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Generalized fundamental matrices as Grassmann tensors

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Abstract Given two linear projections of maximal rank from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$, with $k \geq 3$ and $h_1 + h_2 \geq k + 1$, the *Grassmann tensor* introduced by Hartley and Schaffalitzky (Int J Comput Vis 83(3):274–293, 2009. doi:10.1007/s11263-009-0225-1), turns out to be a *generalized fundamental matrix*. Such matrices are studied in detail and, in particular, their rank is computed. The dimension of the variety that parameterizes such matrices is also determined. An algorithmic application of the generalized fundamental matrix to projective reconstruction is described.

Keywords Multiview geometry · Computer vision · Projective reconstruction · Grassmann tensors

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1 Introduction

In the context of computer vision, the encoding of the essential geometry of situations in which multiple images of the same scene are taken from multiple cameras leads to interesting families of tensors, [1]. These multiview tensors have been introduced, in their most general form, by Hartley and Schaffalitzky [2], and they describe the multiview geometry of scenes

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in higher-dimensional spaces, with views taken as projections onto multiple target spaces of different dimensions. Special cases of these tensors are the classical fundamental matrix for two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, and the trifocal and quadrifocal tensor for, respectively, three and four projections from $\mathbb{P}^3$ to $\mathbb{P}^2$. Properties of the classical fundamental matrix, the trifocal, and quadrifocal tensors, including the computation of their rank, (see [3] for a thorough discussion of the intricacy of the notion of rank of a tensor), and the dimension and equations of the variety that parameterizes these objects, have been investigated by several authors, [4–12]. Nothing is known in general about the rank of these tensors and about the variety that parameterizes them, as soon as the dimension of the ambient space is greater than or equal to 4. This paper is intended as a first step in the investigation of these questions for higher dimensions, and thus it is concerned with the Grassmann tensor for two projections from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$, with $h_1 + h_2 \geq k + 1$, and $k \geq 3$. In this case, the tensor of interest is a matrix, that is a generalization of the classical fundamental matrix. As in the classical case, this matrix gives a correspondence between proper linear subspaces of the two views and thus represents a map between suitable Grassmannians. Theorem 1 shows that, generalizing the classical case, the rank of such a matrix is not maximal and it is given by an explicit formula. In several significant cases, linear relations between rows (or columns) of the matrix, responsible for the drop in rank, are explicitly given. Such relations turn out to be obtained from generalized Plücker relations, [13].

Taking inspiration from [7], the dimension of the variety of generalized fundamental matrices is computed and such variety is shown to be birational to the quotient of an affine cone over a suitable Grassmannian. Moreover, as a generalization of a known result in the classical case, generalized fundamental matrices of rank 2 are completely characterized.

As it is well known, the relevance of Grassmann tensors in computer vision is due to their fundamental role in solving the problem of projective reconstruction, [2,4,14–17]. For this reason, even if this work is primarily concerned with theoretical aspects in a complex projective setting, geometric algorithms to perform reconstruction in some cases are described.

The paper is structured as follows. Section 2 collects simple background material and sets basic notation. In Sect. 3, the generalized fundamental matrix is introduced and studied. In particular, its rank is computed in Theorem 1 and relations among its rows (columns) are investigated in Sect. 3.3. Section 4 studies the variety of generalized fundamental matrices, and, finally, Sect. 5 presents the algorithmic application of the generalized fundamental matrix to projective reconstruction.

2 Notation and background material

In this section, we collect all the necessary notation and background material from classical algebraic geometry and multiview projective geometry in the context of computer vision.

2.1 Projective spaces and Grassmannians

$\mathbb{P}^k$ denotes the k-dimensional complex (or real when specified) projective space. Whenever multiplication by a nonzero scalar is utilized, the scalar will be complex or real accordingly. Once a projective frame is chosen, coordinate vectors $\mathbf{X}$ of points of $\mathbb{P}^k$ are written as columns, and thus their transpose is $\mathbf{X}^T = (X_1, \ldots, X_{k+1})$. For sake of simplicity, sometimes we do not distinguish between points in $\mathbb{P}^k$ and vectors of their coordinates and, in this context, a basis for the underlying vector space $\mathbb{C}^{k+1}$ (or $\mathbb{R}^{k+1}$) will be called a *basis* for $\mathbb{P}^k$. A linear projective subspace $\Lambda \subseteq \mathbb{P}^k$ spanned by $m + 1$ linearly independent points will be

called m-space or subspace of dimension m. By convention, the empty set is considered as a (-1)-space.

For integers $0 \le l \le m$, $G(l, m) = G(l, \mathbb{P}^m)$ denotes the Grassmannian of the l-spaces in $\mathbb{P}^m$, i.e., the set parameterizing these subspaces. Each l-space $L \subset \mathbb{P}^m$ is determined by $l + 1$ independent points $\mathbf{Q_i}$, $i = 1, \ldots, l + 1$. Denoting by $\mathbf{Q_i}^T = q_{ij}$, $j = 1, \ldots, m + 1$, the homogeneous coordinates of $\mathbf{Q_i}$, one can consider the $(l + 1) \times (m + 1)$ matrix $\mathfrak{Q} = [q_{ij}]$ and its $\binom{m+1}{l+1}$ essential $(l + 1) \times (l + 1)$ minors. One usually denotes by $p_{i_1, \ldots, i_{l+1}}$ the $(l + 1) \times (l + 1)$ minor of $\mathfrak{Q}$ corresponding to the $i_1, \ldots i_{l+1}$-columns. Minors $p_{i_1, \ldots, i_{l+1}}$ for which $i_1 < \cdots < i_{l+1}$ (in lexicographic order) are called *Plücker coordinates* of L and determine a point $\mathbf{L}$ in a $\mathbb{P}^{\binom{m+1}{l+1}-1}$. In this projective space, the Plücker coordinates of all l-spaces in $\mathbb{P}^m$ satisfy a set of algebraic equations defining a smooth algebraic variety of dimension $(l + 1)(m - l)$ which is called the *Plücker embedding* of $G(l, m)$ in $\mathbb{P}^{\binom{m+1}{l+1}-1}$. The map $L \mapsto \mathbf{L}$, which associates with each l-space in $\mathbb{P}^m$ the vector of its coordinates as a point of $\mathbb{P}^{\binom{m+1}{l+1}-1}$, is called the *Plücker map* and it is one to one onto its image. For example: $G(1, 3)$ is the Grassmannian of lines in $\mathbb{P}^3$. Its Plücker embedding is a four-dimensional quadric in $\mathbb{P}^5$, and the Plücker map is a one-to-one map from $G(1, 3)$ to this quadric. As a reference for all basic facts on Grassmannians, and more, we suggest [13].

In the sequel, we will need to consider some *Schubert varieties* in $G(l, m)$, for which we introduce the following notation. For a given s-space $S \in \mathbb{P}^m$, $s < l$, we will denote by $\Omega_{(l,m)}(S) \subset G(l, m)$ the variety of all l-spaces in $\mathbb{P}^m$ containing S. It turns out that $\Omega_{(l,m)}(S)$ is a variety of dimension $(m - l)(l - s)$.

