UNIVERSITÀ DEGLI STUDI DI MILANO

CORSO DI DOTTORATO IN SCIENZE MATEMATICHE
XXXIII CICLO

DIPARTIMENTO DI MATEMATICA “FEDERIGO ENRIQUES”

TESI DI DOTTORATO DI RICERCA

Global gradient bounds for solutions of prescribed mean curvature equations on Riemannian manifolds

SSD: MAT/03

Candidato: Giulio Colombo
Matricola: R12112

Tutor: Prof. Marco Rigoli
Coordinatore: Prof. Vieri Mastropietro

A. A. 2019–2020

Contents

CHAPTER 1

Introduction

This thesis is concerned with the study of qualitative properties of solutions of the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0 \tag{1}$$

and of a class of related prescribed mean curvature equations

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=f(x,u,\sqrt{1+|Du|^2}) \tag{2}$$

on complete Riemannian manifolds (M,σ). In particular, we derive global gradient bounds for non-negative (more generally, lower bounded) solutions of such equations under global uniform Ricci lower bounds on M, and we obtain Liouville-type theorems and other rigidity results on Riemannian manifolds with non-negative Ricci curvature. Results presented here have been obtained in collaboration with Marco Magliaro, Luciano Mari and Marco Rigoli, and in large part they appear in [**11**] and [**12**].

We recall some fundamental results on global solution of equation (1) on Euclidean spaces $M=\mathbb{R}^m$. In 1915, Bernstein [**5**] proved that the only solutions of (1) defined on the whole Euclidean plane $\mathbb{R}^2$ (entire solutions) are affine functions. His proof, later perfected by Hopf, [**27**] and Mickle, [**39**], was highly non-trivial and strongly relied on the geometric properties of $\mathbb{R}^2$. Since then, many authors investigated the validity of the analogue of Bernstein's result for higher dimensional Euclidean spaces $\mathbb{R}^m$, $m\geq 3$. By the late 60s, the following sharp form of Bernstein theorem had been extablished:

Entire solutions of (1) on $\mathbb{R}^m$ are affine if and only if $m\leq 7$

through the works of Fleming, [**24**] (new proof for $m=2$), De Giorgi, [**13**] ($m=3$), Almgren, [**2**] ($m=4$), Simons, [**49**] ($m\leq 7$) and Bombieri, De Giorgi, Giusti, [**7**] (counterexamples for $m\geq 8$). A wide variety of further counterexamples was given later by Simon, [**50**].

Further rigidity results have been obtained for solutions of (1) in $\mathbb{R}^m$ under additional a priori assumptions on u. For all dimensions $m\geq 2$, Bombieri, De Giorgi and Miranda, [**8**], obtained a local gradient estimate for minimal graphs $u: B_r(0)\subseteq\mathbb{R}^m\to\mathbb{R}$,

$$|Du(0)|\leq C_1\exp\left(C_2\frac{u(0)-\inf_{B_r}u}{r}\right) \tag{3}$$

with constants $C_i=C_i(m)$, $i=1,2$, thus extending previous results due to Finn, [**23**] and Jenkins, Serrin, [**30, 51**], for $m=2$. A Liouville theorem for equation (1) was then at hand:

Entire positive solutions of (1) on $\mathbb{R}^m$ are constant (for every $m\geq 2$).

Estimate (3) also implies that entire solutions of (1) with negative (or positive) part of at most linear growth have bounded gradient. Moser, [**43**], had previously established that entire solutions of (1) in $\mathbb{R}^m$ *with bounded gradient* are affine functions for every $m\geq 2$. This result is known as Moser's Bernstein theorem. The combination of these results then yielded:

> Entire solutions of (1) on $\mathbb{R}^m$ with at most linear growth on one side are affine functions.

Moser's Bernstein theorem has been sharpened in subsequent years by Bombieri and Giusti, [**9**], and by Farina, [**17**], [**18**], who succeeded in proving that an entire solution of (1) on $\mathbb{R}^m$, $m \geq 8$, is an affine function if and only if it has $m-7$ partial derivatives bounded on one side (not necessarily the same).

The original proof of (3) relied on integral estimates and Sobolev inequalities on minimal graphs due to Miranda, [**40**], [**41**], and based on isoperimetric inequalities for minimal currents in $\mathbb{R}^{m+1}$ introduced by Federer and Fleming, [**22**]. A simplified proof was later given by Trudinger, [**54**], and his technique allowed him ([**55**]) to obtain local gradient estimates of the form (3) also for solutions of the prescribed mean curvature equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = mH(x) \tag{4}$$

on $\mathbb{R}^m$, with constants C_1 and C_2 depending on the C^1 norm of $H \in C^1(\mathbb{R}^m)$. Later, Korevaar, [**31**], [**32**], [**34**], gave another proof of a (non-sharp) local gradient estimate for solutions of (4) using only elementary tools, namely, the finite maximum principle for C^2 functions. His technique also proved effective ([**33**]) in establishing a priori gradient estimates for solutions of equations of prescribed mean curvatures of higher orders.

In recent years, several authors have investigated the possible validity of similar rigidity and regularity results for solutions of equations (1) and (4) on Riemannian manifolds (M, σ), where D, $|\cdot|$ and div are interpreted as gradient, vector norm and divergence associated to the Riemannian metric σ. We recall some of them while presenting the original contributions of this work.

Let (M, σ) be a complete, noncompact Riemannian manifold of dimension $m \geq 2$ with Ricci curvature satisfying $\mathrm{Ric} \geq -(m-1)\kappa^2$ for some $\kappa \geq 0$. We show that entire, non-negative solutions $u : M \to \mathbb{R}^+_0$ of (1) satisfy the global gradient bound

$$\sqrt{1+|Du|^2} \leq e^{\sqrt{m-1}\kappa u} \qquad \text{in } M. \tag{5}$$

As a consequence, for $\kappa = 0$ we deduce the following Liouville-type theorem:

> On complete Riemannian manifolds with $\mathrm{Ric} \geq 0$ entire positive solutions of (1) are constant,

thus extending the aforementioned theorem of Bombieri, De Giorgi, Miranda for $M = \mathbb{R}^m$. The same Liouville-type theorem has been also proved very recently by Ding, [**14**], with completely different techniques. A previous result in this direction was obtained by Rosenberg, Schulze, Spruck, [**46**], under the additional assumption that the sectional curvatures of M are uniformly bounded from below by a negative constant. The gradient estimate (5) is inspired by the one obtained by Yau, [**56**],

$$|Du| \leq (m-1)\kappa u$$

for positive harmonic functions u on complete manifolds with $\mathrm{Ric} \geq -(m-1)\kappa^2$.

Our proof of (5) combines Yau's method for global gradient estimates with the ideas introduced by Korevaar. Yau's and Korevaar's methods are both based on applications of some form of the maximum principle to elliptic equations satisfied by suitable functions of u and $|Du|$. In particular, Korevaar's idea is to apply the finite maximum principle to the Jacobi equation satisfied by $1/\sqrt{1+|Du|^2}$, which involves the Laplace-Beltrami operator Δ_g associated to the graph metric $g = \sigma + \mathrm{d}u^2$. In case of non-compact manifolds, a preliminary localization is required, and this is usually done via cutoff functions obtained from the distance function r from a fixed point $o \in M$. To have a suitable control on second partial derivatives of r, and then on $\Delta_g r$, assumptions on sectional curvatures of M

are needed. Description of this construction is given in Section 3.2. In this work, we obtain the estimate (5) using as starting point, instead of r, a proper function $\psi : M \to [1, +\infty)$ satisfying $\Delta_g \psi \le \psi$, whose existence is obtained via a potential theory result due to Mari and D. Valtorta, [**38**], combined with an estimate on volume growth of geodesic balls in the metric g that is obtained via a calibration argument developed by Trudinger, [**54**]. This allows to suppress assumptions on sectional curvatures of M and to only assume $\mathrm{Ric} \ge -(m-1)\kappa^2$.

We show the validity of gradient bounds of exponential type

$$\sqrt{1+|Du|^2} \le Ae^{Cu}$$

also for non-negative solutions of a class of equations of the form (2) with constants $A \ge 1$, $C \ge 0$ depending on m, κ and on quantitative bounds on $|f|$ and its partial derivatives. The class of nonlinearities $f = f(x, y, w)$ that we consider is comprehensive of expressions of the form

$$f(x, u, \sqrt{1+|Du|^2}) = f_1(x, u) + \frac{f_2(x, u)}{\sqrt{1+|Du|^2}}$$

with $f_1, f_2 \in C^1(M \times \mathbb{R}^+_0)$ such that $|f_i|, |D_x f_i| \le C_0$, $\partial_y f_1 \ge 0$, $\partial_y f_2 \ge -C_0$ for some global constant $C_0 \ge 0$.

Our estimate can be localized on (not necessarily bounded) domains $\Omega \subseteq M$. More precisely, if M is a complete Riemannian manifold, $\Omega \subseteq M$ is an open set and $u \ge 0$ is a solution of (2) in Ω, then we prove

$$\sup_{\Omega} \frac{\sqrt{1+|Du|^2}}{e^{Cu}} \le \max\left\{ A, \limsup_{x \to \partial\Omega} \frac{\sqrt{1+|Du(x)|^2}}{e^{Cu(x)}} \right\} \tag{6}$$

under the assumption that $\mathrm{Ric} \ge -(m-1)\kappa^2$ in Ω and additional requirements on M, and possibly on $\partial\Omega$ or $u_{|\partial\Omega}$. In particular, the conclusion follows by assuming one of the following conditions:

(RΩ) for some $o \in M$ and $\alpha \ge 0$ it holds $\mathrm{Ric} \ge -\alpha^2(1+r^2)$, where $r(x) = \mathrm{dist}_\sigma(o, x)$ is the distance function from o in M and either

a) $u \in C^0(\overline{\Omega})$ and $u_{|\partial\Omega}$ is constant, or

b) $\partial\Omega$ is locally Lipschitz and

$$\liminf_{r \to +\infty} \frac{\log\left(\mathcal{H}^{m-1}(B_r(o) \cap \partial\Omega)\right)}{r^2} < +\infty$$

where $\mathcal{H}^{m-1}$ is $(m-1)$-dimensional Hausdorff measure, or

c) $u \in C^0(\overline{\Omega})$, $\partial\Omega$ is locally Lipschitz and for some $u_0 \in \mathbb{R}$

$$\liminf_{r \to +\infty} \frac{\log \displaystyle\int_{(\partial\Omega) \cap B_r} \min\{r, |u-u_0|\}\, \mathrm{d}\mathcal{H}^{m-1}}{r^2} < +\infty;$$

(K) for some $o \in M$ the sectional curvature K of M satisfies $K \ge -G(r)$ for some continuous, non-decreasing, strictly positive function $G : \mathbb{R}^+_0 \to \mathbb{R}^+$ such that $1/\sqrt{G} \notin L^1(+\infty)$.

Thanks to (6) and an original integral formula inspired by a similar one due to Farina and Valdinoci, [**21**], and later generalized by Farina, Mari, Valdinoci, [**19**], we also obtain the following rigidity result: Let $\Omega \subseteq M$ be a parabolic smooth domain of a complete Riemannian manifold M, let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of the overdetermined problem

$$\begin{cases} \mathrm{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \dfrac{f_2(u)}{\sqrt{1+|Du|^2}} & \text{in } \Omega, \\ u, \partial_\nu u & \text{locally constant on } \partial\Omega \end{cases} \tag{7}$$

for two functions $f_1, f_2 \in C^1(\mathbb{R})$ with $f_1' \geq 0$ and assume that M satisfies (RΩ) or (K) and that $\mathrm{Ric} \geq 0$ in Ω. If $\sup_\Omega |u| < +\infty$, $\sup_{\partial\Omega} |Du| < +\infty$ and $(Du, X) > 0$ in Ω for some Killing field $X \in \mathfrak{X}(\overline{\Omega})$, then Ω is isometric to a product $I \times N$, with $I \subseteq \mathbb{R}$ an interval and N a complete manifold with $\mathrm{Ric}_N \geq 0$, and u only depends on the I-variable. If f_1 and f_2 are constant and $\sup_\Omega |X| < +\infty$, then the conclusion follows by only assuming $(Du, X) \geq 0$, $\not\equiv 0$ on $\partial\Omega$, and if f_2 is a non-negative constant then it is enough to require $\inf_\Omega u > -\infty$. This happens, in particular, if the differential equation in (7) is the minimal surface equation or the constant mean curvature equation.

This result is comparable to others obtained by several authors for overdetermined problems for semilinear equations $\Delta u = f(u)$, in both cases $M = \mathbb{R}^m$ (see for instance [**20**], [**21**] and references therein) and M a Riemannian manifold with $\mathrm{Ric} \geq 0$ ([**19**]). To the best of our knowledge, our result for the differential equation in (7) is new even in cases $M = \mathbb{R}^2, \mathbb{R}^3$.

Our gradient estimate technique also allows to obtain the following generalization of the second aforementioned result of Bombieri, De Giorgi, Miranda: Let M be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and sectional curvature satisfying $K \geq -\alpha(1+r)^{-2}$ for some constant $\alpha \geq 0$, with $r(x) = \mathrm{dist}_\sigma(o, x)$ the distance function from an origin $o \in M$, and let u be a solution of

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \qquad \text{in } M.$$

If $u_-(x) = O(r(x))$ as $r(x) \to +\infty$, then $|Du|$ is bounded in M. If $u_-(x) = o(r(x))$ as $r(x) \to +\infty$, then u is constant. This extends a recent result by Ding, Jost, Xin, [**15**], where the same conclusion is reached with different techniques and more restrictive hypotheses, namely, a two-sided control $|K| \leq \alpha(1+r)^{-2}$ and an Euclidean volume growth condition

$$\lim_{r\to+\infty} \frac{|B_r(o)|}{r^m} > 0,$$

where $m = \dim M$.

CHAPTER 2

Preliminaries

1. Notation

Let (M, σ) be a Riemannian manifold of dimension m. The metric σ will also be denoted with $(\ ,\ )$. We let $|\cdot|$, D, div and Δ denote the vector norm, Levi-Civita connection, divergence and Laplace-Beltrami operator associated to σ. Let $\Omega \subseteq M$ be an open subset and $u : \Omega \to \mathbb{R}$ a twice differentiable function. The graph of u over M is the embedded C^2 hypersurface Σ of $M \times \mathbb{R}$ defined by

$$\Sigma = \Sigma_{u,\Omega} = \{(x, u(x)) \in M \times \mathbb{R} : x \in \Omega\}.$$

The graph map $\Gamma = \Gamma_{u,\Omega} : \Omega \to \Sigma : x \mapsto (x, u(x))$ is a C^2 diffeomorphism. Its inverse is the restriction $\pi_{|\Sigma}$ of the canonical projection $\pi : M \times \mathbb{R} \to M$.

The product manifold $M \times \mathbb{R}$ is given the Riemannian metric $\bar\sigma = \sigma + \mathrm{d}y \otimes \mathrm{d}y$, where y is the canonical coordinate on the $\mathbb{R}$ factor. The ambient metric $\bar\sigma$ induces a Riemannian metric g on Σ by restriction to $T\Sigma \otimes T\Sigma$, that is, by setting $g(X, Y) = \bar\sigma(X, Y)$ for every $X, Y \in T_p\Sigma$, $p \in \Sigma$. As a result, the inclusion map $(\Sigma, g) \hookrightarrow (M \times \mathbb{R}, \bar\sigma)$ is an isometric embedding. The resulting pullback metric on Ω via Γ is

$$\Gamma^* g = \sigma + \mathrm{d}u \otimes \mathrm{d}u.$$

The manifold $(\Omega, \Gamma^* g)$ is isometric to (Σ, g). We let $\|\cdot\|$, ∇, div_g and Δ_g denote the vector norm, Levi-Civita connection, divergence and Laplace-Beltrami operator associated to the metric g. In the following, if not otherwise stated we will regard $\|\cdot\|$, ∇, div_g and Δ_g as acting on functions, vectors or tensor fields defined on Ω, that is, we will almost exclusively work on the manifold $(\Omega, g) := (\Omega, \Gamma^* g)$ obtained by pulling back on Ω the graph metric g, instead of directly working on (Σ, g).

Let $\{x^i\}$ be a local coordinate system on Ω. We write

$$\sigma = \sigma_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad g \equiv \Gamma^* g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

For any function $\varphi \in C^1(\Omega)$ we also write

$$\mathrm{d}\varphi = \varphi_i\, \mathrm{d}x^i, \qquad D\varphi = \varphi^i \frac{\partial}{\partial x^i}$$

so we have that σ_{ij} and g_{ij} are related by

$$g_{ij} = \sigma_{ij} + u_i u_j \qquad \text{for } 1 \le i, j \le m. \tag{8}$$

Let σ^{ij} be the coefficients of the inverse matrix $(\sigma_{ij})^{-1}$, uniquely determined by

$$\sigma^{ik}\sigma_{kj} = \delta^i_j \qquad \text{for } 1 \le i, j \le m$$

with δ the Kronecker symbol. Then the coefficients of $\mathrm{d}\varphi$ and $D\varphi$ are related by

$$\varphi^i = \sigma^{ij}\varphi_j \quad \text{and} \quad \varphi_i = \sigma_{ij}\varphi^j \qquad \text{for } 1 \le i \le m.$$

Similarly, we let g^{ij} be the coefficients of $(g_{ij})^{-1}$, determined by the condition $g^{ik}g_{kj} = \delta^i_j$. A direct computation shows that

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2} \qquad \text{for } 1 \le i, j \le m$$

where

$$W = \sqrt{1 + |Du|^2}.$$

For every $\varphi \in C^1(\Omega)$ we denote by $\nabla\varphi$ its gradient with respect to g, uniquely determined by the condition $\langle \nabla\varphi, \cdot \rangle = \mathrm{d}\varphi$. In local coordinates we have

$$\nabla\varphi = g^{ij}\varphi_j \frac{\partial}{\partial x^i}$$

and by writing down

$$g^{ij}\varphi_j = \sigma^{ij}\varphi_j - \frac{u^i u^j \varphi_j}{W^2}$$

we deduce the intrinsic identity

$$\nabla\varphi = D\varphi - \frac{(Du, D\varphi)}{W^2} Du.$$

In particular, for $\varphi = u$ we get

$$g^{ij}u_j = \frac{u^i}{W^2}, \qquad \text{that is,} \qquad \nabla u = \frac{Du}{W^2}.$$

In general, we have validity of the chain of inequalities

$$\frac{|D\varphi|^2}{W^2} \le \|\nabla\varphi\|^2 \le |D\varphi|^2 \qquad \text{for every } \varphi \in C^1(\Omega). \tag{9}$$

We denote by γ_{ij}^k, Γ_{ij}^k the Christoffel symbols for the metrics σ, g, respectively, associated to local coordinates $\{x^i\}$. They are uniquely determined by conditions

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \gamma_{ij}^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} \qquad \text{for } 1 \le i, j \le m$$

and may be computed as

$$\gamma_{ij}^k = \frac{1}{2}\sigma^{kt}\left(\frac{\partial \sigma_{ti}}{\partial x^j} + \frac{\partial \sigma_{tj}}{\partial x^i} - \frac{\partial \sigma_{ij}}{\partial x^t}\right), \qquad \Gamma_{ij}^k = \frac{1}{2}g^{kt}\left(\frac{\partial g_{ti}}{\partial x^j} + \frac{\partial g_{tj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^t}\right). \tag{10}$$

In particular,

$$\gamma_{ij}^k = \gamma_{ji}^k, \qquad \Gamma_{ij}^k = \Gamma_{ji}^k \qquad \text{for } 1 \le i, j, k \le m. \tag{11}$$

The covariant derivative $D\alpha$ of a 1-form α is defined as the $(0, 2)$ tensor field given by

$$(D\alpha)(X, Y) = X(\alpha(Y)) - \alpha(D_X Y)$$

for every couple of vector fields X, Y. In particular, for 1-forms $\mathrm{d}x^i$ we obtain

$$D_{\partial_{x^i}} \mathrm{d}x^j = -\gamma_{ik}^j \mathrm{d}x^k.$$

More generally, the covariant derivative DT of a tensor field T of type (p, q), $p, q \ge 0$ is the $(p, q+1)$ tensor field given by

$$\begin{aligned}(DT)(X, X_1, \dots, X_q, \alpha^1, \dots, \alpha^p) &= X(T(X_1, \dots, X_q, \alpha^1, \dots, \alpha^p)) \\ &\quad - X\left(\sum_{i=1}^q T(\dots, D_X X_i, \dots, \alpha^1, \dots, \alpha^p)\right) \\ &\quad - X\left(\sum_{j=1}^p T(X_1, \dots, X_q, \dots, D_X \alpha^j, \dots)\right)\end{aligned}$$

for every choice of vector fields $X, X_1, \dots, X_q$ and 1-forms $\alpha^1, \dots, \alpha^p$. If T is expressed in local coordinates as

$$T = T^{i_1 \dots i_p}{}_{j_1 \dots j_q} \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_q} \otimes \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}}$$

then we will write

$$DT = T^{i_1 \dots i_p}{}_{j_1 \dots j_q k} \mathrm{d}x^k \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_q} \otimes \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}},$$

where the coefficients in the above expression are given by

$$T^{i_1\dots i_p}{}_{j_1\dots j_q k} = \frac{\partial}{\partial x^k} T^{i_1\dots i_p}{}_{j_1\dots j_q} - \sum_{s=1}^{q} T^{i_1\dots i_p}{}_{j_1\dots j_{s-1} l j_{s+1}\dots j_q} \gamma^l_{j_s k} + \sum_{t=1}^{p} T^{i_1\dots i_{t-1} l i_{t+1}\dots i_p}{}_{j_1\dots j_q} \gamma^{i_t}_{lk}.$$

Covariant derivatives of 1-forms and tensor fields with respect to the connection ∇ are defined similarly, so that we have

$$\begin{aligned}(\nabla\alpha)(X,Y) &= X(\alpha(Y)) - \alpha(\nabla_X Y),\\ (\nabla T)(X, X_1,\dots,X_q,\alpha^1,\dots,\alpha^p) &= X(T(X_1,\dots,X_q,\alpha^1,\dots,\alpha^p))\\ &\quad - X\left(\sum_{i=1}^{q} T(\dots,\nabla_X X_i,\dots,\alpha^1,\dots,\alpha^p)\right)\\ &\quad - X\left(\sum_{j=1}^{p} T(X_1,\dots,X_q,\dots,\nabla_X\alpha^j,\dots)\right)\end{aligned}$$

for 1-forms $\alpha, \alpha^1, \dots, \alpha^p$, vector fields $X, Y, Y_1, \dots, Y_q$ and (p,q)-type tensor field T. To avoid confusion with notation adopted for DT, in local coordinates the components of a covariant derivative ∇T will bear a semicolon ; as a separator between indices originally pertaining to T and the new lower index, that is, we will write

$$\nabla T = T^{i_1\dots i_p}{}_{j_1\dots j_q;k} \mathrm{d}x^k \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_q} \otimes \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}},$$

with

$$T^{i_1\dots i_p}{}_{j_1\dots j_q;k} = \frac{\partial}{\partial x^k} T^{i_1\dots i_p}{}_{j_1\dots j_q} - \sum_{s=1}^{q} T^{i_1\dots i_p}{}_{j_1\dots j_{s-1} l j_{s+1}\dots j_q} \Gamma^l_{j_s k} + \sum_{t=1}^{p} T^{i_1\dots i_{t-1} l i_{t+1}\dots i_p}{}_{j_1\dots j_q} \Gamma^{i_t}_{lk}.$$

For every $\varphi \in C^2(\Omega)$ the Hessians of φ with respect to the metrics σ and g are defined as the covariant derivatives of $\mathrm{d}\varphi$ with respect to connections D and ∇, respectively. We denote them as

$$\mathrm{Hess}_\sigma(\varphi) = D\mathrm{d}\varphi, \qquad \mathrm{Hess}_g(\varphi) = \nabla\mathrm{d}\varphi.$$

In local coordinates, we write

$$\mathrm{Hess}_\sigma(\varphi) = \varphi_{ij}\mathrm{d}x^j \otimes \mathrm{d}x^i, \qquad \mathrm{Hess}_g(\varphi) = \varphi_{i;j}\mathrm{d}x^j \otimes \mathrm{d}x^i,$$

with

$$\varphi_{ij} = \frac{\partial\varphi_i}{\partial x_j} - \varphi_k\gamma^k_{ij} \equiv \frac{\partial^2\varphi}{\partial x^i\partial x^j} - \frac{\partial\varphi}{\partial x^k}\gamma^k_{ij}, \qquad \varphi_{i;j} = \frac{\partial^2\varphi}{\partial x^i\partial x^j} - \frac{\partial\varphi}{\partial x^k}\Gamma^k_{ij}.$$

From the Schwarz lemma we have $\frac{\partial^2\varphi}{\partial x^i\partial x^j} = \frac{\partial^2\varphi}{\partial x^j\partial x^i}$, hence from (11)

$$\varphi_{ij} = \varphi_{ji}, \qquad \varphi_{i;j} = \varphi_{j;i}$$

and we can write as well

$$\mathrm{Hess}_\sigma(\varphi) = \varphi_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad \mathrm{Hess}_g(\varphi) = \varphi_{i;j}\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

We use (10) to derive a relation between γ_{ij}^k and Γ_{ij}^k. Substituting (8) and using the Schwarz lemma we obtain

$$\begin{aligned}\frac{\partial g_{ti}}{\partial x^j}+\frac{\partial g_{tj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^t}&=\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial^2 u}{\partial x^t\partial x^j}\frac{\partial u}{\partial x^i}+\frac{\partial u}{\partial x^t}\frac{\partial^2 u}{\partial x^i\partial x^j}\\&\quad+\frac{\partial \sigma_{tj}}{\partial x^i}+\frac{\partial^2 u}{\partial x^t\partial x^i}\frac{\partial u}{\partial x^j}+\frac{\partial u}{\partial x^t}\frac{\partial^2 u}{\partial x^j\partial x^i}\\&\quad-\frac{\partial \sigma_{ij}}{\partial x^t}-\frac{\partial^2 u}{\partial x^i\partial x^t}\frac{\partial u}{\partial x^j}-\frac{\partial u}{\partial x^i}\frac{\partial^2 u}{\partial x^j\partial x^t}\\&=\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial \sigma_{tj}}{\partial x^i}-\frac{\partial \sigma_{ij}}{\partial x^t}+2u_t\frac{\partial^2 u}{\partial x^i\partial x^j}\end{aligned}$$

and then

$$\begin{aligned}\Gamma_{ij}^k&=\frac{1}{2}g^{kt}\left(\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial \sigma_{tj}}{\partial x^i}-\frac{\partial \sigma_{ij}}{\partial x^t}+2u_t\frac{\partial^2 u}{\partial x^i\partial x^j}\right)\\&=\frac{1}{2}\left(\sigma^{kt}-\frac{u^k u^t}{W^2}\right)\left(\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial \sigma_{tj}}{\partial x^i}-\frac{\partial \sigma_{ij}}{\partial x^t}\right)+g^{kt}u_t\frac{\partial^2 u}{\partial x^i\partial x^j}\\&=\frac{1}{2}\sigma^{kt}\left(\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial \sigma_{tj}}{\partial x^i}-\frac{\partial \sigma_{ij}}{\partial x^t}\right)-\frac{1}{2}\frac{u^k}{W^2}u_l\sigma^{lt}\left(\frac{\partial \sigma_{ti}}{\partial x^j}+\frac{\partial \sigma_{tj}}{\partial x^i}-\frac{\partial \sigma_{ij}}{\partial x^t}\right)+\frac{u^k}{W^2}\frac{\partial^2 u}{\partial x^i\partial x^j}\\&=\gamma_{ij}^k+\frac{u^k}{W^2}\left(\frac{\partial^2 u}{\partial x^i\partial x^j}-u_l\gamma_{ij}^l\right).\end{aligned}$$

Observing that $\frac{\partial^2 u}{\partial x^i\partial x^j}-u_l\gamma_{ij}^l=u_{ij}$ are the coefficients of $\mathrm{Hess}_\sigma(u)$, we get

$$\Gamma_{ij}^k=\gamma_{ij}^k+\frac{u^k u_{ij}}{W^2}. \tag{12}$$

Hence, for every $\varphi\in C^2(M)$ we have

$$\varphi_{i;j}=\varphi_{ij}+\varphi_k(\gamma_{ij}^k-\Gamma_{ij}^k)=\varphi_{ij}-\frac{\varphi_k u^k}{W^2}u_{ij} \tag{13}$$

and in particular

$$u_{i;j}=\frac{u_{ij}}{W^2}. \tag{14}$$

The upward normal vector field to Σ in $M\times\mathbb{R}$ is given at any point $(x,u(x))\in\Sigma$ by

$$\mathbf{n}_{(x,u(x))}=\frac{\partial_y-Du(x)}{\sqrt{1+|Du(x)|^2}}.$$

Shortly, we write

$$\mathbf{n}=\frac{\partial_y-Du}{W}. \tag{15}$$

Let $\bar{D}$ denote the Levi-Civita connection of $(M\times\mathbb{R},\bar{\sigma})$. The second fundamental form II of the isometric immersion $(\Sigma,g)\hookrightarrow(M\times\mathbb{R},\bar{\sigma})$ is the tensor field $\mathrm{II}:T\Sigma\otimes T\Sigma\to T^\perp\Sigma$ defined by

$$\mathrm{II}(X,Y)=\bar{D}_XY-\nabla_XY$$

for any couple of vector fields $X,Y\in(\Sigma)$. The trace of II with respect to the metric g is the non-normalized mean curvature vector $m\mathbf{H}=\mathrm{Tr}_g(\mathrm{II})$, and the unique function $H:\Sigma\to\mathbb{R}$ such that

$$\mathbf{H}=H\mathbf{n}$$

is the mean curvature (function) of Σ in the direction of $\mathbf{n}$.

A local frame for $T\Sigma$ is given by the collection of vector fields

$$E_i=\frac{\partial}{\partial x^i}+u_i\partial_y \qquad \text{for } 1\le i\le m$$

obtained by pulling back to Σ the local frame $\{\partial_{x^i}\}_{1\le i\le m}$ for Ω via the diffeomorphism $\pi_{|\Sigma}:\Sigma\to\Omega$. The local coframe $\{\omega^i\}$ dual to $\{E_i\}$ is given by

$$\omega^i = \mathrm{d}x^i + u^i\mathrm{d}y \qquad \text{for } 1\le i\le m$$

and is similarly obtained by pulling back the coframe $\{\mathrm{d}x^i\}$. Since $\pi:(\Sigma,g)\to(\Omega,g)$ is an isometry and its differential maps the local frame $\{E_i\}$ to $\{\partial_{x^i}\}$, we have

$$g(E_i,E_j)=g_{ij}, \qquad g=g_{ij}\,\omega^i\otimes\omega^j, \qquad \nabla_{E_i}E_j=\Gamma^k_{ij}E_k.$$

If $U\subseteq\Omega$ is the domain of the local chart $\{x^i\}$, we can extend the vector fields E_i to the cylinder $U\times\mathbb{R}$ by setting $E_i(x,y)=\partial_{x^i}+u_i(x)\partial_y$ for every $(x,y)\in U\times\mathbb{R}$. In this way, we have $\bar{D}_{\partial_y}E_i=0$ for every $1\le i\le m$ and then we can compute

$$\begin{aligned}
\bar{D}_{E_i}E_j=\bar{D}_{\partial_{x^i}}E_j &= D_{\partial_{x^i}}\partial_{x^j}+\frac{\partial u_j}{\partial x^i}\partial_y\\
&=\gamma^k_{ij}\partial_{x^k}+\frac{\partial u_j}{\partial x^i}\partial_y\\
&=\gamma^k_{ij}E_k-\gamma^k_{ij}u_k\partial_y+\frac{\partial u_i}{\partial x^j}\partial_y\\
&=\gamma^k_{ij}E_k+u_{ij}\partial_y
\end{aligned}$$

for every $1\le i,j\le m$. From this we get

$$\begin{aligned}
\mathrm{II}(E_i,E_j)=\bar{D}_{E_i}E_j-\nabla_{E_i}E_j &= (\gamma^k_{ij}-\Gamma^k_{ij})E_k+u_{ij}\partial_y\\
&=-\frac{u^k u_{ij}}{W^2}E_k+u_{ij}\partial_y\\
&=u_{ij}\left(-\frac{u^k\partial_{x^k}}{W^2}-\frac{u^k u_k}{W^2}\partial_y+\partial_y\right)\\
&=u_{ij}\frac{-Du+\partial_y}{W^2}\\
&=\frac{u_{ij}}{W}\mathbf{n}
\end{aligned}$$

and then we can locally express II as

$$\mathrm{II}=\mathrm{II}_{ij}\,\omega^i\otimes\omega^j\otimes\mathbf{n} \qquad\text{with}\qquad \mathrm{II}_{ij}=\frac{u_{ij}}{W}. \tag{16}$$

This yields

$$mH=g^{ij}\mathrm{II}_{ij}=\frac{g^{ij}u_{ij}}{W}. \tag{17}$$

The non-parametric form of the mean curvature equation,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=mH \tag{18}$$

is easily deduced from (17). Indeed, setting $X=\frac{Du}{W}$ and locally writing $X=X^i\partial_{x^i}$, $DX=X^i_{\ j}\mathrm{d}x^j\otimes\partial_{x^i}$, $\mathrm{d}W=W_i\mathrm{d}x^i$, we have

$$\begin{aligned}
W_i&=\frac{u_{ik}u^k}{W},\\
X^i_{\ j}&=\frac{u^i_{\ j}}{W}-\frac{u^iW_j}{W^2}=\frac{1}{W}\left(\sigma^{it}u_{tj}-\frac{u_{jk}u^iu^k}{W^2}\right),\\
\operatorname{div}(X)=\delta^j_iX^i_{\ j}&=\frac{1}{W}\left(\sigma^{it}u_{ti}-\frac{u_{ik}u^iu^k}{W^2}\right)=\frac{1}{W}\left(\sigma^{ij}-\frac{u^iu^j}{W^2}\right)u_{ij}=\frac{g^{ij}u_{ij}}{W}=mH.
\end{aligned}$$

Going back to (12), we can write

$$\Gamma^k_{ij} = \gamma^k_{ij} + \frac{u^k}{W}\mathrm{II}_{ij}$$

and from (13) for every $\varphi \in C^2(\Omega)$ we have

$$\varphi_{i;j} = \varphi_{ij} - \frac{\varphi_k u^k}{W}\mathrm{II}_{ij} \tag{19}$$

and then

$$\Delta_g \varphi = g^{ij}\varphi_{i;j} = g^{ij}\varphi_{ij} - mH\frac{\varphi_k u^k}{W}. \tag{20}$$

For $\varphi = u$, a combination of (17) and (20) yields the parametric form of the mean curvature equation,

$$\Delta_g u = \frac{mH}{W}. \tag{21}$$

We denote the Riemann curvature operators associated to D and ∇ as R and ${}^\nabla R$, respectively. They are tensors of type $(1,3)$ and their action is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,$$
$${}^\nabla R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

for all $X, Y, Z \in \mathfrak{X}(\Omega)$, where $[\cdot,\cdot]$ is the Lie bracket. The components of R are given by

$$R^i{}_{jkt} = \frac{\partial \gamma^i_{jt}}{\partial x^k} + \gamma^i_{sk}\gamma^s_{jt} - \frac{\partial \gamma^i_{jk}}{\partial x^t} - \gamma^i_{st}\gamma^s_{jk}.$$

with the convention that $R(X,Y)Z = R^i{}_{jkt}X^kY^tZ^j\partial_{x^i}$ for every $X = X^i\partial_{x^i}$, $Y = Y^i\partial_{x^i}$, $Z = Z^i\partial_{x^i}$. Similarly, the components of ${}^\nabla R$ are given by

$${}^\nabla R^i{}_{jkt} = \frac{\partial \Gamma^i_{jt}}{\partial x^k} + \Gamma^i_{sk}\Gamma^s_{jt} - \frac{\partial \Gamma^i_{jk}}{\partial x^t} - \Gamma^i_{st}\Gamma^s_{jk}$$

with the agreement that ${}^\nabla R(X,Y)Z = {}^\nabla R^i{}_{jkt}X^kY^tZ^j\partial_{x^i}$.

