ALMOST OVERCOMPLETE AND ALMOST OVERTOTAL SEQUENCES IN BANACH SPACES II

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Abstract

A sequence in a separable Banach space X $\langle$resp. in the dual space $X^*\rangle$ is said overcomplete (OC in short) $\langle$resp. overtotal (OT in short) on $X\rangle$ whenever the linear span of each subsequence is dense in X $\langle$resp. each subsequence is total on $X\rangle$. A sequence in a separable Banach space X $\langle$resp. in the dual space $X^*\rangle$ is said almost overcomplete (AOC in short) $\langle$resp. almost overtotal (AOT in short) on $X\rangle$ whenever the closed linear span of each subsequence has finite codimension in X $\langle$resp. the annihilator (in X) of each subsequence has finite dimension$\rangle$. We provide information about the structure of such sequences. In particular it can happen that, an AOC $\langle$resp. $AOT\rangle$ given sequence admits countably many not nested subsequences such that the only subspace contained in the closed linear span of every of such subsequences is the trivial one $\langle$resp. the closure of the linear span of the union of the annihilators in X of such subsequences is the whole $X\rangle$. Moreover, any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ contains some subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ that is OC in $[\{x_{n_j}\}_{j\in\mathbb{N}}]$; any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ contains some subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$.

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1 Introduction

Throughout the paper we use standard Geometry of Banach Spaces terminology and notation as in [3]. In particular:

- $[S]$ stands for the closure of the linear span of the set S;

- the annihilator in X^* of a subset Γ of the Banach space X is the subspace $\Gamma^\perp \subset X^*$ whose members are the bounded linear functionals on X that vanish on Γ;

- the annihilator in X of a subset Γ of the dual space X^* is the subspace $\Gamma^\top \subset X$, $\Gamma^\top = \cap_{f\in\Gamma} \ker f$;

- a set $\Gamma \subset X^*$ is called total over X whenever $\Gamma^\top = \{0\}$.

Recall that a sequence in a Banach space X is called overcomplete (OC in short) in X whenever the linear span of each of its subsequences is dense in X. It is a well-known fact that overcomplete sequences exist in any separable Banach space. On the basis of this notion, in [1] the first and the fourth authors introduced the following new notions.

- *A sequence in a Banach space X is called almost overcomplete (AOC in short) whenever the closed linear span of each of its subsequences has finite codimension in X.*

- *A sequence in the dual space X^* of the Banach space X is called overtotal on X (OT in short) whenever each of its subsequences is total over X.*

- *A sequence in the dual space X^* of the Banach space X is called almost overtotal (AOT in short) on X whenever the annihilator (in X) of each of its subsequences has finite dimension.*

In [1] some applications have been shown to support the usefulness of these notions. For instance, the fact that bounded AOC as well as AOT sequences must be strongly relatively compact makes it possible to answer quickly in the positive the following questions.

- Must any infinite-dimensional closed subspace of l_∞ contain infinitely many linearly independent elements with infinitely many zero-coordinates? (R. Aron and V.Gurariy, 2003; see Theorem 3.2 in [1].)

- Let $X \subset C(K)$ be an infinite-dimensional subspace of $C(K)$ where K is metric compact. Must a (infinite) sequence $\{t_k\}_{k\in\mathbb{N}}$ exist in K such that $x(t_k) = 0$ for infinitely many linearly independent $x \in X$? (See Theorem 3.1 in [1].)

This paper is a continuation of [1].

Our first aim is to provide information about the structure of AOC and AOT sequences. In particular, for any separable Banach space X the following questions seem to be of interest.

- *Does an AOC sequence exist in X that admits countably many subsequences such that the intersection of their closed linear spans is the origin?*

- *Does an AOT sequence exist on X that admits countably many subsequences such that the closure of the linear span of the union of their annihilators in X is the whole X?*

We answer in the positive both of them, respectively in Section 2 (Proposition 2.3) and Section 3 (Propositions 3.1). It is a remarkable fact that, in both cases, the involved subsequences cannot be nested (Propositions 2.5 and 3.3).

