On very effective hermitian K-theory

Alexey Ananyevskiy, Oliver Röndigs, Paul Arne Østvær

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Abstract

We argue that the very effective cover of hermitian K-theory in the sense of motivic homotopy theory is a convenient algebro-geometric generalization of the connective real topological K-theory spectrum. This means the very effective cover acquires the correct Betti realization, its motivic cohomology has the desired structure as a module over the motivic Steenrod algebra, and that its motivic Adams and slice spectral sequences are amenable to calculations.

1 Introduction

Algebraic and hermitian K-theory have been widely studied since the pioneering works on the Grothendieck-Riemann-Roch theorem [1] and on rings with anti-involutions [10]. Both theories are representable in the stable motivic homotopy category $\mathbf{SH}$ of a field of characteristic $\neq 2$, and more generally over regular noetherian base schemes of finite Krull dimension on which 2 is invertible [6], [25]. Fundamental properties imply that the motivic spectra of algebraic K-theory $\mathbf{KGL}$ and hermitian K-theory $\mathbf{KQ}$ are $(2,1)$- and $(8,4)$-periodic, respectively, with respect to the standard motivic spheres $S^{p,q} := S^{p-q} \wedge \mathbf{G}_{\mathfrak{m}}^{q}$ for $p \geq q$. More precisely, there exist Bott elements in the Grothendieck group $\pi_{2,1}\mathbf{KGL} \cong K_0$ and in the Grothendieck-Witt group $\pi_{8,4}\mathbf{KQ} \cong GW_0$ inducing motivic weak equivalences

$$S^{2,1} \wedge \mathbf{KGL} \xrightarrow{\cong} \mathbf{KGL} \text{ and } S^{8,4} \wedge \mathbf{KQ} \xrightarrow{\cong} \mathbf{KQ}. \tag{1.1}$$

Because of (1.1), $\mathbf{KGL}$ and $\mathbf{KQ}$ are non-connective and should be thought of as "large" motivic spectra. When using K-theoretic invariants to inform the homotopy sheaves of the sphere $\mathbf{1}$ in [20], it is convenient to employ smaller "connective" versions of the motivic K-theory spectra. The geometrical meaning of this notion is still not well understood.

One fascinating aspect of motivic homotopy theory is that it offers different notions of "connectivity" based on:

- Voevodsky's slice filtration for the localizing triangulated subcategory of effective motivic spectra $\mathbf{SH}^{\mathbf{eff}} \subset \mathbf{SH}$ generated under homotopy colimits by motivic $\mathbf{P}^1$-suspension spectra of smooth schemes [26, §2].

- Morel's homotopy t-structure [14, §6.2] characterized over perfect fields by the vanishing of homotopy sheaves, and its extension to general base schemes in [7, §2.1].
- Spitzweck-Østvær's very effective slice filtration for the full subcategory $\mathbf{SH}^{\mathbf{veff}} \subset \mathbf{SH}$ generated under homotopy colimits and extensions by motivic $\mathbf{P}^1$-suspension spectra of smooth schemes [24, Definition 5.5].

The essential difference between the effective and the very effective slice filtrations is that the former records slices with respect to the multiplicative group scheme $\mathbf{G}_{\mathfrak{m}}$ and the latter with respect to the projective line $\mathbf{P}^1$. By construction $\mathbf{SH}^{\mathbf{veff}}$ is a subcategory of $\mathbf{SH}^{\mathbf{eff}}$ closed under the tensor product, but it is not closed under simplicial desuspension and hence not a triangulated subcategory of $\mathbf{SH}$. The sphere $\mathbf{1}$, algebraic cobordism $\mathbf{MGL}$, and quotients thereof such as motivic Moore spectra and the effective cover of $\mathbf{KGL}$ are examples of very effective motivic spectra. Both of the slice filtrations interact well with A_∞- and E_∞-structures [4], but only the very effective one maps to (the even part of) the topological Postnikov filtration under Betti realization [4, §3.3]. Over perfect fields $\mathbf{SH}^{\mathbf{veff}}$ is the nonnegative part of the t-structure on $\mathbf{SH}^{\mathbf{eff}}$, and the identification of the very effective slices of $\mathbf{KQ}$ up to extensions in [2, Theorem 16] makes a strong case for further investigations of the very effective slice filtration.

In this paper we argue that the very effective cover $\mathbf{kq}$ of hermitian K-theory $\mathbf{KQ}$ is a convenient algebro-geometric generalization of the connective cover ko of real topological K-theory KO. The question of whether there is a motivic spectrum with similar properties as ko was first addressed in [9] and [5, Conjecture 5.8] over the fields of complex and real numbers, respectively.

Section 2 begins with some preliminary results on $\mathbf{KGL}$. We identify the effective and very effective covers of $\mathbf{KGL}$ over perfect fields of characteristic $\neq 2$, and similarly for $\mathbf{KGL}/2$ and base schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. Proposition 2.11 relates the very effective covers $\mathbf{kq}$ of $\mathbf{KQ}$ and $\mathbf{kgl}$ of $\mathbf{KGL}$ to the motivic Hopf map $\eta\colon \mathbf{A}^2 \smallsetminus \{0\} \to \mathbf{P}^1$ via the cofiber sequence

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \to \mathbf{kgl}. \tag{1.2}$$

Over the complex numbers, (1.2) has been obtained independently by Bachmann. The same result holds for η, $\mathbf{KQ}$, $\mathbf{KGL}$, and base schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ by [19, Theorem 3.4], but it is plainly false for the effective covers of $\mathbf{KQ}$ and $\mathbf{KGL}$ by the proof of [21, Corollary 5.1]. By using (1.2) we identify the Betti realization of $\mathbf{kq}$ with ko and calculate the mod-2 motivic cohomology $\mathbf{M}\mathbb{Z}/2^\star\mathbf{kq}$ as $\mathcal{A}^\star//\mathcal{A}^\star(1)$; the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^\star$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^\star$-subalgebra generated by Sq^1 and Sq^2 [8], [28]. By dualizing, the mod-2 motivic homology $\mathbf{M}\mathbb{Z}/2_\star\mathbf{kq}$ identifies with $\mathcal{A}_\star\Box_{\mathcal{A}_\star(1)}\mathbf{M}\mathbb{Z}/2_\star$ as an $\mathcal{A}_\star$-comodule algebra, and by change-of-rings the $\mathbf{M}\mathbb{Z}/2$-based Adams spectral sequence for $\mathbf{kq}$ takes the form

$$\mathrm{Ext}^{*,\star}_{\mathcal{A}_\star(1)}(\mathbf{M}\mathbb{Z}/2_\star, \mathbf{M}\mathbb{Z}/2_\star) \Rightarrow \mathbf{kq}_{\star}{}^{\wedge}_{2,\eta}. \tag{1.3}$$

