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Stochastic Dynamic Utilities and Intertemporal Preferences

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Abstract We propose an axiomatic approach which economically underpins the representation of dynamic intertemporal decisions in terms of a utility function, which randomly reacts to the information available to the decision maker throughout time. Our construction is iterative and based on time dependent preference connections, whose characterization is inspired by the original intuition given by Debreu's State Dependent Utilities (1960).

JEL Classification C02 · D81

Keywords intertemporal decisions · stochastic dynamic utility · conditional preferences · sure thing principle

1 Introduction

The *criterium* which leads the decisions of every agent, intervenes in many aspects of real life, determining the economical, political and financial dynamics. For this reason the psychological analysis and the mathematical axiomatization of the agents' behavior has gained a lot of interest, leading to a flourish stream of research literature (see [18] for an exhaustive review). The first key element which comes into play in the decision process is the Subjective Probability, which has been intensively studied since the preliminary contributions by de Finetti [5]. Von Neumann and Morgenstern [29] initiated a systematic

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work on preferences over lotteries, which admit a representation in terms of an expected utility. This intuition dates back to a paper published in 1738 [2], where Bernoulli already realized that any decision is heavily linked to the "particular circumstances of the person making the estimate", which could vary significantly depending on the observed evolution of information. For example a fund manager may start behaving in a risk seeking manner under the stress provoked by a plunge of the financial markets, which is causing severe losses. Debreu [4] gave an axiomatic setup (on a finite state space) to model preference relations which can depend on the future state of nature and can be represented by the so-called state-dependent utility functions (see Theorem 8, in the Appendix). State dependent preferences are sensitive to the random outcomes that may occur in the future, and therefore the agent's subjective utility may be affected by different future scenarios related to the occurrence of specific events. Karni [15] developed measures of risk aversion which allow the partial ordering of state dependend utilities in view of optimal risk sharing analysis. In [30], Wakker and Zank provided an extension of Debreu's result from finite to infinite dimension, for the special case of real-valued outcomes and monotonic preferences. The development of their extended functional, additively decomposable on infinite-dimensional spaces, leads to a numerical representation of the preferences in terms of a state dependent utility u and a probability $\mathbb{P}$ (see Theorem 9, Appendix). The main results in [30] (and [3]) will indeed play a key role in the proof of Theorem 3.
In [16] Kreps and Porteus gave rise to a new axiomatic treatment of the temporal resolution of uncertainty. They consider a discrete time model $t = 0, \dots, T$, where an individual must choose an action d_t constrained to the state x_t occurred at time t. As a random event takes place determining an immediate payoff z_t, the action d_t will affect the probability distribution of (z_t, x_{t+1}) where x_{t+1} is the new state of the world. The result is a dynamic choice behaviour which cannot be represented by a single cardinal utility.
Epstein Zin [11] and Duffie Epstein [7] (see also [10]) constructed a class of recursive preferences over intertemporal consumption lotteries, respectively in discrete and continuous time models. In [11] the recursive utility at time t is given by an aggregating function i.e. $V_t(c) = W(c_t, m_t(V_{t+1}))$ where c_t is the consuption and $m_t(V_{t+1})$ the certainty equivalent at time t of V_{t+1}. Similarly Duffie and Epstein [7] obtained a representation of the recursive utility on consumption streams of the form

$$V_t(c) = \mathbb{E}_{\mathbb{P}}\left[\int_t^T \left(f(c_s, V_s(c)) + \frac{1}{2}A(V_s(c))|\sigma_s|^2\right) ds \big| \mathcal{F}_t\right],$$

where f is an aggregator, A is the variance multiplier and σ is a volatility process. In such context the system of *conditional* preferences between two consumptions is determined by the recursive utility as

$$c \succeq_{\omega,t} c' \iff V_t(c, \omega) \geq V_t(c', \omega).$$

In [8], Epstein and LeBreton showed that the existence of a Bayesian prior is implied by preferences based on beliefs which admit a dynamically consistent updating in response to new information. The effect of consequences by the mean of conditional preferences over acts is introduced by Skiadas in [28]. Given an event F, the preference relation $x \succeq^F y$ "has the interpretation that, ex ante, the decision maker regards the consequences of act x on event F no less desirable than the consequences of act y on the same event "([28, p.350]). Wang [31] axiomatized three updating rules for a class of conditional preferences over consumption-information profiles. A systematic study of conditional preferences is provided in [6]: a *conditional preference order* is a binary relation $\succeq$ which is reflexive, transitive and locally complete. An opportune extension to the conditional setup of the indipendence and Archimedean axioms, led in [6, Theorem 5.2] to the representation of conditional preferences over the set of lotteries in terms of a conditional utility function.
Recursive multiple-priors and dynamic variational preferences (see resp. [9] and [17]) deals with conditional preference relations $\succeq_{t,\omega}$ on consumption streams h. Here $t \in \{0, 1, 2, \ldots, T\}$ is a point in time and ω is the path of the state space observed up to time t. These two classes of preferences can be represented in the form of conditional functionals given respectively by

$$V_t(h) = \inf_{P \in \Delta} \big(E_P \big[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \mid \mathcal{F}_t \big] \big) \tag{1.1}$$

$$V_t(h) = \inf_{P \in \Delta} \big(E_P \big[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \mid \mathcal{F}_t \big] + c_t(p|\mathcal{F}_t) \big), \tag{1.2}$$

where c_t is the recursive ambiguity index, which gurantees time consistency of the preferences under some opportune restrictions ([17, p.14, Axiom 4]). In both papers [9,17] the Dynamic Consistency Axiom plays a fundamental role and inspired the result contained in Proposition 1, Section 3 of the present paper.
Finally we observe that in the recent paper [25], Riedel et al. consider dynamic preferences $\succeq_{t,s}$ on couples (P, f) where P belongs to a set of probabilities and f is an act (imprecise probabilistic framework). An important feature is that Dynamic Consistency of the preferences guarantees that the set of conditional priors is stable under pasting.

From Economics to Finance: the dynamics of decision making. The interplay between Decion Theory and Financial Mathematics had its outbreak after the important contribution given by Merton in [19] and is witnessed by the flourish literature on stochastic optimal control (see [23] for a detailed exposition).
The classical utility maximization problem can be formulated as a stochastic control problem of the form

$$v(t, X) = \sup_{\alpha \in \mathcal{A}(t,X)} E_{\mathbb{P}}[u(V_T(t, X, \alpha)) \mid \mathcal{F}_t],$$

where the sup is to be intended as a $\mathbb{P}$ essential supremum, $\mathcal{A}(t, X)$ is the set of admissible strategies (starting at time t), u is a concave utility function and $V_T(t,X,\alpha)$ is the $\mathcal{F}_T$-measurable final payoff of the strategy α with initial random endowment X (which is $\mathcal{F}_t$-measurable). A primary question is wether a utility maximizer is willing to invest in a strategy α from time t to time T, provided she owns at t the random amount X. The answer to this question is deeply related to the intertemporal comparison between X and the final value of the strategy α given by $V_T(t,X,\alpha)$. One rational solution could be that the agent accepts to invest in the market if she believes to know the optimal solution to the control problem. Namely we can define an intertemporal relation $\succeq_{t,T}$ by

$$X \succeq_{t,T} V_T(t,X,\alpha) \text{ if and only if } v(t,X) \geq E_{\mathbb{P}}[u(V_T(t,X,\alpha)) \mid \mathcal{F}_t], \quad (1.3)$$

where the right hand side insequality is intended $\mathbb{P}$ almost surely. The Dynamic Programming Principle ([23, Theorem 3.3.1]) implies that for any bounded random variable X, $v(t,X) \geq E_{\mathbb{P}}[u(V_T(t,X,\alpha))|\mathcal{F}_t]$ and equality holds whenever α^* is the optimal policy. In this case X is equivalent to $V_T(t,X,\alpha^*)$ from an intertemporal perspective and will be baptized in the following section of this paper Conditional Certainty Equivalent (Definition 1). In this example, v represents the indirect utility and the preference relation $\succeq_{t,T}$ is not a standard binary relation, whose properties need to be introduced carefully, as we shall do in an abstract fashion in Section 3.

This backward approach to utility maximization has recently been argued in a series of paper by Musiela and Zariphopoulou starting from [20, 21], and a novel forward theory has been proposed: the utility function is stochastic, time dependent and moves forwardly. In this theory, the forward performance (which replaces the indirect utility of the classic case) is built through the underlying financial market and must satisfy some appropriate martingale conditions. Namely a Forward Performance is an adapted stochastic process $U(t,x,\omega)$ on a fixed filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ such that: i) $U(0,x,\omega)=u_0(x)$ for any $\omega \in \Omega$; ii) for each t and ω, $U(t,\cdot,\omega)$ is increasing and concave as a function of x; iii) for all $T \geq t$ and each self-financing strategy represented by π, the associated discounted wealth X^π satisfies $E_{\mathbb{P}}[U(T,X_T^\pi) \mid \mathcal{F}_t] \leq U(t,X_t^\pi)$; iv) for all $T \geq t$ there exists a self-financing strategy π^* such that X^{π^*} satisfies the equality in point iii). The first requirement represents the initial *datum* of the problem which determines (together with the market parameters) the evolution of U and motivates our Assumption 1. The remaining properties lead to the existence of U as a solution of a stochastic PDE. A posteriori of this construction we obtain a couple $(\mathbb{P},U)$ which defines an intertemporal relation $\succeq_{s,t}$ by $U(s,\cdot) \geq E_{\mathbb{P}}[U(t,\cdot)|\mathcal{F}_s]$. This intertemporal relation belongs to the class described by Theorem 3 and along the optimal policy π^* the value of the portofolio $X_s^{\pi^*}$ corresponds to the Conditional Certainty Equivalent of $X_t^{\pi^*}$ for any $s \leq t$.
Inspired by this idea, Frittelli and Maggis [13] studied the conditional (dynamic) version of certainty equivalents (as defined in [24]). The preliminary object is a stochastic dynamic utility $u(t,x,\omega)$ representing the evolution of the preferences of the agent. The novelty in [13] is that the (backward) conditional certainty equivalent, represents the time-s-value of the time-t-claim X,

for $0 \leq s \leq t < \infty$, capturing in this way the intertemporal nature of preferences. Unfortunately any axiomatization of intertemporal preferences, which could justify the representation in terms of stochastic dynamic utilities, is still missing in the literature and our aim is to fill this gap.

The aim of this paper. Any decision maker shows a certain level of impatience when compares present and future outcomes, as emotion-based and cognitive-based mechanisms contribute to intertemporal distortions. In [32], Zauberman and Urminsky provide an overview of the psychological determinants of intertemporal choice such as impulsivity, goal completion and reward timing, different evaluation of the future in terms of concreteness, time perception and many other features:

"*In sum, these findings establish that the way people perceive future time itself is an important factor in how they form their intertemporal preferences [...] What is common across the various factors influencing intertemporal preferences is that all these mechanisms influence the relative attractiveness of achieving a present goal compared to a later more distant one.* "[1]

In this paper we aim at characterizing a family of intertemporal preference relations which compare random payoffs whose realizations will be known at different points in time. We will introduce a set of conditional axioms which will lead to the representation of preference in terms of a Stochastic Dynamic Utility $u(t, x, \omega)$ and a Subjective Probability $\mathbb{P}$ on a general state space Ω (see Theorem 3), which can be rephrased as: conditional to the available information, g is preferred at time s to f at time $t \geq s$ if and only if

$$u(s, g) \geq E_{\mathbb{P}}[u(t, f) \mid \mathcal{F}_s] \quad \mathbb{P} - a.s.$$

The Stochastic Dynamic Utility turns out to be a random field adapted to a given filtration which represents the information flow. For this reason $u(t, x, \omega)$ randomly reacts whenever the Decision Maker becomes aware of new sensitive data, such as market behavior, news, catastrophic shocks or any other macro/micro factor which leads to a reconsideration of personal beliefs. Since different random payoffs are defined on different instants in time, the notion of preference relation[2] will satisfy non-standard axioms and will take the name of Intertemporal Preferences (ITP).

The main novelty of our approach is that we provide an abstract axiomatization of Intertemporal Preferences which allows to include in our model "the relative attractiveness of achieving a present goal compared to a later more distant one". Indeed our iterative construction leads to an automatic forward updating of preferences depending on the avaliable information, which satisfies a form of dynamic consinstency. As a byproduct we obtain a theoretical

[1] [32], p. 139.

[2] We point out that the use of the term 'preference'is slightly improper as the ordering will not be a binary relation as it is usually intended.

framework where the theory of Forward Performances [20,21] and the study of Conditional Certainty Equivalent [13] can be embedded.

The key ingredients of ITP can be summarized by four elements: first the information at each time is described by the existence of a filtration $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$, i.e. a family of sigma algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. Second, as ITP compares random payoffs which live at different times, we shall need to introduce a relation $\preceq_{s,t}$ (resp. $\succeq_{s,t}$) for $s < t$ being two points in time. In particular $g \preceq_{s,t} f$ will mean that the $\mathcal{F}_t$-measurable payoff f (which will be fully revealed at time t) is preferred to the $\mathcal{F}_s$-measurable g, conditioned to the knowledge of the information available at time s (Similar for $g \succeq_{s,t} f$). Third the preference relation $\preceq_{s,t}$ is not total if the full information $\mathcal{F}_s$ is not yet disclosed. The notion of conditional preferences, as introduced in [6], becomes an important tool to understand the nature of ITP. Lastly, we will assume that the agent observes real information only through a discretisation of the time line, namely $t_0 = 0 < t_1 < ... < t_n < \ldots$. We observe that in [6] a probability on the conditional sigma algebra was assumed to exist a priori. In our approach this requirement is not necessary, but we rather derive step by step a new probability update which follows directly from the decision theory structure we are choosing.

Some of the techniques involved in the proof of Theorem 3 are inspired by the theory of Conditional Risk Measures (see [12, Chapter 11]). The key element appearing in our construction is indeed the Conditional Certainty Equivalent (see Definition 1) which can be seen as a Conditional Risk Map ([13,14]). Nevertheless, the general class considered in [14] does not necessarily satisfy the Sure-Thing Principle, but rather a weaker regularity assumption ([14, Definition 2.6]) as depicted in the corresponding example of Section 5.

The paper is structured as follows: in Section 2 we provide a description of the notations used in the paper and a toy example to motivate our study; Section 3 is devoted to the introduction of the set of axioms characterizing ITP and to the statement of the main representation result. In Section 4 we prove the result in the unconditioned case (i.e. for trivial initial information). The aim of Section 4 is twofold: on the one hand it will serve as initial step of the induction argument we present in Section 6 to obtain the complete proof of our main Theorem 3. On the other hand, it is written in a self-contained manner, so that it can be read and understood independently from the general conditional setting. Finally in Section 5 we describe applications of our results to cases which go beyond the models described in references [13,20,21].