Our second aim is to give a possible explanation for the following fact. As a consequence of Theorem 3.3 of [1], by using strong relative compactness of bounded AOT sequences we get e. g., as a special case, that any infinite-dimensional closed subspace of l_p contains infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. However, the case $p < \infty$ looks much more complicated to be handled than the case $p = \infty$. In Section 4 we provide an example to show one possible reason for that.

We refer to [1] for general information about AOC and AOT sequences. Here we point out only the evident fact that, if $\{(x_n, x_n^*)\}$ is a countable biorthogonal system, then neither $\{x_n\}$ can be almost overcomplete in $[\{x_n\}]$, nor $\{x_n^*\}$ can be almost overtotal on $[\{x_n\}]$.

2 Almost overcomplete sequences

We start by recalling a simple method, due to Ju. Lyubich, to get an overcomplete sequence in any separable Banach space X. We will use it in the proof of Proposition 2.3.

Fact 2.1 *Let $\{e_k\}_{k\in\mathbb{N}}$ be any bounded sequence such that $[\{e_k\}_{k\in\mathbb{N}}] = X$. Then the sequence*

$$\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$$

is OC in X.

Proof Let $\{y_{m_j}\}_{j=1}^{\infty}$ be any subsequence of $\{y_m\}_{m=2}^{\infty} = \{\sum_{k=1}^{\infty} e_k m^{-k}\}_{m=2}^{\infty}$, let

$$f \in X^* \cap \{y_{m_j}\}^{\perp} \tag{1}$$

and let D be the open unit disk in the complex field. Since the complex function $\phi : D \to \mathbb{C}$ defined by $\phi(t) = \sum_{k=1}^{\infty} f(e_k)t^k$ is holomorphic, from $f(y_{m_j}) = \phi(1/m_j) = 0$ for $j = 1, 2, ...$, it follows $\phi \equiv 0$ that forces $f(e_k) = 0$ for every $k \in \mathbb{N}$. Since f in (1) was arbitrarily chosen, it follows $[\{y_{m_j}\}] = X$. ■

Remark 2.2 A formally different, but substantially equivalent, technique can be used to prove Fact 2.1: see for example the proof of Theorem 2.1.2 in [2]. We will use such technique in the second part of the proof of Proposition 2.3.

Proposition 2.3 *Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: for each $i \in \mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence, that we denote by $\{x_j^i\}_{j\in\mathbb{N}}$ to lighten notation, such that both the following conditions are satisfied*

a) $\text{codim}_X[\{x_j^i\}_{j\in\mathbb{N}}] = i$;

b) $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$.

Proof Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide a normalized M-basis for X. We recall that, by definition, the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ must be total on X. Moreover, it is a well known fact that, at least when A is a finite subset of $\mathbb{N}$, a (topological) complement in X to the subspace $[\{e_k\}_{k\in A}]$ is the subspace $[\{e_k\}_{k\in\mathbb{N}\setminus A}]$. For $i = 1, 2, ...$ put

$$Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,...,2i-1\}}] \tag{2}$$

so $\text{codim}_X Y_i = i$. For each integer $i \in \mathbb{N}$, Y_i is a Banach space itself so, by Fact 2.1, the sequence $\{y_m^i\}_{m\geq 2} \subset Y_i$ defined by

$$y_m^i = \sum_{k=1,k\notin\{i,i+1,i+2,...,2i-1\}}^{\infty} m^{-ik} e_k \qquad i = 1, 2, ...,\ m = 2, 3, ... \tag{3}$$

provides an *OC* sequence in Y_i.

Order in any way the countable set $\cup_{i\in\mathbb{N},m\geq 2}\{y_m^i\}$ as a sequence $\{x_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\{x_p^i\}_{p\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^i\}_{m\geq 2}$: this last sequence being *OC* in Y_i, we have $\text{codim}_X[\{x_p^i\}_{p\in\mathbb{N}}] = \text{codim}_X Y_i = i$. Moreover, since the sequence $\{e_k^*\}_{k\in\mathbb{N}}$ is total on X, it is clear that $\cap_{i=1}^{\infty} Y_i = \{0\}$, so $\cap_{i=1}^{\infty}[\{x_p^i\}_{p\in\mathbb{N}}] = \{0\}$ too.

