma}_\mu(\mathbf{k}, \mathbf{p})$ and of $\hat{\gamma}^5_\mu(\mathbf{k}, \mathbf{p})$ as $\mathbf{k} \to \mathbf{p}^\omega_F$ and $\mathbf{p} \to \mathbf{0}$ divided by $Z v_\mu$, with Z the wave function renormalization and v_μ the dressed velocity,[7] see (2.27), have the physical meaning of (dressed) charges of the quasi-particles around the Weyl nodes. Gauge invariance ensures that the charge of quasi-particles associated with the electromagnetic current is not renormalized by the interaction, that is, its dressed and bare values coincide, see (3.4) below. In contrast, the charge associated with the chiral current is renormalized non-trivially: this should come as no surprise, because the model is not invariant under chiral gauge transformations and, therefore, there is no symmetry protecting such a charge from being dressed by the interaction. The choice of $Z^5_{\mu, \text{bare}}$ is used to impose, via (2.46), the correct physical normalization of the chiral current, namely the one guaranteeing that the charges of the quasi-particles around each Weyl node computed either via the electromagnetic or the chiral current are the same. An analogous phenomenon takes place in QED_4, where it is well known that the axial vertex renormalization is different from the vectorial one, see e.g. [1, eq.(2)]: the discrepancy between the values of the two vertex renormalizations is a manifestation of

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[7] We use the convention that $v_0 = 1$.

the chiral anomaly, since a formal use of chiral gauge invariance would naively suggest their identification.

2) The definition of the components of the chiral current is largely arbitrary, as long as their kernels have the right asymptotic form at the Weyl nodes, see (2.41), and the right discrete symmetries, discussed in Appendix A.1. Changing the specific definition of the chiral current would only affect the specific value of the parameters $Z^5_{\mu,\text{bare}}$. In view of the universality result we prove, this arbitrariness does not affect the quadratic response at dominant order.

3) The introduction of a lattice current for Weyl semimetals, generalizing the proposal of [63] (see also [40]) to lattice interacting models, is an original contribution of this paper. Despite the formal similarity between the electromagnetic and the chiral lattice current, it is important to highlight some basic differences. In the infrared limit (p small and k close to the Weyl nodes), they reduce to the electromagnetic and axial currents of QED_4, respectively. In QED_4, both these currents are conserved, and their conservation is associated with basic gauge symmetries of the model (total and chiral, respectively). On the contrary, in our lattice realization, only the electromagnetic current is associated with a gauge symmetry, from which exact Ward Identities between correlations follow: these imply, in particular, that the dressed charge is not renormalized, see (3.4) below, and that the 4-divergence of the current vanishes even in the presence of an external electromagnetic field, see, e.g., (3.11) below. Neither of these properties holds for the lattice chiral current.

2.6. Main result: condensed matter simulation of the chiral anomaly. We are interested in the response of the expectation of the chiral 4-current $\hat{J}^5_{\mu,p}$ to an adiabatic external gauge field of the form $A_x(t) = e^{\eta t} A_x$, where $\eta > 0$ plays the role of adiabatic parameter. We denote by $\mathcal{H}_L(A(t))$ the Hamiltonian of the interacting system, coupled to the external gauge field via the Peierls substitution (2.28). We will consider the time-evolution of the many-body system from $t = -\infty$ to $t = 0$.

Let $\langle \cdot \rangle_{\beta,L;t} \equiv \text{Tr} \cdot \rho(t)$ be the time-dependent state of the system, where $\rho(t)$ is the solution of the von Neumann equation $i\partial_t \rho(t) = [\mathcal{H}_L(A(t)), \rho(t)]$, with boundary condition $\rho(-\infty) = \rho_{\beta,L}$. Let $\langle \hat{J}^5_{\mu,p}(A(t)) \rangle_{\beta,L;t}$ be the time-dependent average of the chiral lattice current. As discussed in the introduction, we focus our attention on the *quadratic* variation of this quantity with respect to the external field, denoted by $\langle \hat{J}^5_{\mu,p}(A(t)) \rangle^{(2)}_{\beta,L;t}$:

$$\langle \hat{J}^5_{\mu,p}(A(t)) \rangle^{(2)}_{\beta,L;t} = \frac{1}{2} \sum_{i,j=1}^{3} \sum_{p_1,p_2 \in \mathcal{B}_L} \hat{A}_{i,p_1}(t) \hat{A}_{j,p_2}(t) \delta_{p_1+p_2+p,0} \hat{\Pi}^{5;\beta,L}_{\mu,i,j}(\mathbf{p}_1, \mathbf{p}_2),$$

(2.48)

with $\mathbf{p}_i = (\eta, p_i)$. The quadratic response $\hat{\Pi}^{5;\beta,L}_{\mu,i,j}(\mathbf{p}_1, \mathbf{p}_2)$ is the analogue, in our condensed matter context, of the chiral anomaly in QED_4, see (2.51) and following discussion for more details. It can be expressed in terms of equilibrium correlation functions, via second-order time-dependent perturbation theory; see Appendix C, Eq. (C.12). A rigorous application of the Wick rotation, proven in Appendix C, allows us to express the $\beta, L \to \infty$ limit of this quantity in terms of Euclidean correlation functions. Let $\hat{\Pi}^5_{\mu,i,j}(\mathbf{p}_1, \mathbf{p}_2) = \lim_{\beta,L\to\infty} \hat{\Pi}^{5;\beta,L}_{\mu,i,j}(\mathbf{p}_1, \mathbf{p}_2)$. For $\mathbf{p}_1 = (\eta, p_1)$, $\mathbf{p}_2 = (\eta, p_2)$,

$\mathbf{p} = -\mathbf{p}_1 - \mathbf{p}_2$, we have, see Eq. (C.15):

$$\hat{\Pi}^5_{\mu,i,j}(\mathbf{p}_1, \mathbf{p}_2) = \langle \mathbf{T} \, \hat{J}^5_{\mu,\mathbf{p}} \, ; \, \hat{J}_{i,\mathbf{p}_1} \, ; \, \hat{J}_{j,\mathbf{p}_2} \rangle_\infty + \text{Schwinger terms}, \qquad (2.49)$$

where $\langle \, \cdot \, \rangle_\infty = \lim_{\beta,L \to \infty} (\beta L^3)^{-1} \langle \cdot \rangle_{\beta,L}$. We refer the reader to Eq. (C.15) for the precise form of the Schwinger terms. More generally, we denote by $\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ the extension of (2.49) to space-time current insertions, the labels $\nu, \sigma = 0$ corresponding to insertions of a density operator $\hat{J}_{0,\mathbf{p}_i}$.

The next theorem gives the explicit expression of the interacting quadratic response of the chiral current, $\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$, in the limit of low momenta.

Theorem 2.1. *Suppose that the non interacting part of the Hamiltonian satisfies the hypotheses of Sect. 2.2, namely: symmetries (i) to (iv) and Eqs. (2.11) to (2.13). There exists $\lambda_0 > 0$, independent of the choice of ζ in I, such that, for all $\zeta \in I$, there exist functions $\nu, Z^5_{\mu,bare}, v_j, Z$, analytic in $|\lambda| \leq \lambda_0$, for which (2.27) and (2.46) hold. Moreover, for $\eta > 0$, four-vectors $\mathbf{p}_1 = (\eta, p_1)$, $\mathbf{p}_2 = (\eta, p_2)$ such that $P := \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\} \leq |p_F^+|$, and $\mathbf{p} = -\mathbf{p}_1 - \mathbf{p}_2$:*

$$\sum_{\mu=0}^{3} p_\mu \hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{1}{2\pi^2} \sum_{\alpha,\beta=0}^{3} p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\nu\sigma} + R^5_{\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2), \qquad (2.50)$$

where $\varepsilon_{\alpha\beta\nu\sigma}$ is the four dimensional Levi-Civita symbol[8] and $|R^5_{\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)| \leq C_{\theta,\zeta} P^{2+\theta}$, for any $\theta \in (0, 1)$, $\zeta \in I$ and a suitable $C_{\theta,\zeta} > 0$.

The proof of this theorem, presented in the rest of this paper, provides a constructive algorithm for computing the functions ν, Z^5_μ, v_j, as well as the $\beta, L \to \infty$ limit of the Euclidean correlation functions of the system. The construction and proof of analyticity of the staggered chemical potential ν and of the Fermi velocities v_j, $j = 1, 2, 3$, uniformly in ζ, is not new, see [51,52]. An explicit computation at the level of first order perturbation theory shows that the interacting Fermi velocities v_j, $j = 1, 2, 3$, are non-trivial functions of λ and of all the other parameters entering the definition of the Hamiltonian, generically in the choice of H^0 and w. In this sense, the Fermi velocities are non-universal quantities: their value depends on the details of the microscopic Hamiltonian. The same is true for the longitudinal Kubo conductivity, see [51], and for several other physical observables.

The important new piece of information contained in Theorem 2.1 is Eq. (2.50), which shows that the quadratic response of the chiral current is *universal* in the low momentum limit. It plays the same role as the quantized transverse conductivity in quantum Hall fluids. Remarkably, (2.50) is valid for all the choices of the parameter ζ controlling the distance among the Weyl points; it is valid, in particular, arbitrarily close to the critical point $\zeta = \zeta_1$ where the Weyl points merge and the valence and conduction bands touch quadratically, rather than linearly. The situation where the Weyl nodes are very close and the quasi-particle velocities are very small is very common in the actual experimental realization of Weyl semimetals, see e.g. [9, Figs. 2 and 3] and [20, Fig. 1]. In such a situation, the natural parameter measuring the strength of the interaction is the coupling strength divided by the quasi-particle velocity; surprisingly,

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[8] We use the convention that $\varepsilon_{0123} = 1$, and $\varepsilon_{\pi_0,\pi_1,\pi_2,\pi_3} = (-1)^\pi$ for $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ a permutation of $(0, 1, 2, 3)$, with $(-1)^\pi$ the sign of the permutation. As usual, $\varepsilon_{\alpha\beta\nu\sigma} = 0$ whenever the sequence $(\alpha, \beta, \nu, \sigma)$ has at least a repetition.

even if this parameter becomes huge, the interaction still appears not to affect the anomaly coefficient.

Note that the universality of the 'chiral anomaly' in (2.50) holds, provided that the lattice chiral current verifies the condition (2.46). The subtle cancellation underlying our universality result can also be stated in an alternative, yet equivalent, way: let the chiral current be defined as in (2.39), but without apriori fixing $Z_{\mu,\text{bare}}^5$ in any special way; then the divergence of the quadratic response coefficient $\hat{\Pi}_{\mu,\nu,\sigma}^5$ *divided* by the ratio of the axial and vector vertex renormalizations (i.e., by the left side of (2.46) times the chirality index ω) is universal at leading order. The fact that the quadratic response of the axial current to the electromagnetic field *divided* by the axial renormalization is universal (and not the quadratic response itself, as often stated incorrectly) is the very content of the Adler–Bardeen theorem in massless QED_4, see the discussion in [1] after Eqs. (6) and (7). Indeed, a naive perturbative computation of the chiral anomaly in QED_4 would apparently give rise to higher order corrections beyond the chiral triangle graph, see [5]: however, as discussed in [1], these corrections are cancelled exactly by the axial vertex renormalization, which is *different* from the vectorial one.

Note also that, of course, the quadratic response of the electromagnetic, rather than axial, current vanishes, by current conservation.

In order to make the connection between the quadratic response $\hat{\Pi}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2)$ and the transport experiments in Weyl semimetals, we choose the vector potential to be of compact support in space, and set $N_t^5 = i \lim_{\beta,L\to\infty}\langle\hat{J}_{0,0}^5(A(t))\rangle_{\beta,L;t}^{(2)}$. We now want to compute $\partial_t N_t^5$, which represents the quasi-particle flow from the first Weyl node to the second, at quadratic order in A. From the construction of the interacting Gibbs state of the system in the $\beta, L \to \infty$ limit, which Theorem 2.1 is based on, one finds that the time derivative commutes with the $\beta, L \to \infty$ limit. Moreover, note that, by differentiating (2.48) with respect to time, the right side simply gets multiplied by 2η, because its time dependence is all in $\hat{A}_{i,p_1}(t) = e^{\eta t}\hat{A}_{i,p_1}(0)$ and $\hat{A}_{j,p_2}(t) = e^{\eta t}\hat{A}_{j,p_2}(0)$. Note also that 2η is minus the zero-th component of $\mathbf{p} = -(2\eta, 0)$ (recall that in our case $p = -p_1 - p_2 = 0$). Therefore, by using (2.50) in the expression obtained from (2.48) by differentiating with respect to time and by taking $\beta, L \to \infty$, we find:

$$\partial_t N_t^5 = \frac{1}{4\pi^2} \int dx\, \partial_\alpha A_{i,x}(t)\partial_\beta A_{j,x}(t)\varepsilon_{\alpha\beta ij} + \mathfrak{e}_A(t)$$
$$= \frac{1}{2\pi^2} \int dx\, E_x(t) \cdot B_x(t) + \mathfrak{e}_A(t) \tag{2.51}$$

where, in the first line, summation over repeated indices is understood (α, β run from 0 to 3, while i, j from 1 to 3), and ∂_0 denotes the time derivative. The error term $\mathfrak{e}_A(t)$, due to the error term $R_{\nu,\sigma}^5$ in (2.50), collects contributions involving a higher number of derivatives on the vector potential. It is subdominant for a vector potential slowly varying in space. For the second identity, we used the definitions of electric and magnetic fields: $E_{i,x} = -\partial_t A_{i,x}$ and $B_{i,x} = \sum_{j,l=1}^3 \varepsilon_{ijl}\, \partial_j A_{l,x}$. Recent experiments [41,43,45,71,72] in different Weyl semimetals reported evidence for an anomalous flow of quasi-particles between the Weyl nodes, which causes a negative, highly anisotropic, magnetoresistance. For instance, in [71], the resistivity tensor was measured for different values of the angle between E and B, and the response proved to be strongest for E and B parallel, see e.g. [71, Fig. 6A and 6B], in agreement with the $E \cdot B$ dependence in (2.51). Comparing measures at different angles is a way to access directly the quadratic response, which is the one related to the anomaly. We stress that the $E \cdot B$ dependence

found in real (interacting) Weyl semimetals is the same predicted originally for non-interacting systems [63]. This is a first important instance of universality, even though a precise, quantitative, experimental verification of the interaction-independence of the chiral anomaly coefficient has still to come.

The dominant term in the right side of (2.51) is the usual chiral anomaly of QED_4, with the same universal prefactor, irrespective of the presence of the interaction. The interaction independence of the coefficient is the analogue, in our context, of the Adler–Bardeen anomaly non-renormalization theorem [3], which our result is a rigorous version of. In our context, the lattice plays the role of a fixed ultraviolet regularization of an effective QFT model: chiral and Lorentz symmetry are emerging and approximate but nevertheless the chiral anomaly satisfies the AB non-renormalization property. We expect that our methods can be used to prove a generalization of the Adler–Bardeen's theorem for interacting lattice gauge theories at finite cutoff, such as infrared lattice QED_4, see [50,53].

3. Proof of Theorem 2.1

3.1. General strategy and core argument of the proof. The proof of Theorem 2.1 is based on the following strategy:

1. We first compute the generating function of correlations via RG methods; the output of the RG construction is that the correlation functions are expressed in terms of a series expansion in λ and in a sequence of effective parameters, which is convergent, provided that $|\lambda|$ is sufficiently small and the effective parameters are uniformly bounded under the iterations of the RG map.
2. Next, we show that these effective parameters are uniformly bounded, as desired, provided that the staggered chemical potential ν is chosen appropriately. We also show how to fix the bare parameters $Z_{\mu,\text{bare}}^5$ in such a way that (2.46) is verified.
3. The RG expansion also allow us to identify an explicit dominant contribution to the correlations (for momenta close to the Weyl nodes), which can be written explicitly in terms of the 'dressed parameters' of the model. The subdominant contributions to the correlations admit improved dimensional bounds, which imply better regularity properties in momentum space, as compared to their dominant counterparts.
4. Once that the correlations have been written as a dominant contribution, plus a better-behaved remainder, we are in business for proving (2.50): in fact, the left side of (2.50) can be written as the contribution from the dominant part, which can be computed explicitly, plus a remainder, whose value at small momenta is fixed by Ward Identities.

In the next three subsections, Sects. 3.1.1–3.1.3, containing the **core argument** of the proof, we will describe more technically: (1) the outcome of items 1 to 3 for the quadratic response $\hat{\Pi}_{\mu,\nu,\sigma}^5$, see in particular (3.1) below; (2) the proof of our main result, Eq. (2.50), starting from the representation formula (3.1) for $\hat{\Pi}_{\mu,\nu,\sigma}^5$.

The main ingredients involved in these steps are the following:

(i) The decomposition of the quadratic response into an explicit, non-differentiable, term, proportional to the usual chiral triangle graph of QED_4 with momentum cutoff (the function $I_{\mu,\nu,\sigma}$ in (3.1) below), plus a correction (the function $\widetilde{H}_{\mu,\nu,\sigma}^5$ in (3.1) below). **Note**: the correction is not subdominant; rather, it gives a contribution to the chiral anomaly comparable with the one from the first term. However, since it comes from the integration of the infrared-irrelevant terms, it is **more regular** than

the first one: more precisely, it is continuously differentiable in the momenta, and its derivative is Hölder continuous (contrary to the first term, whose derivative is discontinuous at $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{0}$); this allows us to expand it in Taylor series up to first order included. Let us stress that the decomposition (3.1) is, obviously, non-unique: it is natural in a Wilsonian RG computation of the correlation functions, the first term being obtained by neglecting irrelevant terms in the RG sense. Due to the presence of a momentum cutoff, this term breaks gauge invariance, as apparent from (3.7) below. Such a term has the correct tensorial structure (i.e., its divergence with respect to the first index is proportional to $\sum_{\alpha,\beta} p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha,\beta,\nu,\sigma}$, see (2.50)). However, if we neglected the remainder, it would produce an incorrect prefactor in the right side of (2.50) (even in the absence of interactions).

(ii) The explicit computation of the first term in the right side of (3.1), i.e., of the chiral triangle graph of QED_4 with momentum cutoff, leading to (3.6)–(3.7) below.

(iii) The use of Ward Identities, guaranteeing two crucial facts: (1) the vectorial vertex renormalization is proportional to the Fermi velocity, see (3.4); (2) the first order Taylor coefficients of the correction term to the quadratic response, which are well defined thanks to its differentiability, are uniquely fixed by the conservation of the vectorial current, namely by the condition (3.2).

(iv) The normalization of the chiral current, i.e., condition (2.46), which guarantees that the chiral vertex renormalization equals the vectorial one, see (3.5).

Let us now discuss the core argument of the proof in more detail.

3.1.1. The splitting of $\hat{\Pi}^5_{\mu,\nu,\sigma}$ into chiral triangle graph + differentiable correction As an outcome of the RG analysis mentioned in item 1, of the choice of ν mentioned in item 2 and of the splitting of the correlation functions into dominant plus subdominant parts mentioned in item 3, we will prove that the quadratic response of the chiral current to the external electromagnetic field can be written as

$$\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{Z^5_\mu Z_\nu Z_\sigma}{Z^3 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2) + \widetilde{H}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2), \tag{3.1}$$

where: the first term in the right side is defined in terms of the effective parameters Z^5_μ, Z_μ (the chiral/non-chiral vertex renormalizations[9]), of Z (the wave function renormalization, see (2.27)), of v_l (the dressed velocities, see (2.27)), and of the explicit function $I_{\mu,\nu,\sigma}$ (the 'chiral triangle graph' of QED_4 with momentum cutoff, see (3.141) below), computed at $\overline{\mathbf{p}}_1 = (p_{1,0}, v_1 p_{1,1}, v_2 p_{1,2}, v_3 p_{1,3})$, $\overline{\mathbf{p}}_2 = (p_{2,0}, v_1 p_{2,1}, v_2 p_{2,2}, v_3 p_{2,3})$; the second term in the right side is a correction term, which is differentiable in the external momenta in a sufficiently small neighbourhood around the origin, with derivatives that are Hölder continuous of order θ, for all $\theta \in (0, 1)$.

3.1.2. Ward identities and the chiral triangle graph Before we proceed further, let us specify a few additional properties of the quadratic response $\hat{\Pi}^5_{\mu,\nu,\sigma}$ and of the first term in the right side of (3.1).

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[9] These are defined in terms of the amputated vertex functions in (2.44)–(2.45) as follows: $Z_\mu = \lim_{\substack{\mathbf{p}\to\mathbf{0} \\ \mathbf{k}\to\mathbf{p}^\omega_F}} \hat{\gamma}_\mu(\mathbf{k}, \mathbf{p})$, where the limit is understood in the sense specified after (2.46); and $Z^5_\mu = \lim_{\substack{\mathbf{p}\to\mathbf{0} \\ \mathbf{k}\to\mathbf{p}^\omega_F}} \hat{\gamma}^5_\mu(\mathbf{k}, \mathbf{p})$.

The first key property we wish to emphasize is a remarkable consequence of the conservation (2.37), namely the Ward Identity

$$\sum_\nu p_{1,\nu} \hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \sum_\sigma p_{2,\sigma} \hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = 0. \tag{3.2}$$

We stress that there is no similar identity for the contraction of the μ index with the external momenta, due to the lack of lattice *chiral* gauge invariance: while the formal infrared limit of both the electromagnetic and chiral currents are conserved, only the first one is associated with a lattice gauge invariance principle.

Of course, gauge invariance implies the validity of Ward Identities for (infinitely many) other correlations. A crucial one is the 'vertex Ward Identity' relating the vertex function $\hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p})$ in (2.42) with the 2-point function $\hat{S}_2(\mathbf{k})$ in (2.27):

$$\sum_{\mu=0,1,2,3} p_\mu \hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p}) = \hat{S}_2(\mathbf{k}) - \hat{S}_2(\mathbf{k} + \mathbf{p}), \tag{3.3}$$

which implies the following relation between the renormalized parameters:

$$Z_\mu = v_\mu Z, \tag{3.4}$$

with $v_0 := 1$. The condition (3.4) can be interpreted by saying that the electric charge transported by the electromagnetic current is not renormalized, thanks to lattice Ward Identities. The analogous identity does not hold for the lattice chiral current, because there is no lattice chiral symmetry protecting the 'chiral charge' from renormalizing. This is the reason why we need to impose the identity of the charges transported by the electromagnetic and chiral currents via the constants $Z^5_{\mu,\text{bare}}$, as discussed in Sect. 2.5; in fact, by imposing (2.46), we obtain that, for $\mu = 0, 1, 2, 3$,

$$Z^5_\mu = Z_\mu. \tag{3.5}$$

Finally, an explicit computation of the chiral triangle graph $I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ shows that

$$\sum_{\mu=0}^3 (p_{1,\mu} + p_{2,\mu}) I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{6\pi^2} \sum_{\alpha,\beta=0}^3 p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\nu\sigma} + R_{\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) \tag{3.6}$$

$$\sum_{\nu=0}^3 p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{6\pi^2} \sum_{\alpha,\beta=0}^3 p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\mu\sigma} + \widetilde{R}_{\mu,\sigma}(\mathbf{p}_1, \mathbf{p}_2), \tag{3.7}$$

where $\varepsilon_{\alpha\beta\nu\sigma}$ is the Levi-Civita symbol, see footnote 8, and where both R and $\widetilde{R}$ are smaller than $C P^3 / |p_F^+|$ for $P = \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\}$ sufficiently small, as compared with $|p_F^+|$. Note that the fact that the 4-divergences in the left side of (3.7) is non-zero, contrary to (3.2), is ultimately due to the fact that the triangle graph $I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ is computed in the presence of an ultraviolet momentum cutoff, see (3.141) below, which breaks global and chiral gauge symmetries explicitly.

3.1.3. The core argument: proof of (2.50) In view of (3.1), the left side of (2.50) can be rewritten as

$$-\sum_{\mu}(p_{1,\mu}+p_{2,\mu})\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = -\sum_{\mu}(p_{1,\mu}+p_{2,\mu})\frac{Z^5_\mu Z_\nu Z_\sigma}{Z^3 v_1 v_2 v_3}I_{\mu,\nu,\sigma}(\overline{\mathbf{p}}_1,\overline{\mathbf{p}}_2)$$
$$-\sum_{\mu}(p_{1,\mu}+p_{2,\mu})\tilde{H}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2). \qquad (3.8)$$

Let us focus on the first term in the right side of (3.8). Thanks to (3.5), (3.4) and (3.6), recalling that for $a = 1, 2$ and $\mu = 0, 1, 2, 3$ we defined $\overline{p}_{a,\mu} = v_\mu p_{a,\mu}$, we can rewrite it as

$$-\frac{v_\nu v_\sigma}{v_1 v_2 v_3}\Big(\sum_{\alpha,\beta}\frac{1}{6\pi^2}v_\alpha p_{1,\alpha} v_\beta p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma} + R_{\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2)\Big) \qquad (3.9)$$
$$= -\frac{1}{6\pi^2}\sum_{\alpha,\beta}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma} - \frac{v_\nu v_\sigma}{v_1 v_2 v_3}R_{\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2),$$

where, in passing from the left to the right side, we used the fact that $\varepsilon_{\alpha\beta\nu\sigma}\neq 0$ if and only if $(\alpha,\beta,\nu,\sigma)$ is a permutation of $(0, 1, 2, 3)$ (recall that $v_0 = 1$). Plugging this back into (3.8), and recalling that $R_{\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = O(P^3)$, with $P = \max\{|\mathbf{p}_1|,|\mathbf{p}_2|\}$, see the lines following (3.7), we find

$$-\sum_{\mu}(p_{1,\mu}+p_{2,\mu})\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = -\frac{1}{6\pi^2}\sum_{\alpha,\beta}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha,\beta,\nu,\sigma}$$
$$-\sum_{\mu}(p_{1,\mu}+p_{2,\mu})\tilde{H}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) + O(P^3).$$
$$(3.10)$$

Similarly, using (3.7), we find:

$$\sum_{\nu}p_{1,\nu}\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{6\pi^2}\sum_{\alpha,\beta}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\mu\sigma} + \sum_{\nu}p_{1,\nu}\tilde{H}^5_{\mu,\nu,\sigma}(\mathbf{p}_1,\mathbf{p}_2) + O(P^3).$$
$$(3.11)$$

We now use (3.2) and the continuous differentiability of $\tilde{H}^5_{\mu,\nu,\sigma}$ to obtain, after having expanded $\tilde{H}^5_{\mu,\nu,\sigma}$ at first order in Taylor series around the origin,

$$0 = \frac{1}{6\pi^2}\sum_{\alpha,\beta}p_{1,\alpha}p_{2,\beta}\varepsilon_{\alpha\beta\mu\sigma} + \sum_{\nu}p_{1,\nu}\Big(\tilde{H}^5_{\mu,\nu,\sigma}(\mathbf{0},\mathbf{0}) + \sum_{a=1,2}\sum_{\rho}p_{a,\rho}\frac{\partial \tilde{H}^5_{\mu,\nu,\sigma}}{\partial p_{a,\rho}}(\mathbf{0},\mathbf{0})\Big).$$
$$(3.12)$$

Equation (3.12) implies that:

$$\tilde{H}^5_{\mu,\nu,\sigma}(\mathbf{0},\mathbf{0}) = 0, \qquad \frac{\partial \tilde{H}^5_{\mu,\nu,\sigma}}{\partial p_{2,\beta}}(\mathbf{0},\mathbf{0}) = -\frac{1}{6\pi^2}\varepsilon_{\nu\beta\mu\sigma}.$$
$$(3.13)$$

Similarly, using that $\widetilde{H}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \widetilde{H}^5_{\mu,\sigma,\nu}(\mathbf{p}_2, \mathbf{p}_1)$,

$$\frac{\partial \widetilde{H}^5_{\mu,\nu,\sigma}}{\partial p_{1,\beta}}(\mathbf{0}, \mathbf{0}) = -\frac{1}{6\pi^2}\varepsilon_{\sigma\beta\mu\nu}. \tag{3.14}$$

We now go back to (3.11), expand $\widetilde{H}^5_{\mu,\nu,\sigma}$ in Taylor series around the origin up to first order included, and use the identities (3.13)–(3.14), thus finding

$$-\sum_\mu (p_{1,\mu} + p_{2,\mu})\hat{\Pi}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{1}{6\pi^2}\sum_{\alpha,\beta} p_{1,\alpha} p_{2,\beta}\varepsilon_{\alpha\beta\nu\sigma} \tag{3.15}$$

$$+\frac{1}{6\pi^2}\sum_\mu (p_{1,\mu} + p_{2,\mu})\sum_{a=1,2}(-1)^a\sum_\beta p_{a,\beta}\varepsilon_{\nu\beta\mu\sigma} + \sum_\mu (p_{1,\mu} + p_{2,\mu})\widetilde{R}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) + O(P^3),$$

where the term $\widetilde{R}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ comes from the Taylor remainder of $\widetilde{H}^5_{\mu,\nu,\sigma}$. Thanks to the fact that the derivatives of $\widetilde{H}^5_{\mu,\nu,\sigma}$ are Hölder continuous of order θ, for any $0 < \theta < 1$, the remainder $\widetilde{R}^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ is of the order $P^{1+\theta}$ (possibly non-uniformly in θ and in $|p_F^+|$), for $P = \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\}$ sufficiently small. After combining the first two terms in the right side of (3.15), we get (2.50), with $R^5_{\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = O(P^{2+\theta})$.