LEMMA 2.1 (Ricci's commutation relations). *Let $\alpha \in \omega(M)$ be a 1-form on M and let $D^2\alpha = D(D\alpha)$ be its second covariant derivative. For every $X, Y, Z \in \mathfrak{X}(M)$*

$$(D^2\alpha)(Y,X,Z) - (D^2\alpha)(X,Y,Z) = \alpha(R(X,Y)Z).$$

With respect to a local system of coordinates $\{x^i\}$,

$$\alpha_{ijk} - \alpha_{ikj} = \alpha_t R^t{}_{ijk} \tag{22}$$

where $\alpha = \alpha_i \mathrm{d}x^i$ and $D^2\alpha = \alpha_{ijk}\mathrm{d}x^k \otimes \mathrm{d}x^j \otimes \mathrm{d}x^i$.

PROOF. In local coordinates we have

$$D\alpha = \alpha_{ij}\mathrm{d}x^j \otimes \mathrm{d}x^i, \qquad D^2\alpha = \alpha_{ijk}\mathrm{d}x^k \otimes \mathrm{d}x^j \otimes \mathrm{d}x^i$$

with

$$\alpha_{ij} = \frac{\partial \alpha_i}{\partial x^j} - \alpha_s\gamma^s_{ij},$$
$$\alpha_{ijk} = \frac{\partial \alpha_{ij}}{\partial x^k} - \alpha_{sj}\gamma^s_{ik} - \alpha_{is}\gamma^s_{jk}.$$

Substituting the first identity in the RHS of the second one we get

$$\alpha_{ijk} = \frac{\partial^2 \alpha_i}{\partial x^k \partial x^j} - \frac{\partial \alpha_s}{\partial x^k}\gamma^s_{ij} - \alpha_s\frac{\partial \gamma^s_{ij}}{\partial x^k} - \frac{\partial \alpha_s}{\partial x^j}\gamma^s_{ik} + \alpha_t\gamma^t_{sj}\gamma^s_{ik} - \alpha_{is}\gamma^s_{jk}.$$

We rearrange the terms by writing

$$\alpha_{ijk} = \left(\frac{\partial^2 \alpha_i}{\partial x^k \partial x^j} - \alpha_{is}\gamma^s_{jk}\right) - \left(\frac{\partial \alpha_s}{\partial x^k}\gamma^s_{ij} + \frac{\partial \alpha_s}{\partial x^j}\gamma^s_{ik}\right) - \alpha_t\left(\frac{\partial \gamma^t_{ij}}{\partial x^k} - \gamma^t_{sj}\gamma^s_{ik}\right)$$

and then we get

$$\begin{aligned}\alpha_{ijk} - \alpha_{ikj} &= -\alpha_t \left(\frac{\partial \gamma^t_{ij}}{\partial x^k} - \gamma^t_{sj}\gamma^s_{ik} \right) + \alpha_t \left(\frac{\partial \gamma^t_{ik}}{\partial x^j} - \gamma^t_{sk}\gamma^s_{ij} \right) \\ &= \alpha_t \left(\frac{\partial \gamma^t_{ik}}{\partial x^j} - \gamma^t_{sk}\gamma^s_{ij} - \frac{\partial \gamma^t_{ij}}{\partial x^k} + \gamma^t_{sj}\gamma^s_{ik} \right) \\ &= \alpha_t R^t{}_{ijk}.\end{aligned}$$

□

The $(0,4)$-type version of R is defined by setting

$$R(V, Z, X, Y) = \sigma(V, R(X, Y)Z)$$

for every $X, Y, Z, V \in \mathfrak{X}(\Omega)$. In local coordinates we can write

$$R(V, Z, X, Y) = R_{ijkt} V^i Z^j X^k Y^t$$

where the coefficients R_{ijkt} are given by

$$R_{ijkt} = \sigma_{is} R^s{}_{jkt}.$$

For every $X, Y, Z, V \in \mathfrak{X}(\Omega)$ we have

$$R(V, Z, X, Y) = -R(Z, V, X, Y) = -R(V, Z, Y, X) = R(X, Y, V, Z)$$

and, as a consequence, the validity of the first Bianchi identity

$$R(V, Z, X, Y) + R(V, X, Y, Z) + R(V, Y, Z, X) = 0.$$

In local coordinates, the above identities read as

$$R_{ijkt} = -R_{jikt} = -R_{ijtk} = R_{ktij}, \tag{23}$$

$$R_{ijkt} + R_{iktj} + R_{itjk} = 0 \tag{24}$$

for every $1 \le i, j, k, t \le m$.

For every point $p \in M$ and for every couple of linearly independent tangent vectors $X, Y \in T_pM$ we write $X \wedge Y = \operatorname{span}(X, Y)$. The sectional curvature $K(\pi)$ of any 2-plane $\pi \le T_pM$ is defined as

$$K(\pi) = \frac{R(X, Y, X, Y)}{|X|^2|Y|^2 - (X, Y)^2}$$

were $X, Y \in T_pM$ are such that $\pi = X \wedge Y$. This definition is well posed since the value of the quotient on the RHS is independent of the choice of the basis $\{X, Y\} \subseteq T_pM$ for π.

The Ricci tensor Ric is the tensor field of type $(0,2)$ obtained by tracing the $(0,4)$-type version of the Riemann curvature tensor with respect to its first and third arguments (or, equivalently, with respect to the second and fourth ones): for every $p \in M$, $X, Y \in T_pM$ and for any choice of an orthonormal basis $\{V^i\}_{1 \le i \le m}$ for $(T_pM, \sigma|_{T_pM})$ we have

$$\operatorname{Ric}(X, Y) = \sum_{i=1}^m R(V^i, X, V^i, Y).$$

In local coordinates we write

$$\operatorname{Ric} = R_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

where

$$R_{ij} = \sigma^{kt} R_{kitj} = \delta^t_k R^k{}_{itj}.$$

2. The Jacobi equation

Let (M,σ) be a Riemannian manifold, $\Omega \subseteq M$ an open domain, and let $u \in C^3(\Omega)$, $f \in C^1(\Omega)$ be such that

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f. \tag{25}$$

Setting $W = \sqrt{1+|Du|^2}$, the function W^{-1} satisfies the differential identity

$$\Delta_g \frac{1}{W} + \left(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\mathbf{n},\mathbf{n})\right)\frac{1}{W} + \langle \nabla f, \nabla u\rangle = 0, \tag{26}$$

where $\|\mathrm{II}\|^2 = g^{ij}g^{kt}\mathrm{II}_{ik}\mathrm{II}_{jt}$ is the squared length of the second fundamental form of the graph $\Sigma \subseteq M \times \mathbb{R}$, $\mathbf{n}$ is any normal vector field on Σ and $\overline{\mathrm{Ric}}$ is the Ricci tensor of $M \times \mathbb{R}$. For constant f the resulting differential equation satisfied by W^{-1} is also known as Jacobi equation. Identity (26) can be equivalently restated as

$$\Delta_g W = \left(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\mathbf{n},\mathbf{n}) + W\langle \nabla f, \nabla u\rangle\right) W + \frac{2\|\nabla W\|^2}{W}. \tag{27}$$

For every $(x,y) \in M \times \mathbb{R}$ and $V_1 \in T_xM$, $V_2 \in T_y\mathbb{R}$ we have the identity

$$\overline{\mathrm{Ric}}(V,V) = \mathrm{Ric}(V_1,V_1) \qquad \text{for } V = V_1 + V_2,$$

then from (15) we have that (27) can be further expressed as

$$\Delta_g W = \left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du,Du)}{W^2} + W\langle \nabla f, \nabla u\rangle\right) W + \frac{2\|\nabla W\|^2}{W}. \tag{28}$$

We give a derivation of (28).

PROPOSITION 2.2. *Let (M,σ) be a Riemannian manifold, $\Omega \subseteq M$ an open domain, and let $u \in C^3(\Omega)$, $f \in C^1(\Omega)$ be such that*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f. \tag{29}$$

Then the function $W = \sqrt{1+|Du|^2}$ satisfies

$$\Delta_g W = \left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du,Du)}{W^2} + W\langle \nabla f, \nabla u\rangle\right) W + \frac{2\|\nabla W\|^2}{W} \qquad \text{in } \Omega. \tag{30}$$

PROOF. Let $\{x^i\}$ be a local coordinate system on Ω. We have

$$\begin{aligned} W_i &= \frac{u_{ik}u^k}{W}, \\ W_{ij} &= \frac{u_{ik}u^k_j}{W} + \frac{u_{ikj}u^k}{W} - \frac{u_{ik}u^kW_j}{W^2} = \frac{u_{ik}u^k_j}{W} - \frac{u_{ik}u^ku_{jt}u^t}{W^3} + \frac{u_{ikj}u^k}{W} \\ &= g^{kt}\frac{u_{ik}u_{jt}}{W} + \frac{u_{ikj}u^k}{W} \end{aligned}$$

and then

$$g^{ij}W_{ij} = g^{ij}g^{kt}\frac{u_{ik}u_{jt}}{W} + g^{ij}\frac{u_{ikj}u^k}{W}.$$

From Ricci's commutation relations (22) we have

$$g^{ij}u_{ikj}u^k = g^{ij}u_{ijk}u^k + g^{ij}u^tu^kR_{tikj},$$

and from the symmetries (23) of the curvature tensor we get

$$g^{ij}u^tu^kR_{tikj} = \sigma^{ij}u^tu^kR_{tikj} = R_{kt}u^tu^k,$$

hence

$$g^{ij}W_{ij} = g^{ij}g^{kt}\frac{u_{ik}u_{jt}}{W} + \frac{R_{ij}u^iu^j}{W} + \frac{1}{W}g^{ij}u_{ijk}u^k.$$

We differentiate

$$\left(g^{ij}u_{ij}\right)_k = g^{ij}{}_k u_{ij} + g^{ij}u_{ijk}. \tag{31}$$

We compute

$$\begin{aligned} g^{ij}{}_k &= \sigma^{ij}_k - \frac{u^i{}_k u^j}{W^2} - \frac{u^i u^j{}_k}{W^2} + 2\frac{u^iu^jW_k}{W^3} \\ &= 0 - \sigma^{it}\frac{u_{tk}u^j}{W^2} - \sigma^{jt}\frac{u_{tk}u^i}{W^2} + 2\frac{u^iu^ju^tu_{tk}}{W^4} \\ &= -\left(g^{it}u^j + g^{jt}u^i\right)\frac{u_{tk}}{W^2} \end{aligned}$$

and we use the symmetry $u_{ij} = u_{ji}$ to write

$$g^{ij}{}_k u_{ij} = -2g^{it}\frac{u^ju_{ij}}{W}\frac{u_{tk}}{W} = -2g^{it}W_i\frac{u_{tk}}{W}. \tag{32}$$

So, we obtain

$$g^{ij}{}_k u^k u_{ij} = -2g^{it}W_i\frac{u_{tk}u^k}{W} = -2g^{it}W_iW_t$$

and then

$$g^{ij}u_{ijk}u^k = \left(g^{ij}u_{ij}\right)_k u^k - g^{ij}{}_k u^k u_{ij} = \left(g^{ij}u_{ij}\right)_k u^k + 2g^{ij}W_iW_j.$$

This yields

$$g^{ij}W_{ij} = g^{ij}g^{kt}\frac{u_{ik}u_{jt}}{W} + \frac{R_{ij}u^iu^j}{W} + \frac{\left(g^{ij}u_{ij}\right)_k u^k}{W} + \frac{2g^{ij}W_iW_j}{W}.$$

Summing up, we obtain

$$\begin{aligned} \Delta_g W &= g^{ij}W_{ij} - \frac{g^{ij}u_{ij}}{W^2}W_ku^k \\ &= g^{ij}g^{kt}\frac{u_{ik}u_{jt}}{W} + \frac{R_{ij}u^iu^j}{W} + \frac{2g^{ij}W_iW_j}{W} + \left(\frac{\left(g^{ij}u_{ij}\right)_k}{W} - g^{ij}u_{ij}\frac{W_k}{W^2}\right)u^k \\ &= g^{ij}g^{kt}\frac{u_{ik}u_{jt}}{W} + \frac{R_{ij}u^iu^j}{W} + \frac{2g^{ij}W_iW_j}{W} + \left(\frac{g^{ij}u_{ij}}{W}\right)_k u^k. \end{aligned}$$

From (16), (17) and (18) we have $\mathrm{II}_{ij} = W^{-1}u_{ij}$ and $W^{-1}g^{ij}u_{ij} = f$, then we can write

$$\Delta_g W = \|\mathrm{II}\|^2 W + \frac{\mathrm{Ric}(Du, Du)}{W} + \frac{2\|\nabla W\|^2}{W} + (Df, Du)$$

and, since $Du = W^2\nabla u$,

$$\Delta_g W = \|\mathrm{II}\|^2 W + \frac{\mathrm{Ric}(Du, Du)}{W} + \frac{2\|\nabla W\|^2}{W} + W^2\langle\nabla f, \nabla u\rangle.$$

□

Formula (28) is our starting point to derive gradient estimates for non-negative (or lower bounded) solutions of equation (25) via the maximum principle. We outline the main argument behind the proof that will be carried out in Chapter 4: this is essentially Bernstein's method for obtaining a priori gradient bounds for solutions of nonlinear equations, see [**4**]. Let $\eta \in C^2(\Omega)$ be given and set $z = W\eta$. Then in Ω we have

$$\begin{aligned} \nabla z &= W\nabla\eta + \eta\nabla W, \\ \Delta_g z &= W\Delta_g\eta + 2\langle\nabla W, \nabla\eta\rangle + \eta\Delta_g W. \end{aligned}$$

As $W > 0$, we can use the first identity to write

$$\nabla\eta = \frac{\nabla z}{W} - \eta\frac{\nabla W}{W}$$

and then we substitute this into the second one to obtain

$$\Delta_g z = W\Delta_g\eta + 2\frac{\langle\nabla W, \nabla z\rangle}{W} + \eta\left(\Delta_g W - \frac{2\|\nabla W\|^2}{W}\right).$$

Rearranging terms and using (28),

$$\Delta_g z - 2\frac{\langle\nabla W, \nabla z\rangle}{W} = \left(\left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} + W\langle\nabla f, \nabla u\rangle\right)\eta + \Delta_g\eta\right)W. \tag{33}$$

To fix ideas, let us first consider the case where $\eta = e^{-Cu}$ for some constant $C \geq 0$. Recall that we are assuming $u \geq 0$, so $0 < \eta \leq 1$. In this setting we have

$$\Delta_g\eta = \left(-C\Delta_g u + C^2\|\nabla u\|^2\right)\eta$$

and (33) yields

$$\Delta_g z - 2\frac{\langle\nabla W, \nabla z\rangle}{W} = \left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} + W\langle\nabla f, \nabla u\rangle - C\Delta_g u + C^2\|\nabla u\|^2\right)z.$$

If $\overline{\Omega}$ is compact, then either $\sup_\Omega z = \sup_{\partial\Omega} z$ or z attains its global maximum at some point $\bar{x} \in \Omega$. In the second case, from the maximum principle it must be $\nabla z = 0$ and $\Delta_g z \leq 0$ at $\bar{x}$. Using $z > 0$, we obtain

$$\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} + W\langle\nabla f, \nabla u\rangle - C\Delta_g u + C^2\|\nabla u\|^2 \leq 0.$$

Under appropriate assumptions on Ric and f we can ensure that the LHS of this inequality is strictly positive if W exceeds some threshold $A > 1$. Coupling this with condition $\eta \leq 1$ we deduce $z(\bar{x}) \leq W(\bar{x}) \leq A$ and then we obtain a global bound

$$\sup_\Omega z \leq \max\left\{A, \sup_{\partial\Omega} z\right\}, \tag{34}$$

that is, a gradient bound

$$\sup_\Omega \frac{\sqrt{1+|Du|^2}}{e^{Cu}} \leq \max\left\{A, \sup_{\partial\Omega} \frac{\sqrt{1+|Du|^2}}{e^{Cu}}\right\}.$$

If $\overline{\Omega}$ is not compact, then we rely on a localization and approximation argument to derive the a priori bound (34). First, we assume without loss of generality that $\sup_\Omega z > \sup_{\partial\Omega} z$, and we fix $\gamma > 0$ such that $\sup_\Omega z > \gamma > \sup_{\partial\Omega} z$. Then, we set $\Omega_\gamma = \{x \in \Omega : z(x) > \gamma\}$, we let $\psi : \overline{\Omega_\gamma} \to \mathbb{R}_0^+$ be a suitable continuous function with compact sublevel sets and we set $\eta_{\varepsilon,\delta} = e^{-Cu-\varepsilon\psi} - \delta$ for every $\varepsilon, \delta > 0$. In this case we have

$$\Delta_g\eta_{\varepsilon,\delta} = \left(-C\Delta_g u - \varepsilon\Delta_g\psi + \|C\nabla u + \varepsilon\nabla\psi\|^2\right)e^{-Cu-\varepsilon\psi}$$

and then, for the function $z_{\varepsilon,\delta} = W\eta_{\varepsilon,\delta}$,

$$\begin{aligned}\Delta_g z_{\varepsilon,\delta} - 2\frac{\langle\nabla W, \nabla z_{\varepsilon,\delta}\rangle}{W} &= \left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} + W\langle\nabla f, \nabla u\rangle\right)z_{\varepsilon,\delta} \\ &\quad + \left(-C\Delta_g u - \varepsilon\Delta_g\psi + \|C\nabla u + \varepsilon\nabla\psi\|^2\right)We^{-Cu-\varepsilon\psi}.\end{aligned} \tag{35}$$

For every $\varepsilon, \delta > 0$ we have $\eta_{\varepsilon,\delta} < e^{-Cu}$ on Ω_γ, so $z_{\varepsilon,\delta} < z$. On the other hand, $z_{\varepsilon,\delta} \to z$ pointwise on Ω_γ as $(\varepsilon, \delta) \to (0, 0)$. Hence, for every sufficiently small $\varepsilon, \delta > 0$ one has $\sup_{\Omega_\gamma} z_{\varepsilon,\delta} > \gamma \geq \sup_{\partial\Omega_\gamma} z_{\varepsilon,\delta}$. For every $\varepsilon, \delta > 0$ the boundary of

$$\Omega_{\varepsilon,\delta} = \{x \in \overline{\Omega_\gamma} : z_{\varepsilon,\delta} > 0\}$$

is contained in $(\partial\Omega_\gamma) \cup \{z_{\varepsilon,\delta} \leq 0\}$, so we have $\sup_{\partial\Omega_{\varepsilon,\delta}} z_{\varepsilon,\delta} \leq \gamma < \sup_{\Omega_{\varepsilon,\delta}} z_{\varepsilon,\delta}$. Moreover, for every $\varepsilon, \delta > 0$ the set $\Omega_{\varepsilon,\delta}$ is relatively compact in M, being a subset of

$\{\psi \leq \varepsilon^{-1}\log(\delta)\}$. Hence, for every sufficiently small $\varepsilon, \delta > 0$ there exists $x_{\varepsilon,\delta} \in \Omega_{\varepsilon,\delta}$ such that

$$z_{\varepsilon,\delta}(x_{\varepsilon,\delta}) = \max_{\Omega_{\varepsilon,\delta}} z_{\varepsilon,\delta} \equiv \sup_{\Omega_\gamma} z_{\varepsilon,\delta}$$

and by a diagonalization argument

$$\lim_{(\varepsilon,\delta)\to(0,0)} z_{\varepsilon,\delta}(x_{\varepsilon,\delta}) = \sup_{\Omega_\gamma} z \equiv \sup_{\Omega} z.$$

By the maximum principle, at points $x_{\varepsilon,\delta}$ it must be $\nabla z_{\varepsilon,\delta} = 0$, $\Delta_g z_{\varepsilon,\delta} \leq 0$ and then the RHS of (35) must be non-positive. A bit more care is needed in this case to properly bound from below the RHS of this identity, but then we can show again that, for some fixed threshold $A > 1$, for all sufficiently small $\varepsilon, \delta > 0$ it must be $W(x_{\varepsilon,\delta}) \leq A$, and then we conclude $\sup_\Omega z \leq A$. In particular, in the proof of the gradient bound we will need to suitably control the contribution of terms $\varepsilon\Delta_g\psi$ and $\varepsilon^2\|\nabla\psi\|^2$ in inequality (35). For this reason, in Chapter 3 we shall study under different assumptions the possibility of constructing functions $\psi : \overline{\Omega_0} \to \mathbb{R}_0^+$ with compact sublevel sets and with controlled $\|\nabla\psi\|$, $\Delta_g\psi$ on subdomains $\Omega_0 \subseteq \Omega$ such that $\overline{\Omega_0} \subseteq \Omega$. We will call them (good) exhaustion functions.

3. An equation for the directional derivatives of u

Let (M, σ) be a Riemannian manifold, $\Omega \subseteq M$ an open set. We recall that a vector field $X \in \mathfrak{X}(\Omega)$ is said to be a Killing vector field (with respect to the metric σ) if the Lie derivative of the metric σ vanishes along the flow of X,

$$\mathcal{L}_X\sigma = 0$$

a condition that amounts to saying that, for every $x \in \Omega$, the flow of X is a (local) 1-parameter group of (local) isometries in a neighbourhood of x with respect to the metric σ. From the properties of the Levi-Civita connection D we have

$$(\mathcal{L}_X\sigma)(Y, Z) = (D_Y X, Z) + (D_Z X, Y)$$

for every $Y, Z \in \mathfrak{X}(\Omega)$. With respect to a local system of coordinates $\{x^i\}$, this amounts to saying that

$$X_{ij} + X_{ji} = 0 \qquad \text{for } 1 \leq i, j \leq m \tag{36}$$

where X_{ij} are the components of the $(0,2)$-type tensor field $X_{ij}\mathrm{d}x^j \otimes \mathrm{d}x^i$ metrically equivalent to the covariant derivative $DX = X^i{}_j\mathrm{d}x^j \otimes \partial_{x^i}$ of $X = X^i\partial_{x^i}$ (that is, $X_{ij} = \sigma_{ik}X^k{}_j$ for $1 \leq i, j \leq m$).

Let $X \in \mathfrak{X}(\Omega)$ be a Killing vector field and let $u \in C^3(\Omega)$ and $f \in C^1(\Omega)$ satisfy

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f.$$

In the next proposition we derive an expression for $\Delta_g\varphi$, where $\varphi = (Du, X)$ is the directional derivative of u in the direction of X.

Proposition 2.3. *Let $u \in C^3(\Omega)$ satisfy*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f$$

for some given $f \in C^1(\Omega)$ and let $X \in \mathfrak{X}(\Omega)$ be a Killing vector field. Then the function $\varphi = (Du, X)$ satisfies

$$\Delta_g\varphi = W(Df, X) + \frac{2\langle\nabla W, \nabla\varphi\rangle}{W}$$

or, equivalently,

$$W^2 \operatorname{div}_g \left(\frac{\nabla\varphi}{W^2}\right) = W(Df, X).$$

PROOF. In local coordinates we have

$$\varphi = u_k X^k, \tag{37}$$

$$\varphi_i = u_{ki} X^k + u_k X^k_{\ i}, \tag{38}$$

$$\varphi_{ij} = u_{kij} X^k + u_{ki} X^k_{\ j} + u_{kj} X^k_{\ i} + u_k X^k_{\ ij} \tag{39}$$

and then

$$g^{ij}\varphi_{ij} = g^{ij} u_{kij} X^k + 2 g^{ij} u_{ki} X^k_{\ j} + g^{ij} u_k X^k_{\ ij}.$$

By Ricci's commutation relations and (31) we can write

$$\begin{aligned} g^{ij} u_{kij} &= g^{ij}\left(u_{ikj} - u_{ijk}\right) + g^{ij} u_{ijk} = g^{ij} u^t R_{tikj} + \left(g^{ij} u_{ij}\right)_k - g^{ij}_{\ \ k} u_{ij}, \\ X^k_{\ ij} &= \sigma^{kt} X_{tij} = \sigma^{kt}\left(X_{tij} + X_{itj}\right) - \sigma^{kt}\left(X_{itj} - X_{ijt}\right) - \sigma^{kt} X_{ijt} \\ &= \sigma^{kt}\left(X_{tij} + X_{itj}\right) - \sigma^{kt} X^s R_{sitj} - \sigma^{kt} X_{ijt}, \end{aligned}$$

then

$$\begin{aligned} g^{ij} u_{kij} X^k &= g^{ij} u^t X^k R_{tikj} + \left(g^{ij} u_{ij}\right)_k X^k - g^{ij}_{\ \ k} u_{ij} X^k, \\ g^{ij} u_k X^k_{\ ij} &= g^{ij} u^t \left(X_{tij} + X_{itj}\right) - g^{ij} u^t X^s R_{sitj} - u^t g^{ij} X_{ijt} \end{aligned}$$

and we obtain

$$g^{ij}\varphi_{ij} = \left(g^{ij} u_{ij}\right)_k X^k - g^{ij}_{\ \ k} u_{ij} X^k + 2 g^{ij} u_{ik} X^k_{\ j} + g^{ij} u^t \left(X_{tij} + X_{itj}\right) - u^t g^{ij} X_{ijt}.$$

From (32) we can write

$$-g^{ij}_{\ \ k} u_{ij} = 2 g^{it} \frac{W_i}{W} u_{tk}$$

and from (38) we also have

$$u_{tk} X^k = u_{kt} X^k = \varphi_t - u_k X^k_{\ t},$$

hence

$$-g^{ij}_{\ \ k} u_{ij} X^k = 2 g^{it} \frac{W_i}{W} u_{tk} X^k = 2 g^{ij} \frac{W_i}{W} \varphi_j - 2 g^{ij} \frac{W_i}{W} u_k X^k_{\ j}.$$

Moreover,

$$\begin{aligned} -2 g^{ij} \frac{W_i}{W} u_k X^k_{\ j} + 2 g^{ij} u_{ik} X^k_{\ j} &= 2 g^{ij} X^k_{\ j} \left(u_{ik} - \frac{W_i u_k}{W}\right) \\ &= 2 g^{ij} X^k_{\ j} \left(u_{ik} - \frac{u_{it} u^t u_k}{W^2}\right) \\ &= 2 g^{ij} X^k_{\ j} \left(\delta^t_k - \frac{u^t u_k}{W^2}\right) u_{it} \\ &= 2 g^{ij} \sigma^{sk} X_{sj} g^{tl} \sigma_{lk} u_{it} \\ &= 2 g^{ij} g^{ts} X_{sj} u_{it} \\ &= g^{ij} g^{ts} (X_{sj} + X_{js}) u_{it} \end{aligned}$$

where the last equality follows from the symmetries $g^{ij} = g^{ji}$ and $u_{ij} = u_{ji}$. Then,

$$-g^{ij}_{\ \ k} u_{ij} X^k + 2 g^{ij} u_{ik} X^k_{\ j} = 2 g^{ij} \frac{W_i}{W} \varphi_j + g^{ij} g^{ts} (X_{sj} + X_{js}) u_{it}$$

and we obtain

$$\begin{aligned} g^{ij}\varphi_{ij} = {} & \left(g^{ij} u_{ij}\right)_k X^k + 2 g^{ij} \frac{W_i}{W} \varphi_j \\ & + g^{ij} g^{ts} (X_{sj} + X_{js}) u_{it} + g^{ij} u^t \left(X_{tij} + X_{itj}\right) - u^t g^{ij} X_{ijt}. \end{aligned}$$

From (38) we further write

$$g^{ij}u_{ij}\frac{\varphi_k u^k}{W^2} = g^{ij}u_{ij}\frac{u_{kt}u^k X^t}{W^2} + g^{ij}u_{ij}\frac{u_t u^k X^t_{\ k}}{W^2} = g^{ij}u_{ij}\frac{W_t X^t}{W} + g^{ij}u_{ij}\frac{X_{kt}u^k u^t}{W^2}$$

and then

$$\begin{aligned}
\Delta_g\varphi &= g^{ij}\varphi_{ij} - g^{ij}u_{ij}\frac{\varphi_k u^k}{W^2} \\
&= \left(g^{ij}u_{ij}\right)_k X^k - g^{ij}u_{ij}\frac{W_k X^k}{W} + 2g^{ij}\frac{W_i}{W}\varphi_j \\
&\quad + g^{ij}g^{ts}(X_{sj}+X_{js})u_{it} + g^{ij}u^t\left(X_{tij}+X_{itj}\right) - u^t g^{ij}X_{ijt} + g^{ij}u_{ij}\frac{X_{kt}u^k u^t}{W^2} \\
&= W\left(\frac{g^{ij}u_{ij}}{W}\right)_k X^k + 2g^{ij}\frac{W_i}{W}\varphi_j \\
&\quad + g^{ij}g^{ts}(X_{sj}+X_{js})u_{it} + g^{ij}u^t\left(X_{tij}+X_{itj}\right) - u^t g^{ij}X_{ijt} + g^{ij}u_{ij}\frac{X_{kt}u^k u^t}{W^2}.
\end{aligned}$$

From the Killing condition (36), the last four terms in the above identity cancel out and we obtain

$$\Delta_g\varphi = W\left(\frac{g^{ij}u_{ij}}{W}\right)_k X^k + 2g^{ij}\frac{W_i}{W}\varphi_j = W(Df,X) + 2\frac{\langle\nabla W,\nabla\varphi\rangle}{W}.$$

□

Corollary 2.4. *Let $u \in C^3(\Omega)$ satisfy*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \frac{f_2(u)}{W}$$

for some given $f_1, f_2 \in C^1(\mathbb{R})$ and let $X \in \mathfrak{X}(\Omega)$ be a Killing vector field. Then the function $\varphi = (Du, X)$ satisfies

$$\Delta_g\varphi = \left(Wf_1'(u) + f_2'(u)\right)\varphi + \left\langle \frac{2\nabla W}{W} - f_2(u)\nabla u, \nabla\varphi\right\rangle.$$

Equivalently,

$$W^2 e^{-F_2(u)}\operatorname{div}_g\left(\frac{e^{F_2(u)}}{W^2}\nabla\varphi\right) = \left(Wf_1'(u) + f_2'(u)\right)\varphi \tag{40}$$

where F_2 is any primitive of f_2.

