

A SIMPLE CHARACTERIZATION OF DOUBLY TWISTED SPACETIMES

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ABSTRACT. In this note we characterize 1+n doubly twisted spacetimes in terms of ‘doubly torqued’ vector fields. They extend Bang-Yen Chen’s characterization of twisted and generalized Robertson-Walker spacetimes with torqued and concircular vector fields. The result is a simple classification of 1+n doubly-twisted, doubly-warped, twisted and generalized Robertson-Walker spacetimes.

1. INTRODUCTION

Several interesting Lorentzian metrics have a block-diagonal form, with time labelling a foliation with spacelike hypersurfaces. They include doubly-twisted, doubly-warped, twisted, warped, and Robertson-Walker spacetimes [1]. The Frobenius theorem characterizes the vector fields u_i that are hypersurface orthogonal, for which there exist functions λ and f such that, locally, $\lambda u_i = \nabla_i f$ ([2], p.19). This establishes a dual description of such spacetimes: the special form of the metric allows explicit evaluations, the one in terms of the vector field is covariant. While physicists conceive the vector field as a congruence of timelike trajectories, geometers prefer other vectors, as is here illustrated.

Doubly twisted spacetimes were introduced (and named ‘conformally separable’) by Kentaro Yano in 1940:

$$(1) \quad ds^2 = -b^2(\mathbf{q}, t)dt^2 + a^2(t, \mathbf{q})g_{\mu\nu}^*(\mathbf{q})dq^\mu dq^\nu$$

He showed that the metric structure is necessary and sufficient for the hypersurfaces to be totally umbilical [3]. The spacetime is doubly warped if b only depends on \mathbf{q} and a only depends on time t .

Ferrando, Presilla and Morales [4] proved that doubly twisted spacetimes are covariantly characterized by the existence of a timelike unit, shear and vorticity free vector field: $u^i u_i = -1$ and

$$(2) \quad \nabla_i u_j = \varphi(u_i u_j + g_{ij}) - u_i \dot{u}_j$$

where $\dot{u}_j = u^k \nabla_k u_j$ is the acceleration, and $\dot{u}_j u^j = 0$.

In 1979 Bang-Yen Chen introduced twisted spacetimes, eq.(1) with $b = 1$ [5]. Years later he characterized them through the existence of a timelike torqued vector field [6]:

$$(3) \quad \nabla_i \tau_j = \kappa g_{ij} + \alpha_i \tau_j, \quad \alpha_i \tau^i = 0$$

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where κ is a scalar field. We gave the equivalent description in terms of torse-forming time-like unit vectors, eq.(2) with $\dot{u}_i = 0$, and obtained the form of the Ricci tensor [7], and unicity of the vector, up to special cases [8].

Generalized Robertson-Walker (GRW) spacetimes were introduced in 1995 by Alías, Romero and Sánchez [9, 10]:

$$(4) \quad ds^2 = -dt^2 + a^2(t)g_{\mu\nu}^*(\mathbf{q})dq^\mu dq^\nu$$

Bang-Yen Chen characterized them through the existence of a timelike concircular vector field $\nabla_i \tau_j = \kappa g_{ij}$ [11, 12] (the statement can be weakened, see Prop.2.3).

We gave the alternative characterization (2) with $\dot{u}_i = 0$ and $\nabla_i \varphi = -u_i \dot{\varphi}$, and proved the useful property for the Weyl tensor [13]: $u_m C_{jkl}{}^m = 0$ if and only if $\nabla_m C_{jkl}{}^m = 0$. If the Weyl tensor is zero, the spacetime is Robertson-Walker.

All these cases constitute a rich family of manifolds which are mostly studied by geometers. They also appear in physics, as inhomogeneous extensions of the Robertson-Walker metric. In fact, the Einstein equations with a source of imperfect fluid with shear-free and irrotational velocity, lead to doubly-twisted metrics [14, 15, 16, 4, 17, 18]. The Stephani universes fall in this class [2, 19]. The requirement of geodesic flow specialises the metric to twisted, with interesting applications discussed by Coley and McManus [20]. Doubly warped and GRW (or warped) manifolds have an ample geometric literature [21, 22, 23, 24, 12].

In this note we present a simple characterization that includes all such spacetimes, and discuss some properties of doubly torqued vectors.

2. ANOTHER CHARACTERIZATION

Theorem 2.1. *A Lorentzian spacetime is doubly-twisted if and only if it admits a timelike vector field, which we name ‘doubly torqued’:*

$$(5) \quad \nabla_i \tau_j = \kappa g_{ij} + \alpha_i \tau_j + \tau_i \beta_j$$

with $\alpha_i \tau^i = 0$, $\beta_i \tau^i = 0$, and $n\kappa = \nabla_i \tau^i$.