3.1.4. Roadmap In view of the previous subsection, in order to complete the proof of (2.50), we 'just' need to prove the validity of (3.1), with Z_μ, Z^5_μ, Z, v_l satisfying (3.5)–(3.4), and with $I_{\mu,\nu,\sigma}$ satisfying (3.6)–(3.7); we also need to establish (3.2). The proof of all these claims will be given below, together with the proof of the other claims in Theorem 2.1. We will follow the strategy outlined in items 1 to 3 at the beginning of this section. More in detail, the proof in the next sections is organized as follows:

- In Sect. 3.2, we represent the generating function of Euclidean correlations in terms of a Grassmann functional integral. In particular, in Sect. 3.2.1 we state the gauge-invariance property of the Grassmann generating function, which implies a hierarchy of Ward Identities for the correlation functions, including (3.2).
- In Sect. 3.3 we describe the iterative RG computation of the Grassmann generating function, whose output is a convergent expansion for the generating function, in terms of a sequence of effective coupling constants $Z_{\mu,h}$, $Z^5_{\mu,h}$, Z_h, $v_{l,h}$, labelled by the step h of the RG iteration (these are nothing but the finite-h analogues of the parameters Z_μ, Z^5_μ, Z, v_l in (3.1)). If these parameters are suitably bounded, uniformly in h, and if they admit a limit as the number of RG iterations tends to infinity, the expansion for the correlation functions is convergent uniformly in the $\beta, L \to \infty$ limit, for $|\lambda|$ small enough, and the limit of the correlations as $\beta, L \to \infty$ exists. In order for the bounds on the radius of convergence to be uniform in $\zeta \in I$, the RG iteration must be defined differently, depending on whether the momentum scale under consideration in the RG step is larger or smaller than the separation between the Weyl nodes; the two regimes are described in Sects. 3.3.1 and 3.3.2, respectively.
- In Sect. 3.4 we explain how to fix the bare staggered chemical potential ν in such a way that the sequence of effective coupling constants $Z_{\mu,h}$, $Z^5_{\mu,h}$, Z_h, $v_{l,h}$ satisfies the desired bounds. We also show how to fix the bare parameters $Z^5_{\mu,\text{bare}}$ in such a way that (3.5) holds.
- In Sect. 3.5 we explain how to use the convergent expansion provided by the iterative RG computation of the generating function, in order to define a splitting of the main

correlation functions of interest into an explicit dominant part plus a remainder, whose Fourier transforms admit improved dimensional bounds at low external momenta. In particular, we prove (3.1) and (2.27), we show that (3.5) implies (2.46) and, by using the vertex Ward Identity, we prove (3.4).

- In Sect. 3.6 we prove (3.6) and (3.7).

Further technical details are deferred to the appendices: in Appendix A we discuss the symmetries of the Grassmann action and their consequences for the effective action obtained at the h-th step of the RG iteration; in Appendix B we provide further technical details on the 'tree expansion' for the kernels of the effective action and their dimensional bounds; in Appendix C we prove that the quadratic response coefficients we are interested in can be expressed in terms of Euclidean correlation functions.

3.2. Grassmann representation. In this section we introduce a Grassmann integral formulation of the model, which will be the starting point for our RG analysis. In the following, $C, C', \ldots$, denote universal constants (in particular, independent of the distance between the Weyl nodes), whose specific values may change from line to line.

Let γ be any constant larger than 1, α_0 a positive constant, to be fixed later on, and $\chi(s)$ a smooth, even, compactly supported function, such that $\chi(s) = 0$ for $|s| > \alpha_0 \gamma$ and $\chi(s) = 1$ for $|s| < \alpha_0$. For technical convenience, we choose χ to belong to the Gevrey class G^s with $s = 2$, see, e.g., [33, Appendix C]. Recall that $\mathcal{M}_\beta^{\mathrm{f}} = \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$ is the set of fermionic Matsubara frequencies. Given $N \in \mathbb{N}$, we introduce the UV-regularized version of $\mathcal{M}_\beta^{\mathrm{f}}$ as:

$$\mathcal{M}_{\beta,N}^{\mathrm{f}} := \left\{ k_0 \in \mathcal{M}_\beta^{\mathrm{f}} \,\middle|\, \chi(2^{-N}|k_0|) > 0 \right\}. \tag{3.16}$$

Let $\mathcal{B}_L$ be the finite-volume Brillouin zone, Eq. (2.7), and let $\mathbb{D}_{\beta,L,N} := \mathcal{M}_{\beta,N}^{\mathrm{f}} \times \mathcal{B}_L$. We consider the finite Grassmann algebra generated by the Grassmann variables $\{\hat{\psi}_{\mathbf{k},r}^{\pm}\}$ with $\mathbf{k} = (k_0, k) \in \mathbb{D}_{\beta,L,N}$ and $r = 1, 2$.

The Grassmann Gaussian integration $\int P_N(d\psi)(\cdot)$ is a linear functional acting on the Grassmann algebra as follows. Its action on a given Grassmann monomial $\prod_{j=1}^n \hat{\psi}_{\mathbf{k}_j, q_j}^{\varepsilon_j}$ is zero unless $|\{j : \varepsilon_j = +\}| = |\{j : \varepsilon_j = -\}|$, in which case:

$$\int P_N(d\psi) \hat{\psi}_{\mathbf{k},r_1}^- \hat{\psi}_{\mathbf{k}_1',r_1'}^+ \cdots \hat{\psi}_{\mathbf{k}_n,r_n}^- \hat{\psi}_{\mathbf{k}_n',r_n'}^+ = \det[C_{r_j,r_k'}(\mathbf{k}_j, \mathbf{k}_k')]_{j,k=1,\ldots,n}, \tag{3.17}$$

where $C_{r,r'}(\mathbf{k}, \mathbf{k}') = \beta L^3 \delta_{\mathbf{k},\mathbf{k}'} [\hat{g}_{\beta,L,N}(\mathbf{k})]_{r,r'}$ and

$$\hat{g}_{\beta,L,N}(\mathbf{k}) := \frac{\chi_N(k_0)}{-ik_0 + \hat{H}^0(k)}, \qquad \chi_N(k_0) := \chi(2^{-N}k_0). \tag{3.18}$$

We represent the free propagator as:

$$\hat{g}_{\beta,L,N}(\mathbf{k}) = \chi_N(k_0) \begin{pmatrix} -ik_0 + c(k) + d(k) & a(k) - ib(k) \\ a(k) + ib(k) & -ik_0 - c(k) + d(k) \end{pmatrix}^{-1}, \tag{3.19}$$

see (2.8). Recall that $a(p_F^\omega) = b(p_F^\omega) = c(p_F^\omega) = d(p_F^\omega) = 0$ and that a, b, c, d satisfy the parity properties spelled out after (2.8). Moreover, letting $a_R(k) = a(k) - v_1^0 k_1$,

$b_R(k) = b(k) - v_2^0 k_2$, $c_{R,\omega}(k) = c(k) - \omega v_3^0 (k_3 - p_{F,3}^\omega) - \frac{b^0}{2}(k_3 - p_{F,3}^\omega)^2$ and $d_R(k) = d(k)$, these functions satisfy the bounds in (2.14) and (2.15).

The Grassmann Gaussian integration might also be rewritten as:

$$\int P_N(d\psi)(\cdot) = \frac{1}{\mathcal{N}_{\beta,L,N}} \int d\psi \, \exp\left\{ -\frac{1}{\beta L^3} \sum_{\mathbf{k}\in\mathbb{D}_{\beta,L,N}} \sum_{r,r'=1,2} \hat{\psi}_{\mathbf{k},r}^+ [\hat{g}_{\beta,L,N}(\mathbf{k})^{-1}]_{r,r'} \hat{\psi}_{\mathbf{k},r'}^- \right\}(\cdot) \tag{3.20}$$

with $\int d\psi(\cdot)$ is the usual Berezin integration over the Grassmann algebra and $\mathcal{N}_{\beta,L,N}$ a normalization factor:

$$\mathcal{N}_{\beta,L,N} = \prod_{\mathbf{k}\in\mathbb{D}_{\beta,L,N}} (\beta L^3) \det \hat{g}_{\beta,L,N}(\mathbf{k}). \tag{3.21}$$

Let us introduce the configuration space Grassmann field as follows, for $\mathbf{x} \in [0, \beta) \times \Lambda_L$:

$$\psi_{\mathbf{x},r}^\pm := \frac{1}{\beta L^3} \sum_{\mathbf{k}\in\mathbb{D}_{\beta,L,N}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} \hat{\psi}_{\mathbf{k},r}^\pm \, ; \tag{3.22}$$

we then have:

$$\int P_N(d\psi)\psi_{\mathbf{x},r}^- \psi_{\mathbf{y},r'}^+ = \left[g_{\beta,L,N}(\mathbf{x} - \mathbf{y}) \right]_{r,r'},$$

$$g_{\beta,L,N}(\mathbf{x} - \mathbf{y}) := \frac{1}{\beta L^3} \sum_{\mathbf{k}\in\mathbb{D}_{\beta,L,N}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \hat{g}_{\beta,L,N}(\mathbf{k}). \tag{3.23}$$

Denoting by $g_{\beta,L}(\mathbf{x})$ the two-point function of the Fock-space model, it is well-known (and easy to check) that $\lim_{N\to\infty} g_{\beta,L,N}(\mathbf{x}) = g_{\beta,L}(\mathbf{x})$ for $\mathbf{x} \notin \beta\mathbb{N} \times L\mathbb{N}^2$, while for $\mathbf{x} \in \beta\mathbb{N} \times L\mathbb{N}^2$:

$$\lim_{N\to\infty} g_{\beta,L,N}(\mathbf{x}) = \frac{g_{\beta,L}(x_0^+, x) + g_{\beta,L}(x_0^-, x)}{2}. \tag{3.24}$$

Next, we introduce the interaction of the Grassmann field theory as:

$$\mathcal{V}(\psi) := -\lambda \int d\mathbf{x}d\mathbf{y} \sum_{r,r'} n_{\mathbf{x},r} w_{r,r'}(\mathbf{x}, \mathbf{y}) n_{\mathbf{y},r'} - \nu \int d\mathbf{x} \sum_r (-1)^r n_{\mathbf{x},r} \tag{3.25}$$

with the notations: $w_{r,r'}(\mathbf{x}, \mathbf{y}) := \delta(x_0 - y_0) w_{r,r'}(x, y)$; $\int d\mathbf{x}(\cdot) := \int_0^\beta dx_0 \sum_{x\in\Lambda_L}(\cdot)$; and $n_{\mathbf{x},r} = \psi_{\mathbf{x},r}^+ \psi_{\mathbf{x},r}^-$. From now on, we assume that ν is real-analytic in λ, for λ small, and $|\nu| \le C|\lambda|$. A posteriori, we will see that this assumption is compatible with the other requirements on ν we will need. More generally, the interaction of the Grassmann field theory in the presence of complex external fields $A_{\mu,\mathbf{x}}$, $A_{\nu,\mathbf{x}}^5$ (depending now on coordinates $\mathbf{x} = (x_0, x)$, with x_0 an imaginary time in $[0, \beta)$) and of a Grassmann external field $\phi_{\mathbf{x},r}^\pm$ is:

$$\mathcal{V}(A, A^5, \phi, \psi) = \mathcal{V}(\psi) + B(A, \psi) + B^5(A^5, A, \psi) + (\psi^+, \phi^-) + (\phi^+, \psi^-), \tag{3.26}$$

where $(\psi^+, \phi^-) := \sum_{r=1}^2 \int d\mathbf{x} \psi_{\mathbf{x},r}^+ \psi_{\mathbf{x},r}^-$ and

$$B(A, \psi) := (A_0, j_0) + (\psi^+, (H^0 - H^0(A))\psi^-)$$

$$B^5(A^5, A, \psi) := \sum_{\mu=0}^3 (A_\mu^5, j_\mu^5), \tag{3.27}$$

with the understanding that $(A_0, j_0) = \int d\mathbf{x} A_{0,\mathbf{x}} j_{0,\mathbf{x}}$, with $j_{0,\mathbf{x}} = -i \sum_{r=1}^2 \psi_{\mathbf{x},r}^+ \psi_{\mathbf{x},r}^-$, and similarly for (A_μ^5, j_μ^5); here j_μ^5 denotes the Grassmann counterpart of the chiral lattice current, recall Eq. (2.47):

$$j_{\mu,\mathbf{z}}^5(A) = Z_{\mu,\text{bare}}^5 \sum_{x,y \in \Lambda_L} \sum_{r,r'=1}^2 \psi_{(z_0,x),r}^+ \mathfrak{J}_{\mu;r,r'}^5(y-x, z-x) e^{i \int_{x \to y} A_{z_0} \cdot d\ell} \psi_{(z_0,y),r'}^-,$$

where $\int_{x \to y} A_{z_0} \cdot d\ell$ denotes the line integral $\int_0^1 A_{(z_0, x+s(y-x))} \cdot (y-x) ds$. Finally, we introduce the generating functional of the correlations as:

$$\mathcal{W}_{\beta,L}(A, A^5, \phi) := \lim_{N \to \infty} \log \int P_N(d\psi) e^{\mathcal{V}(A, A^5, \phi, \psi)}. \tag{3.28}$$

The existence of the ultraviolet limit $N \to \infty$ is essentially model independent, and has already been proven in a number of places, see *e.g.* [27,29] for the Hubbard model on the honeycomb lattice. It is well-known, see *e.g.* Section 5.1 of [30], that the Fock-space Euclidean correlation functions can be expressed in terms of the derivatives of the generating functional with respect to the external fields. At non-coinciding points:

$$S_{2;r,r'}^{\beta,L}(\mathbf{x}, \mathbf{y}) = \frac{\partial^2 \mathcal{W}_{\beta,L}(A, A^5, \phi)}{\partial \phi_{\mathbf{y},r'}^- \partial \phi_{\mathbf{x},r}^+}\bigg|_{A=A^5=\phi=0},$$

$$\Gamma_{\mu;r,r'}^{\beta,L}(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \frac{\partial^3 \mathcal{W}_{\beta,L}(A, A^5, \phi)}{\partial A_{\mu,\mathbf{z}} \partial \phi_{\mathbf{y},r'}^- \partial \phi_{\mathbf{x},r}^+}\bigg|_{A=A^5=\phi=0},$$

$$\Gamma_{\mu;r,r'}^{5;\beta,L}(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \frac{\partial^3 \mathcal{W}_{\beta,L}(A, A^5, \phi)}{\partial A_{\mu,\mathbf{z}}^5 \partial \phi_{\mathbf{y},r'}^- \partial \phi_{\mathbf{x},r}^+}\bigg|_{A=A^5=\phi=0}, \tag{3.29}$$

$$\Pi_{\mu,\nu,\sigma}^{5;\beta,L}(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \frac{\partial^3 \mathcal{W}_{\beta,L}(A, A^5, \phi)}{\partial A_{\mu,\mathbf{z}}^5 \partial A_{\nu,\mathbf{x}} \partial A_{\sigma,\mathbf{y}}}\bigg|_{A=A^5=\phi=0}.$$

By translation invariance, all the correlation functions in (3.29) only depend on the relative differences of the configuration-space arguments.

Remark. The statement of Theorem 2.1 involves the Fourier transforms of the correlation functions in (3.29),

$$\hat{S}_{2;r,r'}^{\beta,L}(\mathbf{k}) = \int d\mathbf{y} e^{-i\mathbf{k}\mathbf{y}} S_{2;r,r'}^{\beta,L}(\mathbf{0}, \mathbf{y}), \tag{3.30}$$

$$\hat{\Gamma}_{\mu;r,r'}^{\beta,L}(\mathbf{k}, \mathbf{p}) = \int d\mathbf{y} \, d\mathbf{z} \, e^{-i\mathbf{p}\mathbf{z} - i\mathbf{k}\mathbf{y}} \Gamma_{\mu;r,r'}^{\beta,L}(\mathbf{z}, \mathbf{0}, \mathbf{y}), \tag{3.31}$$

$$\hat{\Gamma}_{\mu;r,r'}^{5;\beta,L}(\mathbf{k}, \mathbf{p}) = \int d\mathbf{y} \, d\mathbf{z} \, e^{-i\mathbf{p}\mathbf{z} - i\mathbf{k}\mathbf{y}} \Gamma_{\mu;r,r'}^{5;\beta,L}(\mathbf{z}, \mathbf{0}, \mathbf{y}), \tag{3.32}$$

$$\hat{\Pi}^{5;\beta,L}_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int d\mathbf{x}\, d\mathbf{y} e^{-i\mathbf{p}_1\mathbf{x} - i\mathbf{p}_2\mathbf{y}}\, \Pi^{5;\beta,L}_{\mu,\nu,\sigma}(\mathbf{0}, \mathbf{x}, \mathbf{y}), \tag{3.33}$$

and their $\beta, L \to \infty$ limits, denoted by $\hat{S}_{2;r,r'}(\mathbf{k})$, $\hat{\Gamma}_{\mu;r,r'}(\mathbf{k}, \mathbf{p})$, $\hat{\Gamma}^5_{\mu;r,r'}(\mathbf{k}, \mathbf{p})$, $\hat{\Pi}^5_{\mu,\nu,\sigma}$ $(\mathbf{p}_1, \mathbf{p}_2)$, respectively. For the purpose of proving Theorem 2.1, we can limit ourselves to computing these functions at sufficiently low momenta (more precisely, at $\mathbf{k}$ sufficiently close to the Weyl nodes and $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2$ sufficiently close to $\mathbf{0}$). In particular, in the proof below, we can freely assume that $\hat{A}_{\mu,\mathbf{p}}$ and $\hat{A}^5_{\mu,\mathbf{p}}$ are supported in a sufficiently small neighbourhood of the origin, and we shall do so in the following.

3.2.1. Ward identities Ward identities are nontrivial relations between correlation functions, implied by the conservation of the lattice current (2.37). The Grassmann integral formulation of the model offers a particularly compact way of representing them. The invariance under $U(1)$-local gauge transformation reads:

$$\mathcal{W}_{\beta,L}(A + \partial\alpha, A^5, \phi e^{i\alpha}) = \mathcal{W}_{\beta,L}(A, A^5, \phi), \tag{3.34}$$

for any smooth function $\alpha_{\mathbf{x}}$ on $\mathbb{R}^3$, periodic in x_0 of period β and in x of period L (here $\phi e^{i\alpha}$ is a shorthand for $\{e^{+i\alpha_{\mathbf{x}}}\phi^+_{\mathbf{x},r}, e^{-i\alpha_{\mathbf{x}}}\phi^-_{\mathbf{x},r}\}_{\mathbf{x}\in[0,\beta)\times\Lambda_L}$). Differentiating Eq. (3.34) with respect to α and with respect to the external fields, we obtain a hierarchy of Ward identities. As already emphasized in Sect. 3.1.2, two Ward Identities that are of particular importance for us are (3.2) and (3.3): the first is obtained by deriving once with respect to α, once with respect to A, once with respect to A^5, then setting the external fields A, A^5, ϕ to zero, and taking the $\beta, L \to \infty$ limit; the second is obtained by deriving once with respect to α, twice with respect to ϕ, then setting the external fields A, A^5, ϕ to zero, and taking the $\beta, L \to \infty$ limit.

3.3. Renormalization group analysis. In this section we sketch the RG analysis of the model in the presence of external fields A and A^5. Our analysis follows and extends the one in [51,52], where RG methods have been used to prove the analyticity of the free energy of a specific lattice model of Weyl semimetals within the class of models considered in this paper, and to compute the corresponding two-point function $\hat{S}^{\beta,L}_2$. In the discussion below, we limit ourselves to describe the general scheme and to highlight the main differences compared to [51,52], referring to those papers, and in particular to [52, Sections 2 and 3], for additional technical details. Moreover, for simplicity, we focus on generating functional $\mathcal{W}_{\beta,L}(A, A^5, 0)$. The generalization to the presence of an external fermionic field ϕ is straightforward and will not be explicitly worked out, see, e.g., [25, Section 12] or [29, Section 2.2] for a discussion of the necessary modifications.

The starting point is a scale decomposition of the free propagator, which separates ultraviolet and infrared degrees of freedom:

$$\hat{g}_{\beta,L}(\mathbf{k}) = \hat{g}^{(\leq 0)}(\mathbf{k}) + \hat{g}^{(>0)}(\mathbf{k}) \tag{3.35}$$
$$\hat{g}^{(\leq 0)}(\mathbf{k}) = \chi_0(\mathbf{k})\hat{g}_{\beta,L,N}(\mathbf{k}), \qquad g^{(>0)}(\mathbf{k}) = g_{\beta,L,N}(\mathbf{k}) - \hat{g}^{(\leq 0)}(\mathbf{k}),$$

where $\chi_0(\mathbf{k}) := \chi\left(\sqrt{k_0^2 - \det \hat{H}^0(k)}\right)$. We assume that α_0 is smaller than $\gamma^{-1}\sqrt{c_2}$, with c_2 the constant in (2.17), so that on the support of χ_0 (where, in particular, $0 \leq -\det \hat{H}^0(k) \leq \alpha_0^2\gamma^2$) the momentum k is closer to the Weyl nodes than c_1 and $|d(k)| \leq \frac{1}{2}\sqrt{a^2(k) + b^2(k) + c^2(k)}$, see (2.16). We denote by $g^{(\leq 0)}(\mathbf{x})$ and $g^{(>0)}(\mathbf{x})$

the inverse Fourier transforms of the momentum-space propagators $\hat{g}^{(\leq 0)}(\mathbf{k})$, $\hat{g}^{(>0)}(\mathbf{k})$. Correspondingly, we use the addition principle of Grassmann variables to rewrite the Grassmann Gaussian field $\hat{\psi}^{\pm}$ as:

$$\psi_{\mathbf{x},r}^{\pm} = \psi_{\mathbf{x},r}^{(\leq 0)\pm} + \psi_{\mathbf{x},r}^{(>0)\pm}, \tag{3.36}$$

where $\psi^{(\leq 0)\pm}$, $\psi^{(>0)\pm}$ are independent Grassmann Gaussian fields, with covariances given by $g^{(\leq 0)}$, $g^{(>0)}$ respectively. Due to the fact that the propagator $\hat{g}^{(>0)}(\mathbf{k})$ is supported away from the Weyl nodes $\mathbf{p}_F^{\omega} = (0, p_F^{\omega})$, the configuration-space covariance $g^{(>0)}(\mathbf{x})$ decays faster than any power in $\mathbf{x}$; more precisely, using the assumption that χ is in the Gevrey class of order $s = 2$,

$$\|g^{(>0)}(\mathbf{x})\| \leq C_0 e^{-\kappa_0 \sqrt{|\mathbf{x}|}}, \tag{3.37}$$

for some $\kappa_0 > 0$. This allows to integrate out the ultraviolet degrees of freedom via a simple fermionic cluster expansion, which is largely model independent; see e.g. [27,29]. After the integration of $g^{(>0)}$ we get:

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(0)}(A,A^5)} \int P_{\leq 0}(d\psi^{(\leq 0)})e^{\mathcal{V}^{(0)}(A,A^5,\psi^{(\leq 0)})}, \tag{3.38}$$

where the various objects appearing in Eq. (3.38) have the following meaning. The fermionic Gaussian integration $P_{\leq 0}(d\psi^{(\leq 0)})$ has covariance given by $g^{(\leq 0)}(\mathbf{x} - \mathbf{y})$; the effective interaction on scale zero $\mathcal{V}^{(0)}(A, A^5, \psi^{(\leq 0)})$ has the form:

$$\mathcal{V}^{(0)}(A, A^5, \psi) = \sum_{\substack{n,m_1,m_2 \geq 0 \\ n>0}} \sum_{\underline{r},\underline{\mu},\underline{\nu}} \int d\mathbf{X} d\mathbf{Y} d\mathbf{Z} \Big[\prod_{i=1}^{2n} \psi_{\mathbf{x}_{2i-1},r_{2i-1}}^+ \psi_{\mathbf{x}_{2i},r_{2i}}^- \Big] \cdot \tag{3.39}$$

$$\cdot \Big[\prod_{j=1}^{m_1} A_{\mu_j,\mathbf{y}_j} \Big] \Big[\prod_{\ell=1}^{m_2} A_{\nu_\ell,\mathbf{z}_\ell}^5 \Big] W_{n,m_1,m_2;\underline{r},\underline{\mu},\underline{\nu}}^{(0)}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}),$$

with $\mathbf{X} = \{\mathbf{x}_i\}_{i=1}^{2n}$, $\mathbf{Y} = \{\mathbf{y}_j\}_{j=1}^{m_1}$, $\mathbf{Z} = \{\mathbf{z}_\ell\}_{\ell=1}^{m_2}$; the kernels $W^{(0)}$ are analytic in λ for $|\lambda|$ small enough, and satisfy the following weighted L^1 bound (recall that in our assumptions $\nu = O(\lambda)$):

$$\frac{1}{\beta L^3} \int d\mathbf{Q} \, e^{\frac{\kappa_0}{2} \sqrt{\delta(\mathbf{Q})}} |W_{n,m_1,m_2;\underline{r},\underline{\mu},\underline{\nu}}^{(0)}(\mathbf{Q})| \leq C^{n+m_1+m_2} \Big(\prod_{v=0}^{3} \frac{|Z_{v,\text{bare}}^5|}{|p_{F,3}^+|^{1-\delta_{v,3}}} \Big) |\lambda|^{\max\{\delta_{m_1+m_2,0}, \, n-1\}}, \tag{3.40}$$

where $\mathbf{Q} = (\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, $\delta(\mathbf{Q})$ is the 'Steiner diameter' of $\mathbf{Q}$, i.e., the length of the shortest tree connecting all the points in $\mathbf{Q}$, see [[32], footnote 19], and κ_0 is the same as in (3.37) (the reason for the choice of the stretched exponential weight $e^{\frac{\kappa_0}{2} \sqrt{\delta(\mathbf{Q})}}$ is the comparable stretched exponential decay of $g^{(>0)}$). The generating functional on scale zero $\mathcal{W}^{(0)}(A, A^5)$ has a similar representation, except that only kernels with $n = 0$ (no fermionic external lines) contribute.

Next, we perform the integration of the infrared degrees of freedom, associated to the massless field $\psi^{(\leq 0)}$. To this end, we decompose the field $\psi^{(\leq 0)}$ in terms of single-scale fields $\psi^{(h)}$, $h \leq 0$, that we integrate iteratively. For v_3^0 small, i.e., ζ close to ζ_1, the value at which the Weyl nodes merge, we distinguish two scale regimes, depending on

whether $c(k' + p_F^\omega) \simeq \omega v_3^0 k_3'$ or $c(k' + p_F^\omega) \simeq \frac{b^0}{2} k_3'^2$, recall (2.8), (2.12), (2.14) and (2.15). We define $h_* \leq 0$ as the crossover scale between the two regimes, at which $k_3'^2 \sim \gamma^{h_*} \sim v_3^0 \gamma^{h_*/2}$; more precisely, we let

$$h_* := \min\{0, \lfloor 2 \log_\gamma |v_3^0| \rfloor\}. \tag{3.41}$$

Let $\varepsilon := |p_F^+ - p_F^-|$. Recall that, by assumption, $c_0^{-1}|v_3^0| \leq \varepsilon \leq c_0|v_3^0|$, see (2.13). Thus, $\gamma^{h_*} \sim \varepsilon^2$ for ε small. In the following, we discuss separately the scales $h > h^*$ and the scales $h \leq h^*$.

Remark. If $h_* = 0$, then the infrared analysis simplifies, in that the first regime, discussed in the next subsection, disappears. This is the case when ζ is far from the value ζ_1 at which the Weyl points merge (or, similarly, the case, mentioned after (2.11), in which the Weyl nodes are well separated uniformly in ζ).