2 Preliminaries on Intertemporal preferences.

2.1 Notations

Throughout the paper we shall make use of the notations described in this short section. We fix a measure space $(\Omega, \mathcal{F})$ where Ω is the set of all possible events (*state space*) and $\mathcal{F}$ is a sigma algebra. We shall model information over time by the existence of an arbitrary filtration $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$, with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for every $s \le t$. For any given sigma algebra $\mathcal{G} \subseteq \mathcal{F}$ we denote by $\mathscr{L}(\mathcal{G})$ the space of $\mathcal{G}$-measurable functions taking values in $\mathbb{R}$ (*outcome space*). We shall usually refer to elements $f \in \mathscr{L}(\mathcal{G})$ as random variables (or acts) and denote by $\mathscr{L}^\infty(\mathcal{G})$ its subspace collecting bounded elements i.e. $f \in \mathscr{L}(\mathcal{G})$ such that $|f(\omega)| \le k$ for any $\omega \in \Omega$ and some $k \ge 0$. On $\mathscr{L}(\mathcal{G})$ and $\mathscr{L}^\infty(\mathcal{G})$ we shall consider the usual pointwise order $f \le g$ if and only if $f(\omega) \le g(\omega)$ for every $\omega \in \Omega$ and similarly $f < g$ if and only if $f(\omega) < g(\omega)$ for every $\omega \in \Omega$. Given two elements $f, g \in \mathscr{L}^\infty(\mathcal{G})$ we use the notation $f \vee g$, $f \wedge g$ to indicate respectively the minimum and the maximum between f and g. For a countable family of acts $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathscr{L}^\infty(\mathcal{G})$ we consider the $\inf_n f_n, \sup_n f_n$ the pointwise infimum/supremum of the family and recall that if the family is uniformly bounded then $\inf_n f_n, \sup_n f_n$ are elements of $\mathscr{L}^\infty(\mathcal{G})$. $\mathscr{L}^\infty(\mathcal{G})$ endowed with the sup norm $\|\cdot\|_\infty$ becomes a Banach lattice, where $\|f\|_\infty = \sup_{\omega\in\Omega}|f(\omega)|$. By $\mathbf{1}_A$, $A \in \mathcal{G}$ we indicate the element of $\mathscr{L}^\infty(\mathcal{G})$ such that $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. For $f \in \mathscr{L}^\infty(\mathcal{G})$ and $A \in \mathcal{G}$, $f\mathbf{1}_A$ denotes the restriction of f to A; for any couple $f, g \in \mathscr{L}(\mathcal{G})$ and event $A \in \mathcal{G}$, $f\mathbf{1}_A + g\mathbf{1}_{A^c}$ denotes the random variable that agrees with f on A and with g on A^c. Fix two sigma algebras $\mathcal{G}_1 \subset \mathcal{G}_2$: for any finite partition $\{A_1, ..., A_n\} \subset \mathcal{G}_2$ of Ω and $\{g_j\}_{j=1}^n \subset \mathscr{L}^\infty(\mathcal{G}_1)$, $\sum_{j=1}^n g_j \mathbf{1}_{A_j}$ denotes the element assigning g_j on A_j, $\forall j = 1, ..., n$. This type of random variables can be interpreted as simple acts conditional to $\mathcal{G}_1$ and $\mathcal{S}_{\mathcal{G}_1}(\mathcal{G}_2)$ denotes the space of conditional simple acts. The standard notion of simple acts can be obtained when $\mathcal{G}_1 = \{\emptyset, \Omega\}$ and the corresponding space will be denoted by $\mathcal{S}(\mathcal{G}_2)$.

Whenever a probability $\mathbb{P}$ is given $(\Omega, \mathcal{F}, \mathbb{P})$ becomes a measure space and, as usual, we shall say that a probability $\widetilde{\mathbb{P}}$ is dominated by $\mathbb{P}$ ($\widetilde{\mathbb{P}} \ll \mathbb{P}$) if $\mathbb{P}(A) = 0$ implies $\widetilde{\mathbb{P}}(A) = 0$ for $A \in \mathcal{F}$. Similarly a probability $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ ($\widetilde{\mathbb{P}} \sim \mathbb{P}$) if $\mathbb{P} \ll \widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{P}} \ll \mathbb{P}$. A property holds $\mathbb{P}$ almost surely ($\mathbb{P}$-a.s.), if the set where it fails has 0 probability.

For any given sigma algebra $\mathcal{G} \subseteq \mathcal{F}$ we shall denote with $L^0(\Omega, \mathcal{G}, \mathbb{P})$ the space of equivalence classes of $\mathcal{G}$ measurable random variables that are $\mathbb{P}$ almost surely equal and by $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ the subspace of ($\mathbb{P}$ a.s.) bounded random variables. Formally any $f \in \mathscr{L}(\mathcal{G})$ will be a representative of the class $X := [f]_\mathbb{P} \in L^0(\Omega, \mathcal{G}, \mathbb{P})$. Moreover the essential ($\mathbb{P}$ a.s.) *supremum* of an arbitrary family of random variables $\{X_\lambda\}_{\lambda\in\Lambda} \subseteq L^0(\Omega, \mathcal{G}, \mathbb{P})$ will be simply denoted by $\mathbb{P} - \sup\{X_\lambda \mid \lambda \in \Lambda\}$, and similarly for the essential *infimum* (see [12] Section A.5 for reference).

Let us fix $(\Omega, \mathcal{G}, \mathbb{P})$: given a random field $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for every $f \in \mathscr{L}^\infty(\mathcal{G})$ the map $\omega \mapsto \phi(\omega, f(\omega))$ is $\mathcal{G}$-measurable (see [26] for further details) we introduce the notation

$$L(\mathcal{G}; \phi) = \{[\phi(\cdot, f(\cdot))]_\mathbb{P} \mid f \in \mathscr{L}^\infty(\mathcal{G})\}. \tag{2.4}$$

Indeed $L(\mathcal{G}; \phi)$ represents the range of the random field ϕ in the space $L^0(\Omega, \mathcal{G}, \mathbb{P})$. In order to tackle the issue of continuty of the conditional representation of

preferences, we need to introduce an *ad hoc* definition of continuity for stochastic fields. Consider $(\Omega, \mathcal{G}, \mathbb{P})$ and $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ as for (2.4), we say that ϕ is $\star$-continuous if $\forall f \in \mathscr{L}^\infty(\mathcal{G})$ it holds that $f(\omega)$ belongs to the points of continuity of $\phi(\cdot, \omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ (see Definition 3 in Appendix A for the formal statement).
Finally the space of $\mathbb{P}$ integrable random variables will be denoted by $L^1(\Omega, \mathcal{G}, \mathbb{P})$. We use the standard notation and indicate by $E_{\mathbb{P}}[X]$ the Lebesgue integral of $X \in L^1(\Omega, \mathcal{G}, \mathbb{P})$. Moreover if $\mathcal{H}$ is a sigma algebra contained in $\mathcal{G}$ then $E_{\mathbb{P}}[X \mid \mathcal{H}]$ denotes the conditional expectation of X given $\mathcal{H}$ and $\mathbb{P}_{|\mathcal{H}}$ the restriction of the probability $\mathbb{P}$ on the smaller sigma algebra $\mathcal{H}$.

2.2 State dependent utility and the role of information: a toy example

In this section we propose a toy example to highlight some meaningful aspects arising in our study. We consider a situation where the agent's system of preferences is defined by one-step updates over three times $t \in \{0, 1, 2\}$. Every step corresponds to the disclosure of new information, and therefore the generated filtration becomes a structural underlying foundation. The preferences of the agent are represented by a subjective probability $\mathbb{P}$ and a utility $u(t, x, \omega)$ which reacts to any new input. The agent compares decisions whose consequences are known at different times. Thus the optimal decision between 0 and 2 with respect to the agent's subjective perception of risk and/or gain needs to be resolved by sequential one-step optimizations so that the final output diverges from the classical backward approach.

Two brothers E, Y are inheriting from their old and rich grandmother. The elder brother E is asked to choose between receiving 1 million Euros immediately (at time $t = 0$), or waiting two years (time $t = 2$) when his grandmother will move to the rest home in Sardinia and earn her wonderful villa near the Como Lake. Alternatively E could wait until the intermediate time $t = 1$ to make up his decision, but in any case the younger brother Y will have to accept what is left from E after his decision is taken.
The value of the villa at time 0 is equal to 1 million, but of course it makes little sense to compare the two values today since the villa will be available only at $t = 2$.
Now assume that at time $t = 1$ election for the new Italian Government will take place and the catastrophic event of Italy leaving the European Union (with a consequent default of its economic system) may occur. Call this event A and set $\mathcal{F}_1 = \{\emptyset, \Omega, A, A^c\}$. Brother E knows that if A^c will occur the value of the villa will increase to $1.11 \cdot 10^6$, but in case of default it will fall down to $2 \cdot 10^5$. The probability of the default event A is low but not negligible, say $\mathbb{P}(A) = 0.01$. Finally the probability of defaulting at time 2 (call this event D) knowing that A^c occurred is almost negligible, for instance $\mathbb{P}(D \mid A^c) = 10^{-6}$ (in which case the villa would be worth again $2 \cdot 10^5$). In case that a default did not occur neither at time 1 nor at time 2 then the value of the villa at 2

would jump up to $1.8 \cdot 10^6$. Information at time 2 is therefore described by $\mathcal{F}_2$ the sigma algebra generated by $\{A, D\}$.

Agent E is assumed to be risk neutral as far as Italy is not defaulting i.e. $u(x) = x$. In case of a default (either at time 1 or 2) his utility function would be $\tilde{u}(x) = \frac{1}{2}x$ if $x \geq 0$ or $\tilde{u}(x) = 2x$ if $x < 0$. The naive idea is that once the default has occurred the agent gives more importance in avoiding losses, rather than gaining money. We can synthesize this reasoning by introducing the stochastic dynamic utility as follows

$$u(t, x, \omega) = \begin{cases} u(x) & \text{if } t = 0 \\ \tilde{u}(x)\mathbf{1}_A(\omega) + u(x)\mathbf{1}_{A^c}(\omega) & \text{if } t = 1 \\ \tilde{u}(x)\mathbf{1}_{A\cup D}(\omega) + u(x)\mathbf{1}_{A^c\cap D^c}(\omega) & \text{if } t = 2 \end{cases}$$

We make the following considerations.

- If agent E compares the choice between getting 10^6 today or the villa at time $t = 2$, then he is comparing the utility $u_0(10^6) = 10^6$ with respect to the expected utility of the payoff at time $t = 2$ given by

  $$\text{Expected payoff } = 1.8 \cdot 10^6 \cdot (1 - 10^{-2} - 10^{-6}) + \frac{1}{2} \cdot 2 \cdot 10^5 \cdot (10^{-2} + 10^{-6}).$$

  This Expected payoff is strictly greater than 10^6 and indeed if E neglects the intermediate time $t = 1$ then he will choose for the villa instead of immediate money. But this impulsive strategy would not lead to an optimal solution.
- Assume now that the agent first compare 10^6 with the value of the villa at time $t = 1$. Then

  $$\text{Expected payoff } = 1.11 \cdot 10^6 \cdot 0.9 + \frac{1}{2} \cdot 2 \cdot 10^5 \cdot 0.01 = 10^6.$$

  The expected value of the villa at time 1 equals the cash amount of money that brother E could receive at time 0. Therefore E is indifferent between taking the decision today ($t = 0$) or tomorrow ($t = 1$) and he can take advantage by gaining more information about elections. In the former case E will choose 10^6 which is in fact better than the value of the villa. In the second case he will prefer obtaining the villa at time $t = 2$ rather than 10^6 at time $t = 1$. Clearly this second strategy provides an optimal final profile, since it exploits the additional intermediate information.

Remark 1 Notice that the reasoning would change if the elder brother reckons $\mathbb{P}(A) = 0$. In such a case the additional intermediate information would play no role in the decision process.

3 An axiomatization of intertemporal preferences.

We consider a time interval $[0, +\infty)$, together with a fixed (countable) family of updating times $t_0 = 0 < t_1 < \ldots < t_n < \ldots,$. At each t_i the agent shall

reconsider her preference relations depending on the observed information. In particular at time $t_0 = 0$ no information is available, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Information at each time t is represented by a sigma algebra $\mathcal{F}_t$ and since information increases in time we shall have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for every $s \leq t$.
In the entire paper acts are intended as real valued random variables, matching the framework used in [30][3].

Assumption 1 *We shall always assume throughout the paper that the agent initial preferences are described by the utility function* $u_0 : \mathbb{R} \to \mathbb{R}$. *In addition* u_0 *is supposed to be strictly increasing, continuous and* $u_0(0) = 0$.

The paper could be developed without fixing u_0 as in Assumption 1. The advantage of this choice is twofold: on the one hand fixing a single u_0 gives a sharper uniqueness result in Theorem 3 (see also Remark 7). On the other hand u_0 plays the role of "initial datum", which follows from the idea that u_0 is inherited from the attitude towards decisions shown by the agent in the past (as assumed in [20, 21]). The shape of u_0 is effective: for instance it allows to understand if at the initial time the agent is risk averse or risk seeking and in general how she evaluates variation in the amount of money she owns (quoting [2] "Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.").

The time t_1 represents the first instant when the Decion Maker observes available information which will potentially influence her decision. Random payoffs at time t_1 are described by random variables in $\mathscr{L}^\infty(\mathcal{F}_{t_1})$ and the agent compares these random payoffs with initial sure positions represented by elements in $\mathbb{R}$. In Section 4 we shall provide the representation of an intertemporal preference $\succeq_{0,1}$ connecting the initial time $t_0 = 0$ to t_1. In Proposition 3 we will show the following: if $\succeq_{0,1}$ is complete, transitive, monotone, continuous and satisfies the Sure-Thing Priciple then for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ and $a \in \mathbb{R}$ we have $a \succeq_{0,1} f$ is and only if $u_0(a) \geq \int_\Omega u_1(f(\omega), \omega) d\mathbb{P}_1(\omega)$. This representation is based on Theorem 9 by Wakker and Zank and shows how new inputs will affect the attitude of an agent towards decisions, generating a new utility u_1 which will depend on the state of nature realized. Once time t_1 is reached the Decison Maker will start considering a new aim in the next future, say t_2, and will compare random payoffs, known at time t_1, with those depending on events occurring at t_2. From the t_0 perspective the new intertemporal preference $\preceq_{1,2}$ will be a conditional preference relation which incorporates the further knowledge reached at time t_1. Therefore we shall follow the idea proposed by [6] and make use of similar techniques developed in the conditional setting. This procedure will repeat iteratively at every interval from t_i to t_{i+1} and for this reason our main result will be proved by induction over updating times. Each updating step from t_i to t_{i+1} will be characterized

[3] Indeed this choice is not 'without loss of generality'. Nevertheless, as explained in the Introduction this research is inspired by potential financial applications, and therefore we prefer to choose a more financial friendly setup.

by a preference interconnection $\preceq_{i,i+1}$ (or $\succeq_{i,i+1}$) satisfying conditional transition axioms. Of course since the proof proceeds by induction we will assume that we reached the desired representation up to step t_i and show the representation at the succeeding time t_{i+1}. This will guarantee the existence of a probability $\mathbb{P}_i$ only on the sigma algebra $\mathcal{F}_{t_i}$, which we will need to update to the larger sigma algebra $\mathcal{F}_{t_{i+1}}$, following the Bayesian paradigm.

For the statement of Theorem 3, we fix an arbitrary N and a family of intertemporal preference relations $\succeq_{i,i+1}$ for $i = 0, \dots, N-1$ with the following meaning: for any $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ we say that $g \succeq_{i,i+1} f$ if the agent prefers to hold the gamble g at time t_i than the gamble f at time t_{i+1}, knowing all the information provided at time t_i (similarly for $g \preceq_{i,i+1} f$). As usual we say that $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ is equivalent to $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, namely $g \sim_{i,i+1} f$, if both $g \succeq_{i,i+1} f$ and $g \preceq_{i,i+1} f$, and define the family of *null events* for every $i = 1, \dots, N$ as $\mathcal{N}(\mathcal{F}_{t_i})$ given by

$$\{A \in \mathcal{F}_{t_i} : g \sim_{i-1,i} f \Rightarrow g \sim_{i-1,i} \tilde{g}\mathbf{1}_A + f\mathbf{1}_{A^c}, \forall f, \in \mathscr{L}^\infty(\mathcal{F}_{t_i}), g, \tilde{g} \in \mathscr{L}^\infty(\mathcal{F}_{t_{i-1}})\}. \tag{3.5}$$

An event $A \in \mathcal{F}_{t_i}$ is called *essential* at time t_i if $A \in \mathcal{F}_{t_i} \setminus \mathcal{N}(\mathcal{F}_{t_i})$

The Transition Axiom. We are now ready to introduce the first axiom characterizing the Intertemporal Preferences. In this context the preference ordering $\succeq_{i,i+1}$ is not anymore a binary relation as it is generally understood. For this reason we shall need a reformulation of the axioms which shall be compared to more classical ones. Moreover we work in a conditional setting, which means that the relation $\succeq_{i,i+1}$ is assessed taking into account information available at time t_i. Information are modelled by measurable sets $A \in \mathcal{F}_{t_i}$ and in addition the Decision Maker has a subjective belief concerning sets which are relevant ($A \in \mathcal{F}_{t_i} \setminus \mathcal{N}(\mathcal{F}_{t_i})$) and those which are irrelevant ($A \in \mathcal{N}(\mathcal{F}_{t_i})$). To understand the central role of null sets we refer to the example contained in Section 2.2 (see in particular Remark 1).