Some standard facts concerning duality among Grassmannians will also be useful. Denote by $\check{\mathbb{P}}^m = G(m-1, m)$ the m- projective space which is the dual of $\mathbb{P}^m$. Points of $\check{\mathbb{P}}^m$ represent hyperplanes of $\mathbb{P}^m$. As seen above, a given l-space $L \subset \mathbb{P}^m$ defines a point $\mathbf{L} \in G(l, \mathbb{P}^m)$, but it can be dually described via (the intersection of) the set of all hyperplanes containing it, i.e. as an $(m - l - 1)$-space in $\check{\mathbb{P}}^m$, or, which is the same, as a point of $G(m - l - 1, \check{\mathbb{P}}^m)$. This is the reason why Grassmannians $G(l, m)$ and $G(m - l - 1, m)$ are isomorphic and are said to be *dual* of each other. For the explicit correspondence between usual and dual Plücker coordinates of a given subspace, see [18, Vol I, Book II, p. 292],

2.2 Multiview geometry

For the convenience of the reader, we fix our notation for cameras, centers of projection, and multiple views in the context of projective reconstruction from multiple views in computer vision. A *scene* is a set of N points $\{\mathbf{X_i}\} \in \mathbb{P}^k$, $i = 1 \ldots N$. A *camera* is represented as a central projection P of points in k-space, from a linear center C_P, onto a suitable $\mathbb{P}^h$, $h < k$, where in the traditional setting of real still images it is $k = 3$ and $h = 2$. The target space $\mathbb{P}^h$, which in the traditional setting is usually identified with a physical image, is usually referred to as a *view*. We do not make any formal distinction between the projection *map P* and one of its matrix representations, for which we use the same symbol P. Accordingly, if $\mathbf{X}$ is a point in $\mathbb{P}^k$, we denote its image in the projection equivalently as $P(\mathbf{X})$ or $P \cdot \mathbf{X}$. The center of projection is denoted by C_P. In homogeneous coordinates, the projection mapping $P : \mathbb{P}^k \setminus \{C_P\} \to \mathbb{P}^h$ is described by $\mu \mathbf{x} = P \cdot \mathbf{X}$, where μ is a nonzero constant, P is a $(h + 1) \times (k + 1)$-matrix with rk $(P) = h + 1$, and C_P is the right annihilator of P, hence a $(k - h - 1)$-space. For a given point $\mathbf{X} \in \mathbb{P}^k$, the projecting ray, i.e. the join $\langle C_P, \mathbf{X} \rangle$, is a $(k-h)$-space, as C_P is a $(k-h-1)$-space. Let P^+ denote a pseudoinverse for P, i.e. a matrix such that $P \cdot P^+$ is the identity matrix. Then, for a given $\mathbf{x} \in \mathbb{P}^h$, $P^{-1}(\mathbf{x}) = \langle P^+(\mathbf{x}), C_P \rangle$.

While in many applications one naturally considers projections from $\mathbb{P}^k$ to several spaces all of the same dimension (i.e. one considers a set of multiple images of the same kind of a scene, traditionally 3 or 4 real projections from $\mathbb{P}^3$ to $\mathbb{P}^2$), the general setup allows for projections $Pj : \mathbb{P}^k \setminus C_{Pj} \rightarrow \mathbb{P}^{h_j}$ to spaces of different dimensions. Two different images $Pj(\mathbf{X})$ and $Pm(\mathbf{X})$ of the same point $\mathbf{X}$ are *corresponding points*, while $\langle C_{Pj}, \mathbf{X} \rangle$ and $\langle C_{Pm}, \mathbf{X} \rangle$ are *corresponding rays*. More generally, r linear subspaces $S_i \subset \mathbb{P}^{h_i}$, $i = 1, \ldots, r$ are said to be *corresponding* if there exists at least one point $\mathbf{X} \in \mathbb{P}^k$ such that $Pi(\mathbf{X}) \in S_i$ for $i = 1, \ldots, r$.

Hartley and Schaffalitzky [2] gave a comprehensive theoretical framework for the study of multiview geometry, in any dimension. Here we recall the basic elements of their approach. Consider, again, a set of projections $Pj : \mathbb{P}^k \setminus C_{Pj} \rightarrow \mathbb{P}^{h_j}$, $j = 1, \ldots, r$, $h_j \geq 2$ and a partition $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of $k + 1$, i.e. $1 \leq \alpha_j \leq h_j$ for all j, and $\sum \alpha_j = k + 1$. Let $\{L_j\}$, $j = 1, \ldots, r$, be a set of general s_j-spaces, with $L_j \subset \mathbb{P}^{h_j}$, and $s_j = h_j - \alpha_j$.

Hartley and Schaffalitzky show that imposing to $\{L_j\}$ to be a set of corresponding subspaces, with the assumption that $\sum \alpha_j = k + 1$, one gets a multilinear relation among the Grassmann (Plücker) coordinates of the L_j. The rationale for the above assumption is essentially an application of Grassmann formula. Indeed, if $\sum \alpha_j < k + 1$, then any r-tuple of general subspaces with the given codimension α_j would be corresponding, while if $\sum \alpha_j > k + 1$ imposing to an r-tuple of subspaces to be corresponding would generate more than one multilinear constraint.

The coefficients of these relations are then packaged into a multi-view tensor, called a *Grassmann tensor* with profile $(\alpha_1, \ldots, \alpha_r)$. This Grassmann tensor in the case of two, three, and four views from $\mathbb{P}^3$ to $\mathbb{P}^2$ is the classical *fundamental matrix, trifocal tensor*, and *quadrifocal tensor*, respectively. As mentioned in the introduction, all these objects have been widely studied in different contexts by many authors [4–12]. This work is concerned with the Grassmann tensor in the case of projections from a space of any dimension onto two views of possibly different dimensions.

3 Generalized fundamental matrix

3.1 Definition and explicit construction

Let us now consider two projections $P1 = A$ and $P2 = B$ from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and to $\mathbb{P}^{h_2}$, respectively, where $h_1 + h_2 \geq k + 1$, and where A and B are maximal rank matrices as follows:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\ a_{2,1} & \cdots & \cdots & a_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{h_1+1,1} & \cdots & \cdots & a_{h_2+1,k+1} \end{pmatrix} \quad B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k+1} \\ b_{2,1} & \cdots & \cdots & b_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{h_2+1,1} & \cdots & \cdots & b_{h_2+1,k+1} \end{pmatrix}.$$

Grassmann formula shows that our assumption $h_1 + h_2 \geq k + 1$, in particular, implies that, for generic choices of A and B, the projection centers C_A and C_B do not intersect. Their images $E_1 = A(C_B)$ and $E_2 = B(C_A)$ are subspaces of dimension $k - h_i - 1$ of the view spaces, usually called *epipoles*.

Following [2], we choose a profile (α_1, α_2), where, without loss of generality, $\alpha_1 \geq \alpha_2$, and with $\alpha_1 + \alpha_2 = k + 1$, in order to obtain the constraints necessary to determine the

corresponding tensor. In this case, the tensor is simply a matrix which we will call, following the classical case, *generalized fundamental matrix*.