As indicated in the notation, the filtered target groups of the spectral sequence (1.3) are all $(2,\eta)$-completed. The Ext-algebra over $\mathcal{A}_\star(1)$ appearing in (1.3) is accessible via homological algebra. For explicit calculations with (1.3) we refer to [5] and [9].

Section 3 is concerned with slice calculations. The negative slices of $\mathbf{kq}$ are evidently zero because $\mathbf{kq}$ is an effective motivic spectrum. Over a perfect field of characteristic $\neq 2$ and $i \geq 0$, we show in Theorem 3.2 the calculation

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{2n,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+2,2n}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n-2,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4n,2n}\mathbf{M}\mathbb{Z} & q = 2n, \\ \Sigma^{2n+1,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+3,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n+1,2n+1}\mathbf{M}\mathbb{Z}/2 & q = 2n+1. \end{cases} \tag{1.4}$$

The slices of $\mathbf{kq}$ are considerable "smaller" than those of $\mathbf{KQ}$ [19]. This is a helpful fact which is used in the calculation of the first stable homotopy groups of motivic spheres [20].

An immediate consequence of (1.4) is the explicit form of the slice spectral sequence given by mod-2 motivic cohomology groups $h^\star$ and integral motivic cohomology groups $H^\star$

$$\pi_{p,w}\mathsf{s}_q\mathbf{kq} = \begin{cases} h^{2n-p,2n-w} \oplus \cdots \oplus h^{4n-2-p,2n-w} \oplus H^{4n-p,2n-w} & q = 2n \\ h^{2n+1-p,2n+1-w} \oplus \cdots \oplus h^{4n+1-p,2n+1-w} & q = 2n+1 \end{cases} \Rightarrow \mathbf{kq}_{p,w}. \tag{1.5}$$

In Theorem 3.5 we identify the differentials in (1.5) in terms of motivic Steenrod operations. We also calculate the slices and the slice differentials for the η-inversion of $\mathbf{kq}$.

In Section 4 we identify the 0-line of $\mathbf{kq}$ with the Milnor-Witt K-theory over fields of characteristic not 2, and determine the associated graded for the groups on the 1-line of $\mathbf{kq}$.

Throughout the paper we employ the following assumptions and notations.

F, S	perfect field, finite dimensional separated noetherian scheme
$\mathbf{Sm}_S$	smooth schemes of finite type over S
$S^{s,t}$, $\Omega^{s,t}$, $\Sigma^{s,t}$	motivic (s,t)-sphere, (s,t)-loop space, (s,t)-suspension
$\mathbf{SH}$, $\mathbf{SH}^{\mathbf{eff}}$	motivic and effective motivic stable homotopy categories of S
E, $\mathbf{1} = S^{0,0}$	generic motivic spectrum, the motivic sphere spectrum
Λ, $\mathbf{M}A$	ring, motivic Eilenberg-MacLane spectra of a Λ-module A
$\mathbf{KGL}$, $\mathbf{KQ}$, $\mathbf{KW}$	algebraic and hermitian K-theory, Witt-theory
f_q, $\tilde{\mathsf{f}}_q$, s_q	qth effective cover, very effective cover, and slice

In all results concerning $\mathbf{KQ}$ and $\mathbf{kq}$ we assume that 2 is invertible on the base scheme S, as for $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and following [22] we impose the condition that

$$S \text{ is essentially smooth over a Dedekind domain.} \tag{1.6}$$

Applications will mostly concern perfect fields of characteristic $\neq 2$.

2 Connecting connective K-theories

Definition 2.1: Following [24, §5] we let $\mathbf{kq} \to \mathbf{KQ}$ denote the very effective cover of the hermitian K-theory spectrum $\mathbf{KQ}$ of quadratic forms [6] and let $\mathbf{kgl} \to \mathbf{KGL}$ denote the very effective cover of the algebraic K-theory spectrum $\mathbf{KGL}$ [25].

Remark 2.2: Following [2], [24], and working over F we can identify the very effective cover $\tilde{\mathsf{f}}_q\mathsf{E}$ of E with $\mathsf{f}_0(\mathsf{E}_{\geq 0})$, the effective cover of the connective cover $\mathsf{E}_{\geq 0}$ of E with respect to the homotopy t-structure on **SH** [14].

Lemma 2.3: *If F admits a complex embedding, the Betti realization of* **kgl** *coincides with the connective cover* ku *of the complex topological K-theory spectrum* KU *in the topological stable homotopy category.*

Proof. Recall from [24, Proposition 5.12] that **kgl** is a homotopy quotient of **MGL** under the orientation or Todd genus map, and similarly but easier that ku is a homotopy quotient of **MU**. The Betti realization functor [17, Appendix A] preserves homotopy colimits, and sends **MGL** to **MU**. □

Lemma 2.4: *Over F the effective and very effective covers of* **KGL** *coincide in* **SH**.

Proof. When $\mathsf{char}(F) = 0$ this is shown in [24, Corollary 5.13] by writing the effective cover of **KGL** as a homotopy quotient of **MGL** (the latter is very effective over any base scheme [24, Theorem 5.7]). If $\mathsf{char}(F) > 0$ we follow the proof of [2, Theorem 16] where the effective cover $\mathsf{f}_0\mathbf{KGL} \to \mathbf{KGL}$ is shown to be connective. For $t \geq 0$ the presheaf on $\mathbf{Sm}_F$

$$X \mapsto [\Sigma^{s,t} X_+, \mathsf{f}_0\mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s < 2t$, e.g., for $s - t < 0$ (this holds if X is regular, hence over any regular base scheme S). The case $t = 0$ implies by [2, Proposition 4] that $\mathsf{f}_0\mathbf{KGL}$ is connective, and by [2, Lemma 10] that $\mathsf{f}_0\mathbf{KGL}$ is the very effective cover. □

Remark 2.5: Lemma 2.4 holds more generally for motivic Landweber exact spectra in the sense of [16].

