Einstein manifolds with torsion and nonmetricity

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Manifolds endowed with torsion and nonmetricity are interesting both from the physical and the mathematical points of view. In this paper, we generalize some results presented in the literature. We study Einstein manifolds (i.e., manifolds whose symmetrized Ricci tensor is proportional to the metric) in d dimensions with nonvanishing torsion that has both a trace and a traceless part, and analyze invariance under extended conformal transformations of the corresponding field equations. Then, we compare our results to the case of Einstein manifolds with zero torsion and nonvanishing nonmetricity, where the latter is given in terms of the Weyl vector (Einstein-Weyl spaces). We find that the trace part of the torsion can alternatively be interpreted as the trace part of the nonmetricity. The analysis is subsequently extended to Einstein spaces with both torsion and nonmetricity, where we also discuss the general setting in which the nonmetricity tensor has both a trace and a traceless part. Moreover, we consider and investigate actions involving scalar curvatures obtained from torsionful or nonmetric connections, analyzing their relations with other gravitational theories that appeared previously in the literature. In particular, we show that the Einstein-Cartan action and the scale invariant gravity (also known as conformal gravity) action describe the same dynamics. Then, we consider the Einstein-Hilbert action coupled to a three-form field strength and show that its equations of motion imply that the manifold is Einstein with totally antisymmetric torsion.

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I. INTRODUCTION

In the 19th century, the branches of mathematics and physics experienced an extraordinary progress with the emergence of non-Euclidean geometry. In particular, the development of Riemannian geometry led to many important results, among which is the rigorous mathematical formulation of Einstein's general relativity.

In spite of the success and predictive power of general relativity, there are still some open problems and questions, whose understanding and solution may need the formulation of a new theoretical framework as well as generalizations and extensions of Riemannian geometry. One possible way of generalizing Riemannian geometry consists in allowing for nonvanishing torsion and nonmetricity (metric affine gravity) [1] (see also [2–8] and the recent work [9]). There are several physical (and mathematical) reasons which motivate the introduction of torsion or nonmetricity in the context of gravitational theories (see [1] for details). For instance, nonmetricity is a measure for the violation of local Lorentz invariance, which has been attracting some interest recently. Furthermore, nonmetricity and torsion find applications in the theory of defects in crystals, where, in particular, nonmetricity describes the density of point defects, while torsion is interpreted as density in line defects [10]. Moreover, as shown in [11], incorporating torsion and nonmetricity may allow to explore new physics associated with defects in a hypothetical spacetime microstructure. Recently, in [12–14] the authors discussed the propagation of matter fields in theories with torsion and nonmetricity. Further applications include quantum gravity [15] and cosmology [16–18].

Moreover, torsion is related to the translation group and to the energy-momentum tensor of matter, while nonmetricity is related to the group $GL(4,\mathbb{R})/SO(3,1)$ (in four dimensions) and to the hypermomentum current (see Refs. [5,6,8], where, in particular in the latter, equations of motion in metric affine manifolds were studied); the trace of the nonmetricity (the Weyl vector) is related to the scale group and to the dilation (or scale) current. In particular, in matter the shear and dilation currents couple to nonmetricity, and they are its sources. It is to the dilation current that the Weyl vector is coupled.

Historically, a remarkable generalization of Riemannian geometry was first proposed in 1918 by Weyl (cf. e.g., [19–22] for an introduction), who introduced an additional symmetry in an attempt of unifying electromagnetism with gravity geometrically [23,24]. In Weyl's theory, both the direction and the length of vectors are allowed to vary under parallel transport. However, Weyl's attempt to identify the trace part of the connection associated with stretching and

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contraction with the vector potential of electromagnetism failed, due to observational inconsistencies (see e.g., [25] for details). Subsequently, there were many attempts to adjust the theory. Finally, following [26], Weyl showed that a satisfactory theory of electromagnetism can be achieved if the scale factor is replaced by a complex phase. This was the origin of what is now well known as the U(1) gauge theory.[1]

The trace part of the connection introduced by Weyl is known as the Weyl vector. When it is given by the gradient of a function, there exists a scale transformation (dilatation) that sets the vector to zero. In this case, Weyl geometry is said to be integrable (parallel transported vectors along closed paths return with unaltered lengths) and there exists a subclass of global gauges in which the geometry is Riemannian.

Although Weyl's theory of electromagnetism fails, there has been a renewed interest in it [28,29]. Indeed, there are motivations for seeking a deeper understanding of general relativity formulated within the framework of Weyl geometry (and especially of integrable Weyl geometry), in particular concerning scale invariant general relativity and higher symmetry approaches to gravity involving conformal invariance [25]. Always in Weyl's perspective, conformal (higher curvature) gravity theories were constructed and studied in detail in [30–32]. Furthermore, in [33] an observational constraint to the nonintegrability of lengths in the original Weyl theory was placed for the first time.

A Weyl manifold is a conformal manifold equipped with a torsionless but nonmetric connection, called Weyl connection, preserving the conformal structure. Then, it is said to be Einstein-Weyl if the symmetric, trace-free part of the Ricci tensor of this connection vanishes (and the symmetric part of the Ricci tensor of the Weyl connection is proportional to the metric). Thus, Einstein-Weyl manifolds represent the analog of Einstein spaces in Weyl geometry and are less trivial than the latter, which have necessarily constant curvature in three dimensions.

Einstein-Weyl spaces were studied in [34–46], and they are also relevant in the context of (fake) supersymmetric supergravity solutions [47–52]. Einstein-Weyl geometry is particularly rich in three dimensions [34,35], where it has an equivalent formulation in twistor theory [53], which provides a tool for constructing self-dual four-dimensional geometries. Self-dual conformal four-manifolds play a central role in low-dimensional differential geometry, and a key tool in this context is provided by the so-called Jones-Tod correspondence [54], in which the reduction of the self-duality equation by a conformal vector field is given by the Einstein-Weyl equation together with the linear equation for an Abelian monopole (in other words, the Jones-Tod correspondence is a correspondence between a self-dual space with symmetry and an Einstein-Weyl space with a monopole). Einstein-Weyl structures are also related to certain integrable systems, like the SU(∞) Toda field equations [55] or the dispersionless Kadomtsev-Petviashvili equation [56].

On the other hand, as already mentioned, another generalization of Riemannian geometry is given by the introduction of a nonvanishing torsion, which is the case for the Einstein-Cartan theory [57–61], where the geometrical structure of the manifold is modified by allowing for an antisymmetric part of the affine connection (see also [62] for a recent review on torsional constructions and metric affine gauge theories). Cartan suggested that spacetime torsion is related to the intrinsic angular momentum, before the concept of spin was introduced. Cartan's theory was then reinterpreted as a theory of gravitation with spin and torsion [63–65]. Subsequently, the introduction of a nonvanishing torsion has been widely analyzed in general relativity and in the setting of teleparallel gravities [66–72], as well as in other contexts. In particular, in [73,74] the torsion tensor was related to the Kalb-Ramond field [75]. Furthermore, the relation between torsion and conformal symmetry was studied by several authors, and it turned out that torsion plays an important role in conformal invariance of the action and behaves like an effective gauge field [76,77]. Subsequently it was shown that in the nonminimally coupled metric-scalar-torsion theory, for some special choice of the action, torsion acts as a compensating field and the full theory is conformally equivalent to general relativity at a classical level [78,79]. More recently, in [80] the metric-torsional conformal curvature of four-dimensional spacetime was constructed, and in [81] different types of torsion were investigated, together with their effect on the dynamics and conformal properties of fields. Conformal invariance was also analyzed in generalizations of Einstein-Cartan spaces including nonmetricity [82–85], and in [86] an exhaustive classification of metric affine theories according to their scale symmetries was presented (see also [87]). Finally, in a cosmological context, it was proposed in [88,89] that a nonvanishing torsion can serve as an origin for dark energy. Let us also mention, here, that a generic theory (without matter) involving terms quadratic in torsion and nonmetricity will be classically equivalent at low energy to Einstein's theory, as discussed in [90] and references therein. From a mathematical point of view, Einstein manifolds with skew-symmetric torsion (i.e., totally antisymmetric torsion) were analyzed in [91,92].

Motivated by the fact that nonmetric and torsionful connections are interesting both from the physical and the mathematical point of view, in this paper we generalize some results presented previously in the literature. In particular, we study Einstein manifolds in d dimensions with nonvanishing torsion that has both a trace and a traceless part, and we analyze invariance under extended conformal transformations (see Refs. [78,82], where these

[1]See [27] and references therein for interesting details on "the dawning of gauge theory."

transformations are defined for metric affine spaces) in this context. Then, we compare our results to the case of Einstein spaces with zero torsion and nonvanishing nonmetricity, where the latter is given in terms of the Weyl vector. We find that the trace part of the torsion can alternatively be interpreted as the trace part of the nonmetricity. Subsequently, we extend our analysis to the case of Einstein manifolds with both torsion and nonmetricity (Einstein-Cartan-Weyl spaces), where we allow for both a trace and a traceless part of the nonmetricity tensor. Finally, we construct and investigate actions involving scalar curvatures obtained from torsionful or nonmetric connections, and analyze their relations with other gravitational theories known in the literature. In particular, we consider the Einstein-Cartan action and discuss its relationship with scale invariant gravity (also known as conformal gravity, which is invariant under Weyl transformations) [93–102], showing that they describe the same dynamics. Then, we study the Einstein-Hilbert action coupled to a three-form $H_{\mu\nu\rho}$ and show that its equations of motion imply that the manifold is Einstein with skew-symmetric torsion. Furthermore, it turns out that the equations of motion of Einstein gravity coupled to a three-form may also be retrieved from a constrained action that contains the scalar curvature of a connection with torsion. Let us specify that in this work we will focus on the vacuum, without considering matter.

The remainder of this paper is organized as follows: In Sec. II, we consider Einstein spaces with torsion that has both a trace and a traceless part. In particular, we find the field equations satisfied by an Einstein-Cartan space. Then, the invariance under extended conformal (Weyl) transformations of the latter is studied and the results are compared to the case of Einstein-Weyl manifolds, which have nonvanishing nonmetricity but zero torsion. In Sec. III, we extend the analysis to Einstein-Cartan-Weyl manifolds, and add thereby also a traceless part to the nonmetricity tensor. In Sec. IV, the Weyl invariant Einstein-Cartan action is studied and shown to be equivalent to scale invariant gravity (i.e., conformal gravity), which involves the presence of a scalar field ϕ. Subsequently, in Sec. V we consider the Einstein-Hilbert action coupled to a three-form, and show that the resulting field equations imply that the space is Einstein with torsion, where the latter is proportional to $H_{\mu\nu\rho}$. We conclude our work with some comments and possible future developments. In the Appendix we collect some technical details.

II. EINSTEIN MANIFOLDS WITH TORSION

We first consider a d-dimensional Einstein manifold with metric $g_{\mu\nu}$ and nonvanishing torsion (i.e., a so-called Einstein-Cartan manifold).[2] The connection $\Gamma^\lambda{}_{\mu\nu}$ can be decomposed as

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + N^\lambda{}_{\mu\nu}, \tag{2.1}$$

where $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ are the connection coefficients of the Levi-Civita connection (i.e., the Christoffel symbols) and $N^\lambda{}_{\mu\nu}$ is called the distortion. Here, the latter can be written as[3]

$$N_{\lambda\mu\nu} = \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}), \tag{2.2}$$

where $T^\lambda{}_{\mu\nu} = e_a{}^\lambda T^a{}_{\mu\nu}$ is the torsion,[4] antisymmetric in the last two indices,

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \tag{2.3}$$

Let us also introduce the contorsion (or contortion), antisymmetric in the first two indices,

$$K_{\nu\lambda\mu} = \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}). \tag{2.4}$$

Observe that the distortion (2.2) can then be written as

$$N_{\lambda\mu\nu} = K_{\nu\lambda\mu}. \tag{2.5}$$

In [91,92], Einstein manifolds with skew-symmetric torsion were analyzed. Below, we shall consider a general decomposition of the torsion tensor, which can be decomposed in a traceless and a trace part as

$$T^\lambda{}_{\mu\nu} = \check{T}^\lambda{}_{\mu\nu} + \frac{1}{d-1}(\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu). \tag{2.6}$$

In particular, we have $\check{T}^\nu{}_{\mu\nu} = 0$ and $T_\mu \equiv T^\nu{}_{\mu\nu}$. Notice that $2N^\lambda{}_{[\mu\nu]} = T^\lambda{}_{\mu\nu}$. The distortion (2.5) becomes then

$$\begin{aligned} N_{\lambda\mu\nu} &= \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}) = K_{\nu\lambda\mu} \\ &= \frac{1}{2}(\check{T}_{\nu\lambda\mu} - \check{T}_{\lambda\nu\mu} - \check{T}_{\mu\nu\lambda}) + \frac{1}{d-1}(g_{\mu\nu}T_\lambda - g_{\mu\lambda}T_\nu), \end{aligned} \tag{2.7}$$

and thus (2.1) reads

[2] Our convention for the metric signature is $(-,+,+,\cdots,+)$.

