

Noncommutative deformation of four dimensional Einstein gravity

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Abstract

We construct a model for noncommutative gravity in four dimensions, which reduces to the Einstein-Hilbert action in the commutative limit. Our proposal is based on a gauge formulation of gravity with constraints. While the action is metric independent, the constraints insure that it is not topological. We find that the choice of the gauge group and of the constraints are crucial to recover a correct deformation of standard gravity. Using the Seiberg-Witten map the whole theory is described in terms of the vierbeins and of the Lorentz transformations of its commutative counterpart. We solve explicitly the constraints and exhibit the first order noncommutative corrections to the Einstein-Hilbert action.

Up to now a consistent formulation of four dimensional noncommutative gravity that reduces to the standard Einstein-Hilbert theory in the commutative limit has proven quite difficult to approach [1], [2], [3], [4] [5], [6]. Among others there are problems like finding an invariant measure, solving the inconsistencies of a complex metric, singling out the correct degrees of freedom. In two and three dimensions these difficulties can be avoided since a theory of gravity can be formulated as a gauge theory and we know how to deform gauge transformations in a noncommutative geometry [7], [8], [9], [10].

In this paper we pursue the idea of searching for a description of four dimensional noncommutative gravity as a gauge invariant theory with constraints. For the standard Einstein action with a cosmological term [11], [12] this can be easily done using the $SO(1,4)$ de Sitter group as a start, and then reducing the symmetry to the $SO(1,3)$ Lorentz group via the torsion free constraint. The constraints play an important role: they allow to write an action which, although independent of the metric, is not topological [13]. Moreover their solution eliminates the unphysical, dependent degrees of freedom leaving the metric as the only dynamical field.

When we consider a noncommutative deformation of the theory introducing the Moyal \star -product [14], we have to face the fact that the only consistent gauge groups are the unitary groups. Thus we look for the simplest unitary groups which contain $SO(1,4)$ and $SO(1,3)$ with the aim to deform their algebra with the \star -product. The appropriate groups turn out to be $U_\star(2,2)$ and $U_\star(1,1) \times U_\star(1,1)$ respectively [2]. In fact we find that starting from a gauge theory invariant under $U_\star(2,2)$ we can impose constraints that reduce the symmetry to a subclass contained in $U_\star(1,1) \times U_\star(1,1)$. We name this subalgebra $SO_\star(1,3)$ since it represents *the simplest noncommutative deformation* of the Lorentz algebra $SO(1,3)$. We use the constraints to express the dependent gauge fields in terms of the independent ones and construct an action invariant under $SO_\star(1,3)$. It reduces to the standard action in the commutative limit. Then we show that via the Seiberg-Witten map [16], gauge transformations λ in $SO(1,3)$ turn precisely into $SO_\star(1,3)$ transformations $\hat{\lambda}$. We define our noncommutative theory based on this set $\hat{\lambda}$ of gauge transformations. Thus we are allowed to express the θ -dependence of the fields in the action in terms of the fields in the commutative theory using the Seiberg-Witten map. In this fashion the whole theory is described in terms of the vierbeins and the Lorentz transformations. Finally we solve explicitly the constraints to first order in θ and exhibit the first order noncommutative correction to the Einstein-Hilbert action.

We start by studying how the introduction of the \star product leads to a deformation of the Lorentz group $SO(1,3)$ that we call $SO_\star(1,3)$.

In a noncommutative theory, under an infinitesimal gauge transformation $\tilde{\lambda}$ the gauge connection \tilde{A}_μ transforms as follows:

$$\hat{\delta}_{\tilde{\lambda}} \tilde{A}_\mu = \partial_\mu \tilde{\lambda} + [\tilde{\lambda}, \tilde{A}_\mu]_\star = \partial_\mu \tilde{\lambda} + \tilde{\lambda} \star \tilde{A}_\mu - \tilde{A}_\mu \star \tilde{\lambda} \quad (1)$$

Since

$$[\hat{\delta}_{\tilde{\lambda}_1}, \hat{\delta}_{\tilde{\lambda}_2}] \tilde{A}_\mu = \hat{\delta}_{[\tilde{\lambda}_1, \tilde{\lambda}_2]_\star} \tilde{A}_\mu \quad (2)$$

in order to have a representation of the Lie algebra of the group $[\lambda_1, \lambda_2]_\star$ must be in the algebra.