Over a noetherian scheme S of finite Krull dimension d, the presheaf on $\mathbf{Sm}_S$

$$X \mapsto [\Sigma^{s,t} X_+, \mathbf{KGL}] = KH_{s-2t}(X)$$

is zero for $s - 2t < -d$ by [11], since **KGL** represents homotopy K-theory over S [3]. Thus for $t \geq q$, the presheaf

$$X \mapsto [\Sigma^{s,t} X_+, \mathsf{f}_q\mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s - t + d < q$, and $\mathsf{f}_q\mathbf{KGL}$ is q-connected in the sense of [20, Definition 3.16]. If the very effective slice filtration coincides with the combination of the homotopy t-structure and the effective slice filtration over S, then $\mathsf{f}_0\mathbf{KGL}$ is the very effective cover, i.e., the effective and very effective slices of **KGL** agree. We can argue differently for $\mathbf{KGL}/2$ when 2 is invertible (this proof can also be adapted to motivic Landweber exact spectra).

Lemma 2.6: *Over a base scheme S as in* (1.6) *on which* 2 *is invertible, the effective and very effective covers of* $\mathbf{KGL}/2$ *coincide in* **SH**.

Proof. We claim $\mathbf{KGL}/2$ affords the description as a homotopy quotient of $\mathbf{MGL}/2$ for the generators of the Lazard ring $x_i \in \pi_{2i,i}\mathbf{MGL}$. Since $\mathbf{MGL}$ is effective the orientation map for $\mathbf{KGL}$ factors through

$$\phi\colon \mathbf{MGL} \to \mathsf{f}_0\mathbf{KGL}.$$

For $i \geq 2$ we have $\pi_{2i,i}\phi(x_i) = 0$, so that ϕ admits a factorization

$$\mathbf{MGL}/(x_2, x_3, \dots) \to \mathsf{f}_0\mathbf{KGL}.$$

We claim there is a canonically induced motivic weak equivalence

$$\psi\colon \mathbf{MGL}/(2, x_2, x_3, \dots) \xrightarrow{\cong} \mathsf{f}_0\mathbf{KGL}/2.$$

The map ψ yields an isomorphism on slices by [22, Theorem 11.3] and an appropriate adaption of [23, Proposition 5.4]. We show that $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\psi$ is a map between contractible motivic spectra, i.e., ψ is a map between slice complete spectra. For $\mathbf{KGL}/2$ this follows by the argument prior to Lemma 2.6: By [11] we know $\mathsf{f}_q\mathbf{KGL}$ is q-connected in the sense of [20, Definition 3.16]. Thus $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\mathbf{KGL} \cong *$, and likewise for $\mathsf{f}_0\mathbf{KGL}/2$. The contractibility of $\operatorname{holim}_{q\to\infty} \mathsf{f}_q\mathbf{MGL}/(2, x_2, x_3, \dots)$ follows from the description of the covers $\mathsf{f}_q\mathbf{MGL}$ in the proof of [23, Theorem 4.6]. To conclude for ψ we use that slices detect motivic weak equivalences between slice complete motivic spectra, cf. [7, §8.3]. Recall that $\mathbf{MGL}$ is a very effective motivic spectrum [24, Theorem 5.7]. The lemma follows from the canonically induced motivic weak equivalences in the commutative diagram

$$\begin{array}{ccccc}
\tilde{\mathsf{f}}_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & \tilde{\mathsf{f}}_0\mathsf{f}_0\mathbf{KGL}/2 & \xrightarrow{\cong} & \tilde{\mathsf{f}}_0\mathbf{KGL}/2 \\
\downarrow{\cong} & & \downarrow & & \downarrow \\
\mathsf{f}_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & \mathsf{f}_0\mathsf{f}_0\mathbf{KGL}/2 & \xrightarrow{\cong} & \mathsf{f}_0\mathbf{KGL}/2.
\end{array}$$

□

The Bott element $\mathbf{P}^1 \to \mathbf{KGL}$ lifts canonically to a map $\beta\colon \mathbf{P}^1 \to \mathbf{kgl}$ because $\mathbf{P}^1$ is very effective. Let γ denote the canonical composite

$$\mathbf{kgl} \to \mathsf{f}_0(\mathbf{KGL}) \to \mathsf{s}_0\mathbf{KGL}.$$

Proposition 2.7: *Over F multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl} \xrightarrow{\beta} \mathbf{kgl} \xrightarrow{\gamma} \mathbf{M}\mathbb{Z} \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}.$$

Proof. By Lemma 2.4 we have $\mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \cong \mathbf{kgl}$ and by $(2,1)$-periodicity $\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \cong \Sigma^{-2,-1}\mathbf{kgl}$. Our claim follows from the commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{2,1}\mathsf{f}_0(\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & \mathsf{f}_1(\Sigma^{2,1}\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & \mathsf{f}_1((\Sigma^{2,1}\mathbf{KGL})_{\geq 1}) & \xrightarrow{\cong} & \mathsf{f}_1(\mathbf{KGL}_{\geq 1}) \\
\downarrow{\beta} & & & & & & \downarrow{\beta'} \\
\mathsf{f}_0(\mathbf{KGL}_{\geq 0}) & & & \xrightarrow{\mathrm{id}} & & & \mathsf{f}_0(\mathbf{KGL}_{\geq 0})
\end{array}$$

and the cofiber sequence for the very effective zero slice of **KGL** [2, Lemma 7]

$$\mathsf{f}_1(\mathbf{KGL}_{\geq 1}) \xrightarrow{\beta'} \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \to \mathbf{M}\mathbb{Z},$$

which coincides with the usual zero slice $\mathsf{s}_0\mathbf{KGL} \cong \mathbf{M}\mathbb{Z}$ computed in [12], [27]. □