[3] As we will see in Sec. III, in the case of torsionful, nonmetric connections the distortion is generally defined as $N_{\lambda\mu\nu} = \frac{1}{2}(T_{\nu\lambda\mu} - T_{\lambda\nu\mu} - T_{\mu\nu\lambda}) + \frac{1}{2}(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu})$, where $Q_{\lambda\mu\nu}$ is the nonmetricity tensor (we will introduce and define it later). In the present section we first restrict ourselves to the case of vanishing nonmetricity, namely we consider a metric, torsionful connection. The nonmetric torsion-free case [where $N_{\lambda\mu\nu} = \frac{1}{2}(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu})$] will be discussed at the end of the current section when we will explore Einstein-Weyl spaces.

[4] $e_a{}^\lambda$ denotes the inverse vielbein and early latin indices $a, b, \ldots$ refer to the tangent space. The torsion two-form is defined as $T^a = de^a + \omega^a{}_b \wedge e^b$.

$$\Gamma^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{2}(\check{T}_\nu{}^\lambda{}_\mu - \check{T}^\lambda{}_{\nu\mu} - \check{T}_{\mu\nu}{}^\lambda) + \frac{1}{d-1}(g_{\mu\nu}T^\lambda - \delta_\mu{}^\lambda T_\nu). \tag{2.8}$$

The explicit expression for the Riemann tensor $\tilde{R}^\lambda{}_{\rho\mu\nu} = \partial_\mu\Gamma^\lambda{}_{\nu\rho} - \partial_\nu\Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho}$ of the Einstein-Cartan connection $\Gamma^\lambda{}_{\mu\nu}$ is given in the Appendix [see Eq. (A1)]. There as well as in the following, ∇ denotes the covariant derivative of the Levi-Civita connection. The corresponding Ricci tensor $R_{\rho\nu} = R^\mu{}_{\rho\mu\nu}$ is given by (A2). In particular, one gets

$$R_{[\rho\nu]} = \frac{d-2}{d-1}\nabla_{[\nu}T_{\rho]} - \frac{1}{2}\check{T}_{\mu\sigma[\nu}\check{T}_{\rho]}{}^{\mu\sigma} - \frac{1}{d-1}T^\mu\check{T}_{[\rho\nu]\mu} + \frac{2-d}{2(d-1)}T^\mu\check{T}_{\mu\nu\rho} + \frac{1}{2}\nabla_\mu\check{T}^\mu{}_{\nu\rho}. \tag{2.9}$$

Note that if we set the traceless part of the torsion to zero, $\check{T}^\lambda{}_{\mu\nu} = 0$, we are left with

$$R_{[\rho\nu]} = \frac{d-2}{d-1}\nabla_{[\nu}T_{\rho]} = \frac{d-2}{d-1}\partial_{[\nu}T_{\rho]} \equiv \frac{d-2}{2(d-1)}F_{\nu\rho}, \tag{2.10}$$

where

$$F_{\nu\rho} \equiv \partial_\nu T_\rho - \partial_\rho T_\nu. \tag{2.11}$$

In general, one has thus

$$R_{[\rho\nu]} = \frac{d-2}{2(d-1)}F_{\nu\rho} - \frac{1}{2}\check{T}_{\mu\sigma[\nu}\check{T}_{\rho]}{}^{\mu\sigma} - \frac{1}{d-1}T^\mu\check{T}_{[\rho\nu]\mu} + \frac{2-d}{2(d-1)}T^\mu\check{T}_{\mu\nu\rho} + \frac{1}{2}\nabla_\mu\check{T}^\mu{}_{\nu\rho}. \tag{2.12}$$

One can also construct another Ricci tensor by contracting the second and the third index of the Riemann tensor. However, the Ricci tensor obtained in this way coincides with (A2), since $R_{\lambda\rho\mu\nu} = -R_{\rho\lambda\mu\nu}$ is still valid (while it fails to be for nonmetric connections).

The Ricci scalar reads

$$R = g^{\rho\nu}R_{\rho\nu} = \tilde{R} + \frac{(d-2)(1-d)}{(d-1)^2}T_\mu T^\mu + 2\nabla_\mu T^\mu + \frac{1}{4}\check{T}_{\mu\nu\rho}\check{T}^{\mu\nu\rho} - \frac{1}{2}\check{T}_{\nu\rho\mu}\check{T}^{\mu\nu\rho}. \tag{2.13}$$

Let us now define an Einstein space with torsion by

$$R_{(\rho\nu)} = \lambda g_{\rho\nu} \tag{2.14}$$

for some function λ. Using (A2), this becomes

$$\tilde{R}_{\rho\nu} + \frac{1}{d-1}[g_{\rho\nu}\nabla_\mu T^\mu + (d-2)\nabla_{(\nu}T_{\rho)} + (d-3)T^\mu\check{T}_{(\rho\nu)\mu}] + \frac{1}{(d-1)^2}[(2-d)g_{\rho\nu}T_\mu T^\mu + (d-2)T_\nu T_\rho] + \frac{1}{4}\check{T}_\rho{}^{\mu\sigma}\check{T}_{\nu\mu\sigma} - \frac{1}{2}\check{T}_{\mu\sigma(\rho}\check{T}_{\nu)}{}^{\mu\sigma} - \nabla_\mu\check{T}_{(\rho\nu)}{}^\mu = \lambda g_{\rho\nu}, \tag{2.15}$$

whose trace yields

$$\tilde{R} + 2\nabla_\mu T^\mu - \frac{d-2}{d-1}T_\mu T^\mu + \frac{1}{4}\check{T}^{\mu\rho\sigma}\check{T}_{\mu\rho\sigma} - \frac{1}{2}\check{T}^{\mu\rho\sigma}\check{T}_{\rho\sigma\mu} = \lambda d, \tag{2.16}$$

and thus

$$\lambda = \frac{1}{d}\left(\tilde{R} + 2\nabla_\mu T^\mu - \frac{d-2}{d-1}T_\mu T^\mu + \frac{1}{4}\check{T}^{\mu\rho\sigma}\check{T}_{\mu\rho\sigma} - \frac{1}{2}\check{T}^{\mu\rho\sigma}\check{T}_{\rho\sigma\mu}\right). \tag{2.17}$$

Hence, in terms of Riemannian data, (2.14) becomes

$$\begin{aligned}\tilde{R}_{\rho\nu} &+ \frac{1}{d-1}[(d-2)\nabla_{(\nu}T_{\rho)} + (d-3)T^\mu\check{T}_{(\rho\nu)\mu}] + \frac{1}{(d-1)^2}[(d-2)T_\nu T_\rho] \\ &+ \frac{1}{4}\check{T}_\rho{}^{\mu\sigma}\check{T}_{\nu\mu\sigma} - \frac{1}{2}\check{T}_{\mu\sigma(\rho}\check{T}_{\nu)}{}^{\mu\sigma} - \nabla_\mu\check{T}_{(\rho\nu)}{}^\mu \\ &= \frac{1}{d}g_{\rho\nu}\left[\tilde{R} + \frac{d-2}{d-1}\nabla_\mu T^\mu + \frac{d-2}{(d-1)^2}T_\mu T^\mu + \frac{1}{4}\check{T}^{\mu\tau\sigma}\check{T}_{\mu\tau\sigma} - \frac{1}{2}\check{T}^{\mu\tau\sigma}\check{T}_{\tau\sigma\mu}\right],\end{aligned} \tag{2.18}$$

which is a set of nonlinear partial differential equations characterizing an Einstein manifold with torsion, henceforth termed Einstein-Cartan space.

A. Extended conformal invariance in Einstein-Cartan manifolds

We will now show that (2.14) is invariant under extended conformal transformations discussed in [78]. Thus, let us consider the extended conformal (Weyl) transformations

$$\begin{aligned} g_{\mu\nu} &\mapsto g'_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \\ T^{\lambda}{}_{\mu\nu} &\mapsto T'^{\lambda}{}_{\mu\nu} = T^{\lambda}{}_{\mu\nu} + \delta^{\lambda}{}_{\nu}\partial_{\mu}\omega - \delta^{\lambda}{}_{\mu}\partial_{\nu}\omega, \end{aligned} \tag{2.19}$$

where $\omega = \omega(x)$ is an arbitrary scalar field. Therefore, we have

$$T_{\mu} \mapsto T'_{\mu} = T_{\mu} + (d-1)\partial_{\mu}\omega, \qquad \breve{T}^{\lambda}{}_{\mu\nu} \mapsto \breve{T}'^{\lambda}{}_{\mu\nu} = \breve{T}^{\lambda}{}_{\mu\nu}. \tag{2.20}$$

Moreover, (2.19) leads to the following transformation for the connection:

$$\Gamma^{\rho}{}_{\mu\nu} \mapsto \Gamma'^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \delta^{\rho}{}_{\nu}\partial_{\mu}\omega, \tag{2.21}$$

which is called, specifically, a special projective transformation of the connection (see, for instance, Refs. [86,87]), also known as λ transformation. Let us observe that, actually, the combination of the conformal metric transformation in (2.19) plus the special projective transformation (2.21) of the affine connection is called a frame rescaling (see Refs. [86,87], where frame rescalings have been considered in metric affine spaces, also including Einstein-Cartan ones).

For the Riemann tensor, the Ricci tensor and the scalar curvature, we get respectively

$$\begin{aligned} R^{\sigma}{}_{\rho\mu\nu} &\mapsto R'^{\sigma}{}_{\rho\mu\nu} = R^{\sigma}{}_{\rho\mu\nu}, \\ R_{\rho\nu} &\mapsto R'_{\rho\nu} = R_{\rho\nu}, \\ R &\mapsto R' = e^{-2\omega} R. \end{aligned} \tag{2.22}$$

Now, (2.14) implies $R = \lambda d$, so that (2.14) is equivalent to

$$R_{(\rho\nu)} = \frac{1}{d} R g_{\rho\nu}, \tag{2.23}$$

which is obviously invariant under extended conformal transformations given by (2.19).

B. Comparison with Einstein-Weyl spaces

A Weyl structure on a manifold Σ consists of a conformal structure $[g] = \{fg | f : \Sigma \to \mathbb{R}^+\}$, and a torsion-free connection $\hat{\nabla}$ fulfilling

$$\hat{\nabla}_{\nu} g_{\lambda\mu} = 2\Theta_{\nu} g_{\lambda\mu}, \tag{2.24}$$

for some one-form Θ on Σ (the Weyl vector). The condition (2.24) is invariant under the transformation

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \qquad \Theta_{\mu} \mapsto \Theta'_{\mu} = \Theta_{\mu} + \partial_{\mu}\omega. \tag{2.25}$$

One can then define the nonmetricity tensor, which reads

$$Q_{\mu\nu\lambda} = -\hat{\nabla}_{\nu} g_{\lambda\mu} = -2\Theta_{\nu} g_{\lambda\mu}. \tag{2.26}$$

In this case the distortion is given by

$$\begin{aligned} N^{\lambda}{}_{\mu\nu} &= \frac{1}{2}(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu}) \\ &= -\delta^{\lambda}{}_{\nu}\Theta_{\mu} - \delta^{\lambda}{}_{\mu}\Theta_{\nu} + \Theta^{\lambda} g_{\mu\nu}. \end{aligned} \tag{2.27}$$

A Weyl structure is said to be Einstein-Weyl [20] if the symmetrized Ricci tensor $W_{\rho\nu}$ of $\hat{\nabla}$ is proportional to some metric $g \in [g]$,

$$W_{(\rho\nu)} = \frac{1}{d} g_{\rho\nu} W, \tag{2.28}$$

where W is the scalar curvature of the Weyl connection $\hat{\nabla}$. It is given by[5]

$$W = \tilde{R} + (d-2)(1-d)\Theta_{\mu}\Theta^{\mu} + 2(d-1)\nabla_{\mu}\Theta^{\mu}. \tag{2.29}$$