Let $L \subset \mathbb{P}^{h_1}$, $L' \subset \mathbb{P}^{h_2}$ be linear subspaces of dimension $s_1 = h_1 - \alpha_1$, $s_2 = h_2 - \alpha_2$, respectively, and let $\mathbf{L} \in G(s_1, h_1) \subseteq \mathbb{P}^{\binom{h_1+1}{s_1+1}-1}$ and $\mathbf{L}' \in G(s_2, h_2) \subseteq \mathbb{P}^{\binom{h_2+1}{s_2+1}-1}$ be their associated points in the appropriate Grassmannians.

Choose coordinates $(x_1, \ldots, x_{h_1+1})$ and $(y_1, \ldots, y_{h_2+1})$ in the two views $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$, respectively. Given $s_1 + 1$ points $\mathbf{x^j} = (x_1^j, \ldots, x_{h_1+1}^j)$, $j = 1, \ldots, s_1 + 1$ in $\mathbb{P}^{h_1}$ generating L and $s_2 + 1$ points $\mathbf{y^l} = (y_1^l, \ldots, y_{h_2+1}^l)$, $l = 1, \ldots, s_2 + 1$ in $\mathbb{P}^{h_2}$ generating L', consider the following $(h_1 + h_2 + 2) \times (h_1 + h_2 + 2)$ square matrix:

$$M_{L,L'} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k+1} & x_1^1 & \cdots & x_1^{s_1+1} & 0 & \cdots & 0 \\ a_{2,1} & \cdots & a_{2,k+1} & x_2^1 & \cdots & x_2^{s_1+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{h_1+1,1} & \cdots & a_{h_1+1,k+1} & x_{h_1+1}^1 & \cdots & x_{h_1+1}^{s_1+1} & 0 & \cdots & 0 \\ b_{1,1} & \cdots & b_{1,k+1} & 0 & \cdots & 0 & y_1^1 & \cdots & y_1^{s_2+1} \\ b_{2,1} & \cdots & b_{2,k+1} & 0 & \cdots & 0 & y_2^1 & \cdots & y_2^{s_2+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{h_2+1,1} & \cdots & b_{h_2+1,k+1} & 0 & \cdots & 0 & y_{h_2+1}^1 & \cdots & y_{h_2+1}^{s_2+1} \end{pmatrix}$$

Remark 1 a) If L and L' are corresponding spaces then the linear system

$$M_{L,L'} \cdot \begin{bmatrix} \mathbf{X} \\ \lambda_1 \\ \vdots \\ \lambda_{s_1+1} \\ \mu_1 \\ \vdots \\ \mu_{s_2+1} \end{bmatrix} = \mathbf{0} \tag{1}$$

has a non-trivial solution, hence $\det(M_{L,L'}) = 0$. Indeed L and L' are corresponding if and only if there exists a point $\mathbf{X} \in \mathbb{P}^k$ such that $A \cdot \mathbf{X} \in L$ and $B \cdot \mathbf{X} \in L'$.

b) If either $L \cap E_1 \neq \emptyset$, or $L' \cap E_2 \neq \emptyset$, then $\det(M_{L,L'}) = 0$. Indeed, if, e.g., $\mathbf{e_2} \in L' \cap E_2$, then $\mathbf{e_2} = B \cdot \hat{\mathbf{X}}$, for some $\hat{\mathbf{X}} \in C_A$ and $\mathbf{e_2} = \sum_j \mu_j \mathbf{y^j}$, for some μ_j. Then one can replace $\mathbf{y^{s_2+1}}$ with $\mathbf{e_2}$ in the matrix $M_{L,L'}$. In this case, the last column of $M_{L,L'}$ is $\begin{bmatrix} A \cdot \hat{\mathbf{X}} \\ B \cdot \hat{\mathbf{X}} \end{bmatrix}$ which is clearly linearly dependent from the first $k + 1$ columns.

c) If $L \cap E_1 = \emptyset$, $L' \cap E_2 = \emptyset$ and $\det(M_{L,L'}) = 0$, then L and L' are corresponding. Indeed, with these assumptions, the linear system (1) has a non-trivial solution with, in particular, $\mathbf{X} \neq \mathbf{0}$. Indeed, if $\mathbf{X} = \mathbf{0}$ either $\lambda_j \neq 0$ for some j or $\mu_j \neq 0$ for some j. Either of these cases would contradict the linear independence of the points chosen to generate L and L'.

In view of Remark 1, one sees that $\det(M_{L,L'}) = 0$ is indeed the bilinear constraint that encodes the correspondence between subspaces of the two views.

Let $I = (i_1, \ldots, i_{s_1+1})$, $J = (j_1, \ldots, j_{s_2+1})$, $\hat{J} = (h_1 + 1 + j_1, \ldots, h_1 + 1 + j_{s_2+1})$ with $1 \leq i_1 < \cdots < i_{s_1+1} \leq h_1 + 1$ and $1 \leq j_1 < \cdots < j_{s_2+1} \leq h_2 + 1$. Denote by

I', $\hat{J}'$ the (ordered) sets of complementary indices $I' = \{r \in \{1, \ldots, h_1 + 1\}$ such that $r \notin I\}$ e $\hat{J}' = \{s \in \{h_1 + 2, \ldots, h_1 + h_2 + 2\}$ such that $s \notin \hat{J}\}$. Moreover denote by A_I and B_J, respectively, the matrices obtained from A and B deleting rows $i_1, \ldots, i_{s_1+1}$ and $j_1, \ldots, j_{s_2+1}$, respectively.

Computing $\det(M_{L,L'})$ with an iterated application of the generalized Laplace expansion, one gets:

$$det(M_{L,L'}) = \sum_{I,J} \lambda_I F_{I,J} \lambda'_J$$

where $\lambda_I = p_{i_1,\ldots,i_{s_1+1}}$ are Plücker coordinates of L, and similarly for λ'_J and L', and where the $F_{I,J}$ are given by:

$$F_{I,J} = \epsilon(I, J) \det \begin{pmatrix} A_I \\ B_J \end{pmatrix} \tag{2}$$

where $\epsilon(I, J)$ is $+1$ or -1 according to the parity of the permutation $(I, \hat{J}, I', \hat{J}')$.

Notice that $F_{I,J}$ are suitable maximal minors of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ so that they can be thought of as some of the Plücker coordinates of the k-subspace $\Lambda_{AB} \subset \mathbb{P}^{h_1+h_2+1}$, spanned by the columns of the above matrix. From the construction above, and recalling the relationship between Plücker and dual Plücker coordinates, see [18, Vol I, Book II, p.292], one sees that $F_{I,J}$ are the dual Plücker coordinates $q_{I,\hat{J}}(\Lambda_{AB})$.

The above discussion laid the groundwork for the following definition, using the same notation.

Definition 1 The generalized fundamental matrix for two projections A, B from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$, with profile (α_1, α_2), is, up to a multiplicative nonzero constant, the $\binom{h_1+1}{h_1-\alpha_1+1} \times \binom{h_2+1}{h_2-\alpha_2+1}$ matrix $\mathfrak{F}$, whose entries are $F_{I,J}$ are the dual Plücker coordinates $q_{I,\hat{J}}(\Lambda_{AB})$ with lexicographical order of I for the rows and $\hat{J}$ for the columns.