It remains to show that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is *AOC* in X. Let $\{x_{n_j}\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

A) For some $\bar{i}$, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{\bar{i}}\}_{m\geq 2}$: being $\{y_m^{\bar{i}}\}_{m\geq 2}$ *OC* in $Y_{\bar{i}}$, we have $\text{codim}_X[\{x_{n_j}\}_{j\in\mathbb{N}}] \leq \text{codim}_X Y_{\bar{i}} = \bar{i}$ and we are done.

B) For each i, $\{x_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y^i_m\}_{m\geq 2}$. Take any

$$f \in \{x_{n_j}\}^{\perp}_{j\in\mathbb{N}}. \tag{4}$$

We prove that $f(e_k) = 0$ for every $k \in \mathbb{N}$: it implies $f = 0$, that means that $\{x_{n_j}\}_{j\in\mathbb{N}}$ is complete in X.

Suppose by contradiction that $f(e_{\overline{k}}) \neq 0$ for some index $\overline{k}$: without loss of generality we may assume that $\overline{k}$ is the first of such indexes. For $j \in \mathbb{N}$, let

$$y^{i(j)}_{m(j)} = x_{n_j};$$

put

$$A = \{i : i = i(j),\ j \in \mathbb{N},\ i(j) > \overline{k}\}.$$

Under our assumption $i(j)$ goes to infinity with j, so A is infinite and we have $e_{\overline{k}} \in Y_i$ for every $i \in A$. For $i \in A$, put

$$m_i = \min\{m(j) : i(j) = i,\ y^{i(j)}_{m(j)} \in \{y^i_m\}_{m\geq 2}\}.$$

From (4) it follows that, for each $i \in A$, we have

$$f(e_{\overline{k}}) = -m_i^{i\overline{k}} \sum_{k>\overline{k},\, k\notin\{i,i+1,i+2,\ldots,2i-1\}}^{\infty} m_i^{-ik} f(e_k) \tag{5}$$

hence

$$|f(e_{\overline{k}})| \leq m_i^{i\overline{k}}\, \|f\| \sum_{k>\overline{k},\, k\notin\{i,i+1,i+2,\ldots,2-1\}}^{\infty} m_i^{-ik} \leq \tag{6}$$

$$\leq \|f\| \sum_{k=\overline{k}+1}^{\infty} m_i^{i(\overline{k}-k)} \leq 2\|f\| m_i^{-i} \ \to 0 \text{ as } i \to \infty$$

that forces $f(e_{\overline{k}}) = 0$, so contradicting our assumption. We are done. ■

Our construction above can be modified by replacing (2) with

$$Y_i = [\{e_k\}_{k\neq i}] \tag{7}$$

and modifying (3), (5) and (6) according to that. In this case it is still true that $\bigcap[\{x_{n_j}\}_{j\in\mathbb{N}}] = \{0\}$ as $\{x_{n_j}\}_{j\in\mathbb{N}}$ ranges among all possible subsequences of the AOC sequence $\{x_n\}_{n\in\mathbb{N}}$, but actually the codimension of the closure of the linear span of any subsequence is at most 1. In other words, the following alternative version to Proposition 2.3 holds.

Proposition 2.4 *Any (infinite-dimensional) separable Banach space X contains an AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ with the following property: $\{x_n\}_{n\in\mathbb{N}}$ admits countably many subsequences $\{x_j^i\}_{j\in\mathbb{N}}$, $i = 1, 2, ...$, such that both the following conditions are satisfied*

a) $\operatorname{codim}_X[\{x_j^i\}_{j\in\mathbb{N}}] = 1$ *for each i;*

b) $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$.

By the previous Proposition, it is matter of evidence that actually the conclusion $\bigcap_{i\in\mathbb{N}}[\{x_j^i\}_{j\in\mathbb{N}}] = \{0\}$ is due to the fact that infinitely many pairwise "skew" subsequences can be found of $\{x_n\}_{n\in\mathbb{N}}$. This consideration is stressed by the following proposition.