The condition (2.28) can be rewritten in terms of Riemannian data as

$$\begin{aligned} &\tilde{R}_{\rho\nu} + (d-2)\Theta_{\rho}\Theta_{\nu} + (d-2)\nabla_{(\nu}\Theta_{\rho)} \\ &\quad = \frac{1}{d} g_{\rho\nu}[\tilde{R} + (d-2)\nabla_{\mu}\Theta^{\mu} + (d-2)\Theta_{\mu}\Theta^{\mu}]. \end{aligned} \tag{2.30}$$

The scope of this subsection is to compare the field equations for Einstein manifolds with torsion, (2.18), with the Einstein-Weyl equations (2.30). To this end, let us define

$$A_{\mu} \equiv \frac{T_{\mu}}{d-1}, \tag{2.31}$$

such that, under the first transformation in (2.20), we have

$$A_{\mu} \mapsto A'_{\mu} = A_{\mu} + \partial_{\mu}\omega. \tag{2.32}$$

Using (2.31) in (2.18), one gets

$$\begin{aligned} &\tilde{R}_{\rho\nu} + (d-2)A_{\rho}A_{\nu} + (d-2)\nabla_{(\nu}A_{\rho)} + (d-3)A^{\mu}\breve{T}_{(\rho\nu)\mu} \\ &\quad + \frac{1}{4}\breve{T}_{(\rho}{}^{\mu\sigma}\breve{T}_{\nu)\mu\sigma} - \frac{1}{2}\breve{T}_{\mu\sigma(\rho}\breve{T}_{\nu)}{}^{\mu\sigma} - \nabla_{\mu}\breve{T}_{(\rho\nu)}{}^{\mu} \\ &= \frac{1}{d} g_{\rho\nu}\Big[\tilde{R} + (d-2)\nabla_{\mu}A^{\mu} + (d-2)A_{\mu}A^{\mu} \\ &\quad + \frac{1}{4}\breve{T}^{\mu\tau\sigma}\breve{T}_{\mu\tau\sigma} - \frac{1}{2}\breve{T}^{\mu\tau\sigma}\breve{T}_{\tau\sigma\mu}\Big]. \end{aligned} \tag{2.33}$$

[5] See also the results of sec. III in the case of zero torsion.

Thus, for $\breve{T}^{\lambda}{}_{\mu\nu}=0$, (2.33) exactly coincides with (2.30) if we identify A_μ with Θ_μ, i.e., $T_\mu \to (d-1)\Theta_\mu$. This is actually not surprising, since for $\breve{T}^{\lambda}{}_{\mu\nu}=0$ the torsion two-form is given by

$$T^a{}_{\mu\nu}=\frac{1}{d-1}(e^a{}_\nu T_\mu - e^a{}_\mu T_\nu)=e^a{}_\nu A_\mu - e^a{}_\mu A_\nu \\ = -(e^a\wedge A)_{\mu\nu} \Rightarrow T^a = A\wedge e^a. \tag{2.34}$$

Then, the first Cartan structure equation gives

$$de^a+\omega^a{}_b\wedge e^b = A\wedge e^a = de^a + (\omega^a{}_b - \delta^a{}_b A)\wedge e^b \\ =0. \tag{2.35}$$

We can then define a new connection $\hat{\omega}^{ab}$ as

$$\hat{\omega}^{ab}=\omega^{ab}-\eta^{ab}A, \tag{2.36}$$

which is torsion-free

$$de^a+\hat{\omega}^a{}_b\wedge e^b=0, \tag{2.37}$$

but nonmetric, since $\hat{\omega}^{(ab)}\neq 0$. The trace part of the torsion can thus always be shuffled into a Weyl vector and vice versa. In the latter case, a Weyl structure gets translated into a conformal structure $[g]$ together with a torsionful connection D which is compatible with $[g]$,

$$D_\mu g_{\nu\lambda}=0. \tag{2.38}$$

The torsion of D has only a trace part T_μ, and (2.38) is invariant under the transformation (2.19), (2.20).

Finally, note that a duality between torsion and nonmetricity has also been discussed in [103] in a slightly different context.

III. EINSTEIN MANIFOLDS WITH TORSION AND NONMETRICITY

Let us now consider Einstein spaces with both torsion and nonmetricity (we will call these Einstein-Cartan-Weyl manifolds), and study the Weyl invariance of the corresponding field equations.

With respect to Sec. II, we will in addition allow for a nonmetricity tensor of the form (2.26), where $\hat{\nabla}$ has also torsion. We are thus considering only the trace part of the nonmetricity. The consequences of adding a traceless part will be analyzed at the end of this section. The connection $\hat{\Gamma}^\lambda_{\mu\nu}$ of the Einstein-Cartan-Weyl manifold is given by

$$\hat{\Gamma}^\lambda{}_{\mu\nu}=\tilde{\Gamma}^\lambda{}_{\mu\nu}+N^\lambda{}_{\mu\nu}, \tag{3.1}$$

where the $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ are the Christoffel symbols, and the distortion $N^\lambda{}_{\mu\nu}$ reads

$$N_{\lambda\mu\nu}=\frac{1}{2}(T_{\nu\lambda\mu}-T_{\lambda\nu\mu}-T_{\mu\nu\lambda})+\frac{1}{2}(Q_{\lambda\mu\nu}+Q_{\lambda\nu\mu}-Q_{\mu\lambda\nu}), \tag{3.2}$$

that is, in the present context,

$$N_{\lambda\mu\nu}=\frac{1}{2}(\breve{T}_{\nu\lambda\mu}-\breve{T}_{\lambda\nu\mu}-\breve{T}_{\mu\nu\lambda}) \\ +\frac{1}{d-1}(g_{\mu\nu}T_\lambda - g_{\mu\lambda}T_\nu)+\Theta_\lambda g_{\mu\nu}-\Theta_\mu g_{\lambda\nu}-\Theta_\nu g_{\lambda\mu}. \tag{3.3}$$

The Ricci tensor of $\hat{\nabla}$, that is $\hat{R}_{\rho\nu}=\hat{R}^\mu{}_{\rho\mu\nu}$, is given in the Appendix [see Eq. (A3)]. Note that one can also construct another Ricci tensor $\mathfrak{R}_{\rho\nu}=\hat{R}^\mu{}_{\mu\rho\nu}$ (commonly referred to as the homothetic curvature), since for nonmetric connections the Riemann tensor is not necessarily antisymmetric in the first two indices. In our case we have

$$\mathfrak{R}_{\rho\nu}=d(\nabla_\nu\Theta_\rho-\nabla_\rho\Theta_\nu), \tag{3.4}$$

and thus the Ricci scalar associated with the homothetic curvature is identically zero. On the other hand, the nonvanishing Ricci scalar is given by

$$\hat{R}=g^{\rho\nu}\hat{R}_{\rho\nu} \\ =\tilde{R}+\frac{(d-2)(1-d)}{(d-1)^2}T_\mu T^\mu+2\nabla_\mu T^\mu \\ +\frac{1}{4}\breve{T}_{\mu\nu\rho}\breve{T}^{\mu\nu\rho}-\frac{1}{2}\breve{T}_{\nu\rho\mu}\breve{T}^{\mu\nu\rho}+(d-2)(1-d)\Theta_\mu\Theta^\mu \\ +2(d-1)\nabla_\mu\Theta^\mu+2(2-d)\Theta^\mu T_\mu. \tag{3.5}$$

Observe that, if we define

$$\breve{T}_\mu\equiv T_\mu+(d-1)\Theta_\mu, \tag{3.6}$$

the Ricci scalar (3.5) becomes

$$\hat{R}=\tilde{R}+\frac{(d-2)(1-d)}{(d-1)^2}\breve{T}_\mu\breve{T}^\mu+2\nabla_\mu\breve{T}^\mu+\frac{1}{4}\breve{T}_{\mu\nu\rho}\breve{T}^{\mu\nu\rho} \\ -\frac{1}{2}\breve{T}_{\nu\rho\mu}\breve{T}^{\mu\nu\rho}, \tag{3.7}$$

which corresponds to the Ricci scalar of a metric connection with torsion [cf. Eq. (2.13)], whose trace part is given by $\breve{T}_\mu$.

We define an Einstein-Cartan-Weyl space by

$$\hat{R}_{(\rho\nu)}=\lambda g_{\rho\nu} \tag{3.8}$$

for some function λ. Using (A3), this can be rewritten in the equivalent form,

$$\begin{aligned}
&\tilde{R}_{\rho\nu} + \frac{1}{d-1}[(d-2)\nabla_{(\nu}T_{\rho)} + (d-3)T^{\mu}\breve{T}_{(\rho\nu)\mu}] + \frac{1}{(d-1)^2}[(d-2)T_{\nu}T_{\rho}] \\
&\quad + \frac{1}{4}\breve{T}_{\rho}{}^{\mu\sigma}\breve{T}_{\nu\mu\sigma} - \frac{1}{2}\breve{T}_{\mu\sigma(\rho}\breve{T}_{\nu)}{}^{\mu\sigma} - \nabla_{\mu}\breve{T}_{(\rho\nu)}{}^{\mu} \\
&\quad + (d-2)\Theta_{\nu}\Theta_{\rho} + (d-2)\nabla_{(\nu}\Theta_{\rho)} + \frac{2(d-2)}{d-1}\Theta_{(\nu}T_{\rho)} + (d-3)\Theta^{\mu}\breve{T}_{(\nu\rho)\mu} \\
&= \frac{1}{d}g_{\rho\nu}\left[\tilde{R} + \frac{d-2}{d-1}\nabla_{\mu}T^{\mu} + \frac{d-2}{(d-1)^2}T_{\mu}T^{\mu} + \frac{1}{4}\breve{T}^{\mu\tau\sigma}\breve{T}_{\mu\tau\sigma} - \frac{1}{2}\breve{T}^{\mu\tau\sigma}\breve{T}_{\tau\sigma\mu}\right. \\
&\quad \left. + (d-2)\nabla_{\mu}\Theta^{\mu} + (d-2)\Theta_{\mu}\Theta^{\mu} + \frac{2(d-2)}{d-1}\Theta^{\mu}T_{\mu}\right], \qquad (3.9)
\end{aligned}$$

which is a system of nonlinear partial differential equations characterizing an Einstein-Cartan-Weyl manifold.

A. Extended conformal invariance of the Einstein-Cartan-Weyl equations

Let us now discuss the extended conformal invariance of (3.8). In an affine manifold such as an Einstein-Cartan-Weyl one, the most general extended conformal (Weyl) transformations involving an arbitrary scalar field $\omega = \omega(x)$ which leave the curvature tensor invariant are given by (see [82])

$$\begin{aligned}
g_{\mu\nu} &\mapsto e^{2\omega}g_{\mu\nu}, \\
T^{\lambda}{}_{\mu\nu} &\mapsto T^{\lambda}{}_{\mu\nu} + 2(1-\xi)\delta^{\lambda}{}_{[\nu}\partial_{\mu]}\omega, \\
Q^{\lambda}{}_{\mu\nu} &\mapsto Q^{\lambda}{}_{\mu\nu} - 2\xi\partial_{\mu}\omega\delta^{\lambda}{}_{\nu}, \qquad (3.10)
\end{aligned}$$

where ξ denotes an arbitrary parameter that we are free to include [82,85].[6] In particular, for the one-forms Θ and T and for $\breve{T}^{\lambda}{}_{\mu\nu}$ we find

$$\begin{aligned}
\Theta_{\mu} &\mapsto \Theta_{\mu} + \xi\partial_{\mu}\omega, \\
T_{\mu} &\mapsto T_{\mu} + (1-\xi)(d-1)\partial_{\mu}\omega, \\
\breve{T}^{\lambda}{}_{\mu\nu} &\mapsto \breve{T}^{\lambda}{}_{\mu\nu}, \qquad (3.11)
\end{aligned}$$

and the connection $\hat{\Gamma}$ transforms according to

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} \mapsto \hat{\Gamma}^{\rho}{}_{\mu\nu} + (1-\xi)\delta^{\rho}{}_{\nu}\partial_{\mu}\omega. \qquad (3.12)$$

This ensures the invariance of the curvature tensor due to its special projective invariance (see, for instance, Refs. [86,87]).