Under the \star -operation the Lorentz algebra does not close. To prove this we consider the basis of a Clifford algebra $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, with γ_a 4×4 matrices and η_{ab} the flat Minkowski metric. Then we define $\gamma_{ab} \equiv \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$ as the six generators of the Lorentz group, $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$, and construct the Moyal commutator of two local Lorentz transformations $\lambda_1 = \lambda_1^{ab}\gamma_{ab}$, $\lambda_2 = \lambda_2^{ab}\gamma_{ab}$

$$\begin{aligned} [\lambda_1, \lambda_2]_\star &= \lambda_1^{ab} \star \lambda_2^{cd} \gamma_{ab} \gamma_{cd} - \lambda_2^{cd} \star \lambda_1^{ab} \gamma_{cd} \gamma_{ab} \\ &= \lambda_1^{ab} \star_s \lambda_2^{cd} [\gamma_{ab}, \gamma_{cd}] + \lambda_1^{ab} \star_a \lambda_2^{cd} \{\gamma_{ab}, \gamma_{cd}\} \end{aligned} \quad (3)$$

where [2]

$$\begin{aligned} f \star_s g &\equiv \frac{1}{2}(f \star g + g \star f) = fg + \frac{1}{2!} \left(\frac{i}{2}\right)^2 \theta^{\alpha\beta} \theta^{\gamma\delta} \partial_\alpha \partial_\gamma f \partial_\beta \partial_\delta g \\ &+ \text{even powers in } \theta \\ f \star_a g &\equiv \frac{1}{2}(f \star g - g \star f) = \left(\frac{i}{2}\right) \theta^{\alpha\beta} \partial_\alpha f \partial_\beta g + \\ &+ \frac{1}{3!} \left(\frac{i}{2}\right)^3 \theta^{\alpha\beta} \theta^{\gamma\delta} \theta^{\rho\sigma} \partial_\alpha \partial_\gamma \partial_\rho f \partial_\beta \partial_\delta \partial_\sigma g + \text{odd powers in } \theta \end{aligned} \quad (4)$$

Since

$$\begin{aligned} [\gamma_{ab}, \gamma_{cd}] &= 2(\eta_{ad}\gamma_{bc} + \eta_{cb}\gamma_{ad} - \eta_{ac}\gamma_{bd} - \eta_{bd}\gamma_{ac}) \\ \{\gamma_{ab}, \gamma_{cd}\} &= -2i(\eta_{ac}\eta_{bd}i\mathbb{1} + \epsilon_{abcd}\gamma_5) \end{aligned} \quad (5)$$

then

$$\begin{aligned} [\lambda_1, \lambda_2]_\star &= (\lambda^{ab} + \lambda^{ab(2)} + \dots + \lambda^{ab(2n)} + \dots) \gamma_{ab} + \\ &+ (\lambda^{1(1)} + \lambda^{1(3)} + \dots + \lambda^{1(2n+1)} + \dots) i\mathbb{1} + (\lambda^{5(1)} + \lambda^{5(3)} + \dots + \lambda^{5(2n+1)} + \dots) \gamma_5 \end{aligned} \quad (6)$$

where $\lambda^{(n)}$ is of order θ^n .

The gauge transformations in (6) are not Lorentz transformations but they have a special and rather simple θ -expansion: the terms which are even powers in θ are Lorentz transformations, while the ones odd in θ have non vanishing components on $i\mathbb{1}$ and γ_5 . The set in (6) forms a subclass of the $U_\star(1, 1) \times U_\star(1, 1)$ algebra which is closed under the \star -product: indeed it is easy to prove that if $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ have a θ -expansion as the one in (6), then their Moyal-commutator $[\tilde{\lambda}_1, \tilde{\lambda}_2]_\star$ has again the same kind of θ -expansion, i.e. even powers are proportional to γ_{ab} , odd powers are proportional to $i\mathbb{1}$ and γ_5 . We call this subalgebra $SO_\star(1, 3)$: it should describe the invariance of our noncommutative theory of gravity.