Proposition 2.8: *Over a base scheme S as in* (1.6) *on which* 2 *is invertible, multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl}/2 \xrightarrow{\beta} \mathbf{kgl}/2 \xrightarrow{\gamma} \mathbf{M}\mathbb{Z}/2 \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}/2.$$

Proof. This follows from Lemma 2.6. □

Lemma 2.9: *If* 2 *is invertible on a base scheme S as in* (1.6)*, then the composite*

$$\mathbf{M}\mathbb{Z} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\delta \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{3,1}\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\gamma \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{3,1}\mathbf{M}\mathbb{Z} \wedge \mathbf{M}\mathbb{Z}/2$$

is given by multiplication with the first Milnor operation

$$\mathsf{Q}_1 = \mathsf{Sq}^1\mathsf{Sq}^2 + \mathsf{Sq}^2\mathsf{Sq}^1 \colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{3,1}\mathbf{M}\mathbb{Z}/2.$$

Proof. The proof of Lemma 2.6 shows **KGL** and $\mathsf{f}_0\mathbf{KGL}/2$ are invariant under base change, being homotopy quotients of **MGL**. The same holds for $\mathbf{M}\mathbb{Z}$ and the motivic Steenrod algebra by [22, Section 10, Theorem 11.24]. Hence we may assume $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and reduce to $\mathbb{C}$ by rigidity of the motivic Steenrod algebra [22, Theorem 11.24]. Over $\mathbb{C}$ our claim follows from Lemma 2.3 and the corresponding topological result. □

Remark 2.10: Following [9, Theorem 5.4], Lemma 2.9 shows the mod-2 motivic cohomology $\mathbf{M}\mathbb{Z}/2^{\star}\mathbf{kgl}$ is the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^{\star}$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^{\star}$-subalgebra generated by $\mathsf{Q}_0 = \mathsf{Sq}^1$ and Q_1.

Proposition 2.11: *Over a field of characteristic $\neq 2$, multiplication with the Hopf map η induces a cofiber sequence*

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\mathrm{f}} \mathbf{kgl} \xrightarrow{\mathrm{h}} \Sigma^{2,1}\mathbf{kq}. \tag{2.1}$$

Here f *and* h *are functorially induced by the forgetful and hyperbolic maps between algebraic and hermitian K-theory, respectively.*

Proof. Consider the fiber F of the naturally induced forgetful map $\mathrm{f}_{\geq 0} \colon \mathbf{KQ}_{\geq 0} \to \mathbf{KGL}_{\geq 0}$. Since f_0 is a triangulated functor, $\mathsf{f}_0(\mathsf{F})$ is the fiber of $\mathrm{f} := \mathsf{f}_0(\mathrm{f}_{\geq 0})$. The composite map

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{\mathrm{f}} \mathbf{kgl}$$

is trivial because the first negative algebraic K-group $\pi_{1,1}\mathbf{kgl} = \pi_{1,1}\mathbf{KGL} = K_{-1}$ vanishes over regular schemes. We show there is an induced motivic weak equivalence $\Sigma^{1,1}\mathbf{kq} \to \mathsf{f}_0(\mathsf{F})$

of effective motivic spectra by checking the map of homotopy sheaves $\pi_{s,t}$ is an isomorphism for $t \geq 0$. This follows if (2.1) induces a long exact sequence of sheaves for $t \geq 0$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{s,t}\Sigma^{1,1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{\mathrm{f}} & \pi_{s,t}\mathbf{kgl} & \xrightarrow{\mathrm{h}} & \cdots \\ & & \downarrow \cong & & \downarrow = & & \downarrow = & & \\ \cdots & \longrightarrow & \pi_{s-1,t-1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{\mathrm{f}} & \pi_{s,t}\mathbf{kgl} & \xrightarrow{\mathrm{h}} & \cdots . \end{array} \tag{2.2}$$

By construction, (2.2) is exact for $t \geq 1$ and $s \geq t$ since in the said range it coincides with the long exact sequence

$$\cdots \to \pi_{s,t}\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \pi_{s,t}\mathbf{KQ} \xrightarrow{\mathrm{f}} \pi_{s,t}\mathbf{KGL} \xrightarrow{\mathrm{h}} \cdots$$

induced by the Wood cofiber sequence for η, $\mathbf{KQ}$, and $\mathbf{KGL}$ [19, Theorem 3.4].

If $t \geq 1$ and $s = t$, $\pi_{t,t}(\mathrm{f})\colon \pi_{t,t}\mathbf{kq} \to \pi_{t,t}\mathbf{kgl}$ is surjective since its target is trivial. Thus (2.2) is exact for $t \geq 1$ and all s; recall that $\pi_{s,t}\mathbf{kq} = \pi_{s,t}\mathbf{kgl} = 0$ for all $s < t$.

It remains to consider the case $t = 0$. By [2, Theorem 16] the composite

$$\mathsf{f}_0(\mathbf{KQ}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KQ}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KQ}_{\geq -1})$$

is an equivalence. The canonical map $\mathbf{KQ}_{\geq 0} \to \mathbf{KQ}_{\geq -1}$ is an isomorphism on homotopy sheaves $\pi_{s,t}$ for all $t \geq -1$ and all s. When $s < t-1$ and $s \geq t$ this follows by construction. The case $s = t-1$ holds since $\pi_{t-1,t}\mathbf{KQ} = 0$ for all $t \geq -1$. More precisely, the vanishing for $t \geq 0$ is implied by comparison with Witt theory because $\pi_{t-1,t}\mathbf{KW} = 0$ for all t. The case $t = -1$ follows from the long exact sequence

$$\cdots \to \pi_{0,0}\mathbf{KGL} \xrightarrow{0} \pi_{-2,-1}\mathbf{KQ} \xrightarrow{\eta} \pi_{-1,0}\mathbf{KQ} \xrightarrow{\mathrm{f}} \pi_{-1,0}\mathbf{KGL} = 0,$$

and surjectivity of the rank map $\mathrm{f}\colon \pi_{0,0}\mathbf{kq} \to \pi_{0,0}\mathbf{kgl}$. It follows that there is a canonical motivic weak equivalence

$$\mathsf{f}_0(\mathbf{KQ}_{\geq 0}) \xrightarrow{\cong} \mathsf{f}_{-1}(\mathbf{KQ}_{\geq 0}),$$

which implies exactness of (2.2) for $t = 0$. □

Lemma 2.12: *If 2 is invertible on a base scheme S as in (1.6), then the composite*

$$\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathrm{h} \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{\mathrm{f} \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2$$

is given by multiplication with $\mathsf{Sq}^2\colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{2,1}\mathbf{M}\mathbb{Z}/2$.