Thus, for the Riemann tensor, the Ricci tensor and the scalar curvature, one obtains respectively

$$\hat{R}^{\sigma}{}_{\rho\mu\nu} \mapsto \hat{R}^{\sigma}{}_{\rho\mu\nu}, \qquad \hat{R}_{\rho\nu} \mapsto \hat{R}_{\rho\nu}, \qquad \hat{R} \mapsto e^{-2\omega}\hat{R}. \qquad (3.13)$$

[6]Note that (3.10) implies that $\hat{\nabla}_{\mu}g_{\nu\rho} = 2\Theta_{\mu}g_{\nu\rho}$ transforms covariantly.

Equation (3.8) implies $\hat{R} = \lambda d$, so that (3.8) is equivalent to

$$\hat{R}_{(\rho\nu)} = \frac{1}{d}\hat{R}g_{\rho\nu}, \qquad (3.14)$$

which is clearly invariant under the extended conformal transformations written above.

Let us finally make some comments on two particular cases, namely $\xi = 1$ and $\xi = 0$.

(i) For $\xi = 1$ one has

$$T_{\mu} \mapsto T_{\mu}, \qquad \Theta_{\mu} \mapsto \Theta_{\mu} + \partial_{\mu}\omega. \qquad (3.15)$$

Observe that (3.15) corresponds to the transformation (2.32), for $A_{\mu} = \Theta_{\mu}$, discussed in Sec. II in the context of a Weyl structure (that is with nonmetricity and zero torsion). Moreover, note that this is the only case in which the connection is also invariant, $\hat{\Gamma}^{\rho}{}_{\mu\nu} \mapsto \hat{\Gamma}^{\rho}{}_{\mu\nu}$. In fact, setting $\xi = 1$ into (3.10) and (3.11) leads to a conformal transformation of the metric in an affine space, namely a transformation under which the metric tensor picks up a conformal factor $e^{2\omega}$ while the affine connection is left unchanged (see Refs. [86,87]).

(ii) For $\xi = 0$ we get the extended conformal transformation discussed in [78] in the context of a torsion theory which leads to a special projective transformation for the connection. In particular, in this case we have

$$T_{\mu} \mapsto T_{\mu} + (d-1)\partial_{\mu}\omega, \qquad \Theta_{\mu} \mapsto \Theta_{\mu}, \qquad (3.16)$$

which reproduces exactly the transformation in (2.20) for T_{μ} discussed in Sec. II for manifolds with torsion and vanishing nonmetricity, together with

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} \mapsto \hat{\Gamma}^{\rho}{}_{\mu\nu} + \delta^{\rho}{}_{\nu}\partial_{\mu}\omega, \qquad (3.17)$$

which is a special projective transformation (3.17) for the connection. On the other hand, let us observe that the combination of the conformal metric transformation in (3.10) plus the special projective transformation (3.17) is called, according to [86,87], a frame rescaling.

We can conclude that there are two unique transformations which single out torsion or nonmetricity. This is in agreement with [82]. Note that the same results could have been obtained by considering (3.7), together with the definition (3.6), that is by reabsorbing the nonmetricity and exploiting the transformations of Sec. II for an Einstein-Cartan manifold with torsion and vanishing nonmetricity.

B. Adding a traceless part to the nonmetricity tensor

In the following we extend the above analysis to include a traceless part of the nonmetricity as well. Interestingly, in the case where the latter is totally symmetric, it can be viewed as representing a massless spin-3 field [104,105].

Thus, we decompose

$$Q_{\lambda\mu\nu} = -2\Theta_\mu g_{\nu\lambda} + \breve{Q}_{\lambda\mu\nu}, \tag{3.18}$$

where $\breve{Q}^\nu{}_{\mu\nu} = 0$. Using (2.6) and (3.18) in (3.2), the distortion becomes

$$\begin{aligned} N_{\lambda\mu\nu} &= \frac{1}{2}(\breve{T}_{\nu\lambda\mu} - \breve{T}_{\lambda\nu\mu} - \breve{T}_{\mu\nu\lambda}) + \frac{1}{d-1}(g_{\mu\nu}T_\lambda - g_{\mu\lambda}T_\nu) \\ &\quad + \Theta_\lambda g_{\mu\nu} - \Theta_\mu g_{\lambda\nu} - \Theta_\nu g_{\lambda\mu} + \frac{1}{2}(\breve{Q}_{\lambda\mu\nu} + \breve{Q}_{\lambda\nu\mu} - \breve{Q}_{\mu\lambda\nu}) \\ &= \breve{K}_{\nu\lambda\mu} + \frac{1}{d-1}(g_{\mu\nu}T_\lambda - g_{\mu\lambda}T_\nu) + \Theta_\lambda g_{\mu\nu} - \Theta_\mu g_{\lambda\nu} - \Theta_\nu g_{\lambda\mu} + \breve{M}_{\lambda\mu\nu} \\ &= K_{\nu\lambda\mu} + M_{\lambda\mu\nu}, \end{aligned} \tag{3.19}$$

where we defined the so-called disformation (also known as deflection tensor)

$$\begin{aligned} M_{\lambda\mu\nu} &= \frac{1}{2}(Q_{\lambda\mu\nu} + Q_{\lambda\nu\mu} - Q_{\mu\lambda\nu}) \\ &= \Theta_\lambda g_{\mu\nu} - \Theta_\mu g_{\lambda\nu} - \Theta_\nu g_{\lambda\mu} + \frac{1}{2}(\breve{Q}_{\lambda\mu\nu} + \breve{Q}_{\lambda\nu\mu} - \breve{Q}_{\mu\lambda\nu}) \\ &= \Theta_\lambda g_{\mu\nu} - \Theta_\mu g_{\lambda\nu} - \Theta_\nu g_{\lambda\mu} + \breve{M}_{\lambda\mu\nu}, \end{aligned} \tag{3.20}$$

which is symmetric in the last two indices. $\breve{K}_{\nu\lambda\mu}$ and $\breve{M}_{\nu\lambda\mu}$ are respectively the traceless part of $K_{\nu\lambda\mu}$ and $M_{\nu\lambda\mu}$,

$$\breve{K}_{\nu\lambda\mu} = \frac{1}{2}(\breve{T}_{\nu\lambda\mu} - \breve{T}_{\lambda\nu\mu} - \breve{T}_{\mu\nu\lambda}), \qquad \breve{M}_{\nu\lambda\mu} = \frac{1}{2}(\breve{Q}_{\lambda\mu\nu} + \breve{Q}_{\lambda\nu\mu} - \breve{Q}_{\mu\lambda\nu}). \tag{3.21}$$

From (3.1) one obtains for the connection

$$\begin{aligned} \hat{\Gamma}^\lambda{}_{\mu\nu} &= \tilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{2}(\breve{T}_\nu{}^\lambda{}_\mu - \breve{T}^\lambda{}_{\nu\mu} - \breve{T}_{\mu\nu}{}^\lambda) + \frac{1}{d-1}(g_{\mu\nu}T^\lambda - \delta_\mu{}^\lambda T_\nu) \\ &\quad + \Theta^\lambda g_{\mu\nu} - \Theta_\mu \delta^\lambda{}_\nu - \Theta_\nu \delta^\lambda{}_\mu + \frac{1}{2}(\breve{Q}^\lambda{}_{\mu\nu} + \breve{Q}^\lambda{}_{\nu\mu} - \breve{Q}_\mu{}^\lambda{}_\nu). \end{aligned} \tag{3.22}$$

The explicit expression for the Ricci tensor $\hat{R}_{\rho\nu}$ of $\hat{\nabla}$ is given in the Appendix [see (A4)], and it contains extra contributions from the traceless tensor $\breve{Q}_{\lambda\mu\nu}$. The homothetic curvature is still given by (3.4), while the Ricci scalar is

$$\begin{aligned} \hat{R} &= \tilde{R} + \frac{(d-2)(1-d)}{(d-1)^2}T_\mu T^\mu + 2\nabla_\mu T^\mu \\ &\quad + (d-2)(1-d)\Theta_\mu\Theta^\mu + 2(d-1)\nabla_\mu\Theta^\mu + 2(2-d)\Theta^\mu T_\mu \\ &\quad + \frac{1}{4}(\breve{T}_{\mu\nu\rho}\breve{T}^{\mu\nu\rho} - 2\breve{T}_{\mu\nu\rho}\breve{Q}^{\mu\nu\rho} + \breve{Q}_{\mu\nu\rho}\breve{Q}^{\mu\nu\rho}) - \frac{1}{2}(\breve{T}_{\nu\rho\mu}\breve{T}^{\mu\nu\rho} + \breve{T}_{\mu\nu\rho}\breve{Q}^{\mu\nu\rho} + \breve{Q}_{\nu\rho\mu}\breve{Q}^{\mu\nu\rho}). \end{aligned} \tag{3.23}$$

Observe that, by defining

$$\check{T}_\mu \equiv T_\mu + (d-1)\Theta_\mu, \qquad \check{T}_{\mu\nu\rho} \equiv \breve{T}_{\mu\nu\rho} - \breve{Q}_{\mu\nu\rho}, \tag{3.24}$$

where $\check{T}^{\nu}_{\mu\nu} = 0$, and using the fact that the symmetries of $\check{T}_{\mu\nu\rho}$ and $\check{Q}_{\mu\nu\rho}$ imply

$$\check{T}_{\nu\rho\mu}\check{Q}^{\mu\nu\rho} = 0, \qquad \check{T}^{\mu\nu\rho}\check{Q}_{\nu\rho\mu} = \check{T}^{\mu\nu\rho}\check{Q}_{\mu\rho\nu}, \tag{3.25}$$

one can show that the Ricci scalar (3.23) can be written as

$$\hat{R} = \tilde{R} + \frac{(d-2)(1-d)}{(d-1)^2}\check{T}_\mu\check{T}^\mu + 2\nabla_\mu\check{T}^\mu + \frac{1}{4}\check{T}_{\mu\nu\rho}\check{T}^{\mu\nu\rho} - \frac{1}{2}\check{T}_{\nu\rho\mu}\check{T}^{\mu\nu\rho}, \tag{3.26}$$

which corresponds to the Ricci scalar of a metric connection with nonvanishing torsion, whose trace and traceless parts are now respectively given by $\check{T}_\mu$ and $\check{T}_{\mu\nu\rho}$. This is analogous to the case in which one does not include a traceless contribution for the nonmetricity, cf. Eq. (3.7).

As before, we define an Einstein-Cartan-Weyl space by Eq. (3.8), which becomes in the present context:

$$\begin{aligned}
&\tilde{R}_{\rho\nu} + \frac{1}{d-1}[(d-2)\nabla_{(\nu}T_{\rho)} + (d-3)T^\mu\check{T}_{(\rho\nu)\mu}] + \frac{1}{(d-1)^2}[(d-2)T_\nu T_\rho] \\
&\quad + \frac{1}{4}\check{T}_\rho{}^{\mu\sigma}\check{T}_{\nu\mu\sigma} - \frac{1}{2}\check{T}_{\mu\sigma(\rho}\check{T}_{\nu)}{}^{\mu\sigma} - \nabla_\mu\check{T}_{(\rho\nu)}{}^\mu \\
&\quad + (d-2)\Theta_\nu\Theta_\rho + (d-2)\nabla_{(\nu}\Theta_{\rho)} + \frac{2(d-2)}{d-1}\Theta_{(\nu}T_{\rho)} + (d-3)\Theta^\mu\check{T}_{(\nu\rho)\mu} \\
&\quad + \frac{2-d}{d-1}T^\mu\check{Q}_{\mu(\nu\rho)} + \frac{d-4}{2(d-1)}T_\mu\check{Q}_\rho{}^\mu{}_\nu - (d-2)\Theta^\mu\check{Q}_{\mu(\nu\rho)} + \frac{d-4}{2}\Theta_\mu\check{Q}_\rho{}^\mu{}_\nu \\
&\quad - \frac{1}{4}\check{Q}_{\mu\rho\sigma}\check{Q}^\mu{}_\nu{}^\sigma + 2\nabla^\mu\check{Q}_{\mu(\nu\rho)} - \frac{1}{2}\nabla_\mu\check{Q}_\nu{}^\mu{}_\rho + \frac{1}{2}\check{T}_{\mu(\rho}{}^\sigma\check{Q}^\mu{}_{\nu)\sigma} + \frac{1}{2}\check{T}^\mu{}_{(\rho}{}^\sigma\check{Q}_{\nu)\sigma\mu} - \check{T}_{(\rho}{}^{\mu\sigma}\check{Q}_{\nu)\mu\sigma} \\
&= \frac{1}{d}g_{\rho\nu}\bigg[\tilde{R} + \frac{d-2}{d-1}\nabla_\mu T^\mu + \frac{d-2}{(d-1)^2}T_\mu T^\mu + (d-2)\nabla_\mu\Theta^\mu + (d-2)\Theta_\mu\Theta^\mu + \frac{2(d-2)}{d-1}\Theta^\mu T_\mu \\
&\quad + \frac{1}{4}(\check{T}^{\mu\tau\sigma}\check{T}_{\mu\tau\sigma} + \check{Q}^{\mu\tau\sigma}\check{Q}_{\mu\tau\sigma}) - \frac{1}{2}(\check{T}^{\mu\tau\sigma}\check{T}_{\tau\sigma\mu} + \check{Q}^{\mu\tau\sigma}\check{Q}_{\tau\sigma\mu}) - \check{T}^{\mu\tau\sigma}\check{Q}_{\mu\tau\sigma}\bigg],
\end{aligned} \tag{3.27}$$

which represents a system of nonlinear partial differential equations characterizing an Einstein-Cartan-Weyl manifold with the most general form of torsion and nonmetricity.