Example 1 In the classical case of two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, with $\alpha_1 = \alpha_2 = 2$, the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ has dimension 6×4. The subspace Λ_{AB} is in $G(3, 5) \subset \mathbb{P}^{14}$, hence its 15 Plücker coordinates are q_{r_1,r_2} with $1 \le r_1 < r_2 \le 6$, while the entries of the fundamental matrix are only the nine coordinates q_{i_1,j_1} with $1 \le i_1 \le 3$ and $4 \le j_1 \le 6$, indeed one has:

$$\mathfrak{F} = \begin{pmatrix} q_{1,4} & q_{1,5} & q_{1,6} \\ q_{2,4} & q_{2,5} & q_{2,6} \\ q_{3,4} & q_{3,5} & q_{3,6} \end{pmatrix}.$$

Example 2 Consider two projections from $\mathbb{P}^4$ to $\mathbb{P}^3$ with profile $(3, 2)$. In this case, the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ has dimension 8×5. The subspace Λ_{AB} is in $G(4, 7) \subset \mathbb{P}^{\binom{8}{5}-1}$, and the fundamental matrix $\mathfrak{F}$ is:

$$\mathfrak{F} = \begin{pmatrix} q_{1,5,6} & q_{1,5,7} & q_{1,5,8} & q_{1,6,7} & q_{1,6,8} & q_{1,7,8} \\ q_{2,5,6} & q_{2,5,7} & q_{2,5,8} & q_{2,6,7} & q_{2,6,8} & q_{2,7,8} \\ q_{3,5,6} & q_{3,5,7} & q_{3,5,8} & q_{3,6,7} & q_{3,6,8} & q_{3,7,8} \\ q_{4,5,6} & q_{4,5,7} & q_{4,5,8} & q_{4,6,7} & q_{4,6,8} & q_{4,7,8} \end{pmatrix}.$$

Example 3 Finally, we consider two projections from $\mathbb{P}^5$ to $\mathbb{P}^{h_1} = \mathbb{P}^4$ and $\mathbb{P}^{h_2} = \mathbb{P}^3$ with profile $(3, 3)$. In this case the fundamental matrix $\mathfrak{F}$ has dimension 9×6 and is of the form

$$\mathfrak{F} = \begin{pmatrix} q_{1,2,6} & q_{1,2,7} & q_{1,2,8} & q_{1,2,9} \\ q_{1,3,6} & q_{1,3,7} & q_{1,3,8} & q_{1,3,9} \\ \vdots & \vdots & \vdots & \vdots \\ q_{4,5,6} & q_{4,5,7} & q_{4,5,8} & q_{4,5,9} \end{pmatrix}.$$

The classical fundamental matrix F for two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, besides its role as a bilinear form $\mathbf{x}^T \cdot F \cdot \mathbf{x}'$, vanishing on pairs of corresponding points on the two views, can also be viewed as a map. Indeed F can be interpreted as mapping a point $\mathbf{x} \neq E_1$ in the first view to the line $\mathbf{x}^T \cdot F$, passing through E_2, and containing all points $\mathbf{x}'$ who are corresponding to $\mathbf{x}$. Hence F defines a rational map $\mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ whose image is the pencil of lines through E_2.

Similarly, the *generalized fundamental matrix* is, on the one hand, the matrix of a bilinear form acting on pairs (L, L') where L and L' are corresponding linear spaces in the two views and, on the other hand, can also be viewed as a rational map associating to a linear space L, such that $L \cap E_1 = \emptyset$, the linear space $B(A^{-1}(L))$ which, for a generic L, has dimension $k - \alpha_1$. This interpretation of $\mathfrak{F}$ as a map is investigated in the next section where, in particular, it will be useful in computing rk $(\mathfrak{F})$.

3.2 Geometric determination of rk $(\mathfrak{F})$

The determination of the rank of a tensor is an interesting and usually quite intricate problem. The rank of the classical fundamental matrix is well known to be 2. The rank of the trifocal tensor was determined to be 4, while the rank of the quadrifocal tensor turns out to be 9, [10]. Nothing is known in general about the ranks of generalized Grassmann tensors. In this section, we compute the rank of the generalized fundamental matrix as a first step in this direction. In the same notation as the previous section, one has the following Theorem.

Theorem 1 *The generalized fundamental matrix $\mathfrak{F}$ for two projections of maximal rank and whose centers do not intersect each other, with profile (α_1, α_2), defines a rational map $\Phi : G(s_1, h_1) \dashrightarrow G(k - \alpha_1, h_2)$ whose image is the Schubert variety $\Omega_{(k-\alpha_1, h_2)}(E_2)$. Moreover it is:*

$$\text{rk } (\mathfrak{F}) = \binom{(h_1 - \alpha_1 + 1) + (h_2 - \alpha_2 + 1)}{h_1 - \alpha_1 + 1}.$$

Proof Consider $\mathfrak{F}$ as the matrix of a rational map $\Phi : G(s_1, h_1) \dashrightarrow G(k - \alpha_1, h_2)$ defined as $\Phi(\mathbf{L}) = \hat{\mathbf{L}} \in G(k - \alpha_1, h_2) = G(\alpha_2 + 1, h_2) \subseteq \mathbb{P}^{N_2}$, where $\hat{\mathbf{L}}$ are the Plücker coordinates of $B(A^{-1}(L))$. As we have seen in Remark 1, this map is not defined on the points of $G(h_1 - \alpha_1, h_1)$ which correspond to linear spaces which intersect E_1. Due to the assumption $\alpha_1 + \alpha_2 = k + 1$, the open subset of $G(h_1 - \alpha_1, h_1)$ where the map is defined is not empty.

As in the classical case, the linear space $B(A^{-1}(L))$ contains E_2. This implies that when $\mathbf{L}$ varies in $G(s_1, h_1)$, the image $\Phi(\mathbf{L})$ varies in $\Omega_{(k-\alpha_1, h_2)}(E_2)$, which turns out to be a $(h_2 + \alpha_1 - k)(h_1 - \alpha_1 + 1)$-variety spanning a linear space $\mathbb{P}^\omega$ where $\omega = \binom{(h_1 - \alpha_1 + 1) + (h_2 - \alpha_2 + 1)}{h_1 - \alpha_1 + 1} - 1 = \binom{(h_1 + h_2 - k + 1)}{h_1 - \alpha_1 + 1} - 1$. This dimension can be computed using Proposition 3 in [13] to determine the number of independent linear conditions which define $\Omega_{(k-\alpha_1, h_2)}(E_2)$ inside the Grassmannian.