In order to achieve this goal we start with a $U_\star(2, 2)$ gauge theory and break the symmetry to $SO_\star(1, 3)$ imposing suitable constraints. The procedure is most easily elucidated for the commutative theory [2]: in this case one considers a $U(2, 2)$ gauge theory

and breaks the symmetry down to $SO(1, 3)$, obtaining a gauge formulation for standard gravity. $U(2, 2)$ is the Lie group of complex 4×4 matrices U such that: $U^\dagger \tilde{\eta} U = \tilde{\eta}$ with $\tilde{\eta} = \text{diag}(+ + - -)$. A basis of the Lie algebra is given by 16 linear independent matrices λ satisfying the relation:

$$\lambda^\dagger = -\tilde{\eta} \lambda^\dagger \tilde{\eta}. \quad (7)$$

In the Dirac-Pauli representation with $\gamma_0 = \tilde{\eta}$ we choose the following basis τ^I :

$$(i\mathbb{1}, \gamma_5, i\gamma_a^+, i\gamma_a^-, \gamma_{ab}) \quad (8)$$

where in addition to the generators of $U(1, 1) \times U(1, 1)$ one has $\gamma_a^\pm \equiv \gamma_a(1 \pm \gamma_5)$.

The connection A_μ of the corresponding gauge theory is Lie algebra valued

$$A_\mu = a_\mu i\mathbb{1} + b_\mu \gamma_5 + e_\mu^{a+} i\gamma_a^+ + e_\mu^{a-} i\gamma_a^- + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \quad (9)$$

The field strength is

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \mu \leftrightarrow \nu = F_{\mu\nu}^I \tau_I \quad (10)$$

with components

$$\begin{aligned} F_{\mu\nu}^1 &= \partial_\mu a_\nu - \mu \leftrightarrow \nu \\ F_{\mu\nu}^5 &= \partial_\mu b_\nu + \frac{1}{2}(e_\mu^{a+} e_{\nu a}^- - e_\mu^{a-} e_{\nu a}^+) - \mu \leftrightarrow \nu \\ F_{\mu\nu}^{a+} &= \partial_\mu e_\nu^{a+} - 2b_\mu e_\nu^{a+} + \omega_\mu^{ab} e_{\nu b}^+ - \mu \leftrightarrow \nu \\ F_{\mu\nu}^{a-} &= \partial_\mu e_\nu^{a-} + 2b_\mu e_\nu^{a-} + \omega_\mu^{ab} e_{\nu b}^- - \mu \leftrightarrow \nu \\ F_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} + \omega_{\mu c}^a \omega_\nu^{cb} - 4(e_\mu^{a+} e_\nu^{b-} + e_\mu^{a-} e_\nu^{b+}) - \mu \leftrightarrow \nu \end{aligned} \quad (11)$$

One can show that imposing the constraints

$$\begin{aligned} a_\mu &= b_\mu = 0 \\ e_\mu^{a-} &= \beta e_\mu^{a+} \\ F_{\mu\nu}^{a+} &= 0 \end{aligned} \quad (12)$$

the gauge group $U(2, 2)$ is broken into $SO(1, 3)$ with an additional $U(1)$ global symmetry.

Now we consider the following $SO(1, 3)$ invariant action

$$S = \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr}(\gamma_5 F_{\mu\nu} F_{\rho\sigma}) \quad (13)$$

Using the constraints (12) and the definition

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} + \omega_{\mu c}^a \omega_\nu^{cb} - \mu \leftrightarrow \nu \quad (14)$$

the action (13) becomes

$$S = -\frac{i}{2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} (R_{\mu\nu}^{ab} - 8\beta e_\mu^{a+} e_\nu^{b+})(R_{\rho\sigma}^{cd} - 8\beta e_\rho^{c+} e_\sigma^{d+}) \quad (15)$$

The case $\beta = 0$ gives the topological Gauss-Bonnet term , while $\beta \neq 0$ gives also the classical Einstein action with a cosmological term.

It is worth noticing that the choice of the constant β in (12) determines the value of the cosmological constant of the model. Once the constraints are imposed we are left with $P_a \equiv i\gamma_a^+ + \beta i\gamma_a^-$ and $M_{ab} \equiv \gamma_{ab}$, a basis for the de Sitter or anti de Sitter group depending on the sign of β . For $\beta = 0$ one would obtain the Poincarè group, but in this case the action (13) becomes topological and it cannot be used to describe four dimensional gravity.

Now we turn to the noncommutative case. We consider a $U_\star(2, 2)$ noncommutative gauge theory (i.e. a $U(2, 2)$ -gauge theory in a noncommutative space) and we impose constraints to reduce the symmetry to $SO_\star(1, 3)$. This we want to be the gauge symmetry of our noncommutative gravity, since, as emphasized above, $SO_\star(1, 3)$ is the *natural* noncommutative deformation of the ordinary $SO(1, 3)$ Lorentz algebra.