Finally, we can consider the transformations (3.10). In particular, we have

$$\check{Q}^\lambda{}_{\mu\nu} \mapsto \check{Q}^\lambda{}_{\mu\nu}. \tag{3.28}$$

For the curvature tensors one still has the transformation laws given in (3.13), so that the Einstein-Cartan-Weyl equations (3.8) are again invariant under extended conformal transformations for arbitrary parameter ξ.

IV. EINSTEIN-CARTAN ACTION AND SCALE INVARIANT GRAVITY

Let us consider the action

$$S = \int d^dx\sqrt{-g}\phi^2(R - \kappa\phi^{\frac{4}{d-2}}), \tag{4.1}$$

where R is the Ricci scalar (2.13) of a torsionful but metric connection, ϕ denotes a scalar field, and κ is a constant. Along the same lines of [85], (4.1) can be rewritten as

$$S = \int d^dx\sqrt{-g}\phi^2\bigg(\tilde{R} - \frac{d-2}{d-1}T_\mu T^\mu + 2\nabla_\mu T^\mu + \frac{1}{4}\check{T}_{\mu\nu\rho}\check{T}^{\mu\nu\rho} - \frac{1}{2}\check{T}_{\nu\rho\mu}\check{T}^{\mu\nu\rho} - \kappa\phi^{\frac{4}{d-2}}\bigg), \tag{4.2}$$

with $\tilde{R}$ the scalar curvature of the Levi-Civita connection. One easily shows that (4.2) is invariant under

$$\begin{aligned}
&g_{\mu\nu} \mapsto e^{2\omega}g_{\mu\nu}, \qquad \phi \mapsto e^{\frac{2-d}{2}\omega}\phi, \\
&T_\mu \mapsto T_\mu + (d-1)\partial_\mu\omega, \qquad \check{T}^\lambda{}_{\mu\nu} \mapsto \check{T}^\lambda{}_{\mu\nu}.
\end{aligned} \tag{4.3}$$

Using the traceless part of the contorsion defined in (3.21), the action (4.2) becomes

$$S = \int d^dx\sqrt{-g}\phi^2 \times \bigg(\tilde{R} - \frac{d-2}{d-1}T_\mu T^\mu + 2\nabla_\mu T^\mu - \check{K}_{\nu\rho\mu}\check{K}^{\mu\nu\rho} - \kappa\phi^{\frac{4}{d-2}}\bigg), \tag{4.4}$$

and its variation with respect to T_μ and $\check{K}_{\nu\rho\mu}$ yields respectively

$$T_\mu = -\frac{2(d-1)}{d-2}\frac{\nabla_\mu\phi}{\phi}, \qquad \check{K}_{\mu[\nu\rho]} = 0. \tag{4.5}$$

Notice that T_μ can be eliminated by an extended conformal transformation and is thus pure gauge. Using the definition (3.21) and the fact that the traceless part of the torsion is antisymmetric in the last two indices, we get $\check{T}_{\mu\nu\rho} = 2\check{K}_{\mu[\nu\rho]} = 0$, and therefore also $\check{K}_{\mu\nu\rho} = 0$, in agreement with [79,85].

Varying the action (4.4) with respect to $g_{\mu\nu}$ and ϕ leads to

$$\phi^2\left(\tilde{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\tilde{R}\right) + \frac{2d}{d-2}\nabla_\mu\phi\nabla_\nu\phi - 2\phi\nabla_\nu\nabla_\mu\phi + 2g_{\mu\nu}\phi\nabla_\rho\nabla^\rho\phi - \frac{2}{d-2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi + \frac{1}{2}g_{\mu\nu}\kappa\phi^{\frac{2d}{d-2}} = 0, \tag{4.6a}$$

$$\phi\tilde{R} - \frac{4(d-1)}{d-2}\nabla_\rho\nabla^\rho\phi - \frac{d}{d-2}\kappa\phi^{\frac{d+2}{d-2}} = 0, \tag{4.6b}$$

where we have used the expression for T_ν in (4.5) as well as $\check{K}_{\mu\nu\rho} = 0$. Observe that the trace of (4.6a) implies (4.6b), which can be understood as a consequence of ϕ being pure gauge.

Let us now consider the action

$$S = \int d^dx\sqrt{-g}\left[\phi^2\tilde{R} + \frac{4(d-1)}{d-2}\nabla_\mu\phi\nabla^\mu\phi - \kappa\phi^{\frac{2d}{d-2}}\right], \tag{4.7}$$

which is called scale invariant (also known as conformal gravity). It turns out that the equations of motion following from (4.7) are precisely (4.6a) and (4.6b) obtained from (4.4) after having used the expressions for the torsion. The actions (4.1) and (4.7) describe thus the same dynamics. Notice also that, plugging T_μ [cf. (4.5)] and $\check{K}_{\mu\nu\rho} = 0$ into (4.4), one gets, up to a surface term,[7] the conformal gravity action (4.7) (see also [85]).

One can also show that the action (4.1) implies that the spacetime is Einstein with torsion, which is a completely new result. To see this, observe that Eq. (4.6a) can be rewritten as

$$\tilde{R}_{\mu\nu} + \frac{2d}{d-2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2} - 2\frac{\nabla_\nu\nabla_\mu\phi}{\phi} = \frac{1}{d}g_{\mu\nu}\left(\frac{d}{2}\tilde{R} - 2d\frac{\nabla_\rho\nabla^\rho\phi}{\phi} + \frac{2d}{d-2}\frac{\nabla_\rho\phi\nabla^\rho\phi}{\phi^2} - \frac{d}{2}\kappa\phi^{\frac{4}{d-2}}\right). \tag{4.8}$$

Using also (4.6b), this can be cast into the form

$$\tilde{R}_{\mu\nu} + \frac{2d}{d-2}\frac{\nabla_\mu\phi\nabla_\nu\phi}{\phi^2} - 2\frac{\nabla_\mu\nabla_\nu\phi}{\phi} = \frac{1}{d}g_{\mu\nu}\left(\tilde{R} - 2\frac{\nabla_\rho\nabla^\rho\phi}{\phi} + \frac{2d}{d-2}\frac{\nabla_\rho\phi\nabla^\rho\phi}{\phi^2}\right). \tag{4.9}$$

On the other hand, consider the system (2.18) characterizing an Einstein-Cartan manifold, and use the result (4.5) for the trace part of the torsion as well as $\check{T}_{\mu\nu\rho} = 0$. Then (2.18) boils down precisely to (4.9).

Let us also observe that, as already mentioned in [85], conformal (Weyl) invariance allows to rescale $\phi \mapsto e^{\frac{2-d}{d}\omega}\phi$. One can use this freedom to gauge fix $\phi = 1/(4\sqrt{\pi G})$, where G is Newton's constant. Then the action (4.7) becomes

$$S = \frac{1}{16\pi G}\int d^dx\sqrt{-g}(\tilde{R} - 2\Lambda), \tag{4.10}$$

where we chose $\kappa = 2\Lambda(16\pi G)^{2/(d-2)}$. The Einstein-Hilbert action with cosmological constant can thus be viewed as a gauge fixed version of the action (4.7).

Finally, let us recall that the trace part of the torsion can also be interpreted as the trace part of the nonmetricity (cf. Sec. II B). If we set the traceless part of the torsion to zero, this leads to the action

$$S = \int d^dx\sqrt{-g}\phi^2(W - \kappa\phi^{\frac{4}{d-2}}), \tag{4.11}$$

which is invariant under

$$g_{\mu\nu} \mapsto e^{2\omega}g_{\mu\nu}, \qquad \phi \mapsto e^{\frac{2-d}{2}\omega}\phi, \qquad \Theta_\mu \mapsto \Theta_\mu + \partial_\mu\omega. \tag{4.12}$$

The variation of (4.11) with respect to Θ_μ yields

$$\Theta_\mu = -\frac{2}{d-2}\frac{\nabla_\mu\phi}{\phi}. \tag{4.13}$$

Again, one can easily show that the actions (4.11) and (4.7) describe the same dynamics. Equation (4.11) implies that the spacetime is Einstein-Weyl, where the Weyl vector is given by (4.13), and is thus pure gauge. Notice in this context that there is no known action principle that leads to the Einstein-Weyl equations with nonexact Weyl vector.

V. EINSTEIN-HILBERT ACTION COUPLED TO A THREE-FORM AS EINSTEIN-CARTAN GRAVITY

The Einstein-Hilbert action coupled to a three-form field strength reads

$$S_1 = \int d^d x \sqrt{-g}\left(\tilde{R} - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right), \tag{5.1}$$

where $H_{\mu\nu\rho}$ is given in terms of a gauge potential $B_{\mu\nu}$,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \qquad B_{\mu\nu} = -B_{\nu\mu}. \tag{5.2}$$

The variation of (5.1) with respect to $B_{\mu\nu}$ leads to

$$\nabla^\mu H_{\mu\nu\rho} = 0, \tag{5.3}$$

while varying $g^{\rho\nu}$ gives

$$\tilde{R}_{\rho\nu} - \frac{1}{2} g_{\rho\nu} \tilde{R} + \frac{1}{24} g_{\rho\nu} H_{\mu\tau\sigma} H^{\mu\tau\sigma} - \frac{1}{4} H_\rho{}^{\mu\sigma} H_{\nu\mu\sigma} = 0. \tag{5.4}$$

On the other hand, consider the system (2.18) satisfied by an Einstein manifold with torsion. Assume that $T_\mu = 0$ and take $\breve{T}_{\mu\nu\rho}$ to be completely antisymmetric. Then (2.18) boils down to

$$\tilde{R}_{\rho\nu} - \frac{1}{4} \breve{T}_{\mu\sigma\nu} \breve{T}_\rho{}^{\mu\sigma} = \frac{1}{d} g_{\rho\nu}\left(\tilde{R} - \frac{1}{4} \breve{T}^{\mu\tau\sigma} \breve{T}_{\mu\tau\sigma}\right). \tag{5.5}$$

We would like to compare this with (5.4). To this end, take the trace of (5.4), which leads to

$$\tilde{R} = \frac{d-6}{12(d-2)} H^2, \qquad H^2 \equiv H_{\mu\tau\sigma} H^{\mu\tau\sigma}. \tag{5.6}$$

Now subtract its trace part from (5.4) to obtain

$$\tilde{R}_{\rho\nu} - \frac{1}{d} g_{\rho\nu} \tilde{R} - \frac{1}{4} H_\rho{}^{\mu\sigma} H_{\nu\mu\sigma} + \frac{1}{4d} g_{\rho\nu} H^2 = 0, \tag{5.7}$$

which coincides precisely with (5.5) if we identify $H_{\mu\nu\rho} = \breve{T}_{\mu\nu\rho}$. The equations of motion following from (5.1) can thus be interpreted as implying that the spacetime is Einstein with skew-symmetric torsion $H_{\mu\nu\rho}$ satisfying (5.3). Notice however that the equations (5.4) are more restrictive than (5.5), since they contain in addition the trace part (5.6), while (5.5) is traceless. This is somehow reminiscent of hyper Cauchy-Riemann (hyper-CR, or Gauduchon-Tod) spaces [106], where on top of the (trace-free) Einstein-Weyl equations there is a constraint on the scalar curvature.

