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Lost in Translation? Reading Newton on inverse-cube trajectories

Niccolò Guicciardini

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Abstract This paper examines an annotation in Newton's hand found by H. W. Turnbull in David Gregory's papers in the Library of the Royal Society (London). It will be shown that Gregory asked Newton to explain to him how the trajectories of a body accelerated by an inverse-cube force are determined in a corollary in the *Principia*: an important topic for gravitation theory, since tidal forces are inverse-cube. This annotation opens a window on the more hidden mathematical methods which Newton deployed in his *magnum opus*. The received view according to which the *Principia* are written in a geometric style with no help from calculus techniques must be revised.

Keywords Isaac Newton · David Gregory · central forces

Mathematics Subject Classification (2000) MSC 01A45

1 A standard view on the birth of rational mechanics

Historians like to disprove received views as much as mathematicians like to prove long-standing unsolved conjectures. And in this paper I might seem to be following a pattern that is often adopted by historians in order to arouse bewilderment and elicit praise. I will open my contribution with a view of the reception of Isaac Newton's *Principia* that enjoys wide currency. But let me state from the beginning that my aim is not to disprove: rather, my more modest claim is that the received view needs qualification as it offers too simplistic a narrative.

N. Guicciardini
Facoltà di Scienze Umanistiche, Università degli Studi di Bergamo, via Pignolo 123, 24121, Bergamo BG, Italia
Tel.: +39-035-2052438
Fax: +39-035-2052430
E-mail: niccolo.guicciardini@unibg.it

The standard view is this: Newton's *Principia* revealed to mankind the right physics, but did so in an obsolete mathematical language. Indeed, it is claimed, Newton discovered a true universal law that regulates the motions of bodies, but instead of using the right mathematical language, the calculus, he used geometry. So, the story continues, it is the merit of Continental mathematicians, such as Johann Bernoulli, Pierre Varignon and Leonhard Euler, if the *Principia* got translated into the right language, the differential and integral calculus due to his mathematical enemy, Gottfried Wilhelm Leibniz.[1] In this context, it is often claimed that it is Jacob Hermann and Johann Bernoulli who first applied calculus to central force motion by integrating the equations of motion for inverse-square central forces.[2]

This narrative has several features that deserve praise. It demonstrates once more that in the eighteenth century the notion of "Newtonianism" was a highly latitudinarian concept, since a mathematized Newtonian planetary theory could be advanced in Leibniz's notation and by adopting Leibnizian physical principles, such as conservation of vis viva, by actors who, most often, did not endorse the notion of action at a distance implied by gravitation theory.

However, the standard view is too simplistic. It attributes to a monolithically defined Continental school, based in Basel, Paris and the academies of Berlin and Saint Petersburg, all the advances in the algebraization of mechanics (the supposed translation of Newton's geometrical style of the *Principia* into the language of calculus), forgetting the contributions in this direction provided by British mathematicians such as David Gregory, Roger Cotes, Abraham De Moivre, James Stirling, Thomas Simpson and Colin MacLaurin. Further, the standard view is based on considering what Newton printed in the *Principia*, but ignores what he wrote in his manuscripts and what he circulated through his correspondence.

The standard view on the reception of the *Principia* is rooted in the highly emotional nationalistic terrain of the Newton-Leibniz controversy. At the beginning of the eighteenth century, Leibniz and his supporters, such as Johann Bernoulli, claimed that the absence of calculus in Newton's *magnum opus* was proof positive that its author did not possess the new analysis by 1687. Indeed: if Newton had the calculus, why did he not use it in his mathematical theory of gravitation? This emotional terrain is still fertile nowadays and sometime one

[1] Two classic masterpieces in which this view is defended are [3] and [23].

[2] "Newton solved what was called afterwards for a short time 'the direct Kepler problem' ('le problème direct'): given a curve (e.g. an ellipse) and the center of attraction (e.g. the focus), what is the law of this attraction if Kepler's second law holds? The 'problème inverse' (today: the 'problème direct') was attacked systematically only later, first by Jacob Hermann, then solved completely by Johann Bernoulli in 1710 and following Bernoulli by Pierre Varignon" ([22], 103). However, as we shall see, in writing his *Principia* Newton successfully applied his method of fluxions to the inverse problem for inverse-cube forces. Whether he could do the same for inverse-square forces is an open question. However, Newton's procedure expounded in an annotation to Gregory dated 1694, and which I analyse in this paper, when applied to inverse-square forces leads to a quadrature that Newton could perform. The quadrature occurs in the second catalogue of curves (ordo octavus) of *De methodis* ([18], 3, 252), and a similar quadrature is employed in the solution of Corollary 2, Proposition 91, Book 1: for details see ([13], 282–90).

has the impression that the flames that fueled the Newton-Leibniz controversy have not yet been extinguished!

Let me state in boldface letters that it is not my aim to enter into such controversies, which rather than playing the role of a combative assumption, should constitute a dispassionate object of study for the historian (so one might legitimately ask how British nationalism shaped the Newton-Leibniz controversy, for example). My aim is not to vindicate Newton, but to interpret his work. My aim is not to answer in the positive to the often-posed question "Did Newton use his calculus in the *Principia*?", since this is a bad historical question, in my opinion.

The calculus is not an object, like a pebble (which is, indeed, translated as *calculus* in Latin), that can be discovered by one lucky researcher alone, but consists of a combination of concepts, notations, rules and theorems. It would be difficult to say exactly which of these elements makes the calculus what it is, and certainly it would be simplistic to think that one can find it by inspection, after opening the cover of Newton's *magnum opus*.

I am convinced that historians of mathematics should not ask themselves questions of priority and of attribution of merit. The reasons why credit-attribution so often polarizes the attention of historians of mathematics are to be sought in nationalism and school-partisanship that permeate the ethos of academia (and are sometimes defined as "healthy"). Yet, as Luke Hodgkin puts it: "Awarding points to individuals or civilizations for their excellence in mathematics should not be part of the business of history" ([14], 70). The history of mathematics should try to answer good historical questions, and I will try to identify a few of them at the end of this paper. But what makes a question a "good" historical question? This is very difficult to say, as difficult as it is to tell good mathematical conjectures from bad ones.

2 David Gregory's *Notæ* to the *Principia*

An important document that sheds light on the mathematical methods employed by Newton in the *Principia* is an annotation written by Newton for David Gregory, the nephew of Newton's great contemporary James Gregory. In 1687, when he was the holder of the Chair of Mathematics in Edinburgh, he received a complimentary copy of the *Principia*. He remained quite impressed and began to carefully annotate the *magnum opus* in the early autumn of that year. The result was a 213-page manuscript entitled *Notæ in Newtoni Principia Mathematica Philosophiæ Naturalis*.[3] Probably for political reasons, in 1691 Gregory moved to Oxford, where he was appointed to the Savilian Chair of Astronomy. He soon became a convinced Newtonian, in connection with a group of compatriots of his, whom Anita Guerrini has labelled the "Tory

[3] The original is MS 210 (Royal Society Library, London). There are also three other copies: in Christ Church (Oxford), in the University Library Edinburgh), and in the Gregory Collection of the University of Aberdeen.

Newtonians," and who included Archibald Pitcairne, John Craig, and John Keill [11].

The *Notæ* provide a thorough commentary on the *Principia*, from beginning to end. They were written in three stages, as is evident from the dating, as well as from the paper and ink used. The original manuscript in Gregory's hand is kept in the Royal Society (London), and shelved as MS 210. The first 33 pages, a commentary on the first nine sections of Book 1, were written in Edinburgh and are dated from September 1687 to April 1688. The remaining pages were written in Oxford and are dated from December 23, 1692 to January 29,1694. There are also later additions, sometimes written on slips of paper affixed with paste or wax. The last addition was made in 1708, the year of Gregory's death. At some point Gregory wished to publish the *Notæ* as a running commentary to the *Principia*. He may have communicated this project to Newton at the beginning (4-8) of May 1694, when he was admitted to Newton's chambers in a first of a series of visits.

The May 1694 visit was very important for Gregory, since it occurred after a period of tension between him and Newton. The five-day meeting changed Gregory's scientific life. During these days, Newton showed him his projects for a revision of the *Principia*, and let him study his tracts on fluxions, including an early version of *Tractatus de quadratura curvarum* (the treatise published as an appendix to the *Opticks* in 1704), where Newton developed "quadrature" techniques, that is methods aimed at calculating the area of a surface subtended under a curve. The two men also discussed the Ancients' *prisca sapientia*, religion, astronomy, alchemy, optics, and much more.

We are in a lucky position, since Gregory wrote detailed memoranda of his encounters with Newton that are still extant. Some of these memoranda have the character of rather quick working annotations and in some of them Newton's handwriting is mixed with Gregory's: the pupil seems to have taken as many quick notes as he could with the help and encouragement of his master. And not a few concern the *Principia*.

Indeed, in a memorandum (penned in July 1694) we read (in Turnbull's English translation):

> The second treatise [a draft of *De quadratura*] will contain his [Newton's] Method of Quadratures [...] To this he will subjoin tables [...] on these [tables] depend certain more abstruse parts in his philosophy as hitherto published, such as Corollary 3, Proposition 41 and Corollary 2, Proposition 91.[4] ([17], 386)

[4] The original text in Latin reads: "Secundus Tractatus Methodum suam Quadraturarum continebit quae rem istam mire augebit et promovebit [...] Huic subjungit tabulas pro diversis formis et gradibus figurarum usque ad ordinem decimum [...] Item alias tabulas ad usque classem undecimum ubi spatia non quadrabilia cum coni sectionibus comparantur [...] innituntur quaedam abstrusiora in Philosophia sua hactenus edita ut Corol: 3 prop. XLI et Corol: 2 prop. XCI." [18], 7, p. 197. It is very interesting that Gregory refers to tables in which "spaces which cannot be squared are compared with conic sections," since, as we shall see below, these are the quadrature techniques that allow the most difficult of the quadratures implied in Corollary 3.

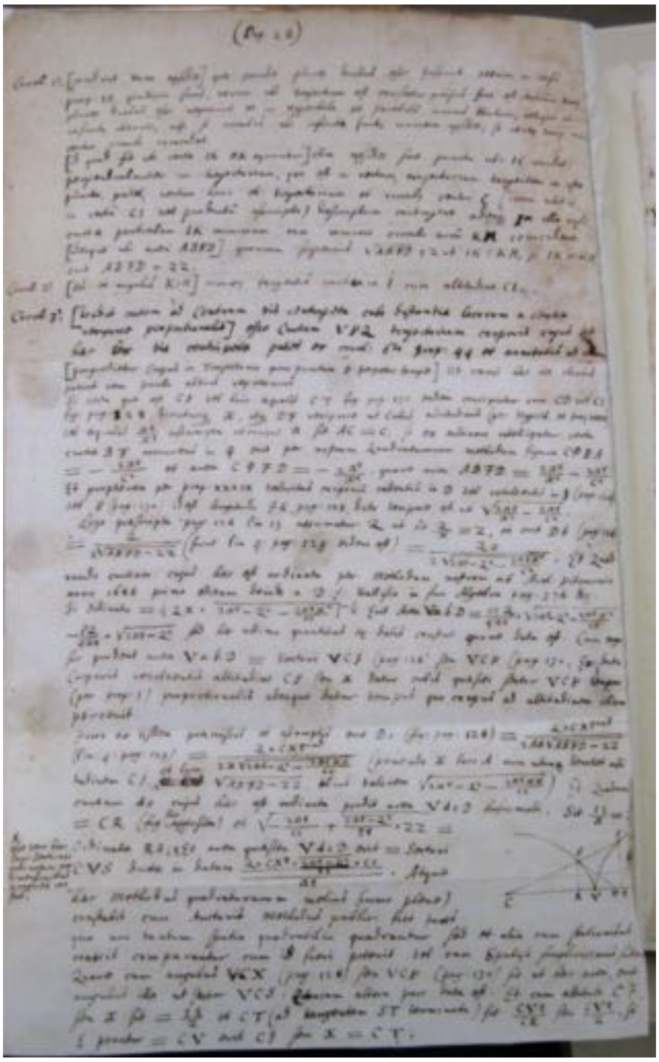

Fig. 1: Pasted sheet with commentary on Corollary 3 in Gregory's *Notæ*. Source: MS 210: 28 (Royal Society Library). ©The Royal Society

Here Gregory is referring to two corollaries of the *Principia* and he is clearly making the observation that Newton's quadrature techniques expressed in "tables" (or "catalogues") inserted in *De quadratura* are employed in these "more abstruse parts" of Newton's work. The first corollary deals with central force motion in an inverse-cube force field (an important topic for gravitation theory, since tidal forces are inverse-cube), the second with the attraction exerted by a homogeneous ellipsoid of revolution on an external point situated on the axis of revolution (a result central for Newton's determination of the Earth's shape). It would be worth looking at both of them, but in this paper I shall concentrate on the first one.