Proof. Let $f = f_1(u) + W^{-1}f_2(u)$. We have

$$Df = \left(f_1'(u) + \frac{f_2'(u)}{W}\right)Du - \frac{f_2(u)}{W^2}DW$$

hence

$$W(Df,X) = \left(Wf_1'(u) + f_2'(u)\right)\varphi - \frac{f_2(u)}{W}(DW,X).$$

In local coordinates we have

$$(DW,X) = \frac{u_{ij}u^j X^i}{W}.$$

From (38) together with the Killing condition (36) we compute

$$u_{ij}u^j X^i = u^j(u_{ij}X^i + u_i X^i_{\ j}) - u^j u_i X^i_{\ j} = u^j\varphi_j - 0$$

and then

$$-\frac{f_2(u)}{W}(DW,X) = -f_2(u)\frac{u^i\varphi_i}{W^2} = -f_2(u)g^{ij}u_i\varphi_j = -f_2(u)\langle\nabla u,\nabla\varphi\rangle.$$

Hence, the conclusion follows from Proposition 2.3. □

CHAPTER 3

Good exhaustion functions

In this chapter we show that if u is a C^2 function defined on an open domain Ω of a complete Riemannian manifold M and if the validity of either condition (RΩ) or (K) from the Introduction is assumed, then for every subdomain $\Omega_0 \subseteq \Omega$ with $\overline{\Omega_0} \subseteq \Omega$ there exists a continuous function $\psi : \overline{\Omega_0} \to \mathbb{R}_0^+$, with $\psi(x) \to +\infty$ as $x \to \infty$ in $\overline{\Omega_0}$, such that $\Delta_g \psi$ and $\|\nabla\psi\|^2$ are suitably controlled from above in Ω_0.[1] This will be essential to carry out the proof of the gradient bound in the next chapter.

1. Basic definitions

Let (N, h) be a Riemannian manifold, $\Omega \subseteq N$ an open set, $f : \Omega \to \mathbb{R}$ a function. Following [**37**], we recall some alternative notions of weak solutions of the differential inequality $\Delta_h u \leq f$.

DEFINITION 3.1. *A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a solution in Ω of the differential inequality $\Delta_h u \leq f$* in the barrier sense *if for every $\bar{x} \in \Omega$, $\varepsilon > 0$ there exist a neighbourhood $U \subseteq \Omega$ of $\bar{x}$ and a function $v \in C^2(U)$ such that*

$$\begin{cases} u \leq v & \text{in } U, \\ u(\bar{x}) = v(\bar{x}), \\ \Delta_h v(\bar{x}) \leq f(\bar{x}) + \varepsilon. \end{cases}$$

In this case, we say that v is a support function *for u at $\bar{x}$.*

This weakened notion of solution was first introduced by Calabi, [**10**], for linear uniformly elliptic operators of second order of the form

$$Lu = a^{ij} u_{ij} + b^i u_i$$

with bounded coefficients a^{ij}, b^i. Indeed, he called such solutions *weak solutions*. As originally showed by Calabi, if u is of class C^2 then it satisfies $\Delta_h u \leq f$ in the barrier sense if and only if it does so in the classical (strong) sense.

DEFINITION 3.2. *A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a* viscosity solution *in Ω of $\Delta_h u \leq f$ if for every $\bar{x} \in \Omega$, for every neighbourhood $U \subseteq \Omega$ of $\bar{x}$ and for every $\phi \in C^2(U)$ satisfying*

$$\begin{cases} \phi \leq u & \text{in } U, \\ \phi(\bar{x}) = u(\bar{x}) \end{cases}$$

it holds

$$\Delta_h \phi(\bar{x}) \leq f(\bar{x}).$$

From the definition itself it follows that if u is a solution of $\Delta_h u \leq f$ in the barrier sense, then it is also a viscosity solution. The converse is not true, in general. The

[1] Hereafter, if M_0 is a subset of a manifold M and $f : M_0 \to \mathbb{R}$ is a function, we say that $f(x) \to +\infty$ as $x \to \infty$ in M_0 if for every $a \in \mathbb{R}$ there exists a compact set $K \subseteq M_0$ such that $f(x) \geq a$ for every $x \in M_0 \setminus K$.

following simple example, taken from [**37**], shows that this may fail even for differentiable functions: in $N = \Omega = \mathbb{R}$, the function

$$u(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

satisfies $u''(0) \leq 0$ in the viscosity sense but not in the barrier sense.

Definition 3.3. *A function $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak (distributional) solution in Ω of the differential inequality $\Delta_h u \leq f$, with $f \in L^1_{\mathrm{loc}}(\Omega)$, if*

$$-\int_\Omega \langle \nabla\phi, \nabla u\rangle \leq \int_\Omega f\phi \qquad \text{for every } 0 \leq \phi \in C^\infty_c(\Omega).$$

From a theorem due to P.-L. Lions [**36**] and H. Ishii [**29**], for continuous functions u, f the inequality $\Delta_h u \leq f$ is satisfied in the viscosity sense if and only if it holds in the distributional sense. In fact, Ishii's theorem is concerned with the notion of viscosity and distributional solutions for differential inequalities of the form $Lu \leq f$ on open subsets $U \subseteq \mathbb{R}^m$, where L is a linear elliptic differential operator of the form

$$L\phi = a^{ij}\phi_{ij} + b^i\phi_i + c\phi \tag{41}$$

and $a^{ij} \in C^{1,1}(U)$, $b^i \in C^{0,1}(U)$, $c, f \in C(U)$, and the equivalence between viscosity and distributional solutions is established under the assumption $\sqrt{\det(a^{ij})} \in C^1(U)$. In every local smooth chart $\{x^i\} : \Omega_0 \subseteq \Omega \to U \subseteq \mathbb{R}^m$ the Laplace-Beltrami operator Δ_h admits a local expression of the form (41) with smooth coefficients. Due to the local nature of the notions of viscosity and distributional solutions, Ishii's theorem directly applies to Δ_h.

We also recall the following global approximation theorem due to Greene-Wu (see Corollary 1 to Theorem 3.2 in [**25**]), that we will need in Section 3.3.

Proposition 3.4 (Greene-Wu's global approximation theorem). *Let (N, h) be a Riemannian manifold, $\Omega \subseteq N$ an open set and let $\eta, \beta, g \in C^0(\Omega)$ be continuous functions, with $\beta, g > 0$. If $u \in C^0(\Omega)$ satisfies*

$$\Delta_h u < \eta \qquad \text{in } \Omega$$

in the distributional sense (equivalently, in the viscosity sense) and if for every $x \in \Omega$ there exist a neighbourhood $U \subseteq \Omega$ and a constant $B \in (0, \beta(x))$ such that

$$|u(y_1) - u(y_2)| \leq B \operatorname{dist}_h(y_1, y_2) \qquad \text{for every } y_1, y_2 \in U,$$

then there exists $v \in C^\infty(\Omega)$ such that

$$\begin{cases} \Delta_h v < \eta & \text{in } \Omega, \\ \|\nabla v\| < \beta & \text{in } \Omega, \\ |u(x) - v(x)| < g & \text{for every } x \in \Omega. \end{cases}$$

2. Constructions via distance functions

Let (M, σ) be a connected, complete Riemannian manifold and let $r(x) = \operatorname{dist}_\sigma(o, x)$ be the distance function from a fixed origin $o \in M$. The function r is Lipschitz continuous on M with Lipschitz constant 1, but in general it is not smooth on M. In fact, we can say that r is smooth on the open set $D_o = M \setminus (\{o\} \cup \operatorname{cut}(o))$, where $\operatorname{cut}(o)$ is the cut locus of o in M, as defined below.

As just anticipated, r is not differentiable at o regardless of the geometry of M. However, it is always possible to find a neighbourhood U of o such that r is smooth on $U \setminus \{o\}$. In particular, the Hessian of the function r has the asymptotic behaviour

$$\operatorname{Hess}(r) = \frac{1}{r}(\sigma - \mathrm{d}r \otimes \mathrm{d}r) + o(1) \qquad \text{as } r \to 0$$

(see [**44**], p. 194) and the function r^2 is of class C^2 in a neighbourhood of o, with

$$\mathrm{Hess}(r^2) = 2\sigma + o(1) \quad \text{as } r \to 0.$$

To introduce the definition of the cut locus $\mathrm{cut}(o)$, let us recall the following notion: a geodesic curve $\gamma : [a,b] \to M$ is said to be a segment if it is length minimizing on $[a,b]$, that is, if

$$\mathrm{dist}_\sigma(\gamma(c_1), \gamma(c_2)) = |c_1 - c_2| \qquad \text{for every } c_1, c_2 \in [a,b].$$

From the Hopf-Rinow theorem, completeness of M implies that every point $x \in M$ is joined to o by at least one segment. A point $x \in M$ is said to be a cut-point for o if there exists a unit speed geodesic $\gamma : \mathbb{R}_0^+ \to M$, with $\gamma(0) = o$, which is a segment between o and x but not between o and $\gamma(r(x) + \varepsilon)$ for any $\varepsilon > 0$. The set of cut points for o is called the cut locus of o in M.

The function r^2 is smooth on $M \setminus \mathrm{cut}(o)$, and r is smooth on $M \setminus (\{o\} \cup \mathrm{cut}(o))$. A procedure introduced by Calabi (Calabi's trick, [**10**]; see also proof of Lemma 7.1.9 in [**44**]) allows to construct families of (smooth) support functions for r at points of $\mathrm{cut}(o)$: if $x_0 \in \mathrm{cut}(o)$ is given and $\gamma : [0, r(x_0)] \to M$ is a segment joining $\gamma(0) = o$ and $\gamma(r(x_0)) = x_0$, then x_0 is not in the cut locus of any point of γ lying between o and x_0, so for every $\varepsilon \in (0, r(x))$ the distance function $r_\varepsilon(x) = \mathrm{dist}_\sigma(o_\varepsilon, x)$ from $o_\varepsilon = \gamma(\varepsilon)$ is smooth in a neighbourhood of x_0. From the triangle inequality we have

$$r(x) \leq r_\varepsilon(x) + \varepsilon$$

for every $x \in M$, with equality for every x lying on γ between o_ε and x_0. In particular, for every sufficiently small $\varepsilon > 0$ the function $r_\varepsilon + \varepsilon$ is smooth in a neighbourhood of x_0 and satisfies

$$\begin{cases} r \leq r_\varepsilon + \varepsilon & \text{in } M, \\ r(x_0) = r_\varepsilon(x_0) + \varepsilon \end{cases}$$

so it is a support function for r at x_0.

The basic tool in the analysis of this section is the following standard Hessian comparison theorem for the distance function from a fixed origin in a Riemannian manifold (see for instance Theorem 2.15 in [**6**] and the previous remarks.)

THEOREM 3.5 (Hessian comparison theorem). *Let (M, σ) be a Riemannian manifold. Having fixed an origin $o \in M$, let $r(x) = \mathrm{dist}_\sigma(o, x)$ be the distance function from o. Let $\gamma : [0, R_0] \to M$ be a segment with $\gamma(0) = o$ and let $G : (0, R_0) \to \mathbb{R}$ be such that*

$$K(\dot\gamma(s) \wedge X) \geq -G(s) \qquad \text{for every } s \in (0, R_0),\ X \perp \dot\gamma(s)$$

If $\phi : (0, R_0) \to \mathbb{R}$ satisfies

$$\begin{cases} \phi' + \phi^2 \geq G & \text{on } (0, R_0), \\ \phi(s) = s^{-1} + o(1) & \text{as } s \to 0 \end{cases}$$

then

$$\mathrm{Hess}\, r(\gamma(s)) \leq \phi(s)\, (\sigma - \mathrm{d}r \otimes \mathrm{d}r) \qquad \text{for every } s \in (0, R_0).$$

From Theorem 3.5 and Calabi's trick we deduce the next Theorem 3.6, whose proof relies on a construction described in the proof of Lemma 2.8 of [**45**]. We recall that if $o \in M$ is given and $x \in M \setminus (\{o\} \cup \mathrm{cut}(o))$, the radial sectional curvature $K_{\mathrm{rad}}(x)$ associated to o is the infimum of the sectional curvatures of tangent 2-planes $\pi \leq T_x M$ such that $Dr \in \pi$.

THEOREM 3.6. *Let (M, σ) be a complete Riemannian manifold. Let $r(x)$ be the distance function from a reference origin $o \in M$, let $G \in C^1(\mathbb{R}_0^+)$ be non-decreasing, with $G(0) = \alpha > 0$, $G'(0) = 0$, and such that the radial sectional curvature satisfies*

$$K_{\mathrm{rad}} \geq -G(r) \qquad \text{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o)).$$

Also let $\Omega \subseteq M$ be an open domain and $u \in C^2(\Omega)$, $f \in C^0(\Omega)$ be such that

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f.$$

Then, the function $\psi : M \to \mathbb{R}_0^+$ defined by

$$\psi(x) = \alpha\left(\int_0^{r(x)} \frac{\mathrm{d}s}{\sqrt{G(s)}}\right)^2 \qquad \text{for every } x \in M$$

is C^2 on $M \setminus \operatorname{cut}(o)$, satisfies

$$\|\nabla\psi\| \le 2\sqrt{\psi}, \quad \Delta_g\psi \le 2\left((m-1)\sqrt{\alpha\psi}\coth\left(\sqrt{\alpha\psi}\right) + \sqrt{\psi}|f| + 1\right) \qquad \text{on } \Omega \setminus \operatorname{cut}(o)$$

and for every $\bar{x} \in \Omega \cap \operatorname{cut}(o)$ there exist sequences of neighbourhoods $\{U_k\}$ of $\bar{x}$ and functions $\psi_k \in C^2(U_k)$ such that

$$\begin{cases} \psi_k \ge \psi & \text{in } U_k, \\ \psi_k(\bar{x}) = \psi(\bar{x}), & \\ \|\nabla\psi_k\|(\bar{x}) \le 2\sqrt{\psi(\bar{x})}, & \\ \limsup_{k\to+\infty} \Delta_g\psi_k(\bar{x}) \le 2\left((m-1)\sqrt{\alpha\psi(\bar{x})}\coth\left(\sqrt{\alpha\psi(\bar{x})}\right) + \sqrt{\psi(\bar{x})}|f| + 1\right). & \end{cases}$$

Proof. Set $H(t) = \int_0^t \sqrt{G(s)}\mathrm{d}s$. The function $\phi(t) = \sqrt{G(t)}\coth(H(t))$ satisfies

$$\phi'(t) = \frac{G'(t)}{2\sqrt{G(t)}}\coth(H(t)) - \frac{\sqrt{G(t)}H'(t)}{\sinh^2(H(t))} \ge -\frac{G(t)}{\sinh^2(H(t))}$$

because of $G' \ge 0$ and $H' = \sqrt{G}$, so it is a solution of

$$\begin{cases} \phi' + \phi^2 \ge G & \text{on } \mathbb{R}^+, \\ \phi(s) = s^{-1} + o(1) & \text{as } s \to 0, \end{cases}$$

where the validity of the second condition can be verified from asymptotic expansions $H(t) = \sqrt{\alpha}t + O(t^3)$, $\coth(t) = t^{-1} + O(t)$, $\sqrt{G(t)} = \sqrt{\alpha} + O(t^2)$ as $t \to 0$. From the Hessian comparison Theorem 3.5, at every point $x \in D_o$ we have

$$\operatorname{Hess} r(x) \le \phi(r(x))(\sigma - \mathrm{d}r \otimes \mathrm{d}r)$$

where inequality is to be intended with respect to the partial ordering of quadratic forms. Having fixed a local coordinate system $\{x^i\}$, this yields

$$g^{ij}r_{ij} \le \phi(r(x))g^{ij}(\sigma_{ij} - r_i r_j)$$

because g is a positive definite quadratic form. The quadratic form $\phi(r)(\sigma - \mathrm{d}r \otimes \mathrm{d}r)$ is also non-negative and we have $(g^{ij}) \le (\sigma^{ij})$, so we can further estimate

$$g^{ij}r_{ij} \le \phi(r(x))\sigma^{ij}(\sigma_{ij} - r_i r_j) = (m-1)\phi(r(x))$$

and by (20) and the Cauchy-Schwarz inequality

$$\Delta_g r \le (m-1)\phi(r) + |f|. \tag{42}$$

We now introduce functions

$$h(t) = \sqrt{\alpha}\int_0^t \frac{\mathrm{d}s}{\sqrt{G(s)}}, \qquad \varphi(x) = h(r(x))$$

in order to write $\psi = \varphi^2$. As $h' \ge 0$ and $h'' \le 0$, we have

$$\Delta_g\varphi \le h'(r)\Delta_g r + h''(r)\|\nabla r\|^2 \le h'(r)\Delta_g r,$$
$$\Delta_g\psi \le 2\varphi\Delta_g\varphi + 2\|\nabla\varphi\|^2 \le 2h'(r)h(r)\Delta_g r + 2h'(r)^2\|\nabla r\|^2.$$

From the monotonicity of G we have $\sqrt{G(t)} \geq \sqrt{\alpha}$ and then we can estimate

$$0 \leq h'(t) = \frac{\sqrt{\alpha}}{\sqrt{G(t)}} \leq 1, \qquad 0 \leq h(t) \leq t, \qquad H(t) \geq \sqrt{\alpha} t \geq \sqrt{\alpha} h(t).$$

From the first two inequalities together with (42) and the definition of ϕ we obtain

$$2h'(r)h(r)\Delta_g r \leq 2(m-1)\sqrt{\alpha} h(r) \coth(H(r)) + 2h(r)|f|$$

and then we can further estimate

$$2h'(r)h(r)\Delta_g r \leq 2(m-1)\sqrt{\alpha} h(r) \coth(\sqrt{\alpha} h(r)) + 2h(r)|f|$$

since the function coth is strictly decreasing on $\mathbb{R}^+$. From $\|\nabla r\| \leq |Dr| = 1$ we also obtain

$$2h'(r)^2 \|\nabla r\|^2 \leq 2, \qquad \|\nabla \psi\| = 2h(r)\|\nabla r\| \leq 2h(r)$$

and then the first part of the thesis follows by observing that $\sqrt{\psi} = \varphi = h(r)$.

We prove the second statement. Let $\bar{x} \in \Omega \cap \mathrm{cut}(o)$. Choose a segment $\gamma : [0, r(\bar{x})] \to M$ such that $\gamma(0) = o$ and $\gamma(r(\bar{x})) = \bar{x}$. Fix $\varepsilon \in (0, r(\bar{x}))$, let $o_\varepsilon = \gamma(\varepsilon)$, $r_\varepsilon(x) = \mathrm{dist}_\sigma(o_\varepsilon, x)$ and define $\gamma_\varepsilon : [0, r_\varepsilon(\bar{x})] \to M$ by setting

$$\gamma_\varepsilon(s) = \gamma(s + \varepsilon) \qquad \text{for every } s \in [0, r_\varepsilon(\bar{x})].$$

The curve γ_ε can be extended to a segment on a slightly larger interval $[0, r_\varepsilon(\bar{x}) + \varepsilon']$, for some $\varepsilon' > 0$, and satisfies $\dot{\gamma}_\varepsilon(s) = \dot{\gamma}(s + \varepsilon)$ for every $0 \leq s \leq r_\varepsilon(\bar{x})$. Then

$$K(\dot{\gamma}_\varepsilon(s) \wedge X) \geq -G(s + \varepsilon) \qquad \text{for every } s \in (0, r_\varepsilon(\bar{x})),\ X \perp \dot{\gamma}_\varepsilon(s).$$

Since the function r_ε is of class C^2 in a neighbourhood of $\bar{x}$, we can repeat the same reasoning as above. We set $G_\varepsilon(s) = G(s + \varepsilon)$, $\alpha_\varepsilon = G_\varepsilon(0)$, $H_\varepsilon(t) = \int_0^t \sqrt{G_\varepsilon(s)} \mathrm{d}s$. The function $\phi_\varepsilon(t) = \alpha_\varepsilon^{-1} \sqrt{G_\varepsilon(s)} \coth(H_\varepsilon(t))$ satisfies

$$\begin{cases} \phi_\varepsilon' + \phi_\varepsilon^2 \geq G_\varepsilon & \text{on } \mathbb{R}^+, \\ \phi_\varepsilon(s) = s^{-1} + O(1) & \text{as } s \to 0 \end{cases}$$

and we are led to

$$\Delta_g r_\varepsilon(\bar{x}) \leq (m-1)\phi_\varepsilon(r_\varepsilon(\bar{x})) + |f|.$$

Then, we define

$$\psi_\varepsilon(x) = h(r_\varepsilon(x) + \varepsilon)^2$$

where h is the same function as above. Since $r_\varepsilon(x) + \varepsilon \geq r(x)$ on M, with equality at $\bar{x}$, and h is non-decreasing, we have $\psi_\varepsilon \geq \psi$ with equality at $\bar{x}$. Estimating as above we obtain

$$\begin{aligned} \Delta_g \psi_\varepsilon &\leq 2h'(r_\varepsilon + \varepsilon) h(r_\varepsilon + \varepsilon) \Delta_g r_\varepsilon + 2h'(r_\varepsilon + \varepsilon)^2 \|\nabla r_\varepsilon\|^2 \\ &\leq 2h'(r_\varepsilon + \varepsilon) h(r_\varepsilon + \varepsilon) \Delta_g r_\varepsilon + 2 \end{aligned}$$

in a neighbourhood of $\bar{x}$. In particular, since $r_\varepsilon(\bar{x}) + \varepsilon = r(\bar{x})$, we have

$$\Delta_g \psi_\varepsilon(\bar{x}) \leq 2h'(r(\bar{x})) h(r(\bar{x})) \Delta_g r_\varepsilon(\bar{x}) + 2.$$

As $\varepsilon \to 0$ we have $\phi_\varepsilon \to \phi$ uniformly on compact subsets of $\mathbb{R}^+$, and $r_\varepsilon(\bar{x}) \to r(\bar{x})$, then $\phi_\varepsilon(r_\varepsilon(\bar{x})) \to \phi(r(\bar{x}))$ and we obtain

$$\limsup_{\varepsilon \to 0} \Delta_g \psi_\varepsilon(\bar{x}) \leq 2\left((m-1)\sqrt{\alpha \psi(\bar{x})} \coth\left(\sqrt{\alpha \psi(\bar{x})}\right) + \sqrt{\psi(\bar{x})}|f| + 1\right).$$

Moreover, for every $\varepsilon > 0$

$$\|\nabla \psi_\varepsilon(\bar{x})\| = 2h(r(\bar{x}))\|\nabla r_\varepsilon\|^2 \leq 2h(r(\bar{x})) = 2\sqrt{\psi(\bar{x})}.$$

Then, the conclusion follows by choosing $\psi_k = \psi_{\varepsilon_k}$ for some sequence $\varepsilon_k \to 0$. □

The second key result of this section, Theorem 3.10 below, is concerned with the case of quadratic decay of the negative part of the curvature tensor. In order to prove it, we need two computational results.

LEMMA 3.7. *The function* $\psi(s) = s\coth(s)$ *satisfies*

$$\psi'(s) > 0, \qquad 1 < \psi(s) < \frac{1+\sqrt{4s^2+1}}{2} \qquad \text{for every } s > 0. \tag{43}$$

PROOF. A straightforward computation yields

$$\psi'(s) = \frac{\sinh(s)\cosh(s) - s}{\sinh^2(s)} = \frac{\sinh(2s) - 2s}{2\sinh^2(s)} > 0 \qquad \text{for } s > 0.$$

Observing that $\psi(s) \to 1$ as $s \to 0$, this implies $1 < \psi(s)$ for every $s > 0$. In view of this, we have equivalence

$$\psi(s) < \frac{1+\sqrt{4s^2+1}}{2} \quad \Leftrightarrow \quad (2\psi(s)-1)^2 < 4s^2+1 \quad \Leftrightarrow \quad \psi(s)^2 - \psi(s) < s^2.$$

By direct computation we have

$$\psi(s) + s^2 - \psi(s)^2 = \frac{s\sinh(s)\cosh(s) - s^2}{\sinh^2(s)} = s\psi'(s) > 0 \qquad \text{for } s > 0$$

and this concludes the proof of the claim. □

The proof of the next Lemma 3.9 relies on the following comparison theorem for Riccati inequalities, drawn from Corollary 2.2 in [**45**].

THEOREM 3.8 (Comparison theorem for Riccati inequalities). *Let* $G \in C^0(\mathbb{R}^+_0)$*, let* $T_1, T_2 > 0$ *and let* $\phi_i \in AC((0, T_i))$*,* $i = 1, 2$*, satisfy*

$$\begin{cases} \phi_1' + \phi_1^2 \le G & \text{on } (0, T_1), \\ \phi_1(t) = t^{-1} + O(1) & \text{as } t \to 0^+, \end{cases} \qquad \begin{cases} \phi_2' + \phi_2^2 \ge G & \text{on } (0, T_2), \\ \phi_2(t) = t^{-1} + O(1) & \text{as } t \to 0^+. \end{cases}$$

Then $T_1 \le T_2$ *and* $\phi_1 \le \phi_2$ *on* $(0, T_1)$.

LEMMA 3.9. *Let* $c > 0$*. The asymptotic Cauchy problem*

$$\begin{cases} \phi'(s) + \phi(s)^2 = \dfrac{c^2}{1+s^2} & \text{for } s \in \mathbb{R}^+, \\ \phi(s) = s^{-1} + O(1) & \text{as } s \to 0 \end{cases} \tag{44}$$

has a global solution $\phi \in C^1(\mathbb{R}^+)$ *satisfying*

$$\phi(s) \le \frac{1+\sqrt{4c^2+1}}{2s} \qquad \text{for every } s > 0.$$

PROOF. The Cauchy problem

$$\begin{cases} h''(s) = \dfrac{c^2}{1+s^2} h(s), \\ h(0) = 0, \ h'(0) = 1 \end{cases}$$

has a global solution $h \in C^1(\mathbb{R}^+_0) \cap C^2(\mathbb{R}^+)$ satisfying $h > 0$ on $\mathbb{R}^+$. The function $\phi = h'/h$ is then a solution of (44). We define functions

$$\phi_0(s) = c\coth(cs), \qquad \phi_1(s) = \frac{c'}{s} \qquad \text{with } c' = \frac{1+\sqrt{4c^2+1}}{2}.$$

A direct computation shows that

$$\phi_0'(s) + \phi_0(s)^2 = c^2 \ge \frac{c^2}{1+s^2}$$

while $\phi(s) = s^{-1} + o(1)$ as $s \to 0$. From the comparison theorem for Riccati inequalities, Theorem 3.8 we deduce $\phi \le \phi_0$ on $\mathbb{R}^+$. Then, by (43) we have $\phi(1) \le \phi_0(1) = c\coth(c) < c' = \phi_1(1)$. We compute

$$\phi_1'(s) + \phi_1(s)^2 = \frac{c'(c'-1)}{s^2} = \frac{c^2}{s^2} \ge \frac{c^2}{1+s^2}$$

and then we apply again the comparison theorem for Riccati inequalities to deduce $\phi \le \phi_1$ on $[1, +\infty)$. From Lemma 3.7, for every $0 < s < 1$ we can estimate

$$\phi_0(s) = c\coth(cs) = \frac{cs\coth(cs)}{s} \le \frac{c\coth(c)}{s} \le \frac{c'}{s} = \phi_1(s)$$

and this, together with $\phi \le \phi_0$, yields $\phi \le \phi_1$ on $\mathbb{R}^+$. □

THEOREM 3.10. *Let (M, σ) be a complete Riemannian manifold. Let $r(x)$ be the distance function from a reference origin $o \in M$ and assume that*

$$K_{\mathrm{rad}} \ge -\frac{c^2}{1+r^2} \qquad \text{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $c \ge 0$. Also let Ω, u, f be as in Theorem 3.6. Then

$$\Delta_g r^2 \le (m-1)\left(1 + \sqrt{4c^2+1}\right) + 2r|f| + 2 \qquad \text{on } \Omega \setminus \mathrm{cut}(o)$$

and for every $\bar{x} \in \Omega \cap \mathrm{cut}(o)$ there exist sequences of neighbourhoods U_k of $\bar{x}$ and a functions $\psi_k \in C^2(U_k)$ such that

$$\begin{cases} \psi_k \ge \psi & \text{in } U_k, \\ \psi_k(\bar{x}) = \psi(\bar{x}), \\ \|\nabla\psi_k\|(\bar{x}) \le 2r(\bar{x}), \\ \limsup_{k\to+\infty} \Delta_g\psi_k(\bar{x}) \le (m-1)\left(1+\sqrt{4c^2+1}\right) + 2r(\bar{x})|f| + 2. \end{cases}$$

PROOF. From the Hessian comparison theorem and Lemma 3.9, and estimating as in the proof of Theorem 3.6, we obtain

$$\Delta_g r \le \frac{(m-1)\left(1+\sqrt{4c^2+1}\right)}{2r} + |f| \qquad \text{on } \Omega \cap D_o,$$
$$\Delta_g r^2 = 2r\Delta_g r + 2\|\nabla r\|^2 \le (m-1)\left(1+\sqrt{4c^2+1}\right) + 2r|f| + 2 \qquad \text{on } \Omega \setminus \mathrm{cut}(o).$$

Let $\bar{x} \in \Omega \cap \mathrm{cut}(o)$. Choose a segment $\gamma : [0, r(\bar{x})] \to M$ such that $\gamma(0) = o$ and $\gamma(r(\bar{x})) = \bar{x}$. Fix $\varepsilon \in (0, r(\bar{x}))$, let $o_\varepsilon = \gamma(\varepsilon)$, $r_\varepsilon(x) = \mathrm{dist}_\sigma(o_\varepsilon, x)$ and define $\gamma_\varepsilon : [0, r_\varepsilon(\bar{x})] \to M$ by setting

$$\gamma_\varepsilon(s) = \gamma(s+\varepsilon) \qquad \text{for every } s \in [0, r_\varepsilon(\bar{x})].$$

The curve γ_ε can be extended to a segment on a slightly larger interval $[0, r_\varepsilon(\bar{x}) + \varepsilon']$, for some $\varepsilon' > 0$, and satisfies $\dot\gamma_\varepsilon(s) = \dot\gamma(s+\varepsilon)$ for every $0 \le s \le r_\varepsilon(\bar{x})$. Then

$$K(\dot\gamma_\varepsilon(s) \wedge X) \ge -\frac{c^2}{1+(s+\varepsilon)^2} \ge -\frac{c^2}{1+s^2} \qquad \text{for every } s \in (0, r_\varepsilon(\bar{x})),\ X \perp \dot\gamma_\varepsilon(s).$$

From the Hessian comparison Theorem 3.5 and Lemma 3.9 we have

$$\Delta_g r_\varepsilon(\bar{x}) \le (m-1)\frac{1+\sqrt{4c^2+1}}{2r_\varepsilon(\bar{x})} + |f|.$$

Setting $\psi_\varepsilon = (r_\varepsilon + \varepsilon)^2$, we have $\psi_\varepsilon \ge r^2$, with equality at $\bar{x}$, and

$$\|\nabla\psi_\varepsilon\|(\bar{x}) \le 2r(\bar{x}), \quad \Delta_g\psi_\varepsilon(\bar{x}) \le (m-1)\left(1+\sqrt{4c^2+1}\right)\frac{r(\bar{x})}{r_\varepsilon(\bar{x})} + 2r(\bar{x})|f| + 2.$$

The desired conclusion then follows by choosing $\psi_k = \psi_{\varepsilon_k}$ for some sequence $\varepsilon_k \to 0$. □

3. Construction via potential theory

Let (M,σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open domain and $u \in C^2(\Omega)$. In this section we show that if the graph $\Sigma = \{(x,u(x)) : x \in \Omega\}$ has bounded mean curvature in $M \times \mathbb{R}$ and M, $\partial\Omega$, $u_{|\partial\Omega}$ do satisfy some mild requirements of global geometric nature, then for any fixed base point $q \in \Omega$ the volume of geodesic balls $B^g_r(q)$ of (Ω, g) (equivalently, the volume of geodesic balls of the graph (Σ, g) centered at $(q,u(q)) \in \Sigma$) satisfies

$$\liminf_{r\to+\infty} \frac{\log |B^g_r(q)|}{r^2} < +\infty. \tag{45}$$

Starting from this fact, we will prove that for every subdomain $\Omega_0 \subseteq \Omega$ with $\overline{\Omega_0} \subseteq \Omega$ and for every $p \in \Omega_0$, $\lambda > 0$ there exists a smooth function $\psi : \overline{\Omega_0} \to [0,+\infty)$ satisfying

$$\begin{cases} \psi(p) = 1, & \\ \psi > 1 & \text{on } \overline{\Omega_0} \setminus \{p\}, \\ \psi(x) \to +\infty & \text{as } \operatorname{dist}_\sigma(p,x) \to \infty, \\ \Delta_g \psi \le \lambda\psi & \text{on } \Omega_0. \end{cases}$$

This will be done by isometrically embedding Ω_0 in a complete Riemannian manifold without boundary (N,h) satisfying a volume growth condition analogous to (45) and by showing that on such manifold, for every $q \in N$, there exists a smooth $\psi_0 : N \to [0,+\infty)$ satisfying

$$\begin{cases} \psi_0(q) = 1, & \\ \psi_0 > 1 & \text{on } N \setminus \{q\}, \\ \psi_0(x) \to +\infty & \text{as } x \to \infty \text{ in } N, \\ \Delta_h \psi_0 \le \lambda\psi & \text{in } N. \end{cases}$$

The first step in this direction is given by Lemma 3.12 below, whose proof relies on a calibration argument due to Trudinger, [**54**], and on a basic inequality proved in the next Lemma 3.11. Hereafter, for any $o \in M$, notation $B_r(o)$ will indicate $B^\sigma_r(o)$, that is, the geodesic ball of radius $r > 0$ and center o in (M,σ).