Proof. As in the proof of Lemma 2.9 it suffices to work over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and hence over $\mathbb{C}$. The result follows from Lemma 2.13 and the corresponding topological statement. □

Lemma 2.13: *If F admits a complex embedding, the Betti realization of* $\mathbf{kq}$ *coincides with the connective cover* ko *of the real topological K-theory spectrum* KO *in the topological stable homotopy category.*

Proof. This follows since the Betti realization sends $\mathbf{KQ}$ to KO, $\mathbf{kgl}$ to ku by Lemma 2.3, and preserves the Wood cofiber sequence. □

Remark 2.14: As in [9, Theorem 5.11], Lemma 2.9 identifies $\mathbf{M}\mathbb{Z}/2^{\star}\mathbf{kq}$ with the quotient of the mod-2 motivic Steenord algebra $\mathcal{A}^{\star}$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^{\star}$-subalgebra generated by Sq^1 and Sq^2, and the homotopy of $\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2$ as a comodule over the dual motivic Steenrod algebra recorded by the $\mathbf{M}\mathbb{Z}/2$-based Adams spectral sequence for $\mathbf{kq}$ (1.3).

Next we observe that $\mathbf{kgl}$ differs from the cover of algebraic K-theory introduced in [9]. By the cofiber sequence

$$\mathbf{kgl} = \mathsf{f}_0(\mathbf{KGL}_{\geq 0}) \to \mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \to \mathsf{s}_{-1}\mathbf{KGL} = \Sigma^{-2,-1}\mathbf{M}\mathbb{Z},$$

we obtain a long exact sequence and an isomorphism

$$\cdots \to \pi_{0,-1}\mathsf{s}_{-1}\mathbf{KGL} \to \pi_{-1,-1}\mathbf{kgl} \overset{\cong}{\to} \pi_{-1,-1}\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \to \pi_{-1,-1}\mathsf{s}_{-1}\mathbf{KGL} \to \cdots . \tag{2.3}$$

The outer terms in (2.3) are trivial. Since $\pi_{-1,-1}\mathsf{f}_{-1}(\mathbf{KGL}_{\geq -1}) \cong \pi_{-1,-1}\mathbf{KGL}$ it follows that $\pi_{-1,-1}\mathbf{kgl} \cong K_1(F) \cong F^{\times}$. Over the complex numbers, this calculation distinguishes $\mathbf{kgl}$ from the (2-complete) positive cellular cover of $\mathbf{KGL}$ in [9] because $\pi_{-1,-1}$ of the latter is trivial by construction.

Finally, we remark that $\mathbf{kq}$ does not coincide with the effective cover $\mathsf{f}_0\mathbf{KQ}$ featuring in the solution of the homotopy limit problem for the C_2-action on $\mathbf{kgl}$ in [21].

3 Slice computations

We shall identify the slices of $\mathbf{kq}$ similarly to the slices of $\mathbf{KQ}$ in [19]. The crucial ingredients are the Wood cofiber sequence (2.11) and the slices of connective algebraic K-theory $\mathbf{kgl}$.

Theorem 3.1: *Over F the canonical map $\mathbf{kgl} \to \mathbf{KGL}$ induces an isomorphism on all nonnegative slices. The negative slices of $\mathbf{kgl}$ are zero.*

Proof. Since $\mathbf{kgl} = \mathsf{f}_0\mathbf{KGL}$ by Lemma 2.4, this follows by construction. □

Identifying the slices of $\mathbf{kq}$ is more involved because $\mathbf{kq} \neq \mathsf{f}_0\mathbf{KQ}$.

Theorem 3.2: *When $\mathsf{char}(F) \neq 2$ the nonnegative slices of $\mathbf{kq}$ are given as*

$$\mathsf{s}_q\mathbf{kq} = \begin{cases} \Sigma^{2n,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+2,2n}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n-2,2n}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4n,2n}\mathbf{M}\mathbb{Z} & q = 2n, \\ \Sigma^{2n+1,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2n+3,2n+1}\mathbf{M}\mathbb{Z}/2 \vee \cdots \vee \Sigma^{4n+1,2n+1}\mathbf{M}\mathbb{Z}/2 & q = 2n+1. \end{cases}$$

The negative slices of $\mathbf{kq}$ are zero. Moreover, the canonical map $\mathbf{kq} \to \mathbf{KQ}$ induces a natural inclusion on slices, and respects the multiplicative structure.

Proof. Since $\mathbf{kq} = \mathsf{f}_0(\mathbf{KQ}_{\geq 0})$ is (very) effective, its negative slices are zero. Applying the slice functor to (2.11) yields a cofiber sequence. The natural isomorphism $\mathsf{s}_q \circ \Sigma^{1,1} \cong \Sigma^{1,1} \circ \mathsf{s}_{q-1}$ of [19, Lemma 2.1] shows the forgetful map $\mathrm{f}\colon \mathbf{kq} \to \mathbf{kgl}$ induces an isomorphism on zero slices

$$\mathsf{s}_0\mathbf{kq} \xrightarrow{\cong} \mathsf{s}_0\mathbf{kgl},$$

and likewise for the unit map $\mathbf{1} \to \mathbf{kq}$.

For the 1-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_0\mathbf{kq} = \Sigma^{1,1}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_1\mathbf{kq} \xrightarrow{\mathsf{s}_1\mathrm{f}} \mathsf{s}_1\mathbf{kgl} = \Sigma^{2,1}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_1\mathrm{h}} \Sigma^{2,1}\mathsf{s}_0\mathbf{kq} = \Sigma^{2,1}\mathbf{M}\mathbb{Z}.$$

Here $\mathsf{s}_1\mathrm{h}$ can be identified with an integer $n \in \mathbb{Z}$. Comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [19, §4.3] shows that $n = 2$, so that $\mathsf{s}_1\mathbf{kq} = \Sigma^{1,1}\mathbf{M}\mathbb{Z}/2$.