When we consult Gregory's *Notæ* at the relevant place (written just before April 1688), we find that Gregory had left a blank space: apparently he was not able to comment Corollary 3 to Proposition 41. We also find a pasted piece of paper that must have been added later (see fig. 1). Indeed, a manuscript page in Newton's hand dated 8 May 1694 (that is, dated to Gregory's stay in Cambridge) is extant amongst Gregory's papers.[5] It is, so it seems highly probable, this manuscript that Gregory later copied and pasted in his *Notæ*, since the two texts correspond almost word for word.[6] What was Newton teaching his pupil in the rooms of Trinity College?

3 Proposition 41, Book 1, and its Third Corollary

3.1 Proposition 41

We have to step back and very briefly consider Proposition 41, Book 1 (Proposition 41, for short) ([19], 529–31). The statement poses a problem:

> Supposing a centripetal force of any kind and granting the quadratures of curvilinear figures, it is required to find the trajectories in which bodies will move and also the times of their motions in the trajectories so found. ([19], 529)

This is a problem (in Newton's times known as the "inverse problem of central forces") that we still teach to our students in the courses on Newtonian mechanics. We ask that this problem be reduced to determine, given initial conditions, the motion of a point mass in a central force field. In this paper we assume that the mass is equal to 1. We ask to reduce this problem to integrations, a choice of polar coordinates being the best for reasons of symmetry.

In a way Newton does the very same thing, to be sure in his peculiar geometrical language. For the sake of brevity, I will not provide an analysis of

[5] This annotation was found by W. H. Turnbull who published it in the third volume of Newton's correspondence ([17], 348–54). It was later included by D. T. Whiteside in ([18], 6, 437–8). Bruce Brackenridge provides a thorough analysis of Newton's annotation for Gregory in [4]. Herman Erlichson discusses Corollary 3 in [9]. Whiteside's commentary in ([18], 6, 352–6) is important. I am deeply indebted to the above works. Another similar, but just sketched, annotation concerning Corollary 3 is at the top of folio Add. 3960.13: 223r (Cambridge University Library) and is edited in [18], VI, 435–7. This annotation might have been written just before the one found in the Library of the Royal Society that I discuss in this paper.

[6] Indeed, in the July 1694 memorandum Gregory wrote: "Most of what in early May of 1694 he [Newton] had corrected or altered in his own copy has been corrected or altered at the respective places in my own copy or in my notes." English translation by Turnbull in [17], 386. In the pasted sheet on p. 28 of the *Notæ* , Gregory not only copied Newton's annotation, but added a few interesting remarks on a method of quadrature he claimed to have found independently from Newton and that was printed in the second volume of Wallis's *Opera* in 1693. See [12], 183; [24], 378.

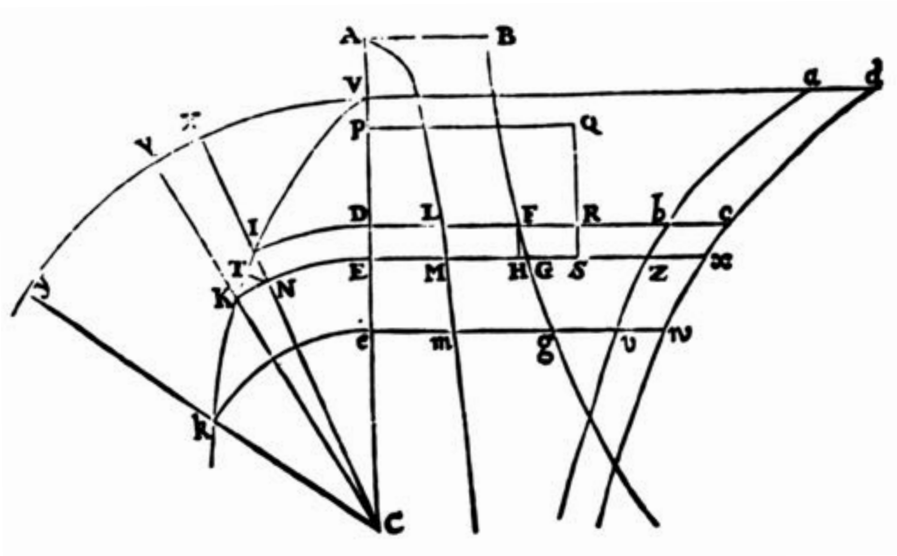

Fig. 2: Diagram for Proposition 41, Book 1, of the *Principia* (1687). Source: Newton, *Philosophiæ Naturalis Principia Mathematica* (1687), p. 128. The diagram was emended in the third (1726) edition.

Newton's demonstration of Proposition 41. I will just consider the result on which its Corollary 3 is based.[7]

Let us have a look at the diagram accompanying this proposition (see fig. 2). Newton considers a "body" setting out from V with a given velocity under the action of a centripetal force. The force's center is C. The trajectory, to be found, is $VIKk$.[8]

As coordinates of the body's position at I Newton uses the distance $CI = A$ and the area of the circle sector VCX. This is very nearly what we would do nowadays by using polar coordinates.

The curve BFG represents the force's intensity in function of distance from the force's center, or in Newton's words "[the ordinate] DF is proportional to the centripetal force in that place $[CI = CD]$ tending towards the centre C" (PM, 525). Thus, the area of the surface subtended to this curve measures what we would call mechanical work.

It should be noted that Newton avails himself of two properties of central force motion that he proved in previous pages. In modern terms, we would understand these two properties as the law of conservation of angular momentum and the law of conservation of mechanical energy. In Newton's terms, the first

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[7] For a careful, step by step, analysis of Proposition 41, I strongly recommend I. B. Cohen's commentary in ([19], 334–45) and [16].

[8] In the third edition (1726) of the *Principia* the two curves to the right meet at point a and one of the two curves has an asymptote. This is not relevant for the present paper. I prefer to consider the text that was in front of Gregory's eyes and that was discussed with Newton in 1694.

property is that the area law holds if and only if the force is central (Propositions 1 and 2 in ([19], 444–7)). Thus the motion is planar and the areal velocity is constant. This first property allows Newton to geometrically represent time as the area of the surface swept by the radius vector. For example, the area VCI is proportional to the time taken by the body to traverse the arc VI. Newton denotes the constant areal velocity with $Q/2$. The second property (that a modern reader would immediately recognize as the law of conservation of mechanical energy) is proved in Propositions 39 and 40 ([19], 524–9).

Starting from these two properties, Newton obtains two curves abz and dcx (defined below in (1) and (2)) that must be "squared" in order to determine the dependence of time (measured by the area VCI) from distance, and the dependence of polar angle (measured by the area of the circular sector VCX) from distance, respectively. "Squaring" a curvilinear surface meant calculating its area.

The ordinate of the curve abz is

$$Db = \frac{Q}{2\sqrt{(ABFD - Q^2/A^2)}}. \tag{1}$$

The ordinate of the curve dcx is

$$Dc = \frac{Q \times CX^2}{2A^2\sqrt{(ABFD - Q^2/A^2)}}. \tag{2}$$

If one squares abz, and thus calculates the functional dependance of the area $VabD$ from the distance $CI = CD$, the functional dependence of time from distance is given.

If one squares dcx, and thus calculates the functional dependance of the area $VdcD$ from the distance $CI = CD$, the functional dependence of polar angle from distance is given.

3.2 A presentist translation of Proposition 41

With some hesitation about the risk always inherent in such translations, when the purpose is historically sensitive interpretation, I dare to offer the reader a translation into modern notation.

Let $A = r$ be the distance from the force centre. Let $Q = h$ be what we call the magnitude of specific angular momentum. Let $ABFD$ be $v_0^2 + 2\int_r^{r_0} F(\rho)d\rho$, where v_0 is the initial speed at $CV = CX = r_0$.[9]

Newton's two quadratures are "equivalent" (in a historically problematic way) to the familiar integrals for the trajectory:

[9] Newton imagines that a second body falls from a rest position A so that during the vertical fall AV it acquires the initial given speed with which the first body is projected at V. Allowing ourselves the use of modern symbolism, Newton states that $v_r^2 = 2ABFD = 2\int_{CD=r}^{CA} Fd\rho = 2\int_r^{CV=r_0} Fd\rho + 2\int_{r_0}^{CA} Fd\rho = 2\int_r^{r_0} Fd\rho + v_0^2$.

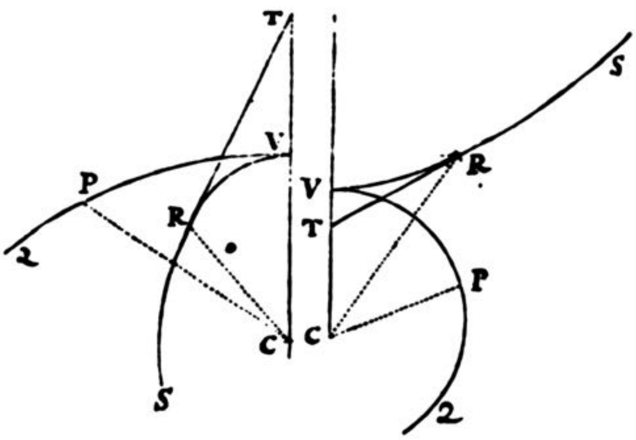


Fig. 3: Diagram for Corollary 3, Proposition 41, Book 1. Source: Newton, *Philosophiæ Naturalis Principia Mathematica* (1687), p. 130. The diagram was modified in the third (1726) edition.

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{\left(v_0^2 + 2\int F \mathrm{d}\rho - h^2/r^2\right)}}, \tag{3}$$

and

$$\mathrm{d}\theta = \frac{\pm h \mathrm{d}r}{r^2\sqrt{\left(v_0^2 + 2\int F \mathrm{d}\rho - h^2/r^2\right)}}. \tag{4}$$

In order to see the equivalence more clearly, I note that Newton chooses the area VCI as a measure of time, and the area VCX as a measure of the polar angle.

Thus: $VCI = (h/2)t = (Q/2)\int dt$ and $VCX = (r_0^2/2)\theta = (CX^2/2)\int d\theta$.

For the historian there are many important differences between Newton's concepts and ours. Yet, we recognize his formulation for the mathematical treatment of central force motion as very similar to ours — in a way, the two are "equivalent." We will discuss these issues of translation in the concluding section.

3.3 Corollary 3: a brief outline

But let us see what happens in Corollary 3 to Proposition 41 (Corollary 3 for short). Newton applies what he has obtained in Proposition 41 to the case in which the force varies with the inverse of the cube of the distance.