Quite remarkably, the equations [(5.3) and (5.4)] can also be retrieved from the constrained action,

$$\begin{aligned} S_2 &= \int d^d x \sqrt{-g}\left[R + \lambda^{\mu\nu\rho}\left(\breve{T}_{\mu\nu\rho} - \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\right)\right] \\ &= \int d^d x \sqrt{-g}\left[\tilde{R} - \frac{d-2}{d-1} T_\mu T^\mu + 2\nabla_\mu T^\mu + \frac{1}{4} \breve{T}_{\mu\nu\rho} \breve{T}^{\mu\nu\rho} - \frac{1}{2} \breve{T}_{\nu\rho\mu} \breve{T}^{\mu\nu\rho} \right. \\ &\quad \left. + \lambda^{\mu\nu\rho}\left(\breve{T}_{\mu\nu\rho} - \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\right)\right], \end{aligned} \tag{5.8}$$

where R denotes the scalar curvature of a torsionful but metric connection [cf. (2.13)], $\lambda^{\mu\nu\rho}$ is a Lagrange multiplier, and $B_{\mu\nu}$ is antisymmetric. The variation of (5.8) with respect to T_μ, $B_{\mu\nu}$, $\lambda^{\mu\nu\rho}$, $\breve{T}_{\mu\nu\rho}$ and $g^{\mu\nu}$ gives respectively

$$T_\mu = 0, \qquad \nabla_\mu \lambda^{[\mu\nu\rho]} = 0, \tag{5.9}$$

$$\breve{T}_{\mu\nu\rho} = \frac{1}{\sqrt{3}}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}), \tag{5.10}$$

$$\lambda^{\mu\nu\rho} = \frac{1}{2}(\breve{T}^{\nu\rho\mu} + \breve{T}^{\rho\mu\nu} - \breve{T}^{\mu\nu\rho}), \tag{5.11}$$

$$\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} + \frac{1}{8} g_{\mu\nu} \breve{T}_{\tau\rho\sigma} \breve{T}^{\tau\rho\sigma} - \frac{3}{4} \breve{T}_\mu{}^{\tau\rho} \breve{T}_{\nu\tau\rho} = 0, \tag{5.12}$$

where we already used $T_\mu = 0$ in (5.12). Equation (5.10) implies that the traceless part of the torsion is completely antisymmetric, and thus (5.11) reduces to

$$\lambda^{\mu\nu\rho} = \frac{1}{2} \breve{T}^{\mu\nu\rho}. \tag{5.13}$$

Plugging this into the last equation of (5.9) leads to

$$\nabla_\mu \breve{T}^{\mu\nu\rho} = 0. \tag{5.14}$$

Finally, using (5.10) in (5.14) and (5.12), one gets precisely (5.3) and (5.4). The actions S_1 and S_2 describe therefore the same dynamics.

VI. DISCUSSION

Motivated by the interest in connections with torsion and nonmetricity both from the physical and the mathematical point of view, we first generalized here some results that appeared previously in the literature. In particular, we considered Einstein spaces with nonvanishing torsion that has both a trace and a traceless part (Einstein-Cartan manifolds), and showed that the resulting field equations are invariant under extended conformal transformations. We then compared our results to Einstein manifolds with zero torsion but nonvanishing nonmetricity, where the latter is given in terms of the Weyl vector Θ_μ (Einstein-Weyl spaces). We saw that, if the traceless part of the torsion is set to zero, then the system of partial differential equations characterizing Einstein-Cartan spaces exactly coincides with the Einstein-Weyl equations if the torsion trace T_μ is replaced by $(d-1)\Theta_\mu$. Subsequently, we extended our analysis to the case of Einstein manifolds with both torsion and nonmetricity (Einstein-Cartan-Weyl spaces), allowing for both a trace and a traceless part of the nonmetricity tensor.

Moreover, we considered actions involving scalar curvatures obtained from torsionful or nonmetric connections, and investigated their relations with other gravitational theories, obtaining completely new results in this context. In particular, we analyzed a conformally (Weyl) invariant action with torsion and its relation with scale invariant gravity, which involves a scalar ϕ, and found that they reproduce the same dynamics. Furthermore, we have shown that the action (4.1) implies that the spacetime is Einstein with torsion. Then, the Einstein-Hilbert action coupled to a three-form field strength $H_{\mu\nu\rho}$ was considered, and it was shown that its equations of motion imply that the manifold is Einstein with skew-symmetric torsion. Furthermore, it turned out that the equations of motion of Einstein gravity coupled to a three-form may also be retrieved from a constrained action that contains the scalar curvature of a connection with torsion. Let us stress that in this paper we concentrated on the vacuum, without considering the presence of matter.

Among the solutions to Einstein's field equations, Einstein spaces are of particular relevance in physics, think for instance of the Kerr-(A)dS solution or of string compactifications on e.g., Sasaki-Einstein manifolds. Since nature could accommodate for torsion and nonmetricity, it seems reasonable to generalize the concept of Einstein spaces to torsionful and nonmetric connections.

The manifolds analyzed in this paper may also have applications in the classification and physical study of (fake) supersymmetric supergravity solutions in the same way as Einstein-Weyl manifolds provide the base space for fake supersymmetric solutions in de Sitter supergravity [47–52]. Under the physical point of view, this analysis is particularly relevant in higher dimensions, since, in $d > 4$, it is highly nontrivial to determine whether a given near-horizon geometry can be extended to a full black hole solution (due to the fact that the strong uniqueness theorems that hold in four dimensions [107–112] break down and there exist different black holes with the same asymptotic charges and different black hole solutions with the same near-horizon geometry). Progress in classifying near-horizon geometries can help to face this problem, as it was proven in [51], where the authors, after having showed that a class of solutions of minimal supergravity in five dimensions is given by lifts of three-dimensional Einstein-Weyl structures of hyper-CR type, considered the task of reconstructing all supersymmetric solutions from such near-horizon geometry, demonstrating that the moduli space of infinitesimal supersymmetric transverse deformations of the near-horizon data is finite dimensional if the spatial section of the horizon is compact.

Always in this context, a new result has recently been obtained in [113], where it has been shown that the horizon geometry for supersymmetric black hole solutions of minimal five-dimensional gauged supergravity is that of a particular Einstein-Cartan-Weyl structure in three dimensions, involving the trace and traceless part of both torsion and nonmetricity, and obeying some precise constraint; in the limit of zero cosmological constant, the set of nonlinear partial differential equations characterizing this Einstein-Cartan-Weyl structure reduces to that of a hyper-CR Einstein-Weyl structure in the Gauduchon gauge, which was shown in [51] to be the horizon geometry in the ungauged BPS (Bogomol'nyi-Prasad-Sommefield) case.

The analysis of this paper might also be extended in other directions. In particular, it would be interesting to generalize the construction of [88] concerning the Chern-Simons formulation of three-dimensional gravity involving torsion and nonmetricity, and the recent results presented in [114] in the context of double field theory. One could also investigate possible generalizations of [104,105].

On the other hand, a future development of our work may consist in possible generalizations of the Jones-Tod correspondence [54] between self-dual conformal four-manifolds with a conformal vector field and Abelian monopoles on Einstein-Weyl spaces in three dimensions. Especially one could ask whether Einstein-Cartan-Weyl manifolds can arise in a similar way by symmetry reduction from higher dimensions.

Finally, a further direction for future research would be a geometrical investigation of the results on unconventional supersymmetry presented recently in [115], where torsion plays a fundamental role, under the perspective developed here.

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APPENDIX: RIEMANN AND RICCI TENSORS

The Riemann tensor of the Einstein-Cartan connection $\Gamma^\lambda{}_{\mu\nu}$ introduced in Sec. II reads

$$\begin{aligned}
R^\lambda{}_{\rho\mu\nu} &= \partial_\mu\Gamma^\lambda{}_{\nu\rho} - \partial_\nu\Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\rho} \\
&= \tilde{R}^\lambda{}_{\rho\mu\nu} + \frac{1}{d-1}[g_{\rho\nu}\nabla_\mu T^\lambda - g_{\rho\mu}\nabla_\nu T^\lambda + \delta_\mu{}^\lambda\nabla_\nu T_\rho - \delta_\nu{}^\lambda\nabla_\mu T_\rho] \\
&\quad + \frac{1}{d-1}\left[\frac{1}{2}\delta_\nu{}^\lambda T^\sigma(-\check{T}_{\sigma\mu\rho} + \check{T}_{\mu\rho\sigma} - \check{T}_{\rho\mu\sigma}) + \frac{1}{2}\delta_\mu{}^\lambda T^\sigma(-\check{T}_{\sigma\nu\rho} + \check{T}_{\nu\rho\sigma} + \check{T}_{\rho\nu\sigma})\right] \\
&\quad + \frac{1}{d-1}[T^\lambda\check{T}_{\rho\mu\nu} - T_\rho\check{T}^\lambda{}_{\mu\nu}] \\
&\quad + \frac{1}{d-1}\left\{\frac{1}{2}T^\sigma[g_{\nu\rho}(\check{T}_\sigma{}^\lambda{}_\mu + \check{T}^\lambda{}_{\mu\sigma} + \check{T}_\mu{}^\lambda{}_\sigma) - g_{\mu\rho}(\check{T}_\sigma{}^\lambda{}_\nu + \check{T}^\lambda{}_{\nu\sigma} + \check{T}_\nu{}^\lambda{}_\sigma)]\right\} \\
&\quad + \frac{1}{(d-1)^2}[g_{\nu\rho}T_\mu T^\lambda - g_{\mu\rho}T_\nu T^\lambda + (g_{\rho\mu}\delta_\nu{}^\lambda - g_{\rho\nu}\delta_\mu{}^\lambda)T_\sigma T^\sigma + T_\rho(\delta_\mu{}^\lambda T_\nu - \delta_\nu{}^\lambda T_\mu)] \\
&\quad + \frac{1}{4}[\check{T}^\lambda{}_\nu{}^\sigma\check{T}_{\mu\rho\sigma} + \check{T}_{\mu\rho}{}^\sigma(\check{T}_\sigma{}^\lambda{}_\nu + \check{T}_\nu{}^\lambda{}_\sigma) - \check{T}_{\sigma\mu\rho}(\check{T}^{\sigma\lambda}{}_\nu + \check{T}^\lambda{}_\nu{}^\sigma + \check{T}_\nu{}^{\lambda\sigma}) + (\check{T}^\lambda{}_\nu{}^\sigma + \check{T}_\nu{}^{\lambda\sigma})\check{T}_{\rho\mu\sigma}] \\
&\quad + \frac{1}{4}[\check{T}_\sigma{}^\lambda{}_\nu\check{T}_{\rho\mu}{}^\sigma - \check{T}_\rho{}^\lambda{}_\mu(\check{T}_{\nu\rho}{}^\sigma + \check{T}_{\rho\nu}{}^\sigma) + \check{T}_{\sigma\nu\rho}(\check{T}^{\sigma\lambda}{}_\mu + \check{T}^\lambda{}_\mu{}^\sigma + \check{T}_\mu{}^{\lambda\sigma})] \\
&\quad - \frac{1}{4}[(\check{T}^\lambda{}_\mu{}^\sigma + \check{T}_\mu{}^{\lambda\sigma})(\check{T}_{\nu\rho\sigma} + \check{T}_{\rho\nu\sigma})] \\
&\quad + \frac{1}{2}[\nabla_\mu\check{T}^\lambda{}_{\nu\rho} + \nabla_\mu\check{T}_\nu{}^\lambda{}_\rho + \nabla_\mu\check{T}_\rho{}^\lambda{}_\nu - \nabla_\nu\check{T}^\lambda{}_{\mu\rho} - \nabla_\nu\check{T}_\mu{}^\lambda{}_\rho - \nabla_\nu\check{T}_\rho{}^\lambda{}_\mu],
\end{aligned} \tag{A1}$$

where $\tilde{R}^\lambda{}_{\rho\mu\nu}$ and ∇ denote respectively the Riemann tensor and the covariant derivative of the Levi-Civita connection. The first line of (A1) follows from the definition $[D_\mu, D_\nu]\omega_\rho + T^\sigma{}_{\mu\nu}D_\sigma\omega_\rho = -R^\lambda{}_{\rho\mu\nu}\omega_\lambda$, where D denotes the connection with coefficients Γ. The corresponding Ricci tensor is given by

$$\begin{aligned}
R_{\rho\nu} = R^\mu{}_{\rho\mu\nu} &= \tilde{R}_{\rho\nu} + \frac{1}{d-1}[g_{\rho\nu}\nabla_\mu T^\mu + (d-2)\nabla_\nu T_\rho] + \frac{1}{(d-1)^2}[(2-d)g_{\nu\rho}T_\mu T^\mu + (d-2)T_\nu T_\rho] \\
&\quad + \frac{1}{d-1}\left\{\frac{1}{2}T^\mu[(2-d)(\check{T}_{\mu\nu\rho} - \check{T}_{\nu\rho\mu}) + (d-4)\check{T}_{\rho\nu\mu}]\right\} \\
&\quad + \frac{1}{4}\check{T}_\nu{}^{\mu\sigma}\check{T}_{\rho\mu\sigma} + \frac{1}{2}(\check{T}_{\mu\nu\sigma}\check{T}_\rho{}^{\mu\sigma} + \nabla_\mu\check{T}^\mu{}_{\nu\rho} - \nabla_\mu\check{T}_{\nu\rho}{}^\mu - \nabla_\mu\check{T}_{\rho\nu}{}^\mu).
\end{aligned} \tag{A2}$$