In the $U_\star(2, 2)$ gauge theory we write the connection \tilde{A}_μ as

$$\tilde{A}_\mu = \tilde{a}_\mu i\mathbb{1} + \tilde{b}_\mu \gamma_5 + \tilde{e}_\mu^{a+} i\gamma_a^+ + \tilde{e}_\mu^{a-} i\gamma_a^- + \frac{1}{4} \tilde{\omega}_\mu^{ab} \gamma_{ab} \quad (16)$$

where all the fields are functions of the space time coordinates and of the noncommutative parameter θ . The corresponding field strength is given by

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \tilde{A}_\mu \star \tilde{A}_\nu - \mu \rightarrow \nu = \tilde{F}_{\mu\nu}^I \tau_I \quad (17)$$

with components

$$\begin{aligned} \tilde{F}_{\mu\nu}^1 &= \partial_\mu \tilde{a}_\nu + i\tilde{a}_\mu \star_a \tilde{a}_\nu - i\tilde{b}_\mu \star_a \tilde{b}_\nu + \frac{i}{2} (\tilde{e}_\mu^{a+} \star_a \tilde{e}_{a\nu}^- + \tilde{e}_\mu^{a-} \star_a \tilde{e}_{a\nu}^+) - \frac{i}{8} \tilde{\omega}_\mu^{ab} \star_a \tilde{\omega}_{\nu ab} - \mu \leftrightarrow \nu \\ \tilde{F}_{\mu\nu}^5 &= \partial_\mu \tilde{b}_\nu + 2i\tilde{a}_\mu \star_a \tilde{b}_\nu + \frac{1}{2} (\tilde{e}_\mu^{a+} \star_s \tilde{e}_{\nu a}^- - \tilde{e}_\mu^{a-} \star_s \tilde{e}_{\nu a}^+) - \frac{i}{8} \epsilon_{abcd} \tilde{\omega}_\mu^{ab} \star_a \tilde{\omega}_\nu^{cd} - \mu \leftrightarrow \nu \\ \tilde{F}_{\mu\nu}^{a+} &= \partial_\mu \tilde{e}_\nu^{a+} - 2\tilde{b}_\mu \star_s \tilde{e}_\nu^{a+} + \tilde{\omega}_\mu^{ab} \star_s \tilde{e}_{\nu b}^+ + 2i\tilde{a}_\mu \star_a \tilde{e}_\nu^{a+} + \frac{i}{2} \epsilon_{bcd}^a \tilde{e}_\mu^{b+} \star_a \tilde{\omega}_\nu^{cd} - \mu \leftrightarrow \nu \\ \tilde{F}_{\mu\nu}^{a-} &= \partial_\mu \tilde{e}_\nu^{a-} + 2\tilde{b}_\mu \star_s \tilde{e}_\nu^{a-} + \tilde{\omega}_\mu^{ab} \star_s \tilde{e}_{\nu b}^- + 2i\tilde{a}_\mu \star_a \tilde{e}_\nu^{a-} - \frac{i}{2} \epsilon_{bcd}^a \tilde{e}_\mu^{b-} \star_a \tilde{\omega}_\nu^{cd} - \mu \leftrightarrow \nu \\ \tilde{F}_{\mu\nu}^{ab} &= \partial_\mu \tilde{\omega}_\nu^{ab} + \tilde{\omega}_{\mu c}^a \star_s \tilde{\omega}_\nu^{cb} - 4(\tilde{e}_\mu^{a+} \star_s \tilde{e}_\nu^{b-} + \tilde{e}_\mu^{a-} \star_s \tilde{e}_\nu^{b+}) + i\epsilon_{cd}^{ab} (\tilde{e}_\mu^{c+} \star_a \tilde{e}_\nu^{d-} - \tilde{e}_\mu^{c-} \star_a \tilde{e}_\nu^{d+}) + \\ &+ 2i\tilde{a}_\mu \star_a \tilde{\omega}_\nu^{ab} + i\epsilon_{cd}^{ab} \tilde{b}_\mu \star_a \tilde{\omega}_\nu^{cd} - \mu \leftrightarrow \nu \end{aligned} \quad (18)$$

Now we want to impose constraints so that the invariance is broken to $SO_\star(1, 3)$. Moreover, in order to recover standard gravity in the commutative limit, the fields must satisfy

$$\begin{aligned} \lim_{\theta \rightarrow 0} \tilde{a}_\mu &= 0 & \lim_{\theta \rightarrow 0} \tilde{b}_\mu &= 0 \\ \lim_{\theta \rightarrow 0} \tilde{e}_\mu^{a-} &= \beta \lim_{\theta \rightarrow 0} \tilde{e}_\mu^{a+} \end{aligned} \quad (19)$$