Proof. We prove the equivalence of (5) with (2).

Let $N = \sqrt{-\tau^i \tau_i}$, and introduce the vector $u_i = \tau_i/N$. Evaluate: $\nabla_i N^2 = -2\tau^j \nabla_i \tau_j = -2\kappa \tau_i + 2\alpha_i N^2$. Then: $\nabla_i N = -\kappa u_i + \alpha_i N$. Next:

$$N \nabla_i u_j = \nabla_i \tau_j - u_j \nabla_i N = \kappa g_{ij} + N \alpha_i u_j + N u_i \beta_j + \kappa u_i u_j - N \alpha_i u_j$$

Therefore: $\nabla_i u_j = (\kappa/N)(u_i u_j + g_{ij}) + u_i \beta_j$. Contraction with u^i shows that $\beta_j = -\dot{u}_j$, and eq.(2) is obtained.

Given (2), the corresponding metric is (1). Define $\beta_j = -\dot{u}_j$ and $\tau_i = S u_i$, where S is to be found. Multiply eq.(2) by S :

$$\nabla_i (S u_j) - u_j \nabla_i S = \varphi S (u_i u_j + g_{ij}) + S u_i \beta_j$$

This is eq.(5) with the vector $\alpha_i = (\nabla_i S)/S + \varphi u_i$.

We must impose the condition: $0 = \alpha_i \tau^i = u^i \nabla_i S - S \varphi$ i.e. $\varphi = u^i \nabla_i \log S$.

In the frame (1) it is $u_0 = -b$, $u_\mu = 0$, and the condition becomes $\varphi = (\partial_t S)/(Sb)$. With the Christoffel symbols in appendix we obtain $\dot{u}_0 = u^0 (\partial_t - \Gamma_{00}^0) u_0 = 0$ and $\dot{u}_\mu = -u^0 \Gamma_{0\mu}^0 u_0 = b_\mu/b$. The time component of (2), $\nabla_0 u_0 = \varphi (u_0^2 + g_{00})$, gives $\varphi = (\partial_t a)/(ab)$. Therefore $S = a$ up to a constant factor. \square

There is some freedom in the choice of the doubly torqued vector: multiplication of eq.(5) by a function λ gives an equation for a vector $\lambda\tau_i$ that is orthogonal to the hypersurfaces and $\nabla_i(\lambda\tau_j) = (\lambda\kappa)g_{ij} + (\alpha_i + \partial_i\lambda/\lambda)(\lambda\tau_j) + (\lambda\tau_i)\beta_j$. It is doubly torqued provided that:

$$(6) \quad \tau^i \partial_i \lambda = 0$$

We show the relation of the special vectors α_i and β_i with the scale functions a and b of the metric (1).

In the coordinate frame (t, \mathbf{q}) , the vectors are $\tau_i = (\tau_0, 0)$, $\alpha_i = (0, \alpha_\mu)$ and $\beta_i = (0, \beta_\mu)$. The equations (5) are: $\partial_0\tau_0 - \Gamma_{00}^0\tau_0 = -\kappa b^2$, $\partial_\mu\tau_0 - \Gamma_{\mu 0}^0\tau_0 = \tau_0\alpha_\mu$, $-\Gamma_{0\mu}^0\tau_0 = \beta_\mu\tau_0$ and $-\Gamma_{\mu\nu}^0\tau_0 = \kappa a^2 g_{\mu\nu}^*$. They can be rewritten as follows:

$$(7) \quad \beta_\mu = -\partial_\mu \log b$$

$$(8) \quad \partial_t \tau_0 = \tau_0 \partial_t \log(ab)$$

$$(9) \quad \partial_\mu \tau_0 = \tau_0 (\alpha_\mu - \beta_\mu)$$

$$(10) \quad \kappa b = -\partial_t(\tau_0/b)$$

The second equation is integrated: $\tau_0(t, \mathbf{q}) = c(\mathbf{q})(ab)(t, \mathbf{q})$, where c is an arbitrary function. Then, the first and third equation give $\alpha_\mu = \partial_\mu \log(ca)$.