Proposition 2.5 *Let $\{x_n\}_{n\in\mathbb{N}}$ be any AOC sequence in any (infinite-dimensional) separable Banach space X and let $\{x_j^1\}_{j\in\mathbb{N}} \supset \{x_j^2\}_{j\in\mathbb{N}} \supset \{x_j^3\}_{j\in\mathbb{N}} \supset ...$ any countable family of nested subsequences of $\{x_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\{\operatorname{codim}_X[\{x_j^i\}_{j\in\mathbb{N}}]\}_{i\in\mathbb{N}}$ is finite (so eventually constant).*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an *AOC* not *OC* sequence in X and let $\{x_j^1\}_{j\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X. Put

$$X_1 = [\{x_j^1\}_{j\in\mathbb{N}}], \qquad p_1 = \operatorname{codim}_X X_1 \geq 1.$$

If $\{x_j^1\}_{j\in\mathbb{N}}$ is *OC* in X_1 we are done; otherwise, let $\{x_{j_k}^1\}_{k\in\mathbb{N}}$ be any of its subsequences whose linear span is not dense in X_1. Put

$$\{x_{j_k}^1\}_{k\in\mathbb{N}} = \{x_j^2\}_{j\in\mathbb{N}}, \qquad X_2 = [\{x_j^2\}_{j\in\mathbb{N}}], \qquad p_2 = \operatorname{codim}_X X_2 > p_1.$$

Now we can continue in this way. Let us prove that this process must stop after finitely many steps. Assume the contrary, i.e. that a nested infinite family

$$\{x_j^1\}_{j\in\mathbb{N}} \supset \{x_j^2\}_{j\in\mathbb{N}} \supset ... \supset \{x_j^i\}_{j\in\mathbb{N}} \supset ...$$

of subsequences of $\{x_n\}_{n\in\mathbb{N}}$ can be found such that $p_i \uparrow \infty$ as $i \uparrow \infty$, where $p_i = \operatorname{codim}_X X_i$ with $X_i = [\{x_j^i\}_{j\in\mathbb{N}}]$.

Under this assumption, we can construct a linearly independent sequence $\{f_i\}_{i=1}^{\infty} \subset X^*$ such that, for each i, $f_i \in X_{i+1}^{\perp} \setminus X_i^{\perp}$. For each i, let y_i be an element of the sequence $\{x_j^i\}_{j\in\mathbb{N}}$ not belonging to the sequence $\{x_j^{i+1}\}_{j\in\mathbb{N}}$ such that $f_i(y_i) \neq 0$ (of course such an element must exist): because of our construction we have $f_k(y_i) = 0$ for each $k < i$. Without loss of generality we may assume $f_i(y_i) = 1$.

Now, following a standard procedure due to Markushevich, put

$$g_1 = f_1, \qquad g_2 = f_2 - f_2(y_1)g_1, \qquad g_3 = f_3 - f_3(y_1)g_1 - f_3(y_2)g_2, ...$$

$$..., \qquad g_k = f_k - \sum_{i=1}^{k-1} f_k(y_i) g_i, ... \ .$$

Clearly we have $g_k(y_i) = \delta_{k,i}$ for each $k, i \in \mathbb{N}$, so actually $\{y_k, g_k\}_{k\in\mathbb{N}}$ is a biorthogonal system with $\{y_k\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$. This is a contradiction since $\{x_n\}_{n\in\mathbb{N}}$ was an *AOC* sequence. ■

As an immediate consequence of Proposition 2.5 we get the following

Corollary 2.6 *Any AOC sequence $\{x_n\}_{n\in\mathbb{N}}$ in a separable Banach space X contains some subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ that is OC in $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ (with, of course, $[\{x_{n_j}\}_{j\in\mathbb{N}}]$ of finite codimension in X).*

3 Almost overtotal sequences

The results shown in the previous section about *AOC* sequences have a dual restatement for *AOT* sequences.