On the other hand, the Ricci tensor of the Einstein-Cartan-Weyl connection $\hat{\Gamma}^\lambda{}_{\mu\nu}$ introduced in Sec. III is

$$\begin{aligned}
\hat{R}_{\rho\nu} = \hat{R}^\mu{}_{\rho\mu\nu} &= \tilde{R}_{\rho\nu} + \frac{1}{d-1}[g_{\rho\nu}\nabla_\mu T^\mu + (d-2)\nabla_\nu T_\rho] + \frac{1}{(d-1)^2}[(2-d)g_{\nu\rho}T_\mu T^\mu + (d-2)T_\nu T_\rho] \\
&\quad + \frac{1}{d-1}\left\{\frac{1}{2}T^\mu[(2-d)(\check{T}_{\mu\nu\rho} - \check{T}_{\nu\rho\mu}) + (d-4)\check{T}_{\rho\nu\mu}]\right\} \\
&\quad + \frac{1}{4}\check{T}_\nu{}^{\mu\sigma}\check{T}_{\rho\mu\sigma} + \frac{1}{2}(\check{T}_{\mu\nu\sigma}\check{T}_\rho{}^{\mu\sigma} + \nabla_\mu\check{T}^\mu{}_{\nu\rho} - \nabla_\mu\check{T}_{\nu\rho}{}^\mu - \nabla_\mu\check{T}_{\rho\nu}{}^\mu) \\
&\quad + (d-2)\Theta_\nu\Theta_\rho + (2-d)\Theta_\mu\Theta^\mu g_{\nu\rho} + g_{\nu\rho}\nabla_\mu\Theta^\mu + (d-1)\nabla_\nu\Theta_\rho - \nabla_\rho\Theta_\nu \\
&\quad + \frac{1}{d-1}[(d-2)\Theta_\rho T_\nu + (d-2)\Theta_\nu T_\rho + 2(2-d)g_{\nu\rho}\Theta^\mu T_\mu] \\
&\quad + \frac{d-2}{2}\Theta^\mu(\check{T}_{\mu\nu\rho} - \check{T}_{\nu\rho\mu}) + \frac{d-4}{2}\Theta^\mu\check{T}_{\rho\nu\mu},
\end{aligned} \tag{A3}$$

where ∇ denotes again the Levi-Civita connection.

Finally, adding a traceless part to the nonmetricity tensor, we have that the Ricci tensor of $\hat{\nabla}$ reads, explicitly,

$$\begin{aligned}
\hat{R}_{\rho\nu} = \hat{R}^{\mu}{}_{\rho\mu\nu} = \tilde{R}_{\rho\nu} &+ \frac{1}{d-1}[g_{\rho\nu}\nabla_\mu T^\mu + (d-2)\nabla_\nu T_\rho] + \frac{1}{(d-1)^2}[(2-d)g_{\nu\rho}T_\mu T^\mu + (d-2)T_\nu T_\rho] \\
&+ \frac{1}{d-1}\left\{\frac{1}{2}T^\mu[(2-d)(\check{T}_{\mu\nu\rho} - \check{T}_{\nu\rho\mu}) + (d-4)\check{T}_{\rho\nu\mu}]\right\} \\
&+ \frac{1}{4}\check{T}_\nu{}^{\mu\sigma}\check{T}_{\rho\mu\sigma} + \frac{1}{2}(\check{T}_{\mu\nu\sigma}\check{T}_\rho{}^{\mu\sigma} + \nabla_\mu \check{T}^\mu{}_{\nu\rho} - \nabla_\mu \check{T}_{\nu\rho}{}^\mu - \nabla_\mu \check{T}_{\rho\nu}{}^\mu) \\
&+ (d-2)\Theta_\nu\Theta_\rho + (2-d)\Theta_\mu\Theta^\mu g_{\nu\rho} + g_{\nu\rho}\nabla_\mu\Theta^\mu + (d-1)\nabla_\nu\Theta_\rho - \nabla_\rho\Theta_\nu \\
&+ \frac{1}{d-1}[(d-2)\Theta_\rho T_\nu + (d-2)\Theta_\nu T_\rho + 2(2-d)g_{\nu\rho}\Theta^\mu T_\mu] \\
&+ \frac{d-2}{2}\Theta^\mu(\check{T}_{\mu\nu\rho} - \check{T}_{\nu\rho\mu}) + \frac{d-4}{2}\Theta^\mu\check{T}_{\rho\nu\mu} \\
&+ \frac{1}{d-1}\left\{\frac{1}{2}T^\mu[(2-d)(\check{Q}_{\mu\nu\rho} + \check{Q}_{\nu\rho\mu}) + (d-4)\check{Q}_{\rho\mu\nu}]\right\} \\
&- \frac{d-2}{2}\Theta^\mu(\check{Q}_{\mu\nu\rho} + \check{Q}_{\nu\rho\mu}) + \frac{d-4}{2}\Theta^\mu\check{Q}_{\rho\mu\nu} \\
&- \frac{1}{4}\check{Q}_{\mu\rho\sigma}\check{Q}^\mu{}_\nu{}^\sigma + \frac{1}{2}(\check{Q}_\nu{}^{\mu\sigma}\check{Q}_{\rho\mu\sigma} - \check{Q}_\nu{}^{\mu\sigma}\check{Q}_{\rho\sigma\mu} - \nabla_\mu\check{Q}_\nu{}^\mu{}_\rho + \nabla_\mu\check{Q}_{\nu\rho}{}^\mu + \nabla_\mu\check{Q}_{\rho\nu}{}^\mu) \\
&+ \frac{1}{2}\check{T}^\mu{}_\rho{}^\sigma\check{Q}_{\mu\nu\sigma} - \frac{1}{2}\check{T}_\rho{}^{\mu\sigma}\check{Q}_{\nu\mu\sigma} - \frac{1}{2}\check{T}_\nu{}^{\mu\sigma}\check{Q}_{\rho\mu\sigma} + \frac{1}{2}\check{T}^\mu{}_\nu{}^\sigma(\check{Q}_{\rho\sigma\mu} - \check{Q}_{\rho\mu\sigma}),
\end{aligned} \tag{A4}$$

which, indeed, now contains extra contributions from the traceless tensor $\check{Q}_{\lambda\mu\nu}$.

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[1] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. **258**, 1 (1995).

[2] E. Schrodinger, *Space-time Structure* (Cambridge University Press, Cambridge, England, 1954).

[3] A. Einstein, *Meaning of Relativity* (Princeton University, Princeton, NJ, 1955).

[4] J. A. Schouten, *Ricci Calculus* (Springer, Berlin, 1954).

[5] *Gauge Theories of Gravitation*, edited by M. Blagojevic and F. Hehl (Imperial College Press, London, 2013).

[6] Y. Ne'eman and F. W. Hehl, Test matter in a space-time with nonmetricity, Classical Quantum Gravity **14**, A251 (1997).

[7] R. F. Sobreiro and V. J. Vasquez Otoya, Aspects of nonmetricity in gravity theories, Braz. J. Phys. **40**, 370 (2010).

[8] D. Puetzfeld and Y. N. Obukhov, Equations of motion in metric-affine gravity: A covariant unified framework, Phys. Rev. D **90**, 084034 (2014).

[9] L. Järv, M. Rünkla, M. Saal, and O. Vilson, Nonmetricity formulation of general relativity and its scalar-tensor extension, Phys. Rev. D **97**, 124025 (2018).

[10] E. Kröner, The differential geometry of elementary point and line defects in Bravais crystals, Int. J. Theor. Phys. **29**, 1219 (1990).

[11] A. Delhom-Latorre, G. J. Olmo, and M. Ronco, Observable traces of nonmetricity: New constraints on metric-affine gravity, Phys. Lett. B **780**, 294 (2018).

[12] P. Luz and V. Vitagliano, Raychaudhuri equation in spacetimes with torsion, Phys. Rev. D **96**, 024021 (2017).

[13] S. Speziale, Raychaudhuri and optical equations for null geodesic congruences with torsion, Phys. Rev. D **98**, 084029 (2018).

[14] D. Iosifidis, C. G. Tsagas, and A. C. Petkou, Raychaudhuri equation in spacetimes with torsion and nonmetricity, Phys. Rev. D **98**, 104037 (2018).

[15] C. Pagani and R. Percacci, Quantum gravity with torsion and nonmetricity, Classical Quantum Gravity **32**, 195019 (2015).

[16] J. Stelmach, Nonmetricity driven inflation, Classical Quantum Gravity **8**, 897 (1991).

[17] E. A. Poberii, Nonmetricity driven inflation, Helv. Phys. Acta **67**, 745 (1994).

[18] J. Beltrán Jiménez and T. S. Koivisto, Modified gravity with vector distortion and cosmological applications, Universe **3**, 47 (2017).

[19] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).

[20] D. M. J. Calderbank and H. Pedersen, Einstein-Weyl geometry, in *Essays on Einstein Manifolds*, edited by