One should note that in this case Newton forces a restriction on initial conditions so that the initial velocity at V is orthogonal to CV (thus, for example, the logarithmic spiral, which is a possible trajectory for inverse-cube central force motion, will not be included in the solution of Corollary 3).

The problem, because of Proposition 41, is reduced to the quadrature of curves abz and dcx, but — and this is most likely what must have been perplexing for Gregory — Newton provides a geometrical construction of the trajectory without giving any detail about how he performed the quadrature. He unhelpfully writes:

> all this follows from the foregoing proposition by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity. ([19], 532)

Newton's construction is as follows (see figure 3). One begins by drawing two conics VRS, an ellipse (left) and a hyperbola (right), with centre C and vertex V. A point R slides along the conic starting from V, and we draw the tangent at R meeting the axis CV at T. We draw the line CR. The trajectory traced by the body is constructed by drawing a segment CP, whose length is equal to CT, and making an angle VCP with the axis CV proportional to the area of the conic sector VCR. The point P traces the sought trajectory ([19], 531–2).[10]

We should appreciate the visual beauty of this construction. One should note that as the point R slides along the conics starting from V, the areas of the conic sectors VCR monotonically increase. In the case of the hyperbola, the tangent at R tends to an asymptote passing through the center C, while in the case of the ellipse the tangent tends to a line parallel to the axis CV as the angle VCR approaches $\pi/2$. This allows us to visualize how in the case of the trajectory constructed via the auxiliary hyperbola (right), the point P spirals towards the center C, since $CT = CP$ tends to zero as the polar angle VCP tends to infinity. On the other hand, in the case of the trajectory constructed via the ellipse (left), $CT = CP$ tends to infinite, as as the polar angle VCP tends to a finite value, and thus the body ascends in a spiral-like trajectory escaping to infinity along an asymptote.

4 Motion in an inverse-cube force field: a modern interlude

A short interlude on the modern treatment of inverse-cube trajectories is now in order. This is done, because after all we look at the past from a present viewpoint, and we might need to be a little bit refreshed on this topic (often an exercise for physics undergraduates).[11]

It is required the determination of the motion (given initial position and velocity) of a point mass ($m = 1$) accelerated by a centripetal and isotropic force

$$\mathbf{F} = -\frac{dU}{dr}\frac{\mathbf{r}}{r}. \tag{5}$$

[10] In the modified diagram printed in the third edition (1726) of the *Principia*, the trajectory associated to the auxiliary hyperbola is more clearly drawn as spiraling towards C.

[11] The simplest approach is via the so-called Binet formula and by employing the exponential form of the trigonometric and hyperbolic functions. See [8], 237–43. The approach I choose here is interesting because of its similarities with Newton's.

Since the mechanical energy E and the angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ are conserved

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} + U(r). \tag{6}$$

We can solve for $\dot{r}$:

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{2(E-U) - h^2/r^2}. \tag{7}$$

Separating the variables, we obtain the equivalent of equation (3):

$$dt = \frac{dr}{\sqrt{2(E-U) - h^2/r^2}}.^{12} \tag{8}$$

Because of conservation of angular momentum,

$$d\theta = (h/r^2)dt, \tag{9}$$

and we obtain the equivalent of equation (4):

$$d\theta = \frac{\pm h dr}{r^2\sqrt{2(E-U) - h^2/r^2}}. \tag{10}$$

The integration of (8) is particularly simple for an inverse-cube force. We set

$$U = -\alpha/r^2, \tag{11}$$

for $\alpha > 0$. Then

$$t = \int \frac{r dr}{\sqrt{2Er^2 + 2\alpha - h^2}} = \frac{1}{2E}\sqrt{2Er^2 + 2\alpha - h^2} + C. \tag{12}$$

The time is thus given by a finite algebraic equation.[13]

The integration of (10) is more complicated, and this might well have been the origin of Gregory's perplexity. For an inverse-cube force, equation (10) is

$$d\theta = \frac{\pm h dr}{r\sqrt{2Er^2 + 2\alpha - h^2}}. \tag{13}$$

The trick consists in changing variable

$$w = 1/r, \tag{14}$$

so that one reduces the problem to the following integral:

$$\theta = \int \frac{\mp h dw}{\sqrt{2E + (2\alpha - h^2)w^2}}. \tag{15}$$

[12] A positive root is chosen. The negative root corresponds to a time reversal $t \to -t$: if $r(t)$ is a solution also $r(-t)$ is a solution (depending on initial conditions).

[13] This is in contrast with the solution for inverse-square forces, a case in which the integration of (8) can only be obtained by means of transcendental functions, as underlined in [4], 327.

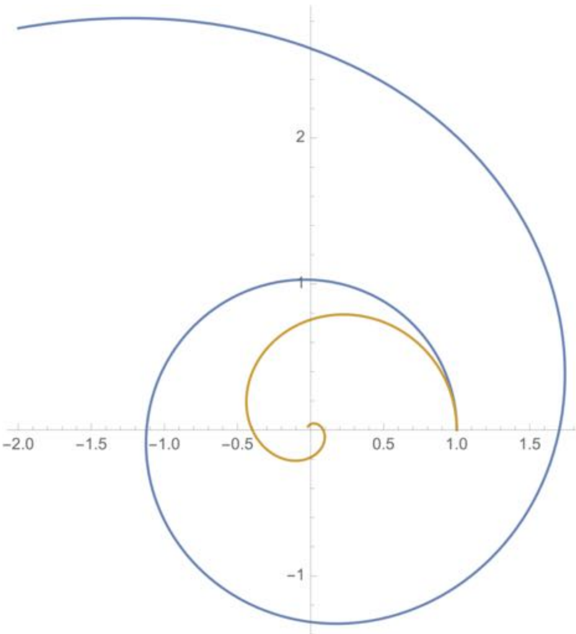


Fig. 4: The trajectories for an attractive inverse-cube force (Cases 1 and 2) constructed in Corollary 3 as VPQ (see fig. 3) for an initial position $\mathbf{r_i}$ equal to $(1,0)$ and an initial velocity $\mathbf{v}_i$ orthogonal to $\mathbf{r}_i$. In Case 1, the point mass traces an ascending spiral-like trajectory under the action of an attractive inverse-cube force (NB: since $\alpha > 0$, $\sqrt{h^2 - 2\alpha}/h < 1$). The point mass escapes to infinity, since for $\theta \to \theta_0 + \pi/2(h/\sqrt{h^2 - 2\alpha})$, r tends to infinite. When $\sqrt{h^2 - 2\alpha}/h > 1$, the force is repulsive ($\alpha < 0$), and the trajectory represents a scattering state with pericenter at $\mathbf{r_i}$. In Case 2 the point mass describes a spiral that tends to the origin as $\theta \to \infty$. The fall to the origin occurs in a finite time calculated by eq. (12). The above plots correspond to $r = (\cos 0.15\,\theta)^{-1}$ and $r = (\cosh 0.5\,\theta)^{-1}$.

Taking into consideration that

$$\int \frac{-dw}{\sqrt{b^2 - w^2}} = \arccos\frac{w}{b} \ , \int \frac{dw}{\sqrt{w^2 - b^2}} = \mathrm{arccosh}\frac{w}{b} \ , \int \frac{dw}{\sqrt{w^2 + b^2}} = \mathrm{arcsinh}\frac{w}{b}, \tag{16}$$

we distinguish, with a suitable choice of the coordinates, θ_0 and r_0, at time $t = 0$, the following five cases.[14] The form of the solution depends on the sign of $2\alpha - h^2$ and E.

[14] We note that in Case 1 and Case 2, $\theta = \theta_0$ corresponds to $r = r_0 = \sqrt{(h^2 - 2\alpha)/2E}$ and $r = r_0 = \sqrt{(2\alpha - h^2)/2|E|}$, respectively. In Case 3, $\theta = \theta_0$ corresponds to the direction of an asymptote. In Case 4 and Case 5, $\theta = 0$ corresponds to $r = r_0$. Asymptotes for Cases 1 and 5 are easily calculated. For example, in Case 5, an asymptote occurs for $\theta = -h/(r_0\sqrt{2E})$.

1. In the first case, $2\alpha - h^2 < 0$ and $E > 0$:

$$\frac{1}{r} = w = \sqrt{\frac{2E}{h^2 - 2\alpha}} \cos\left[\frac{\sqrt{h^2 - 2\alpha}}{h}(\theta - \theta_0)\right].^{15} \tag{17}$$

2. In the second case, $2\alpha - h^2 > 0$ and $E < 0$:

$$\frac{1}{r} = w = \sqrt{\frac{2|E|}{2\alpha - h^2}} \cosh\left[\frac{\sqrt{2\alpha - h^2}}{h}(\theta - \theta_0)\right]. \tag{18}$$

3. In the third case, $2\alpha - h^2 > 0$ and $E > 0$:

$$\frac{1}{r} = w = \sqrt{\frac{2E}{2\alpha - h^2}} \sinh\left[\frac{\sqrt{2\alpha - h^2}}{h}(\theta - \theta_0)\right]. \tag{19}$$

4. In the fourth case, $2\alpha - h^2 > 0$ and $E = 0$:

$$r = r_0 \exp \pm \left(\frac{\sqrt{2\alpha - h^2}}{h}\,\theta\right). \tag{20}$$

5. In the fifth case, $2\alpha - h^2 = 0$ and $E > 0$:

$$\frac{1}{r} = \frac{\sqrt{2E}}{h}\,\theta + \frac{1}{r_0}.^{16} \tag{21}$$

The first three cases are sometimes called "Cotesian" spirals. Case 4 is a logarithmic spiral. Case 5 is a "hyperbolic" spiral. Only the first two cases occur (other than circular trajectories) when the initial velocity is orthogonal to the radius vector, as required in Corollary 3.[17] Cases 1 and 2 are discussed in the caption to Fig. 4. Newton identified Cases 1 and 2 in Corollary 3, and Case 4 in Proposition 9, Book 1. Johann Bernoulli identified Case 5 in [2], 532–3. Roger Cotes classified all five trajectories in [6].

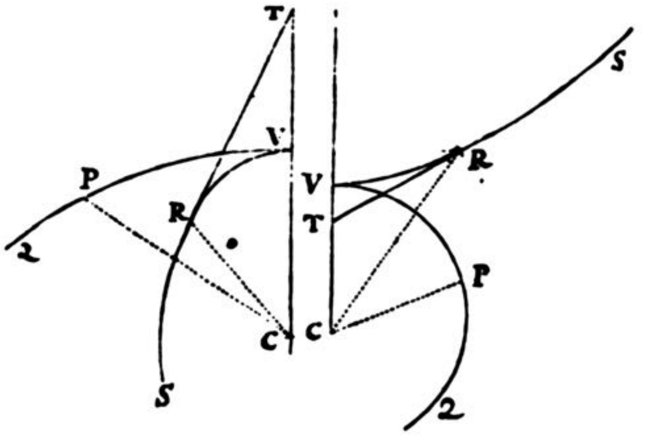
5 Newton's geometric construction of inverse-cube trajectories

It is time to go back to the original words of the master. So we can imagine being, like Gregory, in Newton's chambers with a page of the *Principia* in front of us, but frustratingly with our *Notæ* with a lacuna to fill.

[15] A solution using the sine function is also possible, but this does not generate a new family of trajectories because the sine function can be converted to a cosine by a shift of the polar coordinate. This, of course, is not true for the hyperbolic functions occurring in Cases 2 and 3 below.

[16] In Cases 3, 4, and 5, the particle either spirals into the centre of force or out to infinity, depending on the sign of $\dot{r_0}$.