To this end it is convenient to write the gauge fields in a θ -expanded form

$$\begin{aligned}
\tilde{a}_\mu &= a_\mu^{(1)} + a_\mu^{(2)} + \dots \\
\tilde{b}_\mu &= b_\mu^{(1)} + b_\mu^{(2)} + \dots \\
\tilde{e}_\mu^{a+} &= e_\mu^{a+} + e_\mu^{a+(1)} + e_\mu^{a+(2)} + \dots \\
\tilde{e}_\mu^{a-} &= \beta e_\mu^{a+} + e_\mu^{a-(1)} + e_\mu^{a-(2)} + \dots \\
\tilde{\omega}_\mu^{ab} &= \omega_\mu^{ab} + \omega_\mu^{ab(1)} + \omega_\mu^{ab(2)} + \dots
\end{aligned} \tag{20}$$

where we have taken into account the conditions in (19).

In order to reduce the gauge symmetry we impose the following constraints

$$\tilde{F}^{a+} = F^{a+} + F^{a+(1)} + \dots + F^{a+(n)} + \dots = 0 \tag{21}$$

and, at each order in θ , reflecting the different role played in the Moyal product by the even and the odd powers,

$$e_\mu^{a-(n)} = (-)^n \beta e_\mu^{a+(n)} \tag{22}$$

The constraints in (21) and (22) are just sufficient to break $U_\star(2, 2)$ into $SO_\star(1, 3)$. Indeed $\tilde{F}^{a+} = 0$ requires $\hat{\delta}_\lambda \tilde{F}_{\mu\nu}^{a+} = 0$ i.e.

$$\begin{aligned}
\hat{\delta} \tilde{F}_{\mu\nu}^{a+} &= -2\tilde{F}_{\mu\nu}^5 \star_s \tilde{\lambda}^{a+} + 4\tilde{F}_{\mu\nu}^{ab} \star_s \tilde{\lambda}_b^+ + 2i\tilde{F}_{\mu\nu}^1 \star_a \tilde{\lambda}^{a+} + 2i\epsilon_{bcd}^a \tilde{F}_{\mu\nu}^{bc} \star_a \tilde{\lambda}^{d+} \\
&= 0
\end{aligned} \tag{23}$$

which leads to the condition $\tilde{\lambda}^{a+} = 0$. With this restriction now we consider the constraints in (22) and impose them on the corresponding variations $\delta_\lambda \tilde{e}_\mu^{a+}$ and $\delta_\lambda \tilde{e}_\mu^{a-}$. Since under a gauge transformation the connection transforms as in (1) we have

$$\begin{aligned}
\delta \tilde{e}_\mu^{a+} &= -2\tilde{\lambda}^5 \star_s \tilde{e}_\mu^{a+} - 4\tilde{\lambda}^{ab} \star_s \tilde{e}_{\mu b}^+ + 2i\tilde{\lambda}^1 \star_a \tilde{e}_\mu^{a+} + 2i\epsilon_{bcd}^a \tilde{\lambda}^{bc} \star_a \tilde{e}_\mu^{d+} \\
\delta \tilde{e}_\mu^{a-} &= \partial_\mu \tilde{\lambda}^{a-} + 2\tilde{\lambda}^5 \star_s \tilde{e}_\mu^{a-} - 4\tilde{\lambda}^{ab} \star_s \tilde{e}_{\mu b}^- + 2i\tilde{\lambda}^1 \star_a \tilde{e}_\mu^{a-} - 2i\epsilon_{bcd}^a \tilde{\lambda}^{bc} \star_a \tilde{e}_\mu^{d-} \\
&\quad + 2\tilde{b}_\mu \star_s \tilde{\lambda}^{a-} + \tilde{\omega}_\mu^{ab} \star_s \tilde{\lambda}_b^- + 2i\tilde{a}_\mu \star_a \tilde{\lambda}^{a-} - 2i\epsilon_{bcd}^a \tilde{\omega}_\mu^{bc} \star_a \tilde{\lambda}_\mu^{d-}
\end{aligned} \tag{24}$$

In order to satisfy (22) first we have to impose $\tilde{\lambda}^{a-} = 0$ so that the variations in (24) become

$$\delta \tilde{e}_\mu^{a\pm} = \mp 2\tilde{\lambda}^5 \star_s \tilde{e}_\mu^{a\pm} - 4\tilde{\lambda}^{ab} \star_s \tilde{e}_{\mu b}^\pm + 2i\tilde{\lambda}^1 \star_a \tilde{e}_\mu^{a\pm} \pm 2i\epsilon_{bcd}^a \tilde{\lambda}^{bc} \star_a \tilde{e}_\mu^{d\pm} \tag{25}$$