In coordinates (t, \mathbf{q}) the condition (6) is $\tau^0 \partial_t \lambda = 0$ and implies that λ does not depend on time t . The transformation $(\tau_0, \alpha_\mu, \beta_\mu, \kappa) \rightarrow (\lambda\tau_0, \alpha_\mu + \partial_\mu\lambda/\lambda, \beta_\mu, \lambda\kappa)$ leaves the above equations unchanged. This freedom is used to put $c(\mathbf{q}) = \pm 1$. Then: $\tau_0(t, \mathbf{q}) = -(ab)(t, \mathbf{q})$ (if $\tau^0 > 0$). The other equations give:

$$(11) \quad \alpha_\mu = \frac{\partial}{\partial q^\mu} \log a, \quad \beta_\mu = -\frac{\partial}{\partial q^\mu} \log b, \quad \kappa = \frac{1}{b} \frac{\partial a}{\partial t}$$

and establish a simple relation of the vectors with the metric. An interesting invariant is:

$$(12) \quad \tau^k \tau_k = -\frac{1}{b^2} \tau_0^2 = -a^2$$

For a doubly warped $1+n$ metric, the scale functions b does not depend on time and a does not depend on \mathbf{q} . In this case, the analysis shows that α_i is either zero or a gradient orthogonal to τ , that can be absorbed by a rescaling of τ .

We obtain the characterization:

Theorem 2.2. *A $1+n$ spacetime is doubly warped if and only if there is a timelike vector such that $\nabla_i \tau_j = \kappa g_{ij} + \tau_i \beta_j$ with $\tau^i \beta_i = 0$, and β_i is closed.*

Proof. In a doubly warped spacetime, $b(\mathbf{q})$ and $a(t)$ are given and specialize eq.(5). Eq.(11) gives: $\alpha_\mu = 0$ i.e. $\alpha_i = 0$. Eq.(7) with $\partial_t b = 0$ implies that β_i is closed.

If $\alpha_i = 0$ in (5), then eq.(9), $0 = \partial_\mu \tau_0 / \tau_0 - \partial_\mu \log b$, has solution $\tau_0(t, \mathbf{q}) = F(t)b(t, \mathbf{q})$. The result in (8) gives: $\partial_t \log a = \partial_t F/F$, so that $a = a(t)$. β_i closed becomes $\beta_\mu = -\partial_\mu r(\mathbf{q})$. Eq.(7) gives $b(t, \mathbf{q}) = \exp[r(\mathbf{q}) + s(t)]$. The metric $ds^2 = -e^{2r(\mathbf{q})} e^{2s(t)} dt^2 + a^2(t) g_{\mu\nu}^* dq^\mu dq^\nu$ is doubly warped with a redefinition of time. \square

In twisted $1+n$ spacetimes, a depends on t and \mathbf{q} , and $b = 1$. Then $\beta_i = 0$ and we recover Chen's result (3). In a GRW spacetime the scale function a only depends on t , so that also the vector α_i is zero.

Proposition 2.3. *In a doubly twisted spacetime with doubly-torqued vector τ_i , if $\beta_i = \nabla_i \theta$ and $\tau^i \nabla_i \theta = 0$ then the spacetime is conformally equivalent to a twisted spacetime.*

Proof. Consider the conformal map $\hat{g}_{ij} = e^{2\theta} g_{ij}$. The new Christoffel symbols are

$$\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k \partial_j \theta + \delta_j^k \partial_i \theta - g_{ij} g^{kl} \partial_l \theta$$

The vector $\hat{\tau}_i = e^\theta \tau_i$ solves:

$$\begin{aligned} \hat{\nabla}_i \hat{\tau}_j &= \nabla_i (e^\theta \tau_j) - \hat{\tau}_i \partial_j \theta - \hat{\tau}_j \partial_i \theta + g_{ij} g^{kl} \hat{\tau}_k \partial_l \theta \\ &= \hat{\tau}_j \partial_i \theta + e^\theta (\kappa g_{ij} + \alpha_i \tau_j + \tau_i \partial_j \theta) - \hat{\tau}_i \partial_j \theta - \hat{\tau}_j \partial_i \theta + \hat{g}_{ij} \hat{g}^{kl} \hat{\tau}_k \beta_l \\ &= (e^{-\theta} \kappa) \hat{g}_{ij} + \alpha_i \hat{\tau}_j \end{aligned}$$

The absence of β_i characterizes a twisted spacetime. \square

A consequence of the proof is the following statement (obvious if regarded on the side of the metric):

Proposition 2.4. *Conformal transformations $\hat{g}_{ij} = e^{2\theta} g_{ij}$ map doubly twisted to doubly twisted spacetimes.*

The same conformal transformation, with the condition $\tau^i \nabla_i \theta = 0$, maps doubly warped to doubly warped spacetimes, or twisted to twisted spacetimes.