C. R. LeBrun and M. Wang, Surveys in Differential Geometry Vol. V (International Press, Boston, 1999).
[21] G. B. Folland, Weyl manifolds, J. Diff. Geom. **4**, 145 (1970).
[22] C. Romero, J. B. Fonseca-Neto, and M. L. Pucheu, General relativity and Weyl geometry, Classical Quantum Gravity **29**, 155015 (2012).
[23] H. Weyl, Gravitation and electricity, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1918**, 465 (1918).
[24] H. Weyl, A new extension of relativity theory, Ann. Phys. (Berlin) **364**, 101 (1919); Eine neue erweiterung der relatiritästheorie; Surveys High Energ. Phys. **5**, 237 (1986); Eine neue Erweiterung der Relativitätstheorie, Ann. Phys. (Berlin) **364**, 101 (1919).
[25] J. T. Wheeler, Weyl geometry, Gen. Relativ. Gravit. **50**, 80 (2018).
[26] F. London, Quantum mechanical interpretation of the Weyl theory, Z. Phys. **42**, 375 (1927); Quantenmechanische deutung der theorie von weyl, Surveys High Energ. Phys. **5**, 253 (1986).
[27] L. O'Raifeartaigh, *The Dawning of Gauge Theory* (Princeton University Press, Princeton, NJ, 1997), p. 249.
[28] E. Scholz, The unexpected resurgence of Weyl geometry in late 20th century physics, Einstein Stud. **14**, 261 (2018).
[29] C. Barceló, R. Carballo-Rubio, and L. J. Garay, Weyl relativity: A novel approach to Weyl's ideas, J. Cosmol. Astropart. Phys. 06 (2017) 014.
[30] M. R. Tanhayi, S. Dengiz, and B. Tekin, Weyl-invariant higher curvature gravity theories in n dimensions, Phys. Rev. D **85**, 064016 (2012).
[31] S. Dengiz, E. Kilicarslan, and B. Tekin, Weyl gauging of topologically massive gravity, Phys. Rev. D **86**, 104014 (2012).
[32] S. Dengiz, A noncompact Weyl-Einstein-Yang-Mills model: A semiclassical quantum gravity, Ann. Phys. (Amsterdam) **383**, 560 (2017).
[33] I. P. Lobo and C. Romero, Experimental constraints on the second clock effect, Phys. Lett. B **783**, 306 (2018).
[34] K. P. Tod, Compact 3-dimensional Einstein-Weyl structures, J. Lond. Math. Soc. **s2-45**, 341 (1992).
[35] H. Pedersen and K. P. Tod, Einstein metrics and hyperbolic monopoles, Classical Quantum Gravity **8**, 751 (1991).
[36] H. Pedersen and K. P. Tod, Three-dimensional Einstein-Weyl geometry, Adv. Math. **97**, 74 (1993).
[37] H. Pedersen and A. Swann, Riemannian submersions, four-manifolds and Einstein-Weyl geometry, Proc. London Math. Soc. **s3-66**, 381 (1993).
[38] H. Pedersen and A. F. Swann, Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature, J. Reine Angew. Math. **441**, 99 (1993).
[39] H. Pedersen, Y. S. Poon, and A. F. Swann, The Einstein-Weyl equations in complex and quaternionic geometry, Differential Geometry and its Applications **3**, 309 (1993).
[40] H. Pedersen, Y. S. Poon, and A. F. Swann, The Hitchin-Thorpe inequality for Einstein-Weyl manifolds, Bull. Lond. Math. Soc. **26**, 191 (1994).
[41] H. Pedersen, Y. S. Poon, and A. F. Swann, Einstein-Weyl deformations and submanifolds, Int. J. Math. **7**, 705 (1996).
[42] A. B. Madsen, H. Pedersen, Y. S. Poon, and A. F. Swann, Compact Einstein-Weyl manifolds with large symmetry group, Duke Math. J. **88**, 407 (1997); , Corrigendum, Duke Math. J. **100**, 167 (1999).
[43] D. M. J. Calderbank and P. Tod, Einstein metrics, hypercomplex structures and the Toda field equation, Differential Geometry and its Applications **14**, 199 (2001).
[44] D. M. J. Calderbank, Self-dual Einstein metrics and conformal submersions, arXiv:math/0001041.
[45] D. M. J. Calderbank and M. Dunajski, Scalar-flat Lorentzian Einstein-Weyl spaces, Classical Quantum Gravity **18**, L77 (2001).
[46] D. M. J. Calderbank and H. Pedersen, Self-dual spaces with complex structures, Einstein-Weyl geometry and geodesics, Ann. Inst. Fourier (Grenoble) **50**, 921 (2000).
[47] P. Meessen and A. Palomo-Lozano, Cosmological solutions from fake $N = 2$ EYM supergravity, J. High Energy Phys. 05 (2009) 042.
[48] J. B. Gutowski and W. A. Sabra, Solutions of minimal four-dimensional de Sitter supergravity, Classical Quantum Gravity **27**, 235017 (2010).
[49] J. Grover, J. B. Gutowski, C. A. R. Herdeiro, P. Meessen, A. Palomo-Lozano, and W. A. Sabra, Gauduchon-Tod structures, Sim holonomy and de Sitter supergravity, J. High Energy Phys. 07 (2009) 069.
[50] J. B. Gutowski, A. Palomo-Lozano, and W. A. Sabra, Einstein-Weyl structures and de Sitter supergravity, Classical Quantum Gravity **29**, 105006 (2012).
[51] M. Dunajski, J. Gutowski, and W. Sabra, Einstein-Weyl spaces and near-horizon geometry, Classical Quantum Gravity **34**, 045009 (2017).
[52] M. Randall, Three-dimensional near-horizon metrics that are Einstein-Weyl, arXiv:1704.06433.
[53] N. J. Hitchin, *Complex Manifolds and Einstein Equations, Twistor Geometry and Nonlinear Systems*, edited by H. D. Doebner and T. D. Palev, Lecture Notes in Math Vol. 970 (Springer, Berlin, 1982), p. 79.
[54] P. Jones and K. Tod, Minitwistor spaces and Einstein-Weyl spaces, Classical Quantum Gravity **2**, 565 (1985).
[55] R. S. Ward, Einstein-Weyl spaces and SU(∞) Toda fields, Classical Quantum Gravity **7**, L95 (1990).
[56] M. Dunajski, L. J. Mason, and P. Tod, Einstein-Weyl geometry, the dKP equation and twistor theory, J. Geom. Phys. **37**, 63 (2001).
[57] É. Cartan, Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion, C.R. Hebd. Seances Acad. Sci. **174**, 593 (1922).
[58] É. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie), Ann. Éc. Norm. Sup. **40**, 325 (1923).
[59] É. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (Suite), Ann. Éc. Norm. Sup. **41**, 1 (1924).
[60] É. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée II, Ann. Éc. Norm. Sup. **42**, 17 (1925).
[61] É. Cartan, Sur les variétés à connexion affine et la théorie de la relativité généralisée (Gauthier Villars, 1955). See also On manifolds with an affine connection and the theory

of general relativity, translated by A. Magnon and A. Ashtekar (Bibliopolis, Napoli, 1986).
[62] Y. F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, f(T) teleparallel gravity and cosmology, Rep. Prog. Phys. **79**, 106901 (2016).
[63] D. W. Sciama, *Recent Developments in General Relativity, Festschrift for Infeld* (Pergamon, Oxford, 1962), p. 415.
[64] D. W. Sciama, The Physical structure of general relativity, Rev. Mod. Phys. **36**, 463 (1964); Erratum, Rev. Mod. Phys. **36**, 1103 (1964).
[65] T. W. B. Kibble, Lorentz invariance and the gravitational field, J. Math. Phys. (N.Y.) **2**, 212 (1961).
[66] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, General relativity with spin and torsion: Foundations and prospects, Rev. Mod. Phys. **48**, 393 (1976).
[67] I. L. Shapiro, Torsion: Theory and possible observables, arXiv:hep-th/9811072.
[68] I. L. Shapiro, Physical aspects of the space-time torsion, Phys. Rep. **357**, 113 (2002).
[69] S. Capozziello, G. Lambiase, and C. Stornaiolo, Geometric classification of the torsion tensor in space-time, Ann. Phys. (Berlin) **10**, 713 (2001).
[70] T. Watanabe and M. J. Hayashi, General relativity with torsion, arXiv:gr-qc/0409029.
[71] S. Jensen, *General Relativity with Torsion: Extending Wald's Chapter on Curvature* (2005), https://pdfs.semanticscholar.org/60cf/2adf0400cf08c8d5978e99c6d8f9e8c6d83c.pdf?_ga=2.4773875.1557614665.1580231816-2019410458.1580231816.
[72] A. Golovnev, Introduction to teleparallel gravities, arXiv:1801.06929.
[73] J. Scherk and J. H. Schwarz, Dual models and the geometry of spacetime, Phys. Lett. **52B**, 347 (1974).
[74] A. Saa, Strings in background fields and Einstein-Cartan theory of gravity, Classical Quantum Gravity **12**, L85 (1995).
[75] M. Kalb and P. Ramond, Classical direct interstring action, Phys. Rev. D **9**, 2273 (1974).
[76] Y. N. Obukhov, Conformal invariance and spacetime torsion, Phys. Lett. A **90**, 13 (1982).
[77] J. W. Maluf, Conformal invariance and torsion in general relativity, Gen. Relativ. Gravit. **19**, 57 (1987).
[78] G. German, Brans-Dicke type models with torsion, Phys. Rev. D **32**, 3307 (1985).
[79] J. A. Helayel-Neto, A. Penna-Firme, and I. L. Shapiro, Conformal symmetry, anomaly and effective action for metric-scalar gravity with torsion, Phys. Lett. B **479**, 411 (2000).
[80] L. Fabbri, Metric-torsional conformal gravity, Phys. Lett. B **707**, 415 (2012).
[81] S. Chakrabarty and A. Lahiri, Different types of torsion and their effect on the dynamics of fields, Eur. Phys. J. Plus **133**, 242 (2018).
[82] L. L. Smalley, Brans-Dicke type models with nonmetricity, Phys. Rev. D **33**, 3590 (1986).
[83] T. Dereli and R. W. Tucker, A broken gauge approach to gravitational mass and charge, J. High Energy Phys. 03 (2002) 041.
[84] W. Drechsler and D. Hartley, The role of the internal metric in generalized Kaluza-Klein theories, J. Math. Phys. (N.Y.) **35**, 3571 (1994).
[85] T. Y. Moon, J. Lee, and P. Oh, Conformal invariance in Einstein-Cartan-Weyl space, Mod. Phys. Lett. A **25**, 3129 (2010).
[86] D. Iosifidis and T. Koivisto, Scale transformations in metric-affine geometry, arXiv:1810.12276.
[87] D. Iosifidis, Metric-affine gravity and cosmology/aspects of torsion and non-metricity in gravity theories, arXiv:1902.09643.
[88] S. L. Cacciatori, M. M. Caldarelli, A. Giacomini, D. Klemm, and D. S. Mansi, Chern-Simons formulation of three-dimensional gravity with torsion and nonmetricity, J. Geom. Phys. **56**, 2523 (2006).
[89] A. N. Ivanov and M. Wellenzohn, Einstein-Cartan gravity with torsion field serving as an origin for the cosmological constant or dark energy density, Astrophys. J. **829**, 47 (2016).
[90] R. Percacci, Gravity from a particle physicists' perspective, Proc. Sci., ISFTG2009 (**2009**) 011 [arXiv:0910.5167].
[91] A. C. Ferreira, Einstein four-manifolds with skew torsion, J. Geom. Phys. **61**, 2341 (2011).
[92] I. Agricola and A. C. Ferreira, Einstein manifolds with skew torsion, Q. J. Mech. Appl. Math. **65**, 717 (2014).
[93] P. A. M. Dirac, New basis for cosmology, Proc. R. Soc. A **165**, 199 (1938).
[94] P. A. M. Dirac, Long range forces and broken symmetries, Proc. R. Soc. A **333**, 403 (1973).
[95] P. D. Mannheim, Conformal cosmology with no cosmological constant, Gen. Relativ. Gravit. **22**, 289 (1990).
[96] P. K. Aluri, P. Jain, and N. K. Singh, Dark energy and dark matter in general relativity with local scale invariance, Mod. Phys. Lett. A **24**, 1583 (2009).
[97] L. P. Chimento, A. S. Jakubi, and D. Pavon, An exact model of conformal quintessence, Phys. Rev. D **69**, 083511 (2004).
[98] C. Wetterich, Conformal Fixed Point, Cosmological Constant and Quintessence, Phys. Rev. Lett. **90**, 231302 (2003).
[99] E. Gunzig, A. Saa, L. Brenig, V. Faraoni, T. M. Rocha Filho, and A. Figueiredo, Superinflation, quintessence, and nonsingular cosmologies, Phys. Rev. D **63**, 067301 (2001).
[100] V. Faraoni, Superquintessence, Int. J. Mod. Phys. D **11**, 471 (2002).
[101] B. Kelleher, Scale invariant gravity: Spacetime recovered, Classical Quantum Gravity **21**, 483 (2004).
[102] B. Kelleher, Scale invariance in gravity and implications for the cosmological constant, Classical Quantum Gravity **21**, 2623 (2004).
[103] D. Iosifidis, A. C. Petkou, and C. G. Tsagas, Torsion/non-metricity duality in f(R) gravity, Gen. Relativ. Gravit. **51**, 66 (2019).
[104] N. Boulanger and I. Kirsch, A Higgs mechanism for gravity. Part II. Higher spin connections, Phys. Rev. D **73**, 124023 (2006).
[105] P. Baekler, N. Boulanger, and F. W. Hehl, Linear connections with propagating spin-3 field in gravity, Phys. Rev. D **74**, 125009 (2006).
[106] P. Gauduchon and K. P. Tod, Hyper-Hermitian metrics with symmetry, J. Geom. Phys. **25**, 291 (1998).

[107] S. W. Hawking, Black holes in general relativity, Commun. Math. Phys. **25**, 152 (1972).
[108] W. Israel, Event horizons in static vacuum space-times, Phys. Rev. **164**, 1776 (1967).
[109] B. Carter, Axisymmetric Black Hole has Only Two Degrees of Freedom, Phys. Rev. Lett. **26**, 331 (1971).
[110] D. C. Robinson, Uniqueness of the Kerr Black Hole, Phys. Rev. Lett. **34**, 905 (1975).
[111] W. Israel, Event horizons in static electrovac space-times, Commun. Math. Phys. **8**, 245 (1968).
[112] P. O. Mazur, Proof of uniqueness of the Kerr-Newman black hole solution, J. Phys. A **15**, 3173 (1982).
[113] D. S. Klemm and L. Ravera, Supersymmetric near-horizon geometry and Einstein-Cartan-Weyl spaces, Phys. Lett. B **793**, 265 (2019).
[114] V. A. Penas, Deformed Weitzenböck connections and double field theory, Fortsch. Phys. **67**, 1800077 (2019).
[115] L. Andrianopoli, B. L. Cerchiai, R. D'Auria, and M. Trigiante, Unconventional supersymmetry at the boundary of AdS_4 supergravity, J. High Energy Phys. 04 (2018) 007.