Writing the above relations as θ -expansions

$$\delta \tilde{e}_\mu^{a\pm} = \delta e_\mu^{a\pm} + \delta e_\mu^{a\pm(1)} + \delta e_\mu^{a\pm(2)} + \dots \tag{26}$$

we obtain

$$\begin{aligned}
\delta e_\mu^{a\pm(n)} &= \mp \sum_{p+2k+q=n} 2\lambda^{5(p)} \star_{2k} e_\mu^{a\pm(q)} - 4 \sum_{p+2k+q=n} \lambda^{ab(p)} \star_{2k} e_{\mu b}^{\pm(q)} + \\
&\quad + 2i \sum_{p+2k+1+q=n} \lambda^{1(p)} \star_{2k+1} e_\mu^{a\pm(q)} \pm 2i \sum_{p+2k+1+q=n} \epsilon_{bcd}^a \lambda^{bc(p)} \star_{2k+1} e_\mu^{d\pm(q)}
\end{aligned} \tag{27}$$

where $p, k, q = 0, 1, 2, \dots$ and we used the notation $f \star g = \sum_{k=0}^{\infty} f \star_k g$. At this point it is simple to show that the constraints $e_{\mu}^{a-(n)} = (-)^n \beta e_{\mu}^{a+(n)}$ are satisfied if we impose the additional conditions

$$\begin{aligned}\lambda^{1(2n)} &= \lambda^{5(2n)} = 0 \\ \lambda^{ab(2n+1)} &= 0\end{aligned}\tag{28}$$

Therefore the restricted gauge parameter $\tilde{\lambda}$ belongs to $SO_{\star}(1, 3)$ and this completes our proof.

The action [15] which is invariant under an $SO_{\star}(1, 3)$ transformation $\tilde{\lambda} = \tilde{\lambda}^1 i\mathbb{1} + \tilde{\lambda}^5 \gamma_5 + \tilde{\lambda}^{ab} \gamma_{ab}$ is

$$S_{NC} = \int d^4x \epsilon^{\mu\nu\rho\sigma} Tr(\gamma_5 \tilde{F}_{\mu\nu} \star \tilde{F}_{\rho\sigma})\tag{29}$$

Indeed one immediately obtains

$$\hat{\delta}_{\tilde{\lambda}} S_{NC} = \int d^4x \epsilon^{\mu\nu\rho\sigma} Tr([\gamma_5, \tilde{\lambda}] \star \tilde{F}_{\mu\nu} \star \tilde{F}_{\rho\sigma}) = 0\tag{30}$$

More explicitly (29) can be rewritten as

$$S_{NC} = \frac{i}{2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left(16 \tilde{F}_{\mu\nu}^1 \tilde{F}_{\rho\sigma}^5 - \epsilon_{abcd} \tilde{F}_{\mu\nu}^{ab} \tilde{F}_{\rho\sigma}^{cd} \right)\tag{31}$$

with field strengths as given in (18).

Now we want to use the constraints (21) and (22) in (31) and express the dependent fields in terms of the independent, dynamical ones. First we use

$$\tilde{F}_{\mu\nu}^{a+} = \partial_{\mu} \tilde{e}_{\nu}^{a+} - 2\tilde{b}_{\mu} \star_s \tilde{e}_{\nu}^{a+} + \tilde{\omega}_{\mu}^{ab} \star_s \tilde{e}_{\nu b}^{a+} + 2i\tilde{a}_{\mu} \star_a \tilde{e}_{\nu}^{a+} + \frac{i}{2} \epsilon_{bcd}^a \tilde{e}_{\mu}^{b+} \star_a \tilde{\omega}_{\nu}^{cd} - \mu \leftrightarrow \nu = 0\tag{32}$$

and determine $\tilde{\omega}_{\mu}^{ab}$ order by order in θ , thus obtaining $\omega_{\mu}^{ab(n)} = \omega_{\mu}^{ab(n)}(e_{\mu}^{a+}, \dots, e_{\mu}^{a+(n)}, a_{\mu}^{(1)}, \dots, a_{\mu}^{(n-1)}, b_{\mu}^{(1)}, \dots, b_{\mu}^{(n)})$.