In Plücker coordinates, the map Φ is defined by $\Phi(\mathbf{L}) = \mathbf{L}^T \cdot \mathfrak{F}$ and extends to a rational map $\widetilde{\Phi} : \mathbb{P}^{N_1} \dashrightarrow \mathbb{P}^{N_2}$, where $N_1 = \binom{h_1+1}{h_1-\alpha_1+1} - 1$ and $N_2 = \binom{h_2+1}{h_2-\alpha_2+1} - 1$. The image of $\widetilde{\Phi}$ is contained in $\mathbb{P}^\omega$ because the Grassmannian $G(s_1, h_1)$ is not contained in any hyperplane and hence it spans all of $\mathbb{P}^{N_1}$. Moreover, it is not difficult to see that $\overline{Im(\Phi)} = \Omega_{(k-\alpha_1,h_2)}(E_2)$, hence Φ is dominant on $\Omega_{(k-\alpha_1,h_2)}(E_2)$ and this implies that $\widetilde{\Phi}$ is dominant on $\mathbb{P}^\omega$. Indeed for any $\mathbf{\Lambda} \in \Omega_{(k-\alpha_1,h_2)}(E_2)$ one has $\mathbf{\Lambda} = \Phi(\mathbf{L}) = \widetilde{\Phi}(\mathbf{L})$, for any $\mathbf{L} \subset A(B^{-1}(\Lambda))$, where, according to our notation, $\mathbf{\Lambda}$ denotes the vector of Plücker coordinates of the linear space Λ. This implies that the fundamental matrix has rank: rk$(\mathfrak{F}) = \omega + 1$. □

Remark 2 Notice that, as in the classical case, the generalized fundamental matrix is not of maximal rank. From the geometric point of view, this corresponds to the fact that the fiber $\Phi^{-1}(\mathbf{\Lambda})$, for a given generic $\mathbf{\Lambda} \in \Omega_{(k-\alpha_1,h_2)}(E_2)$, turns out to be the Grassmann variety $G(h_1 - \alpha_1, \alpha_2 - 1)$ of the $(h_1 - \alpha_1)$-spaces contained in $A(B^{-1}(\Lambda))$. This implies that the dimension of the fiber $\Phi^{-1}(\mathbf{\Lambda})$ and, a fortiori, of the fiber $\widetilde{\Phi}^{-1}(\mathbf{\Lambda})$ is positive.

Remark 3 When the hypothesis of Theorem 1 on skew centers of projections and maximal rank of the projection matrices are not satisfied, the generalized fundamental matrix could a priori have rank lower than the expected one. In reality, either $\mathfrak{F}$ is identically zero or it cannot even be constructed.

If $P \in C_A \cap C_B$, one can assume, up to projective transformations in $\mathbb{P}^k$, that $P = (1 : 0 : \cdots : 0)$. In this case, both projection matrices must have a null first column and hence from (2) it follows $\mathfrak{F} = 0$.

In the case of projection matrices with non-maximal rank, one can see that the Grassmann tensor cannot even be constructed. Indeed, for a given profile ($\alpha_1 = h_1 - s_1, \alpha_2 = h_2 - s_2$), with $\alpha_1 + \alpha_2 = k + 1$, as the dimension of the subspaces $\mathbb{P}^{h'_1}$ and $\mathbb{P}^{h'_2}$ of $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$ respectively, which are the images of the two projections, decreases, the corresponding codimensions ($\alpha'_1 = h'_1 - s_1, \alpha'_2 = h'_2 - s_2$) do not satisfy the condition $\alpha'_1 + \alpha'_2 = k + 1$ under which it is possible to define a Grassmann tensor.

3.3 Plücker relations among rows of $\mathfrak{F}$

Theorem 1 shows that $\mathfrak{F}$ does not have maximal rank. It is natural to wonder about the nature of the linear relations existing among rows (or columns) of the generalized fundamental matrix, responsible for the dropping of the rank. In this section we determine such relations, under suitable assumptions on h_i and α_i, showing that they are consequences of well-known generalized Plucker relations (in dual coordinates). In order to do that, we recall, [13, p. 1076], that for a Grassmannian $G(d, n)$, the following relations hold:

$$\sum_\sigma sgn(\sigma) q_{i_1 \ldots i_\lambda, \sigma i_{\lambda+1} \ldots \sigma i_{d+1}} q_{\sigma j_1 \ldots \sigma j_{\lambda+1}, j_{\lambda+2} \ldots j_{d+1}} = 0, \tag{3}$$

where the sum ranges over all permutations σ of $(i_{\lambda+1} \ldots i_{d+1}, j_1 \ldots j_{\lambda+1})$ such that $\sigma i_{\lambda+1} < \cdots < \sigma i_{d+1}$ and $\sigma j_1 < \cdots < \sigma j_{\lambda+1}$.

3.3.1 Case $h_1 = h_2 = h$ and profile (h, h)

In this situation, the matrix $\mathfrak{F}$ is a square matrix of order $h + 1$, of rank 2 (by Theorem 1), whose entry in position (i, j) is $f_{ij} = q_{i,(h+1)+j}$. Considering any three rows of $\mathfrak{F}$:

$$R_a = [q_{a,(h+1)+1}, q_{a,(h+1)+2}, \ldots, q_{a,(h+1)+h+1}]$$
$$R_b = [q_{b,(h+1)+1}, q_{b,(h+1)+2}, \ldots, q_{b,(h+1)+h+1}]$$
$$R_c = [q_{c,(h+1)+1}, q_{c,(h+1)+2}, \ldots, q_{c,(h+1)+h+1}]$$

with $1 \leq a, b, c \leq h + 1$, the generalized Plucker relations (3) imply the relation:

$$q_{b,c} R_a - q_{a,c} R_b + q_{a,b} R_c = 0.$$

Notice that when $k = 3$ and $h = 2$, one has the classical scenario described in Example 1.

3.3.2 Case $h_1 = h_2 = h$ and profile $(h, h - 1)$

In this case, $\mathfrak{F}$ has dimension $(h + 1) \times \frac{h(h+1)}{2}$ and rank 3 (by Theorem 1). It is:

$$\mathfrak{F} = \begin{pmatrix} q_{1,(h+1)+1,(h+1)+2} & q_{1,(h+1)+1,(h+1)+3} & \cdots & q_{1,(h+1)+h,(h+1)+(h+1)} \\ q_{2,(h+1)+1,(h+1)+2} & q_{2,(h+1)+1,(h+1)+3} & \cdots & q_{2,(h+1)+h,(h+1)+(h+1)} \\ \vdots & \vdots & \cdots & \vdots \\ q_{h+1,(h+1)+1,(h+1)+2} & q_{h+1,(h+1)+1,(h+1)+3} & \cdots & q_{h+1,(h+1)+h,(h+1)+(h+1)} \end{pmatrix}$$

For any four rows of $\mathfrak{F}$:

$$R_a = [q_{a,(h+1)+1,(h+1)+2}, q_{a,(h+1)+1,(h+1)+3}, \ldots, q_{a,(h+1)+h,(h+1)+(h+1)}]$$
$$R_b = [q_{b,(h+1)+1,(h+1)+2}, q_{b,(h+1)+1,(h+1)+3}, \ldots, q_{b,(h+1)+h,(h+1)+(h+1)}]$$
$$R_c = [q_{c,(h+1)+1,(h+1)+2}, q_{c,(h+1)+1,(h+1)+3}, \ldots, q_{c,(h+1)+h,(h+1)+(h+1)}]$$
$$R_d = [q_{d,(h+1)+1,(h+1)+2}, q_{d,(h+1)+1,(h+1)+3}, \ldots, q_{d,(h+1)+h,(h+1)+(h+1)}]$$

with $1 \leq a, b, c, d \leq h + 1$, the generalized Plucker relations (3) imply the relation:

$$q_{b,c,d} R_a - q_{a,c,d} R_b + q_{a,b,d} R_c - q_{a,b,c} R_d = 0.$$

Notice that for $k = 4$ and $h = 3$ one has the same situation as in Example 2.