LEMMA 3.11. *Let (M,σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open subset, $u \in C^2(\Omega)$. Let $o \in M$, $p \in \Omega$, $a \in \mathbb{R}$ and set $d = \max\{\operatorname{dist}_\sigma(o,p), |u(p) - a|\}$. For every $d < R$ and for every $\Omega_0 \subseteq \Omega$,*

$$|\Omega_0 \cap B^g_{R-d}(p)|_g \le \int_{A_R} W \, \mathrm{d}x_\sigma \le |\Omega_0 \cap B_R(o)|_\sigma + \int_{A_R} \frac{|Du|^2}{W} \, \mathrm{d}x_\sigma \tag{46}$$

where $A_R = B_R(o) \cap \{x \in \Omega_0 : |u(x) - a| < R\}$ and $|\cdot|_g$, $|\cdot|_\sigma$ denote volume measures induced by g and σ, respectively.

PROOF. The map $\mathrm{id}_\Omega : \Omega \to \Omega$ is distance decreasing from (Ω, g) to (Ω,σ), so we have $B^g_{R-d}(p) \subseteq B_{R-d}(p)$. From triangle inequality and from the definition of d we also have

$$B_{R-d}(p) \subseteq B_{R-d+\operatorname{dist}_\sigma(o,p)}(o) \subseteq B_R(o).$$

Since $\|\nabla u\| < 1$ in Ω, we also have $B^g_{R-d}(p) \subseteq \{x \in \Omega : |u(x) - u(p)| < R - d\}$ and again from triangle inequality and definition of d we obtain

$$\begin{aligned} B_{R-d}(p) &\subseteq \{x \in \Omega : |u(x) - a| < R - d + |u(p) - a|\} \\ &\subseteq \{x \in \Omega : |u(x) - a| < R\}. \end{aligned}$$

The above inclusions yield $\Omega_0 \cap B^g_{R-d}(p) \subseteq A_R$ and then we have

$$|\Omega_0 \cap B^g_{R-d}(p)|_g = \int_{\Omega_0 \cap B^g_{R-d}(p)} 1 \, \mathrm{d}x_g = \int_{\Omega_0 \cap B^g_{R-d}(p)} W \, \mathrm{d}x_\sigma \le \int_{A_R} W \, \mathrm{d}x_\sigma.$$

Observing that $W = \frac{|Du|^2}{W} + \frac{1}{W} \leq \frac{|Du|^2}{W} + 1$ and $A_R \subseteq \Omega_0 \cap B_R(o)$ we further estimate

$$\int_{A_R} W \,\mathrm{d}x_\sigma \leq \int_{A_R} \frac{|Du|^2}{W} \,\mathrm{d}x_\sigma + |A_R|_\sigma \leq \int_{A_R} \frac{|Du|^2}{W} \,\mathrm{d}x_\sigma + |\Omega_0 \cap B_R(o)|_\sigma.$$

□

LEMMA 3.12. *Let (M, σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open subset, $u \in C^2(\Omega)$. Let $o \in M$, $p \in \Omega$, $a \in \mathbb{R}$ and set $d = \max\{\mathrm{dist}_\sigma(o,p), |u(p) - a|\}$. Also let $\Omega_0 \subseteq \Omega$ be a subdomain with smooth boundary and such that $\overline{\Omega_0} \subseteq \Omega$. For every $d < R < R_1$,*

$$\begin{aligned} |\Omega_0 \cap B^g_{R-d}(p)|_g \leq |\Omega_0 \cap B_R(o)|_\sigma + \frac{R}{R_1 - R} |\Omega_0 \cap B_{R_1}(o) \setminus B_R(o)|_\sigma + \\ + R \int_{\Omega_0 \cap B_{R_1}(o)} |f| \,\mathrm{d}x_\sigma + \int_{(\partial\Omega_0) \cap B_{R_1}(o)} \min\{R, |u - a|\} \,\mathrm{d}\mathcal{H}^{m-1}_\sigma \end{aligned} \tag{47}$$

where

$$f = \mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}} \right).$$

PROOF. Consider the functions u_R, ψ defined by

$$u_R = \begin{cases} -R & \text{if } u < a - R, \\ u - a & \text{if } a - R \leq u \leq a + R, \\ R & \text{if } u > a + R, \end{cases}$$

$$\psi(x) = \begin{cases} 1 & \text{if } x \in B_R(o), \\ \dfrac{R_1 - r(x)}{R_1 - R} & \text{if } x \in B_{R_1}(o) \setminus B_R(o), \\ 0 & \text{if } x \in M \setminus B_{R_1}(o). \end{cases}$$

Note that $|\psi u_R| = \psi |u_R| \leq |u_R| = \min\{R, |u - a|\}$. The vector field

$$X = \psi u_R \frac{Du}{W}.$$

is defined and Lipschitz regular in a neighbourhood of $\overline{\Omega_0}$ and is supported in the compact set $\overline{\Omega \cap B_{R_1}(o)}$. Since $\partial\Omega_0$ is smooth, we can apply the divergence theorem with respect to the Riemannian metric σ to obtain

$$\int_{\Omega_0} \mathrm{div}\,(X) \,\mathrm{d}x_\sigma = \int_{\partial\Omega_0} (X, \nu) \,\mathrm{d}\mathcal{H}^{m-1}_\sigma,$$

where ν is the exterior normal to $\partial\Omega_0$. We compute the divergence of X

$$\begin{aligned} \mathrm{div}\,(X) &= \psi u_R \,\mathrm{div}\left(\frac{Du}{W}\right) + \psi \frac{(Du_R, Du)}{W} + u_R \frac{(D\psi, Du)}{W} \\ &= \psi u_R f + \psi \frac{|Du|^2}{W} \mathbf{1}_{\{|u-a|<R\}} - \frac{u_R}{R_1 - R} \frac{(Dr, Du)}{W} \mathbf{1}_{B_{R_1}(o) \setminus B_R(o)} \end{aligned}$$

and then we can write

$$\begin{aligned} \int_{\partial\Omega_0} \psi u_R \frac{(Du, \nu)}{W} \,\mathrm{d}\mathcal{H}^{m-1}_\sigma = &\int_{\{|u-a|<R\}} \psi \frac{|Du|^2}{W} \,\mathrm{d}x_\sigma + \int_{\Omega_0} \psi u_R f \,\mathrm{d}x_\sigma \\ &- \frac{1}{R_1 - R} \int_{\Omega_0 \cap B_{R_1}(o) \setminus B_R(o)} u_R \frac{(Dr, Du)}{W} \,\mathrm{d}x_\sigma. \end{aligned}$$

We rearrange the terms and use Cauchy-Schwarz inequality to write

$$\int_{\{|u-a|<R\}} \psi \frac{|Du|^2}{W}\,\mathrm{d}x_\sigma \le \frac{1}{R_1-R}\int_{\Omega_0\cap B_{R_1}(o)\setminus B_R(o)} |u_R|\frac{|Du|}{W}\,\mathrm{d}x_\sigma + \int_{\Omega_0}\psi|u_R||f|\,\mathrm{d}x_\sigma + \int_{\partial\Omega_0}\psi|u_R|\frac{|Du|}{W}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma$$

Since $\psi \equiv 1$ on $B_R(o)$, $\psi \equiv 0$ on $M \setminus B_{R_1}(o)$ and $0 \le \psi \le 1$ on M, using inequalities $|Du| < W$ and $|u_R| = \min\{R, |u-a|\} \le R$ we obtain

$$\int_{A_R} \frac{|Du|^2}{W}\,\mathrm{d}x_\sigma \le \frac{R}{R_1-R}|\Omega_0\cap B_{R_1}(o)\setminus B_R(o)|_\sigma + R\int_{\Omega_0\cap B_{R_1}(o)}|f|\,\mathrm{d}x_\sigma + \int_{(\partial\Omega_0)\cap B_{R_1}(o)}\min\{R,|u-a|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma$$

where $A_R = B_R(o)\cap\{x\in\Omega_0 : |u(x)-a| < R\}$. Then the desired conclusion follows from Lemma 3.11. □

Theorem 3.13. *Let (M,σ) be a complete Riemannian manifold satisfying*

$$\mathrm{Ric}(Dr,Dr) \ge -\alpha^2(1+r)^2 \qquad \text{on } D_o = M\setminus(\{o\}\cup\mathrm{cut}(o)) \tag{48}$$

for some $\alpha \ge 0$ and some reference origin $o \in M$, where $r(x) = \mathrm{dist}_\sigma(o,x)$. Let $\Omega \subseteq M$ be an open domain and let $u \in C^2(\Omega)$ satisfy

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f$$

for some bounded function $f : \Omega \to \mathbb{R}$. Assume that one of the following conditions is satisfied:

a) $\Omega = M$,
b) $u \in C^0(\overline{\Omega})$ *and* $u_{|\partial\Omega}$ *is constant,*
c) $\partial\Omega$ *is locally Lipschitz and*

$$\liminf_{r\to+\infty}\frac{\log\mathcal{H}^{m-1}_\sigma((\partial\Omega)\cap B_r(o))}{r^2} < +\infty, \tag{49}$$

d) $u \in C^0(\overline{\Omega})$, $\partial\Omega$ *is locally Lipschitz and for some* $u_0 \in \mathbb{R}$

$$\liminf_{r\to+\infty}\frac{\log\displaystyle\int_{(\partial\Omega)\cap B_r}\min\{r,|u-u_0|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma}{r^2} < +\infty. \tag{50}$$

Then, for any $p \in \Omega$

$$\liminf_{r\to+\infty}\frac{\log|B^g_r(p)|_g}{r^2} < +\infty.$$

Proof. Let $C > 0$ be such that $|f| \le C$ on Ω, and let $\Omega_0 \subseteq \Omega$, a, d be as in Lemma 3.12. For almost every $r > 0$ the geodesic ball $B_r(o)$ has Lipschitz regular boundary and from the coarea formula we have

$$\lim_{R_1\to r}\frac{|\Omega_0\cap B_{R_1}(o)\setminus B_r(o)|_\sigma}{R_1-r} = \mathcal{H}^{m-1}_\sigma(\Omega_0\cap\partial B_r(o)).$$

Then, by taking limits for $R_1 \to r$ in (47), for almost every $r > 0$ we have

$$|\Omega_0\cap B^g_{r-d}(p)|_g \le (1+Cr)|\Omega_0\cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma(\Omega_0\cap\partial B_r(o)) + \int_{(\partial\Omega_0)\cap B_r(o)}\min\{r,|u-a|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma.$$

Case a). Let $p = o$, $a = u(o)$, $\Omega_0 = M$. Then $d = 0$ and we have

$$|B^g_r(o)|_g \le (1+Cr)|B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma(\partial B_r(o)). \tag{51}$$

Case b). Let $p \in \Omega$ and let u_0 be the constant value of u on $\partial\Omega$. Let $k \in \mathbb{N}$ be given, choose a regular value $a_k \in (u_0, u_0 + 1/k)$ for u and set $\Omega_k = \{x \in \Omega : u(x) > a_k\}$. With the choice $a = a_k$, $\Omega_0 = \Omega_k$ we have $u = a$ on $\partial\Omega_0$, so

$$|\Omega_k \cap B^g_{r-d}(p)|_g \leq (1 + Cr)|\Omega_k \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_k \cap \partial B_r(o)\right)$$

The sequence $\{\Omega_k\}$ monotonically converges from below to the set $\Omega_+ = \{x \in \Omega : u(x) > u_0\}$, so we obtain

$$|\Omega_+ \cap B^g_{r-d}(p)|_g \leq (1 + Cr)|\Omega_+ \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_+ \cap \partial B_r(o)\right).$$

A similar argument yields

$$|\Omega_- \cap B^g_{r-d}(p)|_g \leq (1 + Cr)|\Omega_- \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_- \cap \partial B_r(o)\right)$$

with $\Omega_- = \{x \in \Omega : u(x) < u_0\}$, and then

$$|\Omega_* \cap B^g_{r-d}(p)|_g \leq (1 + Cr)|\Omega_* \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_* \cap \partial B_r(o)\right)$$

having set $\Omega_* = \{x \in \Omega : u(x) \neq u_0\}$. From Stampacchia's theorem (Theorem 1.56 of [**53**]) we have $|Du| = 0$, and then $W = 1$, $\mathrm{d}x_g = \mathrm{d}x_\sigma$, almost everywhere on $\{u(x) = u_0\}$. Hence,

$$|B^g_{r-d}(p) \setminus \Omega_*|_g = |B^g_{r-d}(p) \setminus \Omega_*|_\sigma \leq |B_r(o) \setminus \Omega_*|_\sigma$$

and we conclude

$$|B^g_{r-d}(p)|_g \leq (1 + Cr)|\Omega \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega \cap \partial B_r(o)\right). \tag{52}$$

Case c). Let $p \in \Omega$, $a = u(p)$. It is possible to find a smooth exhaustion $\{\Omega_k\}$ of Ω, that is, a sequence of open sets with smooth boundaries such that

$$\overline{\Omega_k} \subseteq \Omega_{k+1} \quad \forall k \in \mathbb{N}, \qquad \Omega = \bigcup_{k\in\mathbb{N}} \Omega_k,$$

with the additional property that

$$\lim_{k\to+\infty} \mathcal{H}^{m-1}_\sigma((\partial\Omega_k) \cap B_r(o)) = \mathcal{H}^{m-1}_\sigma((\partial\Omega) \cap B_r(o)).$$

To justify this we refer to [**48**] and Theorem 5.1 in [**16**]: in the neighbourhood $U_{\bar{x}}$ of any point $\bar{x} \in M$ it is possible to find a local chart $\phi : U_{\bar{x}} \to V \subseteq \mathbb{R}^m$ such that $\phi(U_{\bar{x}} \cap \Omega) = \{x \in V : x^m > \psi(x^1, \ldots, x^{m-1})\}$ and $\phi(U_{\bar{x}} \cap \partial\Omega) = \{x \in V : x^m = \psi(x^1, \ldots, x^{m-1})\}$ for some Lipschitz continuous function $\psi : V_0 \to \mathbb{R}$ defined on an open set $V_0 \subseteq \mathbb{R}^{m-1}$ such that $V \subseteq V_0 \times \mathbb{R}$. By the aforementioned Theorem, there exists a sequence $\{\psi_k\}$ of smooth functions $\psi_k : V_0 \to \mathbb{R}$ such that

$$\begin{aligned} \psi_k > \psi \quad & \text{for every } \ k \geq 1 \\ \psi_k \to \psi \quad & \text{uniformly on } \ V_0 \ \text{ as } \ k \to \infty, \\ \partial_{x^i}\psi_k \to \partial_{x^i}\psi \quad & \text{in } \ L^p(V_0), \ \text{ for every } \ p \geq 1, \ \text{ as } \ k \to \infty. \end{aligned}$$

Then the sets $\{x^m > \psi_k(x^1, \ldots, x^{m-1})\}$ do approximate $\phi(U_{\bar{x}} \cap \Omega)$ from the inside, and up to extraction of a subsequence we can assume that they form a monotonically increasing sequence with respect to inclusion. Integration with respect to the Hausdorff measure induced from σ on the hypersurface $\phi^{-1}(\{x \in V : x^m = \psi_k(x^1, \ldots, x^{m-1})\})$ can be represented as integration against $\sqrt{\sigma^{mm} + 2\sum_{i=1}^{m-1} \sigma^{im}\partial_{x^i}\psi_k + \sum_{i,j=1}^{m-1} \sigma^{ij}\partial_{x^i}\psi_k\partial_{x^j}\psi_k}$ in $\mathbb{R}^{m-1}$, and this converges to $\sqrt{\sigma^{mm} + 2\sum_{i=1}^{m-1} \sigma^{im}\partial_{x^i}\psi + \sum_{i,j=1}^{m-1} \sigma^{ij}\partial_{x^i}\psi\partial_{x^j}\psi}$ in $L^1(V_0)$ as $k \to \infty$. In turn, integration with respect to this weight in $\mathbb{R}^{m-1}$ represents integration with respect to Hausdorff measure induced from σ on $\phi^{-1}(\{x \in V : x^m = \psi_k(x^1, \ldots, x^{m-1})\}) = U_{\bar{x}} \cap \partial\Omega$. Coupling this basic construction with a partition of unity one obtains sets Ω_k with the desired properties.

For every $k \in \mathbb{R}$ we can estimate

$$\int_{(\partial\Omega_k)\cap B_r(o)} \min\{r, |u-a|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma \le r\mathcal{H}^{m-1}_\sigma((\partial\Omega_k)\cap B_r(o))$$

and then, choosing $\Omega_0 = \Omega_k$, we have

$$\begin{aligned}|\Omega_k \cap B^g_{r-d}(p)|_g \le (1+Cr)|\Omega_k \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_k \cap \partial B_r(o)\right) \\ + r\mathcal{H}^{m-1}_\sigma((\partial\Omega_k)\cap B_r(o)).\end{aligned}$$

Taking limits of both sides as $k \to +\infty$ we obtain

$$\begin{aligned}|B^g_{r-d}(p)|_g \le (1+Cr)|\Omega \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega \cap \partial B_r(o)\right) \\ + r\mathcal{H}^{m-1}_\sigma((\partial\Omega)\cap B_r(o)).\end{aligned} \tag{53}$$

Case d). Let $p \in \Omega$, $a = u_0$ and let $\{\Omega_k\}$ be again a smooth exhaustion of Ω, with the additional property that the restriction of $\mathcal{H}^{m-1}_\sigma$ to $\partial\Omega_k$ weakly-star converge to the restriction of $\mathcal{H}^{m-1}_\sigma$ to $\partial\Omega$ as $k \to +\infty$. In other words, we are assuming that

$$\lim_{k\to+\infty} \int_{\partial\Omega_k} \varphi\,\mathrm{d}\mathcal{H}^{m-1}_\sigma = \int_{\partial\Omega} \varphi\,\mathrm{d}\mathcal{H}^{m-1}_\sigma \qquad \text{for every } \varphi \in C^0_c(\overline{\Omega}).$$

This is possible by the same argument outlined in the proof of Case c). Then for every $k \in \mathbb{N}$ we have, choosing $\Omega_0 = \Omega_k$,

$$\begin{aligned}|\Omega_k \cap B^g_{r-d}(p)|_g \le (1+Cr)|\Omega_k \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega_k \cap \partial B_r(o)\right) \\ + \int_{(\partial\Omega_k)\cap B_r(o)} \min\{r, |u-a|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma,\end{aligned}$$

and taking limits of both sides we obtain

$$\begin{aligned}|B^g_{r-d}(p)|_g \le (1+Cr)|\Omega \cap B_r(o)|_\sigma + r\mathcal{H}^{m-1}_\sigma\left(\Omega \cap \partial B_r(o)\right) \\ + \int_{(\partial\Omega)\cap B_r(o)} \min\{r, |u-a|\}\,\mathrm{d}\mathcal{H}^{m-1}_\sigma.\end{aligned} \tag{54}$$

By assumption (48), there exist constants $C_1, C_2 > 0$ such that

$$|B_r(o)|_\sigma, \mathcal{H}^{m-1}_\sigma\left(\partial B_r(o)\right) \le C_1 e^{C_2 r^2}$$

for almost every $r > 0$. For a proof of this statement we refer to [**45**], Proposition 2.11. In cases a) or b) this fact together with (51) or (52), respectively, yields

$$|B^g_{r-d}(p)|_g \le (1+(C+1)r)C_1 e^{C_2 r^2} \qquad \text{for every } r > 0$$

and then the desired conclusion follows. In cases c) or d) the same conclusion follows by evaluating inequality (53) or (54) along an appropriate diverging sequence $\{r_k\}$. □

The second step in our construction is the following doubling theorem, whose proof essentially reproduces the one given in [**11**].

THEOREM 3.14. *Let (M_1, g_1) be a connected Riemannian manifold and let $U_1 \subseteq M_1$ be an open, connected set with smooth boundary such that all bounded subsets of $\overline{U_1}$ have compact closure in M_1. Then there exist a connected, complete Riemannian manifold (M_2, g_2), an open subset $U_2 \subseteq M_2$ and a diffeomorphism $\phi : U_1 \to U_2$ with the following properties:*

(a) *$\phi : (U_1, g_1) \to (U_2, g_2)$ is an isometry*
(b) *for every $p \in U_1$ and for every $r \ge 2\,\mathrm{dist}_{g_1}(p, \partial U_1) + 2$*

$$|B^{g_2}_r(\phi(p))|_{g_2} \le 2|U_1 \cap B^{g_1}_{4r}(p)|_{g_1} + 6.$$

Proof. The boundary ∂U_1 is an embedded, smooth, orientable hypersurface in M_1. Let ν be the normal exterior vector field on ∂U_1. There exists a continuous function $t_0 : \partial U_1 \to (0, 1/2)$ such that the normal exponential map $\Psi(x,t) = \exp_x(t\nu(x))$ is a diffeomorphism between the set

$$\mathcal{D} = \{(x,t) \in \partial U_1 \times [0,1] : t < t_0(x)\} \subseteq \partial U_1 \times [0,1]$$

and its image $\Psi(\mathcal{D}) \subseteq M_1$. We write the pull-back metric $\Psi^* g_1$ on $\mathcal{D}$ as $h = \mathrm{d}t^2 + h_t$, so that h_t is the pull-back of the restriction of g_1 to $\Psi((\{t\} \times \partial U_1) \cap \mathcal{D})$. In particular, h_0 is the Riemannian metric induced by g_1 on ∂U_1. We can further assume that t_0 is such that

(i) $h_t(x) \geq \frac{1}{4} h_0(x)$ for every $0 \leq t < t_0(x)$,
(ii) $\int_{\partial U_1} t_0(x) \sqrt{M(x)} \mathrm{d}x \leq 1$, where

$$M(x) = \sup_{0 \leq t \leq \frac{3}{4} t_0(x)} \max \left\{ \frac{\|h_t(x,t)\|_{h_0(x)}^{m-1}}{(m-1)^{(m-1)/2}}, 1 \right\}.$$

We now construct a smooth metric $\tilde{h}$ on the collar $C_1 = \partial U_1 \times [0,1]$ so that the following conditions are satisfied:

$$\tilde{h}(x,t) = \begin{cases} \mathrm{d}t^2 + h_t(x) \equiv h(x,t) & \text{for } t \leq \frac{1}{2} t_0(x), \\ \mathrm{d}t^2 + h_0(x) & \text{for } t \geq 1 - \frac{1}{2} t_0(x), \end{cases} \tag{55}$$

and

$$|C_1|_{\tilde{h}} \leq 3.$$

In order to do so, consider a smooth cutoff function $\varphi : C_1 \to [0,1]$ and a positive smooth function $\eta : C_1 \to (0,1]$ satisfying

$$\varphi(x,t) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2} t_0(x), \\ 0 & \text{if } t \geq \frac{3}{4} t_0(x) \end{cases}$$

and

$$\eta(x,t) \begin{cases} = 1 & \text{for } t \in [0, \frac{1}{2} t_0(x)] \cup [1 - \frac{1}{2} t_0(x), 1], \\ \leq t_0(x)^2 & \text{for } t \in [t_0(x), 1 - t_0(x)], \end{cases}$$

then set

$$\tilde{h}(x,t) = \eta(x,t) \mathrm{d}t^2 + \varphi(x,t) h_t(x) + (1 - \varphi(x,t)) h_0(x).$$

From assumptions on φ and η we immediately have the validity of (55). From the arithmetic mean – geometric mean inequality we have

$$\begin{aligned} \det{}_{\mathrm{d}t^2 + h_0} \tilde{h}(x,t) &= \eta(x,t) \det{}_{h_0} [\varphi(x,t) h_t(x,t) + (1 - \varphi(x,t)) h_0(x,t)] \\ &\leq \eta(x,t) \frac{\|\varphi(x,t) h_t(x,t) + (1 - \varphi(x,t)) h_0(x,t)\|_{h_0}^{m-1}}{(m-1)^{(m-1)/2}} \\ &\leq \eta(x,t) \max \left\{ \frac{\|h_t(x,t)\|_{h_0}^{m-1}}{(m-1)^{(m-1)/2}}, 1 \right\} \\ &\leq \eta(x,t) M(x) \end{aligned}$$

and then we can write

$$|C_1|_{\tilde{h}} \leq \int_{\partial U_1} \sqrt{M(x)} \int_0^1 \sqrt{\eta(x,t)} \, \mathrm{d}t \, \mathrm{d}x \leq 3 \int_{\partial U_1} \sqrt{M(x)} t_0(x) \mathrm{d}x \leq 3.$$

Let $\tilde{U}_1$ be the smooth manifold with boundary obtained by gluing $\overline{U_1}$ and C_1 along their respective boundary components $\partial U_1 \subseteq \overline{U_1}$ and $\partial U_1 \times \{0\} \subseteq C_1$. Also let $\tilde{g}_1$ be the Riemannian metric on $\tilde{U}_1$ given by

$$\tilde{g}_1 = \begin{cases} g_1 & \text{on } U_1, \\ \tilde{h} & \text{on } C_1. \end{cases}$$

Since $\tilde{h} \equiv \Psi^* g_1$ in the intersection of C_1 with a neighbourhood of $\partial U_1 \times \{0\}$, we have that $\tilde{g}_1$ is a smooth Riemannian metric. Moreover, $\tilde{g}_1$ equals the product metric $\mathrm{d}t^2 + h_0(x)$ in a neighbourhood of the boundary $\partial\tilde{U}_1 = \partial U_1 \times \{1\} \subseteq C_1$, so $\partial\tilde{U}_1$ is totally geodesic in $\tilde{U}_1$ and the vector field ∂_t belongs to the kernel of the Riemann curvature operator of $\tilde{U}_1$ in a neighbourhood of $\partial\tilde{U}_1$. By a theorem due to Mori, [**42**], these conditions are sufficient to ensure that the Riemannian manifold (M_2, g_2) obtained by gluing $(\tilde{U}_1, \tilde{g}_1)$ with an isometric copy of itself, say $(\tilde{U}_1', \tilde{g}_1')$, along the common boundary $\partial\tilde{U}_1$ is a smooth Riemannian manifold. (M_2, g_2) is said to be a double of $(\tilde{U}_1, \tilde{g}_1)$.

The isometric embedding $(U_1, g_1) \hookrightarrow (\tilde{U}_1, \tilde{g}_1)$ naturally extends to an isometric embedding $(U_1, g_1) \hookrightarrow (M_2, g_2)$. Choosing U_2 as the image of U_1 under such embedding and letting $\phi : U_1 \to U_2$ be the resulting diffeomorphism, we have that $\phi : (U_1, g_1) \to (U_2, g_2)$ is a Riemannian isometry. It remains to show that (M_2, g_2) is complete and that condition (b) is satisfied.

We first show that (M_2, g_2) is complete. For $i = 1, 2$, let $V_i = \overline{U_i}$ be the closure of U_i in M_i and let dist_{M_i,g_i} and dist_{V_i,g_i} be the length distances induced by g_i on M_i and V_i, respectively. Our hypotheses imply that the space $(V_1, \mathrm{dist}_{V_1,g_1})$ is complete, and the map $\phi : U_1 \to U_2$ continuously extends to a bijection $\overline{\phi} : V_1 \to V_2$ that is a Riemannian isometry between manifolds with boundary, hence $(V_2, \mathrm{dist}_{V_2,g_2})$ is also complete. To show that $(M_2, \mathrm{dist}_{M_2,g_2})$ is complete, we construct a proper Lipschitz retraction $F : M_2 \to V_2$. Let us denote by $f : \tilde{U}_1 \to \tilde{U}_1'$ the isometry between $(\tilde{U}_1, \tilde{g}_1)$ and its copy $(\tilde{U}_1', \tilde{g}_1')$ considered in the construction of M_2. We now regard $\tilde{U}_1$ and $\tilde{U}_1'$ as subsets of M_2. The map $F_0 : M_2 \to \tilde{U}_1$ given by

$$F_0(x) = \begin{cases} x & \text{if } x \in \tilde{U}_1, \\ f^{-1}(x) & \text{otherwise} \end{cases}$$

is a retraction. Let $\pi : C_1 \to \partial U_1$ be the canonical projection onto the first factor. Note that $\pi(C_1) = \partial U_1$ can be identified with the boundary $\partial V_2 \equiv \partial U_2$ of V_2 in M_2. The map $F_1 : \tilde{U}_1 \to V_2$ given by

$$F_1(x) = \begin{cases} x & \text{if } x \in V_2, \\ \pi(x) & \text{otherwise} \end{cases}$$

is also a retraction, and so is the composition $F = F_1 \circ F_0 : M_2 \to V_2$. First, observe that F is proper: indeed, for every compact set $K \subseteq V_2$,

$$F_1^{-1}(K) = K \cup ((K \cap \partial V_2) \times [0, 1])$$

is compact, being a finite union of compact sets, and so is

$$F^{-1}(K) = F_0^{-1}(F_1^{-1}(K)) = F_1^{-1}(K) \cup f(F_1^{-1}(K)).$$

We also claim that F is 2-Lipschitz between $(M_2, \mathrm{dist}_{M_2,g_2})$ and $(V_2, \mathrm{dist}_{V_2,g_2})$. To this aim, let $x, y \in M_2$ and $\varepsilon > 0$. We show that $\mathrm{dist}_{V_2,g_2}(F(x), F(y)) < 2\,\mathrm{dist}_{M_2,g_2}(x, y) + 2\varepsilon$. Let $\gamma : [0, T] \to M_2$ be a curve joining x and y and such that $\ell_{g_2}(\gamma) < \mathrm{dist}_{M_2,g_2}(x, y) + \varepsilon$. Setting $C = C_1 \cup f(C_1)$, by the transversality theorem (Theorem 2.1 in Chapter 3 of [**26**]) we can assume that γ is transversal to ∂C, so in particular $F \circ \gamma : I \to V_2$ is a piecewise smooth curve joining $F(x)$ and $F(y)$ and there exist $0 = s_0 < s_1 < s_2 < \cdots < s_k = T$ such that, letting $I_j = (s_j, s_{j+1})$, for each $0 \le j \le k - 1$

$$\gamma(I_j) \subseteq \mathrm{Int}(C) \quad \text{or} \quad \gamma(I_j) \subseteq U_2 \cup f(U_2)$$

and for each $0 \le j \le k - 2$ the images $\gamma(I_j)$, $\gamma(I_{j+1})$ belong to distinct components of $M_2 \setminus \partial C$. If $\gamma(I_j) \subseteq U_2 \cup f(U_2)$ then

$$\ell_{g_2}((F \circ \gamma)_{|I_j}) = \ell_{g_2}(\gamma_{|I_j}).$$

On the other hand, assume that $\gamma(I_j) \subseteq \mathrm{Int}(C)$. Because of (i) in the definition of t_0,

$$g_2 = \tilde{h} \geq \eta(y,r)^2 \mathrm{d}r^2 + \frac{1}{4} h_0(y) \qquad \text{on } C,$$

so for every tangent vector $V \in TC$ we have

$$g_2(\pi_* V, \pi_* V) = h_0(\pi_* V, \pi_* V) \leq 4 g_2(V, V),$$

hence

$$\ell_{g_2}((F \circ \gamma)_{|I_j}) \leq 2 \ell_{g_2}(\gamma_{|I_j}).$$

Summarizing,

$$\mathrm{dist}_{V_2,g_2}(F(x), F(y)) \leq \ell_{g_2}(F \circ \gamma) \leq 2\ell_{g_2}(\gamma) < 2 \,\mathrm{dist}_{M_2,g_2}(x,y) + 2\varepsilon$$

as claimed. To conclude that (M_2, g_2) is complete, let $\{x_k\}$ be a Cauchy sequence in $(M_2, \mathrm{dist}_{M_2,g_2})$. Then, $\{F(x_k)\}$ is a Cauchy sequence in $(V_2, \mathrm{dist}_{V_2,g_2})$ and therefore converges to some $y \in V_2$ by the above observation. The properness of F implies that $\{x_k\}$ has a limit point in M_2, hence it converges.

We next observe that for every $x, y \in U_2$

$$\begin{aligned} \mathrm{dist}_{M_1,g_1}(\phi^{-1}(x), \phi^{-1}(y)) &\leq \mathrm{dist}_{V_1,g_1}(\phi^{-1}(x), \phi^{-1}(y)) \\ &= \mathrm{dist}_{V_2,g_2}(x,y) \\ &\leq 2\,\mathrm{dist}_{M_2,g_2}(x,y) \\ &\leq 2\,\mathrm{dist}_{V_2,g_2}(x,y), \end{aligned} \tag{56}$$

where the first inequality is obvious since (M_1, g_1) contains more curves joining $\phi^{-1}(x)$ and $\phi^{-1}(y)$ than (V_1, g_1) does, the last inequality follows by similar reason, and the middle inequality is a consequence of 2-Lipschitzianity of F, together with $F_{|V_2} = \mathrm{id}_{V_2}$.