(T.i) Transition Axiom for $\succeq_{i,i+1}$.
Let $A, B \in \mathcal{F}_{t_i}$, $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ then we require for $\succeq_{i,i+1}$ to be

1. locally complete: there exists $A \in \mathcal{F}_{t_i} \setminus \mathcal{N}(\mathcal{F}_{t_i})$ such that either $g\mathbf{1}_A \succeq_{i,i+1} f\mathbf{1}_A$ or $g\mathbf{1}_A \preceq_{i,i+1} f\mathbf{1}_A$.
2. transitive: if $g \succeq_{i,i+1} f$ and $h \preceq_{i,i+1} f$ then $\{g < h\} \in \mathcal{N}(\mathcal{F}_{t_i})$;
3. normalized: if $A, B \in \mathcal{N}(\mathcal{F}_{t_i})$ then $\mathbf{1}_A \sim_{i,i+1} \mathbf{1}_B$.
4. non-degenerate: for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ there exist $g_1, g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $g_1 \preceq_{i,i+1} f$ and $g_2 \succeq_{i,i+1} f$.
5. consistent: if $g\mathbf{1}_A \succeq_{i,i+1} f\mathbf{1}_A$ (resp. $\preceq_{i,i+1}$) and $B \subseteq A$ then $g\mathbf{1}_B \succeq_{i,i+1} f\mathbf{1}_B$ (resp. $\preceq_{i,i+1}$);
6. stable: if $g\mathbf{1}_A \succeq_{i,i+1} f\mathbf{1}_A$ (resp. $\preceq_{i,i+1}$) and $g\mathbf{1}_B \succeq_{i,i+1} f\mathbf{1}_B$ (resp. $\preceq_{i,i+1}$) then $g\mathbf{1}_{A\cup B} \succeq_{i,i+1} f\mathbf{1}_{A\cup B}$ (resp. $\preceq_{i,i+1}$);

Before giving an explanation of (T.i) in its full generality, we specialize it to the unconditioned case ($i = 0$).

(T.0) Transition preference relation $\preceq_{0,1}$.
1. complete: for $a \in \mathbb{R}$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ either $a \succeq_{0,1} f$ or $a \preceq_{0,1} f$;
2. transitive: $a \preceq_{0,1} f$ and $b \succeq_{0,1} f$ implies $a \leq b$;
3. normalized: $0 \sim_{0,1} 0$ (i.e. $0 \succeq_{0,1} 0$ and $0 \preceq_{0,1} 0$).
4. non-degenerate: for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ there exist $y, z \in \mathbb{R}$ such that $y \preceq_{0,1} f$ and $x \succeq_{0,1} f$.

The Axiom (T.0) is composed by four requirements only and the reason of this significant simplification is due to the assumption $\mathcal{F}_0 = \{\emptyset, \Omega\}$. To understand why completeness and transitivity are the natural counterpart suggested by the classical definition of weak order, we observe that Proposition 2 guarantees that under (T.0) for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ there exists a unique $C_{0,1}(f) \in \mathbb{R}$ such that both $C_{0,1}(f) \succeq_{0,1} f$ or $C_{0,1}(f) \preceq_{0,1} f$ hold. We can therefore consider the following induced ordering $\preceq_1$: for any $f, g \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$, $f \preceq_1 g$ if and only if $C_{0,1}(f) \leq C_{0,1}(g)$. Indeed $\preceq_1$ is reflexive and inherits completeness and transitivity from $\preceq_{0,1}$.

We now move to the interpretation of Axiom (T.i): properties 1, 5 and 6 are deeply related and inspired to the notion of conditional preference in [6]. The first property of (T.i) points out that the updating procedure necessarily leads to preferences which are complete in a conditional sense. In particular we shall see in Lemma 2 (which is the counterpart of Lemma 3.2 in [6]) that local completeness allows to compare two acts on three disjoint $\mathcal{F}_{t_i}$ measurable events. Consistency and stability can be understood in terms of information achieved: for example consistency states that if the agent prefer $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ at time t_i rather than $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ at time t_{i+1} knowing that event $A \in \mathcal{F}_{t_i}$ has occurred, than she shall prefer g for any condition $B \in \mathcal{F}_{t_i}$, $B \subseteq A$.
The property 2 in (T.i) is the conditional generalization of its counterpart in (T.0). Normalization (property 3) says that $\mathcal{F}_{t_i}$ null events are preserved in the one step updating. In particular the agent is indifferent between random payoffs which differ from 0 by a negligible $\mathcal{F}_{t_i}$ measurable set (loosely speaking "Holding nothing is indifferent throughout time", up to null events). Non degeneracy (property 4) is the more technical one and guarantees some simplifications in our arguments, since it implies that any random payoff f at time t_{i+1} admits an $\mathcal{F}_{t_i}$-measurable g which is more/less preferred (it is nevertheless a very weak requirement which is satisfied in all the cases of interest).

The definition of Conditional Certainty Equivalent is the basis of our representation results and follows from the idea in [13]. In Section 6 we shall show by induction the existence (and uniqueness) of the Conditional Certainty Equivalent at each time step.

Definition 1 We say $g \sim_{i,i+1} f$ if and only if $g \succeq_{i,i+1} f$ and $g \preceq_{i,i+1} f$. If $g \sim_{i,i+1} f$ then we shall call g the Conditional Certainty Equivalent (CCE) of f and denote the family of all CCEs as $C_{i,i+1}(f)$.

Notation 2 *In what follows we shall use these notations for any* $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ *and* $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$*:*

- $g \sim_{i,i+1} f$ *if both* $g \succeq_{i,i+1} f$ *and* $g \preceq_{i,i+1} f$ *hold;*
- $g \succ_{i,i+1} f$ *if* $g \succeq_{i,i+1} f$ *but* $g\mathbf{1}_A \not\sim_{i,i+1} f\mathbf{1}_A$ $\forall A \in \mathcal{F}_{t_i} \setminus \mathcal{N}(\mathcal{F}_{t_i})$ *(similarly* $g \prec_{i,i+1} f$ *);*
- $g \succ^A_{i,i+1} f$ *if* $g\mathbf{1}_A \succeq_{i,i+1} f\mathbf{1}_A$ *but* $g\mathbf{1}_B \not\sim_{i,i+1} f\mathbf{1}_B$ $\forall B \in \mathcal{F}_{t_i} \setminus \mathcal{N}(\mathcal{F}_{t_i})$ *with* $B \subseteq A$ *(similarly* $g \prec^A_{i,i+1} f$ *);*

Remark 2 We observe that consistency jointly to stability of $\succeq_{i,i+1}$ imply the following pasting properties:

- for any $A, B \in \mathcal{F}_{t_i}$, $g_1, g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ and $f_1, f_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$. If $g_1\mathbf{1}_A \succeq_{i,i+1} f_1\mathbf{1}_A$ and $g_2\mathbf{1}_B \succeq_{i,i+1} f_2\mathbf{1}_B$ then $(g_1+g_2)\mathbf{1}_{A\cap B} \succeq_{i,i+1} (f_1+f_2)\mathbf{1}_{A\cap B}$, $g_1\mathbf{1}_{A\setminus B} \succeq_{i,i+1} f_1\mathbf{1}_{A\setminus B}$ and $g_2\mathbf{1}_{B\setminus A} \succeq_{i,i+1} f_2\mathbf{1}_{B\setminus A}$.
- for a family $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}_{t_i}$ of disjoint events and $A = \cup_n A_n$ we have

$$g\mathbf{1}_A \preceq_{i,i+1} f\mathbf{1}_A \;\Leftrightarrow\; g\mathbf{1}_{A_n} \preceq_{i,i+1} f\mathbf{1}_{A_n} \text{ for every } n.$$

Axiom (T.i) is the key ingredient to obtain the updating construction we are aiming at. In particular assume that at time t_i the agent is characterized by a couple $(\mathbb{P}_i, u_i)$ where $\mathbb{P}_i$ is the subjective probability on measurable events $\mathcal{F}_{t_i}$ and u_i is a state dependent utility such that $u_i(x,\cdot)$ is $\mathcal{F}_{t_i}$-measurable. We shall prove in Proposition 5 that if $\succeq_{i,i+1}$ satisfies (T.i) then for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ there exists a unique Conditional Certainty Equivalent given by $C_{i,i+1}(f) = u_i^{-1}V_{i+1}(f)$, where $V_{i+1}(f) = \mathbb{P}_i - \inf\{u_i(g) \mid g \succeq_{i,i+1} f\}$. Moreover V_{i+1} represents the transition order i.e.

$$g \preceq_{i,i+1} f \Leftrightarrow u_i(g) \le V_{i+1}(f) \quad \mathbb{P}_i\text{-a.s.}$$
$$g \succeq_{i,i+1} f \Leftrightarrow u_i(g) \ge V_{i+1}(f) \quad \mathbb{P}_i\text{-a.s.}$$

Integral representation of Inter Temporal Preferences. We will take into consideration the following axioms: monotonicity, the Sure Thing Principle and a technical continuity, adapted to this conditional setting, which will lead to a representation of the ITP in the desired integral form.

(M.i) Strict Monotonicity.
Given arbitrary $g_1, g_2, g_3, \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, $A \in \mathcal{F}_{t_{i+1}} \backslash \mathcal{N}(\mathcal{F}_{t_{i+1}})$ and $g_1 < g_2$:

$$g_3 \sim_{i,i+1} g_1\mathbf{1}_A + f\mathbf{1}_{A^c} \text{ implies } g_3 \prec^B_{i,i+1} g_2\mathbf{1}_A + f\mathbf{1}_{A^c} \text{ for a } B \in \mathcal{F}_{t_i} \backslash \mathcal{N}(\mathcal{F}_{t_i}),$$

$$g_3 \sim_{i,i+1} g_2\mathbf{1}_A + f\mathbf{1}_{A^c} \text{ implies } g_3 \succ^B_{i,i+1} g_1\mathbf{1}_A + f\mathbf{1}_{A^c} \text{ for a } B \in \mathcal{F}_{t_i} \backslash \mathcal{N}(\mathcal{F}_{t_i}).$$

(ST.i) Sure-Thing Principle.
Given arbitrary $f_1, f_2, h \in \mathcal{S}_{\mathcal{F}_{t_i}}(\mathcal{F}_{t_{i+1}})$, $A \in \mathcal{F}_{t_{i+1}} \setminus \mathcal{N}(\mathcal{F}_{t_{i+1}})$ and $g_1 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$, such that $g_1 \succeq_{i,i+1} f_1\mathbf{1}_A + h\mathbf{1}_{A^c}$ and $g_1 \preceq_{i,i+1} f_2\mathbf{1}_A + h\mathbf{1}_{A^c}$: for any $k \in \mathcal{S}_{\mathcal{F}_{t_i}}(\mathcal{F}_{t_{i+1}})$ there exists $g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $g_2 \succeq_{i,i+1} f_1\mathbf{1}_A + k\mathbf{1}_{A^c}$ and $g_2 \preceq_{i,i+1} f_2\mathbf{1}_A + k\mathbf{1}_{A^c}$.

(C.i) Pointwise continuity.
Consider any uniformly bounded sequence $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$, then for any $g \prec_{i,i+1} f$ (resp. $g \succ_{i,i+1} f$) there exists a partition $\{A_k\}_{k=1}^\infty \subset \mathcal{F}_{t_i}$ such that for any k we have $g\mathbf{1}_{A_k} \preceq_{i,i+1} f_n\mathbf{1}_{A_k}$ (resp. $g\mathbf{1}_{A_k} \succeq_{i,i+1} f_n\mathbf{1}_{A_k}$) for all $n \geq n_k$.

We are now ready to state the main contribution of this paper: Theorem 3 provides the representation of ITP in terms of a unique probability $\mathbb{P}$ and a stochastic field $u(t,x,\omega)$, which describes the random fluctuations of preferences. These were exactly the elements exploited in [13] to determine the dynamics of the Conditional Certainty Equivalent.

Theorem 3 (Representation) *Let Assumption 1 holds and any $\mathcal{F}_{t_i}$ contains three essential disjoint events for every $i = 1, 2, \dots$. The intertemporal preference $\succeq_{i,i+1}$ satisfies (T.i), (M.i), (ST.i) and (C.i) for any $i = 0, \dots, N$ if and only if there exist a probability $\mathbb{P}$ on $\mathcal{F}_{t_N}$ and a Stochastic Dynamic Utility*

$$u(t,x,\omega) = \sum_{i=0}^{N-1} u_i(x,\omega)\mathbf{1}_{[t_i,t_{i+1})}(t) + u_N(x,\omega)\mathbf{1}_{t_N}(t) \tag{3.6}$$

satisfying

(a) $u(t_i,x,\cdot)$ *is* $\mathcal{F}_{t_i}$*-measurable and* $E_{\mathbb{P}}[|u(t_i,x,\cdot)|] < \infty$*, for all* $x \in \mathbb{R}$*;*
(b) $u(t_i,\cdot,\omega)$ *is strictly increasing in* x *and* $u(t_i,0,\omega) = 0$[4]*, for all* $\omega \in \Omega$*;*
(c) $u(t_i,\cdot,\cdot)$ *is* $\star$*-continuous*
(d) $E_{\mathbb{P}}[u_{i+1}(f)|\mathcal{F}_{t_i}] \in L(\mathcal{F}_{t_i};u_i)$[5] *for any* $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ *and*

$$g \succeq_{i,i+1} f \iff u(t_i,g) \geq E_{\mathbb{P}}[u(t_{i+1},f)|\mathcal{F}_{t_i}] \quad \mathbb{P}\text{-a.s.}$$
$$g \preceq_{i,i+1} f \iff u(t_i,g) \leq E_{\mathbb{P}}[u(t_{i+1},f)|\mathcal{F}_{t_i}] \quad \mathbb{P}\text{-a.s.}$$

Relative uniqueness: *the couple* $(\mathbb{P},u)$ *can be replaced by* $(\mathbb{P}^*,u^*)$ *if and only if* $\mathbb{P}$ *is equivalent to* $\mathbb{P}^*$ *on* $\mathcal{F}_{t_N}$ *and for any* $i = 1,\dots,N$ *we have* $\mathbb{P}(u^*(t_i,\cdot,\cdot) = \delta_i u_i) = 1$*, where* δ_i *is the Radon-Nikodym derivative of* $\mathbb{P}_{|\mathcal{F}_{t_i}}$ *with respect to* $\mathbb{P}^*_{|\mathcal{F}_{t_i}}$*.*

Remark 3 At first sight it might seem unexpected that no discount factor appears in the Representation Theorem 3, whereas (1.1) and (1.2) show an explicit dependence on β. However, in the latter examples the presence of a discount factor is motivated by the fact that u is homogeneous in time. Instead, in our framework $u(t,x,\omega)$ varies stochastically in time, and therefore it is in general not possible to disentagle the contribution of discounting from the utility in a unique way. Moreover, the uniqueness of the representation is up to equivalent change of measures, and thus the discount factor would be in any case sensitive to probabilistic measure changes.

[4] This additional requirement is in fact without loss of generality, and allows a useful simplification in the main body of the proof.
[5] Recall the definition in equation (2.4)

Time consistency of intertemporal preferences. The family $\{\succeq_{i,i+1}\}$ of intertemporal preferences is meant to create a link between two successive times t_i and t_{i+1} in order to compare random payoffs whose effects will be known and exploitable at different times. The procedure is a step by step updating and simple inspections show that the following semigroup property holds true for the Conditional Certainty Equivalent

$$C_{s,v}(f) = C_{s,t}(C_{t,v}(f))^6 \quad \forall\, 0 \le s < t < v \text{ and } f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_v) \tag{3.7}$$

where for any $s < t$ the operator $C_{s,t}(\cdot)$ is the ($\mathbb{P}$-a.s. unique) solution of the equation $u(s, C_{s,t}(\cdot)) = E_{\mathbb{P}}[u(t,\cdot) \mid \mathcal{F}_s]$ and u is the Stochastic Dynamic Utility obtained in Theorem 3. As an immediate consequence we can extend the intertemporal preferences to any $s < t$ as follows

$$g \succeq_{s,t} f \iff u(s,g) \ge E_{\mathbb{P}}[u(t,f)|\mathcal{F}_s] \quad \mathbb{P}\text{-a.s.}$$
$$g \preceq_{s,t} f \iff u(s,g) \le E_{\mathbb{P}}[u(t,f)|\mathcal{F}_s] \quad \mathbb{P}\text{-a.s.}$$

where $g \in \mathscr{L}^\infty(\Omega, \mathcal{F}_s)$ and $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_t)$. In virtue of the semigroup property (3.7) we obtain the following time consistency of preferences

Proposition 1 *Let $0 \le s < t < v$ and let $g \in \mathscr{L}^\infty(\Omega, \mathcal{F}_s)$ and $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_v)$ such that $g \succeq_{s,v} f$ (resp. $g \preceq_{s,v} f$). Then $g \succeq_{s,t} h$ (resp. $g \preceq_{s,t} h$) for any $h \in \mathscr{L}^\infty(\Omega, \mathcal{F}_t)$ such that $h \sim_{t,v} f$.*

Since the proof of Theorem 3 will proceed inductively we choose to present the theory in the simpler unconditioned case $\preceq_{0,1}$ (see Section 4). The results in the next section will be therefore necessary to prove the initial step in the induction argument of Theorem 3.
Moreover we stress that the relative uniqueness is sharper than in representation results like those contained in [3,30]. This follows from the fact that the u_0 is fixed a priori (together with the normalization condition $u(t_i, 0, \omega) = 0$) and plays the role of an initial (constraining) condition (see also Proposition 3 for further details).