3.3.1. Regime $h > h^*$ Assume $h_* < 0$ (otherwise the reader can pass directly to next subsection). We inductively assume that, for all $h_* \leq h \leq 0$, the generating function of correlations can be written as:

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h)}(A,A^5)} \int P_{\leq h}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(A,A^5,\psi^{(\leq h)})}. \tag{3.42}$$

The Gaussian Grassmann field $\psi^{(\leq h)}$ has Fourier-space covariance $\hat{g}^{(\leq h)}(\mathbf{k})$ given by:

$$\hat{g}^{(\leq h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{Z_h} A_h(\mathbf{k})^{-1} \tag{3.43}$$

where:

$$A_h(\mathbf{k}) = \begin{pmatrix} -ik_0 + c_h(k) + d_h(k) & a_h(k) - ib_h(k) \\ a_h(k) + ib_h(k) & -ik_0 - c_h(k) + d_h(k) \end{pmatrix}, \tag{3.44}$$

$\chi_h(\mathbf{k}) = \chi\left(\gamma^{-h}\sqrt{k_0^2 - \det \hat{H}^0(k)}\right)$, Z_h is real-analytic in λ, such that $|Z_h - 1| \leq C|\lambda|$, and, recalling that $p_{F,3} \equiv p_{F,3}^+$,

$$a_h(k) = v_{1,h}k_1 + a_R(k)/Z_h, \qquad b_h(k) = v_{2,h}k_2 + b_R(k)/Z_h, \tag{3.45}$$

$$c_h(k) = \left(c(k) + \tfrac{1}{2}\zeta_h(k_3^2 - p_{F,3}^2)\right)/Z_h, \quad d_h(k) = d(k)/Z_h, \tag{3.46}$$

with $v_{l,h}$ real-analytic in λ, such that $|v_{l,h} - v_l^0| \leq C|\lambda|$, for $l = 1, 2$, and $|\zeta_h| \leq C|\lambda|$. The effective potential $\mathcal{V}^{(h)}$ has the form:

$$\mathcal{V}^{(h)}(A, A^5, \psi) = \sum_{\substack{n,m_1,m_2 \geq 0 \\ n > 0}} \sum_{\underline{q},\underline{r},\underline{\mu},\underline{v}} \int d\mathbf{X} d\mathbf{Y} d\mathbf{Z} \left[\prod_{i=1}^{2n} \hat{\partial}^{q_{2i-1}} \psi^+_{\mathbf{x}_{2i-1},r_{2i-1}} \hat{\partial}^{q_{2i}} \psi^-_{\mathbf{x}_{2i},r_{2i}}\right] \cdot \tag{3.47}$$

$$\cdot \left[\prod_{j=1}^{m_1} A_{\mu_j,\mathbf{y}_j}\right]\left[\prod_{\ell=1}^{m_2} A^5_{v_\ell,\mathbf{z}_\ell}\right] W^{(h)}_{n,m_1,m_2;\underline{q};\underline{r},\underline{\mu},\underline{v}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}),$$

which is similar to (3.39), with the difference that now there are the operators $\hat{\partial}^{q_i}$, with $i = 1, \ldots, 2n$, acting on the Grassmann variables; here the labels q_i are multi-indices of the form $q_i = (q_i^0, q_i^1, q_i^2, q_i^3)$ with $q_i^\mu \in \{0, 1, 2, 3, 4\}$, and $\hat{\partial}^{q_i}$ is a pseudo-differential

operator, equal to the identity if $q_i = (0, 0, 0, 0)$, and dimensionally equivalent to the composition of a derivative of order q_i^0 in direction 0, of a derivative of order q_i^1 in direction 1, etc, otherwise[10] (if $h = 0$, the only non-vanishing contribution to the right side of (3.47) is the one with $q_i = (0, 0, 0, 0)$, for all $i = 1, \ldots, 2n$, and (3.47) reduces to (3.39)). The kernels $W_{n,m_1,m_2,\underline{q};\underline{r},\underline{\mu},\underline{v}}^{(h)}$ belong to a suitable weighted L^1 space, see (3.74) below, and are analytic in λ for $|\lambda|$ small enough, uniformly in h (recall that v is assumed to be analytic in λ, as well, and of order λ). We stress that the representation in (3.47) is not unique: the claim is that there exists such a representation, with the kernels satisfying natural dimensional estimates, see (3.74) below. The generating functional of correlations on scale h, $\mathcal{W}^{(h)}$, admits a similar representation, except that only kernels with $n = 0$ contribute. These assumptions are true at scale zero, with $v_{l,0} = v_l^0$, $\zeta_0 = 0$ and $Z_0 = 1$.

The inductive assumption (3.42) is verified at scale $h = 0$, as an outcome of the integration of the ultraviolet degrees of freedom. We now assume that (3.42) is valid at scale $h_* < h \leq 0$ and prove it for $h - 1$. For this purpose, we intend to integrate out the degrees of freedom on scale h, after having properly rewritten the effective potential in the right side of (3.42). As a first step, we split $\mathcal{V}^{(h)}$ as $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{L}$ is the *localization operator*, acting on $\mathcal{V}^{(h)}$ as follows[11]:

$$\mathcal{L}\mathcal{V}^{(h)}(A, A^5, \psi) := \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}_{\mathbf{k}}^+ \big(\mathcal{L}\hat{W}_{1,0,0}^{(h)}(\mathbf{k})\big) \hat{\psi}_{\mathbf{k}}^- \tag{3.48}$$
$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \int \frac{d\mathbf{p}}{(2\pi)^4} \Big[\hat{A}_{\mu,\mathbf{p}} \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \big(\mathcal{L}\hat{W}_{1,1,0;\mu}^{(h)}(\mathbf{k}, \mathbf{p})\big) \hat{\psi}_{\mathbf{k}}^- + \hat{A}_{\mu,\mathbf{p}}^5 \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \big(\mathcal{L}\hat{W}_{1,0,1;\mu}^{(h)}(\mathbf{k}, \mathbf{p})\big) \hat{\psi}_{\mathbf{k}}^- \Big],$$

where $\int \frac{d\mathbf{k}}{(2\pi)^4}$ and $\int \frac{d\mathbf{p}}{(2\pi)^4}$ are shorthands for $(\beta L^3)^{-1} \sum_{\mathbf{k} \in \mathbb{D}_{\beta,L,N}}$ and $(\beta L^3)^{-1} \sum_{\mathbf{p} \in \mathbb{D}_{\beta,L}^b}$, respectively (here $\mathbb{D}_{\beta,L}^b = \frac{2\pi}{\beta}\mathbb{Z} \times \frac{2\pi}{L}\mathbb{Z}^3$), and

$$\mathcal{L}\hat{W}_{1,0,0}^{(h)}(\mathbf{k}) := \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) + \sum_{\mu=0,1,2} k_\mu \partial_\mu \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) + \frac{1}{2} k_3^2 \partial_3^2 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}),$$
$$\mathcal{L}\hat{W}_{1,1,0;\mu}^{(h)}(\mathbf{k}, \mathbf{p}) := \hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) + (k_3 \partial_{k_3} + p_3 \partial_{p_3}) \hat{W}_{1,1,0;\mu}^{(h)}(\mathbf{0}, \mathbf{0}), \tag{3.49}$$
$$\mathcal{L}\hat{W}_{1,0,1;\mu}^{(h)}(\mathbf{k}, \mathbf{p}) := \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) + (k_3 \partial_{k_3} + p_3 \partial_{p_3}) \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}),$$

where $\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$ indicates the $\beta, L \to \infty$ limit of $\hat{W}_{1,0,0}^{(h)}(\mathbf{k})$, and similarly for $\hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{k}, \mathbf{p})$ and $\hat{W}_{1,0,1;\mu}^{(h)}(\mathbf{k}, \mathbf{p})$, whose existence follows from the inductive construction of the kernels and from the corresponding bounds, uniform in β, L described below. The *renormalization* operator is defined as $\mathcal{R} := 1 - \mathcal{L}$. Notice that $\mathcal{R}\mathcal{L} = \mathcal{L}\mathcal{R} = 0$.

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[10] In Fourier space, the Fourier symbol of $\hat{\partial}^q$ is a function of k behaving like $k_0^{q^0} k_1^{q^1} k_2^{q^2} k_3^{q^3}$ for k small. When writing the analogue of (3.47) in Fourier space, we can re-absorb the Fourier symbols of the operators $\hat{\partial}^{q_i}$, with $i = 1, \ldots, 2n$, into the Fourier symbol of the kernel $W_{n,m_1,m_2,\underline{q};\underline{r},\underline{\mu},\underline{v}}^{(h)}$, and we shall do so; after summation over $\underline{q}$, we will denote the resulting modified Fourier symbol of the kernels by $\hat{W}_{n,m_1,m_2;\underline{r},\underline{\mu},\underline{v}}^{(h)}$.

[11] Whenever it will be convenient, from now on, the dependence upon the indices $r_1, r_2, \ldots$, will be left implicit. $\hat{\psi}_{\mathbf{k}}^+$ (resp. $\hat{\psi}_{\mathbf{k}}^-$) will be thought of as a row (resp. column) vector with components $\hat{\psi}_{\mathbf{k},1}^+$, $\hat{\psi}_{\mathbf{k},2}^+$ (resp. $\hat{\psi}_{\mathbf{k},1}^-$, $\hat{\psi}_{\mathbf{k},2}^-$). When writing $\hat{\psi}_{\mathbf{k}}^+ M(\mathbf{k}) \hat{\psi}_{\mathbf{k}}^-$ with $M(\mathbf{k})$ a 2×2 matrix, we will mean $\sum_{r,r'=1}^2 \hat{\psi}_{\mathbf{k},r}^+ M_{r,r'}(\mathbf{k}) \hat{\psi}_{\mathbf{k},r'}^-$.

Remarks.

1) There are potentially other terms in the Taylor expansion of $\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$ that we could include 'for free' in the definition in the first line, namely $k_3 \partial_3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0})$, $\sum_{\mu=0,1,2} k_\mu k_3 \partial_\mu \partial_3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0})$ and $\frac{1}{6} k_3^3 \partial_3^3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0})$: in fact, as stated in the following lemma, these are all zero, by the parity properties of $\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$. Therefore, the reader can think of the right side of the first line in (3.49) as being the Taylor expansion of $\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$ around $\mathbf{k} = \mathbf{0}$, including all terms proportional to 1, $\{k_\mu\}_{\mu=0,1,2,3}$, $\{k_\mu k_3\}_{\mu=0,1,2,3}$, and k_3^3, where some of the terms have not been written explicitly, simply because they are zero. Consequently, $\mathcal{R}\hat{W}_{1,0,0}^{(h)}(\mathbf{k})$ is equal to the corresponding Taylor remainder of $\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$, which is quadratic in $\{k_\mu\}_{\mu=0,1,2}$ and, at $k_0 = k_1 = k_2 = 0$, is *quartic* in k_3; plus a finite size correction, proportional to $\hat{W}_{1,0,0}^{(h)}(\mathbf{k}) - \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{k})$, which is exponentially small in β, L as $\beta, L \to \infty$, and can be bounded as discussed in [34, Appendix B]. We anticipate the fact that the scaling dimension of $\hat{W}_{n,m_1,m_2;\underline{\mu},\underline{\nu}}^{(h),\infty}$ is $\frac{7}{2} - \frac{5}{2}n - m_1 - m_2$, see (3.74) (the convention here is that kernels with positive/zero/negative scaling dimensions correspond to relevant/marginal/irrelevant operators in the RG sense); any additional derivative with respect to k_μ decreases the scaling dimension by 1, if $\mu = 0, 1, 2$, and by $1/2$, if $\mu = 3$. Therefore, from the comments above, it follows that $\mathcal{R}\hat{W}_{1,0,0}^{(h)}(\mathbf{k})$ is irrelevant, with scaling dimension -1.

2) Similarly, from the definitions in (3.49), it follows that $\mathcal{R}\hat{W}_{1,1,0;\mu}^{(h)}(\mathbf{k},\mathbf{p})$ and $\mathcal{R}\hat{W}_{1,0,1;\mu}^{(h)}(\mathbf{k},\mathbf{p})$ are equal to Taylor remainders that are linear in k_μ, p_μ with $\mu = 0, 1, 2$, and quadratic in k_3, p_3, up to a (subdominant) finite size correction. From the formula of the scaling dimension anticipated in the previous item, it follows that they are irrelevant with scaling dimension -1. Note that the linear terms $(k_3 \partial_{k_3} + p_3 \partial_{p_3}) \hat{W}_{1,1,0;\mu}^{(h)}(\mathbf{0},\mathbf{0})$ and $(k_3 \partial_{k_3} + p_3 \partial_{p_3}) \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0},\mathbf{0})$ in the second and third lines of (3.49) are irrelevant with dimension $-1/2$; the reason why we prefer to include them into the local part is to guarantee that the irrelevant part has largest scaling dimension equal to -1. This guarantees that the improved dimensional bound discussed in the paragraph after (3.74) has an additional factor $\gamma^{\theta h}$, with θ any positive constant smaller than 1 (rather than $1/2$), see also Appendix B; this fact is useful in the study of the flow of the effective couplings discussed in Sect. 3.4, see in particular case $\mu = 3$ in (3.107) and case $\mu = 0, 1, 2$ in (3.114), where having $\theta > 1/2$ has important consequences for the resulting flow of $Z_{\mu,h}$ and $Z_{\mu,h}^5$.

The next lemma shows that the lattice symmetries of the model impose non-trivial constraints on the kernels of $\mathcal{LV}^{(h)}$.

Lemma 3.1. *The following identities hold:*

$$\hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = n_h \sigma_3, \quad \partial_\mu \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = \begin{cases} z_{\mu,h}\sigma_\mu & \text{if } \mu = 0, 1, 2, \\ 0 & \text{if } \mu = 3 \end{cases}, \quad \partial_3^2 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = z_{3,h}\sigma_3$$

$$\partial_0 \partial_3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = \partial_1 \partial_3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = \partial_2 \partial_3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = \partial_3^3 \hat{W}_{1,0,0}^{(h),\infty}(\mathbf{0}) = 0, \tag{3.50}$$

where $n_h, z_{i,h}, \tilde{z}_{3,h} \in \mathbb{R}$ and we denoted $\sigma_0 := -i\mathbb{1}_2$. Moreover, letting $p_{F,3} := p_{F,3}^+$:

$$\hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{0},\mathbf{0}) = \begin{cases} Z_{\mu,h}\sigma_\mu & \text{if } \mu = 0, 1, 2, \\ 0 & \text{if } \mu = 3, \end{cases}, \quad \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0},\mathbf{0}) = \begin{cases} 0 & \text{if } \mu = 0, 1, 2, \\ Z_{3,h}^5 \sigma_3 & \text{if } \mu = 3, \end{cases}$$

$$\partial_{k_3} \hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) = 2\Re\left(\partial_{p_3} \hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0})\right) = \begin{cases} 0 & \text{if } \mu = 0, 1, 2, \\ \frac{Z_{3,h}}{p_{F,3}} \sigma_3 & \text{if } \mu = 3, \end{cases} \tag{3.51}$$

$$\partial_{k_3} \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) = 2\Re\left(\partial_{p_3} \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0})\right) = \begin{cases} \frac{Z_{\mu,h}^5}{p_{F,3}} \sigma_\mu & \text{if } \mu = 0, 1, 2, \\ 0 & \text{if } \mu = 3, \end{cases}$$

where $Z_{\mu,h}, Z_{\mu,h}^5 \in \mathbb{R}$, for $\mu = 0, 1, 2, 3$, and we denoted $\Re(M) = (M + M^\dagger)/2$ the Hermitian part of a complex matrix M. Moreover, letting $\Im(M) = -i(M - M^\dagger)/2$ for M a complex matrix,

$$\Im\left(\partial_{p_3} \hat{W}_{1,1,0;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0})\right) = \begin{cases} 0 & \text{if } \mu = 0, 1, 2, \\ \frac{\widetilde{Z}_{3,h}}{p_{F,3}} \sigma_3 & \text{if } \mu = 3, \end{cases} \tag{3.52}$$

$$\Im\left(\partial_{p_3} \hat{W}_{1,0,1;\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0})\right) = \begin{cases} \frac{\widetilde{Z}_{\mu,h}^5}{p_{F,3}} \sigma_\mu & \text{if } \mu = 0, 1, 2, \\ 0 & \text{if } \mu = 3, \end{cases} \tag{3.53}$$

with $\widetilde{Z}_{3,h}, \widetilde{Z}_{\mu,h}^5 \in \mathbb{R}$, for $\mu = 0, 1, 2$.

We refer to Appendix A.2 for the proof of this lemma. Given the definitions of $\mathcal{LV}^{(h)}$ and $\mathcal{RV}^{(h)}$, and the properties of the kernels of $\mathcal{LV}^{(h)}$ spelled out in Lemma 3.1, as well as the definitions of n_h, $z_{\mu,h}$, $Z_{\mu,h}$, $Z_{\mu,h}^5$, we now manipulate the right side of (3.42) as follows: we rewrite $\mathcal{V}^{(h)} = \mathcal{LV}^{(h)} + \mathcal{RV}^{(h)}$, and re-absorb part of $\mathcal{LV}^{(h)}$ in the Grassmann Gaussian integration, thus obtaining:

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h)}(A,A^5)} \int \widetilde{P}_{\le h}(d\psi^{(\le h)}) e^{2^h v_h N_3(\psi^{(\le h)}) + B^{(h)}(A,A^5,\psi^{(\le h)}) + \mathcal{RV}^{(h)}(A,A^5,\psi^{(\le h)})}, \tag{3.54}$$

where, recalling that $p_{F,3} \equiv p_{F,3}^+$, we defined: $2^h v_h := n_h + \frac{1}{2} p_{F,3}^2 z_{h,3}$, $N_3(\psi) := \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}_{\mathbf{k}}^+ \sigma_3 \hat{\psi}_{\mathbf{k}}^-$,

$$B^{(h)}(A, A^5, \sqrt{Z_h}\psi) := \int \frac{d\mathbf{k}}{(2\pi)^4} \int \frac{d\mathbf{p}}{(2\pi)^4} \times \tag{3.55}$$

$$\times \left[\sum_{\mu=0}^{2} Z_{\mu,h} \hat{A}_{\mu,\mathbf{p}} \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \sigma_\mu \hat{\psi}_{\mathbf{k}}^- + (Z_{3,h} \frac{k_3 + p_3/2}{p_{F,3}} + i\widetilde{Z}_{3,h} \frac{p_3}{p_{F,3}}) \hat{A}_{3,\mathbf{p}} \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \sigma_3 \hat{\psi}_{\mathbf{k}}^- \right.$$

$$\left. + \sum_{\mu=0}^{2} (Z_{\mu,h}^5 \frac{k_3 + p_3/2}{p_{F,3}} + i\widetilde{Z}_{\mu,h}^5 \frac{p_3}{p_{F,3}}) \hat{A}_{\mu,\mathbf{p}}^5 \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \sigma_\mu \hat{\psi}_{\mathbf{k}}^- + Z_{3,h}^5 \hat{A}_{3,\mathbf{p}}^5 \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \sigma_3 \hat{\psi}_{\mathbf{k}}^- \right],$$

and the new Grassmann Gaussian integration $\widetilde{P}_h(d\psi^{(\le h)})$ has covariance given by:

$$\tilde{g}^{(\le h)}(\mathbf{k}) = \frac{\chi_h(\mathbf{k})}{Z_{h-1}(\mathbf{k})} \tilde{A}_{h-1}(\mathbf{k})^{-1}, \tag{3.56}$$

where: $Z_{h-1}(\mathbf{k}) := Z_h - \chi_h(\mathbf{k}) z_{0,h}$,

$$\tilde{A}_{h-1}(\mathbf{k}) = \begin{pmatrix} -ik_0 + \tilde{c}_{h-1}(\mathbf{k}) + \tilde{d}_{h-1}(\mathbf{k}) & \tilde{a}_{h-1}(\mathbf{k}) - i\tilde{b}_{h-1}(\mathbf{k}) \\ \tilde{a}_{h-1}(\mathbf{k}) + i\tilde{b}_{h-1}(\mathbf{k}) & -ik_0 - \tilde{c}_{h-1}(\mathbf{k}) + \tilde{d}_{h-1}(\mathbf{k}) \end{pmatrix}, \tag{3.57}$$

and

$$\tilde{a}_{h-1}(\mathbf{k}) = v_{1,h-1}(\mathbf{k})k_1 + a_R(k)/Z_{h-1}(\mathbf{k}), \qquad \tilde{b}_{h-1}(\mathbf{k}) = v_{2,h-1}(\mathbf{k})k_2 + b_R(k)/Z_{h-1}(\mathbf{k}), \quad (3.58)$$

$$\tilde{c}_{h-1}(\mathbf{k}) = \left(c(k) + \tfrac{1}{2}\zeta_{h-1}(\mathbf{k})(k_3^2 - p_{F,3}^2)\right)/Z_{h-1}(\mathbf{k}), \qquad \tilde{d}_{h-1}(\mathbf{k}) = d(k)/Z_{h-1}(\mathbf{k}), \quad (3.59)$$

with

$$v_{l,h-1}(\mathbf{k}) = (Z_h v_{l,h} - \chi_h(\mathbf{k})z_{l,h})/Z_{h-1}(\mathbf{k}), \qquad l = 1, 2, \quad (3.60)$$

$$\zeta_{h-1}(\mathbf{k}) = \zeta_h - \chi_h(\mathbf{k})z_{3,h}. \quad (3.61)$$

Fix $\theta \in (0, 1)$; from now on, we denote by C_θ, C_θ', etc., positive, θ-dependent, constants, possibly divergent as $\theta \to 1^-$. We inductively assume that, for all scales $k \geq h$, $\mu = 0, 1, 2, 3$, and a positive constant C_θ,

$$|z_{\mu,k}| \leq C_\theta|\lambda|\gamma^{\theta k}. \quad (3.62)$$

We will check the validity of these bounds on scale $k = h - 1$. We also assume that, for all $k \geq h$,

$$|v_k| \leq C_\theta|\lambda|\gamma^{\theta k}. \quad (3.63)$$

Eqs. (3.62) imply that

$$|Z_{h-1}(\mathbf{k}) - 1| \leq C|\lambda|, \qquad |\zeta_{h-1}(\mathbf{k})| \leq C|\lambda|, \qquad |v_{l,h-1}(\mathbf{k}) - v_l^0| \leq C|\lambda|, \ \ l = 1, 2. \quad (3.64)$$

We also let

$$Z_{h-1} := Z_{h-1}(\mathbf{0}), \quad \zeta_{h-1} := \zeta_{h-1}(\mathbf{0}), \quad \text{and} \quad v_{l,h-1} := v_{l,h-1}(\mathbf{0}) \quad \text{for} \quad l = 1, 2, 3. \quad (3.65)$$

Obviously, $Z_{h-1}, \zeta_{h-1}, v_{l,h-1}$ satisfy the same bounds as in (3.64). In order to prove the inductive assumption, we decompose the Grassmann field as:

$$\psi^{(\leq h)} = \psi^{(\leq h-1)} + \psi^{(h)}, \quad (3.66)$$

where the Grassmann field $\hat{\psi}^{(\leq h-1)}$ has covariance given by $\hat{g}^{(\leq h-1)}(\mathbf{k})$, defined as in (3.43), (3.44), with h replaced by $h - 1$, while $\psi^{(h)}$ has covariance given by:

$$\hat{g}^{(h)}(\mathbf{k}) := \frac{f_h(\mathbf{k})}{Z_{h-1}(\mathbf{k})} \tilde{A}_{h-1}(\mathbf{k})^{-1}, \quad (3.67)$$

where $f_h(\mathbf{k}) = \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k})$ and we used the fact that, on the support of χ_{h-1}, $Z_{h-1}(\mathbf{k}) = Z_{h-1}$ and $\tilde{A}_{h-1}(\mathbf{k}) = A_{h-1}(\mathbf{k})$. On the support of f_h, $C^{-1}\gamma^{2h} \leq -\det \tilde{A}_{h-1}(\mathbf{k}) \leq C\gamma^{2h}$, $|k_\mu| \leq C\gamma^h$ for $\mu = 0, 1, 2$, and $|k_3| \leq C\gamma^{\frac{h}{2}}$. Therefore, for these values of $\mathbf{k}$, using also the definition of h_* and the bounds (2.13), (2.14), (2.15),

$$\tilde{a}_{h-1}(\mathbf{k}) = v_{1,h-1}(\mathbf{k})k_1 + O(\gamma^{2h}), \quad \tilde{b}_{h-1}(\mathbf{k}) = v_{2,h-1}(\mathbf{k})k_2 + O(\gamma^{2h}), \quad (3.68)$$

$$\tilde{c}_{h-1}(\mathbf{k}) = \tfrac{1}{2}(b^0 + \zeta_{h-1}(\mathbf{k}))k_3^2 + O(\gamma^{h_*} + \gamma^{2h}), \qquad \tilde{d}_{h-1}(\mathbf{k}) = O(\gamma^{2h}), \quad (3.69)$$

which implies $\|\hat{g}^{(h)}(\mathbf{k})\|_\infty \leq C\gamma^{-h}$ and $\|\hat{g}(\mathbf{k})\|_1 \leq C\gamma^{\frac{5}{2}h}$. By a similar discussion, we can also dimensionally bound the derivatives of $\tilde{a}_{h-1}, \tilde{b}_{h-1}, \tilde{c}_{h-1}, \tilde{d}_{h-1}$, thus getting

$$\|\partial_{\mathbf{k}}^\alpha \hat{g}^{(h)}(\mathbf{k})\|_\infty \leq C_{|\alpha|}\gamma^{-h(1+\alpha_0+\alpha_1+\alpha_2+\frac{1}{2}\alpha_3)} \text{ and } \|\partial_{\mathbf{k}}^\alpha \hat{g}(\mathbf{k})\|_1 \leq C_{|\alpha|}\gamma^{h(\frac{5}{2}-\alpha_0-\alpha_1-\alpha_2-\frac{1}{2}\alpha_3)},$$

which, in turn, imply the following bound on $g^{(h)}$, the inverse Fourier transform of $\hat{g}^{(h)}$:

$$\|g^{(h)}(\mathbf{x})\| \leq C_0 \gamma^{\frac{5}{2}h} e^{-\kappa_0 \sqrt{\|\mathbf{x}\|_h}}, \tag{3.70}$$

where

$$\|\mathbf{x}\|_h := \gamma^h(|x_0| + |x_1| + |x_2|) + \gamma^{\frac{h}{2}}|x_3|. \tag{3.71}$$

The constants C_0, κ_0 can be chosen to be the same as in (3.37).

Let us now go back to (3.54). By using the addition principle for Gaussian Grassmann variables, we can rewrite it as:

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h)}(A,A^5)} \int P_{\leq h-1}(d\psi^{(\leq h-1)}) \cdot \tag{3.72}$$

$$\cdot \int P_h(d\psi^{(h)}) e^{\mathcal{R}\mathcal{V}^{(h)}(A,A^5,\psi^{(\leq h)})+2^h v_h N_3(\psi^{(\leq h)})+B^{(h)}(A,A^5,\psi^{(\leq h)})},$$

where $P_h(d\psi^{(h)})$ has covariance given by $g^{(h)}$, $P_{\leq h-1}(d\psi^{(\leq h-1)})$ has covariance given by $g^{(\leq h-1)}$, and, in the exponent in the second line, $\psi^{(\leq h)} = \psi^{(\leq h-1)} + \psi^{(h)}$. We now integrate the field $\psi^{(h)}$ and denote the logarithm of the result of the integration in the second line by $\widetilde{\mathcal{W}}^{(h)}(A, A^5) + \mathcal{V}^{(h-1)}(A, A^5, \psi^{(\leq h-1)})$, so that, letting $\mathcal{W}^{(h-1)}(A, A^5) := \mathcal{W}^{(h)}(A, A^5) + \widetilde{\mathcal{W}}^{(h)}(A, A^5)$,

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h-1)}(A,A^5)} \int P_{\leq h-1}(d\psi^{(\leq h-1)}) e^{\mathcal{V}^{(h-1)}(A,A^5,\psi^{(\leq h-1)})}, \tag{3.73}$$

which reproduces the inductive assumption (3.42) at scale $h-1$. This iterative integration procedure goes on until we reach scale h_*, at which point the procedure is changed, as described in the next subsection. As discussed, e.g., in [25,27,29,52], the effective potential and generating function at scale h, obtained via such an iterative procedure, can be represented as convergent series over Gallavotti-Nicolò (GN) trees, see in particular [52, Section 2], where the tree expansion for a model of Weyl semimetal in the same regime as the one considered here is discussed in detail. The kernels of $\mathcal{V}^{(h)}$ and $\mathcal{W}^{(h)}$ are analytic in λ, for $|\lambda|$ small enough, and satisfy suitable weighted L^1 bounds that, in view of the estimate (3.70) for the single scale propagator, read as follows: for all $h_* \leq h < 0, n \geq 1$, and $|\lambda|$ small enough, if (3.62) and (3.63) are valid for all $k > h$, then, see [52, Lemma 1, Eq. (60)]

$$\frac{1}{\beta L^3} \int d\mathbf{Q} \, e^{\frac{\kappa_0}{2}\sqrt{\delta_h(\mathbf{Q})}} |W_{n,m_1,m_2,\underline{q};\underline{r},\underline{\mu},\underline{\nu}}^{(h)}(\mathbf{Q})| \tag{3.74}$$

$$\leq C^{n+m_1+m_2}(\max_{k>h} \mathfrak{Z}_k)^{m_1}(\max_{k>h} \mathfrak{Z}_k^5)^{m_2} |\lambda|^{\max\{\delta_{m_1+m_2},0,\,n-1\}} \gamma^{h(\frac{7}{2}-\frac{5}{2}n-m_1-m_2-d(\underline{q}))},$$

where: $\mathfrak{Z}_k = \max\{\max_{\mu=0,1,2}|Z_{\mu,k}|, \frac{|Z_{3,k}|}{|p_{F,3}|}, \frac{|\widetilde{Z}_{3,k}|}{|p_{F,3}|}\}$, $\mathfrak{Z}_k^5 = \max\{|Z_{3,k}^5|, \max_{\mu=0,1,2}\{\frac{|Z_{\mu,k}^5|}{|p_{F,3}|}, \frac{|\widetilde{Z}_{\mu,k}^5|}{|p_{F,3}|}\}\}$, $\delta_h(\mathbf{Q})$ is the Steiner diameter measured by using the norm $\|\mathbf{x}\|_h$ in (3.71), and $d(\underline{q}) =$

$\sum_{i=1}^{2n}(q_i^0 + q_i^1 + q_i^2 + \frac{1}{2}q_i^3)$. The kernels of the single-scale contribution to the generating function, $\widetilde{\mathcal{W}}^{(h)}(A, A^5)$ satisfy an estimate analogous to (3.74) with $n = 0$. The combination $\frac{7}{2} - \frac{5}{2}n - m_1 - m_2 - d(\underline{q})$ is the *scaling dimension* of the kernels with $2n$ Grassmann fields, derivatives of order $\underline{q}$ acting on them, and $m_1 + m_2$ external fields of type A or A^5 in the first regime $h \geq h_*$. Note, in particular, that the effective quartic interaction, i.e., the kernel with $n = 2$, no derivatives, and $m_1 = m_2 = 0$, is *irrelevant*, with scaling dimension $-3/2$. The irrelevance of the quartic interaction allows us to derived improved bounds on all the contributions to the kernels associated with GN trees containing at least one interaction endpoint: this is the same as in models of graphene with short range interactions, see [27] and, in particular, [27, Theorem 2]. More precisely, we can split $W_{n,m_1,m_2,\underline{q}}^{(h)} = W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{d}} + W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$, where 'd' stands for dominant and 'r' for remainder, and where the second term collects the contribution of all GN trees with at least one endpoint of type ν_k or on scale 1 (in particular, it includes all the contributions from GN trees with at least one endpoint associated with a quartic interaction, which is necessarily on scale 1); the term $W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$ admits an improved dimensional bound, analogous to (3.74), but with an extra factor $\gamma^{\theta h}$ in the right side, for any fixed θ smaller than 1 (and the constant C replaced by a θ-dependent constant C_θ). The proof of (3.74) and of its improved analogue for $W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$ are completely analogous to those of Eqs. (60) and (61) in Lemma 1 of [52], respectively, modulo a few minor differences discussed in Appendix B.