To conclude, let $p \in U_1$ and $r \geq 2\mathrm{dist}_{M_1,g_1}(p, \partial U_1) + 2$ be given. Let $q = \phi(p) \in U_2$ and let $q' = f(q)$ be the copy of q in $f(U_2)$, let $R = \mathrm{dist}_{M_2,g_2}(q,q')$ and let $B_r^{g_1}(p)$, $B_r^{g_2}(q)$ be the geodesic balls centered at p and q in (M_1, g_1) and (M_2, g_2), respectively. By construction,

$$R \leq 2\mathrm{dist}_{M_1,g_1}(p, \partial U_1) + 2 \leq r$$

and thus

$$\begin{aligned} |B_r^{g_2}(q)|_{g_2} &= |B_r^{g_2}(q) \cap U_2|_{g_2} + |B_r^{g_2}(q) \cap C|_{g_2} + |B_r^{g_2}(q) \cap f(U_2)|_{g_2} \\ &\leq |B_r^{g_2}(q) \cap U_2|_{g_2} + 2|C_1|_{g_2} + |B_{r+R}^{g_2}(q') \cap f(U_2)|_{g_2} \\ &\leq 2|B_{r+R}^{g_2}(q) \cap U_2|_{g_2} + 6 \\ &\leq 2|B_{2r}^{g_2}(q) \cap U_2|_{g_2} + 6. \end{aligned}$$

From (56) we have $\phi^{-1}(B_{2r}^{g_2}(q) \cap U_2) \subseteq B_{4r}^{g_1}(p) \cap U_1$, thus we conclude

$$|B_r^{g_2}(q)|_{g_2} \leq 2|B_{4r}^{g_1}(p) \cap U_1|_{g_1} + 6$$

as required.

□

The third step is represented by Proposition 3.16 and Theorem 3.17 below, whose proofs reproduce the ones given in [**11**]. The proof of Proposition 3.16 is essentially a particular case of a more general construction developed by Mari and Valtorta, [**38**]. The following lemma, which we draw from Theorem 4.1 in [**3**], will be needed in it.

Lemma 3.15. *Let (N,h) be a Riemannian manifold, $\Omega \subseteq N$ a relatively compact open set, $0 \leq \lambda \in L^\infty_{\mathrm{loc}}(\Omega)$ a given function. If $u, v \in H^1_{\mathrm{loc}}(\Omega)$ do satisfy*

$$\begin{cases} \Delta v \leq \lambda v & \text{in } \Omega, \\ \Delta u \geq \lambda u & \text{in } \Omega, \\ u \leq v & \text{on } \partial\Omega \end{cases}$$

then $u \le v$ *on* Ω.

PROPOSITION 3.16. *Let* (N, h) *be a complete Riemannian manifold whose geodesic balls centered at some origin* $o \in N$ *satisfy*

$$\liminf_{r\to+\infty} \frac{\log |B^h_r(o)|_h}{r^2} < +\infty. \tag{57}$$

Then for any $q \in N$ *and* $\lambda > 0$ *there exists* $\psi_0 \in C^\infty(N)$ *satisfying*

$$\begin{cases} \psi_0(q) = 1, & \\ \psi_0 > 1 & \text{on } N \setminus \{q\}, \\ \psi_0(x) \to +\infty & \text{as } x \to \infty \text{ in } N, \\ \Delta_h \psi_0 \le \lambda \psi_0 & \text{on } N. \end{cases} \tag{58}$$

PROOF. Let $\varepsilon > 0$ be small enough so that the geodesic ball $B_{3\varepsilon}(q) \subseteq N$ has compact closure and the exponential map $\exp_q : B_{3\varepsilon}(0_{TN}) \to B_{3\varepsilon}(q)$ is a diffeomorphism. Then $B_R(q)$ has smooth boundary for every $0 < R < 3\varepsilon$ and the distance function from q is smooth in $B_{3\varepsilon}(q) \setminus \{q\}$. Let $\{\Omega_k\}$ be a smooth exhaustion of N, that is, a sequence of relatively compact open subsets with the property that

$$\overline{\Omega_k} \subseteq \Omega_{k+1} \quad \text{for every } k \ge 1, \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = N.$$

Without loss of generality, we assume that $\overline{B_\varepsilon(q)} \subseteq \Omega_1$. For every $k \in \mathbb{N}$ let u_k be the solution of the Dirichlet problem

$$\begin{cases} \Delta u_k = \lambda u_k & \text{in } \Omega_k \setminus \overline{B_\varepsilon(q)}, \\ u_k = 0 & \text{on } \partial B_\varepsilon(q), \\ u_k = 1 & \text{on } \partial\Omega_k. \end{cases}$$

We have $0 \le u_k \le 1$ on $\overline{\Omega_k} \setminus B_\varepsilon(q)$ by Lemma 3.15 applied with couples of functions $(u, v) = (0, u_k)$ and $(u, v) = (u_k, 1)$. The extension $v_k : N \setminus B_\varepsilon(q) \to [0, 1]$ of u_k obtained by setting $v_k \equiv 1$ on $N \setminus \overline{\Omega_k}$ is Lipschitz continuous and satisfies $\Delta v_k \le \lambda v_k$ in the barrier sense on $N \setminus \overline{B_\varepsilon(q)}$, and strongly on $N \setminus (\overline{B_\varepsilon(q)} \cap \partial\Omega_k)$.

From Lemma 3.15 we have that the sequence $\{v_k\}$ is monotone decreasing and then it converges pointwise to some function $v : N \setminus B_\varepsilon(q) \to [0, 1]$. By standard elliptic estimates and a diagonalization argument, up to extraction of a subsequence we have $v_k \to v$ also in the C^2 topology on each compact subset of $N \setminus B_\varepsilon(q)$, and v is a solution of the exterior Dirichlet problem

$$\begin{cases} \Delta v = \lambda v & \text{in } N \setminus \overline{B_\varepsilon(q)}, \\ v = 0 & \text{on } \partial B_\varepsilon(q). \end{cases}$$

From assumption (57), the manifold (N, h) satisfies the weak maximum principle in the sense of Pigola-Rigoli-Setti, see for instance Theorem 4.1 in [**1**], and it must be $v \le 0$. In particular, $v \equiv 0$. Then $\{v_k\}$ is a sequence of non-negative functions converging to 0 in the C^2 topology on each compact subset of $N \setminus B_\varepsilon(q)$. For every $j \ge 1$ we can find $k_j \ge 1$ such that $\|v_{k_j}\|_{C^2(\Omega_j \setminus B_\varepsilon(q))} \le 2^{-j}$. Without loss of generality, we can assume that the sequence $\{k_j\}_j$ is strictly increasing. The series

$$\sum_{j=1}^{+\infty} v_{k_j}$$

converges uniformly on compact subsets of $N \setminus B_\varepsilon(q)$ to some function $w : N \setminus B_\varepsilon(q) \to \mathbb{R}^+_0$. For every $j \ge 1$ we have $v_{k_i} = 1$ on $N \setminus \Omega_i \supseteq N \setminus \Omega_j$ for $1 \le i \le j$, so $w \ge j$ on $N \setminus \Omega_j$. As $\{\Omega_j\}$ is an exhaustion for N, it follows that $w(x) \to +\infty$ as $x \to \infty$. Since each function v_{k_j} is smooth on $N \setminus \{\partial\Omega_k \cup \overline{B_\varepsilon(q)}\}$ and the sets $\partial\Omega_{k_j}$ are pairwise disjoint, the function

w satisfies $\Delta w \le \lambda w$ in the barrier sense (and then also in the viscosity and distributional sense) on $N \setminus \overline{B_\varepsilon(q)}$, and strongly on $N \setminus (\overline{B_\varepsilon(q)} \cup \bigcup_{k\ge1} \partial\Omega_k)$.

Let $a > 0$ be given. The function $w_1 = w + a$ satisfies $\Delta w_1 \le \lambda w = \lambda w_1 - \lambda a < \lambda w_1 - \lambda a/2$. By Greene-Wu approximation Theorem 3.4 we can find a smooth function $\bar w : N \setminus \overline{B_\varepsilon(q)} \to \mathbb{R}$ such that $|w_1 - \bar w| < a/2$ and $\Delta \bar w < \lambda w_1 - \lambda a/2$. Then, in particular, we have

$$\begin{cases} \Delta \bar w \le \lambda \bar w & \text{in } N \setminus \overline{B_\varepsilon(q)}, \\ \bar w > a/2 & \text{in } N \setminus \overline{B_\varepsilon(q)}, \\ \bar w(x) \to +\infty & \text{as } x \to \infty. \end{cases}$$

Let $\phi : N \to [0,1]$ be a smooth function such that $\phi \equiv 1$ on $B_{2\varepsilon}(q)$ and $\phi \equiv 0$ on $N \setminus B_{3\varepsilon}(q)$, then set $z = (1+r^2)\psi + (1-\psi)(1+\bar w)$, with $r(x) = \mathrm{dist}_h(q,x)$ the distance function from q. The function z is smooth and positive on N and satisfies

$$\begin{cases} \Delta z \le C & \text{on } \overline{B_{3\varepsilon}(q)}, \\ \Delta z \le \lambda z & \text{on } N \setminus \overline{B_{3\varepsilon}(q)}, \\ z(q) = 1, & \\ z > 1 & \text{on } N \setminus \{q\}, \\ z(x) \to +\infty & \text{as } x \to \infty, \end{cases}$$

and then the function

$$\psi_0 = \frac{z + C/\lambda}{1 + C/\lambda}$$

satisfies all requirements in the statement. □

THEOREM 3.17. *Let (M,σ) be a connected, complete Riemannian manifold satisfying*

$$\mathrm{Ric}(Dr,Dr) \ge -\alpha^2(1+r)^2 \qquad \text{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $\kappa \ge 0$, where $r(x) = \mathrm{dist}_\sigma(o,x)$ is the distance function from a fixed origin $o \in M$. Let $\Omega \subseteq M$ be an open domain, let $u \in C^2(\Omega)$ satisfy

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f$$

for some bounded function $f : \Omega \to \mathbb{R}$, and assume that one of conditions a), b), c), d) in Theorem 3.13 holds. Then, for every open subset $\Omega_0 \subseteq \Omega$ with smooth boundary and such that $\overline{\Omega_0} \subseteq \Omega$ and for every $p \in \Omega_0$, $\lambda > 0$ there exists a smooth function $\psi : \overline{\Omega_0} \to \mathbb{R}$ satisfying

$$\begin{cases} \psi(p) = 0, & \\ \psi > 0 & \textit{on } \overline{\Omega_0} \setminus \{p\}, \\ \psi(x) \to +\infty & \textit{as } x \to \infty \textit{ in } \overline{\Omega_0}, \\ \Delta_g \psi + \|\nabla\psi\|^2 \le \lambda & \textit{on } \Omega_0. \end{cases} \tag{59}$$

PROOF. By Theorem 3.13, the geodesic balls with center at p in the Riemannian manifold (Ω, g) satisfy the volume growth condition

$$\liminf_{r\to+\infty} \frac{\log |\Omega_0 \cap B^g_r(p)|_g}{r^2} < +\infty$$

and by Theorem 3.14 there exists an isometric embedding $\phi : (\Omega_0, g) \to (N, h)$ of Ω as an open subset of a complete Riemannian manifold (N,h) whose geodesic balls centered at $q = \phi(p)$ satisfy

$$\liminf_{r\to+\infty} \frac{\log |B^h_r(q)|_h}{r^2} < +\infty.$$

Moreover, the embedding ϕ extends up to the boundary to a diffeomorphism $\bar{\phi} : \overline{\Omega_0} \to \overline{\phi(\Omega_0)} \subseteq N$. By Proposition 3.16 there exists $\psi_0 \in C^\infty(N)$ satisfying conditions (58). Then the function $\psi_1 = \psi_0 \circ \bar{\phi} \in C^\infty(\overline{\Omega_0})$ satisfies

$$\begin{cases} \psi_1(p) = 1, \\ \psi_1 > 1 & \text{in } \overline{\Omega_0} \setminus \{p\}, \\ \psi_1(x) \to +\infty & \text{as } x \to \infty \text{ in } \overline{\Omega_0}, \\ \Delta_h \psi_1 \leq \lambda \psi_1 & \text{in } \Omega_0. \end{cases}$$

and therefore $\psi = \log \psi_1$ satisfies (59). $\square$

CHAPTER 4

Global gradient bounds

1. Lower bounded solutions of the prescribed mean curvature equation

Let (M, σ) be a complete Riemannian manifold. In this section we consider a class of prescribed mean curvature equations of the form

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(x, u, \sqrt{1+|Du|^2}) \tag{60}$$

and we derive global gradient bounds for solutions u of (60) defined on open domains $\Omega \subseteq M$ (possibly with $\Omega = M$) and satisfying $u_* = \inf_\Omega u > -\infty$.

If $\Omega = M$ and the Ricci curvature of M satisfies $\mathrm{Ric} \geq -(m-1)\kappa^2$ for some $\kappa \geq 0$, where $m = \dim M$, then we prove that a lower bounded solution $u \in C^3(M)$ of (60) satisfies

$$\sqrt{1+|Du|^2} \leq A_0 e^{C_0(u-u_*)} \qquad \text{on } M$$

for some constants $A_0 > 1$, $C_0 > 0$ only depending on m, κ and on quantitative bounds on f and its gradient. In case $\Omega \neq M$, if $\mathrm{Ric} \geq -(m-1)\kappa^2$ in Ω then for the same constants A_0, C_0 we can show that

$$\frac{\sqrt{1+|Du|^2}}{e^{C_0(u-u_*)}} \leq \max\left\{A_0, \limsup_{x\to\partial\Omega} \frac{\sqrt{1+|Du(x)|^2}}{e^{C_0(u(x)-u_*)}}\right\} \qquad \text{on } \Omega$$

under additional global assumptions on the geometry of M and, possibly, on $\partial\Omega$ and $u_{|\partial\Omega}$. In particular, we reach the desired conclusion under each of the following sets of hypotheses:

(RΩ) For some origin $o \in M$, the Ricci curvature of M satisfies

$$\mathrm{Ric}(Dr, Dr) \geq -\alpha^2(1+r)^2 \qquad \text{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some constant $\alpha \geq 0$, where $r(x) = \mathrm{dist}_\sigma(o, x)$ is the distance function from $o \in M$ and either

a) $\Omega = M$,

b) $u \in C^0(\overline{\Omega})$ and $u_{|\partial\Omega}$ is constant,

c) $\partial\Omega$ is locally Lipschitz regular and

$$\liminf_{r\to+\infty} \frac{\log\left(\mathcal{H}_\sigma^{m-1}(B_r^\sigma(o) \cap \partial\Omega)\right)}{r^2} < +\infty$$

where $\mathcal{H}_\sigma^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure induced by σ, or

d) $u \in C^0(\overline{\Omega})$, $\partial\Omega$ is locally Lipschitz regular and

$$\liminf_{r\to+\infty} \frac{\log \int_{B_r^\sigma(o)\cap\partial\Omega} \min\{r, |u-u_0|\} \mathrm{d}\mathcal{H}_\sigma^{m-1}}{r^2} < +\infty$$

for some fixed constant $u_0 \in \mathbb{R}$.

(K) For some origin $o \in M$, the radial sectional curvature of M satisfies

$$K_{\mathrm{rad}} \geq -G(r) \qquad \text{on } D_o$$

for some positive, continuous, non-decreasing $G : \mathbb{R}_0^+ \to \mathbb{R}^+$ such that

$$\lim_{r\to+\infty} \int_0^r \frac{\mathrm{d}s}{\sqrt{G(s)}} = +\infty.$$

Each of the assumptions above provides sufficient conditions for the existence of appropriate exhaustion functions. In particular:

i) if (M, σ) is complete and satisfies $\mathrm{Ric} \geq -(m-1)\kappa^2$ and the graph of $u \in C^2(M)$ has bounded mean curvature, then from Theorem 3.17 for every $p \in M$ and $\lambda > 0$ there exists $\psi \in C^\infty(M)$ satisfying

$$\begin{cases} \psi(p) = 0, \\ \psi \geq 0 & \text{on } M, \\ \psi(x) \to +\infty & \text{as } x \to \infty \text{ in } M, \\ \Delta_g \psi + \|\nabla\psi\|^2 \leq \lambda & \text{on } M \end{cases}$$

ii) if (RΩ) is satisfied, $\Omega \neq M$ and the graph of $u \in C^2(\Omega)$ has bounded mean curvature, then, from Theorem 3.17 and the validity of either b), c), or d), for every open subset $\Omega_0 \subseteq \Omega$ with $\overline{\Omega_0} \subseteq \Omega$ and for every $p \in \Omega_0$, $\lambda > 0$ there exists $\psi \in C^\infty(\overline{\Omega_0})$ such that

$$\begin{cases} \psi(p) = 0, \\ \psi \geq 0 & \text{on } \overline{\Omega_0}, \\ \psi(x) \to +\infty & \text{as } r(x) \to +\infty,\ x \in \overline{\Omega_0}, \\ \Delta_g \psi + \|\nabla\psi\|^2 \leq \lambda & \text{on } \Omega_0 \end{cases}$$

iii) if (K) is satisfied and the mean curvature of the graph of $u \in C^2(\Omega)$ is bounded in absolute value by $C_0 \geq 0$, then, up to further assuming $G \in C^1(\mathbb{R}_0^+)$ and $G'(0) = 0$, by Theorem 3.6 the function $\psi \in C^2(M \setminus \mathrm{cut}(o)) \cap \mathrm{Lip}(M)$ defined by

$$\psi(x) = \left(\sqrt{G(0)} \int_0^{r(x)} \frac{\mathrm{d}s}{\sqrt{G(s)}}\right)^2$$

satisfies

$$\begin{cases} \psi(o) = 0, \\ \psi \geq 0 & \text{on } M, \\ \psi(x) \to +\infty & \text{as } r(x) \to \infty, \\ \Delta_g \psi \leq 2\left((m-1)\sqrt{G(0)\psi}\coth\left(\sqrt{G(0)\psi}\right) + C_0\sqrt{\psi} + 1\right) & \text{on } \Omega, \\ \|\nabla\psi\|^2 \leq 4\psi & \text{on } \Omega \end{cases}$$

where the last two inequalities hold strongly on $\Omega \setminus \mathrm{cut}(o)$ and in the barrier sense on Ω. More precisely, for every point $x_0 \in \Omega \cap \mathrm{cut}(o)$ we can find a sequence of open neighbourhoods $U_n \subseteq \Omega$ of x_0 and a sequence of functions $\psi_n \in C^2(U_n)$ satisfying

$$\psi_n \geq \psi \quad \text{on } U_n, \qquad \psi_n(x_0) = \psi(x_0)$$

and

$$\Delta_g \psi_n \leq 2\left((m-1)\sqrt{G(0)\psi}\coth\left(\sqrt{G(0)\psi}\right) + C_0\sqrt{\psi} + 1\right) + \frac{1}{n},$$
$$\|\nabla\psi_n\|^2 \leq 4\psi$$

at x_0.

REMARK 4.1. *We observe that in case iii) it is not restrictive to assume $G \in C^1(\mathbb{R}_0^+)$ and $G'(0) = 0$. Indeed, if $G : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and non-decreasing then it is possible to find $G_1 \in C^1(\mathbb{R}_0^+)$ satisfying $G_1' \geq 0$ on $\mathbb{R}_0^+$, $G \leq G_1 \leq G + 1$ on $\mathbb{R}_0^+$ and $G_1'(0) = 0$, and for such a function we still have*

$$K_{\text{rad}} \geq -G_1(r), \qquad \int_0^{+\infty} \frac{\mathrm{d}s}{\sqrt{G_1(s)}} = +\infty.$$

THEOREM 4.2. *Let (M, σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open domain, $I \subseteq \mathbb{R}$ an interval. Let $E = \Omega \times I \times [1, +\infty)$ and let $f \in C^1(E)$ satisfy*

$$\sup_E |f| < +\infty, \quad |D_x f| \leq C_1, \quad \frac{\partial f}{\partial y} \geq -\frac{C_2}{w}, \quad -\frac{C_3}{w^2} \leq \frac{\partial f}{\partial w} \leq \frac{C_4}{w^2} \tag{61}$$

for some constants $C_1, C_2, C_3, C_4 \geq 0$, where (x, y, w) denotes the generic point of E. Let $u : \Omega \to I$, $u \in C^3(\Omega)$, be a solution of equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = f(x, u, \sqrt{1 + |Du|^2}) \qquad \text{in } \Omega.$$

Suppose that $u_* = \inf_\Omega u > -\infty$ and that

$$\operatorname{Ric} \geq -(m-1)\kappa^2 \qquad \text{in } \Omega$$

for some constant $\kappa \geq 0$, where $m = \dim M$. Also assume that either condition (RΩ) or (K) is satisfied. Then there exist $C_0 > C_3$, $A_0 > 1$, only depending on m, κ, C_1, C_2, C_3, such that

$$\sup_\Omega \frac{W}{e^{C_0(u - u_*)}} \leq \max\left\{A_0, \limsup_{x \to \partial\Omega} \frac{W(x)}{e^{C_0(u(x) - u_*)}}\right\}. \tag{62}$$

In particular, (62) holds provided

$$C_0^2 - C_0 C_3 > (m-1)\kappa^2 + C_1 + C_2, \tag{63}$$

$$\inf_{(x,y,w) \in \Omega \times I \times [A_0, +\infty)} \left(\frac{f(x,y,w)^2}{m} - \frac{C_0 f(x,y,w)}{w} + C_5 \frac{w^2 - 1}{w^2} - C_1 \frac{\sqrt{w^2 - 1}}{w}\right) > 0 \tag{64}$$

for some auxiliary parameter C_5 satisfying

$$C_1 < C_5 < C_0^2 - C_0 C_3 - (m-1)\kappa^2 - C_2. \tag{65}$$

REMARK 4.3. *A class of nonlinearities $f = f(x, y, w)$ satisfying (61) is given by functions of the form*

$$f(x, y, w) = f_1(x, y) + \frac{f_2(x, y)}{w}$$

with $f_1, f_2 \in C^1(\Omega \times I)$ such that

$$\sup_E |f_1| < +\infty, \quad -C_4 \leq f_2 \leq C_3, \quad |D_x f_1| + |D_x f_2| \leq C_1, \quad \frac{\partial f_1}{\partial y} \geq 0, \quad \frac{\partial f_2}{\partial y} \geq -C_2.$$

PROOF OF THEOREM 4.2. We divide the proof in two parts. In the first part, we assume the validity of condition (RΩ) and we prove that (62) holds whenever $C_0 > C_3$, $A_0 > 1$ satisfy (63), (64) for some auxiliary C_5 as in (65). In the second part, we assume the validity of condition (K) and we point out the minor modifications needed to repeat the same argument developed in the first part.

Part 1. Assume the validity of (RΩ). Let $C_0 > C_3$ and $C_5 > C_1$ satisfy (63) and (65), then let $A_0 > 1$ be such that (64) is satisfied. Observe that such A_0 indeed exists, as the term in brackets in (64) is larger than or equal to

$$\frac{-C_0 \sup |f|}{w} + C_5 \frac{w^2 - 1}{w^2} - C_1 \frac{\sqrt{w^2 - 1}}{w}$$

and this quantity has a positive limit $C_5 - C_1 > 0$ as $w \to +\infty$. Then, let $\delta_0 > 0$ be such that

$$\frac{f(x,y,w)^2}{m} - \frac{C_0 f(x,y,w)}{w} + C_5 \frac{w^2-1}{w^2} - C_1 \frac{\sqrt{w^2-1}}{w} > \delta_0 C_0 \sup|f| \tag{66}$$

for every $(x, y, w) \in \Omega \times I \times [A_0, +\infty)$, and also let $\tau \in (0,1)$ be small enough so that

$$C_0^2 - C_3 C_0 - 2\tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2} \right)^2 - (m-1)\kappa^2 - C_2 > C_5. \tag{67}$$

Let $z_0 = We^{-C_0 v}$, with $v = u - \inf_\Omega u$. We suppose, by contradiction, that (62) is not satisfied. Then there exists $\gamma > 0$ such that

$$\sup_\Omega z_0 > \gamma > \max \left\{ A_0, \limsup_{x \to \partial\Omega} z_0(x) \right\}$$

and by Sard's theorem we can assume that γ is a regular value for z_0. Then the set $\Omega_\gamma = \{x \in \Omega : z_0(x) > \gamma\}$ has smooth boundary and $\overline{\Omega_\gamma} \subseteq \Omega$. From Theorem 3.17, there exists a smooth function $\psi : \overline{\Omega_\gamma} \to \mathbb{R}_0^+$ satisfying

$$\begin{cases} \psi(x) \to +\infty & \text{as } x \to \infty \text{ in } \overline{\Omega_\gamma}, \\ \Delta_g \psi + \|\nabla \psi\|^2 \le 1 & \text{in } \Omega_\gamma. \end{cases} \tag{68}$$

For any $\varepsilon > 0$, $\delta > 0$ consider functions $\eta_{\varepsilon,\delta} = e^{-C_0 v - \varepsilon\psi} - \delta$, $z_{\varepsilon,\delta} = W\eta_{\varepsilon,\delta}$. For every $\varepsilon, \delta > 0$ we have $\eta_{\varepsilon,\delta} < e^{-C_0 v}$ and then $z_{\varepsilon,\delta} < z_0$, so in particular

$$\sup_{\partial\Omega_\gamma} z_{\varepsilon,\delta} \le \sup_{\partial\Omega_\gamma} z_0 = \gamma. \tag{69}$$

On the other hand, for $(\varepsilon, \delta) \to (0,0)$ we have $\eta_{\varepsilon,\delta} \to e^{-C_0 v}$, $z_{\varepsilon,\delta} \to z_0$ pointwise on $\overline{\Omega_\gamma}$. So, for every sufficiently small $\varepsilon, \delta > 0$ we have

$$\sup_{\overline{\Omega_\gamma}} z_{\varepsilon,\delta} > \gamma. \tag{70}$$

Fix $\varepsilon, \delta > 0$ small enough so that (70) is satisfied together with

$$\frac{1-\tau}{\tau}\varepsilon \le 1, \qquad \varepsilon \le \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2} \right)^2 \frac{A_0^2 - 1}{A_0^2}, \qquad \delta < \delta_0, \tag{71}$$

then set $\eta = \eta_{\varepsilon,\delta}$, $z = z_{\varepsilon,\delta}$.

The function v is non-negative, so $\eta \le e^{-\varepsilon\psi} - \delta$. In particular, $\{x \in \overline{\Omega_\gamma} : \eta(x) \ge 0\}$ is a subset of $\{x \in \overline{\Omega_\gamma} : \psi(x) \le \varepsilon^{-1}\log(1/\delta)\}$, and the latter is a compact set because of the first condition in (68). By continuity, z attains a global maximum on this set at some point $\bar{x}$. Since $z < 0$ whenever $\eta < 0$ and since (70) implies that z is positive somewhere in $\overline{\Omega_\gamma}$, we infer that $z(\bar{x})$ is in fact the (positive) global maximum of z on $\overline{\Omega_\gamma}$. Moreover, from (70) we have $z(\bar{x}) > \gamma$ and then $\bar{x} \in \Omega_\gamma$ due to (69). As $\bar{x}$ is an interior maximum point for z, from the maximum principle we have

$$\nabla z(\bar{x}) = 0, \qquad \Delta_g z(\bar{x}) \le 0. \tag{72}$$

From (35) we have that z satisfies the differential equation

$$\begin{aligned} \Delta_g z - \frac{2\langle \nabla W, \nabla z \rangle}{W} &= \left(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\mathbf{n}, \mathbf{n}) + W\langle \nabla f, \nabla u \rangle \right) W\eta + W\Delta_g \eta \\ &= \left(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\mathbf{n}, \mathbf{n}) + W\langle \nabla f, \nabla u \rangle \right) z + \\ &\quad + \left(-C_0 \Delta_g v - \varepsilon \Delta_g \psi + \|C_0 \nabla v + \varepsilon \nabla \psi\|^2 \right) W(\eta + \delta). \end{aligned}$$

At points where $\eta > 0$ we can rewrite

$$\Delta_g z - \frac{2\langle \nabla W, \nabla z\rangle}{W} = \left(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\mathbf{n},\mathbf{n}) + W\langle \nabla f, \nabla u\rangle\right) z + \\ + \left(1 + \frac{\delta}{\eta}\right)\left(-C_0\Delta_g v - \varepsilon \Delta_g \psi + \|C_0 \nabla v + \varepsilon \nabla \psi\|^2\right) z$$

and then, from identities $\nabla v = \nabla u$, $\Delta_g v = \Delta_g u = W^{-1} f$ and estimates

$$\|\mathrm{II}\|^2 \geq \frac{\mathrm{Tr}_g(\mathrm{II})^2}{m} = \frac{f^2}{m},$$
$$\overline{\mathrm{Ric}}(\mathbf{n},\mathbf{n}) = \frac{\mathrm{Ric}(Du, Du)}{W^2} \geq -(m-1)\kappa^2 \frac{|Du|^2}{W^2}$$

we can further rewrite

$$\Delta_g z - \frac{2\langle \nabla W, \nabla z\rangle}{W} \geq \left(\frac{f^2}{m} - (m-1)\kappa^2 \frac{|Du|^2}{W^2} + W\langle \nabla f, \nabla u\rangle\right) z + \\ + \left(1 + \frac{\delta}{\eta}\right)\left(-\frac{C_0 f}{W} - \varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2\right) z.$$

Since $z(\bar{x}) > 0$, from this inequality and (72) we deduce

$$0 \geq \frac{f^2}{m} - (m-1)\kappa^2 \frac{|Du|^2}{W^2} + W\langle \nabla f, \nabla u\rangle + \\ + \left(1 + \frac{\delta}{\eta}\right)\left(-\frac{C_0 f}{W} - \varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2\right) \qquad \text{at } \bar{x},$$

that is, after some rearrangements and recalling that $\eta W = z$,

$$\frac{\delta C_0 f}{z} \geq \frac{f^2}{m} - \frac{C_0 f}{W} - (m-1)\kappa^2 \frac{|Du|^2}{W^2} + W\langle \nabla f, \nabla u\rangle + \\ + \left(1 + \frac{\delta}{\eta}\right)\left(-\varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2\right). \tag{73}$$

We now proceed to estimate the RHS of (73) from below. We start from the term $W\langle \nabla f, \nabla u\rangle$. In local coordinates $\{x^i\}$ around $\bar{x}$ we have

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y} u_i + \frac{\partial f}{\partial w} W_i\right) \mathrm{d}x^i =: f_i \mathrm{d}x^i$$

and then

$$\langle \nabla u, \nabla f\rangle = g^{ij}\left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y} u_i + \frac{\partial f}{\partial w} W_i\right) u_j.$$

By $g^{ij} u_j = W^{-2} \sigma^{ij} u_j$ and from (61) we can estimate

$$g^{ij} \frac{\partial f}{\partial x^i} u_j = \frac{1}{W^2} \sigma^{ij} \frac{\partial f}{\partial x^i} u_j \geq -\frac{|D_x f||Du|}{W^2} \geq -C_1 \frac{|Du|}{W^2},$$
$$g^{ij} \frac{\partial f}{\partial y} u_i u_j = \frac{\partial f}{\partial y} \frac{|Du|^2}{W^2} \geq -\frac{C_2 |Du|^2}{W^3}.$$

Recalling that $\nabla z = 0$ at $\bar{x}$, we have

$$\mathrm{d}W = -\frac{W}{\eta} \mathrm{d}\eta = W\left(1 + \frac{\delta}{\eta}\right)(C_0 \mathrm{d}u + \varepsilon \mathrm{d}\psi)$$

and thus

$$g^{ij} \frac{\partial f}{\partial w} W_i u_j = W \frac{\partial f}{\partial w}\left(1 + \frac{\delta}{\eta}\right)\langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi\rangle.$$

Summing up, at $\bar{x}$ we have

$$W\langle \nabla u, \nabla f\rangle \geq -C_1 \frac{|Du|}{W} - C_2 \frac{|Du|^2}{W^2} + W^2 \frac{\partial f}{\partial w}\left(1 + \frac{\delta}{\eta}\right)\langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi\rangle$$

and then from (73) we obtain

$$
\begin{aligned}
\frac{\delta C_0 f}{z} \geq{}& \frac{f^2}{m} - \frac{\delta C_0 f}{W} - (m-1)\kappa^2 \frac{|Du|^2}{W^2} - C_1 \frac{|Du|}{W} - C_2 \frac{|Du|^2}{W^2} + \\
&+ \left(1 + \frac{\delta}{\eta}\right)\left(-\varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2 + W^2 \frac{\partial f}{\partial w} \langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi \rangle \right).
\end{aligned}
\tag{74}
$$

We now turn our attention to the last pair of brackets. Direct computation and an application of Cauchy-Schwarz's and Young's inequalities yield

$$
\begin{aligned}
&\|\varepsilon \nabla \psi + C_0 \nabla u\|^2 + W^2 \frac{\partial f}{\partial w} \langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi \rangle = \\
&\quad = \varepsilon^2 \|\nabla \psi\|^2 + \left(C_0^2 + C_0 W^2 \frac{\partial f}{\partial w}\right) \|\nabla u\|^2 + \left(2C_0 + W^2 \frac{\partial f}{\partial w}\right) \varepsilon \langle \nabla u, \nabla \psi \rangle \\
&\quad \geq \varepsilon^2 \|\nabla \psi\|^2 + \left(C_0^2 + C_0 W^2 \frac{\partial f}{\partial w}\right) \|\nabla u\|^2 - \tau \left(C_0 + \frac{W^2}{2}\frac{\partial f}{\partial w}\right)^2 \|\nabla u\|^2 - \frac{1}{\tau}\varepsilon^2 \|\nabla \psi\|^2.
\end{aligned}
$$

From (61) we have

$$
C_0^2 + C_0 W^2 \frac{\partial f}{\partial w} \geq C_0^2 - C_0 C_3, \qquad \left(C_0 + \frac{W^2}{2}\frac{\partial f}{\partial w}\right)^2 \leq \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2,
$$

then

$$
\begin{aligned}
&\|\varepsilon \nabla \psi + C_0 \nabla u\|^2 + W^2 \frac{\partial f}{\partial w} \langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi \rangle \geq \\
&\qquad \geq \left(C_0^2 - C_0 C_3 - \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2\right) \|\nabla u\|^2 - \frac{1-\tau}{\tau}\varepsilon^2 \|\nabla \psi\|^2
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&-\varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2 + W^2 \frac{\partial f}{\partial w} \langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi \rangle \geq \\
&\quad \geq \left(C_0^2 - C_0 C_3 - \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2\right) \|\nabla u\|^2 - \varepsilon \left(\Delta_g \psi + \frac{1-\tau}{\tau}\varepsilon \|\nabla \psi\|^2\right).
\end{aligned}
$$

From the second condition in (68) and the first two conditions in (71), we can estimate

$$
\varepsilon \left(\Delta_g \psi + \frac{1-\tau}{\tau}\varepsilon \|\nabla \psi\|^2\right) \leq \varepsilon \left(\Delta_g \psi + \|\nabla \psi\|^2\right) \leq \varepsilon \leq \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2 \frac{A_0^2 - 1}{A_0^2}.
$$

Now recall that $z(\bar{x}) > \gamma > A_0$. Since $\eta \leq 1$, this implies $W(\bar{x}) > A_0$, that is,

$$
\|\nabla u\|^2 = \frac{|Du|^2}{W^2} = \frac{W^2 - 1}{W^2} \geq \frac{A_0^2 - 1}{A_0^2} \qquad \text{at } \bar{x}.
$$

Then we can estimate

$$
\varepsilon \left(\Delta_g \psi + \frac{1-\tau}{\tau}\varepsilon \|\nabla \psi\|^2\right) \leq \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2 \|\nabla u\|^2
$$

and consequently

$$
\begin{aligned}
&-\varepsilon \Delta_g \psi + \|C_0 \nabla u + \varepsilon \nabla \psi\|^2 + W^2 \frac{\partial f}{\partial w} \langle \nabla u, C_0 \nabla u + \varepsilon \nabla \psi \rangle \geq \\
&\qquad \geq \left(C_0^2 - C_0 C_3 - 2\tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2}\right)^2\right) \frac{|Du|^2}{W^2} \\
&\qquad > \left(C_5 + (m-1)\kappa^2 + C_2\right) \frac{|Du|^2}{W^2}
\end{aligned}
$$

where we have also used (67). Since the last term of this chain of inequalities is positive, we further have

$$\left(1+\frac{\delta}{\eta}\right)\left(-\varepsilon\Delta_g\psi+\|C_0\nabla u+\varepsilon\nabla\psi\|^2+W^2\frac{\partial f}{\partial w}\langle\nabla u,C_0\nabla u+\varepsilon\nabla\psi\rangle\right)>$$
$$>\left(C_5+(m-1)\kappa^2+C_2\right)\frac{|Du|^2}{W^2}.$$

Substituting this into (74) we obtain

$$\frac{\delta C_0 f}{z}>\frac{f^2}{m}-\frac{C_0 f}{W}+C_5\frac{|Du|^2}{W^2}-C_1\frac{|Du|}{W}\qquad\text{at }\bar{x}. \tag{75}$$

Since $z(\bar{x})>A_0>1$, from the third inequality in (71) we have

$$\frac{\delta C_0 f}{z}\le\delta C_0\sup|f|\le\delta_0 C_0\sup|f|\qquad\text{at }\bar{x}. \tag{76}$$

Since $W(\bar{x})\ge z(\bar{x})>A_0$ and $|Du|=\sqrt{W^2-1}$, from (66) we also have

$$\frac{f^2}{m}-\frac{C_0 f}{W}+C_5\frac{|Du|^2}{W^2}-C_1\frac{|Du|}{W}>\delta_0 C_0\sup|f|\qquad\text{at }\bar{x} \tag{77}$$

and comparing (75), (76) and (77) we obtain the desired contradiction.