4 Unconditioned intertemporal preference

We consider a Decision Maker who compares an initial amount of some good, whose value is surely determined (and its benefit is immediate) with respect to bounded random payoffs (e.g. bets, assets, future value of goods) at a fixed time t_1 represented by elements in the space $\mathscr{L}^\infty(\mathcal{F}_{t_1})$. We say that the agent is initially naive, as the initial information are represented by the trivial $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and therefore the space $\mathscr{L}^\infty(\mathcal{F}_0)$ is isometric to the real line $\mathbb{R}$.

Consider the transition preference $\preceq_{0,1}$ (or $\succeq_{0,1}$) which connects $\mathscr{L}^\infty(\mathcal{F}_{t_1})$ to $\mathscr{L}^\infty(\mathcal{F}_0)$. As already observed in the case $i = 0$ the first Axiom (T.0) is

[6] Abuse of notation: the precise formulation should be $C_{s,v}(f) = C_{s,t}(g)$ where $g \in \mathscr{L}(\mathcal{F}_t)$ is a version of $C_{t,v}(f)$.

composed only by four requirements: completeness, transitivity, normalization and non degeneracy (which is the more technical requirement we shall use in Lemma 1).

Remark 4 From Notation 2 we can easily deduce the meaning of the symbols $\sim_{0,1}$, $\succ_{0,1}$, $\prec_{0,1}$. We also recall that the set of null events induced by $\preceq_{0,1}$ is given by

$$\mathcal{N}(\mathcal{F}_{t_1}) = \{A \in \mathcal{F}_{t_1} : a \sim_{0,1} f \Rightarrow a \sim_{0,1} b\mathbf{1}_A + f\mathbf{1}_{A^c}, \forall f \in \mathscr{L}^\infty(\mathcal{F}_{t_1}), a, b \in \mathbb{R}\}.$$

Definition 2 If $a \sim_{0,1} f$ then we shall call a the (Conditional) Certainty Equivalent of f and denote the family of all CCEs as $C_{0,1}(f)$.

We now show that under (T.0) the CCE exists and is unique. Notice that this notion of certainty equivalent matches the dynamic generalization introduced by [13]. The CCE will also provide a natural representation of the intertemporal preference $\preceq_{0,1}$ (see the following Proposition 2).
Consider the maps

$$V_1^-(f) = \sup\{u_0(a) \mid a \preceq_{0,1} f\} \text{ and } V_1^+(f) = \inf\{u_0(a) \mid a \succeq_{0,1} f\},$$

where u_0 will be always supposed to fullfill Assumption 1.

We note that in the definition of $V_1^\pm(f)$, u_0 needs not to be fixed. Indeed if we consider the total ordering on $\mathscr{L}^\infty(\mathcal{F}_{t_1})$ induced by the functionals $V_1^\pm(\cdot)$ (i.e. $f_1 \preceq_1 f_2$ if and only if $V_1^\pm(f_1) \leq V_1^\pm(f_2)$) this would not be affected by the choice of u_0. Nevertheless, as previously explained we prefer to think u_0 as an initial data characterizing the decision maker, with the advantage of obtaining a sharper notion of uniqueness.

Lemma 1 *Under (T.0) and Assumption 1 the maps V_1^+, V_1^- are well defined from $\mathscr{L}^\infty(\mathcal{F}_{t_1})$ to $\mathbb{R}$. Moreover $V_1^+(f) = V_1^-(f)$ for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$.*

Proof From completeness $V_1^\pm$ are well defined and taking values in $\mathbb{R} \cup \{\pm\infty\}$. The fact that $V_1^\pm(f)$ are finite for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ follows from non degeneracy in Axiom (T.0).
For any $a, b \in \mathbb{R}$ such that $a \preceq_{0,1} f$ and $b \succeq_{0,1} f$ we have $u_0(a) \leq u_0(b)$, and therefore $V_1^-(f) \leq V_1^+(f)$. Now assume by contradiction $V_1^-(f) < V_1^+(f)$: since u_0 is strictly increasing and continuous there exists c such that $u_0(c) \in (V_1^-(f), V_1^+(f))$. From completeness either $c \succeq_{0,1} f$ or $c \preceq_{0,1} f$ getting in both cases a contradiction since

$$\sup\{u_0(a) \mid a \preceq_{0,1} f\} < u_0(c) < \inf\{u_0(a) \mid a \succeq_{0,1} f\}.$$

Notation 4 *From now on, whenever (T.0) and Assumption 1 are in force we shall denote $V_1 := V_1^+ \equiv V_1^-$.*

Proposition 2 *Let (T.0) and Assumption 1 hold. Then for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ there exists a unique Conditional Certainty Equivalent given by $C_{0,1}(f) =$*

$u_0^{-1}V_1(f)$. *Moreover* V_1 *takes values in the range of* u_0 *and represents the transition order i.e.*

$$a \preceq_{0,1} f \Leftrightarrow u_0(a) \leq V_1(f) \tag{4.8}$$

$$a \succeq_{0,1} f \Leftrightarrow u_0(a) \geq V_1(f) \tag{4.9}$$

Proof Existence and uniqueness follow from the previous Lemma 1 and Assumption 1. Notice that $a \preceq_{0,1} f$ (resp. $a \succeq_{0,1} f$) obviously implies $u_0(a) \leq V_1(f)$ (resp. $u_0(a) \geq V_1(f)$). For the reverse implication we can observe that $u_0(a) = V_1(f)$ implies $a \sim_{0,1} f$. If instead $u_0(a) < V_1(f)$ (resp. $u_0(a) > V_1(f)$), then necessary $a \prec_{0,1} f$ (resp. $a \succ_{0,1} f$) as $V_1(f) = \inf\{u_0(a) \mid a \succeq_{0,1} f\}$ (resp. $V_1(f) = \sup\{u_0(a) \mid a \preceq_{0,1} f\}$).

We will take into consideration the following axioms: monotonicity, the Sure Thing Principle and a technical continuity, which we recall here to clarify their meaning in this simplified unconditioned case.

(M.0) Strict Monotonicity: for all $a, b, c \in \mathbb{R}$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ $A \in \mathcal{F}_{t_1} \setminus \mathcal{N}(\mathcal{F}_{t_1})$ and $a < b$ we have $c \sim_{0,1} a\mathbf{1}_A + f\mathbf{1}_{A^c}$ implies $c \prec_{0,1} b\mathbf{1}_A + f\mathbf{1}_{A^c}$ (resp. $c \sim_{0,1} b\mathbf{1}_A + f\mathbf{1}_{A^c}$ implies $c \succ_{0,1} a\mathbf{1}_A + f\mathbf{1}_{A^c}$).

(ST.0) Sure-Thing Principle: consider arbitrary $f, g, h \in \mathcal{S}(\mathcal{F}_{t_1})$, $A \in \mathcal{F}_{t_1} \setminus \mathcal{N}(\mathcal{F}_{t_1})$ and $a \in \mathbb{R}$ such that $a \succeq_{0,1} f\mathbf{1}_A + h\mathbf{1}_{A^c}$ and $a \preceq_{0,1} g\mathbf{1}_A + h\mathbf{1}_{A^c}$ then for any $k \in \mathcal{S}(\mathcal{F}_{t_1})$ there exists $b \in \mathbb{R}$ such that $b \succeq_{0,1} f\mathbf{1}_A + k\mathbf{1}_{A^c}$ and $b \preceq_{0,1} g\mathbf{1}_A + k\mathbf{1}_{A^c}$.

(C.0) Pointwise continuity: consider any uniformly bounded sequence $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_1})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$, then for all $a \prec_{0,1} f$ (resp. $a \succ_{0,1} f$) there exists N such that $a \preceq_{0,1} f_n$ (resp. $a \succeq_{0,1} f_n$) for $n > N$.

Remark 5 In the classical Decision Theory (see [22]) the Sure-Thing Principle is a sort of independence principle: it says that the preference between two acts, f and g, should only depend on the values of f and g when they differ. If f and g differ only on an event A, if A does not occur f and g result in the same outcome exactly. In our intertemporal framework the interpretation is exactly the same, even though we need to deal with the comparison at time 0.

Remark 6 In the present context the Sure-Thing Principle (ST.0) easily implies for arbitrary $f, g \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ and $A \in \mathcal{F}_{t_1}$: $V_1(f\mathbf{1}_A) \leq V_1(g\mathbf{1}_A)$ and $V_1(f\mathbf{1}_{A^c}) \leq V_1(g\mathbf{1}_{A^c})$ then $V_1(f) \leq V_1(g)$.

We can now state the main representation result of this section which will be proved in Section 6.

Proposition 3 *Assume that* $\mathcal{F}_{t_1}$ *contains at least three disjoint essential events and Assumption 1 is in force. Axioms (T.0), (M.0), (ST.0) and (C.0) hold if and only if there exists a probability* $\mathbb{P}_1$ *on* Ω *and a function* $u_1(\cdot, \omega) : \mathbb{R} \to \mathbb{R}$, *strictly increasing* $\forall \omega \in \Omega$ *and* $\star$*-continuous such that the functional* V_1

$$V_1(f) = \int_\Omega u_1(f(\omega), \omega) d\mathbb{P}_1 \tag{4.10}$$

represents the preference $\preccurlyeq_{0,1}$ *(in the sense of* (4.8) *and* (4.9)*) and takes values in the range of* u_0*.*
The following uniqueness holds for (4.10) *:* $(\mathbb{P}_1, u_1)$ *can be replaced by* $(\mathbb{P}^*, u^*)$ *if and only if* $\mathbb{P}_1$ *is equivalent to* $\mathbb{P}^*$ *and* $\mathbb{P}_1(u^* = \delta u_1 + \tau) = 1$*, where* δ *is the Radon-Nikodym derivative of* $\mathbb{P}_1$ *with respect to* $\mathbb{P}^*$ *and* $\tau \in \mathscr{L}(\mathcal{F}_{t_1})$ *with* $E_{\mathbb{P}^*}[\tau] = 0$*.*

Remark 7 Even though Proposition 3 shows many similarities with Theorem 9, some work needs to be done to show that Axioms (T.0), (M.0), (ST.0) and (C.0) are sufficient to apply the results in [3,30]. Moreover as V_1 is defined via a fixed u_0, we shall show that the coefficient $\sigma > 0$ appearing in Theorem 9 is necessarily equal to 1.

Remark 8 We observe that even if not mentioned explicitly, necessarily the random variable $u_1(x, \cdot)$ is integrable with respect to $\mathbb{P}_1$ for any $x \in \mathbb{R}$.
Moreover if we impose the normalization requirement $u_1(0, \omega) = 0$ for every $\omega \in \Omega$ then τ is equal to 0 $\mathbb{P}$-a.s..

5 Applications to Financial Economics.

Dynamic updating of state dependent utilities. In the previous Section 4 we proposed the simplified situation in which the Decion Maker is considering only two points in time (present and future). From the one hand Proposition 3 leads to a representation in terms of a state dependend utility as in [30]. On the other the main message of our approach differs significantly also in the unconditioned case. Indeed in the framework of [30] the certainty equivalent of $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ is the value $x \in \mathbb{R}$ such that $E_{\mathbb{P}_1}[u_1(f)] = E_{\mathbb{P}_1}[u_1(x)]$ i.e. the monetary sure amount that will be equivalent to the random payoff f at the future time t_1. From our perspective instead $C_{0,1}(f)$ is the value that is equivalent today to the future random payoff f.
We illustrate here how intertemporal preferences can act as a dynamic updating of state dependent utilities in a simplified case to avoid issues of measurability related to the stochastic nature of our approach.
On $(\Omega, \mathcal{F})$ consider an information process $\{I_{t_i}\}_{i=0}^N$ such that $I_{t_i} : \Omega \to A_i \subset \mathbb{R}^d$ is $\mathcal{F}$ measurable, $I_0 = x \in \mathbb{R}^d$ and A_i is at most countable for simplicity. Let $\mathcal{F}_{t_i}$ be the σ-algebra generated by $I_{t_1}, \dots, I_{t_i}$. We can notice that $\mathcal{F}_{t_i}$ is generated by atoms of the form

$$\Sigma_{t_i}^{\mathbf{x}} = \{\omega \in \Omega \mid I_{t_k} = x_k, \forall\, k = 1, \dots, i\},$$

where $\mathbf{x} = (x_1, \dots, x_i) \in A_1 \times \ldots \times A_i$. We denote by $\succeq_{i,i+1}^{\Sigma_{t_i}^{\mathbf{x}}}$ (in agreement with Notation 2) the ITP $\succeq_{i,i+1}$ conditioned to the occurence of the event $\Sigma_{t_i}^{\mathbf{x}}$ for some observed information $\mathbf{x} \in A_1 \times \ldots \times A_i$. Then $\succeq_{i,i+1}^{\Sigma_{t_i}^{\mathbf{x}}}$ falls into the class described in the previous Section 4. In particular this means that $\succeq_{i,i+1}$ generates a standard state dependent utility when it is localized to the atom $\Sigma_{t_i}^{\mathbf{x}}$. The power of Theorem 3 is that it allows to "paste"together in a measurable way all this local systems of preferences to obtain a global structure which can be automatically updated forwardly in time. This result is reached for general measurable spaces without any restriction on cardinality of Ω.

Conditional Risk Maps. We refer to [12] for an extensive introduction to the theory of Risk Measures and here only recall few notions from the general setup in [14]. For a fixed filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_i}\}, \mathbb{P})$ we consider for $i = 0, \dots, N-1$ a class of conditional maps $\Phi_i : L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ such that $\Phi_i(0) = 0$. We say that the maps are regular (REG) if: for every $X, Y \in L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$ and $A \in \mathcal{F}_{t_i}$ $\Phi_i(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \Phi_i(X)\mathbf{1}_A + \Phi_i(Y)\mathbf{1}_{A^C}$. The class $\{\Phi_i\}$ naturally induces a family of ITP as follows: for any $g \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ and $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$

$$g \preceq_{i,i+1} f \text{ if and only if } [g]_{\mathbb{P}} \leq \Phi_i([f]_{\mathbb{P}}) \quad \mathbb{P}_i\text{-a.s.}.$$

Notice that by (REG) $\preceq_{i,i+1}$ satisfies Axiom (T.i). Clearly any additional assumption on the maps Φ_i reflects on properties of $\preceq_{i,i+1}$.
In particular if the maps are monotone (MON) i.e. for every $i = 0, \dots, N$, $X, Y \in L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$, $X \leq Y$ implies $\Phi_i(X) \leq \Phi_i(Y)$, then $\preceq_{i,i+1}$ satisfies (M.i).
Similarly if for every $X, Y \in L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$ and $A \in \mathcal{F}_{t_{i+1}}$, $\Phi_i(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \Phi_i(X\mathbf{1}_A) + \Phi_i(Y\mathbf{1}_{A^C})$ (we refer at this property as (LIN)), then $\preceq_{i,i+1}$ satisfies (ST.i)[7]. Finally we can notice that the continuity Axiom (C.i) follows immediately from the usual "Lebesgue property"of Risk Measures (see [12, Section 4.3]), namely: for any sequence $\{X_n\}$, bounded in $L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$, such that $X_n \to X$ $\mathbb{P}$-a.s. we have $\Phi_i(X_n) \to \Phi_i(X)$ $\mathbb{P}$-a.s..