At this level we have obtained a theory in terms of the fields \tilde{e}_{μ}^{a+} , \tilde{a}_{μ} and \tilde{b}_{μ} invariant under transformations $\tilde{\lambda} \in SO_{\star}(1, 3)$. It represents a noncommutative deformation of Einstein gravity which contains the vierbeins plus an infinite number of additional fields which enter at all orders in the θ -expansion of \tilde{e}_{μ}^{a+} , \tilde{a}_{μ} and \tilde{b}_{μ} . Now we attempt to reduce the number of independent fields employing the Seiberg-Witten map [16].

In general the map allows to express the gauge connection of a noncommutative theory \hat{A}_{μ} as a θ -expansion of standard gauge theory variables A_{μ} . In the present case we want to identify the fields \tilde{e}_{μ}^{a+} , \tilde{a}_{μ} and \tilde{b}_{μ} with the corresponding ones obtained via the Seiberg-Witten maps $\hat{e}_{\mu}^{a+}(e_{\mu}^{a+})$, $\hat{a}_{\mu}(e_{\mu}^{a+})$ and $\hat{b}_{\mu}(e_{\mu}^{a+})$.

As we will show the procedure is consistent with the choice of the constraints (21) and (22). The only dynamical fields of the theory turn out to be the vierbeins e_{μ}^{a+} in terms of which we can define the space-time metric

$$g_{\mu\nu} = e_{\mu}^{a+} e_{\nu a}^{+}\tag{33}$$

In order to implement consistently this procedure it is crucial to prove that the gauge group $SO_*(1,3)$ of the noncommutative theory is related to the $SO(1,3)$ commutative theory precisely via the Seiberg-Witten map. Then we can determine the functions $\tilde{e}_\mu^{a+}(e_\mu^{a+})$, $\tilde{a}_\mu(e_\mu^{a+})$ and $\tilde{b}_\mu(e_\mu^{a+})$ solving the equations [16]

$$\begin{aligned}\hat{\delta}_\lambda \tilde{e}_\mu^{a+}(e_\mu^{a+}) &= \tilde{e}_\mu^{a+}(e_\mu^{a+} + \delta_\lambda e_\mu^{a+}) - \tilde{e}_\mu^{a+}(e_\mu^{a+}) \\ \hat{\delta}_\lambda \tilde{a}_\mu(e_\mu^{a+}) &= \tilde{a}_\mu(e_\mu^{a+} + \delta_\lambda e_\mu^{a+}) - \tilde{a}_\mu(e_\mu^{a+}) \\ \hat{\delta}_\lambda \tilde{b}_\mu(e_\mu^{a+}) &= \tilde{b}_\mu(e_\mu^{a+} + \delta_\lambda e_\mu^{a+}) - \tilde{b}_\mu(e_\mu^{a+})\end{aligned}\tag{34}$$

where λ belongs to the Lorentz algebra, i.e. $\lambda = \lambda^{ab}\gamma_{ab}$ (the $SO(1,3)$ gauge group of the commutative limit), while $\hat{\lambda}$ belongs to the $SO_*(1,3)$ gauge group of the corresponding (via the Seiberg-Witten map) noncommutative theory.

Thus let us show that if we start from an $SO(1,3)$ gauge theory and use the Seiberg-Witten map to construct the corresponding noncommutative one, the gauge group is precisely mapped into $SO_*(1,3)$. We describe the commutative theory through its connection $A_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ and gauge parameters $\lambda = \lambda^{ab}\gamma_{ab}$ and the corresponding noncommutative one through $\hat{A}(A)$ and $\hat{\lambda}(\lambda, A)$ defined as [16]

$$\delta_\lambda \hat{A}_\mu = \hat{A}_\mu(A + \delta_\lambda A) - \hat{A}_\mu(A)\tag{35}$$

where

$$\begin{aligned}\delta_\lambda A_\mu &= \partial_\mu \lambda + \lambda A_\mu - A_\mu \lambda \\ \delta_\lambda \hat{A}_\mu &= \partial_\mu \hat{\lambda} + \hat{\lambda} \star \hat{A}_\mu - \hat{A}_\mu \star \hat{\lambda}\end{aligned}\tag{36}$$

The solution of (35) is equivalent to

$$\begin{aligned}\delta \hat{A}_\mu(\theta) &= -\frac{i}{4}\delta\theta^{\alpha\beta}\{\hat{A}_\alpha, (\partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu})\}_\star \\ \delta \hat{\lambda}(\theta) &= \frac{i}{4}\delta\theta^{\alpha\beta}\{\partial_\alpha \lambda, A_\beta\}_\star.\end{aligned}\tag{37}$$