For the 2-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_1\mathbf{kq} = \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \xrightarrow{\eta} \mathsf{s}_2\mathbf{kq} \xrightarrow{\mathsf{s}_2\mathrm{f}} \mathsf{s}_2\mathbf{kgl} = \Sigma^{4,2}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_2\mathrm{h}} \Sigma^{2,1}\mathsf{s}_1\mathbf{kq} = \Sigma^{3,2}\mathbf{M}\mathbb{Z}/2.$$

Hence $\mathsf{s}_2\mathrm{h} = 0$, the cofiber sequence splits, and we get $\mathsf{s}_2\mathbf{kq} = \Sigma^{2,2}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4,2}\mathbf{M}\mathbb{Z}$. Moreover, $\mathsf{s}_2\mathrm{f}$ is the projection map onto $\Sigma^{4,2}\mathbf{M}\mathbb{Z}$.

For the 3-slices there is a cofiber sequence

$$\Sigma^{1,1}\mathsf{s}_2\mathbf{kq} = \Sigma^{3,3}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{5,3}\mathbf{M}\mathbb{Z} \xrightarrow{\eta} \mathsf{s}_3\mathbf{kq} \xrightarrow{\mathsf{s}_3\mathrm{f}} \mathsf{s}_3\mathbf{kgl} = \Sigma^{6,3}\mathbf{M}\mathbb{Z} \xrightarrow{\mathsf{s}_3\mathrm{h}} \Sigma^{2,1}\mathsf{s}_2\mathbf{kq}.$$

Here $\mathsf{s}_3\mathrm{h}$ maps trivially to $\Sigma^{4,3}\mathbf{M}\mathbb{Z}/2$, while the component of $\mathsf{s}_3\mathrm{h}$ mapping to $\Sigma^{6,3}\mathbf{M}\mathbb{Z}$ can be identified with an integer $n \in \mathbb{Z}$. We deduce $n = 2$ by comparison with the hyperbolic map $\mathbf{KGL} \to \mathbf{KQ}$ in [19, §4.3]. Hence we obtain $\mathsf{s}_3\mathbf{kq} \cong \Sigma^{3,3}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{5,3}\mathbf{M}\mathbb{Z}/2$.

Iterating these arguments produces the claimed calculation. □

Remark 3.3: Contrary to the calculation of the slices of $\mathbf{KQ}$ in [19] there is no "mysterious summand" appearing in Theorem 3.2 thanks to the connectivity of $\mathbf{kq}$. Each slice of $\mathbf{kq}$ is a finite sum of motivic Eilenberg-MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. The odd slices of $\mathbf{kq}$ are cellular of finite type for every F [20, §3.3], and likewise for all the slices when $\mathsf{char}(F) = 0$.

The multiplicative structure on the graded slices $\mathsf{s}_*\mathbf{kq}$ can be identified similarly to $\mathsf{s}_*\mathbf{KQ}$ as in [18, Theorem 3.3]. In more details, there is a motivic weak equivalence

$$\mathsf{s}_*\mathbf{kq} \cong \mathbf{M}\mathbb{Z}[\eta, \sqrt{\alpha}]/(2\eta = 0, \eta^2 \xrightarrow{\delta} \sqrt{\alpha}).$$

Here η has bidegree $(1,1)$ and $\sqrt{\alpha}$ is a class of bidegree $(4,2)$ arising from the $(8,4)$-periodicity operator on $\mathbf{KQ}$ mentioned in the introduction. Moreover, the action of the Hopf map η on the slices of $\mathbf{kq}$ can be read off from the proof of Theorem 3.2, giving us the next result.

Theorem 3.4: *When $\mathsf{char}(F) \neq 2$ the slices of $\mathbf{kq}[\frac{1}{\eta}] = \mathbf{KW}_{\geq 0}$ are given by*

$$\mathsf{s}_q(\mathbf{KW}_{\geq 0}) = \Sigma^{q,q}\Big(\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{2,0}\mathbf{M}\mathbb{Z}/2 \vee \Sigma^{4,0}\mathbf{M}\mathbb{Z}/2 \vee \cdots\Big),$$

and

$$\mathsf{s}_*(\mathbf{KW}_{\geq 0}) \cong \mathbf{M}\mathbb{Z}[\eta, \sqrt{\alpha}]/(2\eta = 2\sqrt{\alpha} = 0, \eta^2 \xrightarrow{\mathsf{Sq}^1} \sqrt{\alpha}).$$

The canonical map $\mathbf{KW}_{\geq 0} \to \mathbf{KW}$ *induces the natural inclusion on slices, and respects the multiplicative structure.*

Let $\mathbf{d}_1^{\mathbf{kq}}(q)\colon \mathsf{s}_q\mathbf{kq} \to \Sigma^{1,0}\mathsf{s}_{q+1}\mathbf{kq}$ denote the first slice differential as a map of motivic spectra, and similarly for $\mathbf{KW}_{\geq 0}$. By Theorem 3.2, $\mathbf{d}_1^{\mathbf{kq}}(q)$ is a map between finite sums of motivic Eilenberg-MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. Thus $\mathbf{d}_1^{\mathbf{kq}}(q)$ can be described via its restriction $\mathbf{d}_1^{\mathbf{kq}}(q,i)$ to the summand corresponding to the unique suspension $\Sigma^{q+i,q}$. We note that $\mathbf{d}_1^{\mathbf{kq}}(q,i)$ splits into at most three nontrivial components.

Let $\tau \in h^{0,1} \cong \mu_2(F)$ and $\rho \in h^{1,1} \cong F^\times/2$ denote the classes represented by $-1 \in F$; $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group of F in degree p and weight q. There are canonical maps $\mathrm{pr}\colon \mathbf{M}\mathbb{Z} \to \mathbf{M}\mathbb{Z}/2$ and $\partial\colon \mathbf{M}\mathbb{Z}/2 \to \Sigma^{1,0}\mathbf{M}\mathbb{Z}$.