Part 2. We assume the validity of (K). We repeat verbatim the initial section of Part 1, up to the definition of set Ω_γ. In particular, we let δ_0 and τ be as in (66) and (67). From Theorem 3.6 we have the existence of a function $\psi: M\to\mathbb{R}_0^+$ satisfying

$$\begin{cases}\psi(x)\to+\infty & \text{as } r(x)\to\infty,\\ \Delta_g\psi\le 2\left((m-1)\sqrt{\alpha\psi}\coth\left(\sqrt{\alpha\psi}\right)+\sqrt{\psi}\sup_E|f|+1\right) & \text{on }\Omega_\gamma,\\ \|\nabla\psi\|\le 2\sqrt{\psi} & \text{on }\Omega_\gamma\end{cases} \tag{78}$$

in the barrier sense, for some $\alpha>0$. For every $t>0$, let

$$\begin{aligned}&\varepsilon(t)=t^{-3/4},\\ &\delta(t)=e^{-t^{1/4}},\\ &Q(t)=2\varepsilon(t)\left((m-1)\sqrt{\alpha t}\coth(\sqrt{\alpha t})+\sqrt{t}\sup_E|f|+1\right)+4\frac{1-\tau}{\tau}\varepsilon(t)^2t.\end{aligned}$$

As $t\to+\infty$ we have

$$\coth(\sqrt{\alpha t})\to1,\qquad\varepsilon(t)\sqrt{t}\to0,\qquad\delta(t)\to0$$

so there exists $T_0>0$ such that

$$Q(t)<\tau\left(C_0+\frac{\max\{C_3,C_4\}}{2}\right)^2\frac{A_0^2-1}{A_0^2},\qquad\delta(t)<\delta_0$$

for every $t>T_0$.

For every $t>0$ let us also set

$$\eta_t=e^{-C_0v-\varepsilon(t)\psi}-\delta(t),\qquad z_t=W\eta_t,\qquad\Omega_{\gamma,t}=\{x\in\Omega_\gamma:\psi(x)<t\}.$$

We have $z_t\le z_0$ in $\overline{\Omega_\gamma}$ and $z_t\to z_0$ pointwise as $t\to+\infty$. Then, for every $t>0$

$$\sup_{\partial\Omega_\gamma}z_t\le\gamma$$

and there exists $t>T_0$ such that

$$\sup_{\overline{\Omega_\gamma}}z_t>\gamma.$$

Since $\gamma>0$, in fact one has

$$\sup_{\overline{\Omega_\gamma}}z_t=\sup_{\{z_t>0\}}z_t=\sup_{\{\eta_t>0\}}z_t$$

and since $\eta_t \leq e^{-\varepsilon(t)\psi} - \delta(t) = e^{-t^{-3/4}\psi} - e^{-t^{1/4}}$ we have $\{\eta_t > 0\} \subseteq \{\psi < t\} = \Omega_{\gamma,t}$. The closure $\overline{\Omega_{\gamma,t}}$ is compact by the properness of ψ, so there exists a point $\bar{x} \in \overline{\Omega_{\gamma,t}}$ such that

$$z_t(\bar{x}) = \max_{\overline{\Omega_{\gamma,t}}} z_t = \sup_{\overline{\Omega_\gamma}} z_t > \gamma.$$

In particular, $\bar{x} \in \Omega_{\gamma,t}$. Let $k \in \mathbb{N}$ be such that

$$Q(t) + \frac{1}{k} < \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2} \right)^2 \frac{A_0^2 - 1}{A_0^2}. \tag{79}$$

Since ψ satisfies (78) in the barrier sense, there exist a neighbourhood $U \subseteq \Omega_{\gamma,t}$ of $\bar{x}$ and a function $\psi_k \in C^2(U)$ satisfying

$$\begin{cases} \psi_k \geq \psi & \text{in } U, \\ \psi_k(\bar{x}) = \psi(\bar{x}), \\ \varepsilon(t)\Delta_g \psi_k(\bar{x}) + \dfrac{1-\tau}{\tau}\varepsilon(t)^2 \|\nabla \psi_k(\bar{x})\|^2 < Q(t) + \dfrac{1}{k}. \end{cases} \tag{80}$$

Fix $\varepsilon = \varepsilon(t)$, $\delta = \delta(t)$. The function $z = e^{-C_0 v - \varepsilon \psi_k} - \delta$ satisfies

$$z \leq z_t \leq z_t(\bar{x}) = z(\bar{x}) \quad \text{in } U$$

so $\bar{x}$ is an interior maximum point for z in U. The function z is of class $C^2(U)$, so from the maximum principle we have

$$\nabla z(\bar{x}) = 0, \qquad \Delta_g z(\bar{x}) \leq 0.$$

Also observe that ψ_k satisfies

$$\varepsilon \Delta_g \psi_k(\bar{x}) + \frac{1-\tau}{\tau}\varepsilon^2 \|\nabla \psi_k(\bar{x})\|^2 < \tau \left(C_0 + \frac{\max\{C_3, C_4\}}{2} \right)^2 \frac{A_0^2 - 1}{A_0^2}$$

as a consequence of (79) and (80). From this point on, the argument proceeds exactly as in Part 1. □

2. Liouville theorems and other consequences

In this section we derive some consequences from the general gradient bound given in Theorem 4.2. Let us recall the definition of conditions (RΩ) and (K).

(RΩ) For some origin $o \in M$, the Ricci curvature of M satisfies

$$\mathrm{Ric}(Dr, Dr) \geq -\alpha^2(1 + r^2) \qquad \text{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $\alpha \geq 0$, where $r(x) = \mathrm{dist}_\sigma(o, x)$ is the distance function from $o \in M$, and, given $\Omega \subseteq M$ and $u \in C^2(\Omega)$, one of conditions a), b), c), d) of Theorem 3.13 is satisfied.

(K) For some origin $o \in M$, the radial sectional curvature of M satisfies

$$K_{\mathrm{rad}} \geq -G(r) \qquad \text{on } D_o$$

for some continuous, non-decreasing $G : \mathbb{R}_0^+ \to \mathbb{R}^+$ such that $1/\sqrt{G} \notin L^1(+\infty)$.

2.1. Bounded solutions have bounded gradient. The first, more immediate consequence of Theorem 4.2 is that bounded entire solutions of equation (60), with f as in (61), have bounded gradient, and bounded solutions defined on proper subdomains $\Omega \subsetneq M$ have bounded gradient in Ω if their gradient is uniformly bounded in a neighbourhood of $\partial\Omega$.

COROLLARY 4.4. *Let (M,σ) be a complete Riemannian manifold, $\Omega\subseteq M$ an open domain, $I\subseteq\mathbb{R}$ an interval. Let $E=\Omega\times I\times[1,+\infty)$ and let $f\in C^1(E)$ satisfy*

$$\sup_E|f|<+\infty,\quad |D_xf|\le C_1,\quad \frac{\partial f}{\partial y}\ge-\frac{C_2}{w},\quad -\frac{C_3}{w^2}\le\frac{\partial f}{\partial w}\le\frac{C_4}{w^2}$$

for some constants $C_1,C_2,C_3,C_4\ge 0$. Let $u:\Omega\to I$, $u\in C^3(\Omega)$, be a solution of equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=f(x,u,\sqrt{1+|Du|^2})\qquad in\ \Omega$$

and suppose that either (RΩ) or (K) holds. If

$$\sup_\Omega|u|<+\infty,\qquad \limsup_{x\to\partial\Omega}|Du(x)|<+\infty$$

and

$$\mathrm{Ric}\ge-(m-1)\kappa^2\qquad in\ \Omega$$

for some constant $\kappa\ge0$, then

$$\sup_\Omega|Du|<+\infty.$$

To illustrate other consequences of Theorem 4.2, we need to establish a preliminary lemma. Roughly speaking, our aim is to precise under which conditions we will be able to let $C_0\searrow C$ in the estimate (62), with $C\ge C_3$ satisfying

$$C^2-CC_3=(m-1)\kappa^2+C_1+C_2,$$

while keeping A_0 uniformly bounded.

LEMMA 4.5. *Let $C_1,C_2,C_3,K\ge0$ be real numbers and let $C\ge C_3$ satisfy*

$$C^2-CC_3=K+C_1+C_2. \tag{81}$$

i) *If $C_1=0$ then there exists $\varepsilon_0>0$ with the following property: for every $\varepsilon\in(0,\varepsilon_0)$ there exist $C_0\in(C,C+\varepsilon)$ and $A_0\in(1,1+\varepsilon)$ such that*

$$\inf_{s\le0,w\ge A_0}\left(\frac{s^2}{m}-\frac{C_0s}{w}+C_5\frac{w^2-1}{w^2}\right)>0$$

for every $0<C_5<C_0^2-C_0C_3-K-C_2$.

ii) *If $C_1=C_2=C_3=K=0$ then there exist $\varepsilon_0>0$ and $A\ge1$ with the following property: for every $\varepsilon\in(0,\varepsilon_0)$ there exist $C_0\in(0,\varepsilon)$ and $C_5\in(C_0^2/2,C_0^2)$ such that*

$$\inf_{s\ge0,w\ge A}\left(\frac{s^2}{m}-\frac{C_0s}{w}+C_5\frac{w^2-1}{w^2}\right)>0.$$

iii) *Let $0<H_0\le H_1<+\infty$. Then there exist $\varepsilon_0>0$ and $A\ge1$ with the following property: for every $\varepsilon\in(0,\varepsilon_0)$ there exists $C_0\in(C,C+\varepsilon)$ such that*

$$\inf_{H_0\le|s|\le H_1,w\ge A}\left(\frac{s^2}{m}-\frac{C_0s}{w}+C_5\frac{w^2-1}{w^2}-C_1\frac{\sqrt{w^2-1}}{w}\right)>0$$

for every $C_1<C_5<C_0^2-C_0C_3-K-C_2$.

PROOF. Statement i) follows from the observation that, for any $C_0>0$, $C_5>0$, $A_0>1$ and for every $s\le0$, $w\ge A_0$

$$\frac{s^2}{m}-\frac{C_0s}{w}+C_5\frac{w^2-1}{w^2}\ge C_5\frac{w^2-1}{w^2}\ge C_5\frac{A_0^2-1}{A_0^2}>0.$$

ii) Let $A > \sqrt{1+\frac{m}{2}}$. For every $C_0 > 0$, $C_5 > C_0^2/2$ and for every $s \geq 0$, $w \geq A$ we have

$$\begin{aligned}\frac{s^2}{m} - \frac{C_0 s}{w} + C_5 \frac{w^2-1}{w^2} &\geq \frac{s^2}{m} - \frac{C_0 s}{w} + \frac{C_0^2}{2}\frac{w^2-1}{w^2} \\ &= m\left(\frac{s}{m} - \frac{C_0}{2w}\right)^2 - \frac{mC_0^2}{4w^2} + \frac{C_0^2}{2}\frac{w^2-1}{w^2} \\ &\geq -\frac{mC_0^2}{4A^2} + \frac{C_0^2}{2}\frac{A^2-1}{A^2} = \frac{C_0^2}{2A^2}\left(A^2 - 1 - \frac{m}{2}\right) > 0\end{aligned}$$

iii) Let $\varepsilon_0 > 0$. There exists $A \geq 1$ such that

$$\frac{H_0^2}{m} - \frac{(C+\varepsilon_0)H_1}{A} + C_1\left(\frac{\sqrt{A^2-1}}{A} - 1\right) > 0.$$

For every $C_0 \in (C, C+\varepsilon_0)$ and $H_0 \leq |s| \leq H_1$, $w \geq A$ we can estimate

$$\frac{s^2}{m} - \frac{C_0 s}{w} \geq \frac{H_0^2}{m} - \frac{C_0 H_1}{A} \geq \frac{H_0^2}{m} - \frac{(C+\varepsilon_0)H_1}{A}$$

and for every $C_5 \geq C_1$, $w \geq A$

$$\begin{aligned}C_5\frac{w^2-1}{w^2} - C_1\frac{\sqrt{w^2-1}}{w} &= (C_5 - C_1)\frac{w^2-1}{w^2} + C_1\left(\frac{w^2-1}{w^2} - \frac{\sqrt{w^2-1}}{w}\right) \\ &= (C_5 - C_1)\frac{w^2-1}{w^2} + C_1\frac{\sqrt{w^2-1}}{w}\left(\frac{\sqrt{w^2-1}}{w} - 1\right) \\ &\geq C_1\left(\frac{\sqrt{w^2-1}}{w} - 1\right) \geq C_1\left(\frac{\sqrt{A^2-1}}{A} - 1\right).\end{aligned}$$

where inequalities follow from observation that $\sqrt{A^2-1}/A \leq \sqrt{w^2-1}/w < 1$. Then,

$$\begin{aligned}\inf_{H_0 \leq |s| \leq H_1, w \geq A_0}&\left(\frac{s^2}{m} - \frac{C_0 s}{w} + C_5\frac{w^2-1}{w^2} - C_1\frac{\sqrt{w^2-1}}{w}\right) \geq \\ &\geq \frac{H_0^2}{m} - \frac{(C+\varepsilon_0)H_1}{A} + C_1\left(\frac{\sqrt{A^2-1}}{A} - 1\right) > 0.\end{aligned}$$

□

2.2. Lower bounded solutions with bounded gradient.

COROLLARY 4.6. *Let (M,σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open set, $I \subseteq \mathbb{R}$ be an interval and let $f \in C^1(I \times [1, +\infty))$ satisfy*

$$-\Lambda \leq f \leq \Lambda, \qquad \frac{\partial f}{\partial y} \geq 0, \qquad 0 \leq \frac{\partial f}{\partial w} \leq \frac{\Lambda}{w^2}$$

for some constant $\Lambda \geq 0$. Let $u : \Omega \to I$, $u \in C^3(\Omega)$ be a solution of equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(u, \sqrt{1+|Du|^2}) \qquad \text{in } \Omega.$$

Suppose that $u_* = \inf_\Omega u > -\infty$ and that Ric ≥ 0 *in Ω. If $\Omega \neq M$, then also assume that either (RΩ) or (K) is satisfied and that* $\limsup_{x\to\partial\Omega} |Du(x)| < +\infty$. *Then*

$$\sup_\Omega |Du| < +\infty.$$

Proof. Set $C_1 = C_2 = C_3 = K = 0$ and $C_4 = \Lambda$. By statements i) and ii) in Lemma 4.5, there exist $A \geq 1$ and $\varepsilon_0 > 0$ such that, for every $C_0 \in (0, \varepsilon_0)$ and for every $C_5 \in (C_0^2/2, C_0^2)$,

$$\inf_{s\in\mathbb{R}, w\geq A} \left(\frac{s^2}{m} - \frac{C_0 s}{w} + C_5 \frac{w^2-1}{w^2} \right) > 0.$$

Then, for $A_0 = A$ and for every sufficiently small $C_0 > 0$, conditions (63) and (64) in Theorem 4.2 are satisfied for some auxiliary parameter C_5 satisfying (65) and we deduce

$$\frac{W}{e^{C_0(u-u_*)}} \leq \max\left\{ A, \limsup_{x\to\partial\Omega} \frac{W(x)}{e^{C_0(u(x)-u_*)}} \right\}.$$

Since $C_0(u-u_*) \geq 0$, we further obtain

$$\frac{W}{e^{C_0(u-u_*)}} \leq \max\left\{ A, \limsup_{x\to\partial\Omega} W(x) \right\}.$$

The LHS of this inequality converges pointwise to W on Ω as $C_0 \to 0$, so we get

$$\sup_\Omega W \leq \max\left\{ A, \limsup_{x\to\partial\Omega} W(x) \right\},$$

that is,

$$\sup_\Omega |Du| \leq \max\left\{ \sqrt{A^2-1}, \limsup_{x\to\partial\Omega} |Du(x)| \right\}$$

and then the desired conclusion follows. □

2.3. Liouville theorems.

Corollary 4.7. *Let (M,σ) be a complete, connected Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $I \subseteq \mathbb{R}$ be an interval and $f \in C^1(I \times [1,+\infty))$ satisfy*

$$-\Lambda \leq f \leq 0, \quad \frac{\partial f}{\partial y} \geq 0, \quad 0 \leq \frac{\partial f}{\partial w} \leq \frac{\Lambda}{w^2}$$

for some constant $\Lambda \geq 0$. Let $u : M \to I$, $u \in C^3(M)$ be a solution of equation

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = f(u, \sqrt{1+|Du|^2}) \qquad \text{in } M.$$

If $u_* = \inf_M u > -\infty$, then u is constant.

In particular,

Theorem 4.8. *Let (M,σ) be a complete, connected Riemannian manifold with* $\mathrm{Ric} \geq 0$. *If $u \geq 0$ is a solution of*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = 0 \qquad \text{in } M$$

then u is constant.

Proof of Corollary 4.7. Set $C_1 = C_2 = C_3 = K = 0$ and $C_4 = \Lambda$. By statement i) in Lemma 4.5, for every $\varepsilon > 0$ we can find $0 < C_0 = C_0(\varepsilon) < \varepsilon$ and $1 < A_0 < 1+\varepsilon$ such that

$$\inf_{s\leq 0, w\geq A_0} \left(\frac{s^2}{m} - \frac{C_0 s}{w} + C_5 \frac{w^2-1}{w^2} \right) > 0$$

for every $0 < C_5 < C_0$. From Theorem 4.2 we get

$$W \leq (1+\varepsilon) e^{C_0(\varepsilon)(u-u_*)} \qquad \text{on } M.$$

The RHS of this inequality tends to 1 pointwise on M as $\varepsilon \to 0$, so we get $W \leq 1$ on M, that is, $W \equiv 1$. Equivalently, $|Du| \equiv 0$ on M, and then we conclude that u is constant by connectedness of M. □

2.4. Minimal and CMC graphs.

COROLLARY 4.9. *Let (M, σ) be a complete Riemannian manifold, $\Omega \subseteq M$ an open set and let $0 \le u \in C^3(\Omega)$ be a solution of equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = mH \qquad in\ \Omega$$

for some constant $H \in \mathbb{R}$. Suppose that $\mathrm{Ric} \ge -(m-1)\kappa^2$ in Ω for some $\kappa \ge 0$. If $\Omega \ne M$, then also assume that either (RΩ) or (K) is satisfied. Then

$$\sup_\Omega \frac{\sqrt{1+|Du|^2}}{e^{\sqrt{m-1}\kappa u}} \le \max\left\{A,\ \limsup_{x\to\partial\Omega} \frac{\sqrt{1+|Du(x)|^2}}{e^{\sqrt{m-1}\kappa u(x)}}\right\} \tag{82}$$

for some $A \ge 1$ only depending on m, H, κ. In particular, if $H \le 0$ then (82) holds with $A = 1$.

PROOF. Let $C_1 = C_2 = C_3 = C_4 = 0$ and $K = (m-1)\kappa^2$. Then $C = \sqrt{m-1}\kappa$ from formula (81). From either i) or ii) in Lemma 4.5 we have that for some $A \ge 1$ (with $A = 1$ in case $H \le 0$) and for every sufficiently small $\varepsilon > 0$ there exist $C < C_0 = C_0(\varepsilon) < C + \varepsilon$ and $A < A_0 < A + \varepsilon$ such that

$$\inf_{s\le 0, w\ge A_0}\left(\frac{s^2}{m} - \frac{C_0 s}{w} + C_5\frac{w^2-1}{w^2}\right) > 0$$

for every $0 < C_5 < C_0^2 - C^2$. From Theorem 4.2 we get

$$\frac{\sqrt{1+|Du|^2}}{e^{C_0(\varepsilon)u}} \le \max\left\{A+\varepsilon,\ \limsup_{x\to\partial\Omega}\frac{\sqrt{1+|Du(x)|^2}}{e^{C_0(\varepsilon)u(x)}}\right\} \qquad \text{in } \Omega.$$

Since $C_0(\varepsilon)u \ge Cu = \sqrt{m-1}\kappa u$, we can bound

$$\frac{\sqrt{1+|Du|^2}}{e^{C_0(\varepsilon)u}} \le \max\left\{A+\varepsilon,\ \limsup_{x\to\partial\Omega}\frac{\sqrt{1+|Du(x)|^2}}{e^{\sqrt{m-1}\kappa u(x)}}\right\} \qquad \text{in } \Omega$$

and by letting $\varepsilon \to 0$ we obtain the desired conclusion. □

3. Minimal graphic functions with negative part of linear growth

In this section we adapt the argument of the proof of Theorem 4.2 to obtain a global gradient bound for minimal graphic functions, that is, solutions of the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0, \tag{83}$$

on complete Riemannian manifolds with $\mathrm{Ric} \ge 0$ and satisfying a quadratic decay condition on the negative part of the curvature tensor. In particular, we will obtain that on such manifolds a solution of (83) satisfying a one-sided linear growth bound has globally bounded gradient.

THEOREM 4.10. *Let (M, σ) be a complete Riemannian manifold of dimension $m \ge 2$, let $r(x) = \mathrm{dist}_\sigma(o, x)$ be the distance function from a reference origin $o \in M$ and assume that the radial sectional curvature K_{rad} satisfies*

$$K_{\mathrm{rad}} \ge -\frac{\gamma^2}{1+r^2} \quad on\ D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $\gamma \geq 0$. *Then, for every* $a > 0$ *it is possible to find* $C_{1,\gamma}(a), C_{2,\gamma}(a) > 1$, *with* $C_{i,\gamma}(a) \to 1$ *as* $a \to 0$ *for* $i = 1, 2$, *such that the following is true: if* $\Omega \subseteq M$ *is an open set where* $\mathrm{Ric} \geq 0$ *and* u *is a solution in* Ω *of the equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

satisfying $u(x) \geq -ar(x)$ *for every* $x \in \Omega$, *then*

$$W \leq C_{1,\gamma}(a) \max\left\{C_{2,\gamma}(a), \limsup_{x\to\partial\Omega} W(x)\right\} \qquad \text{on } \Omega. \tag{84}$$

Proof. We will show that inequality (84) holds true for

$$C_{1,\gamma}(a) = \frac{1-e^{-C_1}}{e^{-aqC_1} - e^{-C_1}}, \qquad C_{2,\gamma}(a) = \sqrt{1+L}$$

provided L, C_1, q are positive numbers satisfying conditions

$$\frac{1-\tau}{1+L}\left(q^2 L - \frac{4}{\tau}\right) C_1 > (m-1)\left(1+\sqrt{4\gamma^2+1}\right) + 2, \qquad q < 1/a \tag{85}$$

together with some parameter $\tau \in (0, 1)$. We remark that for every $\gamma \geq 0$, $a > 0$ it is possible to find L, C_1, q, τ satisfying these requirements. Indeed, for any fixed $0 < \tau < 1$ and $0 < q < 1/a$ we can choose L large enough so that $q^2 L > 4/\tau$, and then C_1 large enough so that the first inequality is verified. Moreover, for $0 < a < 1$ conditions (85) are satisfied, for instance, by

$$\tau = \frac{1}{2}, \quad q = \frac{1}{\sqrt{a}}, \quad L = 10a, \quad C_1 = (2+10a)\left((m-1)\left(1+\sqrt{4\gamma^2+1}\right)+2\right)$$

and the resulting values of $C_{i,\gamma}(a)$, $i = 1, 2$, do converge to 1 as $a \to 0$.

Let L, C_1, q and τ be given satisfying the above requirements. Let $R > 0$ and set

$$C = \frac{qC_1}{R}, \quad \varepsilon = \frac{C_1}{R^2}, \quad u_R = u + aR, \quad \eta_R = e^{-Cu_R - \varepsilon r^2} - e^{-C_1}, \quad z_R = W\eta_R.$$

We denote by $B_R = B^\sigma_R(o)$ the geodesic ball of (M, σ) of radius R centered at o. Note that on $\Omega \cap \overline{B_R}$ we have $u_R \geq a(R-r) \geq 0$ and then $\eta_R \leq e^{-\varepsilon r^2} - e^{-C_1}$. In particular

$$\eta_R \leq 1 - e^{-C_1} \quad \text{on } \Omega \cap \overline{B_R}, \qquad \eta_R \leq 0 \quad \text{on } \Omega \cap \partial B_R. \tag{86}$$

We will show that

$$z_R \leq (1-e^{-C_1}) \max\left\{\sqrt{1+L}, \limsup_{x\to\partial\Omega} W(x)\right\} \quad \text{on } \Omega \cap B_R.$$

Without loss of generality, we can assume that $\Omega_R = \{x \in \Omega \cap B_R : z_R(x) > 0\}$ is non-empty. By compactness of $\overline{\Omega_R}$, there exists a sequence $\{x_n\} \subseteq \Omega_R$ satisfying

$$\lim_{n\to+\infty} z_R(x_n) = \sup_{\Omega\cap B_R} z_R > 0 \qquad \text{and} \qquad x_n \to \bar{x}$$

for some $\bar{x} \in \overline{\Omega_R}$.

Suppose that $\bar{x} \in \partial(\Omega \cap B_R)$. We have inclusion

$$\partial(\Omega \cap B_R) \subseteq \left(\overline{\Omega} \cap \partial B_R\right) \cup \left(\overline{B_R} \cap \partial\Omega\right) \equiv (\Omega \cap \partial B_R) \cup \left(\overline{B_R} \cap \partial\Omega\right)$$

where equivalence follows by observing that $\left(\overline{\Omega} \cap \partial B_R\right) \setminus (\Omega \cap \partial B_R) = \partial\Omega \cap \partial B_R$ is already contained in $\overline{B_R} \cap \partial\Omega$. It must be $\bar{x} \in \partial\Omega$. If this were not the case, then we would have $\bar{x} \in \Omega \cap \partial B_R$. From continuity of z_R in Ω it would then be $z_R(\bar{x}) > 0$ and, therefore, $\eta_R(\bar{x}) > 0$, contradicting the above observation that $\eta_R \leq 0$ on $\Omega \cap \partial B_R$. Having established $\bar{x} \in \partial\Omega$, we infer

$$\sup_{\Omega\cap B_R} z_R = \lim_{n\to+\infty} z_R(x_n) \leq (1-e^{-C_1}) \limsup_{n\to+\infty} W(x_n) \leq (1-e^{-C_1}) \limsup_{x\to\partial\Omega} W(x),$$

where the first inequality follows from (86).

Suppose now that $\bar{x} \in \Omega \cap B_R$. Then $z_R(\bar{x}) > 0$ by continuity of z_R in Ω. If $\bar{x} \in D_o$, then z_R is of class C^2 and by the maximum principle must satisfy $\nabla z_R = 0$, $\Delta_g z_R \leq 0$ at $\bar{x}$. Since $\mathrm{Ric} \geq 0$ on Ω and $z_R > 0$ at $\bar{x}$, from (35) we have

$$\Delta_g z_R \geq \left(-\varepsilon \Delta_g r^2 + \|\varepsilon \nabla r^2 + C\nabla u\|^2\right) W e^{-Cu_R - \varepsilon r^2}$$

and then it must be

$$-\varepsilon \Delta_g r^2 + \|\varepsilon \nabla r^2 + C\nabla u\|^2 \leq 0 \tag{87}$$

at $\bar{x}$. From Theorem 3.10 we have

$$-\varepsilon \Delta_g r^2 \geq -\frac{C_1}{R^2}\left((m-1)\left(1+\sqrt{4\gamma^2+1}\right)+2\right)$$

and from (9), together with Young's inequality, we can estimate

$$\|\varepsilon \nabla r^2 + C\nabla u\|^2 \geq \frac{|\varepsilon D r^2 + C Du|^2}{W^2} \geq \frac{1}{W^2}\left((1-\tau)C^2|Du|^2 + \left(1-\frac{1}{\tau}\right)\varepsilon^2 |Dr^2|^2\right).$$

We use $1-\tau > 0$, $r \leq R$ and the definitions of C and ε to further write

$$\|\varepsilon \nabla r^2 + C\nabla u\|^2 \geq \frac{1-\tau}{W^2}\left(C^2|Du|^2 - \frac{4\varepsilon^2 R^2}{\tau}\right) = \frac{1-\tau}{W^2}\frac{C_1^2}{R^2}\left(q^2|Du|^2 - \frac{4}{\tau}\right).$$

We can now conclude that $|Du(\bar{x})|^2 \leq L$, since otherwise we would get

$$\|\varepsilon \nabla r^2 + C\nabla u\|^2 \geq \frac{1-\tau}{1+L}\frac{C_1^2}{R^2}\left(q^2 L - \frac{4}{\tau}\right)$$

and then, from (85),

$$\begin{aligned} &-\varepsilon \Delta_g r^2 + \|\varepsilon \nabla r^2 + C\nabla u\|^2 \geq \\ &\geq \frac{C_1}{R^2}\left(\frac{1-\tau}{1+L}\left(q^2 L - \frac{4}{\tau}\right)C_1 - (m-1)\left(1+\sqrt{4\gamma^2+1}\right) - 2\right) > 0, \end{aligned}$$

contradicting (87). From $|Du(\bar{x})|^2 \leq L$ we obtain

$$\sup_{\Omega \cap B_R} z_R = z_R(\bar{x}) \leq (1-e^{-C_1})\sqrt{1+L}.$$

If $\bar{x} \in \Omega \cap \mathrm{cut}(o)$ then z_R may not be of class C^2 in a neighbourhood of $\bar{x}$ and we can not directly apply the above argument. However, r^2 satisfies conditions

$$\Delta_g r^2 \leq (m-1)\left(1+\sqrt{4\gamma^2+1}\right)+2, \qquad |Dr^2|^2 \leq 4r^2$$

in the barrier sense on Ω. In particular, by Theorem 3.10, in a neighbourhood of $\bar{x}$ we can find a smooth support function ψ for r^2 at $\bar{x}$ satisfying

$$\Delta_g \psi < \frac{1-\tau}{1+L}\left(q^2 L - \frac{4}{\tau}\right)C_1, \qquad |D\psi|^2 \leq 4r(\bar{x})^2$$

and then we can repeat the above argument with ψ in place of r^2, as outlined in the proof of Part 2 of Theorem 4.2.