[17] We note that circular trajectories are possible when $2\alpha = h^2$ and $E = 0$, but they are unstable. The reader may consider the effective potential energy for inverse cube forces, $U_{eff} = h^2/(2r^2) + U(r) = h^2/(2r^2) - \alpha/r^2$. When $2\alpha = h^2$, the effective potential energy U_{eff} is flat and any r is a possible radius for an unstable circular trajectory ($\dot{r} = 0$) with velocity $|v| = \sqrt{2\alpha}/r$.



5.1 Newton's construction

We recall that Newton's construction as printed in the *Principia* (see figure 3 reproduced above) consists of the following.

1. Draw two conics VRS, an ellipse (left) and an hyperbola (right), with centre C and vertex V.
2. A point R slides along the conic starting from V: draw the tangent at R meeting the axis CV at T.
3. Draw the line CR.
4. Draw a segment CP whose length is equal to CT and making an angle VCP with the axis CV proportional to the area of the conic sector VCR.
5. The point P traces the sought trajectory.

Or in Newton's august words:

> If with center C and principal vertex V, any conic VRS is described, and from any point R of it the tangent RT is drawn so as to meet the axis CV, indefinitely produced, at point T; and joining CR there is drawn the straight line CP, which is equal to the abscissa CT and makes an angle VCP proportional to the sector VCR; then, if a centripetal force inversely proportional to the cube of the distance of places from the center tends towards the center C, and the body leaves the place V with the proper velocity along a line perpendicular to the straight line CV, the body will move forward in the trajectory VPQ, which point P continually traces out; and therefore, if the conic VRS is a hyperbola, the body will descend to the center. But if the conic is an ellipse, the body will ascend continually, and will go off to infinity.
>
> And, conversely, if the body leaves the place V with any velocity and, depending on whether the body has begun either to descend obliquely to the center or to ascend obliquely from it, the figure VRS is either a hyperbola or an ellipse, the trajectory can be found by increasing or diminishing the angle VCP in some given ratio. But also if the centripetal force is changed into a centrifugal force, the body will ascend obliquely in the trajectory VPQ, which is found by taking the angle VCP proportional to the elliptic sector VCR, and by taking the length CP equal to the

length CT, as above. All this follows from the foregoing proposition [41], by means of the quadrature of a certain curve, the finding of which, as being easy enough, I omit for the sake of brevity. ([19], 531–2)

5.2 An algebraic translation

But we are still fluctuating between past and present, and we might be tempted, before immersing ourselves in the mathematical culture of the late seventeenth century, to impart (for the sake of brevity, so to say) to Gregory a crash course in modern calculus and trigonometry to notice that the above construction delivers the solutions we named Cases 1 and 2 above.

Let us use as Cartesian coordinates of the point $R(x, v)$ the abscissa x and the ordinate v (the origin is in the force centre C and the x-axis is vertical).

As auxiliary conics I introduce:

$$x^2 + v^2 = 1, \tag{22}$$

and

$$x^2 - v^2 = 1.^{18} \tag{23}$$

By construction, the polar coordinates of the body's position $P(r, \theta)$ are $r = CP = CT$, and $\theta = VCP = (2/k)VCR$ (for some constant k).[19]

One readily obtains

$$r = CT = x - v\frac{dx}{dv} = \frac{1}{x}.^{20} \tag{24}$$

Equation (24) corresponds to the solutions we have classed as Case 1 and Case 2 in section 4.

Indeed, for the circle $x = \cos(2VCR) = \cos k\theta$ and for the hyperbola $x = \cosh(2VCR) = \cosh k\theta$. Thus, for $k = 0.15$ and $k = 0.5$ we obtain the trajectories illustrated in fig. 4.

5.3 An omitted method

If we are to believe Newton, he achieved this construction thanks to "a method for squaring curvilinear figures" illustrated in the "foregoing Proposition [41]." By reading Corollary 3, we learn that it is this omitted method that allows him to square the curves dcx and abz whose abscissa is the distance CD from the force centre and whose ordinates are Dc and Db (see figure 2, and equations (1) and (2)). We also learn from figure 3 that Newton is making use of auxiliary conic sections.

[18] The choice of the auxiliary conics is crucial. Conics with different parameters give rise to different trajectories.

[19] The trajectory is expressed in polar coordinates. So θ is an angle. Instead, we take VCR as the area of the circle and hyperbolic sectors, respectively.

[20] This must have been evident for Newton from Apollonius, *Conics*, I.37. [1], 65–7.

In the next section, we look at certain quadrature techniques that Newton developed in the mid 1660s, systematized in two "catalogues" dating from 1670, and then printed as "tables" in *De quadratura*. In a few words, we have to check whether these "more abstruse parts" in the *Principia*, as Gregory recorded in his July 1694 *memorandum*, depend upon those tables that aroused his admiration.[21]

6 Some of Newton's quadrature techniques

6.1 A digression into Newton's quadratures of conic sections

Conic sections, as we shall see, played a prominent role in Newton's "quadrature techniques." The object of these techniques was to "square curvilinear figures," that is, to calculate the area of the surface bounded by plane curves.

Since his youthful studies in the *anni mirabiles* (1664–6), Newton knew how to square conic sections via the binomial theorem and via an application of what nowadays we would call "term-wise integration."

6.1.1 Logarithms

So, for example, for the hyperbola, Newton wrote

$$v = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots , \tag{25}$$

a result that he considered valid when x is small. Newton knew that the area under the hyperbola and over the interval $[0, x]$ for $x > 0$ (and the negative of this area when $-1 < x < 0$) is a measure for $\ln(1 + x)$. Thus, by (to speak anachronistically) "term-wise integration," Newton could express $\ln(1 + x)$ as a power series:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots . \tag{26}$$

He did not, however, write that the above series is equal to "$\ln(1 + x)$." Until the mid eighteenth century, European mathematicians rarely used a notation for transcendental functions. They represented them via geometrical constructions, such as the hyperbolic surface in the case of the logarithm. Leibniz and Johann Bernoullli were pioneers in using symbols for the logarithm and the exponential function, but with the trigonometric or hyperbolic functions the general policy, before Euler, was to use geometric constructions [15].

[21] This is an innovative element of my paper, since below I will detail the relationships between Corollary 3 and Newton's quadrature techniques of the *De methodis* and *De quadratura*. These techniques date back to work Newton carried out in the 1660s.

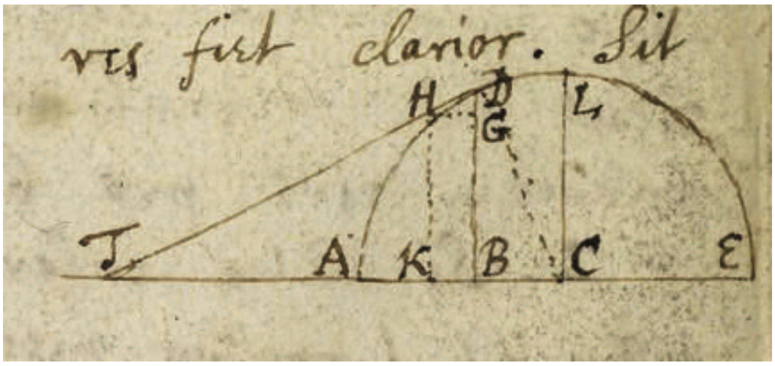


Fig. 5: Quadrature of the circle from Newton's *De analysi* (1669). Source: MS 81.2: 6v (Royal Society Library). Edited in ([18], 2, 232–33. ©The Royal Society

6.1.2 Transcendental functions and geometrical constructions

We begin to understand why Newton expressed the solutions for Corollary 3 as a geometric construction. These solutions imply a functional dependence of the radius CP and the conic sector VCR expressed by transcendental relationships (the cosine and the hyperbolic cosine), and these were constructed geometrically until at least Euler's times

The connection between quadratures and "transcendental functions"[22] was profound. Seventeenth-century mathematicians realized that while the slope of an algebraic curve is algebraic, most often the curvilinear area bounded by an algebraic curve (or the arc-length) is not: it had to be expressed via an infinite series, as we have seen above.

6.1.3 The arcsin

Let us consider another example, this time a rectification of an algebraic curve. Given a unit circle with equation

$$x^2 + v^2 = 1, \tag{27}$$

we have to calculate the arc-length $s = LD$ (see fig. 5). Newton envisaged the circumference $ADLE$ as being generated by the continuous "flow" of a point from E to A. The infinitesimal arc DH generated in a "moment" of time o, he called the "moment of the arc," and he denoted it as $\dot{s}o$.[23] By geometrical reasoning, Newton knew that the moment of the arc DH is to the moment of the abscissa GH, as the radius $DC = 1$ is to the ordinate $DB = v = \sqrt{1 - x^2}$.

[22] This is Leibniz's term, as Newton would use the Cartesian terminology "mechanical curves," and this (indeed) is a terminological distinction of great significance for the historian.

[23] Actually, Newton introduced the dot notation in the 1690s.

In symbols:

$$\frac{\dot{s}}{\dot{x}} = \frac{1}{\sqrt{1-x^2}}. \tag{28}$$

Referring to what nowadays we would call the "fundamental theorem of the calculus," Newton now stated that the arc s is measured by the area of the surface bounded by the curve $v^{-1} = 1/\sqrt{1-x^2}$. This area (we would write $\int dx/\sqrt{1-x^2}$) could be calculated via the binomial theorem and term-wise integration as:

$$x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + O(x^9). \tag{29}$$

We would say that this series is equal to "arcsin(x)," but, as we know, this is not what Newton, Leibniz, or Johann Bernoulli would do. The mathematicians active in late seventeenth-century Europe would say that the above series "approximates the arc whose sine is x." What came first was not the formula, but a geometrical functional relationship between geometrical magnitudes that was given by a construction, namely: by drawing a circle, by drawing a point on its circumference, and by tracing the abscissa x and the arc-length s. Newton noticed that as x flows (what he called the "correlate quantity"), so does s (the "relate quantity") in a functionally related way. Once you have this geometrical relationship between magnitudes embedded in the given figure (a unit circle), you can ask yourself how to calculate the arc-length $s = LD$ (or equivalently the area of the circle sector $s/2 = LCD$) given the sine $x = CB$. Since the functional relationship between the arc-length and the sine is not algebraic, Newton made recourse to infinite series.

6.1.4 The arccosh

The quadrature of the circle that we have just reviewed is of great importance for the treatment of inverse-cube orbits, as is apparent if we consider the integrals (16). Further, Newton could extend the quadrature of the circle to the unit hyperbola

$$x^2 - v^2 = 1. \tag{30}$$

If we set the hyperbolic sector VCR (figure 3) as equal to $s/2$, a result analogous to (28) follows:

$$\frac{\dot{s}}{\dot{x}} = \frac{1}{\sqrt{x^2-1}}. \tag{31}$$

From this we would derive in modern notation $\int dx/\sqrt{x^2-1} = \text{arccosh}(x)$. The integrals (16) follow from (28) and (31) by a simple substitution of variables.[24] To repeat: Newton did not use an integral sign or a notation for hyperbolic functions. For him formula (31) expressed a geometrical proportionality between the infinitesimal increments of the sector VCR and of the abscissa x. This proportionality allowed him to express the relationship between the

[24] In modern notation, we provide the following example: $\int dx/\sqrt{b^2-x^2} = (1/b)\int dx/\sqrt{1-(x/b)^2} = \int d\lambda/\sqrt{1-\lambda^2} = \arcsin\lambda$, for $\lambda = x/b$.