Theorem 3.5: *When* $\mathsf{char}(F) \neq 2$ *the* $\mathbf{d}_1$*-differential in the slice spectral sequence for* $\mathbf{kq}$ *is given by*

$$\mathbf{d}_1^{\mathbf{kq}}(q,i) = \begin{cases} (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2, 0) & q-1 > i \equiv 0 \bmod 4 \\ (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \tau) & q-1 > i \equiv 2 \bmod 4 \end{cases}$$

$$\mathbf{d}_1^{\mathbf{kq}}(q,q) = \begin{cases} (0, \mathsf{Sq}^2 \circ \mathrm{pr}, 0) & q \equiv 0 \bmod 4 \\ (0, \mathsf{Sq}^2 \circ \mathrm{pr}, \tau \circ \mathrm{pr}) & q \equiv 2 \bmod 4 \end{cases}$$

$$\mathbf{d}_1^{\mathbf{kq}}(q,q-1) = \begin{cases} (\partial\mathsf{Sq}^2\mathsf{Sq}^1, \mathsf{Sq}^2, 0) & q \equiv 1 \bmod 4 \\ (\partial\mathsf{Sq}^2\mathsf{Sq}^1, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \tau) & q \equiv 3 \bmod 4. \end{cases}$$

Proof. Use Theorem 3.2 and the identification of $\mathbf{d}_1^{\mathbf{KQ}}$ for $\mathbf{KQ}$ in [19, Theorem 5.5]. □

Theorem 3.6: *When* $\mathsf{char}(F) \neq 2$ *the* $\mathbf{d}_1$*-differential in the slice spectral sequence for* $\mathbf{KW}_{\geq 0}$ *is given by*

$$\mathbf{d}_1^{\mathbf{KW}_{\geq 0}}(q,i) = \begin{cases} (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2, 0) & i \equiv 0 \bmod 4 \\ (\mathsf{Sq}^3\mathsf{Sq}^1, \mathsf{Sq}^2 + \rho\mathsf{Sq}^1, \tau) & i \equiv 2 \bmod 4. \end{cases}$$

Proof. This follows from Theorem 3.4 and the identification of $\mathbf{d}_1^{\mathbf{KW}}$ for $\mathbf{KW}$ recorded in [19, Theorem 5.3]. □

Following [18, §4] we calculate the first slice differentials for $\mathbf{kq}$ and $\mathbf{KW}_{\geq 0}$ in terms of the multiplicative generators for their slices.

We note that $\mathsf{d}_1^{\mathbf{kq}}(\sqrt{\alpha}^m\eta^n)$ is given by

$$\begin{cases} \tau\sqrt{\alpha}^{m-1}\eta^{n+3} + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m\equiv 1(2), n>1 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m\equiv 0(2), n>1 \\ \tau\sqrt{\alpha}^{m-1}\eta^4 + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^2 + \delta\mathsf{Sq}^2\mathsf{Sq}^1\sqrt{\alpha}^{m+1} & m\equiv 1(2), n=1 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^2 + \delta\mathsf{Sq}^2\mathsf{Sq}^1\sqrt{\alpha}^{m+1} & m\equiv 0(2), n=1 \\ \tau\mathrm{pr}\sqrt{\alpha}^{m-1}\eta^3 + \mathsf{Sq}^2\mathrm{pr}\sqrt{\alpha}^m\eta & m\equiv 1(2), n=0 \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta & m\equiv 0(2), n=0, \end{cases} \tag{3.1}$$

while $\mathsf{d}_1^{\mathbf{KW}_{\geq 0}}(\sqrt{\alpha}^m\eta^n)$ is given by

$$\begin{cases} \tau\sqrt{\alpha}^{m-1}\eta^{n+3} + (\mathsf{Sq}^2+\rho\mathsf{Sq}^1)\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m\equiv 1(2) \\ \mathsf{Sq}^2\sqrt{\alpha}^m\eta^{n+1} + \mathsf{Sq}^3\mathsf{Sq}^1\sqrt{\alpha}^{m+1}\eta^{n-1} & m\equiv 0(2). \end{cases} \tag{3.2}$$

Remark 3.7: The corresponding formula for $\mathsf{d}_1^{\mathbf{KQ}}(\sqrt{\alpha}^m\eta^n)$ in [18, §4] contains a typo when $m\equiv 1(2), n=0$. We thank Bert Guillou for pointing this out to us.

Remark 3.8: Bachmann [2] determined the very effective slices of **KQ** and hence of **kq** up to extensions. Additional work is needed to identify the corresponding first very effective slice differentials. A first step is to calculate the endomorphisms of the very effective zero slice of **KQ**. The very effective slices of $\mathbf{KW}_{\geq 0}$ were determined up to extensions in [2, Lemma 6].

4 Homotopy computations

First we identify the target of the slice spectral sequences for the sphere and very effective hermitian K-theory.

Theorem 4.1: *Over a field F of characteristic $\neq 2$ there are conditionally convergent slice spectral sequences*

$$\pi_\star\mathsf{s}_*\mathbf{1} \Longrightarrow \pi_\star\mathbf{1}^\wedge_\eta, \tag{4.1}$$

and

$$\pi_\star\mathsf{s}_*\mathbf{kq} \Longrightarrow \pi_\star\mathbf{kq}^\wedge_\eta. \tag{4.2}$$

Proof. Here (4.1) is shown in [20, §3]. The only issue in (4.2) is to identify the quotient of **kq** by η with a slice complete spectrum [21, §4]. This follows directly from Lemma 2.4, Proposition 2.11 and [21, Lemma 3.11]. □

To formulate our identification of the 0-line of **kq** we recall the definition of Milnor-Witt K-theory $K^{MW}_*(F)$ in [15]. It is the quotient of the free associative integrally graded ring on the set of symbols $[F^\times] := \{[u] \mid u\in F^\times\}$ in degree 1 and η in degree -1 by the homogeneous ideal enforcing the relations

(1) $[uv] = [u] + [v] + \eta[u][v]$ (η-twisted logarithm),

(2) $[u][v] = 0$ for $u + v = 1$ (Steinberg relation),

(3) $[u]\eta = \eta[u]$ (commutativity), and

(4) $(2 + [-1]\eta)\eta = 0$ (hyperbolic relation).