Summing up, we have shown that for every $R > 0$

$$z_R \leq (1-e^{-C_1})\max\left\{\sqrt{1+L}, \limsup_{x\to\partial\Omega} W(x)\right\} \quad \text{on } \Omega \cap B_R.$$

As $R \to +\infty$ we have $\eta_R \to e^{-aqC_1} - e^{-C_1}$ pointwise on Ω, so we conclude that

$$(e^{-aqC_1} - e^{-C_1})W \leq (1-e^{-C_1})\max\left\{\sqrt{1+L}, \limsup_{x\to\partial\Omega} W(x)\right\} \qquad \text{on } \Omega.$$

□

COROLLARY 4.11. *Let* (M, σ) *be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let* $r(x) = \mathrm{dist}_\sigma(o, x)$ *be the distance function from a reference origin* $o \in M$ *and assume that the radial sectional curvature* K_{rad} *satisfies*

$$K_{\mathrm{rad}} \geq -\frac{\gamma^2}{1+r^2} \quad \textit{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $\gamma \geq 0$. *If* u *is a solution in* M *of equation*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

then

i) *if* $u_-(x) = O(r(x))$ *then* u *has bounded gradient,*
ii) *if* $u_-(x) = o(r(x))$ *then* u *is constant.*

COROLLARY 4.12. *Let* (M, σ) *be a complete Riemannian manifold, let* $r(x) = \mathrm{dist}_\sigma(o, x)$ *be the distance function from a reference origin* $o \in M$ *and assume that the radial sectional curvature* K_{rad} *satisfies*

$$K_{\mathrm{rad}} \geq -\frac{\gamma^2}{1+r^2} \quad \textit{on } D_o = M \setminus (\{o\} \cup \mathrm{cut}(o))$$

for some $\gamma \geq 0$. *Let* $\Omega \subseteq M$ *be an open set where* $\mathrm{Ric} \geq 0$ *and let* u *be a solution in* Ω *of the equation*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$$

satisfying

$$\Lambda := \limsup_{x \to \partial\Omega} |Du(x)| < +\infty.$$

Then

i) *if* $u_-(x) = O(r(x))$ *then* u *has bounded gradient,*
ii) *if* $u_-(x) = o(r(x))$ *then* $|Du| \leq \Lambda$ *on* Ω.

CHAPTER 5

Applications to splitting theorems

1. Splitting for solutions of overdetermined problems

Let (M, σ) be a complete Riemannian manifold and $\Omega \subseteq M$ an open subset with smooth boundary and exterior normal ν. In this section we prove splitting results for solutions of overdetermined Dirichlet problems of the form

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \dfrac{f_2(u)}{\sqrt{1+|Du|^2}} & \text{in } \Omega \\ u, \partial_\nu u & \text{locally constant on } \partial\Omega \end{cases} \tag{88}$$

under assumption that $\mathrm{Ric} \geq 0$ in Ω, that either condition (RΩ) or (K) is satisfied and that Ω is a parabolic domain, in the sense that we are going to precise right now. First, let us recall that a Riemannian manifold with boundary (N, h) is said to be parabolic if its Neumann Laplacian is parabolic, that is, if every (weak) solution $v \in C(N) \cap H^1_{\mathrm{loc}}(N)$ of

$$\begin{cases} \Delta_h v \geq 0 & \text{in } \operatorname{int} N, \\ \partial_\nu v \leq 0 & \text{on } \partial N, \\ \sup_N v < +\infty \end{cases} \tag{89}$$

is constant, where ν is the exterior normal of ∂N in N and v is said be a weak solution of (89) if

$$\int_N h(\nabla_h v, \nabla_h \phi)\, \mathrm{d}x_h \leq 0 \qquad \text{for every } 0 \leq \phi \in C^\infty_c(N).$$

Definition 5.1. *Let (M, σ) be a complete Riemannian manifold without boundary. An open, connected subset $\Omega \subseteq M$ with smooth boundary is said to be a parabolic domain if $(\overline{\Omega}, \sigma)$ is a parabolic manifold with boundary.*

From [**28**] we have the following characterization: a Riemannian manifold with boundary (N, h) is parabolic if and only if each compact subset $K \subseteq N$ with non-empty interior has zero capacity, where the capacity $\mathrm{cap}(K)$ is defined as

$$\mathrm{cap}(K) = \inf\left\{ \int_N |\nabla_h \phi|_h^2 \, \mathrm{d}x_h : \phi \in \mathrm{Lip}_c(N),\ \phi \geq 1 \text{ on } K \right\}.$$

The above definition and characterization can be extended to weighted Laplace operators: if (N, h) is a Riemannian manifold with boundary and $f \in C^1(N)$, we define the weighted Laplace-Beltrami operator $\Delta_{h,f}$ by

$$\Delta_{h,f}\phi := e^f \operatorname{div}_h(e^{-f}\nabla_h \phi) \equiv \Delta_h \phi - h(\nabla_h f, \nabla_h \phi)$$

for every $\phi \in C^2(N)$. The operator $\Delta_{h,f}$ is symmetric with respect to the weighted volume measure $e^{-f}\mathrm{d}x_h$ and we say that it is parabolic on N if every (weak) solution $v \in C(N) \cap H^1_{\mathrm{loc}}(N)$ of

$$\begin{cases} \Delta_{h,f} v \geq 0 & \text{in } \operatorname{int} N, \\ \partial_\nu v \leq 0 & \text{on } \partial N, \\ \sup_N v < +\infty \end{cases} \tag{90}$$

is constant, where in this case v is said be a weak solution of (90) if

$$\int_N h(\nabla_h v, \nabla_h \phi)\, e^{-f} \mathrm{d}x_h \le 0 \qquad \text{for every } 0 \le \phi \in C_c^\infty(N).$$

The proof of Theorem 1.5 in [**28**] extends verbatim to showing that $\Delta_{h,f}$ is parabolic on (N, h) if and only if for every compact set $K \subseteq N$ with non-empty interior the weighted capacity

$$\mathrm{cap}_f(K) = \inf \left\{ \int_N |\nabla_h \phi|_h^2\, e^{-f} \mathrm{d}x_h : \phi \in \mathrm{Lip}_c(N),\ \phi \ge 1 \text{ on } K \right\}.$$

is zero.

The proof of the splitting Theorem 5.5 relies on a weighted geometric Poincaré inequality for solutions of (88) that are strictly monotone in the direction of some Killing vector field $X \in \mathfrak{X}(\overline{\Omega})$. This inequality is inspired by an analogous one for monotone solutions of semilinear equations $\Delta u = f(u)$ first introduced by Farina and Valdinoci in [**21**] in Euclidean space, and later extended to the context of Riemannian manifolds by Farina, Mari, Valdinoci, [**19**]. The key feature is that the support of the test function in the Poincaré inequality is allowed to intersect the boundary $\partial\Omega$. This is made possible by cancellations in integration, first observed in [**21**], due to the identity (91) below, which is a consequence of the overdetermined condition in (88).

Lemma 5.2. *Let (M, σ) be a Riemannian manifold and $\Omega \subseteq M$ an open subset with C^1 boundary. Let $u \in C^2(\overline{\Omega})$, $X \in \mathfrak{X}(\overline{\Omega})$ be a Killing field. If u and $\partial_\nu u$ are locally constant on $\partial\Omega$, then the function $v = (Du, X)$ satisfies*

$$\langle vW\nabla W - |Du|^2 \nabla v, \nu \rangle = 0 \qquad \text{on } \partial\Omega \tag{91}$$

for any vector ν normal to $\partial\Omega$.

Proof. On $\partial\Omega$ we have $Du = (\partial_\nu u)\nu$ because u is locally constant. With respect to a local coordinate system $\{x^i\}$ we write

$$WW_i = \left(|Du|^2/2\right)_i = u_{ij}u^j, \qquad v_i = u_{ij}X^j + X_{ij}u^j$$

and then

$$u^i(vWW_i - |Du|^2 v_i) = vu_{ij}u^i u^j - |Du|^2 u_{ij}u^i X^j - |Du|^2 X_{ij}u^i u^j.$$

Since $|Du| = |\partial_\nu u|$ is constant along $\partial\Omega$, we have $(D|Du|^2, Y) = 2u_{ij}u^i Y^j = 0$ for every vector field $Y = Y^j e_j$ orthogonal to ν. In particular, this is true for $Y = vDu - |Du|^2 X = (Du, X)Du - |Du|^2 X$, with components $Y^j = vu^j - |Du|^2 X^j$, hence

$$u^i(vWW_i - |Du|^2 v_i) = vu_{ij}u^i Y^j - |Du|^2 X_{ij}u^i u^j = -|Du|^2 X_{ij}u^i u^j = 0$$

having used the Killing condition $X_{ij} + X_{ji} = 0$. So, we have

$$(vWDW - |Du|^2 Dv, Du) = 0$$

or, equivalently,

$$\langle vW\nabla W - |Du|^2 \nabla v, \nabla u \rangle = 0.$$

In case $Du \ne 0$, from $Du = (\partial_\nu u)\nu$ and $\nabla u = W^{-2}Du$ we conclude

$$(vWDW - |Du|^2 Dv, \nu) = \langle vW\nabla W - |Du|^2 \nabla v, \nu \rangle = 0.$$

In case $Du = 0$ the same conclusion simply follows from $v = 0 = |Du|$. □

Before stating and proving the next result, let us fix some notation. If Ω is an open set and $u \in C^2(\Omega)$, then for every $x \in \Omega$ where $\mathrm{d}u \ne 0$ the level set $\Sigma_x = \{y \in \Omega : u(y) = u(x)\}$ is an embedded regular hypersurface in a neighbourhood of x. We let A be its second fundamental form in (Ω, g) and for any $\phi \in C^1(\Omega)$ we let

$$\nabla_\top \phi = \nabla\phi - \left\langle \nabla\phi, \frac{\nabla u}{\|\nabla u\|} \right\rangle \frac{\nabla u}{\|\nabla u\|}$$

be the tangential gradient of ϕ on Σ_x, that is, the orthogonal projection of $\nabla\phi$ onto the tangent subspace to Σ_x. Then, along Σ_x, the remainder in the classical Kato inequality is made explicit by the following inequality from [**52**],

$$\|\operatorname{Hess}_g(u)\|^2 - \|\nabla\|\nabla u\|\|^2 = \|\nabla_\top\|\nabla u\|\|^2 + \|\nabla u\|^2\|A\|^2. \tag{92}$$

Note that $\phi = \|\nabla u\|$ is C^1 in the set $\{\mathrm{d}u \neq 0\}$. Moreover, by (14) and (16) we have

$$\|\operatorname{Hess}_g(u)\|^2 = \frac{\|\mathrm{II}\|^2}{W^2}$$

and then

$$\|\mathrm{II}\|^2 - W^2\|\nabla\|\nabla u\|\|^2 = W^2\left(\|\nabla_\top\|\nabla u\|\|^2 + \|\nabla u\|^2\|A\|^2\right). \tag{93}$$

Theorem 5.3. *Let (M,σ) be a Riemannian manifold and $\Omega \subseteq M$ an open subset with C^1 boundary. Let $f_1, f_2 \in C^1(\mathbb{R})$ be given functions and let $u \in C^3(\Omega)\cap C^2(\overline{\Omega})$ be a solution of*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \frac{f_2(u)}{\sqrt{1+|Du|^2}} \qquad \text{in } \Omega$$

with u and $\partial_\nu u$ locally constant on $\partial\Omega$. If $X \in \mathfrak{X}(\overline{\Omega})$ is a Killing vector field and $v = (Du, X) > 0$ in $\overline{\Omega}$, then

$$\int_\Omega e^{F_2(u)}\left(W^2\left(\|\nabla_\top\|\nabla u\|\|^2 + \|\nabla u\|^2\|A\|^2\right) + \frac{\operatorname{Ric}(Du,Du)}{W^2}\right)\varphi^2 \mathrm{d}x_g =$$
$$= \int_\Omega e^{F_2(u)}\|\nabla u\|^2\|\nabla\varphi\|^2\mathrm{d}x_g - \int_\Omega e^{F_2(u)}\frac{v^2}{W^2}\left\|\nabla\frac{\varphi|Du|}{v}\right\|^2\mathrm{d}x_g$$

for every $\varphi \in \operatorname{Lip}_c(\overline{\Omega})$, where $F_2' = f_2$.

Proof. Consider the vector fields

$$Y = \varphi^2 e^{F_2(u)}\frac{\nabla W}{W}, \qquad Z = \varphi^2|Du|^2 e^{F_2(u)}\frac{\nabla v}{W^2 v}$$

and compute

$$\operatorname{div} Y = \varphi^2 W \operatorname{div}\left(e^{F_2(u)}\frac{\nabla W}{W^2}\right) + e^{F_2(u)}\left(\varphi^2\frac{\|\nabla W\|^2}{W^2} + \frac{\langle\nabla\varphi^2,\nabla W\rangle}{W}\right),$$
$$\operatorname{div} Z = \varphi^2\frac{|Du|^2}{v}\operatorname{div}\left(e^{F_2(u)}\frac{\nabla v}{W^2}\right) - e^{F_2(u)}\left(\varphi^2\frac{|Du|^2}{W^2}\frac{\|\nabla v\|^2}{v^2} - \frac{\langle\nabla(\varphi^2|Du|^2),\nabla v\rangle}{W^2 v}\right).$$

We recall the differential identity

$$\phi^2\frac{\|\nabla v\|^2}{v^2} - \frac{\langle\nabla\phi^2,\nabla v\rangle}{v} = v^2\left\|\nabla\frac{\phi}{v}\right\|^2 - \|\nabla\phi\|^2 \tag{94}$$

which can be easily deduced dividing both sides of

$$\phi^2\|\nabla v\|^2 - v\langle\nabla\phi^2,\nabla v\rangle = \|\phi\nabla v - v\nabla\phi\|^2 - v^2\|\nabla\phi\|^2$$

by v^2. We apply (94) with the choice $\phi = \varphi|Du|$ to get

$$\operatorname{div} Z = \varphi^2\frac{|Du|^2}{v}\operatorname{div}\left(e^{F_2(u)}\frac{\nabla v}{W^2}\right) - e^{F_2(u)}\left(\frac{v^2}{W^2}\left\|\nabla\frac{\varphi|Du|}{v}\right\|^2 - \frac{\|\nabla(\varphi|Du|)\|^2}{W^2}\right).$$

From the previous Lemma we have $\langle Y - Z, \nu\rangle = 0$ on $\partial\Omega$, hence an application of the divergence theorem yields

$$\int_\Omega (\operatorname{div} Y - \operatorname{div} Z)\,\mathrm{d}x_g = 0$$

and we obtain

$$\int_\Omega \varphi^2 \left(W \operatorname{div}\left(e^{F_2(u)} \frac{\nabla W}{W^2} \right) - \frac{|Du|^2}{v} \operatorname{div}\left(e^{F_2(u)} \frac{\nabla v}{W^2} \right) \right) =$$

$$= \int_\Omega \frac{e^{F_2(u)}}{W^2} \left(\|\nabla(\varphi|Du|)\|^2 - \varphi^2 \|\nabla W\|^2 - \langle \nabla\varphi^2, W\nabla W\rangle - v^2 \left\| \nabla \frac{\varphi|Du|}{v} \right\|^2 \right).$$

From (28) and (40) we have

$$W \operatorname{div}\left(e^{F_2(u)} \frac{\nabla W}{W^2} \right) = e^{F_2(u)} \left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} + (W f_1'(u) + f_2'(u))\|\nabla u\|^2 \right),$$

$$\frac{|Du|^2}{v} \operatorname{div}\left(e^{F_2(u)} \frac{\nabla v}{W^2} \right) = e^{F_2(u)} \left(W f_1'(u) + f_2'(u) \right) \|\nabla u\|^2,$$

and by direct computation (note that $|Du|$, $\|\nabla u\|$ are positive C^2 functions in Ω, because of $u \in C^3(\Omega)$ and since $\mathrm{d}u \neq 0$ in Ω as a consequence of condition $(Du, X) > 0$)

$$\begin{aligned}
\|\nabla(\varphi|Du|)\|^2 - \varphi^2\|\nabla W\|^2 - \langle \nabla\varphi^2, W\nabla W\rangle &= \\
&= \||Du|\nabla\varphi + \varphi\nabla|Du|\|^2 - \varphi^2\|\nabla W\|^2 - 2\langle \varphi\nabla\varphi, |Du|\nabla|Du|\rangle \\
&= |Du|^2\|\nabla\varphi\|^2 + \varphi^2\|\nabla|Du|\|^2 - \varphi^2\|\nabla W\|^2 \\
&= |Du|^2\|\nabla\varphi\|^2 + \varphi^2\left(\|\nabla\sqrt{W^2-1}\|^2 - \|\nabla W\|^2 \right) \\
&= |Du|^2\|\nabla\varphi\|^2 + \varphi^2\left(\left(\frac{W}{\sqrt{W^2-1}} \right)^2 - 1 \right) \|\nabla W\|^2 \\
&= |Du|^2\|\nabla\varphi\|^2 + \varphi^2 \frac{\|\nabla W\|^2}{|Du|^2} \\
&= |Du|^2\|\nabla\varphi\|^2 + \varphi^2 W^4 \|\nabla\|\nabla u\|\|^2
\end{aligned}$$

where the last equality follows from the identity

$$\frac{\nabla W}{|Du|} = W^2 \nabla\|\nabla u\|,$$

which in turn can be checked by direct computation

$$\begin{aligned}
\nabla\|\nabla u\| = \nabla\frac{|Du|}{W} &= \frac{\nabla|Du|}{W} - \frac{|Du|\nabla W}{W^2} \\
&= \frac{\nabla W}{|Du|} - \frac{|Du|\nabla W}{W^2} \\
&= \frac{(W^2 - |Du|^2)\nabla W}{|Du|W^2} = \frac{\nabla W}{|Du|W^2}
\end{aligned}$$

where in the middle equality we have used the identity $\frac{\nabla|Du|}{W} = \frac{\nabla W}{|Du|}$, that is, $\nabla|Du|^2 = \nabla W^2$, whose validity follows from the very definition $W^2 = 1 + |Du|^2$. Hence,

$$\int_\Omega e^{F_2(u)} \left(\|\mathrm{II}\|^2 - W^2\|\nabla\|\nabla u\|\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} \right) =$$

$$= \int_\Omega e^{F_2(u)} \|\nabla u\|^2 \|\nabla\varphi\|^2 - \int_\Omega e^{F_2(u)} \frac{v^2}{W^2} \left\| \nabla \frac{\varphi|Du|}{v} \right\|^2$$

and by (93) we reach the desired conclusion. □

THEOREM 5.4. *Let (M, σ) be a Riemannian manifold and $\Omega \subseteq M$ an open subset with C^1 boundary. Let $f_1, f_2 \in C^1(\mathbb{R})$ be given functions and let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a* *solution of*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \frac{f_2(u)}{\sqrt{1+|Du|^2}} \qquad \text{in } \Omega$$

with u and $\partial_\nu u$ locally constant on $\partial\Omega$. If $X \in \mathfrak{X}(\overline{\Omega})$ is a Killing vector field and $v = (Du, X) > 0$ in Ω, then

$$\int_\Omega e^{F_2(u)}\left(W^2\left(\|\nabla_\top\|\nabla u\|\|^2 + \|\nabla u\|^2\|A\|^2\right) + \frac{\operatorname{Ric}(Du,Du)}{W^2}\right)\varphi^2 \mathrm{d}x_g \leq$$
$$\leq \int_\Omega e^{F_2(u)}\|\nabla u\|^2\|\nabla\varphi\|^2 \mathrm{d}x_g - \int_\Omega e^{F_2(u)}\frac{v^2}{W^2}\left\|\nabla\frac{\varphi|Du|}{v}\right\|^2 \mathrm{d}x_g$$

for every $\varphi \in \operatorname{Lip}_c(\overline{\Omega})$, where $F_2' = f_2$.

Proof. Let $\varepsilon > 0$ and set

$$Y = \varphi^2 e^{F_2(u)}\frac{\nabla W}{W}, \qquad v_\varepsilon = v + \varepsilon, \qquad Z_\varepsilon = \varphi^2|Du|^2 e^{F_2(u)}\frac{\nabla v_\varepsilon}{W^2 v_\varepsilon}.$$

Observing that $\nabla v_\varepsilon = \nabla v$, from the divergence theorem and Lemma 5.2 we have

$$\begin{aligned}\int_\Omega \operatorname{div}(Y - Z_\varepsilon)\mathrm{d}x_g &= \int_{\partial\Omega}\varphi^2 e^{F_2(u)}\left\langle\frac{\nabla W}{W} - \frac{|Du|^2}{W^2}\frac{\nabla v}{v_\varepsilon}, \nu\right\rangle \mathrm{d}\mathcal{H}_g^{m-1}\\ &= \int_{\partial\Omega}\varphi^2 e^{F_2(u)}\left(1 - \frac{v}{v_\varepsilon}\right)\frac{\langle\nabla W,\nu\rangle}{W}\mathrm{d}\mathcal{H}_g^{m-1}\\ &= \int_{\partial\Omega}\varphi^2 e^{F_2(u)}\frac{\varepsilon}{v_\varepsilon}\frac{\langle\nabla W,\nu\rangle}{W}\mathrm{d}\mathcal{H}_g^{m-1}.\end{aligned}$$

Repeating the computations in proof of Theorem 5.3 we obtain

$$\begin{aligned}&\int_\Omega \operatorname{div}(Y - Z_\varepsilon)\mathrm{d}x_g =\\ &\quad = \int_\Omega e^{F_2(u)}\left[\left(\|\mathrm{II}\|^2 - W^2\|\nabla\|\nabla u\|\|^2 + \frac{\operatorname{Ric}(Du,Du)}{W^2}\right)\varphi^2 - \|\nabla u\|^2\|\nabla\varphi\|^2\right]\mathrm{d}x_g\\ &\quad + \int_\Omega e^{F_2(u)}\left(Wf_1'(u) + f_2'(u)\right)\|\nabla u\|^2\frac{\varepsilon}{v_\varepsilon}\varphi^2 \mathrm{d}x_g + \int_\Omega \frac{e^{F_2(u)}}{W^2}v_\varepsilon^2\left\|\nabla\frac{\varphi|Du|}{v_\varepsilon}\right\|^2 \mathrm{d}x_g.\end{aligned}$$

Shortly, we write

$$\int_\Omega e^{F_2(u)}\left(\|\mathrm{II}\|^2 - W^2\|\nabla\|\nabla u\|\|^2 + \frac{\operatorname{Ric}(Du,Du)}{W^2}\right)\varphi^2 \mathrm{d}x_g + I_1(\varepsilon) + I_2(\varepsilon) = I_3(\varepsilon)$$

with

$$\begin{aligned}I_1(\varepsilon) &= \int_\Omega \frac{e^{F_2(u)}}{W^2}v_\varepsilon^2\left\|\nabla\frac{\varphi|Du|}{v_\varepsilon}\right\|^2 \mathrm{d}x_g,\\ I_2(\varepsilon) &= \int_\Omega e^{F_2(u)}\left(Wf_1'(u) + f_2'(u)\right)\|\nabla u\|^2\frac{\varepsilon}{v_\varepsilon}\varphi^2 \mathrm{d}x_g,\\ I_3(\varepsilon) &= \int_\Omega e^{F_2(u)}\|\nabla u\|^2\|\nabla\varphi\|^2 \mathrm{d}x_g + \int_{\partial\Omega}\varphi^2 e^{F_2(u)}\frac{\varepsilon}{v_\varepsilon}\frac{\langle\nabla W,\nu\rangle}{W}\mathrm{d}\mathcal{H}_g^{m-1}.\end{aligned}$$

From Fatou's lemma we have

$$\liminf_{\varepsilon\to 0^+} I_1(\varepsilon) \geq \int_\Omega \lim_{\varepsilon\to 0^+}\left(v_\varepsilon^2\frac{e^{F_2(u)}}{W^2}\left\|\nabla\frac{\varphi|Du|}{v_\varepsilon}\right\|^2\right)\mathrm{d}x_g = \int_\Omega \frac{e^{F_2(u)}}{W^2}v^2\left\|\nabla\frac{\varphi|Du|}{v}\right\|^2 \mathrm{d}x_g.$$

Since $0 < \varepsilon/v_\varepsilon < 1$ for every $\varepsilon > 0$, by applying Lebesgue's dominated convergence theorem with dominating function $e^{F_2(u)}|Wf_1'(u) + f_2'(u)|\|\nabla u\|^2\varphi^2$ we obtain

$$\lim_{\varepsilon\to 0^+} I_2(\varepsilon) = \int_\Omega \lim_{\varepsilon\to 0^+}\left(e^{F_2(u)}\left(Wf_1'(u)+f_2'(u)\right)\|\nabla u\|^2\frac{\varepsilon}{v_\varepsilon}\varphi^2\right)\mathrm{d}x_g = 0,$$

and by similar arguments we also have

$$\lim_{\varepsilon\to 0^+} I_3(\varepsilon) = \int_\Omega e^{F_2(u)}\|\nabla u\|^2\|\nabla\varphi\|^2\mathrm{d}x_g.$$

Then, the conclusion follows. □

We are now in position to prove

THEOREM 5.5. *Let (M,σ) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and let $\Omega \subseteq M$ be a parabolic domain with smooth boundary. Let $f_1, f_2 \in C^1(\mathbb{R})$ be given functions, with $f_1' \geq 0$. Let $u \in C^3(\Omega)\cap C^2(\overline{\Omega})$ be a solution of*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \frac{f_2(u)}{\sqrt{1+|Du|^2}} \qquad \text{in } \Omega$$

satisfying

$$\begin{cases} u, \partial_\nu u & \text{locally constant on } \partial\Omega,\\ \sup_\Omega |u| < +\infty, & \\ \sup_{\partial\Omega}|Du| < +\infty, & \\ (Du, X) > 0 & \text{in } \Omega \end{cases}$$

for some Killing vector field $X \in \mathfrak{X}(\overline{\Omega})$, and assume that either condition (RΩ) or (K) is satisfied. Then Ω is isometric to the Riemannian product of an open interval $I \subseteq \mathbb{R}$ and a complete manifold N with $\mathrm{Ric}_N \geq 0$*, the function u only depends on the I-variable, and (X, ∂_t) is constant in Ω, where ∂_t is the unit tangent vector of the family of curves $I \times \{\xi\}$, $\xi \in N$.*

PROOF. From Corollary 4.4 we have that $\sup_\Omega W < +\infty$. Then (Ω, g) is quasi-isometric to (Ω,σ) and therefore $(\overline{\Omega}, g)$ is a parabolic manifold with boundary. Moreover, since u is bounded we have that $e^{F_2(u)}$ is bounded for any primitive F_2 for f_2 on $\mathbb{R}$.

By Theorem 1.5 in [**28**], from the parabolicity of $(\overline{\Omega}, g)$ we have existence of a sequence $\{\varphi_n\} \subseteq \mathrm{Lip}_c(\overline{\Omega})$ satisfying

$$\varphi_n \to 1 \quad \text{in } W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}), \qquad \int_\Omega \|\nabla\varphi_n\|^2\mathrm{d}x_g \to 0$$

as $n \to +\infty$. From Theorem 5.4 and condition $\mathrm{Ric} \geq 0$, for every $n \geq 0$ we have inequality

$$\int_\Omega e^{F_2(u)}W^2\left(\|\nabla_\top\|\nabla u\|\|^2 + \|\nabla u\|^2\|A\|^2\right)\varphi_n^2\mathrm{d}x_g + \int_\Omega e^{F_2(u)}\frac{v^2}{W^2}\left\|\nabla\frac{\varphi_n|Du|}{v}\right\|^2\mathrm{d}x_g \leq$$
$$\leq \int_\Omega e^{F_2(u)}\|\nabla u\|^2\|\nabla\varphi_n\|^2\mathrm{d}x_g.$$

By Cauchy-Schwarz and Young's inequalities we can estimate

$$\begin{aligned}\left\|\nabla\frac{\varphi_n|Du|}{v}\right\|^2 &= \varphi_n^2\left\|\nabla\frac{|Du|}{v}\right\|^2 + \frac{|Du|^2}{v^2}\|\nabla\varphi_n\|^2 + 2\left\langle\varphi_n\nabla\frac{|Du|}{v}, \frac{|Du|}{v}\nabla\varphi_n\right\rangle\\ &\geq \left(1-\frac{1}{2}\right)\varphi_n^2\left\|\nabla\frac{|Du|}{v}\right\|^2 + (1-2)\frac{|Du|^2}{v^2}\|\nabla\varphi_n\|^2\end{aligned}$$

and then we get

$$\int_\Omega e^{F_2(u)} W^2 \left(\|\nabla_\top \|\nabla u\|\|^2 + \|\nabla u\|^2 \|A\|^2 \right) \varphi_n^2 \mathrm{d}x_g + \frac{1}{2} \int_\Omega e^{F_2(u)} \frac{v^2}{W^2} \left\| \nabla \frac{|Du|}{v} \right\|^2 \varphi_n^2 \mathrm{d}x_g \leq$$
$$\leq 2 \int_\Omega e^{F_2(u)} \|\nabla u\|^2 \|\nabla \varphi_n\|^2 \mathrm{d}x_g,$$

having used $|Du|^2/W^2 = \|\nabla u\|^2$. Since $e^{F_2(u)}$ is bounded, the RHS of this inequality tends to 0 as $n \to +\infty$. Then, by Fatou's lemma we obtain

$$\|\nabla_\top \|\nabla u\|\|^2 + \|\nabla u\|^2 \|A\|^2 \equiv 0, \qquad \nabla \frac{|Du|}{v} \equiv 0 \qquad \text{in } \Omega. \tag{95}$$

This is the starting point for the proof of the splitting. We reproduce the argument given in [**12**], which in turn follows the line of the one in [**19**]. From $v > 0$ we deduce that $Du \neq 0$ on Ω, so the vector field $Y = \nabla u/\|\nabla u\|$ is well defined on Ω and level sets of u are regular embedded hypersurfaces in Ω. For $x \in \Omega$, let $\{V_i\}_{1 \leq i \leq m}$ be a local g-orthonormal frame for $T\Omega$ in a neighbourhood of x, with $V_m = Y$. Then $\{V_i\}_{1 \leq i \leq m-1}$ is a local frame for the tangent subspace of the hypersurface $\{u = u(x)\}$. We have

$$\langle \nabla \|\nabla u\|, V_i \rangle = \mathrm{Hess}_g(u)(Y, V_i), \qquad A(V_i, V_j) = \frac{\mathrm{Hess}_g(u)(V_i, V_j)}{\|\nabla u\|}$$

for $1 \leq i, j \leq m-1$, with A the second fundamental form of $\{u = u(x)\}$ in (Ω, g). From these identities we deduce that the only nonzero component of $\mathrm{Hess}_g(u)$ is the one in the direction of $\mathrm{d}u \otimes \mathrm{d}u$. Since $\mathrm{Hess}_\sigma(u) = W^2 \,\mathrm{Hess}_g(u)$ by (14), we infer

$$\mathrm{Hess}_\sigma(u) = \mathrm{Hess}_\sigma(u) \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) \frac{\mathrm{d}u}{|Du|} \otimes \frac{\mathrm{d}u}{|Du|} \qquad \text{in } \Omega. \tag{96}$$

From this identity we deduce that $|Du|$ is locally constant on level sets of u, that integral curves of $Du/|Du|$ are geodesics in M and that level sets of u are totally geodesic in Ω with respect to both metrics σ and g. Since u is locally constant on $\partial\Omega$, in the limit we obtain that each connected component of $\partial\Omega$ is a totally geodesic hypersurface in (M, σ).