Therefore any class of conditional maps $\{\Phi_i\}$ which satisfies the aforementioned properties can be represented as a Conditional Certainty Equivalent by Theorem 3 (See also [13, Section 2.3] and [14, Section 1.1]).

Stochastic control. In the Introduction (pp. 3-4) we proposed the classical framework of utility maximization, which can help understanding the meaning of Axiom (T.i) 1,5,6. In this paragraph we eleborate on the class of intertemporal preferences which arises in discrete time stochastic optimization problems for conditional maps in the spirit of recursive preferences [11,16]. In this example we consider an agent who is asked to choose between two random endowments whose outcome will be revealed at two different times. To this aim she adopts the criterium of the best expected value where the conditional "expectation"operator Φ_i is not necessarily linear.
For a fixed $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_i}\})$ we consider a set of controlled processes $\{X_{t_i}(\mathbf{a})\}_{i=0}^N$ with $X_{t_i}(\mathbf{a}) \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ for every control $\mathbf{a}$. The control $\mathbf{a} = \{\alpha_{t_i}\}_{i=0}^N$ is also a stochastic process which is often assumed to be predictable (i.e. $\alpha_{t_{i+1}}$ is $\mathcal{F}_{t_i}$-measurable). This is for example the case of $X_{t_i}(\mathbf{a})$ being the discrete time stochastic integral representing the payoff of a self-financing strategy (see

[7] This property in general holds only for few classes of Risk Measures.

[12, Chapter 5]). Indeed the controlled process $X_{t_i}(\mathbf{a})$ will be a function of $\mathbf{a}$ only up to time t_i i.e. $X_{t_i}(\mathbf{a}) = X_{t_i}(\alpha_0, \ldots, \alpha_{t_i})$. Set $t_N = T$ and fix a random endowment $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ then the backward stochastic control problem assume the following form

$$U_{t_i}(f) := \mathbb{P} - \sup_{(\alpha_{t_i},\ldots,\alpha_T)} \Phi_i\big([f + X_T(\alpha_{t_i}, \ldots, \alpha_T)]_{\mathbb{P}}\big)$$

where $\mathbb{P} - \sup$ represents the *essential supremum* and $\Phi_i : L^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ is a conditional map such that $\Phi_i(0) = 0$. A standard example that can be considered is $\Phi_i(\cdot) = E_{\mathbb{P}}[\phi(\cdot) \mid \mathcal{F}_{t_i}]$ so that we obtain the usual Merton problem.
Similarly to the previous paragraph the stochastic control problem induces a family of ITP as follows: for any $g \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ and $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_{i+1}}, \mathbb{P})$

$$g \preceq_{i,i+1} f \text{ if and only if } U_{t_i}(g) \leq U_{t_{i+1}}(f) \quad \mathbb{P}\text{-a.s.}.$$

Moreover if Φ_i satisfies (REG) (MON) and (LIN) then $\preceq_{i,i+1}$ satisfies Axiom (T.i), (M.i) and (ST.i).

Dynamic Variational Preferencs. In the theory of Dynamic Variational Preferences two consumption streams can be compared by the mean of the functional $V_t(h)$ defined in (1.2). Given two consumption plans which start from two different times $h = (h_{t_i}, \ldots, h_{t_N})$, $k = (k_{t_{i+1}}, \ldots, k_{t_N})$, we can define the family of ITP by

$$h \succeq_{i,i+1} k \text{ if and only if } V_{t_i}(h) \geq V_{t_{i+1}}(k) \ \mathbb{P}\text{-a.s.}.$$

If we set the notation $h\mathbf{1}_A = (h_{t_i}\mathbf{1}_A, \ldots, h_{t_N}\mathbf{1}_A)$, $k\mathbf{1}_A = (k_{t_{i+1}}\mathbf{1}_A, \ldots, k_{t_N}\mathbf{1}_A)$ then Axiom (T.i) is automatically satisfied. Simple inspections show that Axiom (M.i) can be also adapted to this context but indeed the Sure-Thing Principle fails to hold unless the ambiguity index c_t weights only a single model (i.e. there exists $p \in \Delta$ such that for every t, $c_t(\bar{p} \mid \mathcal{F}_t) = 0$ and $c_t(p \mid \mathcal{F}_t) = +\infty$ for any $p \in \Delta$, $p \neq \bar{p}$).

6 Inductive proof of Theorem 3

This section is entirely devoted to the proof of the main Theorem of this paper. The first step is the proof of Proposition 3 which also plays the role of “first step ”in the inductive argument.

6.1 Proof of Proposition 3

Observe that the hypothesis of Proposition 2 are satisfied. Hence, the representation $a \succcurlyeq_{0,1} f \iff u_0(a) \geq V_1(f)$ holds where V_1 is defined as in Lemma 1. Furthermore, for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ the CCE $C_{0,1}(f)$ exists and is uniquely given by $u_0^{-1}(V_1(f))$. The existence of the CCE for every act f directly implies that the range of the function V_1 is contained in the range of u_0.

Proof of (⇒) We define a weak order on $\mathscr{L}^\infty(\mathcal{F}_{t_1})$ as $f \preceq g$ if and only if $V_1(f) \leq V_1(g)$. (T.0) implies $\preceq$ is complete, reflexive and transitive (i.e. satisfies (A1) in the Appendix).
Let $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ and outcomes $x > y$: indeed (M.0) implies $V_1(x\mathbf{1}_A + f\mathbf{1}_{A^c}) > V_1(y\mathbf{1}_A + f\mathbf{1}_{A^c})$, for all nonnull events $A \in \mathcal{F}_{t_1}$ and thus $\preceq$ is strictly monotone in the sense of (A2). Similarly (ST.0) implies that $\preceq$ satisfies (A3).
Let now $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_1})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$ and $\|f_n\|_\infty < k$ for all $n \in \mathbb{N}$. Let now $g \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ such that $g \succ f$ and consider $a = C_{0,1}(g)$ (which exists by Proposition 2). Then $a \succ_{0,1} f$ and by (C.0) we can find $\bar{n}$ such that for all $n \geq \bar{n}$ we have $a \succ_{0,1} f_n$. Therefore $V_1(g) = u_0(a) > V_1(f_n)$ (similarly for the opposite inequality) showing that (A4) holds for $\preceq$.

We can therefore apply Theorem 9 and find the desired representation (4.10) namely $V_1(f) = \int_\Omega u_1(f(\omega),\omega)d\mathbb{P}_1 = E_{\mathbb{P}_1}[u_1(f)]$ and its uniqueness. Let therefore $\mathbb{P}^*$ and $u^*(\cdot,\cdot) = \tau + \sigma\delta u_1(\cdot,\cdot)$ obtained by Theorem 9. Observe that $V_1(0) = u_0(0) = 0$ implies $E_{\mathbb{P}^*}[\tau] = 0$. Moreover as $u_0(C_{0,1}(f)) = V_1(f) = E_{\mathbb{P}^*}[u^*(f)] = E_{\mathbb{P}_1}[\sigma u_1(f)]$, we have necessarily $\sigma = 1$.

We now show that the utility u_1 is $\star$-continuous on $(\Omega, \mathcal{F}_{t_1}, \mathbb{P}_1)$. To this end consider any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$. It is sufficient to show that $\mathbb{P}_1(LD_f) = 0$ where LD_f is the set defined in Appendix A replacing ϕ with u_1. Indeed, with an analogous argument one obtains $\mathbb{P}_1(RD_f) = 0$. Ultimately, the thesis follows from the observation that $\mathbb{P}_1(D_f) = \mathbb{P}_1(LD_f \cup RD_f) = 0$ and the arbitrariness of f.
As consequence of Lemma 4, $LD_f \in \mathcal{F}_{t_1}$, so either $\mathbb{P}_1(LD_f) = 0$ or $\mathbb{P}_1(LD_f) > 0$. Suppose, by contradiction, that there exists $f^* \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ such that $\mathbb{P}_1(LD_{f^*}) > 0$. Set $B := LD_{f^*}$ and let $f = f^*\mathbf{1}_B$ and $f_n = (f - \frac{1}{n})\mathbf{1}_B$. By construction $f, f_n \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ for each $n \in \mathbb{N}$, $f_n(\omega) \to f(\omega)$ for each $\omega \in \Omega$ and $\sup_n \|f_n\|_\infty \leq \|f\|_\infty + 1$. Furthermore, by the definition of B, $u_1(f(\omega),\omega) > \sup_n u_1(f_n(\omega),\omega)$ for each $\omega \in B$ while $u_1(f(\omega),\omega) = u_1(f_n(\omega),\omega)$ for each $\omega \in B^C$. Since $\mathbb{P}_1(B) > 0$ and $x \mapsto u_1(x,\omega)$ is increasing, by Monotone Convergence Theorem we have:

$$\lim_n E_{\mathbb{P}_1}[u_1(f_n,\cdot)] = E_{\mathbb{P}_1}[\sup_n u_1(f_n,\cdot)] < E_{\mathbb{P}_1}[u_1(f,\cdot)]$$

By continuity and strict monotonicity of u_0, there exists $a \in \mathbb{R}$ such that

$$\sup_n E_{\mathbb{P}_1}[u_1(f_n,\cdot)] < u_0(a) < E_{\mathbb{P}_1}[u_1(f,\cdot)]$$

that is $a \succ_{0,1} f_n$ $\forall n$ while $a \prec_{0,1} f$. This contradicts Axiom (C.0); hence we conclude that $\mathbb{P}_1(B)$ equals zero.

Proof of (⇐) Viceversa, we assume that the preference $\succcurlyeq_{0,1}$ is given by:

$$a \succcurlyeq_{0,1} f \iff u_0(a) \geq V_1(f)$$

for $a \in \mathbb{R}$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$, with $V_1(f) = E_{\mathbb{P}_1}[u_1(f,\cdot)]$, where u_1 and $\mathbb{P}_1$ are given as in Proposition 3. We want to show that that $\succcurlyeq_{0,1}$ satisfies Axioms (T.0), (M.0), (ST.0) and (C.0).

Let $a \in \mathbb{R}$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$. Clearly either $u_0(a) \le V_1(f)$ or $u_0(a) \ge V_1(f)$, and therefore $\succcurlyeq_{0,1}$ is complete. Consider $a, b \in \mathbb{R}$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ satisfying $a \preccurlyeq_{0,1} f$ and $b \succcurlyeq_{0,1} f$. This means that $u_0(a) \le V_1(f) \le u_0(b)$. From the fact that u_0 is strictly increasing it follows that $b \ge a$, that is that $\succcurlyeq_{0,1}$ is transitive. Clearly $0 \sim_{0,1} 0$ since $u_0(0) = 0 = E_{\mathbb{P}_1}[u_1(0,\cdot)]$. Finally, let $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$. By assumption the range of V_1 is contained in the range of u_0 so that there exists $b \in \mathbb{R}$ such that $u_0(b) \ge V_1(f)$ and (equivalently) $b \succcurlyeq_{0,1} f$. For the same reason there exists $a \in \mathbb{R}$ such that $u_0(a) \le V_1(f)$, that is $a \preccurlyeq_{0,1} f$. This means that $\succcurlyeq_{0,1}$ is non-degenerate concluding the proof that Axiom (T.0) holds.

Let $a, b, c \in \mathbb{R}$ with $a < b$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_1})$ and $A \in \mathcal{F}_{t_1}$ being non-null. Suppose that $c \sim_{0,1} a\mathbf{1}_A + f\mathbf{1}_{A^C}$, that is $u_0(c) = E_{\mathbb{P}_1}[u_1(a\mathbf{1}_A + f\mathbf{1}_{A^C},\cdot)] = E_{\mathbb{P}_1}[u_1(a,\cdot)\mathbf{1}_A + u_1(f,\cdot)\mathbf{1}_{A^C})]$.
Now, since $u_1(\cdot,\omega)$ is strictly increasing for each ω, then $E_{\mathbb{P}_1}[u_1(a,\cdot)\mathbf{1}_A] < E_{\mathbb{P}_1}[u_1(b,\cdot)\mathbf{1}_A]$. Then $u_0(c) < E_{\mathbb{P}_1}[u_1(b,\cdot)\mathbf{1}_A + u_1(f,\cdot)\mathbf{1}_{A^C})] = E_{\mathbb{P}_1}[u_1(b\mathbf{1}_A + f\mathbf{1}_{A^C},\cdot)]$ which means $c \prec_{0,1} b\mathbf{1}_A + f\mathbf{1}_{A^C}$. The same argument can be used for $c \sim_{0,1} b\mathbf{1}_A + f\mathbf{1}_{A^C}$ leading to $c \succ_{0,1} a\mathbf{1}_A + f\mathbf{1}_{A^C}$. Thus, $\succcurlyeq_{0,1}$ satisfies (M.0).

(ST.0) follows from the simple fact that $E_{\mathbb{P}_1}[u_1(f\mathbf{1}_A + h\mathbf{1}_{A^c},\cdot)] \le E_{\mathbb{P}_1}[u_1(g\mathbf{1}_A + h\mathbf{1}_{A^c},\cdot)]$ if and only if $E_{\mathbb{P}_1}[u_1(f\mathbf{1}_A + k\mathbf{1}_{A^c},\cdot)] \le E_{\mathbb{P}_1}[u_1(g\mathbf{1}_A + k\mathbf{1}_{A^c},\cdot)]$ whatever the choice of $A \in \mathcal{F}_{t_1} \setminus \mathcal{N}(\mathcal{F}_{t_1})$ and $f, g, h, k \in \mathscr{L}(\mathcal{F}_{t_1})$.

Finally, let $(f_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_1})$ be uniformly bounded and converging pointwise to f for each $\omega \in \Omega$. Let $K := \sup_n \|f_n\|_\infty \in \mathbb{R}^+$. Since the integral representation is pointwise continuous (on uniformly bounded sequences) we have:

$$E_{\mathbb{P}_1}[u_1(f_n,\cdot)] \to E_{\mathbb{P}_1}[u_1(f,\cdot)] \tag{6.11}$$

Now let $a \in \mathbb{R}$ such that $a \prec_{0,1} f$ and call $\varepsilon := E_{\mathbb{P}_1}[u_1(f,\cdot)] - u_0(a) > 0$. Then by (6.11) there exists $N \in \mathbb{N}$ such that $|E_{\mathbb{P}_1}[u_1(f,\cdot)] - E_{\mathbb{P}_1}[u_1(f_n,\cdot)]| < \varepsilon$ $\forall n > N$. The triangular inequality implies that $u_0(a) < E_{\mathbb{P}_1}[u_1(f_n,\cdot)]$ $\forall n > N$, that is $a \prec_{0,1} f_n$. The same argument applies to $a \succ_{0,1} f$. Hence, (C.0) holds, concluding the proof.

On the direct implication ($\Rightarrow$). We shall proceed by induction. In fact if $N = 1$ Theorem 3 reduces to Proposition 3, which is proved in the first paragraph of this section.

Assumption 5 *[Induction] We assume that the statement is true up to i. In particular it means that we can guarantee the existence of a probability $\mathbb{P}_i$ on $\mathcal{F}_{t_i}$ and state-dependent utilities $\{u_k\}_{k=1}^i$, where $u_k(x,\cdot)$ is $\mathcal{F}_{t_k}$-measurable, integrable, strictly increasing in x, $\star$-continuous, $u_k(0,\cdot) = 0$ and*

$$g \succeq_{k-1,k} f \iff u_{k-1}(g) \ge E_{\mathbb{P}_i}[u_k(f)|\mathcal{F}_{t_{k-1}}] \quad \mathbb{P}_i\text{-a.s.}$$
$$g \preceq_{k-1,k} f \iff u_{k-1}(g) \le E_{\mathbb{P}_i}[u_k(f)|\mathcal{F}_{t_{k-1}}] \quad \mathbb{P}_i\text{-a.s.}$$

for any $k = 1, \ldots, i$, $f \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_{k-1}})$, $g \in \mathscr{L}^\infty(\Omega, \mathcal{F}_{t_k})$.

Under this assumption we shall now prove that the representation can be forwardly updated to time t_{i+1}.