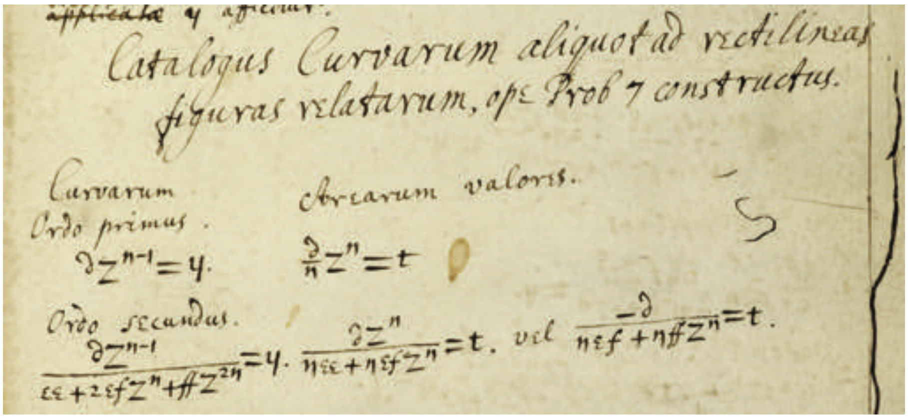


Fig. 6: The incipit of the first catalogue. Source: Add. 3960.14: 77. Reproduced by kind permission of the the Syndics of the Cambridge University Library.

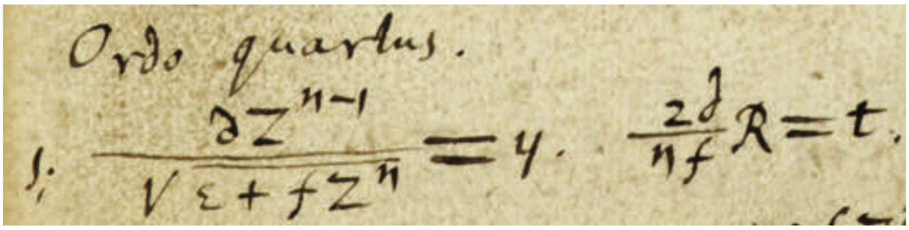


Fig. 7: First case of *ordo quartus* in the first catalogue. Source: Add. 3960.14: 77. Reproduced by kind permission of the Syndics of the Cambridge University Library.

abscissa (what we call the hyperbolic cosine) of point R and the hyperbolic sector VCR as an infinite series.

6.1.5 A question

Can we provide more evidence that Newton made a connection between his method of fluxions and an important proposition of the *Principia* such as Corollary 3? Indeed we can, by looking attentively to a table of quadratures ("integrals," in Leibnzian terms) that he wrote in 1670 and later included in the draft version of *De quadratura* that Gregory inspected in 1694.

6.2 Newton's quadratures "by means of finite equations"

The so-called *De methodis serierum et fluxionum*, composed in 1670–1 ([18], 3, 32–328), contains two "catalogues of curves" in which one finds tabulated the equations of several curves, divided into different "orders," and the "values of their areas." These catalogues were reprinted, with some variants, in *De* *quadratura curvarum*,which Newton published as an appendix to the *Opticks* in 1704.[25]

6.2.1 The fundamental theorem of calculus

The first catalogue (see fig. 6) is based on Newton's understanding of what we call the fundamental theorem of calculus ([18], 3, 236–41). In the mid 1660s Newton, most probably inspired by Isaac Barrow, proved that, given a plane curve, the fluxion of the area bounded by this curve, by the abscissa, and by the ordinate, is to the fluxion of the abscissa as the ordinate is to 1. So, for example, with reference to Proposition 41, Newton would say that the fluxion of the area $VabD$ is to the fluxion of the abscissa CD as the ordinate Db is to 1 (see fig. 2). This insight immediately set him on the project of tabulating curves and their areas.

The first catalogue of the *De methodis* summarizes some of Newton's results on quadratures (see fig. 6). According to Newton's conventions, δ, e, f, g are constants, η can be a "positive or negative integer or fraction." The curves to be squared have abscissa z and ordinate y, while the areas of the surfaces they subtend are denoted by t or τ. Most of the equations of the curves in the first catalogue involve radicals of the form $R = \sqrt{e + fz^\eta}$ or $R = \sqrt{e + fz^\eta + gz^{2\eta}}$.

6.2.2 A simple example

Let us consider the first tabulated curve (see fig. 6). Since we know that the fluxion of $\tau = (\delta/\eta)z^\eta$ is $\dot{\tau} = \delta z^{\eta-1}$ (assuming $\dot{z} = 1$), Newton concluded that the "value of the area" of the curve $y = \delta z^{\eta-1}$ is $\tau = (\delta/\eta)z^\eta$. A simple consequence of the fundamental theorem.

6.2.3 A quadrature for Corollary 3

For our purposes, the relevant quadrature occurs as the first in *ordo quartus* (see fig. 7). For $\eta = 2$ and $\delta = 1$, this translates into the statement that, if a curve has equation

$$y = \frac{z}{\sqrt{fz^2 + e}}, \tag{32}$$

then its area is

$$\tau = \frac{1}{f}\sqrt{fz^2 + e}. \tag{33}$$

This quadrature is what we need in order to square curve abz in Prop. 41 that determines the time in function of the radius for inverse-cube trajectories (set $y\dot{z} = ydz = dt$, $z = r$, $f = 2E$ and $e = 2\alpha - h^2$ and you obtain equation (12)).

[25] I will not discuss the variants between the *Catalogi* divided into *ordines* of the *De methodis* and the *Tabulae* divided into *formae* of *De quadratura*, since they are not relevant for the thesis defended in this paper. According to an expert judge such as Whiteside: "The tables of integrals which Gregory saw were in fact [. . .] those which Newton introduced more than twenty years earlier into his general 1671 tract on fluxions and infinite series [the *De methodis*], rather than their lightly refashioned equivalent which he had much more recently appended to his revised text 'De quadratura curvarum'." [18], 7, 197.

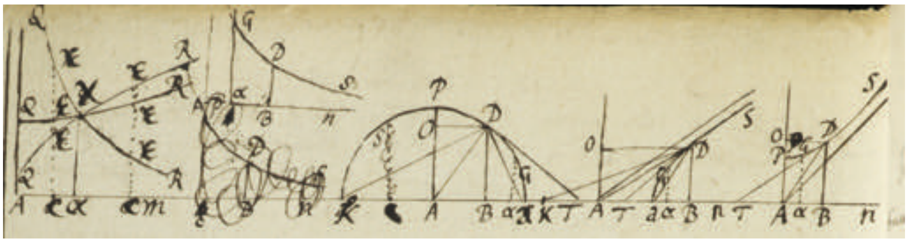


Fig. 8: Auxiliary conics for the second catalogue of curves. Source: Add. 3960.14: 75. Reproduced by kind permission of the Syndics of the Cambridge University Library.

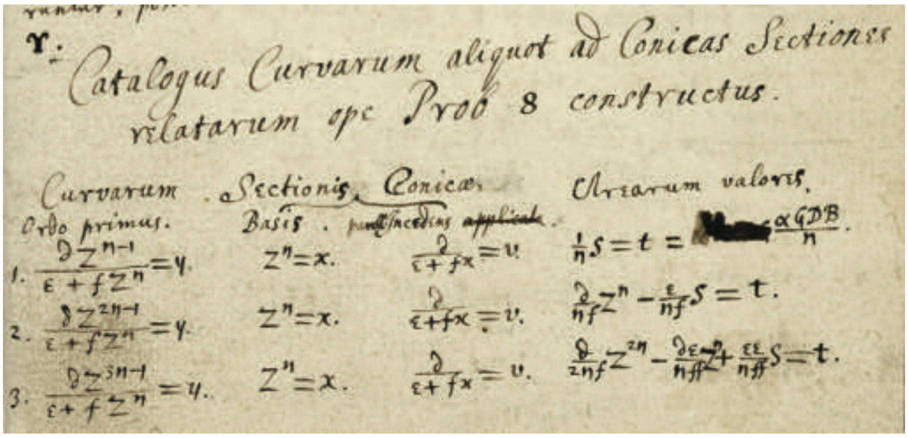


Fig. 9: The incipit of the second catalogue. Source: Add. 3960.14: 79. Reproduced by kind permission of the Syndics of the Cambridge University Library.

6.3 Newton's quadratures "by means of conic sections"

A second catalogue of "curves related to conic sections" occurs in the *De methodis* a few pages later ([18], 3, 241–55). This catalogue too was republished, with some variants, in *De quadratura*, and Gregory must have seen a draft for the latter work. Basically, by appropriate substitutions of variables, Newton reduces the quadrature of a series of curves divided into several "orders" to the quadrature of conic sections.

6.3.1 Auxiliary conics

The conics appear at the top of the table ([18], 3, 242), and we immediately recognize the auxiliary conics that Newton used in Corollary 3 of his *Principia* (see fig. 8). The third figure from left is the auxiliary ellipse, the last figure to the right is the auxiliary hyperbola. The lettering is different, but we easily realize that the relevant conic sectors equivalent to VCR in Corollary 3 are here αGDA (ellipse) and PAD (hyperbola). We are on the right track.

6.3.2 Substitution of variables

We see (fig. 9) that in this second catalogue there are four columns.The first column lists, as in the first catalogue, the equations of the curves to be squared divided into orders. As before, their abscissa is z, ordinate y, and area τ. Then we have a second column where the reader learns how to change variable (today we would write $x = f(z)$). The next column lists the equations of conic sections, with abscissa x, ordinate v, and area s. The sought area τ is expressed in terms of the conic area s in the fourth column (in Leibniz's notation, one would write $\tau = \int ydz$ and $s = \int vdx$).

As we know, Newton could square conic sections by means of infinite series that we would understand as the Taylor series of log, trigonometric and hyperbolic functions. Indeed, the whole gist of *De quadratura* is the calculation of curvilinear areas (the "quadrature of curves") by means of infinite series.[26]

6.3.3 A simple example

This is best explained by an example: the simplest is the first case of the first order see (fig 9). We have to square

$$y = \frac{\delta z^{\eta-1}}{e + fz^{\eta}}. \tag{34}$$

With a change of variable $z^{\eta} = x$, the quadrature of the curve is reduced to the quadrature of a hyperbola $v = \delta/(e + fx)$. Newton would write:

$$y\dot{z} = \frac{\delta}{x(e+fx)} \frac{x\dot{x}}{\eta} = \frac{1}{\eta} \frac{\delta}{e+fx} \dot{x} = \frac{1}{\eta} v\dot{x}, \tag{35}$$

and from this conclude that

$$\tau = \frac{1}{\eta} s = \frac{\alpha GDB}{\eta}.^{27} \tag{36}$$

Newton, of course, knew that s is the logarithm $(\delta/f)\log(fx+e)$, but he would express the area as an infinite series:

$$\tau = \frac{\delta}{\eta f}(1 + \frac{fx}{e} - \frac{f^2x^2}{2e^2} + \frac{f^3x^3}{3e^3} - \ldots), \tag{37}$$

and rather than write a symbol for the logarithm he would represent this transcendental magnitude geometrically in terms of the area subtended under an hyperbola αGDB (see the hyperbola, second from left, in fig. 8).

6.3.4 A quadrature for Corollary 3

For our purposes, the relevant quadrature occurs as the first in *ordo septimus* (see fig. 10).[28] For $\eta = 2$, this translates into the statement that, if a curve

[26] This might be missed by a superficial reading.

[27] In Leibnizian notation: $\tau = \int ydz = (1/\eta) \int vdx = (1/\eta)s$.

[28] In the second *Tabula* of *De quadratura*, this corresponds to the first case of the fourth *Forma*.