By using these bounds, we can also prove the inductive assumptions (3.62). In fact, assume that (3.62) and (3.63) are valid for $k \geq h$. Then the bound (3.74) is valid at scale $h - 1$, and similarly for its improved analogue for $W_{n,m_1,m_2,\underline{q}}^{(h-1);\mathrm{r}}$. By using the definitions of $z_{\mu,h-1}$ with $\mu = 0, 1, 2, 3$, and the bound on $W_{1,0,0,\underline{q}}^{(h-1);\mathrm{r}}$, we find that (3.62) is valid with $k = h - 1$, as well.

Concerning the proof of (3.63), let $\beta_h^\nu := \nu_{h-1} - 2\nu_h$ be the *beta function* for ν_h. From the bound on $W_{1,0,0}^{(h-1);\mathrm{int}}$, we find that

$$|\beta_h^\nu| \leq C_\theta |\lambda| \gamma^{\theta h}, \tag{3.75}$$

for any $\theta \in (0, 1)$. This does not imply that *any* solution of the beta function equation

$$\nu_{h-1} = 2\nu_h + \beta_h^\nu \tag{3.76}$$

with $\nu_0 = O(\lambda)$ satisfies (3.63). However, it is enough that we find *one* special choice of ν_0 (or, equivalently, of the staggered chemical potential ν in (2.19)) for which the solution satisfies such a bound. We temporarily proceeding by assuming the existence of such a good initial datum, and we will prove this assumption in Sect. 3.4. In that section, we will also derive bounds on the effective couplings $Z_{\mu,h}, Z_{\mu,h}^5$.

3.3.2. Regime $h \leq h^*$ In order to integrate the remaining scales, we proceed as follows. We choose α_0 in the definition of χ (see beginning of Sect. 3.2) small enough that the support of $\chi_{h_*}(\mathbf{k})$ consists of two 'well-separated' sets,[12] centered at $\mathbf{p}_F^+$ and $\mathbf{p}_F^-$, respectively. We decompose the fermionic field in terms of two independent *quasiparticle fields*, associated with the two Weyl nodes (as in the previous section, we will

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[12] By 'well-separated' we mean that the corridor between these two sets centered at $\mathbf{p}_F^\pm$ has diameter comparable with the diameter of the sets themselves.

often leave the r indices implicit, and think $\psi^+_{\mathbf{x},\omega}$ resp. $\psi^-_{\mathbf{x},\omega}$ as two-component row resp. column vectors):

$$\psi^{(\leq h_*)\pm}_{\mathbf{x}} := \sum_{\omega=\pm} e^{\pm i \mathbf{p}^\omega_F \mathbf{x}} \psi^{(\leq h_*)\pm}_{\mathbf{x},\omega},$$

$$\psi^{(\leq h_*)\pm}_{\mathbf{x},\omega} := \frac{1}{\beta L^3} \sum_{\mathbf{k}'} e^{\pm i \mathbf{k}' \mathbf{x}} \hat{\psi}^{(\leq h_*)\pm}_{\mathbf{k}',\omega}, \quad \hat{\psi}^{(\leq h_*)\pm}_{\mathbf{k}',\omega} := \hat{\psi}^{(\leq h_*)\pm}_{\mathbf{p}^\omega_F+\mathbf{k}'}. \quad (3.77)$$

where in the second line the sum over $\mathbf{k}'$ runs over the four-dimensional vectors such that $\mathbf{k}' + \mathbf{p}^\omega_F$ is in $\mathbb{D}_{\beta,L,N} \cap \mathrm{supp}(\chi_{h_*})$ and $|k'_3| < |p^\omega_F|$. The quasi-particle fields $\hat{\psi}^{(\leq h_*)\pm}_{\mathbf{k}',\omega}$ have covariance $\hat{g}^{(\leq h_*)}_\omega(\mathbf{k}') = \chi_{h_*,\omega}(\mathbf{k}') Z^{-1}_{h_*} A_{h_*,\omega}(\mathbf{k}')^{-1}$, with $\chi_{h_*,\omega}(\mathbf{k}') := \chi_{h_*}(\mathbf{k}' + \mathbf{p}^\omega_F) \mathbb{1}(|k'_3| < |p^\omega_F|)$, and $A_{h_*,\omega}(\mathbf{k}') = A_{h_*}(\mathbf{k}' + \mathbf{p}^\omega_F)$.

As in the previous regime, we integrate the scales $h \leq h_*$ in a multiscale fashion: at each step, we decompose the quasi-particle Grassmann field $\psi^{(\leq h)}_\omega$ as the sum of two independent fields, one describing the fluctuations at scale h, and the other at smaller scales: $\psi^{(\leq h)}_\omega = \psi^{(\leq h-1)}_\omega + \psi^{(h)}_\omega$; we decompose the effective potential as the sum of a localized part plus an 'irrelevant' remainder; we combine part of the localized part of the effective potential with the Grassmann measure; we integrate out the field at scale h; we iterate. More precisely, we inductively assume that, for any $h_\beta \leq h \leq h_*$, with $h_\beta = \lfloor \log_\gamma(\pi/\beta) \rfloor$, the generating functional of the correlations can be written as:

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h)}(A,A^5)} \int P_{\leq h}(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(A,A^5,\psi^{(\leq h)})}. \quad (3.78)$$

Let us explain the meaning of the various objects involved: $P_{\leq h}(d\psi^{(\leq h)})$ denotes a Gaussian Grassmann integration with covariance

$$\int P_{\leq h}(d\psi^{(\leq h)}) \hat{\psi}^{-(\leq h)}_{\omega_1,\mathbf{k}'_1,r_1} \hat{\psi}^{+(\leq h)}_{\omega_2,\mathbf{k}'_2,r_2} = \beta L^3 \delta_{\omega_1,\omega_2} \delta_{\mathbf{k}'_1,\mathbf{k}'_2} \left[\hat{g}^{(\leq h)}_{\omega_1}(\mathbf{k}'_1) \right]_{r_1,r_2}, \quad (3.79)$$

where

$$\hat{g}^{(\leq h)}_\omega(\mathbf{k}') := \frac{\chi_{h,\omega}(\mathbf{k}')}{Z_h} A_{h,\omega}(\mathbf{k}')^{-1} \quad (3.80)$$

and: $\chi_{h,\omega}(\mathbf{k}') := \chi_h(\mathbf{k}' + \mathbf{p}^\omega_F) \mathbb{1}(|k'_3| < |p^\omega_F|)$, Z_h is a real-analytic function of λ, to be inductively defined below,

$$A_{h,\omega}(\mathbf{k}') = \begin{pmatrix} -ik'_0 + c_{h,\omega}(\mathbf{k}') + d_{h,\omega}(\mathbf{k}') & a_{h,\omega}(\mathbf{k}') - ib_{h,\omega}(\mathbf{k}') \\ a_{h,\omega}(\mathbf{k}') + ib_{h,\omega}(\mathbf{k}') & -ik'_0 - c_{h,\omega}(\mathbf{k}') + d_{h,\omega}(\mathbf{k}') \end{pmatrix}, \quad (3.81)$$

with

$$a_{h,\omega}(\mathbf{k}') = v_{1,h}k'_1 + a_R(\mathbf{k}' + \mathbf{p}^\omega_F)/Z_h, \quad b_{h,\omega}(\mathbf{k}') = v_{2,h}k'_2 + b_R(\mathbf{k}' + \mathbf{p}^\omega_F)/Z_h, \quad (3.82)$$

$$c_{h,\omega}(\mathbf{k}') = \omega v_{3,h}k'_3 + \tilde{c}_{R,\omega}(\mathbf{k}')/Z_h d_{h,\omega}(\mathbf{k}') = d(\mathbf{k}' + \mathbf{p}^\omega_F)/Z_h, \quad (3.83)$$

and, recalling that $c_{R,\omega}$ was defined by (2.12), we let $\tilde{c}_{R,\omega}(k') := \frac{b^0+\zeta_{h_*}}{2}(k'_3)^2 + c_{R,\omega}(k' + p_F^\omega)$. Moreover, the function $\mathcal{V}^{(h)}$ in the right side of (3.78) is the effective potential, which can be represented in a way analogous to (3.47), namely

$$\mathcal{V}^{(h)}(A, A^5, \psi) = \sum_{\substack{n,m_1,m_2\geq 0 \\ n>0}} \sum_{\substack{\underline{q},\underline{r},\underline{\omega}, \\ \underline{\mu},\underline{\nu}}} \int d\mathbf{X}d\mathbf{Y}d\mathbf{Z} \Big[\prod_{i=1}^{2n} \hat\partial^{q_{2i-1}} \psi^+_{\mathbf{x}_{2i-1},\omega_{2i-1},r_{2i-1}} \hat\partial^{q_{2i}} \psi^-_{\mathbf{x}_{2i},\omega_{2i},r_{2i}} \Big] \cdot$$

$$\cdot \Big[\prod_{j=1}^{m_1} A_{\mu_j,\mathbf{y}_j} \Big] \Big[\prod_{\ell=1}^{m_2} A^5_{\nu_\ell,\mathbf{z}_\ell} \Big] W^{(h)}_{n,m_1,m_2,\underline{q};\underline{\omega},\underline{r},\underline{\mu},\underline{\nu}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}), \tag{3.84}$$

and $\mathcal{W}^{(h)}(A, A^5)$ is the finite-scale contribution to the generating function, which admits a representation analogous to (3.84), with the difference that only terms with $n = 0$ contribute to the sums.

The inductive assumption (3.78) and following equations is verified at scale $h = h_*$, as an outcome of the integration of the first regime, provided we let

$$v_{3,h_*} = v_3^0 + \zeta_{h_*} p_{F,3} \tag{3.85}$$

(note that the representation (3.84) at scale h_* follows from (3.47) at the same scale, in combination with the definition of quasi-particle fields (3.77)). We now assume that (3.78) is valid at scale h and prove it for $h - 1$. We decompose $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, with $\mathcal{L}$ the localization operator, defined as follows:

$$\mathcal{L}\mathcal{V}^{(h)}(A, A^5, \psi) := \sum_{\omega=\pm} \int \frac{d\mathbf{k}'}{(2\pi)^4} \hat\psi^+_{\mathbf{k}',\omega}\big(\mathcal{L}\hat{W}^{(h)}_{1,0,0;(\omega,\omega)}(\mathbf{k}')\big)\hat\psi^-_{\mathbf{k}',\omega} \tag{3.86}$$

$$+ \sum_{\omega=\pm} \int \frac{d\mathbf{k}'}{(2\pi)^4} \int \frac{d\mathbf{p}}{(2\pi)^4} \Big[\hat{A}_{\mu,\mathbf{p}}\hat\psi^+_{\mathbf{k}'+\mathbf{p},\omega}\big(\mathcal{L}\hat{W}^{(h)}_{1,1,0;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})\big)\hat\psi^-_{\mathbf{k}',\omega}$$

$$+ \hat{A}^5_{\mu,\mathbf{p}}\hat\psi^+_{\mathbf{k}'+\mathbf{p},\omega}\big(\mathcal{L}\hat{W}^{(h)}_{1,0,1;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})\big)\hat\psi^-_{\mathbf{k}',\omega} \Big],$$

where $\int \frac{d\mathbf{k}}{(2\pi)^4}$ and $\int \frac{d\mathbf{p}}{(2\pi)^4}$ are shorthands for $(\beta L^3)^{-1} \sum_{\mathbf{k}'}$ and $(\beta L^3)^{-1} \sum_{\mathbf{p}}$, and

$$\mathcal{L}\hat{W}^{(h)}_{1,0,0;(\omega,\omega)}(\mathbf{k}') := \hat{W}^{(h),\infty}_{1,0,0;(\omega,\omega)}(\mathbf{0}) + \sum_{\mu=0}^{3} k'_\mu \partial_\mu \hat{W}^{(h),\infty}_{1,0,0;(\omega,\omega)}(\mathbf{0}), \tag{3.87}$$

$$\mathcal{L}\hat{W}^{(h)}_{1,1,0;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p}) := \hat{W}^{(h),\infty}_{1,1,0;(\omega,\omega),\mu}(\mathbf{0}, \mathbf{0}), \qquad \mathcal{L}\hat{W}^{(h)}_{1,0,1;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p}) := \hat{W}^{(h),\infty}_{1,0,1;(\omega,\omega),\mu}(\mathbf{0}, \mathbf{0}),$$

where $\hat{W}^{(h),\infty}_{1,0,0;(\omega,\omega)}(\mathbf{k}')$ indicates the $\beta, L \to \infty$ limit of $\hat{W}^{(h)}_{1,0,0;(\omega,\omega)}(\mathbf{k}')$, and similarly for $\hat{W}^{(h)}_{1,1,0;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})$ and $\hat{W}^{(h)}_{1,0,1;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})$.

Remark. We anticipate the fact that the scaling dimension of $\hat{W}^{(h),\infty}_{n,m_1,m_2;\underline{\omega},\underline{\mu},\underline{\nu}}$ in this second regime is $4 - \frac{3}{2}n - m_1 - m_2$, see (3.102); any additional derivative with respect to k_μ decreases the scaling dimension by 1, so that, from the definitions, it follows that $\mathcal{R}\hat{W}^{(h)}_{1,0,0}(\mathbf{k}')$, $\mathcal{R}\hat{W}^{(h)}_{1,1,0;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})$ and $\mathcal{R}\hat{W}^{(h)}_{1,0,1;(\omega,\omega),\mu}(\mathbf{k}', \mathbf{p})$ are all irrelevant, with scaling dimension -1. The reader may recognize that the scaling dimensions are the same as those of QED_4: our theory can be seen as a lattice realization of lattice infrared QED_4, with the important difference that the speed of light is replaced by the anisotropic

running velocities $v_{\mu,h}$. Since $v_{3,h}$ (or, equivalently, v_3^0) vanishes in the limit as the Weyl nodes merge, the dimensional bounds on the effective potentials, see (3.102) below, are potentially affected by dangerous $1/v_{3,h}$ factor, which may a priori have an impact on the convergence properties of the infrared expansion of the observables of interest. However, by carefully tracking the loss and gain of factors $v_{3,h}$, due to the non-uniform dependence of the propagator upon $v_{3,h}$ and to the difference of scaling dimensions between the first and second regimes, one finds that the bad and good dependences upon these factors compensates, and lead to the overall factor $|v_3^0|^{n-1+\|\underline{q}^3\|_1}$ in (3.102) below; see [52, Section 3] and Appendix B.

As in the regime $h > h_*$, we shall use the localized term $\mathcal{L}\mathcal{V}^{(h)}$ to redefine the Grassmann integration and the coupling of the fermions with the external fields. The next lemma establishes important symmetry properties of the kernels of $\mathcal{L}\mathcal{V}^{(h)}$ (see Appendix A.2 for the proof).

Lemma 3.2. *The following identities hold:*

$$\hat{W}_{1,0,0;(\omega,\omega)}^{(h),\infty}(\mathbf{0}) = n_h\sigma_3, \qquad \partial_\mu \hat{W}_{1,0,0;(\omega,\omega)}^{(h),\infty}(\mathbf{0}) = z_{\mu,h}\sigma_{\mu,\omega}, \tag{3.88}$$

with $n_h, z_{\mu,h} \in \mathbb{R}$, and

$$\sigma_{\mu,\omega} := \begin{cases} \sigma_\mu & \text{if } \mu = 0, 1, 2, \\ \omega\sigma_3 & \text{if } \mu = 3. \end{cases} \tag{3.89}$$

Moreover,

$$\hat{W}_{1,1,0;(\omega,\omega),\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) = Z_{\mu,h}\sigma_{\mu,\omega}, \quad \hat{W}_{1,0,1;(\omega,\omega),\mu}^{(h),\infty}(\mathbf{0}, \mathbf{0}) = \omega Z_{\mu,h}^5\sigma_{\mu,\omega}, \tag{3.90}$$

with $Z_{\mu,h}, Z_{\mu,h}^5 \in \mathbb{R}$.

We now: rewrite $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$ in the right side of (3.78); then write $\mathcal{L}\mathcal{V}^{(h)}$ explicitly, in view of Lemma 3.2, in terms of n_h, $z_{\mu,h}$, $Z_{\mu,h}$ and $Z_{\mu,h}^5$; then re-absorb the quadratic part of $\mathcal{L}\mathcal{V}^{(h)}$ associated with $\sum_\mu k_\mu' \partial_\mu \hat{W}_{1,0,0;\omega}^{(h),\infty}(\mathbf{0}) \equiv \sum_\mu k_\mu' \sigma_{\mu,\omega}z_{\mu,h}$ in the Gaussian Grassmann integration, thus getting

$$e^{\mathcal{W}_{\beta,L}(A,A^5,0)} = e^{\mathcal{W}^{(h)}(A,A^5)} \int \widetilde{P}_{\leq h}(d\psi^{(\leq h)})e^{2^h v_h N_3(\psi^{(\leq h)})+B^{(h)}(A,A^5,\psi^{(\leq h)})+\mathcal{R}\mathcal{V}^{(h)}(A,A^5,\psi^{(\leq h)})}, \tag{3.91}$$

where: $2^h v_h \equiv n_h$, $N_3(\psi) = \sum_\omega \int \frac{d\mathbf{k}'}{(2\pi)^4} \hat{\psi}_{\mathbf{k}',\omega}^+ \sigma_3 \hat{\psi}_{\mathbf{k}',\omega}^-$,

$$B^{(h)}(A, A^5, \psi) = \sum_{\omega,\mu} \int \frac{d\mathbf{k}'}{(2\pi)^4} \frac{d\mathbf{p}}{(2\pi)^4} (Z_{\mu,h}\hat{A}_{\mu,\mathbf{p}} + \omega Z_{\mu,h}^5\hat{A}_{\mu,\mathbf{p}}^5)\hat{\psi}_{\mathbf{k}'+\mathbf{p},\omega}^+\sigma_{\mu,\omega}\hat{\psi}_{\mathbf{k}',\omega}^-, \tag{3.92}$$

and the new Grassmann Gaussian integration $\widetilde{P}_{\leq h}(d\psi^{(\leq h)})$ has covariance given by:

$$\tilde{g}_\omega^{(\leq h)}(\mathbf{k}') = \frac{\chi_{h,\omega}(\mathbf{k}')}{Z_{h-1,\omega}(\mathbf{k}')}\tilde{A}_{h-1,\omega}(\mathbf{k}')^{-1}, \tag{3.93}$$

where: $Z_{h-1,\omega}(\mathbf{k}') = Z_h - \chi_{h,\omega}(\mathbf{k}')\, z_{0,h}$,

$$\tilde{A}_{h-1,\omega}(\mathbf{k}') = \begin{pmatrix} -ik_0' + \tilde{c}_{h-1,\omega}(\mathbf{k}') + \tilde{d}_{h,\omega}(\mathbf{k}') & \tilde{a}_{h-1,\omega}(\mathbf{k}') - i\tilde{b}_{h-1,\omega}(\mathbf{k}') \\ \tilde{a}_{h-1,\omega}(\mathbf{k}') + i\tilde{b}_{h-1,\omega}(\mathbf{k}') & -ik_0' - \tilde{c}_{h-1,\omega}(\mathbf{k}') + \tilde{d}_{h,\omega}(\mathbf{k}') \end{pmatrix}, \tag{3.94}$$

and

$$\tilde{a}_{h-1,\omega}(\mathbf{k}') = v_{1,h-1,\omega}(\mathbf{k}')k_1' + \frac{a_R(k' + p_F^\omega)}{Z_{h-1,\omega}(\mathbf{k}')}, \qquad \tilde{b}_{h-1,\omega}(\mathbf{k}') = v_{2,h-1,\omega}(\mathbf{k}')k_2' + \frac{b_R(k' + p_F^\omega)}{Z_{h-1,\omega}(\mathbf{k}')},$$

$$\tilde{c}_{h-1,\omega}(\mathbf{k}') = \omega\, v_{3,h-1,\omega}(\mathbf{k}')k_3' + \frac{\tilde{c}_{R,\omega}(k')}{Z_{h-1,\omega}(\mathbf{k}')}, \qquad \tilde{d}_{h-1,\omega}(\mathbf{k}') = \frac{d(k' + p_F^\omega)}{Z_{h-1,\omega}(\mathbf{k}')},$$

with

$$v_{l,h-1,\omega}(\mathbf{k}') = \frac{Z_h v_{l,h} - \chi_{h,\omega}(\mathbf{k}')z_{l,h}}{Z_{h-1,\omega}(\mathbf{k}')}, \qquad l = 1, 2, 3. \tag{3.95}$$

Similarly to the previous regime, we inductively assume that, for all scales $h \leq k \leq h_*$ and any $\theta \in (0, 1)$:

$$|z_{\mu,k}| \leq C_\theta |\lambda|\, |v_\mu^0|\gamma^{\theta k}, \tag{3.96}$$

where $v_0^0 := 1$. We will check the validity of these bounds on scale $k = h - 1$. We also assume that ν_k satisfies (3.63), for all $h \leq k \leq h_*$. Notice that (3.96), in combination with the bounds on Z_h, $v_{l,h}$ derived in Sect. 3.3.1, implies:

$$|Z_{h-1,\omega}(\mathbf{k}') - 1| \leq C|\lambda|, \qquad |v_{l,h-1,\omega}(\mathbf{k}') - v_l^0| \leq C|\lambda|\,|v_l^0|. \tag{3.97}$$

We also let

$$Z_{h-1} := Z_{h-1,\omega}(\mathbf{0}) \qquad \text{and} \qquad v_{l,h-1} := v_{l,h-1,\omega}(\mathbf{0}),$$

and, of course, these effective parameters satisfy the same bounds as (3.97). To check the inductive assumption, we decompose the Grassmann field as:

$$\psi_\omega^{(\leq h)} = \psi_\omega^{(\leq h-1)} + \psi_\omega^{(h)}, \tag{3.98}$$

where $\hat{\psi}_\omega^{(\leq h-1)}$ has covariance given by $\hat{g}_\omega^{(\leq h-1)}(\mathbf{k})$, defined as in (3.80), (3.81), with h replaced by $h - 1$, while $\psi_\omega^{(h)}$ has covariance:

$$\hat{g}_\omega^{(h)}(\mathbf{k}') = \frac{f_{h,\omega}(\mathbf{k}')}{Z_{h-1,\omega}(\mathbf{k}')} \tilde{A}_{h-1,\omega}(\mathbf{k}')^{-1}, \tag{3.99}$$

with $f_{h,\omega}(\mathbf{k}') = \chi_{h,\omega}(\mathbf{k}') - \chi_{h-1,\omega}(\mathbf{k}')$ and we used the fact that, on the support of $\chi_{h-1,\omega}$, $Z_{h-1,\omega}(\mathbf{k}') = Z_{h-1}$ and $\tilde{A}_{h-1}(\mathbf{k}') = A_{h-1}(\mathbf{k}')$. On the support of $f_{h,\omega}$,

$$C^{-1}\gamma^{2h} \leq -\det \tilde{A}_{h-1,\omega}(\mathbf{k}') \leq C\gamma^{2h},$$

$|k_\mu'| \leq C\gamma^h$ for $\mu = 0, 1, 2$, and $|k_3'| \leq C\gamma^h/|v_3^0|$. Therefore, for these values of $\mathbf{k}'$, recalling the definition of h_*,

$$\tilde{a}_{h-1}(\mathbf{k}) = v_{1,h-1,\omega}(\mathbf{k}')k_1' + O(\gamma^{2h}), \quad \tilde{b}_{h-1}(\mathbf{k}) = v_{2,h-1,\omega}(\mathbf{k}')k_2' + O(\gamma^{2h}),$$

$$\tilde{c}_{h-1}(\mathbf{k}) = \omega v_{3,h-1,\omega}(\mathbf{k}')k_3' + O(\gamma^{2h-h_*}), \qquad \tilde{d}_{h-1}(\mathbf{k}) = O(\gamma^{2h}),$$

which implies $\|\hat{g}^{(h)}(\mathbf{k})\|_\infty \leq C\gamma^{-h}$ and $\|\hat{g}(\mathbf{k})\|_1 \leq C\gamma^{3h}/|v_3^0|$. By a similar discussion, we can also dimensionally bound the derivatives of $\tilde{a}_{h-1}, \tilde{b}_{h-1}, \tilde{c}_{h-1}, \tilde{d}_{h-1}$, thus getting

$$\|\partial_{\mathbf{k}'}^\alpha \hat{g}_\omega^{(h)}(\mathbf{k}')\|_\infty \leq C_{|\alpha|}|v_3^0|^{|\alpha_3|}\gamma^{-h(1+|\alpha|)}, \qquad \|\partial_{\mathbf{k}'}^\alpha \hat{g}_\omega(\mathbf{k}')\|_1 \leq C_{|\alpha|}\gamma^{h(3-|\alpha|)}/|v_3^0|, \tag{3.100}$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = \sum_\mu |\alpha_\mu|$. These in turn imply the following bound in configuration space:

$$\|g_\omega^{(h)}(\mathbf{x})\| \leq (C_0/|v_3^0|)\gamma^{3h}e^{-\kappa_0\sqrt{\gamma^h d(\mathbf{x})}} \tag{3.101}$$

for suitable $C_0, \kappa_0 > 0$, where $d(\mathbf{x}) := |x_0| + |x_1| + |x_2| + |x_3|/|v_3^0|$.

At this point, we integrate the field $\psi^{(h)}$ out, as done in Eqs. (3.72) and (3.73), thus obtaining the analogue of (3.73), where the effective action $\mathcal{V}^{(h-1)}$ can be represented as in (3.84), with h replaced by $h-1$. Also in this case, the kernels of the effective potential can be represented in terms of a convergent GN tree expansion, and are analytic in λ, for $|\lambda|$ small, *uniformly in* ζ, i.e., in the distance between the Weyl nodes. They satisfy the following weighted L^1 estimates: for all $h_\beta \leq h < h_*, n \geq 1$, and $|\lambda|$ small enough, uniformly in ζ, if (3.96) and (3.63) are valid for $h < k \leq h_*$, then, see [52, Lemma 2, Eq. (93)],

$$\frac{1}{\beta L^3} \int d\mathbf{Q}\, e^{\frac{\kappa_0}{2}\sqrt{\gamma^h \delta_*(\mathbf{Q})}} |W_{n,m_1,m_2,\underline{q};\underline{\omega},\underline{r},\underline{\mu},\underline{v}}^{(h)}(\mathbf{Q})| \leq \tag{3.102}$$

$$\leq C^{n+m_1+m_2}|v_3^0|^{n-1+\|\underline{q}^3\|_1}(\max_{k>h} \mathfrak{Z}_k)^{m_1}(\max_{k>h}\mathfrak{Z}_k^5)^{m_2}|\lambda|^{\max\{\delta_{m_1+m_2},0,\,n-1\}}\gamma^{h(4-3n-m_1-m_2-\|\underline{q}\|_1)},$$

where $\delta_*(\mathbf{Q})$ is the Steiner diameter measured by using the norm $d(\mathbf{x})$ defined after (3.101), $\|\underline{q}^3\|_1 = \sum_{i=1}^{2n}|q_i^3|$, $\|\underline{q}\|_1 = \sum_{i=1}^{2n}\sum_{\mu=0}^3|q_i^\mu|$, and, if $k \leq h_*$, $\mathfrak{Z}_k = \max_\mu |Z_{\mu,k}|$ and $\mathfrak{Z}_k^5 = \max_\mu |Z_{\mu,k}^5|$ (while, if $k > h_*$, $\mathfrak{Z}_k$ and $\mathfrak{Z}_k^5$ were defined after (3.74)). The kernels of the single-scale contribution to the generating function, $\widetilde{\mathcal{W}}^{(h)}(A, A^5)$ satisfy an estimate analogous to (3.102) with $n = 0$. The combination $4 - 3n - m_1 - m_2 - \|\underline{q}\|_1$ is the *scaling dimension* of the kernels with $2n$ Grassmann fields, derivatives of order $\underline{q}$ acting on them, and $m_1 + m_2$ external fields of type A or A^5 in the second regime $h < h_*$. Also in this case, the quartic interaction ($n = 2, m_1 = m_2 = 0, \|\underline{q}\|_1 = 0$) is irrelevant, and its scaling dimension is -2. Also in the second regime, we can obtain improved bounds on all the contributions to the kernels associated with GN trees containing at least one interaction endpoint: more precisely, if we split $W_{n,m_1,m_2,\underline{q}}^{(h)} = W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{d}} + W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$, where the second term collects the contribution of all GN trees with at least one endpoint of type ν_k or one endpoint on scale $h_* + 1$ (including, in particular, all the contributions from GN trees with endpoints associated with quartic interactions), the term $W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$ admits an improved dimensional bound, analogous to (3.74), but with an extra factor in the right side equal to: $\gamma^{\theta h}$, if $m_1 = m_2 = 0$ or $n \geq 2$; $\gamma^{\theta(h-h_*)}$, if $m_1 + m_2 \geq 1$ and $n = 1$ (again, θ can be chosen to be any positive exponent smaller than 1, at the cost of replacing the constant C in the right side of (3.102) by C_θ, which possibly diverges as $\theta \to 1^-$). The proof is completely analogous to that of Eq. (94) of Lemma 2 in [52], modulo a few minor differences discussed in Appendix B.