Since $\partial\Omega$ has at most countably many connected components and u is constant on each of them, the set $B = u(\partial\Omega) \subseteq \mathbb{R}$ consists of at most countably many points. As u is non-constant in Ω, we can find $b \in u(\Omega) \setminus B$. Let N be a connected component of the level set $\{u = b\} \subseteq \Omega$. By the implicit function theorem, N is a properly embedded hypersurface in M and is a manifold without boundary, complete with respect to the metric σ_N induced from σ. We denote with $\Phi(t, x)$ the flow of $Du/|Du|$ starting from N, defined on the connected set

$$\mathcal{D} \subseteq \mathbb{R} \times N, \qquad \mathcal{D} = \{(t, x) : x \in N,\ t \in (t_1(x), t_2(x))\}$$

where, for every $x \in N$, $t_1(x) \in [-\infty, 0)$ and $t_2(x) \in (0, +\infty]$ are the extrema of the largest open interval $I_x = (t_1(x), t_2(x))$ such that for every $t \in I_x$ the point $\Phi(t, x)$ is well defined and belongs to Ω. If $t_1(x) > -\infty$ (respectively, $t_2(x) < +\infty$) then the curve $t \mapsto \Phi(t, x)$ converges to a point of $\partial\Omega$ as $t \searrow t_1(x)$ (resp., $t \nearrow t_2(x)$) which we shall denote as $x_* = \Phi(t_1(x)^+, x)$ (resp., $x^* = \Phi(t_2(x)^-, x)$). The function t_1 is upper semi-continuous on N, that is, for every $x \in N$ we have

$$\limsup_{n \to +\infty} t_1(x_n) \leq t_1(x)$$

for every sequence $\{x_n\} \subseteq N$ converging to x: otherwise, we could find $t \in (t_1(x), 0]$ and a sequence $\{x_n\}$ converging to x such that $t_1(x_n) \to t$, yielding $\partial\Omega \ni (x_n)_* \to \Phi(t, x) \in \Omega$, absurd. Similarly, the function t_2 is lower semi-continuous on N. Hence, $\mathcal{D}$ is open in $\mathbb{R} \times N$.

From (96) we deduce

$$\left(D_V \frac{Du}{|Du|}, Z \right) = 0 \qquad \text{for every } V, Z \in T_x\Omega,\ x \in \Omega$$

thus $Du/|Du|$ is a parallel vector field. Then the induced metric on $\mathcal{D}$ by Φ is the product metric $\mathrm{d}t^2 + \sigma_N$. Let $c_0 > 0$ be the constant value of $|Du|$ on N and let β be the maximal solution of the Cauchy problem

$$\begin{cases} y'(s) = \left(f_1(s) + \dfrac{f_2(s)}{\sqrt{1+y(s)^2}} \right) \dfrac{(1+y(s)^2)^{3/2}}{y(s)}, \\ y(b) = c_0. \end{cases}$$

Since u is strictly increasing along the curves $t \mapsto \Phi(t,x)$ and $|Du|$ is locally constant on level sets of u, for every $x \in \Omega$ there exists a neighbourhood $U_x \subseteq \Omega$ and a C_2 real function β_x such that

$$|Du| = \beta_x(u) \qquad \text{on } U_x.$$

As $Du/|Du|$ is parallel, on U_x we have

$$\begin{aligned} f_1(u) + \frac{f_2(u)}{\sqrt{1+\beta_x(u)^2}} &= \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = \operatorname{div}\left(\frac{|Du|}{\sqrt{1+|Du|^2}} \frac{Du}{|Du|} \right) \\ &= D_{Du/|Du|} \frac{|Du|}{\sqrt{1+|Du|^2}} = \beta_x(u) \left(\frac{\beta_x}{\sqrt{1+\beta_x^2}} \right)'(u) \\ &= \frac{\beta_x(u)\beta_x'(u)}{(1+\beta_x(u)^2)^{3/2}} \end{aligned}$$

that is, β_x is a solution of the Cauchy problem

$$\begin{cases} y'(s) = \left(f_1(s) + \dfrac{f_2(s)}{\sqrt{1+y(s)^2}} \right) \dfrac{(1+y(s)^2)^{3/2}}{y(s)}, \\ y(u(x)) = |Du(x)|. \end{cases}$$

Without loss of generality, we can assume that β_x is the maximal solution of this problem. For points $x \in N$, this yields $\beta_x = \beta$ by uniqueness. Hence, for every $x_1, x_2 \in \Phi(\mathcal{D})$ belonging to the same curve $t \mapsto \Phi(t,x)$, $x \in N$, it must be $\beta_{x_1} = \beta_{x_2}$. Therefore, $\beta_x = \beta$ for every $x \in \Phi(\mathcal{D})$, that is,

$$|Du| = \beta(u) \qquad \text{on } \Phi(\mathcal{D}).$$

We claim that $\Phi(\mathcal{D}) = \Omega$. The map Φ is a diffeomorphism and $\mathcal{D}$ is open in $\mathbb{R} \times N$, so $\Phi(\mathcal{D})$ is open in Ω. We check that $\Phi(\mathcal{D})$ is also closed in Ω, thus deducing $\Phi(\mathcal{D}) = \Omega$ by connectedness of Ω.

First, we show that t_1 and t_2 are constant on N. We prove this for t_1, the proof for t_2 being analogous. Set

$$\rho(s) = \int_b^s \frac{\mathrm{d}\tau}{\beta(\tau)} \qquad \text{for every } s \in u(\Omega).$$

Note that by integrating $\frac{\mathrm{d}}{\mathrm{d}t} u(\Phi) = |Du|(\Phi) = \beta(u(\Phi))$ we get

$$t = \int_b^{u(\Phi(t,x))} \frac{\mathrm{d}\tau}{\beta(\tau)} = \rho(u(\Phi(t,x))) \qquad \text{for every } (t,x) \in \mathcal{D}.$$

We show that t_1 is lower semi-continuous on N. Suppose, by contradiction, that for some $x \in N$ and for some sequence $\{x_n\} \subseteq N$ converging to x we have

$$\lim_{n \to +\infty} t_1(x_n) < t_1(x).$$

Fix $\bar{t}$ such that

$$\lim_{n\to+\infty} t_1(x_n) < \bar{t} < t_1(x), \qquad \bar{t} \notin B = u(\partial\Omega).$$

Then, $\{\Phi(\bar{t}, x_n)\} \subseteq \Omega$ converges to a point $\bar{x} \in \partial\Omega$. Along this sequence, u has the constant value $\rho^{-1}(\bar{t})$, so by continuity it must be $u(\bar{x}) = \rho^{-1}(\bar{t})$. But $\rho^{-1}(\bar{t}) \notin u(\partial\Omega)$ and we have reached a contradiction. Since we had already shown that t_1 is upper semi-continuous, we conclude that t_1 is continuous on N. For every $x \in N$ we either have $t_1(x) = -\infty$ or $t_1(x) \in (-\infty, 0)$. In the second case, the endpoint $x_* = \lim_{t\to t_1(x)^+} \Phi(t, x)$ belongs to $\partial\Omega$ and by continuity $t_1(x) = \rho(u(x_*))$. So, $t_1(N) \subseteq \rho(B) \cup \{-\infty\}$. Since this set consists of at most countably many elements, it contains no open intervals. As t_1 is continuous on the connected set N, we conclude that t_1 is constant.

Let $T_1 \in [-\infty, 0)$ and $T_2 \in (0, +\infty]$ be the constant values of t_1 and t_2 on N, so that

$$\mathcal{D} = (T_1, T_2) \times N.$$

For every $\bar{t} \in (T_1, T_2)$, the image $N_{\bar{t}} = \Phi(\{\bar{t}\} \times N) \subseteq \Omega$ is a connected open subset of the embedded submanifold $\{u = \rho^{-1}(\bar{t})\} \subseteq \Omega$. The restriction $\Phi_{|\{\bar{t}\}\times N} : \{\bar{t}\} \times N \to N_{\bar{t}}$ is a local Riemannian isometry and $\{\bar{t}\} \times N$ is complete, so $\Phi_{|\{\bar{t}\}\times N}$ is a Riemannian covering map and therefore $N_{\bar{t}}$ is also complete with respect to the intrinsic geodesic distance, that we shall denote by $d_{\bar{t}}$ (see [**44**], Lemma 5.6.4 and Proposition 5.6.3).

We prove that $\Phi(\mathcal{D})$ is closed in Ω. Let $\{p_n\} \subseteq \Phi(\mathcal{D})$ be a given sequence converging to some point $\bar{p} \in \Omega$. We have to show that $\bar{p} \in \Phi(\mathcal{D})$. Set $\bar{t} = \rho(u(\bar{p}))$. For every n we can find $(t_n, x_n) \in \mathcal{D}$ such that $p_n = \Phi(t_n, x_n)$. By continuity, $t_n = \rho(u(p_n)) \to \rho(u(\bar{p})) = \bar{t}$, hence $T_1 \le \bar{t} \le T_2$. Both inequalities are strict, otherwise either $\{(x_n)_*\} = \{\Phi(T_1^+, x_n)\}$ or $\{(x_n)^*\} = \{\Phi(T_2^-, x_n))\}$ would be a sequence of points of $\partial\Omega$ converging to $\bar{p} \in \Omega$, absurd. Setting $q_n = \Phi(\bar{t}, x_n)$ for every n, we have that $\{q_n\}$ is a sequence of points of $N_{\bar{t}}$ converging to $\bar{p}$ in M, since $d_\sigma(p_n, q_n) \le |\bar{t} - t_n| \to 0$. Hence, $\{q_n\}$ is a Cauchy sequence in M. By completeness of M, any two points q_n, $q_{n'}$ are joined by a minimizing geodesic arc in M. Since $(N_{\bar{t}}, d_{\bar{t}})$ is complete and totally geodesic, every geodesic in M joining two points of $N_{\bar{t}}$ must lie in $N_{\bar{t}}$. So, $\{q_n\}$ is a Cauchy sequence in $N_{\bar{t}}$ and therefore converges to some point $\bar{q} \in N_{\bar{t}}$. Since $N_{\bar{t}}$ is embedded in M, $\bar{q} = \bar{p}$ and we conclude $\bar{p} \in N_{\bar{t}} \subseteq \Phi(\mathcal{D})$, as desired. This shows that $\Phi(\mathcal{D})$ is closed in Ω.

As already stated, since $\Phi(\mathcal{D})$ is non-empty and both open and closed in the connected set Ω, we have $\Phi(\mathcal{D}) = \Omega$. Thus, Φ realizes an isometry between Ω and the product manifold

$$(T_1, T_2) \times N.$$

Furthermore, u only depends on the variable t because

$$u(\Phi(t, x)) = \rho^{-1}(t) \qquad \text{for every } (t, x) \in (T_1, T_2) \times N.$$

The second identity in (95) implies

$$v = cW\|\nabla u\| \qquad \text{on } \Omega$$

for some constant $c > 0$. Since $v = (X, Du)$ and $W\|\nabla u\| = |Du|$, this identity rewrites as $(X, Du) = c|Du|$, that is,

$$(X, \partial_t) = c.$$

□

For particular choices of f_1, f_2, the conclusion of Theorem 5.5 can be reached under weakened assumptions. In Theorem 5.6 we show that if f_1, f_2 are non-positive and non-decreasing, then boundedness of u can be relaxed to one-sided boundedness (more precisely, $\inf_\Omega u > -\infty$). In Theorem 5.7 we show that if f_1 and f_2 are constant then the monotonicity condition $(Du, X) > 0$ of u in the direction of the Killing field X can be a-priori deduced from the request that $(Du, X) \ge 0$, $\not\equiv 0$ on $\partial\Omega$, under assumption that $|X|$ is bounded in Ω.

Theorem 5.6. *Let (M,σ) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and let $\Omega \subseteq M$ be a parabolic domain with smooth boundary. Let $f_1, f_2 \in C^1(\mathbb{R})$ be given functions, with $f_i \leq 0$, $f_i' \geq 0$ for $i = 1,2$. Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f_1(u) + \frac{f_2(u)}{\sqrt{1+|Du|^2}} \qquad \text{in } \Omega$$

satisfying

$$\begin{cases} u, \partial_\nu u & \text{locally constant on } \partial\Omega, \\ \inf_\Omega u > -\infty, & \\ \sup_{\partial\Omega} |Du| < +\infty, & \\ (Du, X) > 0 & \text{in } \Omega \end{cases}$$

for some Killing vector field $X \in \mathfrak{X}(\overline{\Omega})$ and assume that (RΩ) or (K) holds true. Then Ω and u split as in Theorem 5.5.

Proof. If $f_i \leq 0$ and $f_i' \geq 0$ then we have $-\infty < f_i(u_*) \leq f_i(u) \leq 0$ for $i = 1,2$. Setting $I = [u_*, +\infty)$, the function $f : I \times [1,+\infty)$ given by

$$f(y,w) = f_1(y) + \frac{f_2(y)}{w}$$

satisfies

$$-\Lambda \leq f(y,w) \leq 0, \quad \frac{\partial f}{\partial y}(y,w) \geq 0, \quad 0 \leq \frac{\partial f}{\partial w}(y,w) \equiv -\frac{f_2(y)}{w^2} \leq \frac{\Lambda}{w^2}$$

for every $(y,w) \in I \times [1,+\infty)$, with $\Lambda = |f_1(u_*)| + |f_2(u_*)|$. Then, from Corollary 4.6 we obtain that $\sup_\Omega |Du| < +\infty$ under the sole assumption that $\inf_\Omega u > -\infty$. Then, (Ω, g) is parabolic. Moreover, for any primitive F_2 of f_2 we have $F_2(u) \leq F_2(u_*) < +\infty$ due to $f_2 \leq 0$, so $e^{F_2(u)}$ is bounded on Ω. Having established these facts, one can repeat the proof of Theorem 5.5. □

Theorem 5.7. *Let (M,σ) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and let $\Omega \subseteq M$ a parabolic domain with smooth boundary. Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ satisfy*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H_1 + \frac{H_2}{\sqrt{1+|Du|^2}} \qquad \text{in } \Omega$$

for some constants $H_1, H_2 \in \mathbb{R}$. Assume that (RΩ) or (K) is satisfied and that

$$\begin{cases} u, \partial_\nu u & \text{locally constant on } \partial\Omega, \\ \sup_{\partial\Omega} |Du| < +\infty, & \\ (Du, X) \geq 0, \not\equiv 0 & \text{on } \partial\Omega \end{cases}$$

for some Killing vector field $X \in \mathfrak{X}(\overline{\Omega})$ with $\sup_\Omega |X| < +\infty$. If either

$$(i)\ \sup_\Omega |u| < +\infty \qquad \text{or} \qquad (ii) \begin{cases} \inf_\Omega u > -\infty, \\ H_2 \leq 0, \end{cases}$$

then Ω and u split as in Theorem 5.5.

Proof. If (i) is satisfied, then we have $\sup_\Omega W < +\infty$ as already observed in the proof of Theorem 5.5. If (ii) is satisfied, then the function $f : \mathbb{R} \times [1,+\infty) \to \mathbb{R}$ given by

$$f(y,w) = H_1 + \frac{H_2}{w}$$

satisfies

$$-|H_1| - |H_2| \leq f \leq |H_1| + |H_2|, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial w} = -\frac{H_2}{w^2} = \frac{|H_2|}{w^2},$$

so the conditions in Corollary 4.6 are satisfied for $\Lambda = |H_1| + |H_2|$ and, again, we deduce $\sup_\Omega W < +\infty$. In both cases, $(\overline{\Omega}, g)$ is a parabolic manifold with boundary.

Let f_1, f_2 be the constant functions on $\mathbb{R}$ given by $f_i \equiv H_i$ for $i = 1, 2$. A primitive F_2 for f_2 is the function $F_2(t) = H_2 t$. If (i) is satisfied, then $e^{F_2(u)} = e^{H_2 u}$ is bounded in Ω. If (ii) is satisfied, then $e^{F_2(u)} = e^{H_2 u} \leq e^{H_2 u_*} < +\infty$ is again bounded in Ω. So, in both cases we have $\sup_\Omega e^{F_2(u)} < +\infty$.

The function $v = (Du, X)$ is bounded on Ω as a consequence of Cauchy-Schwarz inequality

$$|v| \leq |Du||X| \leq \left(\sup_\Omega |Du|\right)\left(\sup_\Omega |X|\right) < +\infty$$

and satisfies

$$W^2 e^{-F_2(u)} \operatorname{div}_g \left(e^{F_2(u)} \frac{\nabla v}{W^2}\right) = 0$$

by (40). The weight $e^{F_2(u)}/W^2$ is bounded in Ω, so the weighted operator

$$\mathscr{L}\phi = W^2 e^{-F_2(u)} \operatorname{div}_g \left(e^{F_2(u)} \frac{\nabla \phi}{W^2}\right)$$

is parabolic on Ω due to parabolicity of Δ_g and the characterization of parabolicity recalled at the beginning of the section. Hence, the bounded, $\mathscr{L}$-harmonic function v must satisfy

$$\inf_\Omega v = \inf_{\partial\Omega} v \geq 0, \qquad \sup_\Omega v = \sup_{\partial\Omega} v > 0.$$

If $\inf_\Omega v > 0$, then $v > 0$ in Ω. If $\inf_\Omega v = 0$, then v is not constant in Ω and, by the strong maximum principle, it cannot attain the value $0 = \inf_\Omega v$ in Ω. So again it must be $v > 0$, that is,

$$(Du, X) > 0 \qquad \text{in } \Omega.$$

Having obtained $\sup_\Omega W < +\infty$, $\sup_\Omega e^{F_2(u)} < +\infty$ and $(Du, X) > 0$ in Ω, from this point on we can repeat again the argument in the proof of Theorem 5.5. ☐

2. Splitting on parabolic manifolds

In this section we prove a splitting theorem for parabolic, complete Riemannian manifolds with non-negative Ricci curvature and negative part of the sectional curvature decaying quadratically, in presence of non-constant, entire minimal graphic functions of linear growth. The proof of Theorem 5.8 parallels that of Li's splitting theorem for complete, parabolic manifolds of non-negative Ricci curvature supporting non-constant harmonic functions of linear growth, [**35**].

THEOREM 5.8. *Let (M, σ) be a complete parabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and with sectional curvature satisfying*

$$K \geq -\frac{\gamma^2}{1 + r^2}$$

for some $\gamma \geq 0$, where $r(x) = \mathrm{dist}_\sigma(o, x)$ is the distance from a fixed origin $o \in M$. If there exists a non-constant solution u of equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0$$

satisfying $u(x) \geq -ar(x)$ for some constant $a > 0$, then M is isometric to the Riemannian product $\mathbb{R} \times N$ for some complete and parabolic manifold N, with $\mathrm{Ric}_N \geq 0$, *and u is an affine function of the $\mathbb{R}$-coordinate.*

Proof. From Theorem 4.10, u has bounded gradient on M. As a consequence, the identity map is a quasi-isometry between the Riemannian manifolds (M, σ) and (M, g), and so the operator Δ_g is parabolic on M. We have

$$\Delta_g \frac{1}{W} = -\left(\|\mathrm{II}\|^2 + \frac{\mathrm{Ric}(Du, Du)}{W^2} \right) \frac{1}{W} \leq 0,$$

that is, $1/W$ is a bounded Δ_g-superharmonic function and as such it must be constant. This fact, coupled with the Jacobi equation itself and with inequalities $1/W > 0$, $\|\mathrm{II}\| \geq 0$, $\mathrm{Ric} \geq 0$, leads to the conclusion $\|\mathrm{II}\| \equiv 0$ and $\mathrm{Ric}(Du, Du) \equiv 0$. Since W is constant, so is $|Du|$. Since u is assumed to be non-constant, we have $|Du| \equiv c_0$ for some positive constant $c_0 > 0$. From this fact and $\mathrm{II} \equiv 0$ it is possible to show that M splits isometrically as a product $\mathbb{R} \times N$, with N a complete parabolic manifold with $\mathrm{Ric}_N \geq 0$ and u only depending on the $\mathbb{R}$-coordinate (and then necessarily being an affine function of it). Indeed, by (16) we have that $\mathrm{II} \equiv 0$ is equivalent to $\mathrm{Hess}_\sigma(u) \equiv 0$, so the validity of identity (96) is established and then one can repeat the argument of the proof of Theorem 5.5. □

As a consequence of Theorem 5.8 we also have the following

Theorem 5.9. *Let (M_0, σ_0) be a complete parabolic Riemannian manifold with non-negative sectional curvatures and let $(M, \sigma) = (\mathbb{R} \times M_0, \mathrm{d}t^2 + \sigma_0)$ with t the canonical coordinate on $\mathbb{R}$. If u is a solution in M of the equation*

$$\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0$$

satisfying $u(x) \geq -ar(x)$ for some constant $a > 0$, with $r(x) = \mathrm{dist}_\sigma(o, x)$ the distance from a reference origin $o \in M$, then either

i) *u is an affine function of $t \in \mathbb{R}$, where (t, ξ) denotes the generic point of $M = \mathbb{R} \times M_0$, or*
ii) *$M_0 = \mathbb{R} \times N_0$ for some complete and parabolic manifold N_0 with $\mathrm{Ric}_{N_0} \geq 0$ and u is an affine function of $(t, s) \in \mathbb{R}^2$, where (t, s, ζ) denotes the generic point of $M = \mathbb{R}^2 \times N_0$.*

Proof. The product manifold (M, σ) has non-negative sectional curvature and from Theorem 4.10 we deduce that u has bounded gradient on M. Then the weighted Laplacian $\mathscr{L} = \Delta_{g, 2\log W}$ defined by

$$\mathscr{L}\psi = W^2 \,\mathrm{div}_g \left(\frac{\nabla \psi}{W^2} \right) = \Delta_g \psi - \frac{2\langle \nabla W, \nabla \psi \rangle}{W}$$

is a uniformly elliptic differential operator in divergence form with bounded weight, with respect to the Riemannian metric of non-negative Ricci curvature σ. By a result of Saloff-Coste ([**47**], Theorem 7.4), this operator then satisfies a Liouville property: the only bounded solutions of equation $\mathscr{L}\psi = 0$ on M are constant functions.

The manifold M carries a global parallel vector field ∂_t, whose integral curves are the lines $\mathbb{R} \times \{\xi\}$, $\xi \in M_0$. In particular, ∂_t is a Killing vector field and the function $v = \partial_t u \equiv (Du, \partial_t)$ is a solution of $\mathscr{L}v = 0$ on M. From Cauchy-Schwarz's inequality we have $|v| \leq |Du|$, hence v is bounded on M and must be constant.

Let c_0 be the constant value of $\partial_t u$. The reference origin $o \in M$ can be expressed as $o = (t_0, \xi_0)$ for some $t_0 \in \mathbb{R}$, $\xi_0 \in M_0$. We define $u_0(\xi) = u(t_0, \xi)$ for every $\xi \in M_0$. For every (t, ξ) we can write

$$u(t,\xi) = u_0(\xi) + c_0(t - t_0), \quad Du(t, \xi) = D_0 u_0(\xi) + c_0 \partial_t, \quad W(t, \xi) = \sqrt{1 + c_0^2 + |D_0 u_0(\xi)|_0^2}$$

with D_0, $|\cdot|_0$ the connection and vector norm of M_0, respectively. From the isometric splitting $\mathbb{R} \times M_0 = M$ we have the following expression for the function r,

$$r(t, \xi) = \sqrt{|t - t_0|^2 + r_0(\xi)^2},$$

where $r_0(\xi) = \text{dist}_{\sigma_0}(\xi_0, \xi)$ is the distance from ξ_0 in M_0, and then we deduce that

$$u_0(\xi) \geq -ar_0(\xi)$$

for every $\xi \in M_0$. From $D\partial_t = 0$ and $\partial_t W = 0$ we get

$$\text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \text{div}_0\left(\frac{D_0u_0}{\sqrt{1+c_0^2+|D_0u_0|_0^2}}\right)$$

with div_0 the divergence of M_0. Setting $u_1 = u_0/\sqrt{1+c_0^2}$, we can rewrite

$$\frac{D_0u_0}{\sqrt{1+c_0^2+|D_0u_0|_0^2}} = \frac{D_0u_1}{\sqrt{1+|D_0u_1|_0^2}}$$

and then we conclude that u_1 is a solution in M_0 of

$$\text{div}_0\left(\frac{D_0u_1}{\sqrt{1+|D_0u_1|_0^2}}\right) = 0$$

satisfying $u_1(\xi) \geq -ar_0(\xi)/\sqrt{1+c_0^2}$. If u_1 is constant then we have conclusion i), otherwise from Theorem 5.8 we deduce that M_0 splits as described in ii) and that u_1 is an affine function of the variable s, say $u_1(s,\zeta) = c_1 s + c_2$, $c_1, c_2 \in \mathbb{R}$, where (s,ζ) denotes the generic point in the product $\mathbb{R} \times N_0$. This yields the expression

$$u(t,s,\zeta) = c_0 t + c_1\sqrt{1+c_0^2}s + c_2\sqrt{1+c_0^2}$$

completing the proof of ii). □

Bibliography

1. J. L. Alías, P. Mastrolia, M. Rigoli, *Maximum principles and geometric applications.* Springer Monographs in Mathematics. Springer, Cham, 2016. xvii+570 pp. ISBN: 978-3-319-24335-1; 978-3-319-24337-5. MR3445380
2. F. J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem.* Ann. of Math. (2) **84** (1966), 277–292. MR0200816
3. P. Antonini, D. Mugnai, P. Pucci, *Quasilinear elliptic inequalities on complete Riemannian manifolds.* J. Math. Pures Appl. (9) **87** (2007), no. 6, 582–600. MR2335088
4. S. Bernstein, *Sur la généralisation du problème de Dirichlet.* Math. Ann. **62** (1906), no. 2, 253–271. MR1511375
5. S. Bernstein, *Sur un théorème de géomètrie et son application aux équations aux dérivées partielles du type elliptique.* Comm. Soc. Math. de Kharkov 2 (15) (1915-1917), 38–45; German translation: *Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus.* Math. Z. 26 (1) (1927), 551–558.
6. B. Bianchini, L. Mari, M. Rigoli, *On some aspects of oscillation theory and geometry.* Mem. Amer. Math. Soc. **225** (2013), no. 1056, vi+195 pp. ISBN: 978-0-8218-8799-8. MR3112813
7. E. Bombieri, E. De Giorgi, E. Giusti, *Minimal cones and the Bernstein problem.* Invent. Math. **7** (1969), 243–268. MR0250205
8. E. Bombieri, E. De Giorgi, M. Miranda, *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche.* Arch. Rational Mech. Anal. **32** (1969), 255–267. MR0248647
9. E. Bombieri, E. Giusti, *Harnack's inequality for elliptic differential equations on minimal surfaces.* Invent. Math. **15** (1972), 24–46. MR0308945
10. E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry.* Duke Math. J. **25** (1958), 45–56. MR0092069
11. G. Colombo, M. Magliaro, L. Mari, M. Rigoli, *Bernstein and half-space properties for minimal graphs under Ricci lower bounds.* (2019) Preprint available at `arXiv:1911.12054`
12. G. Colombo, L. Mari, M. Rigoli, *A splitting theorem for capillary graphs under Ricci lower bounds.* (2020) Preprint available at `arXiv:2007.15143`
13. E. De Giorgi, *Una estensione del teorema di Bernstein.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **19** (1965), 79–85. MR0178385
14. Q. Ding, *Liouville type theorems for minimal graphs over manifolds.* (2019) Preprint available at `arXiv:1911.10306`
15. Q. Ding, J. Jost, Y. Xin, *Minimal graphic functions on manifolds of nonnegative Ricci curvature.* Comm. Pure Appl. Math. **69** (2016), no. 2, 323–371. MR3434614
16. P. Doktor, *Approximation of domains with Lipschitzian boundary.* Časopis Pěst. Mat. **101** (1976), no. 3, 237–255. MR0461122
17. A. Farina, *A Bernstein-type result for the minimal surface equation.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), no. 4, 1231–1237. MR3467654
18. A. Farina, *A sharp Bernstein-type theorem for entire minimal graphs.* Calc. Var. Partial Differential Equations **57** (2018), no. 5, paper no. 123, 5 pp. MR3832988
19. A. Farina, L. Mari, E. Valdinoci, *Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds.* Comm. Partial Differential Equations **38** (2013), no. 10, 1818–1862. MR3169764
20. A. Farina, B. Sciunzi, E. Valdinoci, *Bernstein and De Giorgi type problems: new results via a geometric approach.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 741–791. MR2483642
21. A. Farina, E. Valdinoci, *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems.* Arch. Rational Mech. Anal. **195** (2010), no. 3, 1025–1058. MR2591980
22. H. Federer, W. H. Fleming, *Normal and integral currents.* Ann. of Math. (2) **72** (1960), 458–520. MR0123260
23. R. Finn, *New estimates for equations of minimal surface type.* Arch. Rational Mech. Anal. **14** (1963), 337–375. MR0157096

24. W. H. Fleming, *On the oriented Plateau problem.* Rend. Circ. Mat. Palermo (2) **11** (1962), 69–90. MR0157263
25. R. E. Greene, H. Wu, *C^∞ approximations of convex, subharmonic, and plurisubharmonic functions.* Ann. Sci. École Norm. Sup. $4^{\text{ème}}$ série **12** (1979), no. 1, 47–84. MR532376
26. M. W. Hirsch, *Differential topology.* Graduate Texts in Mathematics, 33. Springer-Verlag, New York, 1994. x+222 pp. ISBN: 0-387-90148-5. MR1336822
27. E. Hopf, *On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature.* Proc. Amer. Math. Soc. **1** (1950), 80–85. MR0035803
28. D. Impera, S. Pigola, A. G. Setti, *Potential theory for manifolds with boundary and applications to controlled mean curvature graphs.* J. Reine Angew. Math. **733** (2017), 121–159. MR3731326
29. H. Ishii, *On the equivalence of two notions of weak solutions, viscosity solutions and distribution solutions.* Funkcial. Ekvac. **38** (1995), no. 1, 101–120. MR1341739
30. H. Jenkins, J. Serrin, *Variational problems of minimal surface type. I.* Arch. Rational Mech. Anal. **12** (1963), 185–212. MR0145194
31. N. Korevaar, *The normal variations technique for studying the shape of capillary surfaces.* Astérisque No. 118 (1984), 189–195. MR0761748
32. N. Korevaar, *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation*, in F. E. Browder (ed.), Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983), 81–89, Proc. Sympos. Pure Math., **45**, Part 2, Amer. Math. Soc., Providence, RI, 1986. MR0843597
33. N. Korevaar, *A priori interior gradient bounds for solutions to elliptic Weingarten equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), no. 5, 405–421. MR0921546
34. N. Korevaar, *Maximum principle gradient estimates for the capillary problem.* Comm. Partial Differential Equations **13** (1988), no. 1, 1–31. MR0914812
35. P. Li, *Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature.* Ann. of Math. (2) **124** (1986), no. 1, 1–21. MR0847950
36. P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness.* Comm. Partial Differential Equations **8** (1983), no. 11, 1229–1276. MR0709162
37. C. Mantegazza, G. Mascellani, G. Uraltsev, *On the distributional Hessian of the distance function.* Pacific J. Math. **270** (2014), no. 1, 151–166. MR3245852
38. L. Mari, D. Valtorta, *On the equivalence of stochastic completeness and Liouville and Khas'minskii conditions in linear and nonlinear settings.* Trans. Amer. Math. Soc. **365** (2013), no. 9, 4699–4727. MR3066769
39. E. J. Mickle, *A remark on a theorem of Serge Bernstein.* Proc. Amer. Math. Soc. **1** (1950), 86–89. MR0035804 f
40. M. Miranda, *Diseguaglianze di Sobolev sulle ipersuperfici minimali.* Rend. Sem. Mat. Univ. Padova **38** (1967), 69–79. MR0221350
41. M. Miranda, *Una maggiorazione integrale per le curvature delle ipersuperfici minimali.* Rend. Sem. Mat. Univ. Padova **38** (1967), 91–107. MR0222778
42. H. Mori, *The Riemannian double of a manifold and eigenspaces of the Laplacian.* Math. J. Toyama Univ. **14** (1991), 67–80. MR1145141
43. J. Moser, *On Harnack's theorem for elliptic differential equations.* Comm. Pure Appl. Math. **14** (1961), 577–591. MR0159138
44. P. Petersen, *Riemannian geometry.* Third edition. Graduate Texts in Mathematics, 171. Springer, Cham, 2016. xviii+499 pp. ISBN: 978-3-319-26652-7; 978-3-319-26654-1. MR3469435
45. S. Pigola, M. Rigoli, A. G. Setti, *Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique.* Progress in Mathematics, 266. Birkhäuser Verlag, Basel, 2008. xiv+282 pp. ISBN: 978-3-7643-8641-2. MR2401291
46. H. Rosenberg, F. Schulze, J. Spruck, *The half-space property and entire positive minimal graphs in $M \times \mathbb{R}$.* J. Differential Geom. **95** (2013), no. 2, 321–336. MR3128986
47. L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds.* J. Differential Geom. **36** (1992), no. 2, 417–450. MR1180389
48. T. Schmidt, *Strict interior approximation of sets of finite perimeter and functions of bounded variation.* Proc. Amer. Math. Soc. **143** (2015), no. 5, 2069–2084. MR3314116
49. J. Simons, *Minimal varieties in riemannian manifolds.* Ann. of Math. (2) **88** (1968), 62–105. MR0233295
50. L. Simon, *Entire solutions of the minimal surface equation.* J. Diff. Geom. **30** (1989), 643–688. MR1021370
51. J. Serrin, *A priori estimates for solutions of the minimal surface equation.* Arch. Rational Mech. Anal. **14** (1963), 376–383. MR0158160
52. P. Sternberg, K. Zumbrun, *A Poincaré inequality with applications to volume-constrained area-minimizing surfaces.* J. Reine Angew. Math. **503** (1998), 63–85. MR1650327

53. G. M. Troianello, *Elliptic differential equations and obstacle problems*. The University Series in Mathematics. Plenum Press, New York, 1987. xiv+353 pp.
54. N. S. Trudinger, *A new proof of the interior gradient bound for the minimal surface equation in n dimensions*. Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 821–823. MR0296832
55. N. S. Trudinger, *Gradient estimates and mean curvature*. Math. Z. **131** (1973), 165–175. MR0324187
56. S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228. MR0431040