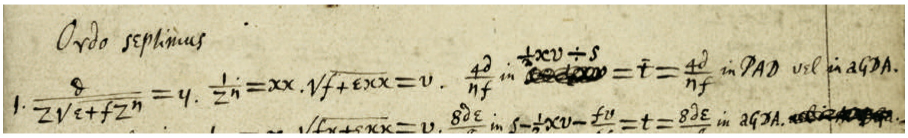


Fig. 10: First case of *ordo septimus* in the second catalogue. Source: Add. 3960.14: 81.Reproduced by kind permission of the Syndics of the Cambridge University Library.

has equation

$$y = \frac{\delta}{z\sqrt{fz^2 + e}}, \tag{38}$$

then its area is

$$\tau = \frac{2\delta}{f} PAD \text{ or } \frac{2\delta}{f} \alpha GDA, \tag{39}$$

where PAD and αGDA, are the conic sectors in fig. 8.

This quadrature is what we need in order to square curve dcx in Prop. 41 that determines the polar angle in function of the radius for inverse-cube trajectories (set $y\dot{z} = ydz = d\theta$, $z = r$, $\delta = h$, $f = 2E$, and $e = 2\alpha - h^2$, and you obtain equation (13)).

Note that the change of variable in the second column is very much the one we employed in the modern interlude in section 4 (equation (14)): Newton sets $x = 1/z$ (equivalent to our $w = 1/r$).

In the third column, we find the equation of the auxiliary conics $v = \sqrt{ex^2 + f}$ ($e < 0$ for the ellipse, $e > 0$ for the hyperbola).

Via the change of variable indicated in second column the required quadrature is reduced to a simpler form as follows:

$$y\dot{z} = \frac{\delta\dot{z}}{z\sqrt{fz^2 + e}} = \frac{-\delta x}{\sqrt{f/x^2 + e}} \frac{\dot{x}}{x^2} = -\frac{\delta\dot{x}}{\sqrt{ex^2 + f}}. \tag{40}$$

In modern notation, this corresponds to reducing the sought integral, via substitution of variable, to an integral that we have already encountered in section 4 in our modern solution (see equation (15):

$$\delta \int \frac{dz}{z\sqrt{fz^2 + e}} = -\delta \int \frac{dx}{\sqrt{ex^2 + f}}. \tag{41}$$

We integrated this in terms of the trigonometric and hyperbolic functions (see equations (16)). Thus, for example, we would write (for $e > 0$):

$$\delta \int \frac{dx}{\sqrt{ex^2 + f}} = \frac{\delta}{\sqrt{e}} \text{arccosh}\left(\sqrt{\frac{e}{f}} x\right) + C. \tag{42}$$

But Newton, his contemporaries and immediate successors, as we know, did not express this quadrature in symbolic terms. Rather, they made recourse to a construction in terms of the auxiliary conics.

The quadrature is provided in the fourth column as:

$$\tau = \frac{2\delta}{f}\left|\frac{1}{2}xv - s\right| = \frac{2\delta}{f}PAD, \tag{43}$$

where s is the area of the surface subtended under the conic whose abscissa is x and ordinate is v.[29] Indeed, the absolute value of the difference between the area of the triangle with sides x and v and the area of the region subtended under the conic is equal to the area of the conic sector PAD, in terms of which Newton performs the required quadrature. Or, in modern symbols:

$$\tau = \frac{2\delta}{f}\left|\frac{1}{2}x\sqrt{ex^2+f} - \int \sqrt{ex^2+f}dx\right|.^{30} \tag{44}$$

We note that for the conic $v^2 = ex^2 + f$ (when again we take the hyperbola as our example, $e > 0$) the sector PAD has area equal to

$$PAD = \frac{1}{2}\sqrt{f}\sqrt{\frac{f}{e}}\text{arccosh}\left(\sqrt{\frac{e}{f}}x\right), \tag{45}$$

which makes Newton's geometric quadrature (43) equivalent to the modern analytic solution (42).

We find the quadrature as tabulated in *ordo septimus* of the second catalogue, the very same curve, the very same substitution of variable, and the solution expressed in terms of the very same auxiliary conics in Newton's annotation for Gregory, written on May 8, 1694, to which we now turn at last. Newton was using his quadrature techniques in order to explain to his new friend a difficult part of the *Principia*.

7 Newton's annotation for Gregory

I provide an English translation (slightly adapted from Newton's *Correspondence*) and a commentary of Newton's annotation for Gregory dated 8 May 1694 ([17], 351). It should be recalled that this is a quick note that was taken at the end of five very busy days. It was not meant for publication, but just

[29] Newton did not use the modern symbol for the absolute value $|\frac{1}{2}xv - s|$ but rather one that he found in Barrow's works. Newton wrote ÷ for "the Difference of two Quantities, when it is uncertain whether the latter should be subtracted from the former, or the former from the latter."

[30] To recapitulate. The first case of the seventh order translated into Leibnizian notation is as follows. For $\eta = 2$, Newton evaluates the integral $\int \delta/(z\sqrt{e+fz^2})\,dz$ (δ, e, f constants). By substitution of variables $z = x^{-1}$, he reduces it to the conic area $s = \int v dx = \int \sqrt{f+ex^2}\,dx$. Namely,

$$\tau = \int \frac{\delta}{z\sqrt{fz^2+e}}\,dz = \frac{2\delta}{f}\left|\frac{1}{2}xv - s\right| + C = \frac{2\delta}{f}\left|\frac{1}{2}x\sqrt{f+ex^2} - \int\sqrt{f+ex^2}\,dx\right| + C.$$

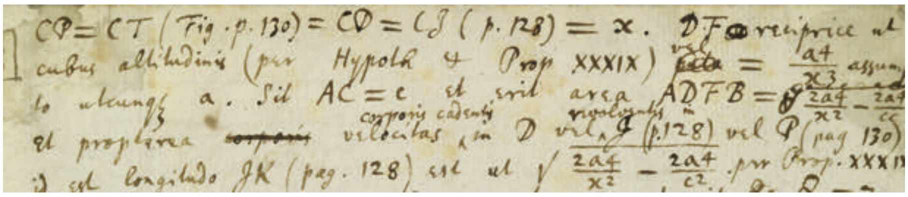


Fig. 11: The incipit of Newton's annotations for Gregory on Corollary 3. Source: Gregory MS: 163 (Royal Society Library). ©The Royal Society

to give a competent colleague a sufficiently detailed hint on how to proceed in order to fill the gap in the *Notæ*. I take the liberty of altering Newton's words in a few inconsequential details. Namely, I use the symbol r, instead of x, and ζ, instead of z. This is done to spare my reader a headache, since the variables x and z have already appeared in this paper with different meanings.

7.1 Translation of Proposition 41 into algebra

Newton opens the annotation as follows (see fig. 11):

> $CP = CT$ (Fig. p. 130) $= CD = CI$ (p. 128) $= r$. DF is reciprocally as the cube of the altitude (by Hypothesis and Prop. 34) or $= a^4/r^3$ where any arbitrary a is assumed. Let $AC = c$; then the area $ADFB = 2a^4/r^2 - 2a^4/cc$. And on that account the velocity of the falling body at D, or the revolving one at I (p. 128) or P (p. 130), that is the length IK (p. 128), is as $\sqrt{2a^4/r^2 - 2a^4/c^2}$ by Prop. 39.

Here Newton is referring to two figures: one accompanying Proposition 41 on page 128 (see fig. 2) and the other accompanying Corollary 3 on page 130 (see fig. 3) of his *Principia*. He is thus associating algebraic symbols with the lengths of the segments occurring in these two figures. The "altitude," which is the distance from the force centre, is denoted by x, but in the translation above we change it to r. The ordinate DF of the curve BFG, which in fig. 2 represents the force's intensity, is set equal to a^4/r^3 (it is thus an inverse-cube force). By the fundamental Propositions 39 and 40, Book 1, Newton derives (by a very elementary quadrature) that the speed is proportional to $\sqrt{2a^4/r^2 - 2a^4/c^2}$ (where $c = CA$ is the position from which the body falls from rest in order to reach the initial speed at CV). That is, speed is proportional to the area $ABFD$ (actually the factor 2 should appear in the denominator!).

We note the following. Newton is here "preparing" his Proposition 41 in algebraic language in order to employ his quadrature techniques (below) in a more expeditious way. He uses the expression a^4/r^3 for the ordinate DF (an inverse-cube force). The constant a is raised to the fourth power, so that the ratio a^4/r^3 has the dimension of a length. Thus, Newton's algebraic language is attentive to geometrical interpretation. The last statement of this first excerpt is equivalent to what nowadays we understand as the principle of conservation of mechanical energy.

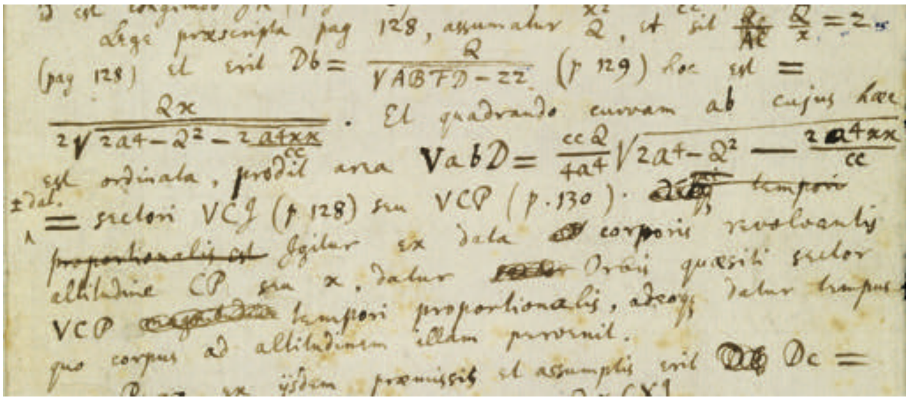


Fig. 12: Newton's annotations for Gregory on Corollary 3 (detail). Source: Gregory MS: 163 (Royal Society Library). ©The Royal Society

7.2 Algebraic quadrature of curve abz

The annotation continues as follows (see fig. 12):

> By the law prescribed on p. 128 let Q be assumed and let $Q/r = Z$ (p. 128): then will (p. 129)
>
> $$Db = \frac{Q}{[2]\sqrt{ABFD - ZZ}}$$
>
> that is
>
> $$= \frac{Qr}{2\sqrt{2a^4 - Q^2 - 2a^4rr/cc}}.$$
>
> And by squaring the curve $ab[z]$ of which this is the ordinate, we get
>
> $$\text{area}\, VabD = \frac{ccQ}{4a^4}\sqrt{2a^4 - Q^2 - \frac{2a^4rr}{cc}} \pm \text{a given}$$
>
> = the sector VCI (p. 128) or VCP (p. 130) ~~proportional to the time at~~ Therefore from CP or r, the given altitude of the revolving body, is given the sector VCP of the orbit which is sought, proportional to the time; and so the time is given in which the body attains that altitude.

Here Newton translates the fundamental "law" for the time in function of the radius (see equation (1)). The time is given by the quadrature of the curve abz with ordinate Db (see fig. 2) on page 128 of the *Principia*.