The definition of $z_{\mu,h-1}$ and the bound on $W^{(h-1);\mathrm{r}}_{n,m_1,m_2,q;\underline{\omega},\underline{r},\underline{\mu},\underline{\nu}}$, readily imply the validity of (3.96) at scale $k = h - 1$, as desired. Concerning $\overline{(3.63)}$, also in this second regime we find that ν_h satisfies the beta function equation (3.76), with β^ν_h satisfying (3.75) even for $h < h_*$. The choice of the initial datum ν_0 (or, equivalently, of the staggered chemical potential ν) for which ν_k satisfies (3.63) for all $h \le 0$ will be discussed in the next section, where we will also derive bounds on the flow of the effective couplings $Z_{\mu,h}, Z^5_{\mu,h}$.

The iterative RG integration of the second regime goes on until we reach scale h_β. At that point, we integrate 'in one shot' all the remaining Grassmann degrees of freedom, thus obtaining the desired generating function of correlations, in the form of a sum of single-scale contributions from $h = h_\beta$ to $h = 0$ (the covariance of $\psi^{(\le h_\beta)}_\omega$ admits the same dimensional bounds as the one of $\psi^{(h_\beta+1)}_\omega$; therefore, the result of the integration of $\psi^{(\le h_\beta)}_\omega$ admits the same qualitative bounds as the one at scale $h_\beta + 1$, so that the last iteration step does not give any additional difficulty).

The uniformity in β, L and ζ of the convergence of the GN tree expansion for the single-scale contributions to the generating function, as well as the dimensional bounds (3.74) and (3.102), imply that the correlation functions at fixed space-time positions are analytic functions of λ and $\{\nu_h\}_{h\le 0}$, in a sufficiently small neighbourhood of the origin, uniformly in β, L and ζ. Elaborating on this and on the fact that all the Taylor coefficients of such convergent expansions admit a limit as $\beta, L \to \infty$ implies the existence of the $\beta, L \to \infty$ limit of the Euclidean correlations, as well as the analyticity of the limits, stated in Theorem 2.1, provided that the sequence $\{\nu_h\}_{h\le 0}$ satisfies the promised bounds, see (3.63), and is itself analytic in λ. This will be proved in the next section, together with the bounds on the flow of the effective vertex functions Z_h, Z^5_h.

3.4. The flow of effective couplings. In this section we discuss the RG flow of the effective parameters $\nu_h, Z_{\mu,h}, Z^5_{\mu,h}$ (as well as the one of $\tilde{Z}_{3,h}$ and $\tilde{Z}^5_{\mu,h}$ with $\mu = 0, 1, 2$, which appear in the first regime). In particular, we explain how to fix the bare staggered chemical potential ν in (2.19) in such a way that the running staggered chemical potential ν_h satisfies (3.63) at all scales $h \le 0$. We also explain how to fix $Z^5_{\mu,\mathrm{bare}}$, see (2.39), in such a way that their dressed counterpart, $Z^5_{\mu,h}$, flow to any prescribed 4-tuple of values as $h \to -\infty$. The bounds discussed in the following use a few properties of the GN trees contributing to the effective potential, discussed in [52, Sections 2 and 3], see also Appendix B.

3.4.1. Flow of ν Starting from the beta function equation for ν_h, (3.76), we find

$$\nu_0 = 2^k \nu_k - \sum_{k < h \le 0} 2^{h-1} \beta^\nu_h, \qquad (3.103)$$

for any $k < 0$. Here ν_0 is an analytic function of λ and ν that, assuming ν to be of order λ, is of the form $\nu_0 = \nu + O(\lambda^2)$. If we now require that ν_k tends to zero as $k \to -\infty$, we get

$$\nu_0 = -\sum_{h \le 0} 2^{h-1} \beta^\nu_h, \qquad (3.104)$$

and, more in general,

$$\nu_k = -\sum_{h \leq k} 2^{h-k-1} \beta_h^\nu. \tag{3.105}$$

Recall that β_h^ν is a function of λ and of the effective parameters $\nu_{h'}$ on scales $h' \geq h$. Therefore, we can regard the right side of (3.105) as a function of the whole sequence $\underline{\nu} := \{\nu_k\}_{k \leq 0}$, which we denote by $T_k(\underline{\nu}, \lambda)$, so that (3.105) can be read as a fixed point equation $\nu_k = T_k(\underline{\nu}, \lambda)$ for the sequence $\underline{\nu}$. By proceeding as done in many other papers before, see e.g. [34, Section 6.4.2], we look for a fixed point in the Banach space of sequences $\underline{\nu}$ such that $\|\underline{\nu}\|_\theta := \sup_{h \leq 0} |\nu_h| \gamma^{-\theta h} \leq K|\lambda|$, for $\theta = 3/4$ (say) and K a suitable (sufficiently large) constant. Following the same strategy as [34, Section 6.4.2] (in a much simpler setting), the reader can check that the map $T : \underline{\nu} \to \{T_k(\underline{\nu}, \lambda)\}_{k \leq 0}$ is a contraction on such a Banach space, which implies the existence of a unique fixed point in that space. The value of ν_0 of such a fixed point sequence corresponds to the 'right' initial datum to assign in order for (3.63) to be satisfied at all scales. Finally, note that fixing ν_0 is equivalent to fixing ν (recall that $\nu_0 = \nu_0(\nu, \lambda)$ is analytic in ν, λ and that $\nu_0(\nu, \lambda) = \nu_0 + O(\lambda^2)$; appealing to the implicit function theorem, we can analytically invert ν_0 with respect to ν). The value of the bare staggered chemical potential ν fixed via this strategy is the one stated in Theorem 2.1.

3.4.2. Flow of $Z_{\mu,h}$

In the *first regime*, recalling that $Z_{\mu,h}$ is defined via (3.51), we write

$$Z_{\mu,h-1} = Z_{\mu,h} + \beta_{\mu,h}, \tag{3.106}$$

where $\beta_{\mu,h}$ includes the contributions from GN trees that have at least one endpoint of type ν_h or one endpoint on scale 1 of order λ. Therefore, $\beta_{\mu,h}$ is bounded in the same way as $W_{1,1,0,\underline{q}}^{(h);r}$, with $\|\underline{q}\|_1 = \|\underline{q}^3\|_1$ equal to 0 or 1, depending on whether $\mu = 0, 1, 2$ or $\mu = 3$, respectively; see the paragraph after (3.74). We thus get, for any $0 < \theta < 1$,

$$|\beta_{\mu,h}| \leq C_\theta |\lambda| \gamma^{\theta h} \begin{cases} \sup_{h \leq k \leq 0} |Z_{\mu,k}| & \text{if } \mu = 0, 1, 2 \\ \gamma^{-h/2} \sup_{h \leq k \leq 0} (|Z_{3,k}| + |\tilde{Z}_{3,k}|) & \text{if } \mu = 3 \end{cases} \tag{3.107}$$

where $Z_{\mu,0}$ for $\mu = 0, 1, 2, 3$ and $\tilde{Z}_{3,0}$ are analytic functions of λ, bounded as $|Z_{\mu,0} - v_\mu^0| \leq C|\lambda| |v_\mu^0|$ and $|\tilde{Z}_{3,0}| \leq C|\lambda| |v_3^0|$.

Remark. The importance of having a gain factor $\gamma^{\theta h}$ with θ any positive constant smaller than 1 (rather than 1/2, as other simpler choices of the localization operator would have implied), is apparent from (3.107). In fact, choosing θ larger than 1/2 makes all the components of the beta function summable in h, uniformly in v_3^0. A posteriori, this motivates the definitions (3.49), see also the remarks following it. Similar considerations are valid for the beta function for $Z_{\mu,h}^5$, see in particular (3.114) below.

Similarly, the flow equation for $\tilde{Z}_{3,h}$ is

$$\tilde{Z}_{3,h-1} = \tilde{Z}_{3,h} + \tilde{\beta}_{3,h}, \tag{3.108}$$

with

$$|\tilde{\beta}_{3,h}| \leq C_\theta |\lambda| \gamma^{\theta h} \gamma^{-h/2} \sup_{h \leq k \leq 0} (|Z_{3,k}| + |\tilde{Z}_{3,k}|). \tag{3.109}$$

From these bounds on the beta function, we readily find

$$|Z_{\mu,h} - Z_{\mu,0}| \leq C|\lambda|\,|Z_{\mu,0}|, \qquad |\tilde{Z}_{3,h}| \leq C|\lambda|\,|Z_{3,0}|, \tag{3.110}$$

for any $h \geq h_*$, uniformly in h.

In the *second regime*, the flow of $Z_{\mu,h}$ is controlled by a flow equation of the same form as (3.106), where $\beta_{\mu,h}$ is bounded in the same way as $W_{1,1,0,\underline{0}}^{(h);\mathrm{r}}$, see the paragraph after (3.102):

$$|\beta_{\mu,h}| \leq C|\lambda|\gamma^{\theta(h-h_*)} \sup_{h \leq k \leq h_*} |Z_{\mu,k}|. \tag{3.111}$$

Using this bound on the beta function and (3.110), we readily find that

$$|Z_{\mu,h} - Z_{\mu,0}| \leq C|\lambda|\,|Z_{\mu,0}|, \tag{3.112}$$

for any $h \leq 0$, that $Z_\mu := \lim_{h \to -\infty} Z_{\mu,h}$ exists and is analytically close to $Z_{\mu,0}$, and that the limit is reached at an exponential rate: $|Z_{\mu,h} - Z_\mu| \leq C|\lambda|\,|Z_\mu|\gamma^{\theta(h-h_*)}$.

3.4.3. Flow of $Z_{\mu,h}^5$ The discussion is very similar to that of the flow of $Z_{\mu,h}$. In the *first regime*, recalling that $Z_{\mu,h}^5$ is defined via (3.51), we write

$$Z_{\mu,h-1}^5 = Z_{\mu,h}^5 + \beta_{\mu,h}^5, \tag{3.113}$$

where $\beta_{\mu,h}^5$ includes the contributions from GN trees that have at least one endpoint of type ν_h or one endpoint on scale 1 of order λ. Therefore, $\beta_{\mu,h}^5$ is bounded in the same way as $W_{1,0,1,\underline{q}}^{(h);\mathrm{r}}$, with $\|\underline{q}\|_1 = \|\underline{q}^3\|_1$ equal to 0 or 1, depending on whether $\mu = 3$, or $\mu = 0, 1, 2$, respectively; see the paragraph after (3.74). We thus get, for any $0 < \theta < 1$,

$$|\beta_{\mu,h}^5| \leq C|\lambda|\gamma^{\theta h} \begin{cases} \gamma^{-h/2} \sup_{h \leq k \leq 0}(|Z_{\mu,k}^5| + |\tilde{Z}_{\mu,k}^5|) & \text{if } \mu = 0, 1, 2 \\ \sup_{h \leq k \leq 0} |Z_{3,k}^5| & \text{if } \mu = 3 \end{cases} \tag{3.114}$$

where $Z_{\mu,0}^5$ for $\mu = 0, 1, 2, 3$ and $\tilde{Z}_{\mu,0}^5$ for $\mu = 0, 1, 2$ are analytic functions of λ, bounded as $|Z_{\mu,0}^5 - Z_{\mu,\mathrm{bare}}^5| \leq C|\lambda|\,|Z_{\mu,\mathrm{bare}}^5|$ and $|\tilde{Z}_{\mu,0}^5| \leq C|\lambda|\,|Z_{\mu,\mathrm{bare}}^5|$. Similarly, the flow equation for $\tilde{Z}_{\mu,h}^5$ for $\mu = 0, 1, 2$ is

$$\tilde{Z}_{\mu,h-1}^5 = \tilde{Z}_{\mu,h}^5 + \tilde{\beta}_{\mu,h}^5, \tag{3.115}$$

with

$$|\tilde{\beta}_{\mu,h}^5| \leq C|\lambda|\gamma^{\theta h}\gamma^{-h/2} \sup_{h \leq k \leq 0} (|Z_{\mu,k}^5| + |\tilde{Z}_{\mu,k}^5|). \tag{3.116}$$

From these bounds on the beta function, we readily find

$$|Z_{\mu,h}^5 - Z_{\mu,\mathrm{bare}}^5| \leq C|\lambda|\,|Z_{\mu,\mathrm{bare}}^5|, \qquad |\tilde{Z}_{\mu,h}^5| \leq C|\lambda|\,|Z_{\mu,\mathrm{bare}}^5|, \tag{3.117}$$

for any $h \geq h_*$, uniformly in h.

In the *second regime*, the flow of $Z_{\mu,h}^5$ is controlled by a flow equation of the same form as (3.113), where $\beta_{\mu,h}^5$ is bounded in the same way as $W_{1,0,1,0}^{(h);r}$, see the paragraph after (3.102):

$$|\beta_{\mu,h}^5| \leq C|\lambda|\gamma^{\theta(h-h_*)} \sup_{h \leq k \leq h_*} |Z_{\mu,k}^5|. \tag{3.118}$$

Using this bound on the beta function and (3.117), we readily find that

$$|Z_{\mu,h}^5 - Z_{\mu,\text{bare}}^5| \leq C|\lambda| \, |Z_{\mu,\text{bare}}|, \tag{3.119}$$

for any $h \leq 0$, that $Z_\mu^5 := \lim_{h \to -\infty} Z_{\mu,h}^5$ exists and is analytically close to $Z_{\mu,\text{bare}}^5$, and that the limit is reached at an exponential rate: $|Z_{\mu,h}^5 - Z_\mu^5| \leq C|\lambda| \, |Z_\mu^5|\gamma^{\theta(h-h_*)}$. Note that these facts imply that the relation between Z_μ^5 and $Z_{\mu,\text{bare}}^5$ can be inverted into $Z_{\mu,\text{bare}}^5 = (1 + O(\lambda))Z_\mu^5$, thus allowing us to fix the dressed chiral renormalizations as desired. In particular, in the following we will need $Z_\mu^5 \equiv Z_\mu$; therefore, we will fix the bare chiral parameters in such a way that this condition is satisfied.

3.5. Asymptotic infrared behaviour of the correlation functions.
In this section we use the RG construction described in the previous sections to study the $\beta, L \to \infty$ limit of the correlation functions in Equations (3.30) to (3.33). Our goal is to isolate the dominant part, at large distances and/or momenta close to the Weyl nodes, which has an explicit structure in terms of the dressed parameters

$$Z := \lim_{h \to -\infty} Z_h, \quad v_l := \lim_{h \to -\infty} v_{l,h}, \quad Z_\mu := \lim_{h \to -\infty} Z_{\mu,h}, \quad Z_\mu^5 := \lim_{h \to -\infty} Z_{\mu,h}^5, \tag{3.120}$$

plus a subdominant part, which has better regularity properties in momentum space.

The quadratic response coefficient to the chiral current. We start by analyzing $\hat{\Pi}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2)$, which we rewrite as

$$\hat{\Pi}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2) = \hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2) + \hat{\Pi}_{\mu,\nu,\sigma}^{5;r}(\mathbf{p}_1, \mathbf{p}_2), \tag{3.121}$$

where the first term in the right side is, by definition, the sum of the dominant parts of the single-scale contributions to the generating function $\widetilde{\mathcal{W}}^{(h)}(A, A_5)$ of order $m_1 = 2$ in A and $m_2 = 1$ in A^5, with $h < h_*$, in the $\beta, L \to \infty$ limit. More precisely, using the same convention of Fourier transform as in (3.33),

$$\hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2) := \sum_{h < h_*} \hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;d}(\mathbf{p}_1, \mathbf{p}_2), \tag{3.122}$$

where $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;d}(\mathbf{p}_1, \mathbf{p}_2) = \lim_{\beta,L \to \infty} \hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h);d}(\mathbf{p}_1, \mathbf{p}_2)$, and $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;d}(\mathbf{p}_1, \mathbf{p}_2)$ collects the contributions from GN trees with one endpoint of type $A^5\psi^+\psi^-$ and two endpoints of type $A\psi^+\psi^-$, all on scales smaller than h_*. In terms of the decomposition (2.49), the Schwinger terms are all included in the remainder term $\hat{\Pi}_{\mu,\nu,\sigma}^{5;r}(\mathbf{p}_1, \mathbf{p}_2)$, while $\hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2)$ corresponds to the 'infrared contribution' (i.e., the dominant one from the scales smaller than h_*) to $\langle \mathbf{T} \, \hat{J}_{\mu,\mathbf{p}}^5 ; \hat{J}_{\nu,\mathbf{p}_1} ; \hat{J}_{\sigma,\mathbf{p}_2} \rangle_\infty$.

Thanks to the bound (3.102) (or, better to its analogue with $n = 0$) and to the boundedness of $\mathfrak{Z} := \sup_{k \leq 0} \mathfrak{Z}_k$ and of $\mathfrak{Z}^5 := \sup_{k \leq 0} \mathfrak{Z}_k^5$, proved in the previous section, we see that the h-th term in the sum in the right side of (3.122) is bounded by $C\mathfrak{Z}^5(\mathfrak{Z})^2\gamma^h/|v_3^0|$. Therefore, the sum in the right side of (3.122) is absolutely convergent. Moreover, each term in the sum is continuous with respect to $\mathbf{p}_1, \mathbf{p}_2$, uniformly in h. Hence, (3.122) is continuous as a function of the external momenta.

Note, however, that the bound (3.102) does not imply *differentiability* of (3.122) in the external momenta. In fact, it implies that the derivatives in $\mathbf{p}_1$ or $\mathbf{p}_2$ of $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;d}(\mathbf{p}_1, \mathbf{p}_2)$ are bounded by $C\mathfrak{Z}^5(\mathfrak{Z})^2|v_3^0|^{-1}$, which is clearly non summable over h.

Let us now consider the remainder term,

$$\hat{\Pi}_{\mu,\nu,\sigma}^{5;r}(\mathbf{p}_1, \mathbf{p}_2) := \sum_{h < h_*} \hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;r}(\mathbf{p}_1, \mathbf{p}_2) + \sum_{h \geq h_*} \hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty}(\mathbf{p}_1, \mathbf{p}_2). \quad (3.123)$$

The summands and their derivatives are bounded via (3.74), (3.102), and its improved version for the remainder term, discussed after (3.102). These bounds imply that (3.123) is not only continuous in the external momenta, but also differentiable: in fact, they imply that the derivatives in $\mathbf{p}_1$ or $\mathbf{p}_2$ of $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;r}(\mathbf{p}_1, \mathbf{p}_2)$ with $h < h_*$ are bounded by $C_\theta\mathfrak{Z}^5(\mathfrak{Z})^2|v_3^0|^{-1}\gamma^{\theta(h-h_*)}$, and that those of $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty}(\mathbf{p}_1, \mathbf{p}_2)$ with $h \geq h_*$ are bounded by $C\mathfrak{Z}^5(\mathfrak{Z})^2\gamma^{-h/2}$. Therefore, the derivatives in $\mathbf{p}_1$ or $\mathbf{p}_2$ of (3.123) are finite, and bounded by $C\mathfrak{Z}^5(\mathfrak{Z})^2|v_3^0|^{-1}$. Moreover, they are continuous in a sufficiently small neighbourhood of the origin; more precisely, the bound proportional to $\gamma^{\theta(h-h_*)}$ on the derivatives of $\hat{W}_{0,2,1;\mu,\nu,\sigma}^{(h),\infty;r}(\mathbf{p}_1, \mathbf{p}_2)$ with $h < h_*$ implies that the derivatives in $\mathbf{p}_1, \mathbf{p}_2$ of (3.123) are Hölder continuous of exponent $\theta > 0$, for any $\theta < 1$.

Let us now go back to $\hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2)$, which we want to further decompose into an explicitly computable contribution plus an additional differentiable remainder. We write:

$$\hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2) = \hat{\Pi}_{\mu,\nu,\sigma}^{5;rel}(\mathbf{p}_1, \mathbf{p}_2) + \hat{\Pi}_{\mu,\nu,\sigma}^{5;nr}(\mathbf{p}_1, \mathbf{p}_2), \quad (3.124)$$

where the label 'rel' stands for relativistic and 'nr' for non-relativistic. The first term in the right side is defined via the same GN tree expansion as the one for $\hat{\Pi}_{\mu,\nu,\sigma}^{5;d}(\mathbf{p}_1, \mathbf{p}_2)$, with the difference that the values of the GN trees are computed by replacing:

(i) the single scale propagators $\hat{g}_\omega^{(h)}(\mathbf{k}')$ in (3.99) by its asymptotic, relativistic, counterpart, namely by

$$\hat{g}_\omega^{(h);rel}(\mathbf{k}') = \frac{f_{h,\omega}(\mathbf{k}')}{Z} \begin{pmatrix} -ik_0' + v_3\omega k_3' & v_1 k_1' - iv_2 k_2' \\ v_1 k_1' + iv_2 k_2' & -ik_0' - v_3\omega k_3' \end{pmatrix}^{-1}, \quad (3.125)$$

with Z, v_l as in (3.120);

(ii) the parameters $Z_{\mu,h}$ and $Z_{\mu,h}^5$ associated with the endpoints of type $A_\mu\psi^+\psi^-$ and $A_\mu^5\psi^+\psi^-$, respectively, by their $h \to -\infty$ limits Z_μ and Z_μ^5, respectively, see (3.120).

From the bounds on $Z_h, v_{l,h}, Z_{\mu,h}, Z_{\mu,h}^5$ derived in Sects. 3.3.2 and 3.4, we know that

$$|Z_h - Z| \leq C|\lambda|\gamma^{\theta h}, \quad |v_{l,h} - v_l| \leq C|\lambda||v_l|\gamma^{\theta h}, \quad (3.126)$$

$$|Z_{\mu,h} - Z_\mu| \leq C|\lambda||Z_\mu|\gamma^{\theta(h-h_*)}, \quad |Z_{\mu,h}^5 - Z_\mu^5| \leq C|\lambda||Z_\mu^5|\gamma^{\theta(h-h_*)}, \quad (3.127)$$

from which it follows that the difference between $\hat{g}_\omega^{(h)}(\mathbf{k}')$ and $\hat{g}_\omega^{(h);\mathrm{rel}}(\mathbf{k}')$ admits an improved dimensional bound with an extra factor $\gamma^{\theta h}$, as compared with the bounds (3.100). This, combined with the bounds in (3.127), implies that the non-relativistic term $\hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{nr}}(\mathbf{p}_1, \mathbf{p}_2)$ in the right side of (3.124) is a sum of single-scale contributions, each of which is bounded by $C_\theta 3^5 (3)^2 |v_3^0|^{-1} \gamma^h \gamma^{\theta(h-h_*)}$, and whose derivatives with respect to $\mathbf{p}_1$ or $\mathbf{p}_2$ are bounded by $C_\theta 3^5 (3)^2 |v_3^0|^{-1} \gamma^{\theta(h-h_*)}$. Hence, by summing over $h < h_*$, we find that the non-relativistic term in the right side of (3.124) is differentiable in $\mathbf{p}_1, \mathbf{p}_2$, with θ-Hölder continuous derivatives, in a sufficiently small neighbourhood of $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{0}$.

In conclusion, we can rewrite the response function $\hat{\Pi}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2)$ as:

$$\hat{\Pi}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2) = \hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{rel}}(\mathbf{p}_1, \mathbf{p}_2) + \hat{H}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2), \tag{3.128}$$

where $\hat{H}_{\mu,\nu,\sigma}^5(\mathbf{p}_1, \mathbf{p}_2) = \hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{r}}(\mathbf{p}_1, \mathbf{p}_2) + \hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{nr}}(\mathbf{p}_1, \mathbf{p}_2)$, which is differentiable with respect to the external momenta, with derivatives that are Hölder continuous of exponent θ, for any $0 < \theta < 1$, in a sufficiently small neighbourhood of the origin. Instead, the first term coincides with the contribution due to a *non-interacting* relativistic model, with dressed parameters (equal to the limiting parameters $Z_\mu, v_l, Z_\mu, Z_\mu^5$) and an ultraviolet cutoff on scale h_*. Eq. (3.128) is almost the desired representation for the quadratic response, see (3.1). However, in order to prove (3.1) starting from (3.128) we need to compute the relativistic term $\hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{rel}}(\mathbf{p}_1, \mathbf{p}_2)$ more explicitly, which will be done in the next section.

The dressed propagator and the vertex functions. We now consider the dressed propagator $\hat{S}_2(\mathbf{k})$ and the vertex functions $\hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p})$ and $\hat{\Gamma}_\mu^5(\mathbf{k}, \mathbf{p})$, which are obtained by an RG construction analogous to the one described in Sect. 3.3, in the presence of the Grassmann source field ϕ, see (3.26) and (3.28). We decompose $\hat{S}_2(\mathbf{k})$ (resp. $\hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p})$, resp. $\hat{\Gamma}_\mu^5(\mathbf{k}, \mathbf{p})$) in a way analogous to (3.121): that is, we distinguish a dominant part, analogous to (3.122), which consists of the sum of the single-scale contributions from the GN trees with one endpoint of type $\phi^+\psi^-$ and one of type $\psi^+\phi^-$ (resp. one endpoint of type $A\psi^+\psi^-$, one of type $\phi^+\psi^-$, one of type $\psi^+\phi^-$, resp. one endpoint of type $A^5\psi^+\psi^-$, one of type $\phi^+\psi^-$, one of type $\psi^+\phi^-$), all on scales smaller than h_*, from a remainder term, analogous to (3.123), which includes all the other contributions (those from GN trees with root on scale smaller than h_* and at least one endpoint on scale $h_* + 1$, and those from GN with root on scale h_* or larger). The dominant term is further decomposed in analogy with (3.124) into a 'relativistic' contribution, obtained via the substitutions spelled in items (i) and (ii) after (3.124), plus an additional remainder. The relativistic contribution is explicit: for the dressed propagator with $\mathbf{k}$ close to $\mathbf{p}_F^\omega$, letting $\mathbf{k}' = \mathbf{k} - \mathbf{p}_F^\omega$, it is equal to

$$\hat{S}_2^{\mathrm{rel}}(\mathbf{k}) = \sum_{h < h_*} \hat{g}_\omega^{(h);\mathrm{rel}}(\mathbf{k}') \equiv \chi_{h_*-1,\omega}(\mathbf{k}') \hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}'), \tag{3.129}$$

where $\chi_{h,\omega} = \sum_{k \leq h} f_{h,\omega}$, and $\hat{g}_\omega^{\mathrm{rel}}(\mathbf{k})$ is defined in the same way as (3.125), with $f_{h,\omega}(\mathbf{k})$ replaced by 1, that is,

$$\hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}') = \frac{1}{Z} \left(\sum_{\mu=0}^3 v_\mu k_\mu' \sigma_{\mu,\omega} \right)^{-1}, \tag{3.130}$$

where $v_0 = 1$ and $\sigma_{\mu,\omega}$ was defined in (3.89) (note, a posteriori, that the cutoff function $\chi_{h_*-1,\omega}$ can be dropped from the right side of (3.129) for $\mathbf{k}'$ small enough, because $\chi_{h_*-1,\omega} \equiv 1$ in a sufficiently small neighbourhood of the origin). Similarly, for the vertex functions with $\mathbf{k}$ close to $\mathbf{p}_F^\omega$, letting again $\mathbf{k}' = \mathbf{k} - \mathbf{p}_F$, the relativistic contribution reads

$$\hat{\Gamma}_\mu^{\mathrm{rel}}(\mathbf{k}+\mathbf{p}) = \hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}'+\mathbf{p}) Z_\mu \sigma_{\mu,\omega} \hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}'), \tag{3.131}$$

$$\hat{\Gamma}_\mu^{5;\mathrm{rel}}(\mathbf{k}+\mathbf{p}) = \hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}'+\mathbf{p}) Z_\mu^5 \omega \sigma_{\mu,\omega} \hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}'). \tag{3.132}$$

On the other hand, the remainder terms (the analogue of (3.123) and the analogue of the non-relativistic term in (3.124)) admit an improved dimensional bound: if $\mathbf{k} - \mathbf{p}_F^\omega$, $\mathbf{p}$ and $\mathbf{k} + \mathbf{p} - \mathbf{p}_F^\omega$ are all small and of the order γ^h, the remainder terms have an additional factor $\gamma^{\theta h}$, as compared to the corresponding dominant, relativistic, term.