3.3.3 Case $h_1 = h_2 = h$, $k = h + 1$ and any profile $(\alpha_1, \alpha_2) = (h - t, t + 2)$

In this case, we can consider any profile $(\alpha_1, \alpha_2) = (h - t, t + 2)$, with $0 \leq t \leq \frac{h-2}{2}$. Notice that the assumption $k = h + 1$ is equivalent to the condition that the number of rows of $\mathfrak{F}$, $\binom{h+1}{h-\alpha_1+1}$, is exactly the number $\binom{d+2}{\lambda+1}$ of summands appearing in relations (3). Hence, in this case, each Plucker relation involves all of the rows of $\mathfrak{F}$.

Theorem 1 gives $rk(\mathfrak{F}) = \binom{h}{t+1}$. Here we show, by induction on t, that all relations among the rows of $\mathfrak{F}$ are generated by generalized Plucker relations as in (3). Notice that one can consider relations (3) as linear equations in the entries of the fundamental matrix $q_{\sigma j_1 \ldots \sigma j_{\lambda+1}, j_{\lambda+2} \ldots j_{d+1}}$, with coefficients $q_{i_1 \ldots i_\lambda, \sigma i_{\lambda+1} \ldots \sigma i_{d+1}}$, which are themselves Plucker coordinates, and which do not depend on $j_{\lambda+2} \ldots j_{d+1}$, so that they are invariant for all the elements of a given column of $\mathfrak{F}$.

Hence relations (3) give rise to linear relations among the rows of $\mathfrak{F}$. More precisely, under the above assumptions, for a profile $(h - t, t + 2)$, the matrix $\mathfrak{F}$ has dimension $\binom{h+1}{h-t} \times \binom{h+1}{t+2}$ and its entries are $f_{I,J}$ with $I = (i_1, \ldots, i_{t+1})$ and $J = (j_1, \ldots, j_{h-t-1})$, so that, in the generalized Plucker relations (3), with $\lambda = t$ and $d = h - 1$, the entries of $\mathfrak{F}$ can be identified with the elements $q_{\sigma j_1 \ldots \sigma j_{t+1}, j_{t+2} \ldots j_{h+2}}$ and the coefficients of (3) can be organized in the following $\binom{2h+2}{t+2} \times \binom{h+1}{t+3}$ matrix $\mathfrak{R}(\mathfrak{F}) = [q_{i_1 \ldots i_t, \sigma i_{t+1} \ldots \sigma i_{h+2}}]$. The rows of $\mathfrak{R}(\mathfrak{F})$ depend on

the choice of the indices $i_1 \ldots i_t$ among $2h + 2$ elements corresponding to the rows of the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$. The proof hinges on the fact that, if we restrict the choice only to the $h+1$ rows of the matrix A, we obtain a $\binom{h+1}{t+2} \times \binom{h+1}{t+3} = \binom{h+1}{h-t-1} \times \binom{h+1}{t+3}$ submatrix $\mathfrak{R}_o$ which comes out to be the fundamental matrix corresponding to the two projections B and A, in reversed order, with profile $\alpha_1 = h - (t - 1)$, $\alpha_2 = (t - 1) + 2$. Hence one can apply induction on t to prove that $rk(\mathfrak{F}_t) = \binom{h}{t+1}$, where $\mathfrak{F}_t$ denotes the generalized fundamental matrix for the profile $(h - t, t + 2)$. Analogously, let us denote by $\mathfrak{R}_{ot}$ the matrix of the Plücker relations considered above, for the profile $(h - t, t + 2)$.

- If $t = 0$, then $\mathfrak{F}_0$ has dimension $(h + 1) \times \binom{h+1}{2}$ and $\mathfrak{R}_{o0}$ is a row vector, as $\lambda = 0$. Hence this Plucker relation is enough to show that the rank of $\mathfrak{F}_0$ is h.
- By induction, at step $t - 1$, we assume the statement is true for the profile $(h - (t - 1), (t - 1) + 2)$, i.e. $rk(\mathfrak{F}_{t-1}) = \binom{h}{t}$.
  At step t we recall that $rk(\mathfrak{R}_{ot}) = rk(\mathfrak{F}_{t-1}) = \binom{h}{t}$, so that

$$rk(\mathfrak{F}_t) = \binom{h + 1}{t + 1} - rk(\mathfrak{R}_{ot}) = \binom{h + 1}{t + 1} - \binom{h}{t} = \binom{h}{t + 1}.$$

4 The variety of generalized fundamental matrices

In this section, we want to investigate the geometrical properties of the variety, $\mathfrak{X} = \mathfrak{X}_{(\alpha_1, \alpha_2)}$, parameterizing the fundamental matrices corresponding to a given profile (α_1, α_2) for two projections from $\mathbb{P}^k$ to $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$ with $k = \alpha_1 + \alpha_2 - 1$, $1 \leq \alpha_1 \leq h_1$, $1 \leq \alpha_2 \leq h_2$, $h_1 + h_2 \geq k + 1$, and $\alpha_1 \geq \alpha_2$. In particular, we prove the following Proposition:

Proposition 1 *In the notation and setting above, $\mathfrak{X}$ is an algebraic variety with:*

$$dim(\mathfrak{X}) = k(h_1 + h_2) + h_1 + h_2 - k^2.$$

Proof The construction of this variety can be accomplished drawing inspiration from the setting of Aholt and Oeding in [7]. Let U_1 and U_2 be the vector spaces of dimension respectively $h_1 + 1$ and $h_2 + 1$ of the rows of A and of B, and let W be the vector space $U_1 \oplus U_2$, of dimension $h_1 + h_2 + 2$. The matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ can be thought of as a point in the Grassmann variety, $G(k, \mathbb{P}(W)) \subset \mathbb{P}(\bigwedge^{k+1} W)$.

As the projection matrices are defined up to independent projective transformations of the two views, one is naturally lead to consider the group $S = SL(U_1) \times SL(U_2) \subset SL(W)$ consisting of the unit determinant blocks of dimensions $(h_1 + 1) \times (h_1 + 1)$ and $(h_2 + 1) \times (h_2 + 1)$ on the diagonal of a $(h_1 + h_2 + 2) \times (h_1 + h_2 + 2)$ matrix, and its action on the affine cone $C(G(k, \mathbb{P}(W)))$ over $G(k, \mathbb{P}(W))$.

In the decomposition of $\bigwedge^{k+1} W$ as a S−module:

$$\bigwedge^{k+1} W = \bigoplus_{a=0}^{k+1} \left(\bigwedge^a U_1 \otimes \bigwedge^{k+1-a} U_2 \right), \tag{4}$$

the summand corresponding to the minors of $\begin{bmatrix} A \\ B \end{bmatrix}$ appearing in the fundamental matrix $\mathfrak{F}$ is $\bigwedge^{\alpha_1} U_1 \otimes \bigwedge^{\alpha_2} U_2$.