For an inverse-cube force, this quadrature is particularly simple (in modern terms it is equivalent to equation (12)). It takes the form:

$$y = Db = \frac{\delta r}{\sqrt{fr^2 + e}}, \tag{46}$$

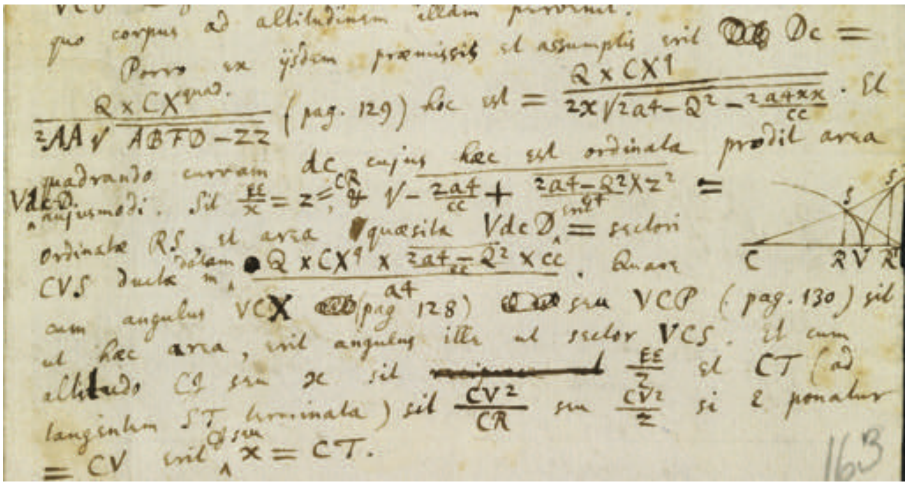


Fig. 13: Newton's annotations for Gregory on Corollary 3 (detail). Source: Gregory MS: 163 (Royal Society Library). ©The Royal Society

for $\delta = Q/2$, $f = -2a^4/c^2$, and $e = 2a^4 - Q^2$. This quadrature is elementary. But notice that Newton included it in the *ordo quartus* of his first catalogue of curves (see fig. 7). This first catalogue is based on the idea that, as we would say nowadays, differentiation and integration are inverse operations. It is interesting to note that Newton does not forget to add a constant of integration.

7.3 Geometric construction of quadrature of curve *dcx*

The annotation ends somewhat hurriedly with the following lines (see fig. 13). Gregory, who in his *memoranda* of the previous days often refers with admiration to Newton's higher-quadrature techniques, needed just this hint in order to connect this "abstruse" part of the *Principia* to the quadrature that Newton "omitted" in the printed text of the *Principia*. We read:

> Furthermore from the same premises and assumptions
>
> $$Dc = \frac{Q \times CX^{quad}}{2AA\sqrt{ABFD - ZZ}} \text{ (pag. 129)}$$
>
> that is
>
> $$= \frac{Q \times CX^{q}}{2r\sqrt{2a^4 - Q^2 - 2a^4 rr/cc}}.$$
>
> And by squaring the curve $dc[x]$ of which this is the ordinate we get the area $VdcD$, in this way. Let
>
> $$\frac{\epsilon\epsilon}{r} = \zeta = CR,$$

and

$$\sqrt{-\frac{2a^4}{cc}+\frac{2a^4-Q^2}{\epsilon^4}\zeta^2} = \text{ordinate}\, RS,$$

and $VdcD$, the area sought, will be equal to the sector VCS multiplied by the given

$$\frac{Q\times CX^q \times (2a^4-Q^2/\epsilon\epsilon)\times cc}{a^4}.^{31}$$

Wherefore, since the angle VCX (p. 128) or VCP (p. 130) is proportional to this area, that angle will be proportional to the sector VCS. And since the altitude CI or r is $\epsilon\epsilon/\zeta$, and CT (terminated at the tangent ST) is CV^2/CR or CV^2/ζ, then, if ϵ is put $= CV$, CI or $r = CT$.

Here Newton translates the second fundamental "law" for the polar angle in function of the radius (see equation (2)) that occurs on page 128 of the *Principia* (see fig. 2). Even at the beginning of the eighteenth century, this quadrature would have been considered a difficult one,

For an inverse-cube force, Newton needs to square a curve with abscissa r and ordinate Dc. Since Q, CX, a and c, are constants, the curve that Newton has to square (an equivalent of the differential equation (13)) has the form:

$$y = Dc = \frac{\delta}{r\sqrt{fr^2+e}}, \tag{47}$$

for $\delta = (Q/2)CX^2$, $f=-2a^4/c^2$, $e = 2a^4-Q^2$. This form belongs to the *ordo septimus* of the second catalogue for $\eta = 2$ (see fig. 10).

Thus, Newton (as in the second column) proceeds with the variable substitution

$$\zeta = \frac{\epsilon^2}{r}, \tag{48}$$

where ϵ is set equal to the semi-major axis CV. This variable substitution is equivalent to our $w = 1/r$ in section 4 (equation (14)).

Next, he introduces (as in the third column) the auxiliary conics (drawn at the bottom of the page (see fig. 13)) with abscissa $CR=\zeta$ and ordinate

$$v = RS = \sqrt{-\frac{2a^4}{cc}+\frac{2a^4-Q^2}{\epsilon^4}\zeta^2}. \tag{49}$$

Newton concludes with the construction (which we would express in terms of trigonometric and hyperbolic functions) prescribed in the fourth column of the seventh order of the second catalogue. Namely:

1. the radius r is set equal to CT (and inversely proportional to $\zeta = CR$),
2. the polar angle VCP is set proportional to the sector VCS.[32]

[31] Whiteside notes that the given constant should be $Qc^2/2a^4$ ([18], 6, 438).
[32] In symbolic terms: $r=\epsilon^2/\zeta = \epsilon^2(\cos k\theta)^{-1}$ and $r=\epsilon^2/\zeta=\epsilon^2(\cosh k\theta)^{-1}$.

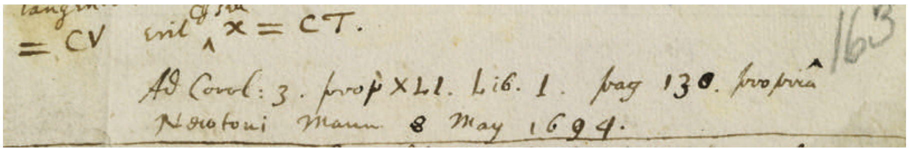


Fig. 14: Newton's annotations for Gregory on Corollary 3 (detail). Gregory adds "Ad Corol: 3 prop XLI Lib 1. pag. 130 propria Newtoni manu 8 May 1694." Source: Gregory MS: 163 (Royal Society Library). ©The Royal Society

It should be stressed that Newton's recourse to a geometric construction for this "mechanical" ("transcendental," in Leibniz's terms) quadrature is typical of the mathematical culture adopted in Europe until Euler. For example, as Nauenberg has shown, Johann Bernoulli's solution of the inverse problem for inverse-square forces ends with a construction in terms of an auxiliary circle ([16], 291–2).

7.4 Desinit

The annotation ends with a line in Gregory's handwriting (see fig. 14):

> To Corollary 3, Proposition 41, Book 1, pag. 130, in Newton's own hand
> 8 May 1694.

Gregory made a copy of this annotation that he later pasted in his *Notæ*, filling a blank half-page that he had left in correspondence to Corollary 3 (see fig. 1). The master had taught his disciple a lesson that was to remain hidden in Gregory's hands or available to his correspondents. There are reasons to believe that Gregory shared the content of his mathematical memoranda. For example, he may have circulated a short manuscript treatise on fluxions he composed in the autumn of 1694, another effect of his May encounter with Newton ([13], 334). As late as 1714 the blind Lucasian Professor, Nicholas Saunderson, was writing about a proposal for publishing the *Notae* ([20], 1, 264–5).

8 On translating and interpreting

8.1 Two questions

Newton's annotation for Gregory on Corollary 3 gives us some important information about the mathematical methods of the *Principia.* These methods were not exclusively based on a supposed "Newtonian style" framed in terms of "geometrical" limiting procedures.[33] In some instances, when dealing with

[33] The notion of a "Newtonian style" characterizing the *Principia* was forcefully defended by François De Gandt in his enjoyable and learned monograph [7].

central force motion, the attraction of extended bodies, the oscillation of pendulums, resisted motion, the solid of least resistance, and in many astronomical parts of the third book, Newton deployed techniques that were an integral part of his algebraic method of fluxions.[34] It should be stressed, however, that the methods employed in Corollary 3, and in the other instances listed above, are not representative of the whole *Principia*, a work that is characterized by a great variety of mathematical methods: some of these, but not all, were part of the method of series and fluxions.

In the case considered in this paper, we have seen Newton employing quadrature techniques. These cannot be considered a marginal element of the method of fluxions. Seventeenth-century mathematicians were in possession of quadrature techniques based on approximation methods reminiscent of Archimedean exhaustion proofs. But this is not the case with the quadratures employed in Corollary 3: in the second half of the seventeenth century these would have been considered as quite advanced and up-to-date.

Indeed, the quadratures employed in Corollary 3 imply several features that we associate with "calculus" and "rational mechanics." These features are: the understanding of the fundamental theorem of calculus implied in the first catalogue,the algorithm for finding the fluxion of (differentiating) irrational equations (irrational functions) displayed in the first catalogue, the techniques of quadrature (integration) by means of substitution of variables displayed in the second catalogue, the calculation of slopes of tangents to curves and the use of the so-called characteristic triangle (see fig. 5), the handling of transcendental magnitudes in terms of infinite series such as the binomial theorem, the representation of the variation of physical magnitudes (such as the force's intensity) in terms of plane curves (graphs), and the use of infinitesimals "moments" (differentials) referred to the continuous variation of physical magnitudes. All the above elements are necessary ingredients that enabled Newton to write the annotation for Gregory. One might even claim that Newton knew how to solve the problem of central forces in terms of integrations!

Two interesting questions remain open.

[34] Gregory's *Notæ* are a treasure trove of information on the relationships between the method of fluxions and Newton's *Principia*, something that might be missed upon superficial inspection. In many places Gregory refers to quadrature methods. Further, the Aberdeen exemplar of the *Notæ* includes pages on Newton's celebrated treatment of the solid of least resistance, an extraordinary application of fluxions to a proposition of the second book of the *Principia* that Gregory also included in the manuscript treatise on fluxions he circulated in the mid 1690s ("Isaaci Newtoni Methodus Fluxionum; ubi Calculus Differentialis Leibnitij, et Methodus Tangentium Barrovij explicantur, et exemplis plurimis omnis generis illustrantur. Auctore Davide Gregorio M. D. Astronomiæ Professore Saviliano Oxoniæ." A fair copy is in Christ Church (Oxford). The original is in St. Andrews University Library (MS QA 33G8/D12). Other copies are in the Cambridge University Library, Lucasian Papers [Res. 1894]: No. 13 and in the Macclesfield Collection, Add. 9597.9.3 and Add. 9597.9.4). Other relevant information on Newton's use of fluxions in the *Principia* can be gathered, most notably, from Newton's manuscripts and his correspondence with Nicolas Fatio de Duillier and Roger Cotes.

1. In what sense do Newton's methods differ from those deployed by Leibniz, Varignon, Johann Bernoulli, and Euler?[35]
2. Why did Newton omit to explicitly formulate the fluxional techniques he employed in the *Principia*?[36]

8.2 Translations

The first question depends on a defining feature of the history of mathematics. As historians of mathematics, we are always, as is apparent from this paper, wrestling with the problem of translation. We are in possession of a contemporary mathematical language into which we can translate the language of the past. Indeed, we cannot forget our mathematics when we look at the mathematics of the past, or that of other cultures. It is a fact, depending on the robustness and universality of mathematics, that such translations are possible. Yet, as historians we are very much interested in those elements of past actors' languages that are "lost in translation," being peculiar to their idiosyncratic conception of mathematics.

There are many differences between our mathematical language and Newton's. In physics we use equations rather than proportionalities, and this leads us to deploy dimensional constants which the experimenters have to measure. We talk in terms of functions defined on number domains, rather than relations between geometrical magnitudes. We do not reduce problems to geometric "quadratures of curves," but to analytic integrations. We relate solutions of differential equations to initial conditions and ask ourselves questions of existence and uniqueness that in Newton's times were left to intuition. One might even claim that Newton's notion of "solution" was different from ours: in tackling the problem of central forces, he sought a construction of the trajectory, whereas we seek a class of functions (a general integral). We use a vector notation and algebra for directed magnitudes, while until the beginning of the nineteenth century mathematicians referred to and manipulated directed magnitudes via geometrical diagrams (for example, we express the conservation of angular momentum in terms of a vector equation, whereas Newton demonstrated the conservation of areal velocity and the conservation of the orbital plane for central force motion in geometric terms). We deal with central forces in terms of a scalar potential energy U that is not to be found in Newton.