Summarizing, letting $\mathbf{k}' = \mathbf{k} - \mathbf{p}_F^\omega$ and assuming that $|\mathbf{k}'|, |\mathbf{k}' + \mathbf{p}|, |\mathbf{p}|$ are all in $[c\gamma^h, C\gamma^h]$, for some $h < h_*$ and $c, C > 0$, we find that $\hat{S}_2(\mathbf{k}' + \mathbf{p}_F^\omega)$ satisfies (2.27), with $\|R_\omega(\mathbf{k}')\| \leq C_{\theta,h_*} \gamma^{\theta h}$, while

$$\hat{\Gamma}_\mu(\mathbf{k}, \mathbf{p}) = Z_\mu \hat{S}_2(\mathbf{k}+\mathbf{p}) \sigma_{\mu,\omega} \hat{S}_2(\mathbf{k})(1 + \hat{R}_\mu(\mathbf{k}, \mathbf{p})), \tag{3.133}$$

$$\hat{\Gamma}_\mu^5(\mathbf{k}, \mathbf{p}) = Z_\mu^5 \hat{S}_2(\mathbf{k}+\mathbf{p}) \omega \sigma_{\mu,\omega} \hat{S}_2(\mathbf{k})(1 + \hat{R}_\mu^5(\mathbf{k}, \mathbf{p})), \tag{3.134}$$

with $\|\hat{R}_\mu(\mathbf{k}, \mathbf{p})\|, \|R_\mu^5(\mathbf{k}, \mathbf{p})\| \leq C_{\theta,h_*} \gamma^{\theta h}$.

By plugging (3.133) and (2.27) into the vertex Ward Identity (3.3), we obtain Eq. (3.4), that is, $Z_\mu = v_\mu Z$. As already mentioned, this identity says that the charge carried by the electromagnetic current is not renormalized by the interaction.

There is no analogue of the vertex Ward Identity for the chiral vertex $\hat{\Gamma}_\mu^5(\mathbf{k}, \mathbf{p})$. Therefore, the condition (2.46) must be enforced by properly fixing $Z_{\mu,\mathrm{bare}}^5$. By using (3.133) and (3.134), we find that the left side of (2.46) equals $(Z_\mu^5/Z_\mu)\omega \mathbb{1}_2$, so that (2.46) reduces to (3.5), $Z_\mu^5 = Z_\mu$. This condition is enforced by fixing the bare parameters $Z_{\mu,\mathrm{bare}}^5$ in the way discussed at the end of Sect. 3.4.3.

3.6. The relativistic contribution to the quadratic response $\hat{\Pi}_{\mu,\nu,\sigma}^5$.

In this section we first prove (3.1), starting from (3.128). Next, we prove (3.6)–(3.7). For this purpose, we need to compute the relativistic contribution to the quadratic response $\hat{\Pi}_{\mu,\nu,\sigma}^5$ more explicitly. The definitions imply that, for $\mathbf{p}_1, \mathbf{p}_2$ sufficiently close to $\mathbf{0}$, recalling (3.129) and (3.130),

$$\hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{rel}}(\mathbf{p}_1, \mathbf{p}_2) = -Z_\mu^5 Z_\nu Z_\sigma \sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \chi_{h_*-1,\omega}(\mathbf{k}) \, \chi_{h_*-1,\omega}(\mathbf{k}+\mathbf{p}_1) \chi_{h_*-1,\omega}(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)$$

$$\cdot \mathrm{Tr}\{\hat{g}_\omega^{\mathrm{rel}}(\mathbf{k})\omega\sigma_{\mu,\omega}\hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)\sigma_{\sigma,\omega}\hat{g}_\omega^{\mathrm{rel}}(\mathbf{k}+\mathbf{p}_1)\sigma_{\nu,\omega}\} + \left[(\mathbf{p}_1, \nu) \leftrightarrow (\mathbf{p}_2, \sigma)\right]. \tag{3.135}$$

For the purpose of computing the dominant contribution to $\hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{rel}}(\mathbf{p}_1, \mathbf{p}_2)$, we can freely replace $\chi_{h_*-1,\omega}(\mathbf{k})$ by $\chi(\gamma^{-h_*}\|\mathbf{k}\|_v)$, where $\|\mathbf{k}\|_v^2 = \sum_\mu v_\mu^2 k_\mu^2$. In fact, if we denote by $\hat{\Pi}_{\mu,\nu,\sigma}^{5;*}(\mathbf{p}_1, \mathbf{p}_2)$ the analogue of the right side of (3.135) with $\chi_{h_*-1,\omega}(\mathbf{k})$ replaced by $\chi(\gamma^{-h_*}\|\mathbf{k}\|_v)$, it is easy to check that the difference $\hat{\Pi}_{\mu,\nu,\sigma}^{5;\mathrm{rel}}(\mathbf{p}_1, \mathbf{p}_2) - \hat{\Pi}_{\mu,\nu,\sigma}^{5;*}(\mathbf{p}_1, \mathbf{p}_2)$ is continuously differentiable in $\mathbf{p}_1, \mathbf{p}_2$ in a small neighbourhood of

the origin. After rescaling $k_\mu \to k_\mu/v_\mu$, we find, letting $\chi_*(\mathbf{k}) := \chi(\gamma^{-h_*}|\mathbf{k}|)$, $\overline{\mathbf{p}}_1 = (p_{1,0}, v_1 p_{1,1}, v_2 p_{1,2}, v_3 p_{1,3})$, and similarly for $\overline{\mathbf{p}}_2$,

$$\hat{\Pi}^{5;*}_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{Z_\mu^5 Z_\nu Z_\sigma}{Z^3 v_1 v_2 v_3} \int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi_*(\mathbf{k})\,\chi_*(\mathbf{k}+\overline{\mathbf{p}}_1)\chi_*(\mathbf{k}+\overline{\mathbf{p}}_1+\overline{\mathbf{p}}_2)}{|\mathbf{k}|^2\,|\mathbf{k}+\overline{\mathbf{p}}_1|^2\,|\mathbf{k}+\overline{\mathbf{p}}_1+\overline{\mathbf{p}}_2|^2} \cdot \tag{3.136}$$

$$\cdot \sum_{\mu_1,\mu_2,\mu_3=0}^{3} k_{\mu_1}(k_{\mu_2}+\overline{p}_{1,\mu_2}+\overline{p}_{2,\mu_2})(k_{\mu_3}+\overline{p}_{1,\mu_3}) \sum_{\omega=\pm} \mathrm{Tr}\{\sigma^\dagger_{\mu_1,\omega}(\omega\sigma_{\mu,\omega})\sigma^\dagger_{\mu_2,\omega}\sigma_{\sigma,\omega}\sigma^\dagger_{\mu_3,\omega}\sigma_{\nu,\omega}\}$$

$$+\Big[(\mathbf{p}_1,\nu) \leftrightarrow (\mathbf{p}_2,\sigma)\Big],$$

The sum over ω of the trace of $\{\sigma^\dagger_{\mu_1,\omega}(\omega\sigma_{\mu,\omega})\sigma^\dagger_{\mu_2,\omega}\sigma_{\sigma,\omega}\sigma^\dagger_{\mu_3,\omega}\sigma_{\nu,\omega}\}$ can be conveniently written in terms of four-dimensional Euclidean gamma matrices. In fact, after performing the trace-preserving transformations

$$\sigma_{\mu,+} \to i\sigma_{\mu,+} = \begin{cases} \mathbb{1}_2 & \text{if } \mu = 0 \\ i\sigma_\mu & \text{if } \mu > 0 \end{cases} \quad \text{and} \quad \sigma_{\mu,-} \to i\sigma_3\sigma_{\mu,-}\sigma_3 = \begin{cases} \mathbb{1}_2 & \text{if } \mu = 0 \\ -i\sigma_\mu & \text{if } \mu > 0 \end{cases},$$

we recognize that

$$\sum_{\omega=\pm} \mathrm{Tr}\{\sigma^\dagger_{\mu_1,\omega}(\omega\sigma_{\mu,\omega})\sigma^\dagger_{\mu_2,\omega}\sigma_{\sigma,\omega}\sigma^\dagger_{\mu_3,\omega}\sigma_{\nu,\omega}\} = -\mathrm{Tr}\{\gamma_{\mu_1}\gamma_\mu\gamma_5\gamma_{\mu_2}\gamma_\sigma\gamma_{\mu_3}\gamma_\nu\}, \tag{3.137}$$

where

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & i\sigma_j \\ -i\sigma_j & 0 \end{pmatrix} \quad \text{for} \quad j = 1, 2, 3, \tag{3.138}$$

and

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{3.139}$$

Note that the gamma matrices are Hermitian and satisfy the anticommutation rules $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$, for all $\mu, \nu \in \{0, 1, 2, 3, 5\}$. In conclusion,

$$\hat{\Pi}^{5;\mathrm{rel}}_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{Z_\mu^5 Z_\nu Z_\sigma}{Z^3 v_1 v_2 v_3} I_{\mu,\nu,\sigma}(\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2) + R^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2), \tag{3.140}$$

where $R^5_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ is smooth in $\mathbf{p}_1, \mathbf{p}_2$ in a sufficiently small neighbourhood of the origin, and, letting $\not{k} := \sum_{\mu=0}^{3} \gamma_\mu k_\mu$,

$$I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\Big\{\frac{\chi_*(\mathbf{k})}{\not{k}}\gamma_\mu\gamma_5\frac{\chi_*(\mathbf{k}+\mathbf{p}_1+\mathbf{p}_2)}{\not{k}+\not{p}_1+\not{p}_2}\gamma_\sigma\frac{\chi_*(\mathbf{k}+\mathbf{p}_1)}{\not{k}+\not{p}_1}\gamma_\nu\Big\} + \Big[(\mathbf{p}_1,\nu) \leftrightarrow (\mathbf{p}_2,\sigma)\Big],$$
$$\tag{3.141}$$

which is nothing but the chiral triangle graph of QED_4 in the presence of an ultraviolet cutoff χ_*. By plugging (3.140) into (3.128), we get (3.1), with $\widetilde{H}^5_{\mu,\nu,\sigma} = \hat{H}^5_{\mu,\nu,\sigma} + R^5_{\mu,\nu,\sigma}$.

3.6.1. The chiral triangle graph with momentum cutoff Let us now prove (3.6)–(3.7). For ease of notation, in the present subsection we drop the label $*$ from χ_*, and denote it simply by χ. We start from (3.6), whose left side equals, letting $\mathbf{q} := \mathbf{p}_1 + \mathbf{p}_2$,

$$\sum_{\mu=0}^{3} q_\mu I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) \tag{3.142}$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \slashed{q} \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\nu \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\} + [(\nu, \mathbf{p}_1) \longleftrightarrow (\sigma, \mathbf{p}_2)]$$

$$\equiv T_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) + T_{\sigma\nu}(\mathbf{p}_2, \mathbf{p}_1). \tag{3.143}$$

Using the anti-commutativity of the fifth gamma matrix,

$$T_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \slashed{q} \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\nu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\}. \tag{3.144}$$

Now we use the following rewriting:

$$\frac{\chi(\mathbf{k})}{\slashed{k}} \slashed{q} \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} = \left[\frac{\chi(\mathbf{k})}{\slashed{k}} - \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \right] + \frac{\chi(\mathbf{k})}{\slashed{k}} C(\mathbf{k}, \mathbf{q}) \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}}, \tag{3.145}$$

where, for $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ in the support of $\chi(\mathbf{k})\chi(\mathbf{k}+\mathbf{q})$:

$$C(\mathbf{k}, \mathbf{q}) = \slashed{k}(\chi^{-1}(\mathbf{k}) - 1) - (\slashed{k}+\slashed{q})(\chi^{-1}(\mathbf{k}+\mathbf{q}) - 1). \tag{3.146}$$

The contribution to $T_{\mu\nu}(\mathbf{p}_1, \mathbf{p}_2)$ due to the difference in square brackets in (3.145) is:

$$\int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \gamma_\nu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\} - \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\nu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\}. \tag{3.147}$$

Changing integration variable $\mathbf{k} \to \mathbf{k} - \mathbf{p}_2$ in the second integral, and using the cyclicity of the trace we get:

$$(3.147) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \gamma_\nu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\} - \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \gamma_\sigma \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\slashed{k}+\slashed{p}_1} \gamma_\nu \gamma_5 \right\}, \tag{3.148}$$

which gives zero contribution to (3.142). Therefore,

$$\sum_{\mu=0}^{3} q_\mu I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = T_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) + T_{\sigma\nu}^C(\mathbf{p}_2, \mathbf{p}_1), \tag{3.149}$$

with

$$T_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) := \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\chi(\mathbf{k})}{\slashed{k}} C(\mathbf{k}, \mathbf{q}) \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\nu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\}.$$

Using (3.146) we find

$$T_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \left[(1 - \chi(\mathbf{k})) \chi(\mathbf{k}+\mathbf{q}) \chi(\mathbf{k}+\mathbf{p}_2) \mathrm{Tr} \left\{ \frac{1}{\slashed{k}+\slashed{q}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \right\} \right.$$

$$- (1 - \chi(\mathbf{k} + \mathbf{q})) \chi(\mathbf{k}) \chi(\mathbf{k} + \mathbf{p}_2) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k} + \slashed{p}_2} \gamma_\sigma \Big\} \Big].$$

We now shift the integration variable $\mathbf{k} \to \mathbf{k} - \mathbf{p}_2$ in the term in the first line and use the cyclicity of the trace, thus finding

$$T_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \Big[(1 - \chi(\mathbf{k} - \mathbf{p}_2)) \chi(\mathbf{k} + \mathbf{p}_1) \chi(\mathbf{k}) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\sigma \gamma_5 \frac{1}{\slashed{k} + \slashed{p}_1} \gamma_\nu \Big\}$$

$$- (1 - \chi(\mathbf{k} + \mathbf{q})) \chi(\mathbf{k}) \chi(\mathbf{k} + \mathbf{p}_2) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k} + \slashed{p}_2} \gamma_\sigma \Big\} \Big], \quad (3.150)$$

so that, after plugging back this expression into (3.149) and exchanging names $(\mathbf{p}_1, \nu) \longleftrightarrow (\mathbf{p}_2, \sigma)$ in one of the terms contributing to $T_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)$, we get

$$\sum_{\mu=0}^{3} q_\mu I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) + \tilde{T}_{\sigma\nu}^C(\mathbf{p}_2, \mathbf{p}_1), \quad (3.151)$$

with

$$\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2) := \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}) \chi(\mathbf{k} + \mathbf{p}_2) (\chi(\mathbf{k} + \mathbf{q}) - \chi(\mathbf{k} - \mathbf{p}_1)) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k} + \slashed{p}_2} \gamma_\sigma \Big\}.$$

$$(3.152)$$

Note that $\tilde{T}_{\nu\sigma}^C(\mathbf{0}, \mathbf{0}) = 0$. Let us now expand $\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)$ in Taylor series around $(\mathbf{p}_1, \mathbf{p}_2) = (\mathbf{0}, \mathbf{0})$ and let us focus on the terms of order 1 and 2 in the momenta, to be denoted by $[\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)]^{(1)}$ and $[\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)]^{(2)}$, respectively. It is easy to check that, for $P = \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\}$ sufficiently small, as compared to γ^{h_*}, the Taylor remainder of order 3 is bounded by[13] $C\gamma^{-h_*} P^3$. Moreover, it is straightforward to check that $[\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)]^{(1)} = 0$, by parity. Let us now consider $[\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)]^{(2)}$, which consists of several terms:

$$[\tilde{T}_{\nu\sigma}^C(\mathbf{p}_1, \mathbf{p}_2)]^{(2)} = A_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) + B_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) + C_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) + D_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2),$$

$$(3.153)$$

where, using the convention that repeated indices are summed from 0 to 3,

$$A_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2} (q_\mu q_\lambda - p_{1,\mu} p_{1,\lambda}) \int \frac{d\mathbf{k}}{(2\pi)^4} \chi^2(\mathbf{k}) \partial_\mu \partial_\lambda \chi(\mathbf{k}) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k}} \gamma_\sigma \Big\}$$

$$B_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = p_{2,\mu} (q_\lambda + p_{1,\lambda}) \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}) \partial_\mu \chi(\mathbf{k}) \partial_\lambda \chi(\mathbf{k}) \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k}} \gamma_\sigma \Big\}$$

$$C_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = (q_\mu + p_{1,\mu}) \int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi^2(\mathbf{k}) \partial_\mu \chi(\mathbf{k})}{|\mathbf{k}|^2} \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \slashed{p}_2 \gamma_\sigma \Big\}$$

$$D_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -2(q_\mu + p_{1,\mu}) p_{2,\lambda} \int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi^2(\mathbf{k}) \partial_\mu \chi(\mathbf{k}) k_\lambda}{|\mathbf{k}|^2} \mathrm{Tr} \Big\{ \frac{1}{\slashed{k}} \gamma_\nu \gamma_5 \frac{1}{\slashed{k}} \gamma_\sigma \Big\}.$$

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[13] In order to prove this, we bound the Taylor remainder of order 3 by $P^3 D_3$, with D_3 an upper bound on the third derivative with respect to $\mathbf{p}_1, \mathbf{p}_2$ of the right side of (3.161). Next, we note that, due to the structure of the right side of (3.152), at least one of such derivatives acts on one of the cutoff functions χ; therefore, for $|\mathbf{p}_1|, |\mathbf{p}_2|$ small enough (as compared with the support of χ), D_3 can be dimensionally bounded by $(\text{const.}) \int_{S_*} d\mathbf{k} |\mathbf{k}|^{-5}$, where $S_* = \{\mathbf{k} : c\gamma^{h_*} \le |\mathbf{k}| \le C\gamma^{-h_*}\}$.

Now, $A_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = B_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = D_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = 0$ by simple parity reasons: in fact, after the computation of the trace, letting χ' be the derivative of χ with respect to $|\mathbf{k}|$, $\hat{k}_\mu = k_\mu/|\mathbf{k}|$, and $\epsilon_{\alpha\nu\beta\sigma}$ the Levi-Civita symbol (see footnote 8),

$$A_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = 2(q_\mu q_\lambda - p_{1,\mu} p_{1,\lambda})\varepsilon_{\alpha\nu\beta\sigma} \int \frac{d\mathbf{k}}{(2\pi)^4} \chi^2(\mathbf{k})\big[\chi''(\mathbf{k})\hat{k}_\mu \hat{k}_\lambda$$

$$+\frac{\chi'(\mathbf{k})}{|\mathbf{k}|}(\delta_{\mu\lambda} - \hat{k}_\mu\hat{k}_\lambda)\big]\frac{k_\alpha k_\beta}{|\mathbf{k}|^4},$$

$$B_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = 4p_{2,\mu}(q_\lambda + p_{1,\lambda})\varepsilon_{\alpha\nu\beta\sigma} \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k})(\chi'(\mathbf{k}))^2\hat{k}_\mu\hat{k}_\lambda \frac{k_\alpha k_\beta}{|\mathbf{k}|^4},$$

$$D_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -8(q_\mu + p_{1,\mu})p_{2,\lambda}\varepsilon_{\alpha\nu\beta\sigma} \int \frac{d\mathbf{k}}{(2\pi)^4} \chi^2(\mathbf{k})\chi'(\mathbf{k}) \frac{\hat{k}_\mu k_\lambda k_\alpha k_\beta}{|\mathbf{k}|^6},$$

which are all zero by the anti-symmetry in $\alpha \longleftrightarrow \beta$. The only non-trivial term we are left with is

$$C_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = 4(q_\mu + p_{1,\mu})p_{2,\beta}\varepsilon_{\alpha\nu\beta\sigma} \int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi^2(\mathbf{k})\chi'(\mathbf{k})\hat{k}_\mu k_\alpha}{|\mathbf{k}|^4}, \qquad (3.154)$$

where, using that the angular integration in the 4D integral over $\mathbf{k}$ gives $2\pi^2$, we can rewrite

$$\int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi^2(\mathbf{k})\chi'(\mathbf{k})\hat{k}_\mu k_\alpha}{|\mathbf{k}|^4} = \frac{\delta_{\mu\alpha}}{4} \int \frac{d\mathbf{k}}{(2\pi)^4} \frac{\chi^2(\mathbf{k})\chi'(\mathbf{k})}{|\mathbf{k}|^3} = \frac{\delta_{\mu\alpha}}{4} \frac{1}{(2\pi)^4} 2\pi^2 \Big(-\frac{1}{3}\Big) = -\frac{\delta_{\mu\alpha}}{96\pi^2}$$

where in the second equality we used that $\int_0^\infty \chi^2(\rho)\chi'(\rho)d\rho = -1/3$, where, with some abuse of notation, we denoted $\chi(|\mathbf{k}|) \equiv \chi(\mathbf{k})$. Plugging this back into (3.154) and using again the anti-symmetry in $\alpha \longleftrightarrow \beta$, we find (recalling that $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2$)

$$C_{\nu\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{1}{12\pi^2} p_{1,\alpha} p_{2,\beta}\varepsilon_{\alpha\nu\beta\sigma}, \qquad (3.155)$$

so that, putting things together,

$$q_\mu I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{1}{6\pi^2} p_{1,\alpha} p_{2,\beta}\varepsilon_{\alpha\nu\beta\sigma} + O(\gamma^{-h_*}P^3), \qquad (3.156)$$

which proves (3.6) (note that the order of the indices in $\varepsilon_{\alpha\beta\nu\sigma}$ in the right side of (3.6) is different from the one in the right side of (3.156), which explains the different sign).

Let us now prove (3.7). We compute:

$$p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\Big\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \gamma_\mu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \slashed{p}_1 \frac{\chi(\mathbf{k}+\mathbf{p}_2)}{\slashed{k}+\slashed{p}_2} \gamma_\sigma \Big\}$$

$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\Big\{ \frac{\chi(\mathbf{k})}{\slashed{k}} \gamma_\mu \gamma_5 \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\sigma \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\slashed{k}+\slashed{p}_1} \slashed{p}_1 \Big\}. \qquad (3.157)$$

In the first term we rename $\mathbf{k} \to -\mathbf{k} - \mathbf{q}$ and use the cyclicity of the trace; in the second we rename $\mathbf{k} \to -\mathbf{k} - \mathbf{p}_1$, use the cyclicity of the trace and the anti-commutation properties of γ_5; we get:

$$-p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\Big\{ \frac{\chi(\mathbf{k}+\mathbf{q})}{\slashed{k}+\slashed{q}} \gamma_\mu \gamma_5 \frac{\chi(\mathbf{k})}{\slashed{k}} \slashed{p}_1 \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\slashed{k}+\slashed{p}_1} \gamma_\sigma \Big\}$$

$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\left\{ \frac{\chi(\mathbf{k}-\mathbf{p}_2)}{\mathbf{k}-\mathbf{\not{p}}_2} \gamma_\sigma \gamma_5 \frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} \mathbf{\not{p}}_1 \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1} \gamma_\mu \right\}. \tag{3.158}$$

Now, in both integrals we rewrite $\frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} \mathbf{\not{p}}_1 \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1}$ by using (3.145):

$$\frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} \mathbf{\not{p}}_1 \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1} = \left[\frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} - \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1} \right] + \frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} C(\mathbf{k}, \mathbf{p}_1) \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1}. \tag{3.159}$$

It is easy to see that the contribution to $p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)$ coming from the term in square brackets in (3.159) vanishes. Therefore, we are left with

$$-p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\left\{ \frac{\chi(\mathbf{k}+\mathbf{q})}{\mathbf{k}+\mathbf{\not{q}}} \gamma_\mu \gamma_5 \frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} C(\mathbf{k}, \mathbf{p}_1) \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1} \gamma_\sigma \right\}$$

$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \mathrm{Tr}\left\{ \frac{\chi(\mathbf{k}-\mathbf{p}_2)}{\mathbf{k}-\mathbf{\not{p}}_2} \gamma_\sigma \gamma_5 \frac{\chi(\mathbf{k})}{\mathbf{\not{k}}} C(\mathbf{k}, \mathbf{p}_1) \frac{\chi(\mathbf{k}+\mathbf{p}_1)}{\mathbf{k}+\mathbf{\not{p}}_1} \gamma_\mu \right\} \tag{3.160}$$

that, using the explicit form of $C(\mathbf{k}, \mathbf{p}_1)$ becomes:

$$-p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}+\mathbf{q})(1 - \chi(\mathbf{k}))\chi(\mathbf{k}+\mathbf{p}_1) \mathrm{Tr}\left\{ \frac{1}{\mathbf{k}+\mathbf{\not{q}}} \gamma_\mu \gamma_5 \frac{1}{\mathbf{k}+\mathbf{\not{p}}_1} \gamma_\sigma \right\}$$

$$- \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}+\mathbf{q})\chi(\mathbf{k})(1 - \chi(\mathbf{k}+\mathbf{p}_1)) \mathrm{Tr}\left\{ \frac{1}{\mathbf{k}+\mathbf{\not{q}}} \gamma_\mu \gamma_5 \frac{1}{\mathbf{\not{k}}} \gamma_\sigma \right\}$$

$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}-\mathbf{p}_2)(1 - \chi(\mathbf{k}))\chi(\mathbf{k}+\mathbf{p}_1) \mathrm{Tr}\left\{ \frac{1}{\mathbf{k}-\mathbf{\not{p}}_2} \gamma_\sigma \gamma_5 \frac{1}{\mathbf{k}+\mathbf{\not{p}}_1} \gamma_\mu \right\}$$

$$- \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k}-\mathbf{p}_2)\chi(\mathbf{k})(1 - \chi(\mathbf{k}+\mathbf{p}_1)) \mathrm{Tr}\left\{ \frac{1}{\mathbf{k}-\mathbf{\not{p}}_2} \gamma_\sigma \gamma_5 \frac{1}{\mathbf{\not{k}}} \gamma_\mu \right\}.$$

If we now rename $\mathbf{k} \to \mathbf{k} - \mathbf{p}_1$ in the first line, and $\mathbf{k} \to \mathbf{k} + \mathbf{p}_2$ in the third and fourth lines, we can rewrite this as

$$-p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k})\chi(\mathbf{k}+\mathbf{p}_2)(\chi(\mathbf{k}+\mathbf{q}) - \chi(\mathbf{k}-\mathbf{p}_1)) \mathrm{Tr}\left\{ \frac{1}{\mathbf{\not{k}}} \gamma_\sigma \gamma_5 \frac{1}{\mathbf{k}+\mathbf{\not{p}}_2} \gamma_\mu \right\}$$

$$+ \int \frac{d\mathbf{k}}{(2\pi)^4} \chi(\mathbf{k})\chi(\mathbf{k}+\mathbf{q})(\chi(\mathbf{k}+\mathbf{p}_1) - \chi(\mathbf{k}+\mathbf{p}_2)) \mathrm{Tr}\left\{ \frac{1}{\mathbf{\not{k}}} \gamma_\sigma \gamma_5 \frac{1}{\mathbf{k}+\mathbf{\not{q}}} \gamma_\mu \right\}. \tag{3.161}$$

The expression in the right side vanishes at $(\mathbf{p}_1, \mathbf{p}_2) = (\mathbf{0}, \mathbf{0})$. Also in this case, we expand it in Taylor series around the origin and focus on the terms of order 1 and 2 in the momenta, the Taylor remainder of order 3 being smaller than $C\gamma^{-h_*}P^3$, for $P = \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\}$ small enough, see footnote 13. Also in this case, it is straightforward to check that the term of order 1 vanishes, by parity. After having computed the trace, we find that the term of order 2 can be rewritten as

$$p_{1,\nu}[I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2)]^{(2)} = -4\big[(q_\nu + p_{1,\nu})p_{2,\beta} + (p_{1,\nu} - p_{2,\nu})q_\beta\big]\varepsilon_{\alpha\beta\mu\sigma} \int \frac{d\mathbf{k}}{(2\pi)^4} \chi^2(\mathbf{k})\chi'(\mathbf{k}) \frac{\hat{k}_\nu k_\alpha}{|\mathbf{k}|^4}. \tag{3.162}$$

Recalling that $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2$ and computing the integral over $\mathbf{k}$ we finally get:

$$p_{1,\nu} I_{\mu,\nu,\sigma}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{6\pi^2} p_{1,\alpha} p_{2,\beta} \varepsilon_{\alpha\beta\mu\sigma} + O(\gamma^{-h_*}P^3), \tag{3.163}$$

which proves (3.7). As anticipated in (3.1.4), this concludes the proof of Theorem 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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A. Symmetries

In this appendix we first reformulate the symmetries (ii) to (iv) of Sect. 2.2, as well as the Hermitian conjugation symmetry, in terms of the Grassmann variables used in the RG construction of the generating functional of correlations. Next, we discuss the implications of the symmetries, including the proof of Lemmas 3.1 and 3.2.