Proof. Given the vector τ_i , consider the vector $\hat{\tau}_i = e^{2\theta} \tau_i$ in the new metric. It solves the equation of a doubly torqued vector:

$$\hat{\nabla}_i \hat{\tau}_j = (\kappa + \hat{\tau}^k \partial_k \theta) \hat{g}_{ij} + (\alpha_i + \hat{h}_i{}^k \partial_k \theta) \hat{\tau}_j + \hat{\tau}_i (\beta_j - \hat{h}_j{}^k \partial_k \theta)$$

where $\hat{h}_{ij} = \hat{g}_{ij} - \hat{\tau}_i \hat{\tau}_j / \hat{\tau}^2$ is a projection.

If instead the vector $\hat{\tau}_i = e^\theta \tau_i$ is considered, in the new metric it solves

$$\hat{\nabla}_i \hat{\tau}_j = (e^{-\theta} \kappa + \hat{\tau}^k \partial_k \theta) \hat{g}_{ij} + \alpha_i \hat{\tau}_j + \tau_i (\beta_j - \partial_j \theta)$$

If the space is doubly warped ($\alpha_i = 0$ and β closed) then, with the condition $\tau^i \partial_i \theta = 0$, the vector $\hat{\tau}_i$ solves the equation for a doubly warped spacetime. The same is true for twisted spacetimes ($\beta_i = 0$). \square

3. CONCLUSION

The introduction of doubly torqued vectors, defined by the equation (5), has the virtue of covariantly describing a class of $1 + n$ spacetimes in simple manner: doubly-twisted ($\alpha \neq 0$, $\beta \neq 0$), doubly-warped ($\alpha = 0$, $\beta \neq 0$ closed), twisted ($\alpha \neq 0$, $\beta = 0$), generalized Robertson-Walker ($\alpha = \beta = 0$).

We provide some examples from the physics literature.

Stephani universes [19] are conformally flat solutions of the Einstein field equations with a perfect fluid source. As in the Robertson-Walker space-time, the hypersurfaces orthogonal to the matter world-lines have constant curvature, but now its value k , and even its sign, changes from one hypersurface to another. The line element is doubly twisted:

$$ds^2 = -D(t, \mathbf{x}) dt^2 + \frac{R^2(t)}{V^2(t, \mathbf{x})} (dx^2 + dy^2 + dz^2)$$

with $V = 1 + \frac{1}{4} \|\mathbf{x} - \mathbf{x}_0(t)\|^2$, $D = F(t) [\dot{V}/V - \dot{R}/R]$, $k = [C^2(t) - 1/F^2(t)] R^2(t)$, arbitrary functions of time C , F , R and \mathbf{x}_0 .

Another example is the solution by Banerjee et al. [15] for a matter field with shear and vorticity free velocity and heat transfer, in a conformally flat metric:

$$ds^2 = -V^2(t, \mathbf{x})dt^2 + \frac{1}{U^2(t, \mathbf{x})}(dx^2 + dy^2 + dz^2)$$

with $UV = A(t)\|\mathbf{x}\|^2 + \mathbf{A}(t)\cdot\mathbf{x} + A_4(t)$, $U = B(t)\|\mathbf{x}\|^2 + \mathbf{B}(t)\cdot\mathbf{x} + B_4t$, where A , B , A_i and B_i are arbitrary functions of time.

Coley studied the case with no acceleration, $V = 1$, which makes the above metric twisted ([20] eq.3.12). The same paper contains an example of GRW spacetime (eq. 1.2), which is also Bianchi VI₀:

$$ds^2 = -dt^2 + X(t)^2(dx^2 + e^{-2x}dy^2 + e^{2x}dz^2)$$

For a discussion and examples of doubly warped metrics, see [21].

APPENDIX

These are the Christoffel symbols for the doubly-twisted metric (1):

$$\begin{aligned} \Gamma_{00}^0 &= \frac{\partial_t b}{b}, & \Gamma_{\mu,0}^0 &= \frac{b_{,\mu}}{b}, & \Gamma_{0,0}^\mu &= \frac{bb^{\mu}}{a^2}, & \Gamma_{\mu,0}^\rho &= \frac{\partial_t a}{a}\delta_\mu^\rho, & \Gamma_{\mu,\nu}^0 &= \frac{a\partial_t a}{b^2}g_{\mu\nu}^*, \\ \Gamma_{\mu,\nu}^\rho &= \Gamma_{\mu,\nu}^{*\rho} + \frac{a_{,\nu}}{a}\delta_\mu^\rho + \frac{a_{,\mu}}{a}\delta_\nu^\rho - \frac{a^\rho}{a}g_{\mu\nu}^* \end{aligned}$$

where $a_{,\mu} = \partial_\mu a$ and $a^\mu = g^{*\mu\nu}a_{,\nu}$, and the same is for b .

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