To some twenty first-century readers the annotation for Gregory we have analysed cannot but appear as a strange mixture of geometrical practice, phys-

[35] The classic works by Michel Blay [3] and Clifford Truesdell [23], for example, are certainly very helpful for finding an answer. For a recent assessment see [16].

[36] As a matter of fact, Newton contemplated ending the *Principia* with an appendix on quadratures in the 1690s, when revising the text, and again in the 1710s, when preparing the second edition of 1713. But in the end he resolved not to do so. It is only in the posthumous translation of 1729 due to Andrew Motte that we find two quadrature methods (for the attraction of an ellipsoid of revolution and for the solid of least resistance) printed as appendices "communicated by a friend," who may have been David Gregory.

ical insights, and dangerously ungrounded algebraic manipulations. The problem of translation, this switching back and forth between present and past languages, even somewhat frustrates our conviction about the universality of mathematics.

However — and this makes history of mathematics an even more demanding enterprise — there are also other mathematical languages that soon clutter the historian's desk: ones such as that of Euler, for example, which differ from, and yet are "equivalent to," Newton's. It is a worthy historiographical enterprise to attempt to understand how Euler and Lagrange improved on Newton, in what progress was made in passing from the *Principia* (1687) to the *Mechanica* (1736), and then from the *Mechanica* to the *Méchanique Analitique* (1788). From this perspective, the robustness of the standard view must be recognized. Eighteenth-century mathematicians had to abandon Newton's quadratures and geometrical representation of transcendental magnitudes. The problems they tackled required them to do so: they transformed mechanics in such a way that the *Principia* became an object of curiosity, rather than a source of scientific inspiration.

8.3 Comparisons

As a telling example of the controversial issues of interpretation we encounter in comparing Newton's fluxional method to the Leibnizian calculus, I might refer to a passage in our modern treatment of inverse-cube trajectories in section 4, namely: the separation of variables that we perform in deducing equation (8) from equation (7). I will rewrite this passage, which is so standard for a modern reader that it is easy to forget the importance of this algorithm.

Given (equation 7)

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{2(E-U) - h^2/r^2},$$

we wrote (equation 8)

$$dt = \frac{dr}{\sqrt{2(E-U) - h^2/r^2}}.$$

It is this substitution of the dot-notation with the differential one that allowed us to separate the variables.

As Blay has taught us in his magisterial works, Leibniz, Johann Bernoulli, Hermann and Varignon (among others) algebraised time, velocity and acceleration in terms of the Leibnizian calculus [3]. It is this algebrisation (in this case, the representation of radial speed as dr/dt) that allows the above separation of variables.

Yet, we should recall the three following points.

1. Newton could "square" the area $VabD$ for an inverse cube force (see fig. 2), and thus obtain the algebraic function that expresses the functional

dependence of time with radial distance in the inverse-cube case (as we have seen in section 7). Thus, in his own way, he could tackle the inverse problem of central forces in algebraic terms.

2. Further, in his method of fluxions, Newton often understood a symbol such as $\dot{x}$ to stand not for a finite speed, but rather for an infinitesimal increment, or "moment," of the pertinent variable x (an increment Leibniz would express with dx). This allowed Newton's immediate British followers, such as David Gregory, Roger Cotes and Colin Maclaurin, who trod in Newton's steps, to write formulas that correspond to the ones employed by the Leibnizians. So, for example, the early practitioners of the fluxional method, would represent, in Cartesian coordinates (x, y), the infinitesimal increment of the arc of a plane curve with $\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2}$ (which, of course, would correspond to $ds = \sqrt{dx^2 + dy^2}$). In this way, they were able to reproduce all the results on integration published by mathematicians such as Johann Bernoulli and Varignon in the *Acta Eruditorum.* In this context, it should be noted that very soon those who employed Newton's notation learned to translate the Leibnizian differential representation of the velocity $v = dx/dt$ in Newton's notation as $v = \dot{x}/\dot{t}$. This notation allowed the early fluxionists to perform the separation of variables that we have just seen above.[37] The idea that the Newtonian calculus lagged behind the Continental one because of an adherence to Newton's notation is a refutable myth.[38]
3. Finally, one should not forget that the mathematicians working with Leibniz's notation in the early decades of the eighteenth century, remained anchored to a geometric interpretation of the differential magnitudes they manipulated. For example, they understood a symbol such as dy/dx as representing a ratio between two geometric infinitesimal magnitudes. Their concern with geometric interpretation is revealed by the fact that, when writing a differential equation, they paid due attention to the geometric homogeneity of the left- and right-hand sides. To depict the fluxional method as "geometrical," in opposition to an "algebraic" differential calculus is an over-simplification.[39]

Let us continue our little exercise of "comparative mathematical literature," so to speak. From a modern view-point, one might tackle central force

[37] For example in Thomas Simpson's treatise on the method of fluxions, first printed in 1750, we read that "the Time wherein the Space $\dot{x}$ would be uniformly described is known to be as $\dot{x}/v$," (where v is the velocity) and a few pages later this is applied to rectilinear accelerated motion in order to calculate the time T "by finding the fluent" of "$\dot{T} = \dot{x}/v$." This is, of course, in Leibnzian notation equivalent to $dt = dx/v$. See [21], pp. 244–6.

[38] The absence of a symbol equivalent to Leibniz's $\int$ is a well-known weakness of Newton's notation. In a few instances, Newton used to draw a square $\Box$, in most cases he used words such as "the fluent of" or the "area of." Some eighteenth-century British mathematicians mixed the two notations by writing, for example, $\int y\dot{x}$ for $\int ydx$. The use of an elongated f for "fluent of," rather than Leibniz's elongated s, $\int$, for "summa" is also documented. [5], 2, 244–6.

[39] On the "dual" character, algebraic and geometric, of differentiation in the early Leibnizian calculus, see [10].

motion as follows. The integration of equation (8)

$$t(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{2(E-U) - h^2/r^2}},$$

defines t in function of r (where we choose $t(r_0) = 0$). By inversion, the radial position in function of time, $r(t)$, can be obtained. By using the conservation of angular momentum (9), the solution for θ is obtained as

$$\theta(t) = h \int_0^t \frac{dt}{r^2(t)} + \theta_0.$$

This leads to a complete solution of the determination of the trajectories ($r(t)$, $\theta(t)$) in a central force field.

The above calculation is so familiar that we tend to forget that in the eighteenth century specific algorithmic tools and new concepts were developed to facilitate such manipulations. In this case, what makes the above procedure easy are the concept and notation for functions (such as $r(t)$, $\theta(t)$), and the expression of the law of areas (or, conservation of angular momentum) as a differential equation (9), rather than as a statement concerning the rate of increase of the area spanned by the radius vector.[40] It is a fact, well-known to historians of eighteenth-century mathematics, that the notion and notation for functions emerged in the middle of the century, mostly in relation to issues related to the integration of partial differential equations (such as the vibrating string equation).

In the absence of a concept of function, Newton achieved the same result, but in a more cumbersome way. He required the squaring of two curves, corresponding to equations (8) and (10). In his characteristic geometrical way of representing functional relationships, Newton had to refer to two areas $VabD$ and $VdcD$, and then state that the quadrature of these areas gives the time (represented by the area VCI swept by the radius vector) in function of the distance from the force centre, and the polar angle (represented by the area of the circle sector VCX) in function of the distance from the force centre.

8.4 Re-enactments

As I said above, the historian wishes to recapture, to re-enact, what has been lost in the progress so masterfully recounted by Blay and Truesdell. It is undeniable that Leibniz, Varignon, Bernoulli and Euler produced an algorithm and obtained results in the integration of differential equations applied to mechanics that are superior to Newton's. However, Newton's mixing of geometrical

[40] We note that angular momentum is a vector quantity, so it is also the direction of the vector (or equivalently the fact that the motion is planar) that must be taken into consideration. Before the advent of vector notation at the beginning of the nineteenth century (another conceptual shift to which corresponds the introduction of yet another mathematical language), the constancy in orientation of the angular momentum could only be expressed in geometric terms.

intuition and algebra possesses its own beauty and conceptual depth. There was progress in passing from Newton to, say, Lagrange, but also a loss. In writing the history of the mathematization of mechanics, as historians we are interested in recapturing what has been "lost in translation." This is important mostly because what is lost is often what is peculiar and idiosyncratic: what is revealing of the author's intent and agenda. In our interpretative work we have thus to translate and compare the languages of past actors, in an attempt to understand what they wished to achieve, as Newton did, by choosing an austere Euclidean style when printing a volume on the mathematical principles of natural philosophy, when addressing a question posed by a frustrated visitor, or when omitting a quadrature "for the sake of brevity."

8.5 Interpretations

As to the second question. It seems as though Newton used a public language in which he printed his *magnum opus*, and a private language with his acolytes, whenever they asked for clarifications. This bifurcation between esoteric and exoteric mathematics is a fascinating aspect of Newton's policy of publication that cannot be dealt within the limits of this paper. Tentative answers can be reached only by broadening the field of historical enquiry. One has to consider what the intended audience of the author of the *Principia* might have been, how the choice of a public mathematical style became part of his self-fashioning in the republic of letters as a staunch anti-Cartesian, how and why the algebraic method of fluxions became a source of anxiety for a natural philosopher who regarded himself as an heir to the Ancients, rather than as a follower of the "men of recent times."[41]

8.6 Invitation

One of the aims of this paper is to suggest that the vexed question of Newton's use of calculus in the *Principia* still deserves our attention and might lead to "good" historical questions, such as the two we have just considered, which bypass as irrelevant the animosity and partisanship that have muddied the waters since the Newton-Leibniz controversy giving rise to a "standard view" that polarizes Newton's use of "geometry" in the *Principia* and the Continental "rational mechanics" in an all too simplistic way.

Questions of priority have often polarized the attention of historians of mathematics. These questions are, however, ill-chosen since they are based on the idea that in the development of mathematics there exist objects (such as theories, theorems, concepts, methods) that can be found by a single discoverer. From this point of view, the historian's task would be to lend credit to single individuals. But the development of mathematics is a much more complex phenomenon, and the application of calculus to mechanics is no exception.

[41] I have discussed this issue in [13].

It would be a gross exaggeration to say that Newton founded, or anticipated, the mechanics of Euler and Lagrange. Yet, it would be equally wrong to think that Newton and his acolytes were extraneous to a treatment of mechanics in terms of calculus.

If our program is aimed at "rediscovering the rational mechanics of the Age of Reason," as Truesdell put it in his seminal paper that initiated the influential history of this journal [23], then we have to accept a more complex image of the establishment of rational mechanics, and include Newton and the Newtonians within a picture that the standard view depicts as a purely Continental story spanning from Leibniz to Euler. Indeed, Newton and his British followers often contributed results in analytic mechanics that were shared with the Continentals, sometimes in a polemic context, as even a fleeting perusal of the Bernoulli correspondence at the Universität Bibliothek in Basel reveals. One might cite, without any claim to exhaustiveness: Cotes's study on the inverse-cube trajectories, Taylor's work on the vibrating string, Stirling's and Maclaurin's theorems on the gravitational attraction of ellipsoids. The viewpoint I suggest leads to further questions with which it may be apposite to conclude this paper: What did Newton mean by calculus, or by the method of series and fluxions? How did he apply it to mechanics? What publication policy did he follow in printing the *Principia*? How did he privately communicate his results to his acolytes? What can we learn about Newton's mathematical culture when we consider his views on mathematical method and publication practices? A serious study of these questions might shed light on an important chapter of the history of mathematics.

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