A.1. Symmetries of the Grassmann action. Consider the Grassmann action $S_0(\psi, A) + \mathcal{V}(\psi) + \sum_\mu (A_\mu^5, j_\mu^5)$, where $\mathcal{V}(\psi)$ was defined in (3.25), (A_μ^5, j_μ^5) in (3.27), and, letting $\int \frac{d\mathbf{k}}{(2\pi)^4}$ is a shorthand for $\frac{1}{\beta L^3} \sum_{\mathbf{k} \in \mathbb{D}_{\beta,L,N}}$,

$$S_0(\psi, \mathbf{A}) := -\int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}_{\mathbf{k}}^+ [\hat{g}_{\beta,L,N}(\mathbf{k})]^{-1} \hat{\psi}_{\mathbf{k}}^- + (A_0, j_0) + (\psi^+, (H_0 - H_0(A))\psi^-),$$
$$(A.1)$$

with $\hat{g}_{\beta,L,N}(\mathbf{k})$ as in (3.20), and $(A_0, j_0) + (\psi^+, H_0 - H_0(A)\psi^-)$ as in (3.27). The reader can easily check that all the terms in Grassmann action are separately invariant under the following symmetries.

(i) *Hermitian conjugation.*

$$\psi_{\mathbf{x}}^\pm \to \pm \psi_{(-x_0,x)}^{\mp,T}, \quad A_\mu(\mathbf{x}) \to (-1)^{\delta_{\mu,0}} A_\mu(-x_0, x), \quad A_\mu^5(\mathbf{x}) \to (-1)^{\delta_{\mu,0}} A_\mu^5(-x_0, x),$$

and $c \to c^*$, where c is a generic numerical constant appearing in the action. In Fourier space, these transformations read: $\hat{\psi}_{\mathbf{k}}^\pm \to \pm \hat{\psi}_{(-k_0,k)}^{\mp,T}$ (or, in the quasi-particle representation, $\hat{\psi}_{\omega,\mathbf{k}}^\pm \to \pm \hat{\psi}_{\omega,(-k_0,k)}^{\mp,T}$), $\hat{A}_{\mu,\mathbf{p}} \to (-1)^{\delta_{\mu,0}} \hat{A}_{\mu,(p_0,-p)}$, $\hat{A}_{\mu,\mathbf{p}}^5 \to (-1)^{\delta_{\mu,0}} \hat{A}_{\mu,(p_0,-p)}^5$, and $c \to c^*$.

(ii) *Inversion symmetry.*

$$\psi_{\mathbf{x}}^{-} \to \sigma_3 \psi_{(x_0,-x)}^{-}, \psi_{\mathbf{x}}^{+} \to \psi_{(x_0,-x)}^{+}\sigma_3,$$
$$A_\mu(\mathbf{x}) \to (-1)^{1-\delta_{\mu,0}} A_\mu(x_0,-x), \qquad A_\mu^5(\mathbf{x}) \to (-1)^{\delta_{\mu,0}} A_\mu^5(x_0,-x).$$

In Fourier space, these transformations read: $\hat{\psi}_{\mathbf{k}}^{-} \to \sigma_3 \hat{\psi}_{(k_0,-k)}^{-}, \hat{\psi}_{\mathbf{k}}^{+} \to \hat{\psi}_{(k_0,-k)}^{+}\sigma_3$ (or, in the quasi-particle representation, $\hat{\psi}_{\omega,\mathbf{k}}^{-} \to \sigma_3 \hat{\psi}_{-\omega,(k_0,-k)}^{-}, \hat{\psi}_{\omega,\mathbf{k}}^{+} \to \hat{\psi}_{-\omega,(k_0,-k)}^{+}\sigma_3$), $\hat{A}_{\mu,\mathbf{p}} \to (-1)^{1-\delta_{\mu,0}} \hat{A}_{\mu,(p_0,-p)}$, and $\hat{A}_{\mu,\mathbf{p}}^5 \to (-1)^{\delta_{\mu,0}} \hat{A}_{\mu,(p_0,-p)}^5$

(iii) *Reflection about a horizontal plane.*

$$\psi_{\mathbf{x}}^{\pm} \to \psi_{(x_0,x_1,x_2,-x_3)}^{\pm}, \qquad A_\mu(\mathbf{x}) \to (-1)^{\delta_{\mu,3}} A_\mu(x_0,x_1,x_2,-x_3),$$
$$A_\mu^5(\mathbf{x}) \to (-1)^{1-\delta_{\mu,3}} A_\mu^5(x_0,x_1,x_2,-x_3).$$

In Fourier space, these transformations read: $\hat{\psi}_{\mathbf{k}}^{\pm} \to \hat{\psi}_{(k_0,k_1,k_2,-k_3)}^{\pm}$ (or, in the quasi-particle representation, $\hat{\psi}_{\omega,\mathbf{k}}^{\pm} \to \hat{\psi}_{-\omega,(k_0,k_1,k_2,-k_3)}^{\pm}$), $\hat{A}_{\mu,\mathbf{p}} \to (-1)^{\delta_{\mu,3}} \hat{A}_{\mu,(p_0,p_1,p_2,-p_3)}$, and $\hat{A}_{\mu,\mathbf{p}}^5 \to (-1)^{1-\delta_{\mu,3}} \hat{A}_{\mu,(p_0,p_1,p_2,-p_3)}^5$.

(iv) *Reflection about a vertical plane + color exchange.*

$$\psi_{\mathbf{x}}^{-} \to -\sigma_1 \psi_{\bar{\mathbf{x}}}^{-}, \psi_{\mathbf{x}}^{+} \to \psi_{\bar{\mathbf{x}}}^{+}\sigma_1,$$
$$A_\mu(\mathbf{x}) \to (-1)^{\delta_{\mu,0}+\delta_{\mu,1}} A_\mu(\bar{\mathbf{x}}), \qquad A_\mu^5(\mathbf{x}) \to (-1)^{\delta_{\mu,0}+\delta_{\mu,1}} A_\mu^5(\bar{\mathbf{x}}),$$

where $\bar{\mathbf{x}} = (-x_0, -x_1, x_2, x_3)$. In Fourier space, these transformations read: $\hat{\psi}_{\mathbf{k}}^{-} \to -\sigma_1 \hat{\psi}_{\bar{\mathbf{k}}}^{-}, \hat{\psi}_{\mathbf{k}}^{+} \to \hat{\psi}_{\bar{\mathbf{k}}}^{+}\sigma_1$ (or, in the quasi-particle representation, $\hat{\psi}_{\omega,\mathbf{k}}^{-} \to -\sigma_1 \hat{\psi}_{\omega,\bar{\mathbf{k}}}^{-}$, $\hat{\psi}_{\omega,\mathbf{k}}^{+} \to \hat{\psi}_{\omega,\bar{\mathbf{k}}}^{+}\sigma_1$), $\hat{A}_{\mu,\mathbf{p}} \to (-1)^{\delta_{\mu,0}+\delta_{\mu,1}} \hat{A}_{\mu,\bar{\mathbf{p}}}$, and $\hat{A}_{\mu,\mathbf{p}}^5 \to (-1)^{\delta_{\mu,0}+\delta_{\mu,1}} \hat{A}_{\mu,\bar{\mathbf{p}}}^5$.

A.2. Consequences of the symmetries. The symmetries listed above are preserved by the multiscale RG construction described in Sect. 3.3. This implies, in particular, that the kernels of the effective potential on scale h are invariant under the symmetries, and their local parts as well. Let us then discuss the implications of the symmetries on the structure of the local parts of the effective action, separately for the two scale regimes $h \le h_*$ and $h > h_*$. We will thus prove the symmetry properties listed in Lemmas 3.1 and 3.2.

A.2.1. The local $\psi^+\psi^-$ term Regime $h \ge h_*$. Let us consider a quadratic term, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}_{\mathbf{k}}^{+} M \hat{\psi}_{\mathbf{k}}^{-}, \tag{A.2}$$

where M is a complex 2×2 matrix. The symmetries (i), (ii), (iii), (iv) imply that:

$$M = M^\dagger = \sigma_3 M \sigma_3 = -\sigma_1 M \sigma_1. \tag{A.3}$$

If we now expand M in the 'Pauli basis', $M = a_0 \mathbb{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, we see that (A.3) implies that $M = a_3\sigma_3$, with $a_3 \in \mathbb{R}$. This proves the first identity in the first line of (3.50).

Regime $h < h_*$. Next, let us consider a quadratic term, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}^+_{\omega,\mathbf{k}} M_\omega \hat{\psi}^-_{\omega,\mathbf{k}}, \tag{A.4}$$

with M_ω a complex 2×2 matrix. Imposing symmetry (iii) we find that $M_\omega = M_{-\omega} \equiv M$. The symmetries (i), (ii), (iii), (iv) imply that:

$$M = M^\dagger = \sigma_3 M \sigma_3 = -\sigma_1 M \sigma_1. \tag{A.5}$$

Writing $M = a_0 \mathbb{1} + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3$, we see that (A.5) implies that $M = a_3 \sigma_3$, with $a_3 \in \mathbb{R}$. This proves the first identity in (3.88).

A.2.2. The local $\psi^+ \partial_\mu \psi^-$ terms Regime $h \geq h_*$. Consider a quadratic term, invariant under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}^+_{\mathbf{k}} k_\mu M_\mu \hat{\psi}^-_{\mathbf{k}}, \tag{A.6}$$

where M_μ are complex 2×2 matrices, with $\mu = 0, 1, 2, 3$. Symmetry (iii) implies that $M_3 = -M_3 = 0$, while it does not have any implications for M_μ with $\mu = 0, 1, 2$. Imposing the validity of symmetries (i), (ii), (iv), we find:

$$M_0 = -M_0^\dagger = \sigma_3 M_0 \sigma_3 = \sigma_1 M_0 \sigma_1, \tag{A.7}$$

$$M_1 = M_1^\dagger = -\sigma_3 M_1 \sigma_3 = \sigma_1 M_1 \sigma_1, \tag{A.8}$$

$$M_2 = M_2^\dagger = -\sigma_3 M_2 \sigma_3 = -\sigma_1 M_2 \sigma_1, \tag{A.9}$$

If we now expand M_μ with $\mu = 0, 1, 2$ in the 'Pauli basis', $M_\mu = a_0^\mu \mathbb{1} + a_1^\mu \sigma_1 + a_2^\mu \sigma_2 + a_3^\mu \sigma_3$, we see that (A.7)–(A.9) imply that $M_0 = a_0^0 \mathbb{1}$, $M_1 = a_1^1 \sigma_1$, $M_2 = a_2^2 \sigma_2$, with $a_0^0 \in i\mathbb{R}$ and $a_1^1, a_2^2 \in \mathbb{R}$. This proves the second identity in the first line of (3.50).

Regime $h < h_*$ Let us consider a quadratic term, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\sum_{\omega=\pm} \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{\psi}^+_{\omega,\mathbf{k}} k_\mu M_{\mu,\omega} \hat{\psi}^-_{\omega,\mathbf{k}}, \tag{A.10}$$

where $M_{\mu,\omega}$ are complex 2×2 matrices, with $\mu = 0, 1, 2, 3$. Imposing symmetry (iii) we find that $M_{\mu,\omega} = (-1)^{\delta_{\mu,3}} M_{\mu,-\omega}$, so we let $M_{\mu,\omega} \equiv M_\mu$, for $\mu = 0, 1, 2$, and $M_{3,\omega} \equiv \omega M_3$. Imposing the validity of symmetries (i), (ii), (iv), we find:

$$M_0 = -M_0^\dagger = \sigma_3 M_0 \sigma_3 = \sigma_1 M_0 \sigma_1, \tag{A.11}$$

$$M_1 = M_1^\dagger = -\sigma_3 M_1 \sigma_3 = \sigma_1 M_1 \sigma_1, \tag{A.12}$$

$$M_2 = M_2^\dagger = -\sigma_3 M_2 \sigma_3 = -\sigma_1 M_2 \sigma_1, \tag{A.13}$$

$$M_3 = M_3^\dagger = \sigma_3 M_3 \sigma_3 = -\sigma_1 M_3 \sigma_1. \tag{A.14}$$

If we now expand M_μ in the 'Pauli basis', $M_\mu = a_0^\mu \mathbb{1} + a_1^\mu \sigma_1 + a_2^\mu \sigma_2 + a_3^\mu \sigma_3$, we see that (A.11)–(A.14) imply that $M_0 = a_0^0 \mathbb{1}$, $M_1 = a_1^1 \sigma_1$, $M_2 = a_2^2 \sigma_2$, $M_3 = a_3^3 \sigma_3$, with $a_0^0 \in i\mathbb{R}$ and $a_1^1, a_2^2, a_3^3 \in \mathbb{R}$. This proves the second identity in (3.88).

A.2.3. The local $\psi^+\partial_\mu\partial_3\psi^-$ and $\psi^+\partial_3^3\psi^-$ terms Regime $h \geq h_*$. Let us consider the following quadratic terms, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1:

$$\int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{\psi}_{\mathbf{k}}^+ k_\mu k_3 Q_\mu \psi_{\mathbf{k}}^- \quad \text{and} \quad \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{\psi}_{\mathbf{k}}^+ k_3^3 \widetilde{Q}_3 \psi_{\mathbf{k}}^-, \tag{A.15}$$

where Q_μ with $\mu = 0, 1, 2, 3$ and $\widetilde{Q}_3$ are complex 2×2 matrices. Symmetry (iii) implies that $Q_0 = Q_1 = Q_2 = \widetilde{Q}_3 = 0$, which proves the second line of (3.50). On the other hand, symmetries (i), (ii), (iv) imply that

$$Q_3 = Q_3^\dagger = \sigma_3 Q_3 \sigma_3 = -\sigma_1 Q_3 \sigma_1, \tag{A.16}$$

from which we find $Q_3 = b\sigma_3$ with $b \in \mathbb{R}$. This proves the third identity in the first line of (3.50).

A.2.4. The local $A\psi^+\psi^-$ and $A\psi^+\partial_3\psi^-$ terms Regime $h \geq h_*$. Let us consider the following terms, quadratic in ψ and linear in A, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1:

$$\int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{A}_{\mu,\mathbf{p}} \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ \Gamma_\mu \hat{\psi}_{\mathbf{k}}^- \quad \text{and}$$

$$\int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{A}_{\mu,\mathbf{p}} \hat{\psi}_{\mathbf{k}+\mathbf{p}}^+ (k_3 \Gamma_\mu^3 + p_3 \widetilde{\Gamma}_\mu^3) \hat{\psi}_{\mathbf{k}}^-,$$

where $\Gamma_\mu, \Gamma_\mu^3, \widetilde{\Gamma}_\mu^3$ are complex 2×2 matrices. Symmetry (iii) implies that $\Gamma_3 = 0$ and $\Gamma_\mu^3 = \widetilde{\Gamma}_\mu^3 = 0$ for $\mu = 0, 1, 2$. Symmetries (i), (ii), (iv) imply that Γ_μ satisfy the same as (A.7)–(A.9), with M_μ replaced by Γ_μ. Therefore, $\Gamma_0 = ic_0\mathbb{1}, \Gamma_1 = c_1\sigma_1, \Gamma_2 = c_2\sigma_2$, with $c_1, c_1, c_2 \in \mathbb{R}$, which proves the first identity in the first line of (3.51). Symmetry (i) implies that $\Gamma_3^3 = (\Gamma_3^3)^\dagger$, while $\widetilde{\Gamma}_3^3 + (\widetilde{\Gamma}_3^3)^\dagger = \Gamma_3^3$. Moreover, symmetries (ii) and (iv) imply that $\Gamma_3^3 = \sigma_3\Gamma_3^3\sigma_3 = -\sigma_1\Gamma_3^3\sigma_1$ and $\widetilde{\Gamma}_3^3 = \sigma_3\widetilde{\Gamma}_3^3\sigma_3 = -\sigma_1\widetilde{\Gamma}_3^3\sigma_1$. In conclusion, $\Gamma_3^3 = a\sigma_3$ and $\widetilde{\Gamma}_3^3 = b\sigma_3$, with $a \in \mathbb{R}$, $b \in \mathbb{C}$ and $\Re(b) = a/2$. This proves the second line of (3.51) and (3.52).

Regime $h < h_*$. Let us now consider a term quadratic in ψ and linear in A, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\sum_{\omega=\pm} \int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{A}_{\mu,\mathbf{p}} \hat{\psi}_{\omega,\mathbf{k}+\mathbf{p}}^+ \Gamma_{\mu,\omega} \hat{\psi}_{\omega,\mathbf{k}}^-,$$

where $\Gamma_{\mu,\omega}$ are complex 2×2 matrices. Imposing symmetry (iii) we find that $\Gamma_{\mu,\omega} = (-1)^{\delta_{\mu,3}}\Gamma_{\mu,-\omega}$, so we let $\Gamma_{\mu,\omega} \equiv \Gamma_\mu$, for $\mu = 0, 1, 2$, and $\Gamma_{3,\omega} \equiv \omega\Gamma_3$. Imposing the validity of symmetries (i), (ii), (iv), we find the same as (A.11)–(A.14), with M_μ replaced by Γ_μ. Therefore, $\Gamma_0 = ic_0\mathbb{1}, \Gamma_1 = c_1\sigma_1, \Gamma_2 = c_2\sigma_2, \Gamma_3 = c_3\sigma_3$, with $c_1, c_1, c_2, c_3 \in \mathbb{R}$. This proves the first identity in (3.90).

A.2.5. The local $A_5\psi^+\psi^-$ and $A_5\psi^+\partial_3\psi^-$ terms Regime $h \geq h_*$. Let us consider the following terms, quadratic in ψ and linear in A^5, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1:

$$\int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{A}^5_{\mu,\mathbf{p}} \hat{\psi}^+_{\mathbf{k}+\mathbf{p}} \Gamma^5_\mu \hat{\psi}^-_{\mathbf{k}}$$

$$\text{and} \quad \int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \, \hat{A}_{\mu,\mathbf{p}} \hat{\psi}^+_{\mathbf{k}+\mathbf{p}} (k_3 \Gamma^{5;3}_\mu + p_3 \widetilde{\Gamma}^{5;3}_\mu) \hat{\psi}^-_{\mathbf{k}},$$

$\Gamma^5_\mu, \Gamma^{5;3}_\mu, \widetilde{\Gamma}^{5;3}_\mu$ are complex 2×2 matrices. Imposing the symmetry (iii), we get $\Gamma^5_\mu = 0$ for $\mu \neq 3$, and $\Gamma^{5;3}_3 = \widetilde{\Gamma}^{5;3}_3 = 0$. Symmetries (i), (ii), (iv) imply that $\Gamma^5_3 = (\Gamma^5_3)^\dagger = \sigma_3 \Gamma^5_3 \sigma_3 = -\sigma_1 \Gamma^5_3 \sigma_1$, which gives $\Gamma^5_3 = a\sigma_3$, with $a \in \mathbb{R}$. This proves the second identity in the first line of (3.51). Symmetry (i) implies that $\Gamma^{5;3}_0 = -(\Gamma^{5;3}_0)^\dagger$ and $\widetilde{\Gamma}^{5;3}_0 - (\widetilde{\Gamma}^{5;3}_0)^\dagger = \Gamma^{5;3}_0$, while $\Gamma^{5;3}_\mu = (\Gamma^{5;3}_\mu)^\dagger$ and $\widetilde{\Gamma}^{5;3}_\mu + (\widetilde{\Gamma}^{5;3}_\mu)^\dagger = \Gamma^{5;3}_\mu$ for $\mu = 1, 2$. Moreover, symmetries (ii) and (iv) imply that: $\Gamma^{5;3}_0 = \sigma_3 \Gamma^{5;3}_0 \sigma_3 = \sigma_1 \Gamma^{5;3}_0 \sigma_1$ and the same $\widetilde{\Gamma}^{5;3}_0$; $\Gamma^{5;3}_1 = -\sigma_3 \Gamma^{5;3}_1 \sigma_3 = \sigma_1 \Gamma^{5;3}_1 \sigma_1$ and the same $\widetilde{\Gamma}^{5;3}_1$; $\Gamma^{5;3}_2 = -\sigma_3 \Gamma^{5;3}_2 \sigma_3 = -\sigma_1 \Gamma^{5;3}_2 \sigma_1$ and the same $\widetilde{\Gamma}^{5;3}_2$. In conclusion: $\Gamma^{5;3}_0 = ia_0 \mathbb{1}_2$ and $\widetilde{\Gamma}^{5;3}_0 = ib_0 \mathbb{1}_2$; $\Gamma^{5;3}_1 = a_1 \sigma_1$ and $\widetilde{\Gamma}^{5;3}_1 = b_1 \sigma_1$; $\Gamma^{5;3}_2 = a_2 \sigma_2$ and $\widetilde{\Gamma}^{5;3}_2 = b_2 \sigma_2$; where $a_\mu \in \mathbb{R}, b_\mu \in \mathbb{C}$, and $\Re(b_\mu) = a_\mu/2$, for all $\mu = 0, 1, 2$. This proves the third line of (3.51) and (3.53).

Regime $h < h_*$. Let us consider a term quadratic in ψ and linear in A^5, symmetric under the symmetries (i), (ii), (iii), (iv) of Appendix A.1, of the form

$$\int \frac{d\mathbf{p}}{(2\pi)^4} \int \frac{d\mathbf{k}}{(2\pi)^4} \hat{A}^5_{\mu,\mathbf{p}} \hat{\psi}^+_{\omega,\mathbf{k}+\mathbf{p}} \Gamma^5_{\mu,\omega} \hat{\psi}^-_{\omega,\mathbf{k}},$$

where $\Gamma^5_{\mu,\omega}$ are complex 2×2 matrices. Imposing symmetry (iii) we find that $\Gamma^5_{\mu,\omega} = (-1)^{1-\delta_{\mu,3}} \Gamma^5_{\mu,-\omega}$, so we let $\Gamma^5_{\mu,\omega} \equiv \omega \Gamma^5_\mu$, for $\mu = 0, 1, 2$, and $\Gamma^5_{3,\omega} \equiv \Gamma^5_3$. Imposing the validity of symmetries (i), (ii), (iv), we find the same as (A.11)–(A.14), with M_μ replaced by Γ^5_μ. Therefore, $\Gamma^5_0 = ic_0 \mathbb{1}, \Gamma^5_1 = c_1 \sigma_1, \Gamma^5_2 = c_2 \sigma_2, \Gamma^5_3 = c_3 \sigma_3$, with $c_1, c_1, c_2, c_3 \in \mathbb{R}$. This proves the second identity in (3.90).

B. Dimensional Bounds on the Kernels of the Effective Potential

In this appendix we provide some additional details on the proofs of (3.74), (3.102) and of their improved analogues, discussed after (3.74) and (3.102), respectively. As anticipated above, the proofs of these bounds are completely analogous to those of Lemma 1 and Lemma 2 in [52], where a specific model of Weyl semimetal within the class of models considered in this paper was analyzed. Since there a few minor differences in the bounds (3.74), (3.102) and in their improved analogues, as compared to those stated in [52, Lemma 1 and 2], here we highlight where these differences rely and explain their origin. A first macroscopic difference is that [52] explicitly treats only the free energy (i.e., $A = A^5 = 0$), which obviously has an impact on the definition of the iterative step. In addition to this, there are other more technical differences, which we discuss separately for the first and second regimes.

First regime. The RG construction in the first regime is virtually the same as the one described in [52, Section 2], modulo a slightly different definition of localization (compare

[52, Eq. (43)] with the definition in (3.48)–(3.49)) that, in particular, takes into account the presence of the external fields A and A^5. At each scale $h_* \leq h \leq 0$, the effective potential can be expressed in terms of a GN tree expansion virtually equivalent to the one described in [52, Section 2.1]. As already mentioned, the proof of (3.74) and of its improved analogue (see discussion after (3.74)) goes along the same lines as the proof of Eqs. (60) and (61) of [52], respectively. However, there are a few technical differences that deserve comments:

- The L^1 norm in the left side of (3.74) has a stretched exponential weight, contrary to the one used in [52, Eq. (60)]. However, the inclusion of the stretched exponential weight involving the tree distance can be accomodate without extra difficulties, thanks to the stretched exponential decay of the propagator (3.101). See [8,32] for two recent works on fermionic RG where similar exponentially weighted L^1 norms are studied via the same kind of methods of this paper.
- Eq. (3.74) takes into account the possibile presence of the derivative labels $\underline{q}$, which was neglected for simplicity in [52, Eq. (60)] (the proof of the bound in the presence of such indices remains unaltered).
- Most importantly, the improved dimensional bound on $W_{n,m_1,m_2,\underline{q}}^{(h);\mathrm{r}}$ has an additional factor $\gamma^{\theta h}$, as compared with (3.74), where θ is *any* positive constant smaller than 1; see discussion in the paragraph after (3.74). This should be compared with the dimensional gain $\gamma^{h/8}$ stated in [52, Eq. (61)]. Actually, in [52, Eq. (61)], the factor $\gamma^{h/8}$ can be replaced 'for free' by $\gamma^{\theta h}$ with θ a positive constant strictly smaller than $1/2$, simply because in that case the largest scaling dimension of an irrelevant operator is $-1/2$ (see the comment "*Note that, if v is not an endpoint, $5\frac{|P_v|}{4} - \frac{7}{2} + z(P_v) \geq \frac{1}{2}$ by the definition of $\mathcal{R}$*" after [52, Eq. (70)], and note that, thanks to this fact, the factors $\gamma^{-(h_v - h_{v'})/4}$ in [52, Eqs. (71), (73), (74)] can be replaced by $\gamma^{-\theta(h_v - h_{v'})}$ with $0 < \theta < 1/2$, thus leading to the claimed improvement). In our case, thanks to the definition of $\mathcal{L}$, see (3.48)–(3.49), and to the cancellation properties stated in the Remark after (3.49) and in Lemma 3.1, the largest scaling dimension of an irrelevant operator is -1, rather than $-1/2$; therefore, by the same line of reasoning, the GN trees with at least one endpoint on scale 1 admit a bound with an additional 'short memory factor' $\gamma^{\theta h}$ with θ a positive constant strictly smaller than 1 (rather than 'just' $1/2$).

Second regime. The RG construction in the first regime is virtually the same as the one described in [52, Section 3], modulo a slightly different definition of localization (compare [52, Eq. (88)] with the definition in (3.86)–(3.87)) that, in particular, takes into account the presence of the external fields A and A^5. At each scale $h < h_*$, the effective potential can be expressed in terms of a GN tree expansion virtually equivalent to the one described in [52, Section 3.1]. As already mentioned, the proof of (3.102) and of its improved analogue (see discussion in the paragraph after (3.102)) goes along the same lines as the proof of Eqs. (93) and (94) of [52], respectively. Also in this case, there are a few technical differences that deserve comments:

- As in the first regime, in (3.102) we use a weighted L^1 norm, rather than the standard one, and we take into account the possibile presence of the derivative labels $\underline{q}$; as commented above, the proof of the bound remains essentially unaltered by these two modifications.
- Concerning (3.102) and its improved version, an important difference with respect to [52, Eqs. (93)–(94)] is the presence of the pre-factor $|v_3^0|^{n-1+\|\underline{q}^3\|_1}$. Even though such

factor does not appear in [52, Eqs. (93)–(94)], its presence is actually proved in [52] (in the case $\|q^3\|_1 = 0$, but the proof in the general case is essentially unchanged), see l.1 after [52, Eq. (107)].

- Concerning the improved dimensional bound on $W_{n,m_1,m_2,\underline{q}}^{(h);r}$, in the paragraph after (3.102) we claim that it has an additional factor $\gamma^{\theta h}$ if $m_1 = m_2 = 0$ or $n \geq 2$, or $\gamma^{\theta(h-h_*)}$ if $m_1 + m_2 \geq 1$ and $n = 1$, with θ any positive constant strictly smaller than 1; this has to be compared with the factor $\gamma^{\frac{1}{8}(h-h_*)}$ stated in [52, Eq. (94)]. First of all, for the same reasons explained above for the first regime (see in particular the third item of the dotted list), the factor $\gamma^{\frac{1}{8}(h-h_*)}$ stated in [52, Eq. (94)] can be straightforwardly improved to $\gamma^{\theta(h-h_*)}$, for any $0 < \theta < 1$, because the largest scaling dimension of an irrelevant operator is -1. We still need to discuss the origin of an additional gain factor $\gamma^{\theta h_*}$ for the terms with $m_1 = m_2 = 0$ or $n \geq 2$. The point is that these terms are defined in terms of GN trees with at least one endpoint on scale $h_* + 1$ with $m_1 = m_2 = 0$ or $n \geq 2$; in turn, each such endpoint has a kernel generated by the tree expansion of the first regime, and comes from GN trees with at least one endpoint on scale 1: therefore, it is bounded via the improved dimensional bound discussed in the paragraph after (3.74), which is characterized by an additional factor $\gamma^{\theta h_*}$ as compared to the 'basic' bound (3.74), as desired.

C. The Quadratic Response

In this appendix we compute the quadratic response coefficient in the right side of (2.48) and prove that its $\beta, L \to \infty$ limit can be expressed in terms of Euclidean correlation functions.