Milnor-Witt K-theory is ε-commutative for $\varepsilon = -(1 + [-1]\eta)$. By work of Morel [13] there is an isomorphism with the graded ring of endomorphisms of the sphere

$$K_*^{MW}(F) \cong \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}\mathbf{1}.$$

Moreover, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt ring of quadratic forms with its standard presentation, inverting η in $K_*^{MW}(F)$ yields the ring of Laurent polynomials $W(F)[\eta^{\pm 1}]$ over the Witt ring, and $K_*^{MW}(F)/\eta = K_*^{M}(F)$, the Milnor K-theory ring of F.

Theorem 4.2: *Over a field F of characteristic $\neq 2$ the unit map $\mathbf{1} \to \mathbf{kq}$ induces an isomorphism on 0-lines*

$$K_*^{MW}(F) \xrightarrow{\cong} \bigoplus_{n\in\mathbb{Z}} \pi_{n,n}\mathbf{kq}. \tag{4.3}$$

Proof. Recall from [20, §5] the short exact sequence

$$0 \to \pi_{n,n}\mathbf{1} \to \pi_{n,n}\mathbf{1}^{\wedge}_{\eta} \oplus \pi_{n,n}\mathbf{1}[\tfrac{1}{\eta}] \to \pi_{n,n}\mathbf{1}^{\wedge}_{\eta}[\tfrac{1}{\eta}] \to 0. \tag{4.4}$$

Similarly, following [20, §3], the η-arithmetic square

$$\begin{array}{ccc} \mathbf{kq} & \longrightarrow & \mathbf{kq}[\tfrac{1}{\eta}] \\ \downarrow & & \downarrow \\ \mathbf{kq}^{\wedge}_{\eta} & \longrightarrow & \mathbf{kq}^{\wedge}_{\eta}[\tfrac{1}{\eta}] \end{array}$$

for very effective K-theory yields a short exact sequence

$$0 \to \pi_{n,n}\mathbf{kq} \to \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta} \oplus \pi_{n,n}\mathbf{kq}[\tfrac{1}{\eta}] \to \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}[\tfrac{1}{\eta}] \to 0. \tag{4.5}$$

Here we use the vanishing of $\pi_{n+1,n}\mathbf{kq}^{\wedge}_{\eta}[\frac{1}{\eta}]$ and $\pi_{n-1,n}\mathbf{kq}$. On the terms contributing to the 0-line, the map from (4.1) to (4.2) is an isomorphism. Theorem 3.5 combined with the same computations as in [20, §4] show the said isomorphism persists to the E^{∞}-page. By invoking Theorem 4.1 we conclude $\pi_{n,n}\mathbf{1}^{\wedge}_{\eta} \xrightarrow{\cong} \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}$ and $\pi_{n,n}\mathbf{1}^{\wedge}_{\eta}[\frac{1}{\eta}] \xrightarrow{\cong} \pi_{n,n}\mathbf{kq}^{\wedge}_{\eta}[\frac{1}{\eta}]$. As noted above, by [13] we have $\pi_{n,n}\mathbf{1}[\frac{1}{\eta}] \xrightarrow{\cong} \pi_{n,n}\mathbf{kq}[\frac{1}{\eta}] \cong \pi_{n,n}\mathbf{KW}_{\geq 0} \cong W(F)$. A straightforward comparison between (4.4) and (4.5) allows us to deduce (4.3). □

Remark 4.3: It was pointed to us by Bachmann that the results of [2] yield an isomorphism of the zeroth generalized slices $\tilde{\mathsf{s}}_0\mathbf{1} \cong \tilde{\mathsf{s}}_0\mathbf{KQ}$. This gives another proof for Theorem 4.2.

We note the isomorphism $\pi_{n+1,n}\mathbf{kq} \xrightarrow{\cong} \pi_{n+1,n}\mathbf{kq}^{\wedge}_{\eta}$ follows as in [20, Proposition 5.3]. Thus for the purpose of identifying the 1-line of **kq** we may use Theorem 3.5 and computations as in [20, §4] to deduce:

Proposition 4.4: *The only nontrivial terms in* (4.2) *contributing to* $\pi_{n+1,n}\mathbf{kq}$ *are*

$$E^{\infty}_{n+1,q,n}(\mathbf{kq}) = \begin{cases} h^{-n+1,-n+2}/\mathsf{Sq}^2(h^{-n-1,-n+1}) & q=2 \\ h^{-n,-n+1}/\mathsf{Sq}^2\mathrm{pr}(H^{-n-2,-n}) & q=1 \\ H^{-n-1,-n} & q=0. \end{cases}$$

Here $h^{i,j}$ and $H^{i,j}$ denote the mod-2 and integral motivic cohomology groups of F in degree i and weight j. This determines the 1-line of **kq** up to extensions. When $n>1$ we read off the vanishing $\pi_{n+1,n}\mathbf{kq}=0$. The first nontrivial group on the 1-line is $\pi_{2,1}\mathbf{kq} \cong \mu_2(F) \cong \mathbb{Z}/2$. When $n=0$ we obtain $\pi_{1,0}\mathbf{kq} \cong \pi_{1,0}\mathbf{KQ} \cong F^{\times}/2 \oplus \mu_2(F)$. Furthermore, there is a short exact sequence

$$0 \to h^{2,3}/\mathsf{Sq}^2(h^{0,2}) \to \pi_{0,-1}\mathbf{kq} \to h^{1,2} \to 0. \tag{4.6}$$

When $n \leq -2$ the group $\pi_{n+1,n}\mathbf{kq}$ surjects onto the integral motivic cohomology group $H^{-n-1,-n}$, with kernel described by Proposition 4.4.

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St. Petersburg Department, Steklov Math. Institute, and
Chebyshev Laboratory, St. Petersburg State University, Russia.
e-mail: alseang@gmail.com

Institut für Mathematik, Universität Osnabrück, Germany.
e-mail: oliver.roendigs@uni-osnabrueck.de

Department of Mathematics, University of Oslo, Norway.
e-mail: paularne@math.uio.no