Consider the projection $\pi : \mathbb{P}(\bigwedge^{k+1}(U_1 \oplus U_2)) \dashrightarrow \mathbb{P}(\bigwedge^{\alpha_1} U_1 \otimes \bigwedge^{\alpha_2} U_2)$. Then

$$\mathfrak{X} = \overline{\pi(G(k, \mathbb{P}(W)))}.$$

As in the classical case of projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ [7,10], we have that the projection π induces a birational map

$$\Pi : C(G(k, \mathbb{P}(W)))_{/(\mathbb{C}^* \oplus \mathbb{C}^*)} \dashrightarrow \mathfrak{X} \hookrightarrow \mathbb{P}(\bigwedge^{k+1}(U_1 \oplus U_2)).$$

Indeed Hartley and Shaffalitzky [2], prove that, but for the case $h_1 = h_2 = 1$, a Grassmann tensor completely determines the set of projection matrices up to projective transformations. The birationality of the map Π allows us to compute the dimension of $\mathfrak{X}$ via counting the number of essential parameters needed to determine A e B. This gives:

$$\begin{aligned} \dim(\mathfrak{X}) &= ((h_1 + 1)(k + 1) - 1) + (h_2 + 1)(k + 1) - 1) - ((k + 1)^2 - 1) \\ &= k(h_1 + h_2) + h_1 + h_2 - k^2. \end{aligned}$$

□

Remark 4 Note that, as expected, $\dim \mathfrak{X}$ is also the dimension of the quotient $C(G(k, \mathbb{P}(W)))_{/(\mathbb{C}^* \oplus \mathbb{C}^*)}$.

4.1 A characterization of $\mathfrak{X}$ with profile (h_1, h_2)

In particular, in the case of profile $(\alpha_1, \alpha_2) = (h_1, h_2)$, under the usual assumptions of projections of maximal ranks and skew centers, we get the following:

Theorem 2 *A matrix $\mathfrak{G}$ of dimension $(h_1 + 1) \times (h_2 + 1)$ is the fundamental matrix for a suitable pair of projections from $\mathbb{P}^{h_1+h_2-1}$ to $\mathbb{P}^{h_1}$ and $\mathbb{P}^{h_2}$ with profile (h_1, h_2) and skew centers if and only if $rank(\mathfrak{G}) = 2$.*

Proof Under the current assumptions, Theorem 1 gives rk $(\mathfrak{F}) = 2$. On the other hand, the dimension of the variety of rank 2 matrices of dimension $(h_1+1) \times (h_2+1)$ is $2(h_1+h_2)-1 = dim(\mathfrak{X})$. □

This generalizes the well-known result for the classical case, [10, Theorem 4.1]. Notice that, in general, one cannot expect to extend this result to other profiles. Indeed, in the previous section we have shown that $dim(\mathfrak{X}) = k(h_1 + h_2) + h_1 + h_2 - k^2$.

On the other hand, the dimension, δ_r, of the variety of the $(N_1 + 1) \times (N_2 + 1)$-matrices of rank r is

$$\delta_r = (N_1 + N_2 + 2)r - r^2 - 1.$$

As $N_1 = \binom{h_1+1}{\alpha_1} - 1$ and $N_2 = \binom{h_2+1}{\alpha_2} - 1$, generically this dimension turns out to be greater than $dim(\mathfrak{X})$. For example, if $h_1 = h_2 = h$, and $(\alpha_1, \alpha_2) = (h, h - 1)$, which implies rk $(\mathfrak{F}) = r = 3$, one has $dim(\mathfrak{X}) = 6h - 4$ and $\delta_r = \frac{3}{2}h^2 + \frac{9}{2}h - 7$.

5 Application to reconstruction in computer vision

From the point of view of computer vision, the *generalized fundamental matrix* is relevant because of its application to reconstruction problems.

Given multiple images of an unknown scene, taken from unknown cameras, the goal of the *reconstruction problem* is to determine the positions of the cameras and of the scene points. Reconstruction problems can be posed in several geometric settings as metric, affine, or projective. As this work has been conducted entirely within the projective framework, reconstruction will always be assumed to be achieved up to projective transformations.

Within a projective setting, the center is the only property of a camera which is preserved under homographies of the view spaces (e.g. see [19]), and hence projective reconstruction of cameras consists only of the determination of their centers.

Therefore, given r views of a scene $\{\mathbf{X_i}\} \subset \mathbb{P}^k$, the reconstruction problem has two stages: reconstructing the camera centers and reconstructing the scene, i.e. the position of the points $\{\mathbf{X_i}\}$ in $\mathbb{P}^k$, once cameras have been reconstructed.

Assuming that the scene to be reconstructed consists of a large enough number of points, in general enough mutual positions, a first natural question is to determine the numerical conditions on k, h_j, for $j = 1 \ldots r$, under which projective reconstruction is possible, both for the cameras Pj and for the scene $\{\mathbf{X_i}\}$ in $\mathbb{P}^k$.

Both these conditions are implicitly given in [2] in the general setting and they are explicitly highlighted in [20, Propositions 3.1, 3.2 and [21], Sect. 4.1] in the case $h_1 = \cdots = h_r$.

We now recall these conditions in our more general setting, and we sketch the geometric rationale for the statements.

a) *Let $\mathbb{P}^{h_j}$, $j = 1, \ldots, r$ be r target views for projections from $\mathbb{P}^k$, in which a large enough set of corresponding subspaces, of dimensions $s_j = h_j - \alpha_j$, are given. Then the centers of projections can be determined if and only if $\sum_{j=1}^{r} \alpha_j \geq k + 1$.*
   The projective reconstruction of the associated cameras (hence of the centers of projection) is possible only if requiring the existence of a point in the intersection of all rays projecting the given corresponding subspaces impose an actual non-trivial constraint on the position of the centers in $\mathbb{P}^k$. In this case, knowing enough sets of corresponding subspaces, one can deduce the equations of the centers. For this to happen, the intersection of r general $(k - \alpha_i)$-spaces, for $i = 1, \ldots r$, must be empty. Grassmann formula shows that the dimension of the iterated intersection of such $(k - \alpha_i)$-spaces drops by α_i at each step. Therefore $\sum_{i=1}^{t} \alpha_i \geq k + 1$.

b) *Let $\mathbb{P}^{h_j}$, $j = 1, \ldots, r$ be r target views for projections from $\mathbb{P}^k$, in which a large enough set of corresponding subspaces, of dimensions $s_j = h_j - \alpha_j$, are given. Assume camera centers are known. Then a scene $\{\mathbf{X_i}\}$ can be reconstructed if and only if $\sum_{i=1}^{t} \alpha_i \geq k$.*
   A scene point $\mathbf{X}$ is determined by the intersection of a suitable number of corresponding rays. Hence the reconstruction of a point is possible if and only if the intersection of r generic $(k - \alpha_i)$-spaces, for $i = 1, \ldots r$, is at most one point. Using again Grassmann formula, we get $\sum_{i=1}^{r} \alpha_i \geq k$.