C.1. Second-order time-dependent perturbation theory. Consider a time-dependent vector potential $A_x(t) = e^{\eta t} A_x$ with $\eta > 0$. The evolution of the state for $t \leq 0$ is determined by the von Neumann equation,

$$i\partial_t \rho(t) = [\mathcal{H}_L(A(t)), \rho(t)], \qquad \rho(-\infty) = \rho_{\beta,L}, \tag{C.1}$$

with $\rho_{\beta,L}$ as in (2.20). Given an observable $\mathcal{O}$, we denote by $\mathcal{O}(A(t))$ its coupling to the external gauge field, via the Peierls substitution. We are interested in computing $\mathrm{Tr}\, \mathcal{O}(A(t))\rho(t) \equiv \langle \mathcal{O}(A(t)) \rangle_{\beta,L;t}$ at second order in the external field, in the case that $\mathcal{O} = \hat{J}_{\mu,p}^5$. We set:

$$\begin{aligned}
\langle \mathcal{O}(A(t)) \rangle_{\beta,L;t} - \langle \mathcal{O} \rangle_{\beta,L} &= (\langle \mathcal{O}(A(t)) \rangle_{\beta,L;t} - \langle \mathcal{O}(A(t)) \rangle_{\beta,L}) + (\langle \mathcal{O}(A(t)) \rangle_{\beta,L} \\
&\quad - \langle \mathcal{O} \rangle_{\beta,L}) \\
&\equiv \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{C.2}$$

We compute I and II separately, at second order in the external field. To this end, it is convenient to study the evolution of the state in the interaction picture. Let $\rho^{\mathrm{int}}(t) := e^{i\mathcal{H}_L t} \rho(t) e^{-i\mathcal{H}_L t}$. Then:

$$i\partial_t \rho_t^{\mathrm{int}} = [\mathcal{H}_{L,t}(A(t)) - \mathcal{H}_L, \rho^{\mathrm{int}}(t)]. \tag{C.3}$$

where $\mathcal{H}_{L,t}(A(t)) = e^{i\mathcal{H}_L t} \mathcal{H}_L(A(t)) e^{-i\mathcal{H}_L t}$. In general, in this appendix, the subscript t stands for the evolution at time t in the interaction picture: for any observable O, we

let $O_t = e^{i\mathcal{H}_L t} O e^{-i\mathcal{H}_L t}$ (we also let $\mathcal{O}_t(A(t)) \equiv [\mathcal{O}(A(t))]_t = e^{i\mathcal{H}_L t} \mathcal{O}(A(t)) e^{-i\mathcal{H}_L t}$, etc.)

Term I. A simple computation gives:

$$I = \langle \mathcal{O}(A(t)) \rangle_{\beta,L;t} - \langle \mathcal{O}(A(t)) \rangle_{\beta,L} = -i \int_{-\infty}^{t} ds \, \mathrm{Tr} \, \mathcal{O}_t(A(t))[\mathcal{P}_s(A(s)), \rho^{\mathrm{int}}(s)]$$

$$= -i \int_{-\infty}^{t} ds \, \mathrm{Tr}[\mathcal{O}_t(A(t)), \mathcal{P}_s(A(s))] \rho^{\mathrm{int}}(s), \tag{C.4}$$

where $\mathcal{P}(A(t)) := \mathcal{H}_L(A(t)) - \mathcal{H}_L$. and in the second identity we used the cyclicity of the trace. A one-step iteration of this formula gives:

$$I = -i \int_{-\infty}^{t} ds \, \langle \mathcal{O}_t(A(t)), \mathcal{P}_s(A(s)) \rangle_{\beta,L} \tag{C.5}$$

$$- \int_{-\infty}^{t} ds \int_{-\infty}^{s} ds_1 \mathrm{Tr}[[\mathcal{O}_t(A(t)), \mathcal{P}_s(A(s))], \mathcal{P}_{s_1}(A(s_1))] \rho^{\mathrm{int}}(s_1).$$

We denote by $I^{(2)}$ the second order contribution in A. If we write $\mathcal{O}_t^{(k)}(A)$ for the k-th order of $\mathcal{O}_t(A)$ in A, and similarly for $\mathcal{P}_t^{(k)}(A)$, we get:

$$I^{(2)} = -i \int_{-\infty}^{t} ds \, \langle [\mathcal{O}_t, \mathcal{P}_s^{(2)}(A(s))] \rangle_{\beta,L} - i \int_{-\infty}^{t} ds \, \langle [\mathcal{O}_t^{(1)}(A(t)), \mathcal{P}_s^{(1)}(A(s))] \rangle_{\beta,L}$$

$$- \int_{-\infty}^{t} ds \int_{-\infty}^{s} ds_1 \langle [[\mathcal{O}_t, \mathcal{P}_s^{(1)}(A(s))], \mathcal{P}_{s_1}^{(1)}(A(s_1))] \rangle_{\beta,L} \equiv I_1^{(2)} + I_2^{(2)} + I_3^{(2)}. \tag{C.6}$$

Letting $\int \frac{dp}{(2\pi)^3}$ be a shorthand for $L^{-3} \sum_{p \in \frac{2\pi}{L} \mathbb{Z}^3}$, we have

$$\mathcal{P}^{(1)}(A) = -\sum_{k=1,2,3} \int \frac{dp}{(2\pi)^3} \, \hat{A}_{k,p} \hat{J}_{k,p},$$

$$\mathcal{P}^{(2)}(A) = -\frac{1}{2} \sum_{k,k'=1,2,3} \int \frac{dp_1}{(2\pi)^3} \int \frac{dp_2}{(2\pi)^3} \, \hat{A}_{k,p_1} \hat{A}_{k',p_2} \hat{\Delta}_{k,k'}(p_1, p_2).$$

where $\hat{J}_{k,p}$ is defined as in (2.31) and the second line should be understood as the definition of $\hat{\Delta}_{k,k'}(p_1, p_2)$. Let us now set $\mathcal{O} = \hat{J}_{\mu,p}^5$, in which case we write:

$$\mathcal{O}^{(1)}(A) = \sum_{k=1,2,3} \int \frac{dp_1}{(2\pi)^3} \, \hat{A}_{k,p_1} \hat{\Delta}_{\mu,k}^5(p, p_1), \tag{C.7}$$

which should be understood as the definition of $\hat{\Delta}_{\mu,k}^5(p, q)$. With these definitions, the three terms in the right side of (C.6) take the following explicit form:

$$I_1^{(2)} = \frac{i}{2} \sum_{k,k'=1,2,3} \int \frac{dp_1}{(2\pi)^3} \int \frac{dp_2}{(2\pi)^3} \, \hat{A}_{k,p_1} \hat{A}_{k',p_2} \int_{-\infty}^{t} ds \, e^{2\eta s} \langle [\hat{J}_{\mu,p,t}^5, \hat{\Delta}_{k,k',s}(p_1, p_2)] \rangle_{\beta,L}$$

$$\equiv \frac{i}{2} \sum_{k,k'} \int \frac{dp_1}{(2\pi)^3} e^{2\eta t} \hat{A}_{k,p_1} \hat{A}_{k',-p-p_1} \int_{-\infty}^0 ds \, e^{2\eta s} \langle [\hat{J}^5_{\mu,p}, \hat{\Delta}_{k,k',s}(p_1,-p-p_1)] \rangle_{\beta,L},$$

(C.8)

where in the last step we used space and time translation invariance of the Gibbs state,

$$I_2^{(2)} = i \sum_{k,k'} \int \frac{dp_1}{(2\pi)^3} e^{2\eta t} \hat{A}_{k,p_1} \hat{A}_{k',-p-p_1} \int_{-\infty}^0 ds \, e^{\eta s} \langle [\hat{\Delta}^5_{\mu,k}(p,p_1), \hat{J}_{k',-p-p_1,s}] \rangle_{\beta,L},$$

(C.9)

and

$$I_3^{(2)} = -\sum_{k,k'} \int \frac{dp_1}{(2\pi)^3} e^{2\eta t} \hat{A}_{k,p_1} \hat{A}_{k',-p-p_1} \int_{-\infty}^0 ds \int_{-\infty}^s ds_1 \, e^{\eta s + \eta s_1} \langle [[\hat{J}^5_{\mu,p}, \hat{J}_{k,p_1,s}], \hat{J}_{k',-p-p_1,s_1}] \rangle_{\beta,L}.$$

(C.10)

Term II. For $\mathcal{O} = \hat{J}^5_{\mu,p}$ we write:

$$\mathcal{O}^{(2)}(A) = \frac{1}{2} \sum_{k,k'=1,2,3} \int \frac{dp_1}{(2\pi)^3} \int \frac{dp_2}{(2\pi)^3} \hat{A}_{k,p_1} \hat{A}_{k',p_2} \hat{\Delta}^5_{\mu,k,k'}(p,p_1,p_2),$$

which should be understood as the definition of $\hat{\Delta}^5_{\mu,k,k'}(p,p_1,p_2)$, from which we get:

$$\text{II}^{(2)} = \frac{1}{2} \sum_{k,k'} \int \frac{dp_1}{(2\pi)^3} e^{2\eta t} \hat{A}_{k,p_1} \hat{A}_{k',-p-p_1} \langle \hat{\Delta}^5_{\mu,k,k'}(p,p_1,-p-p_1) \rangle_{\beta,L}. \quad \text{(C.11)}$$

In conclusion, $\langle \mathcal{O}(A(t)) \rangle^{(2)}_{\beta,L;t} = $ (C.8)$+$(C.9)$+$(C.10)$+$(C.11), from which we find that the quadratic response coefficient in the right side of (2.48), recalling that $\mathbf{p}_1 = (\eta, p_1)$ and $\mathbf{p}_2 = (\eta, p_2)$, is given explicitly by

$$\hat{\Pi}^{5;\beta,L}_{\mu,k,k'}(\mathbf{p}_1, \mathbf{p}_2)$$

$$= -\frac{1}{L^3} \int_{-\infty}^0 ds_1 \int_{-\infty}^{s_1} ds_2 \, e^{\eta s_1 + \eta s_2} \left\{ \langle [[\hat{J}^5_{\mu,p}, \hat{J}_{k,p_1,s_1}], \hat{J}_{k',p_2,s_2}] \rangle_{\beta,L} + [(k,p_1) \leftrightarrow (k',p_2)] \right\}$$

$$+ \frac{i}{L^3} \int_{-\infty}^0 ds \left\{ e^{2\eta s} \langle [\hat{J}^5_{\mu,p}, \hat{\Delta}_{k,k',s}(p_1,p_2)] \rangle_{\beta,L} + e^{\eta s} \left[\langle [\hat{\Delta}^5_{\mu,k}(p,p_1), \hat{J}_{k',p_2,s}] \rangle_{\beta,L} \right. \right.$$

$$+ \langle [\hat{\Delta}^5_{\mu,k'}(p,p_2), \hat{J}_{k,p_1,s}] \rangle_{\beta,L} \left] \right\} + \frac{1}{L^3} \langle \hat{\Delta}^5_{\mu,k,k'}(p,p_1,p_2) \rangle_{\beta,L}. \quad \text{(C.12)}$$

The next task is to Wick-rotate to imaginary times the various terms. The Wick rotation of the second term in the right side (the one expressed in terms of an integral over s) can be performed as discussed in [6,30,56], see in particular [6, Appendix B] and [56, Appendix B], and gives

$$\lim_{\beta,L\to\infty} \frac{i}{L^3} \int_{-\infty}^0 ds \left\{ e^{2\eta s} \langle [\hat{J}^5_{\mu,p}, \hat{\Delta}_{k,k',s}(p_1,p_2)] \rangle_{\beta,L} + e^{\eta s} \left[\langle [\hat{\Delta}^5_{\mu,k}(p,p_1), \hat{J}_{k',p_2,s}] \rangle_{\beta,L} + \right. \right.$$

$$+ \langle [\hat{\Delta}^5_{\mu,k'}(p,p_2), \hat{J}_{k,p_1,s}] \rangle_{\beta,L} \left] \right\} =$$

(C.13)

$$= \langle \mathbf{T} \, \hat{J}^5_{\mu,\mathbf{p}} \, ; \, \hat{\Delta}_{k,k'}(\mathbf{p}_1, \mathbf{p}_2) \rangle_\infty + \langle \mathbf{T} \, \hat{\Delta}^5_{\mu,k}(\mathbf{p}, \mathbf{p}_1) \, ; \, \hat{J}_{k',\mathbf{p}_2} \rangle_\infty + \langle \mathbf{T} \, \hat{\Delta}^5_{\mu,k'}(\mathbf{p}, \mathbf{p}_2) \, ; \, \hat{J}_{k,\mathbf{p}_1} \rangle_\infty,$$

where: $\mathbf{p}_i = (\eta, p_i) \equiv (p_{i,0}, p_i)$, $\mathbf{p} = (-2\eta, -p_1 - p_2) \equiv (p_0, p)$,

$$\hat{J}^5_{\mu,\mathbf{p}} = \int_0^\beta dx_0 e^{-ip_0 x_0} \hat{J}^5_{\mu,(x_0,p)} \quad \text{and} \quad \hat{\Delta}_{k,k'}(\mathbf{p}_1, \mathbf{p}_2) = \int_0^\beta dx_0 e^{-i(p_{1,0}+p_{2,0})x_0} [\hat{\Delta}_{k,k'}(p_1, p_2)]_{x_0},$$

with $\hat{J}^5_{\mu,(x_0,p)} = e^{x_0 \mathcal{H}_L} \hat{J}^5_{\mu,p} e^{-x_0 \mathcal{H}_L}$ and $[\hat{\Delta}_{k,k'}(p_1, p_2)]_{x_0} = e^{x_0 \mathcal{H}_L} \hat{\Delta}_{k,k'}(p_1, p_2) e^{-x_0 \mathcal{H}_L}$
($\hat{J}_{k,\mathbf{p}_1}$ and $\hat{\Delta}^5_{\mu,k'}(\mathbf{p}, \mathbf{p}_2)$ are defined analogously). Moreover, $\mathbf{T}$ is the imaginary-time-ordering operator, ordering the operators in decreasing imaginary-time order, and $\langle \cdot \rangle_\infty = \lim_{\beta,L\to\infty} (\beta L^3)^{-1} \langle \cdot \rangle_{\beta,L}$. The semicolon symbol in the right-hand side of (C.13) denotes truncation in the correlation functions. To introduce it, we used that in the left side we can freely subtract to every observable in the commutators the corresponding statistical average (of course, the subtraction leaves the commutator invariant).
We are left with discussing the Wick rotation of the first term in the right side of (C.12). This will be done in the next subsection, where we prove that

$$-\lim_{\beta,L\to\infty} \frac{1}{L^3} \int_{-\infty}^0 ds_1 \int_{-\infty}^{s_1} ds_2\, e^{\eta s_1 + \eta s_2} \left\{ \langle [[\hat{J}^5_{\mu,p}, \hat{J}_{k,p_1,s_1}], \hat{J}_{k',p_2,s_2}] \rangle_{\beta,L} + [(k, p_1) \leftrightarrow (k', p_2)] \right\} =$$
$$= \langle \mathbf{T} \hat{J}^5_{\mu,\mathbf{p}} \,;\, \hat{J}_{k,\mathbf{p}_1} \,;\, \hat{J}_{k',\mathbf{p}_2} \rangle_\infty, \tag{C.14}$$

with the same notations as in (C.13). All in all, we get:

$$\hat{\Pi}^5_{\mu,k,k'}(\mathbf{p}_1, \mathbf{p}_2) = \langle \mathbf{T} \hat{J}^5_{\mu,\mathbf{p}} \,;\, \hat{J}_{k,\mathbf{p}_1} \,;\, \hat{J}_{k',\mathbf{p}_2} \rangle_\infty + \langle \mathbf{T} \hat{J}^5_{\mu,\mathbf{p}} \,;\, \hat{\Delta}_{k,k'}(\mathbf{p}_1, \mathbf{p}_2) \rangle_\infty \tag{C.15}$$
$$+ \langle \mathbf{T} \hat{\Delta}^5_{\mu,k}(\mathbf{p}, \mathbf{p}_1) \,;\, \hat{J}_{k',\mathbf{p}_2} \rangle_\infty + \langle \mathbf{T} \hat{\Delta}^5_{\mu,k'}(\mathbf{p}, \mathbf{p}_2) \,;\, \hat{J}_{k,\mathbf{p}_1} \rangle_\infty + \lim_{\beta,L\to\infty} \langle \hat{\Delta}^5_{\mu,k,k'}(p, p_1, p_2) \rangle_{\beta,L}.$$

C.2. Wick rotation for correlations of three observables. In this section we prove (C.14). Let

$$T_{\beta,L}(\eta) = -\frac{1}{L^3} \int_{-\infty}^0 ds_1 \int_{-\infty}^{s_1} ds_2\, e^{\eta s_1 + \eta s_2} \left\{ \langle [[\hat{J}^5_{\mu,p}, \hat{J}_{k,p_1,s_1}], \hat{J}_{k',p_2,s_2}] \rangle_{\beta,L} + [(k, p_1) \leftrightarrow (k', p_2)] \right\}$$

be the function of interest, thought of as a function of η. We add and subtract $T_{\beta,L}(\eta_\beta)$, where $\eta_\beta = \frac{2\pi}{\beta} \lceil \frac{\beta \eta}{2\pi} \rceil \in \frac{2\pi}{\beta} \mathbb{N}$. We will prove below that $T_{\beta,L}(\eta) - T_{\beta,L}(\eta_\beta)$ is bounded by (const.)β^{-1}, uniformly in L. As for $T_{\beta,L}(\eta_\beta)$, it is equal to a double integral over imaginary times of the appropriate Euclidean correlation function, as implied by the following proposition.

Proposition A.1. *Let A, B, C be bounded fermionic operators, of the form:*

$$O = \sum_{X \subset \Lambda_L} O_X, \tag{C.16}$$

with O_X even in the fermionic operators, commuting with the total number operator. We assume that the sum in (C.16) runs over subsets X such that $|X|$ is bounded uniformly in L. Let $A(z) = e^{i\mathcal{H}_L z} A e^{-i\mathcal{H}_L z}$, for $z \in \mathbb{C}$. Let $\eta_i \in (2\pi/\beta)\mathbb{N}$ (with the convention that $\mathbb{N}$ is the set of positive integers) and consider:

$$\mathcal{I}^{\beta,L}_{ABC} := -\int_{-\infty}^0 ds_1 \int_{-\infty}^{s_1} ds_2 \left(e^{\eta_1 s_1 + \eta_2 s_2} \langle [[C, A(s_1)], B(s_2)] \rangle_{\beta,L} + e^{\eta_1 s_2 + \eta_2 s_1} \langle [[C, B(s_1)], A(s_2)] \rangle_{\beta,L} \right). \tag{C.17}$$

One has $\mathcal{I}_{ABC}^{\beta,L} = \mathcal{J}_{ABC}^{\beta,L}$, where

$$\mathcal{J}_{ABC}^{\beta,L} := \int_0^\beta ds_1 \int_0^\beta ds_2 \, e^{-is_1\eta_1 - is_2\eta_2} \langle \mathbf{T}\, A(-is_1)B(-is_2)C\rangle_{\beta,L}, \qquad \text{(C.18)}$$

with $\mathbf{T}$ the time-ordering operator, which orders the operator in the decreasing imaginary-time order.

Before giving the proof of Proposition A.1, let us explain how to adapt it to the case at hand. We let: $C = \hat{J}_{\mu,p}^5 - \langle \hat{J}_{\mu,p}^5 \rangle_{\beta,L}$, $A(s_1) = \hat{J}_{k,p_1,s_1} - \langle \hat{J}_{k,p_1}\rangle_{\beta,L}$, and $B(s_2) = \hat{J}_{k',p_2,s_2} - \langle \hat{J}_{k,p_2}\rangle_{\beta,L}$, so that, for $\eta_1 = \eta_2 = \eta_\beta$, $T_{\beta,L}(\eta_\beta) = \mathcal{I}_{ABC}^{\beta,L}$. Using the proposition, we get

$$T_{\beta,L}(\eta_\beta) = \frac{1}{L^3} \int_0^\beta dx_{1,0} \int_0^\beta dx_{2,0} \, e^{-i\eta_\beta(x_{1,0}+x_{2,0})} \langle \mathbf{T}\, \hat{J}_{k,(x_{1,0},p_1)};\, \hat{J}_{k',(x_{2,0},p_2)};\, \hat{J}_{\mu,p}^5 \rangle_{\beta,L}$$

$$\equiv \frac{1}{\beta L^3} \langle \mathbf{T}\, \hat{J}_{k,\tilde{\mathbf{p}}_1};\, \hat{J}_{k',\tilde{\mathbf{p}}_2};\, \hat{J}_{\mu,\tilde{\mathbf{p}}}^5 \rangle_{\beta,L}, \qquad \text{(C.19)}$$

with $\tilde{\mathbf{p}}_i = (\eta_\beta, p_i)$ and $\mathbf{p} = (-2\eta_\beta, -p_1 - p_2)$. If we now take $\beta, L \to \infty$ with $\eta_\beta \to \eta$, this expression tends to $\langle \mathbf{T}\, \hat{J}_{k,\mathbf{p}_1};\, \hat{J}_{k',\mathbf{p}_2};\, \hat{J}_{\mu,\mathbf{p}}^5 \rangle_\infty = \langle \mathbf{T}\, \hat{J}_{\mu,\mathbf{p}}^5;\, \hat{J}_{k,\mathbf{p}_1};\, \hat{J}_{k',\mathbf{p}_2} \rangle_\infty$, as desired (existence of the limit follows from the construction of the Euclidean correlation functions of Sect. 3.3).

Proof. By using the definition of $\mathbf{T}$, we rewrite (C.18) as:

$$\mathcal{J}_{ABC}^{\beta,L} = \int_0^\beta ds_1 \int_0^{s_1} ds_2 \, e^{-is_1\eta_1 - is_2\eta_2} \langle A(-is_1)B(-is_2)C\rangle_{\beta,L}$$

$$+ \int_0^\beta ds_1 \int_{s_1}^\beta ds_2 \, e^{-is_1\eta_1 - is_2\eta_2} \langle B(-is_2)A(-is_1)C\rangle_{\beta,L} \equiv \mathrm{I} + \mathrm{II}.$$

$$\text{(C.20)}$$

For notational convenience, we denote:

$$A_{\eta_1}(z) := e^{z\eta_1} A(z), \quad B_{\eta_2}(z) := e^{z\eta_2} B(z), \qquad z \in \mathbb{C}. \qquad \text{(C.21)}$$

Consider I. We apply Cauchy theorem to rewrite the integral over s_2 as follows:

$$\int_0^{s_1} ds_2 \, \langle A_{\eta_1}(-is_1)B_{\eta_2}(-is_2)C\rangle_{\beta,L} = -i \int_{-\infty}^0 dt_2 \, \langle A_{\eta_1}(-is_1)B_{\eta_2}(t_2)C\rangle_{\beta,L} \qquad \text{(C.22)}$$

$$+ i \int_{-\infty}^0 dt_2 \, \langle A_{\eta_1}(-is_1)B_{\eta_2}(t_2 - is_1)C\rangle_{\beta,L},$$

where we used the fact that $B_{\eta_2}(z) \to 0$ as $\mathrm{Re}\, z \to -\infty$, thanks to the factor $e^{\eta_2 \mathrm{Re} z}$ and the fact that $\eta_2 > 0$; see Fig. 1.

We now rewrite the integral over s_1 in a similar way, thus getting

$$\mathrm{I} = -\int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \left(\langle A_{\eta_1}(t_1)B_{\eta_2}(t_2)C\rangle_{\beta,L} - \langle A_{\eta_1}(t_1 - i\beta)B_{\eta_2}(t_2)C\rangle_{\beta,L} \right. \quad \text{(C.23)}$$

$$\left. - \langle A_{\eta_1}(t_1)B_{\eta_2}(t_1 + t_2)C\rangle_{\beta,L} + \langle A_{\eta_1}(t_1 - i\beta)B_{\eta_2}(t_1 + t_2 - i\beta)C\rangle_{\beta,L} \right). \qquad \text{(C.24)}$$

Recalling that $e^{i\eta_1\beta} = 1$, we have $\langle A_{\eta_1}(t_1 - i\beta)B\rangle_{\beta,L} = \langle BA_{\eta_1}(t_1)\rangle_{\beta,L}$, and similarly for $B_{\eta_2}(t_1 + t_2 - i\beta)$, so that (C.23) can be rewritten as:

$$I = -\int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \left(\langle [A_{\eta_1}(t_1), B_{\eta_2}(t_2)C]\rangle_{\beta,L} + \langle [C, A_{\eta_1}(t_1)B_{\eta_2}(t_1 + t_2)]\rangle_{\beta,L} \right) \tag{C.25}$$

By proceeding in the same way, we can rewrite II analogously:

$$II = -\int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \left(\langle [CB_{\eta_2}(t_2), A_{\eta_1}(t_1)]\rangle_{\beta,L} + \langle [B_{\eta_2}(t_1 + t_2)A_{\eta_1}(t_1), C]\rangle_{\beta,L} \right). \tag{C.26}$$

If we now take the sum of I and II, use the fact that $[A, BC] = B[A, C] + [A, B]C$ and recall the Jacobi identity $[[A, B], C]] + [[C, A], B] + [[B, C], A] = 0$, we find:

$$\begin{aligned} \mathcal{J}_{ABC}^{\beta,L} &= \int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \left(\langle [[B_{\eta_2}(t_2), C], A_{\eta_1}(t_1)], \rangle_{\beta,L} + \langle [[A_{\eta_1}(t_1), B_{\eta_2}(t_1 + t_2)], C]\rangle_{\beta,L} \right) \\ &= \int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \, \langle [[B_{\eta_2}(t_2), C], A_{\eta_1}(t_1)], \rangle_{\beta,L} \\ &\quad + \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \, \langle [[A_{\eta_1}(t_1), B_{\eta_2}(t_2)], C]\rangle_{\beta,L}. \end{aligned}$$

In the first term in the second line we split the integral over t_2 as $\int_{-\infty}^0 dt_2 = \int_{-\infty}^{t_1} dt_2 + \int_{t_1}^0 dt_2$. If we now combine together the two terms with the integral over t_2 from $-\infty$ to t_1 and use the Jacobi identity again, we get:

$$\begin{aligned} \mathcal{J}_{ABC}^{\beta,L} &= \int_{-\infty}^0 dt_1 \int_{t_1}^0 dt_2 \, \langle [[B_{\eta_2}(t_2), C], A_{\eta_1}(t_1)], \rangle_{\beta,L} \\ &\quad - \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \, \langle [[C, A_{\eta_1}(t_1)], B_{\eta_2}(t_2)]\rangle_{\beta,L}, \end{aligned}$$

which is the same as the right side of (C.17). $\qquad\square$

In view of the discussion at the beginning of this subsection, as well as of (C.19) and following lines, in order to conclude the proof of (C.14) we are left with proving that $T_{\beta,L}(\eta) - T_{\beta,L}(\eta_\beta)$ is bounded from above by $(\text{const.})\beta^{-1}$, uniformly in L. Using the fact that $0 \le \eta_\beta - \eta \le (2\pi)/\beta$, we find that

$$|T_{\beta,L}(\eta) - T_{\beta,L}(\eta_\beta)| \le \tag{C.27}$$

$$\le \frac{2\pi}{\beta} \frac{1}{L^3} \int_{-\infty}^0 ds_1 \int_{-\infty}^{s_1} ds_2 \, (|s_1| + |s_2|) e^{\eta(s_1 + s_2)} \left\{ \left\| [[C, A(s_1)], B(s_2)] \right\| + (A \leftrightarrow B) \right\},$$

with A, B, C as defined before (C.19). By the Lieb-Robinson bounds for multi-commutators [17], see in particular [17, item (ii) of Corollary 4.12], one finds that there exist $C_{ABC} > 0$ independent of L such that the norm $\left\| [[C, A(s_1)], B(s_2)] \right\|$ is bounded from above by $C_{ABC}L^3(1 + |s_1| + |s_2|)^6$, where 6 should be understood as twice the spatial dimension. Therefore,
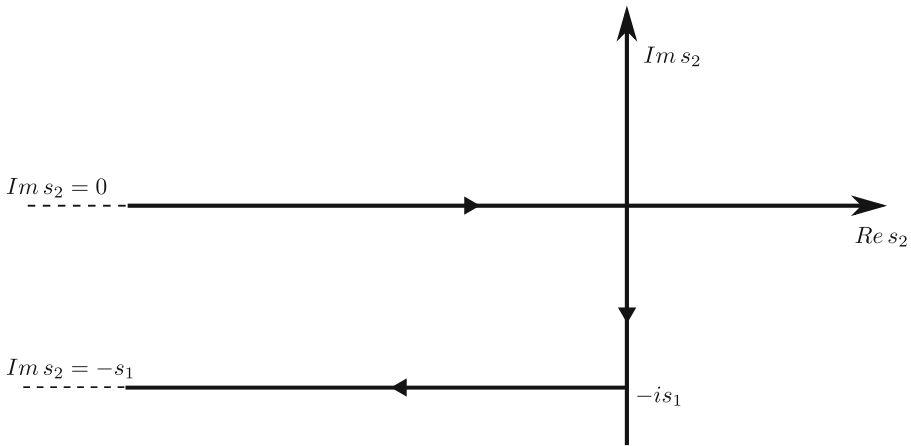
Fig. 1. The integral over the complex contour is zero

$$\left| T_{\beta,L}(\eta) - T_{\beta,L}(\eta_\beta) \right| \leq \frac{4\pi C_{ABC}}{\beta} \int_{-\infty}^{0} ds_1 \int_{-\infty}^{s_1} ds_2 \, (1 + |s_1| + |s_2|)^7 e^{\eta(s_1+s_2)},$$
(C.28)

which vanishes in the limit as $\beta \to \infty$, uniformly in L, for any $\eta > 0$. This concludes the proof of (C.14).

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