Proposition 3.1 *Let X be any (infinite-dimensional) separable Banach space. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ that is AOT on X and, for each $i \in \mathbb{N}$, admits a subsequence $\{f_j^i\}_{j\in\mathbb{N}}$ such that both the following conditions are satisfied*
a) $\dim\{f_j^i\}_{j\in\mathbb{N}}^\top = i$;
b) $[\bigcup_{i\in\mathbb{N}}\{f_j^i\}_{j\in\mathbb{N}}^\top] = X$.

Proof The idea for the proof is the same as for the proof of Proposition 2.3, so we confine ourselves to sketch the fundamental steps.

Let the biorthogonal system $\{e_k, e_k^*\}_{k\in\mathbb{N}} \subset X \times X^*$ provide an M-basis for X with $\{e_k^*\}_{k\in\mathbb{N}}$ a norm-one sequence in X^*. For $i = 1, 2, ...$ put

$$Z_i = [\{e_k\}_{k=i}^{2i-1}], \quad Y_i = [\{e_k\}_{k\notin\{i,i+1,i+2,...,2i-1\}}], \quad {}^*Y_i = [\{e_k^*\}_{k\notin\{i,i+1,i+2,...,2i-1\}}].$$

Clearly $X = Z_i \oplus Y_i$ and ${}^*Y_i^\top = Z_i$, so $\dim {}^*Y_i^\top = i$ for $i = 1, 2, ...$. For each integer $i \in \mathbb{N}$, the sequence $\{y_m^{*i}\}_{m\geq 2} \subset {}^*Y_i$ defined by

$$y_m^{*i} = \sum_{k=1, k\notin\{i,i+1,i+2,...,2i-1\}}^{\infty} m^{-ik} e_k^* \qquad i = 1, 2, ..., \ m = 2, 3, ...$$

being overcomplete in the Banach space ${}^*Y_i$, is overtotal on Y_i.

Order in any way the countable set $\cup_{i\in\mathbb{N},m\geq 2}\{y_m^{*i}\}$ as a sequence $\{f_n\}_{n\in\mathbb{N}}$. For each i, select a subsequence $\{f_p^i\}_{p\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ whose terms belong to $\{y_m^{*i}\}_{m\geq 2}$: since this last sequence is overtotal on Y_i, we have $\{f_p^i\}_{p\in\mathbb{N}}^{\top} = Z_i$ too, so $\dim\{f_p^i\}_{p\in\mathbb{N}}^{\top} = i$. Moreover, since the sequence $\{e_k\}_{k\in\mathbb{N}}$ is complete in X, we have $[\cup_{i=1}^{\infty} Z_i] = X$.

It remains to show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is AOT on X. Let $\{f_{n_j}\}_{j\in\mathbb{N}}$ be any of its subsequences. Two cases are possible.

A) For some $\bar{i}$, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains infinitely many terms from $\{y_m^{*\bar{i}}\}_{m\geq 2}$: being $\{y_m^{*\bar{i}}\}_{m\geq 2}$ OT on $Y_{\bar{i}}$, we have $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top} \subset Z_{\bar{i}}$, $\dim\{f_{n_j}\}_{j\in\mathbb{N}}^{\top} \leq \bar{i}$ and we are done.

B) For each i, $\{f_{n_j}\}_{j\in\mathbb{N}}$ contains at most finitely many terms from $\{y_m^{*i}\}_{m\geq 2}$. Take any $x \in \{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$: by proceeding exactly as in *B)* of the proof of Proposition 2.3, just interchanging the roles of points and functionals, we get $e_k^*(x) = 0$ for every $k \in \mathbb{N}$. $\{e_k^*\}_{k\in\mathbb{N}}$ being total on X, it follows $x = 0$. It means that $\{f_{n_j}\}_{j\in\mathbb{N}}$ too is total on X and again we are done.

The proof is complete. ■

As we did for AOC sequences, with obvious modifications in the previous proof we can obtain for AOT sequences the following alternative version to Proposition 3.1: it is the dual version to Proposition 2.4.