Remark 9 We point out that $\mathcal{N}(\mathcal{F}_{t_i}) = \{A \mid \exists B \in \mathcal{F}_{t_i},\ \mathbb{P}_i(B) = 0 \text{ and } A \subseteq B\}$, where $\mathcal{N}(\mathcal{F}_{t_i})$ are the null sets induced by the relations $\preceq_{i-1,i}$, $\succeq_{i-1,i}$ as in (3.5).

Although a conditional preference is not total, the following lemma, which is inspired by Lemma 3.2 in [6], shows that local completeness allows to derive for every two acts an $\mathcal{F}_{t_i}$-measurable partition on which a comparison can be achieved.

Lemma 2 *Consider any $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$. If Assumption 5 holds and $\succeq_{i,i+1}$ satisfies (T.i) then there exists a pairwise disjoint family of events $A, B, C \in \mathcal{F}_{t_i}$ such that $\mathbb{P}_i(A \cup B \cup C) = 1$ and*

$$g\mathbf{1}_A \sim_{i,i+1} f\mathbf{1}_A,$$
$$g \succ^B_{i,i+1} f$$
$$g \prec^C_{i,i+1} f.$$

Proof Fix $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$, $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, and define $\mathcal{E} := \{\tilde{A} \in \mathcal{F}_{t_i} : g\mathbf{1}_{\tilde{A}} \sim_{i,i+1} f\mathbf{1}_{\tilde{A}}\}$, $S := \sup_{\tilde{A}\in\mathcal{E}} \mathbb{P}_i(\tilde{A})$. We can find $\{A_n\}_n \subseteq \mathcal{E}$ such that $\mathbb{P}_i(A_n) \to S$: we have $\mathbb{P}_i(\cup_n A_n) \geq \mathbb{P}_i(A_n)$ for every n which implies $\mathbb{P}_i(\cup_n A_n) = S$ (from (T.i) and Remark 2 we have $\cup_n A_n \in \mathcal{E}$). We finally show that up to null events $\cup_n A_n$ represents the largest event on which g is conditionally equivalent to f: let $\tilde{A} \in \mathcal{E}$ and $B = \tilde{A} \setminus (\cup_n A_n)$. Then $B \cup (\cup_n A_n) \in \mathcal{E}$ and $\mathbb{P}_i(B \cup (\cup_n A_n)) = \mathbb{P}_i(B) + S$. Necessarily $\mathbb{P}_i(B) = 0$.
We therefore set $A := \cup_n A_n$ and consider $\mathcal{U} := \{\tilde{B} \in \mathcal{F}_{t_i},\ \tilde{B} \subseteq A^c : g\mathbf{1}_{\tilde{B}} \succeq_{i,i+1} f\mathbf{1}_{\tilde{B}}\}$. Notice that from the construction of A if we find $\tilde{B} \in \mathcal{U}$ such that $g\mathbf{1}_{\tilde{B}} \sim_{i,i+1} f\mathbf{1}_{\tilde{B}}$ then $\mathbb{P}_i(\tilde{B}) = 0$. Following the same argument as in the previous step we construct a maximal $B \in \mathcal{U}$ such that $\mathbb{P}_i(B) \geq \mathbb{P}_i(\tilde{B})$ for all $\tilde{B} \in \mathcal{U}$: indeed it is not possible the finding of $B' \subset B$ with $\mathbb{P}_i(B') > 0$ such that $g\mathbf{1}_{B'} \preceq_{i,i+1} f\mathbf{1}_{B'}$, and therefore $g \succ^B_{i,i+1} f$.
Finally we can consider $\mathcal{D} := \{\tilde{C} \in \mathcal{F}_{t_i},\ \tilde{C} \subseteq (A \cup B)^c : g\mathbf{1}_{\tilde{C}} \preceq_{i,i+1} f\mathbf{1}_{\tilde{C}}\}$ and following the same reasoning we can find $C \in \mathcal{D}$ such that $\mathbb{P}_i(C) \geq \mathbb{P}_i(\tilde{C})$ for all $\tilde{C} \in \mathcal{D}$ and $g \prec^C_{i,i+1} f$.
By construction $\mathbb{P}_i(A \cup B \cup C) = 1$ and the probability of the intersections is always 0.

Consider for any $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ the upper and lower level sets $\mathcal{C}^u_g = \{f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}}) \mid g \preceq_{i,i+1} f\}$ and $\mathcal{C}^l_g = \{f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}}) \mid g \succeq_{i,i+1} f\}$ and the maps

$$V^-_{i+1}(f) = \mathbb{P}_i - \sup\{u_i(g) \mid f \in \mathcal{C}^u_g\} = \mathbb{P}_i - \sup\{u_i(g) \mid g \preceq_{i,i+1} f\}$$
$$V^+_{i+1}(f) = \mathbb{P}_i - \inf\{u_i(g) \mid f \in \mathcal{C}^l_g\} = \mathbb{P}_i - \inf\{u_i(g) \mid g \succeq_{i,i+1} f\}$$

Lemma 3 *Let Assumption 5 holds and $\succeq_{i,i+1}$ satisfies (T.i). The maps $V^+_{i+1}(f): \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}}) \to L^0(\Omega,\mathcal{F}_{t_i},\mathbb{P}_i)$, $V^-_{i+1}(f): \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}}) \to L^0(\Omega,\mathcal{F}_{t_i},\mathbb{P}_i)$ are well defined. Moreover, as $u_i(\omega,\cdot)$ is strictly increasing and $\star$-continuous (Assumption 5), then $V^+_{i+1}(f) = V^-_{i+1}(f)$ for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$.*

Notation 6 *We shall often use the notation $u_i^{-1}V_{i+1}(f)$ to indicate the function mapping $\omega \to u_i^{-1}(V_{i+1}(f)(\omega),\omega)$* [8].

Proof Let $g_1, g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $\mathbb{P}_i(g_1 = g_2) = 1$. We have from Remark 2 that $\mathcal{C}^u_{g_1} = \mathcal{C}^u_{g_2}$ and $\mathcal{C}^l_{g_1} = \mathcal{C}^l_{g_2}$, and therefore V^+_{i+1}, V^-_{i+1} are well defined.
From now on we fix $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$: for any $g_1, g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $g_1 \preceq_{i,i+1} f$ and $g_2 \succeq_{i,i+1} f$ then the set $\{g_1 > g_2\} \in \mathcal{N}(\mathcal{F}_{t_i})$. From the monotonicity of u_i we have $\mathbb{P}_i(u_i(g_1) \le u_i(g_2)) = 1$, and therefore $V^-_{i+1}(f) \le V^+_{i+1}(f)$, $\mathbb{P}_i$ almost surely.
To prove that $V^-_{i+1}(f) = V^+_{i+1}(f)$ we need to find $g_-, g_+ \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $u_i(g_\pm) = V^\pm_{i+1}(f)$ $\mathbb{P}_i - a.s.$. We prove the existence of g_+, then the same argument works also for g_-. Take a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(\mathcal{F}_{t_i})$ satisfying $g_n \succeq_{i,i+1} f$, $g_{n+1}(\omega) \le g_n(\omega)$ $\forall n \in \mathbb{N}, \omega \in \Omega$ and $u_i(g_n(\omega),\omega) \downarrow (V^+_{i+1}(f))(\omega)$ for each $\omega \in A$ for some event $A \in \mathcal{F}_{t_i}$ with $\mathbb{P}_i(A) = 1$. The existence of such sequence is guaranteed by the definition of $\mathbb{P}_i - \inf$ and the fact that the set $\{g \in \mathscr{L}^\infty(\mathcal{F}_{t_i}) : g \succeq_{i,i+1} f\}$ is downward directed [9]. Since the prefererence relation $\preceq_{i,i+1}$ is non degenerate, there exists an act $h \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $h \preceq_{i,i+1} f$ implying that the event $\{\omega \in \Omega : h(\omega) > g_n(\omega)\} \in \mathcal{F}_{t_i}$ is null for each $n \in \mathbb{N}$. This means that the sequence $(g_n)_n$ is decreasing and has a $\mathbb{P}_i - a.s.$ finite lower bound, so there exists an event $B \in \mathcal{F}_{t_i}$ with $\mathbb{P}_i(B) = 1$ and an act $g_+ \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $g_n(\omega) \downarrow g_+(\omega) \in \mathbb{R}$ for all $\omega \in B$. The $\star$-contintuity of u_i ensures that $g_n(\omega)$ and $g_+(\omega)$ belong to the points of (right) continuity of $u_i(\cdot,\omega)$ for each $\omega \in C$ for some $C \in \mathcal{F}_{t_i}$ with $\mathbb{P}_i(C) = 1$. This leads to:

$$(V^+_{i+1}(f))(\omega) = \lim_n u_i(g_n(\omega),\omega) = u_i(g_+(\omega),\omega)$$

for each $\omega \in A \cap B \cap C$ and $\mathbb{P}_i(A \cap B \cap C) = 1$.
Consider now $\bar{A} \in \mathcal{F}_{t_i}$ defined by $\bar{A} := \{g_- < g_+\}$. For $\lambda \in (0,1)$ define the convex combination $g_\lambda := \lambda g_+ + (1-\lambda) g_-$. Indeed $\bar{A} = \{g_\lambda < g_+\} = \{u_i(g_\lambda) < u_i(g_+)\} = \{g_\lambda > g_-\} = \{u_i(g_\lambda) > u_i(g_-)\}$.
Observe that if $\mathbb{P}_i(\bar{A}) = 0$ we have the thesis. Otherwise we claim that for any $B \subseteq \bar{A}$, $B \in \mathcal{F}_{t_i}$, $\mathbb{P}_i(B) > 0$ neither $g_\lambda \mathbf{1}_B \preceq_{i,i+1} f\mathbf{1}_B$ nor $g_\lambda \mathbf{1}_B \succeq_{i,i+1} f\mathbf{1}_B$ occur. This claim indeed contradicts local completeness in (T.i).
To show the claim we consider first the case $g_\lambda \mathbf{1}_B \preceq_{i,i+1} f\mathbf{1}_B$ for some $B \subseteq \bar{A}$, $B \in \mathcal{F}_{t_i}$ and $\mathbb{P}_i(B) > 0$, since the other follows in a similar way. As a consequence of Remark 2 we have $g_\lambda \mathbf{1}_B + g_- \mathbf{1}_{B^c} \preceq_{i,i+1} f$. From the construction $B \backslash \{u_i(g_\lambda \mathbf{1}_B + g_- \mathbf{1}_{B^c}) > V^-(f)\}$ is null, but from the definition $\mathbb{P}_i - \sup$ we necessarily have $\{u_i(g_\lambda \mathbf{1}_B + g_- \mathbf{1}_{B^c}) > V^-(f)\} \in \mathcal{N}(\mathcal{F}_{t_i})$. Hence, $g_\lambda \mathbf{1}_B \preceq_{i,i+1} f\mathbf{1}_B$

[8] This function is well defined and measurable as $u_i(\cdot,\omega)$ is strictly increasing for any $\omega \in \Omega$.

[9] A set $\mathcal{A}$ is downward directed if for any $f, g \in \mathcal{A}$ the minimum $f \wedge g \in \mathcal{A}$. The existence of a minimizing sequence is proved in Appendix A.5 of [12]

cannot occur for any for $B \subseteq \bar{A}$, $B \in \mathcal{F}_{t_i}$ and $\mathbb{P}_i(B) > 0$. Similarly we can obtain that $g_\lambda \mathbf{1}_B \succeq_{i,i+1} f\mathbf{1}_B$ cannot occur for any for $B \subseteq \bar{A}$, $B \in \mathcal{F}_{t_i}$ and $\mathbb{P}_i(B) > 0$, concluding the proof of the claim.

Notation 7 *From now on we shall denote* $V_{i+1} := V^+_{i+1} = V^-_{i+1}$.

Proposition 4 *Let Assumption 5 holds and* $\succeq_{i,i+1}$ *satisfies (T.i). Then for any* $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ *there exists a unique Conditional Certainty Equivalent given by* $C_{i,i+1}(f) = u_i^{-1} V_{i+1}(f) \in L^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P}_i)$. *Moreover* V_{i+1} *represents the transition order i.e.*

$$g \preceq_{i,i+1} f \Leftrightarrow u_i(g) \leq V_{i+1}(f) \quad \mathbb{P}_i\text{-a.s.} \tag{6.12}$$

$$g \succeq_{i,i+1} f \Leftrightarrow u_i(g) \geq V_{i+1}(f) \quad \mathbb{P}_i\text{-a.s.} \tag{6.13}$$

and necessarily $V_{i+1}(f) \in L^1(\Omega, \mathcal{F}_{t_i}, \mathbb{P}_i)$.

Proof In this proof we denote (with a slight abuse of notation) by $V_{i+1}(f)$ any of its $\mathcal{F}_{t_i}$-measurable version. Existence and uniqueness follow from the previous Lemma 3. We only need to show that $C_{i,i+1}(f) = u_i^{-1} V_{i+1}(f) \in L^\infty(\Omega, \mathcal{F}_{t_i}, \mathbb{P}_i)$. For any couple $g_1, g_2 \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $g_1 \preceq_{i,i+1} f$ and $g_2 \succeq_{i,i+1} f$ we can observe that $u_i(g_1) \leq V_{i+1}(f) \leq u_i(g_2)$, $\mathbb{P}_i$ almost surely, which automatically implies $V_{i+1}(f) \in L^1(\Omega, \mathcal{F}_{t_i}, \mathbb{P}_i)$ (we are assuming $u_i(\cdot, x)$ is integrable for any x). At the same time from u_i strictly increasing in x we can deduce $g_1 \leq C_{i,i+1}(f) \leq g_2$, $\mathbb{P}_i$ almost surely.

To show the representation property (6.12) and (6.13), we consider the case $g \preceq_{i,i+1} f$ as $g \succeq_{i,i+1} f$ follows in a similar fashion. Obviously $g \preceq_{i,i+1} f$ implies $\mathbb{P}_i(u_i(g) \leq V_{i+1}(f)) = 1$ (from the definition of $V^+_{i+1} = V_{i+1}$).
For the reverse implication notice that on the set $A = \{u_i(g) = V_{i+1}(f)\}$ we necessarily have $g\mathbf{1}_A \sim_{i,i+1} f\mathbf{1}_A$. If instead we consider $A = \{u_i(g) < V_{i+1}(f)\}$ then either $\mathbb{P}_i(A) = 0$ or necessary $g\mathbf{1}_A \preceq_{i,i+1} f\mathbf{1}_A$ and $g\mathbf{1}_B \nsim_{i,i+1} f\mathbf{1}_B$ for any $B \subset A$, $B \in \mathcal{F}_{t_i}$ as $V_{i+1}(f)$ is by definition $\mathbb{P}_i - \inf\{u_i(g) \mid g \succeq_{i,i+1} f\}$ (This can be easily verified applying (T.i)).

Corollary 1 *Let Assumption 5 holds and* $\succeq_{i,i+1}$ *satisfies (T.i). For any* $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ *and* $A \in \mathcal{F}_{t_i}$ *we have* $V_{i+1}(f\mathbf{1}_A) = V_{i+1}(f)\mathbf{1}_A$, $\mathbb{P}_i$ *almost surely.*

Proof From the previous construction we have $u_i^{-1} V_{i+1}(f\mathbf{1}_A) \sim_{i,i+1} f\mathbf{1}_A$. Moreover from (T.i) we also have that $u_i^{-1} V_{i+1}(f) \sim_{i,i+1} f$ implies

$$u_i^{-1} V_{i+1}(f)\mathbf{1}_A \sim_{i,i+1} f\mathbf{1}_A.$$

Hence, from transitivity we deduce $u_i^{-1} V_{i+1}(f)\mathbf{1}_A = u_i^{-1} V_{i+1}(f\mathbf{1}_A)$, $\mathbb{P}_i$ almost surely, and thus the thesis.