In our case with two views, under the assumption of Sect. 3, it is $\alpha_1 + \alpha_2 = k + 1$, and hence reconstruction of both cameras and scene points is indeed possible.

The role of the (generalized) fundamental matrix is better highlighted in the description of the algorithms involved in projective reconstruction. In the classical case of two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, the reconstruction procedure follows the following steps:

1) *Fundamental matrix from correspondences*: If a sufficient number of corresponding points is known, one can determine the fundamental matrix F by solving the linear system $\mathbf{x}^T \cdot F \cdot \mathbf{x}' = 0$. Note that, generically, 7 pairs of corresponding points are sufficient, see [4], and a finer analysis can be found in [17].
2) *Projections from the fundamental matrix*: Once F is determined, projection matrices can be reconstructed as in [4, Section 8.5.3].

3) *Scene from projection matrices and correspondences*: Once projection matrices are known, the scene points are determined intersecting corresponding rays.

In the following section, we describe the geometric-exact algorithmic procedures for performing the above steps.

5.1 Reconstruction algorithms

All algorithms presented below are described for the general case of views of different dimensions, but implemented, for the sake of simplicity, with $h_1 = h_2 = h$. Furthermore, as seen in the previous sections, different profiles (α_1, α_2) could be chosen in the setup of projective reconstruction. Our implementation uses $\alpha_1 = h$ and $\alpha_2 = k - h + 1$. MATLAB®code is available from the authors upon request.

5.1.1 Generalized fundamental matrix from correspondences

Recall that, for the profile $(h_1, k - h_1 + 1)$, the generalized fundamental matrix is defined to be the matrix $\mathfrak{F}$ such that $\mathbf{x}^T \mathfrak{F} \mathbf{W} = 0$, for any corresponding pair $(\mathbf{x}, W)$, where $\mathbf{x}$ is a point in the first view, W is a $(h_1 + h_2 - k - 1)-$space of the second view, and $\mathbf{W}$ is the vector of Plücker coordinates of W. Assume a large enough set of pairs $\{(\mathbf{x_i}, W_i)\}$ of corresponding spaces in the two views are given. For each of them, one has a constraint $\mathbf{x_i}^T \mathfrak{F} \mathbf{W_i} = 0$, where the entries f_{rs} of $\mathfrak{F}$ are unknown. The set of these constraints gives a linear system whose solutions are entries of $\mathfrak{F}$.

5.1.2 Projection matrices from the generalized fundamental matrix

Assume the generalized fundamental matrix $\mathfrak{F}$ is given and one wants to recover a pair of projections A, B corresponding to it. As it is standard in this context, everything being defined up to projective transformations, one can assume $A = [I|O]$ where I is the identity matrix of order $h_1 + 1$ and O denotes a $(h_1 + 1) \times (k - h_1)$ zero-matrix. Thus one has only to recover the matrix B.

First notice that, due to the results of Sect. 3, from $\mathfrak{F}$ one can extract a set of generators for the epipole E_2 in the second view. Indeed we have seen that a set of generators for the projective space $\mathbb{P}^\omega$ spanned by $\Omega_{(k-\alpha_1, h_2)}(E_2)$ is $\{\mathbf{b}_1^T \mathfrak{F}, \dots, \mathbf{b}_{h_1+1}^T \mathfrak{F}\}$, where $\{\mathbf{b}_1, \dots, \mathbf{b}_{h_1+1}\}$ denotes the standard basis for $\mathbb{P}^{h_1}$. For the chosen profile, a dimension count shows that $\Omega_{(k-\alpha_1, h_2)}(E_2) = \mathbb{P}^\omega$, hence each $\mathbf{b}_i^T \mathfrak{F}$ is the vector of the Plücker coordinates of a $(k - \alpha_1)$-subspace of the second view containing E_2. From the Plücker coordinates of $\mathbf{b}_i^T \mathfrak{F} = \mathbf{W}_i$, one can then recover a set of generators for the corresponding linear space W_i, with standard techniques.

For any pair of corresponding point and space $(\mathbf{x_t}, W_s)$, the constraint $\mathbf{x_t}^T \mathfrak{F} \mathbf{W_s} = 0$ can be written as $det(M_{t,s}) = 0$, where $M_{t,s} = M(\mathbf{x_t}, \mathbf{W_s})$ is a $(h_2 + 1) \times (h_2 + 1)-$ matrix whose columns $M(1), \dots, M(h_2 + 1)$ are:

- $M(j) = \mathbf{e}_j$, for $j = 1, \dots, k - h_1$, where $\mathbf{e}_1 \dots \mathbf{e}_{k-h_1}$ are vectors spanning the epipole E_2;
- $M(k - h_1 + 1) = \mathbf{y}$, where $\mathbf{y} = B(\mathbf{p})$ with $\mathbf{p} = A^+(\mathbf{x_r})$, and $A \cdot A^+ = I$;
- $M(i) = \mathbf{z}_i$ for $i = k - h_1 + 2, \dots, h_2 + 1$, where $\mathbf{z}_i$ are points of $\mathbb{P}^{h_2}$ spanning W.

Then, up to a constant, one has $f_{lm} = \det(M_{l,m}) = \mathbf{x_l}^T \mathfrak{F} \mathbf{W_m}$ where $\mathbf{x_l}$ is the l-th element of the standard basis of $\mathbb{P}^{h_1}$ and $\mathbf{W_m}$ is the m-th element of the standard basis of $\mathbb{P}^{N_2}$. Hence,

one can compute $\det(M_{l,m})$ and all the elements f_{lm} of $\mathfrak{F}$, by using generalized Laplace expansion, considering the last $s_2 + 1$ columns as a block, so that the minors of this block represent the Plücker coordinates of W_m.

Hence, for a given pair $(\mathbf{x}, W)$, all the columns of M are determined by $\mathfrak{F}$ but the $(k - h_1 + 1)$-th which depends linearly on the entries b_{uv} of B.

Letting $\mathbf{x_l}$ vary among elements of the standard basis of $\mathbb{P}^h$ and $\mathbf{W_m}$ among elements of a basis of the projective space $\mathbb{P}^{N_2}$, one gets a linear system $f_{lm} = \det(M_{l,m})$ of $(h_1 + 1) \times (N_2 + 1)$ equations in $(h_2 + 1) \times (k + 1)$ unknowns b_{uv}, which solves the problem. Existence of solutions for the above system is guaranteed by the birationality of the map Π, introduced in the proof of Proposition 1. Moreover, in case of an underdetermined system with infinitely many solutions, the birationality of Π also implies that all solutions are projectively equivalent.

5.1.3 Scene from projection matrices and correspondences

Once one has recovered a pair of projection matrices A and B, given a pair of corresponding spaces $\{(\mathbf{x}, W)\}$, it is possible to determine the scene point $\mathbf{X}$ in $\mathbb{P}^k$. More explicitly one simply solves the system (1) where $M_{L,L'}$ has been populated with A, B, $\mathbf{x}$ and a set of generators for W.

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