Proposition 3.2 *Let X be any (infinite-dimensional) separable Banach space. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ that is AOT on X and admits countably many subsequences $\{f_j^i\}_{j\in\mathbb{N}}$, $i = 1, 2, ...$, such that both the following conditions are satisfied*

a) $\dim\{f_j^i\}_{j\in\mathbb{N}}^{\top} = 1$ *for each i;*

b) $[\bigcup_{i\in\mathbb{N}}\{f_j^i\}_{j\in\mathbb{N}}^{\top}] = X$.

We point out that, though the existence of an AOT sequence on a Banach space X does not imply X to be separable (one of the significant applications of this concept we have shown in [1] was to the space l_∞), the results we have shown in Propositions 3.1 and 3.2, as they have been stated, must concern only separable spaces. In fact, the annihilator of any subsequence of any AOT sequence being finite-dimensional, the closed linear span of the union of countably many of such annihilators must be separable too.

Finally we notice that also Proposition 2.5 has its dual version that shows that the countably many subsequences in the statement of Proposition 3.2 cannot be assumed to be nested. The proof can be carried on exactly like the proof of Proposition 2.5, just interchanging the roles of points and functionals, so we omit it.

Proposition 3.3 *Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence AOT on any (infinite-dimensional) Banach space X and let $\{f_j^1\}_{j\in\mathbb{N}} \supset \{f_j^2\}_{j\in\mathbb{N}} \supset \{f_j^3\}_{j\in\mathbb{N}} \supset \ldots$ any countable family of nested subsequences of $\{f_n\}_{n\in\mathbb{N}}$. Then the increasing sequence of integers $\{\dim\{f_j^i\}_{j\in\mathbb{N}}^{\top}\}_{i\in\mathbb{N}}$ is finite (so eventually constant).*

As an immediate consequence of Proposition 3.3 we get the following

Corollary 3.4 *Any AOT sequence $\{f_n\}_{n\in\mathbb{N}}$ on a Banach space X contains some subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ that is OT on any subspace of X complemented to $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$ (with, of course, $\{f_{n_j}\}_{j\in\mathbb{N}}^{\top}$ of finite dimension).*

4 A counterexample on compact operators

This Section is devoted to provide an example that may be of interest in Operator theory. In [1] it was proved e. g. that any infinite-dimensional closed subspace of l_p contain infinitely many elements with infinitely many zero-coordinates not only when $p = \infty$, as we mentioned at the beginning, but for any $p \geq 1$. In fact the following much more general results have been proved there.

Theorem 4.1 ([1], Theorem 3.2) *Let X be a separable infinite-dimensional Banach space and $T : X \to l_\infty$ be a one-to-one bounded non compact linear operator. Then there exist an infinite-dimensional subspace $Y \subset X$ and a strictly increasing sequence $\{n_k\}$ of integers such that $e_{n_k}(Ty) = 0$ for any $y \in Y$ and for any k (e_n the "n-coordinate functional" on l_∞).*

Theorem 4.2 ([1], Theorem 3.3) *Let X, Y be infinite-dimensional Banach spaces. Let Y have an unconditional basis $\{u_i\}_{i=1}^{\infty}$ with $\{e_i\}_{i=1}^{\infty}$ as the sequence of the associated coordinate functionals. Let $T : X \to Y$ be a one-to-one bounded non compact linear operator. Then there exist an infinite-dimensional subspace $Z \subset X$ and a strictly increasing sequence $\{k_l\}$ of integers such that $e_{k_l}(Tz) = 0$ for any $z \in Z$ and any $l \in \mathbb{N}$.*

To prove both the Theorems, the fundamental tool was the fact that bounded *AOT* sequences are strongly relatively compact ([1], Theorem 2.3). However, despite Theorem 4.1 was then obtained as a quite easy consequence of the Ascoli-Arzelà Theorem, the proof of Theorem 4.2 has required some additional delicate tools. One could expect that Theorem 4.2 should be proved in a simple way by the following argument.

"Under notation as in the statement of Theorem 4.2, assume by contradiction that for each sequence of integers $\{i_j\}$ we have $\dim(\{T^*e_{i_j}\}^\top) < \infty$. Then the sequence $\{T^*e_i\} \subset X^*$ is almost overtotal on X, so $\{T^*e_i\}$ is relatively norm-compact in X^*. $\{e_i\}$ being the sequence of the coordinate functionals associated to the (unconditional) basis $\{u_i\}$ of Y, that forces T to be a compact operator, contradicting our assumption."