Remark 10 Let Assumption 5 holds and $\succeq_{i,i+1}$ satisfies (T.i). For any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ and $A \in \mathcal{F}_{t_i}$ we have $g \prec^A_{i,i+1} f$ (resp. $g \succ^A_{i,i+1} f$) implies $\{u_i(g) \geq V_{i+1}(f)\} \cap A \in \mathcal{N}(\mathcal{F}_{t_i})$ (resp. $\{u_i(g) \leq V_{i+1}(f)\} \cap A \in \mathcal{N}(\mathcal{F}_{t_i})$)

Last step of the proof for (⇒): Let Assumption 5 holds and $\succeq_{i,i+1}$ satisfies all the Axioms (T.i), (M.i), (ST.i) and (C.i). In order to conclude the proof we show that there exist a probability $\mathbb{P}_{i+1}$ on $(\Omega, \mathcal{F}_{t_{i+1}})$ which agrees with $\mathbb{P}_i$ on $\mathcal{F}_{t_i}$ and a state-dependent utility $u_{i+1}(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ strictly increasing $\forall \omega \in \Omega$, such that

$$V_{i+1}(f) = E_{\mathbb{P}_{i+1}}[u_{i+1}(\cdot, f) \mid \mathcal{F}_{t_i}] \quad \mathbb{P}_i\text{-a.s.}. \tag{6.14}$$

We define an intertemporal preference relation between time 0 and t_{i+1} as $a \preceq_{0,i+1} f$ (resp. $a \succeq_{0,i+1} f$) if and only if $u_0(a) \leq E_{\mathbb{P}_i}[V_{i+1}(f)]$ (resp. $u_0(a) \geq E_{\mathbb{P}_i}[V_{i+1}(f)]$) for any $a \in \mathbb{R}$ and $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$.
Simple inspections show that $\preceq_{0,i+1}$ satisfies (T.0), (M.0) and (ST.0).
We now prove the continuity (C.0) of $\preceq_{0,i+1}$: consider any uniformly bounded sequence $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$. Consider $a \prec_{0,i+1} f$ (the case $a \succ_{0,i+1} f$ follows in a similar way) so that we necessarily have $u_0(a) < E_{\mathbb{P}_i}[V_{i+1}(f)]$. It is possible to find $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i})$ such that $u_i(g) < V_{i+1}(f)$ and $u_0(a) < E_{\mathbb{P}_i}[u_i(g)]$ [10].
Since $g \prec_{i,i+1} f$ we apply (C.i) and find a sequence of indexes $\{n_k\}_{k=1}^\infty$ and a partition $\{A_k\}_{k=1}^\infty \subset \mathcal{F}_{t_i}$ such that for any k we have $g\mathbf{1}_{A_k} \preceq_{i,i+1} f_n\mathbf{1}_{A_k}$ for all $n \geq n_k$.

For $B_N = \cup_{i=1}^N A_i$ and $d = \sup_n \|f_n\|_\infty$ consider the CCE $C_{i,i+1}(-d)$. The sequence $\{u_i(g\mathbf{1}_{B_N} + C_{i,i+1}(-d)\mathbf{1}_{B_N^c}\}_{N\in\mathbb{N}}$ is dominated by the integrable function $|u_i(g)| + |u_i(C_{i,i+1}(-d))|$ and pointwise converges to $u_i(g)$. From Dominated Convergence Theorem we can find $\bar{N}$ such that

$$E_{\mathbb{P}_i}[u_i(g\mathbf{1}_{B_{\bar{N}}} + C_{i,i+1}(-d)\mathbf{1}_{B_{\bar{N}}^c})] > u_0(a),$$

so that from (T.i) we can deduce $u_i(g\mathbf{1}_{B_{\bar{N}}} + C_{i,i+1}(-d)\mathbf{1}_{B_{\bar{N}}^c}) \leq V_{i+1}(f_n\mathbf{1}_{B_{\bar{N}}} - d\mathbf{1}_{B_{\bar{N}}^c})$ for $n > \bar{N}$ and

$$E_{\mathbb{P}_i}[V_{i+1}(f_n)] \geq E_{\mathbb{P}_i}[V_{i+1}(f_n\mathbf{1}_{B_{\bar{N}}} - d\mathbf{1}_{B_{\bar{N}}^c})] > u_0(a), \quad \forall\, n > \bar{N},$$

which shows (C.0) of $\preceq_{0,i+1}$.

Given that $\preceq_{0,i+1}$ satisfies (T.0), (M.0), (ST.0) and (C.0) premise we can apply Proposition 3 and find a probability $\widetilde{\mathbb{P}}$ on $\mathcal{F}_{t_{i+1}}$ and a state-dependent utility $\widetilde{u}$ such that $E_{\mathbb{P}_i}[V_{i+1}(f)] = E_{\widetilde{\mathbb{P}}}[\widetilde{u}(f)]$ for any $f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$.
Notice from (T.i) point 3 that $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}_i$ on $\mathcal{F}_{t_i}$. For $\widetilde{\mathbb{P}}_{|\mathcal{F}_{t_i}}$ being the restriction of $\widetilde{\mathbb{P}}$ on $\mathcal{F}_{t_i}$ define $Z = \frac{d\mathbb{P}_i}{d\widetilde{\mathbb{P}}_{|\mathcal{F}_{t_i}}}$, which is an $\mathcal{F}_{t_i}$-measurable random variable. For any $A \in \mathcal{F}_{t_{i+1}}$ set $\mathbb{P}_{i+1}(A) := E_{\widetilde{\mathbb{P}}}[Z\mathbf{1}_A]$, $u_{i+1}(\omega, x) = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}_{i+1}}\widetilde{u}(\omega, x)$ and notice that $\mathbb{P}_{i+1}(A) = \mathbb{P}_i(A)$ for any $A \in \mathcal{F}_{t_i}$. We have

[10] To show the existence of such g we need to consider for any $\varepsilon > 0$, $C_{i,i+1}(f) - \varepsilon$ so that $u_i(C_{i,i+1}(f) - \varepsilon) < u_i(C_{i,i+1}(f)) = V_{i+1}(f)$; observing that $u_i(C_{i,i+1}(f) - \varepsilon)$ increases monotonically to $u_i(C_{i,i+1}(f))$ (for any $\omega \in \Omega$) we can find an $\bar{\varepsilon}$ such that $u_0(a) < E_{\mathbb{P}_i}[u_i(C_{i,i+1}(f) - \bar{\varepsilon})] < E_{\mathbb{P}_i}[u_i(C_{i,i+1}(f))] = E_{\mathbb{P}_i}[V_{i+1}(f)]$.

$$E_{\mathbb{P}_i}[V_{i+1}(f)] = E_{\widetilde{\mathbb{P}}}[\widetilde{u}(f)] = E_{\mathbb{P}_{i+1}}[u_{i+1}(f)] = E_{\mathbb{P}_i}[E_{\mathbb{P}_{i+1}}[u_{i+1}(f) \mid \mathcal{F}_{t_i}]],$$

so that we can obtain for every $A \in \mathcal{F}_{t_i}$ that

$$\begin{aligned} E_{\mathbb{P}_i}[V_{i+1}(f)\mathbf{1}_A] = E_{\mathbb{P}_i}[V_{i+1}(f\mathbf{1}_A)] &= E_{\mathbb{P}_i}[E_{\mathbb{P}_{i+1}}[u_{i+1}(f\mathbf{1}_A) \mid \mathcal{F}_{t_i}]] \\ &= E_{\mathbb{P}_i}[E_{\mathbb{P}_{i+1}}[u_{i+1}(f) \mid \mathcal{F}_{t_i}]\mathbf{1}_A], \end{aligned}$$

which implies the representation (6.14).

We finally show the $\star$-continuity of u_{i+1}. As for the unconditional case, it is enough to show that for each $f \in \mathscr{L}(\mathcal{F}_{t_{i+1}})$ it holds that $\mathbb{P}_{i+1}(LD_f) = 0$ where LD_f is defined in Appendix A with respect to the stochastic field u_{i+1}. Notice that, as consequence of Lemma 4, $\forall f \in \mathscr{L}(\mathcal{F}_{t_{i+1}})$ then $LD_f \in \mathcal{F}_{t_{i+1}}$, so either $\mathbb{P}_{i+1}(LD_f) = 0$ or $\mathbb{P}_{i+1}(LD_f) > 0$. By contradiction, we assume that there exists an act $f^* \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ for which $\mathbb{P}_{i+1}(LD_{f^*}) > 0$. In order to simplify the notation we set $B := LD_f$ and, since the probability $\mathbb{P}_{i+1}$ is fixed, we denote $\mathbb{P}_{i+1} - \sup(\mathcal{A})$ simply with $\sup(\mathcal{A})$ for any family $\mathcal{A} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$. Define $f := f^*\mathbf{1}_B$ and $f_n := \left(f - \frac{1}{n}\right)\mathbf{1}_B$ for each $n \in \mathbb{N}$. Clearly $f_n \to f$ in $\mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$, $\|f_n\| \leq \|f\| + 1 < +\infty$ $\forall n$ and $f_n(\omega) = f(\omega) = 0$ $\forall \omega \in B^C$. By definition of B, it hods that $u_{i+1}(f(\omega),\omega) > \sup_n u_{i+1}(f_n(\omega),\omega)$ for $\mathbb{P}_{i+1}$-a.e. $\omega \in B$ and, so, we have:

$$\mathbb{P}_i\left(\mathbb{E}_{\mathbb{P}_{i+1}}[u_{i+1}(f,\cdot)|\mathcal{F}_{t_i}] > \sup_n \mathbb{E}_{\mathbb{P}_{i+1}}[u_{i+1}(f_n,\cdot)|\mathcal{F}_{t_i}]\right) > 0 \tag{6.15}$$
$$\mathbb{E}_{\mathbb{P}_{i+1}}[u_{i+1}(f,\cdot)|\mathcal{F}_{t_i}] \geq \sup_n \mathbb{E}_{\mathbb{P}_{i+1}}[u_{i+1}(f_n,\cdot)|\mathcal{F}_{t_i}] \quad \mathbb{P}_{i+1} - a.s.$$

Define now $g_n := C_{i,i+1}(f_n)$ and $g := C_{i,i+1}(f)$. Observe that $\{g_n\}_n$ is an increasing sequence as $\{f_n\}_n$ increases and $u_i(\cdot,\omega)$ is strictly increasing for each ω by Assumption 5 and has g as upper bound. If $g_n(\omega) \to g(\omega)$ for $\mathbb{P}_i$-a.e. $\omega \in \Omega$ then, by the $\star$-continuity of u_i, it would happen that $u_i(g_n(\omega),\omega) \to u_i(g(\omega),\omega)$ for $\mathbb{P}_i$-a.e. $\omega \in \Omega$ in contradiction with (6.15). Hence, there exists $A \in \mathcal{F}_{t_i}$ with $\mathbb{P}_i(A) > 0$ such that $\sup_n g_n(\omega) < g(\omega)$ for each $\omega \in A$. Take now $\lambda \in (0,1)$ and consider $g_\lambda := \lambda g + (1-\lambda)\sup_n g_n$. It holds that:

$$\sup_n g_n(\omega) \leq g_\lambda \leq g(\omega) \quad \text{for } \mathbb{P}_i - a.e.\ \omega \in \Omega$$
$$\sup_n g_n(\omega) < g_\lambda < g(\omega) \quad \text{for each } \omega \in A$$

Therefore, it follows that $g_\lambda \succeq_{i,i+1} f_n$ and $g_\lambda \succ^A_{i,i+1} f_n$ $\forall n$, while $g_\lambda \preceq_{i,i+1} f$ and $g_\lambda \prec^A_{i,i+1} f$ which is in contradiction with axiom (C.i).

On the reverse implication ($\Leftarrow$). We now assume that there exist a probability $\mathbb{P}$ on $\mathcal{F}_{t_N}$ and a Stochastic Dynamic Utility $u(t,x,\omega)$ in the form of (3.6) with properties (a) (b) (c) and (d). Then, for any $i = 1,\ldots,N-1$, it is easy to show that the intertemporal preferences $\succeq_{i,i+1}, \preceq_{i,i+1}$ satisfy Axioms (T.i), (M.i), (ST.i), from the properties of the conditional expectation and the monotonicity of the Stochastic Dynamic Utility.

The only critical point is showing property (C.i). To this aim let $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ be a uniformly bounded sequence, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$. Choose any $g \prec_{i,i+1} f$ then necessarily $\mathbb{P}(u(t_i, g) \geq E_{\mathbb{P}}[u(t_{i+1}, f)|\mathcal{F}_{t_i}]) = 0$.
As $\sup_n \|f_n\|_\infty < d$ for some $d > 0$ we build the increasing sequence $l_n := \inf_{k\geq n} f_k \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ and notice $l_n \leq f_n$ and $l_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$. Moreover $\|l_n\|_\infty < d$ for all $n \in \mathbb{N}$ and consequently $|u(t_{i+1}, l_n)| \leq |u(t_{i+1}, d)|$ which is integrable. We can apply the Dominated Convergence Theorem for conditional expectation and obtain $E_{\mathbb{P}}[u(t_{i+1}, l_n)|\mathcal{F}_{t_i}](\omega) \to E_{\mathbb{P}}[u(t_{i+1}, f)|\mathcal{F}_{t_i}](\omega)$ for any $\omega \in \Omega$ (by choosing an opportune version of the conditional expectation). Consider the sequence of sets $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{t_i}$ defined by

$$B_k := \{u(t_i, g) < E_{\mathbb{P}}[u(t_{i+1}, l_k)|\mathcal{F}_{t_i}]\}.$$

Indeed $\cup_k B_k = \Omega$ from the pointwise convergence and we deduce that the pairwise disjoint family $A_1 := B_1, \dots, A_k := B_k \setminus (\cup_{i=1}^{k-1} A_i)$ satisfies again $\cup_k A_k = \Omega$, and therefore forms a partition of Ω. We conclude by observing that for any $n \geq k$ we have $f_n \geq l_k$, and therefore $u(t_i, g)(\omega) < E_{\mathbb{P}}[u(t_{i+1}, f_n)|\mathcal{F}_{t_i}](\omega)$ for any $\omega \in A_k$. Finally for every $n \geq k$ we deduce $g\mathbf{1}_{A_k} \preceq_{i,i+1} f_n\mathbf{1}_{A_k}$, as the follwoing identies $u(t_i, g)\mathbf{1}_{B_k} = u(t_i, g\mathbf{1}_{B_k})$ and $E_{\mathbb{P}}[u(t_{i+1}, f_n)|\mathcal{F}_{t_i}]\mathbf{1}_{B_k} = E_{\mathbb{P}}[u(t_{i+1}, f_n\mathbf{1}_{B_k})|\mathcal{F}_{t_i}]$ hold $\mathbb{P}$-a.s.. The argument repeats in the same way when $g \succ_{i,i+1} f$.