In fact this argument does not work since the last conclusion T being forced to be compact is false, as the following example shows.

Example 4.3 *There exist a Banach space Y with an unconditional basis $\{u_i\}_{i\in\mathbb{N}}$, $\{e_i\}_{i\in\mathbb{N}}$ being the sequence of the associated coordinate functionals, and a non-compact operator $T : c_0 \to Y$ such that $T^*e_i \to 0$ as $i \to \infty$ (so the sequence $\{T^*e_i\}$ is relatively norm compact).*

Proof. Let $\{u_i^k\}_{i=1}^k$ be the natural (algebraic) basis of $\mathbb{R}^k$. For $k \in \mathbb{N}$, define $T_k : \mathbb{R}^k \to \mathbb{R}^k$ in the following way

$$T_k(\sum_{i=1}^k a_i u_i^k) = \sum_{i=1}^k a_i u_i^k/k, \quad a_i \in \mathbb{R} \text{ for } i = 1, ..., k.$$

Let l_∞^k $\langle$resp. $l_1^k\rangle$ be the $k-$dimensional space $\mathbb{R}^k$ endowed with the max-norm $\langle$resp. the 1-norm$\rangle$. If we consider $T_k : l_\infty^k \to l_1^k$, we easily get $\|T_k\| = 1$ for every $k \in \mathbb{N}$.

For a sequence $\{X_k, \|\cdot\|_{X_k}\}_{k=1}^\infty$ of Banach spaces, consider the Banach space $(\oplus_{k=1}^\infty X_k)_{c_0}$ (the linear space, under the usual algebraic operations, whose elements are the sequences $\{x_k\}_{k=1}^\infty$, $x_k \in X_k$ for each k, such that $\|x_k\|_{X_k} \to 0$ as $k \to \infty$, endowed with the norm $\|\{x_k\}_{k=1}^\infty\| = \max_k \|x_k\|_{X_k}$).

Clearly we have

$$c_0 = (\oplus_{k=1}^\infty l_\infty^k)_{c_0}. \tag{8}$$

Put

$$Y = (\oplus_{k=1}^\infty l_1^k)_{c_0}.$$

Order the set $\cup_{k=1}^\infty \{u_i^k\}_{i=1}^k$ in the natural way and rename it as

$$\{u_1^1,\, u_1^2,\, u_2^2,\, ..., u_1^k, ...,\, u_k^k,\, ...\} = \{u_1,\, u_2,\, u_3,\, ...\}. \tag{9}$$

Of course $\{u_i\}_{i=1}^{\infty}$ is an unconditional basis both for c_0 and for Y. Call P_k the natural norm-one projection of c_0 onto l_{∞}^k suggested by (8) and define $T : c_0 \to Y$ in the following way

$$Tx = \sum_{i=0}^{\infty} T_k P_k x, \quad x \in c_0.$$

T is a (linear) non-compact operator, since $\|T(\sum_{i=1}^{k} u_i^k)\| = 1$ and $\sum_{i=1}^{k} u_i^k$ is weakly null as $k \to \infty$. However, if we denote by $\{e_i\}_{i=1}^{\infty}$ the sequence of the coordinate functionals associated to the basis $\{u_i\}_{i=1}^{\infty}$ of Y, it is true that $T^* e_i \to 0$ in X^* as $i \to \infty$. In fact, for $x = \sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k u_j^k \in B_{c_0}$ the following holds

$$|x_j^k| \leq 1 \quad 1 \leq j \leq k, \quad k = 1, 2, ...$$

so, if we denote by $u_{j_i}^{k_i}$ the element u_i as identified by (9), we have

$$|(T^* e_i)(x)| = |e_i(Tx)| = |e_i(\sum_{k=1}^{\infty} \sum_{j=1}^{k} x_j^k u_j^k / k)| = |x_{j_i}^{k_i}| / k_i \leq 1/k_i.$$

Since $k_i \to \infty$ with i, we are done. ■

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