On the uniqueness. To conclude the proof we need to show the relative uniqueness. Consider the new couple $(\mathbb{P}^*, u^*)$ such that $\mathbb{P}$ is equivalent to $\mathbb{P}^*$ on $\mathcal{F}_{t_N}$ and for any $i = 1, \dots, N$ we have $\mathbb{P}(u^*(t_i, \cdot, \cdot) = \delta_i u_i) = 1$, where δ_i is the Radon-Nikodym derivative of $\mathbb{P}_{|\mathcal{F}_{t_i}}$ with respect to $\mathbb{P}^*_{|\mathcal{F}_{t_i}}$. We show for any arbitrary $i = 1, \dots, N-1$, $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i}), f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$ the first of the following equivalences

$$u^*(t_i, g) \geq E_{\mathbb{P}^*}[u^*(t_{i+1}, f)|\mathcal{F}_{t_i}] \quad \mathbb{P}^*\text{-a.s.} \iff u(t_i, g) \geq E_{\mathbb{P}}[u(t_{i+1}, f)|\mathcal{F}_{t_i}] \quad \mathbb{P}\text{-a.s.}$$
$$u^*(t_i, g) \leq E_{\mathbb{P}^*}[u^*(t_{i+1}, f)|\mathcal{F}_{t_i}] \quad \mathbb{P}^*\text{-a.s.} \iff u(t_i, g) \leq E_{\mathbb{P}}[u(t_{i+1}, f)|\mathcal{F}_{t_i}] \quad \mathbb{P}\text{-a.s.},$$

as the second one is similar. To this aim we recall the martingality property

$$\delta_i = E_{\mathbb{P}^*}\left[\frac{d\mathbb{P}}{d\mathbb{P}^*} \mid \mathcal{F}_{t_i}\right] = E_{\mathbb{P}^*_{|\mathcal{F}_{t_{i+1}}}}[\delta_{i+1} \mid \mathcal{F}_{t_i}] \quad \mathbb{P}^*\text{-a.s..}$$

and the conditional change of measure

$$\frac{E_{\mathbb{P}^*}[\delta_{i+1} u_{i+1}(f) \mid \mathcal{F}_{t_i}]}{E_{\mathbb{P}^*}[\delta_{i+1} \mid \mathcal{F}_{t_i}]} = E_{\mathbb{P}}[u_{i+1}(f) \mid \mathcal{F}_{t_i}] \quad \mathbb{P}\text{-a.s..} \tag{6.16}$$

Moreover the equivalence between $\mathbb{P}$ and $\mathbb{P}^*$ allows to write the following inequalities indifferently in the $\mathbb{P}/\mathbb{P}^*$ almost sure sense so that we obtain

$$\begin{aligned} u^*(t_i, g) \geq E_{\mathbb{P}^*}[u^*(t_{i+1}, f)|\mathcal{F}_{t_i}] &\iff \delta_i u_i(g) \geq E_{\mathbb{P}^*}[\delta_{i+1} u_{i+1}(f)|\mathcal{F}_{t_i}] \\ &\iff \delta_i u_i(g) \geq E_{\mathbb{P}}[u_{i+1}(f)|\mathcal{F}_{t_i}] \cdot E_{\mathbb{P}^*}[\delta_{i+1} \mid \mathcal{F}_{t_i}] \\ &\iff u(t_i, g) \geq E_{\mathbb{P}}[u(t_{i+1}, f)|\mathcal{F}_{t_i}]. \end{aligned}$$

On the contrary suppose that $(\mathbb{P}^*, u^*)$ are given in a way such that for $i = 1, \ldots, N$: $E_{\mathbb{P}^*}[|u^*(t_i, x, \cdot)|] < \infty$, for all $x \in \mathbb{R}$, $u^*(t_i, \cdot, \omega)$ is strictly increasing in x, $u^*(t_i, 0, \omega) = 0$ for all $\omega \in \Omega$ and

$$u^*(t_{i-1}, g) \geq E_{\mathbb{P}^*}[u^*(t_i, f)|\mathcal{F}_{t_{i-1}}]\, \mathbb{P}^*\text{-a.s.} \Leftrightarrow u(t_{i-1}, g) \geq E_{\mathbb{P}}[u(t_i, f)|\mathcal{F}_{t_{i-1}}]\, \mathbb{P}\text{-a.s.}$$
$$u^*(t_{i-1}, g) \leq E_{\mathbb{P}^*}[u^*(t_i, f)|\mathcal{F}_{t_{i-1}}]\, \mathbb{P}^*\text{-a.s.} \Leftrightarrow u(t_{i-1}, g) \leq E_{\mathbb{P}}[u(t_i, f)|\mathcal{F}_{t_{i-1}}]\, \mathbb{P}\text{-a.s.},$$

for any arbitrary $g \in \mathscr{L}^\infty(\mathcal{F}_{t_i}), f \in \mathscr{L}^\infty(\mathcal{F}_{t_{i+1}})$. The equivalence of $\mathbb{P}$ and $\mathbb{P}^*$ follows immediately. Moreover it is important to observe that the preferences $\succeq_{i-1,i}$ induced by $(\mathbb{P}, u)$ and $(\mathbb{P}^*, u^*)$ are the same and satisfy all the axioms (in virtue of the previous point of the proof), which in particular implies that the CCE always exists. Moreover for any $\omega \in \Omega$ we imposed $u(t_i, 0, \omega) = u^*(t_i, 0, \omega) = 0$. For $i = 1$ we already know $\mathbb{P}(u^*(t_1, \cdot, \cdot) = \delta_1 u_1) = 1$ from Proposition 3. Let $\delta_i = E_{\mathbb{P}^*}\left[\frac{d\mathbb{P}}{d\mathbb{P}^*} \mid \mathcal{F}_{t_i}\right]$ as before and consider the first $i = 2, \ldots, N$ such that either the set $A = \{\omega \in \Omega \mid u^*(t_i, \cdot, \omega) > \delta_i u_i(\cdot, \omega)\}$ or $A = \{\omega \in \Omega \mid u^*(t_i, \cdot, \omega) < \delta_i u_i(\cdot, \omega)\}$ have positive probability. Let $C_{i-1,i}(\mathbf{1}_A)$ be the CCE of $\mathbf{1}_A$, which is the equal under $(\mathbb{P}, u)$ or $(\mathbb{P}^*, u^*)$. Hence,

$$\begin{aligned} u^*(t_{i-1}, C_{i-1,i}(\mathbf{1}_A)) &= E_{\mathbb{P}^*}[u^*(t_i, \mathbf{1}_A)|\mathcal{F}_{t_{i-1}}] \quad \mathbb{P}^*\text{-a.s. and} \\ u(t_{i-1}, C_{i-1,i}(\mathbf{1}_A)) &= E_{\mathbb{P}}[u(t_i, \mathbf{1}_A)|\mathcal{F}_{t_{i-1}}] \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{6.17}$$

By performing a conditional change of measure as in (6.16), the second equation can be rewritten as

$$\delta_{i-1} u(t_{i-1}, C_{i-1,i}(\mathbf{1}_A)) = E_{\mathbb{P}^*}[\delta_i u(t_i, \mathbf{1}_A)|\mathcal{F}_{t_{i-1}}] \quad \mathbb{P}^*\text{-a.s..}$$

Subtracting this last equation and (6.17), would lead to a contradiction since the left hand side is always equal to 0 ($\mathbb{P}$-a.s.) whereas the right hand side is not. Therefore $\mathbb{P}(A)$ is necessarily 0.

A On $\star$-continuity

Throughout this section we fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a random field $\phi : \mathbb{R} \times \Omega \to \mathbb{R}$ such that for each $f \in \mathscr{L}^\infty(\mathcal{G})$ the map $\omega \mapsto \phi(f(\omega), \omega)$ is $\mathcal{G}$-measurable and for any ω, $x \mapsto \phi(x, \omega)$ is non decreasing. For any $f \in \mathscr{L}^\infty(\mathcal{G})$ we set

$$\phi(f(\omega)^+, \omega) = \inf_{n \in \mathbb{N}} \phi(f(\omega) + 1/n, \omega) \text{ and } \phi(f(\omega)^-, \omega) = \sup_{n \in \mathbb{N}} \phi(f(\omega) - 1/n, \omega)$$

and define the following sets:

$$\begin{aligned} RD_f &= \{\omega \in \Omega : \big(\phi(f(\omega)^+, \omega) - \phi(f(\omega), \omega)\big) > 0\} \\ LD_f &= \{\omega \in \Omega : \big(\phi(f(\omega), \omega) - \phi(f(\omega)^-, \omega)\big) > 0\} \\ D_f &= \{\omega \in \Omega : \big(\phi(f(\omega)^+, \omega) - \phi(f(\omega)^-, \omega)\big) > 0\} \end{aligned}$$

We now prove a useful lemma which allows to give a well-posed definition of continuity for random fields.

Lemma 4 *For each $f \in \mathscr{L}^\infty(\mathcal{G})$ the sets RD_f, LD_f, D_f, defined above, are $\mathcal{G}$-measurable.*

Proof Observe that the set RD_f can be written as:

$$\begin{aligned} RD_f &= \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \left\{\omega \in \Omega : \left(\phi\left(f(\omega)+\frac{1}{n},\omega\right)-\phi(f(\omega),\omega)\right) > \frac{1}{m}\right\} \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \left[\phi\left(f(\cdot)+\frac{1}{n},\cdot\right)-\phi\left(f(\cdot),\cdot\right)\right]^{-1}\left(\frac{1}{m},+\infty\right) \end{aligned}$$

which is $\mathcal{G}$-measurable by measurability of the function

$$\omega \to \phi_n(\omega) = \phi\left(f(\omega)+\frac{1}{n},\omega\right)-\phi(f(\omega),\omega).$$

Clearly a similar argument shows that $LD_f \in \mathcal{G}$. Finally, $D_f = LD_f \cup RD_f \in \mathcal{G}$.

Definition 3 The random fields ϕ is $\star$-continuous if $\mathbb{P}(D_f) = 0$ for every $f \in \mathscr{L}^\infty(\mathcal{G})$.

Remark 11 Observe that the set D_f defined in Lemma 4 can be interpreted as:

$$D_f = \{\omega \in \Omega : f(\omega) \text{ is a point of discontinuity of the function } \phi(\cdot,\omega)\}$$

In particular for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathscr{L}^\infty(\mathcal{G})$ such that $f_n(\omega) \to f(\omega)$ we have $\phi(f_n(\omega),\omega) \to \phi(f(\omega),\omega)$ for any $\omega \in D_f$. Moreover it follows that the definition of $\star$-continuity is well posed as the set D_f is measurable by Lemma 4.
Notice also that taking $f \equiv x \in \mathbb{R}$ then $D_x = \{\omega \in \Omega : \phi(\cdot,\omega) \text{ is discontinuous in } x\}$. Therefore, the condition $\mathbb{P}(D_x) = 0$ means that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the map $\phi(\cdot,\omega)$ is continuous in x . On the other hand, if ϕ is $\mathbb{P}-a.s.$ continuous and satisfies the measurability condition of Lemma 4 then it is also $\star$-continuous. Hence, the $\star$-continuity is a notion of continuity which is deeply related to the probability space (in particular, to the σ-algebra) and is weaker than the $\mathbb{P}$-a.s. continuity of the trajectories but stronger than the $\mathbb{P}$-a.s. continuity at fixed points.

B State dependent utilities

As in the rest of the paper $(\Omega, \mathcal{F})$ denotes a measurable space and $\mathscr{L}^\infty(\mathcal{F})$ is the space of all acts, represented by real valued $\mathcal{F}$-measurable and bounded random variables. We here use the term "act "in order to match the terminology adopted in [30] on which this Appendix is based. This term must be used with care in order to avoid confusion with the general notion of Anscombe-Aumann acts. Indeed in [1] acts are functions from the state space $(\Omega, \mathcal{F})$ to a convex set of lotteries over a consequence set.

In this appendix the preference relation is a binary relation $\succeq$ on $\mathscr{L}^\infty(\mathcal{F})$: for f, g $\in \mathscr{L}^\infty(\mathcal{F})$, if f is preferred to g, write $f \succeq g$. The preference relation satisfies the following axiom:

(A1) Preference order: if it is reflexive ($\forall f \in \mathscr{L}^\infty(\mathcal{F})$, $f \sim f$), complete ($\forall f, g \in \mathscr{L}^\infty(\mathcal{F})$, $f \succeq g$ or $f \preceq g$) and transitive ($\forall f, g, h \in \mathscr{L}^\infty(\mathcal{F})$ such that $f \succeq g$ and $g \succeq h$ then $f \succeq h$)

Definition 4 A representing function of the preference relation is a function $V : \mathscr{L}^\infty(\mathcal{F}) \to \mathbb{R}$ which is order-preserving, i.e.,

$$f \succeq g \iff V(f) \geq V(g).$$

We use the standard conventions: $f \preceq g$ if $g \succeq f$; $f \sim f$ if both $g \succeq f$ and $f \succeq g$; $g \nsim f$ if either $g \not\succeq f$ or $f \not\succeq g$; $g \succ f$ if $g \succeq f$ but $f \not\succeq g$.

Definition 5 An event $A \in \mathcal{F}$ is null if $f\mathbf{1}_A + g\mathbf{1}_{A^c} \sim g \ \forall f, g \in \mathscr{L}^\infty(\mathcal{F})$.We shall denote by $\mathcal{N}(\mathcal{F})$ be the set of null events.
As a consequence a $\succeq$-atom is an element $A \in \mathcal{F}$ such that for every $B \in \mathcal{F}$ with $\varnothing \neq B \subset A$ either B or $A \setminus B$ is null.
An event is essential if it belongs to $\mathcal{F} \setminus \mathcal{N}(\mathcal{F})$.

We can consider the following additional Axioms:

(A2) Strictly monotone if $x\mathbf{1}_A + f\mathbf{1}_{A^c} \succ y\mathbf{1}_A + f\mathbf{1}_{A^c}$, for all nonnull events $A \in \mathcal{F}$, for all $f \in \mathscr{L}^\infty(\mathcal{F})$ and outcomes $x > y$.
(A3) Sure-thing principle: consider arbitrary $f, g, h \in \mathscr{L}^\infty(\mathcal{F})$ and $A \in \mathcal{F}$ such that $f\mathbf{1}_A + h\mathbf{1}_{A^c} \preceq g\mathbf{1}_A + h\mathbf{1}_{A^c}$ then for every $c \in \mathscr{L}^\infty(\mathcal{F})$ we have $f\mathbf{1}_A + c\mathbf{1}_{A^c} \preceq g\mathbf{1}_A + c\mathbf{1}_{A^c}$.
(A3) holds on $\mathcal{S}(\mathcal{F})$ if we substitute in the previous statement $\mathscr{L}^\infty(\mathcal{F})$ with $\mathcal{S}(\mathcal{F})$ (as defined in the paragraph Notations).
(A4') Norm continuity if $\forall f \in \mathscr{L}^\infty(\mathcal{F})$ the sets $\{g \in \mathscr{L}^\infty(\mathcal{F}) : g \succeq f\}$ and $\{g \in \mathscr{L}^\infty(\mathcal{F}) : f \succeq g\}$ are $\|\cdot\|_\infty$-closed.

Theorem 8 (Debreu 1960, state-dependent expected utility for finite state space) *Let $\mathscr{L}^\infty(\mathcal{F})$ the set of acts and $\succeq$ a preference relation on it. Let the state space $\Omega = \{\omega_1, ..., \omega_n\}$, where at least three states are nonnull. Then the following two statements are equivalent:*

(i) *There exist n continuous functions $V_j : \mathbb{R} \to \mathbb{R}$, $j = 1, ..., n$, that are strictly increasing for all nonnull states and constant for all null states, and such that $\succeq$ is represented by*

$$V(f) = \sum_{j=1}^{n} V_j(f(\omega_j)). \tag{B.18}$$

(ii) *$\succeq$ is a norm continuous, strictly monotonic preference order that satisfies the sure thing principle.*

The following uniqueness holds for (1) : $W(f) = \sum_{j=1}^{n} W_j(f(\omega_j))$ represent $\succeq$ if and only if there exist $\tau_1, ..., \tau_n \in \mathbb{R}$ and $\sigma > 0$ such that $W_j = \tau_j + \sigma V_j \ \forall j$, implying that $W = \tau + \sigma V$ for $\tau = \tau_1 + ... + \tau_n$.

In [30] the previous Theorem is generalized to an infinite state spaces Ω when Ω contains no atoms. We here recall the integral reformulation of the Debreu representation given in [3] under pointwise continuity.

Definition 6 A preference order is

(A4) Pointwise continuous if for any uniformly bounded sequence $\{f_n\} \subseteq \mathscr{L}^\infty(\mathcal{F})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$ then $\forall g \in \mathscr{L}^\infty(\mathcal{F})$ such that $g \succ f$ (resp. $g \prec f$) $\exists J \in \mathbb{N}$ such that $g \succ f^j$ (resp. $g \prec f^j$) $\forall j > J$.

Theorem 9 ([30], Theorem 12 and [3], Theorem 5) *Let $\mathscr{L}^\infty(\mathcal{F})$ be the set of acts and $\succeq$ the preference relation on it. Assume that $\mathcal{F}$ contains at least three disjoint essential events. Then the following two statements are equivalent:*

(i) *There exists a countably additive measure $\mathbb{P}$ on Ω and a function (the state-dependent utility) $u(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ strictly increasing $\forall \omega \in \Omega$, such that $\succeq$ is represented by the pointwise continuous integral*

$$f \to \int_\Omega u(\omega, f(\omega)) d\mathbb{P}.$$

(ii) *$\succeq$ satisfies: (A1), (A2), (A3) on $\mathcal{S}(\mathcal{F})$, (A4).*

The following uniqueness holds: the couple $(\mathbb{P}, u)$ can be replaced by $(\mathbb{P}^*, u^*)$ if and only if $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent and $\mathbb{P}(u^* = \tau + \sigma\delta u) = 1$, where $\tau : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, $\sigma > 0$ and δ is the Radon-Nikodym density function of $\mathbb{P}$ with respect to $\mathbb{P}^*$.

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