Our goal is to prove that the gauge parameter $\hat{\lambda}$ belongs to $SO_*(1,3)$. We look for a solution of (37) in a θ -expanded form:

$$\begin{aligned}\hat{A}_\mu &= A_\mu + A_\mu^{(1)} + A_\mu^{(2)} + \dots \\ \hat{\lambda} &= \lambda + \lambda^{(1)} + \lambda^{(2)} + \dots\end{aligned}\tag{38}$$

and we want to prove that

$$\begin{aligned}A_\mu^{(n)}(A) &= A_\mu^{(n)I}(A)t_{I(n)} \\ \lambda^{(n)}(A) &= \lambda^{(n)I}(A)t_{I(n)}\end{aligned}\tag{39}$$

where $t_{I(n)} = \gamma_{ab}$ if n is even, while $t_{I(n)} \in (i\mathbb{1}, \gamma_5)$ if n is odd. We prove (39) by induction first for $A_\mu^{(n)}$. We begin with the $n = 1$ case. As emphasized above in the

commutative theory we have $A_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$, $\lambda = \lambda^{ab}\gamma_{ab}$ and $F_{\mu\nu} = \frac{1}{4}F_{\mu\nu}^{ab}\gamma_{ab}$. Thus inserting $\delta A_\mu^{(1)} = A_{\mu\alpha\beta}\delta\theta^{\alpha\beta}$ in the first equation (37) we obtain the result for $n = 1$

$$A_{\mu\alpha\beta} = -\frac{1}{2}A_\alpha^{ab}(\partial_\beta A_\mu^{cd} + F_{\beta\mu}^{cd})(\eta_{ac}\eta_{bd}i\mathbb{1} + \epsilon_{abcd}\gamma_5) \quad (40)$$

Now assuming the result is true for general n , we prove it for $n + 1$. We define

$$A_\mu^{(n)} = \frac{1}{n}A_{\mu\alpha_1\beta_1,\dots,\alpha_n\beta_n}\theta^{\alpha_1\beta_1}\dots\theta^{\alpha_n\beta_n} \quad (41)$$

so that

$$\delta A_\mu^{(n)}(\theta) = A_{\mu\alpha_1\beta_1,\dots,\alpha_n\beta_n}\theta^{\alpha_1\beta_1}\dots\delta\theta^{\alpha_n\beta_n} \quad (42)$$

Inserting the above expressions in the first equation (37) we find

$$\begin{aligned} \delta A_\mu^{(n+1)}(\theta) &= -\frac{i}{4}\delta\theta^{\alpha\beta}\sum_{p+2k+q=n}A_\alpha^{(p)I}\star_{2k}(\partial_\beta A_\mu^{(q)J} + F_{\beta\mu}^{(q)J})\{t_{I(p)}, t_{J(q)}\} + \\ &- \frac{i}{4}\delta\theta^{\alpha\beta}\sum_{p+2k+1+q=n}A_\alpha^{(p)I}\star_{2k+1}(\partial_\beta A_\mu^{(q)J} + F_{\beta\mu}^{(q)J})[t_{I(p)}, t_{J(q)}] \end{aligned} \quad (43)$$

Using the standard commutation relations among the generators we end up with $A_\mu^{(2n)} = A_\mu^{(2n)ab}\gamma_{ab}$ and $A_\mu^{(2n+1)} = A_\mu^{(2n+1)1}i\mathbb{1} + A_\mu^{(2n+1)5}\gamma_5$. This result on the θ -structure of $\hat{A}_\mu(\theta)$, and the second of the Seiberg and Witten equations (37) lead to the conclusion that the parameter $\hat{\lambda}(\theta)$ belongs to $SO_\star(1, 3)$.

As anticipated above now we can safely determine the independent fields of our non-commutative theory through the Seiberg-Witten map. We apply this procedure and explicitly compute the first order noncommutative correction to the standard gravity action. The action is expanded in powers of θ

$$\begin{aligned} S_{NC} &= \frac{i}{2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\left(16\tilde{F}_{\mu\nu}^1\tilde{F}_{\rho\sigma}^5 - \epsilon_{abcd}\tilde{F}_{\mu\nu}^{ab}\tilde{F}_{\rho\sigma}^{cd}\right) \\ &= S + S^{(1)} + S^{(2)} + \dots \end{aligned} \quad (44)$$

and the first noncommutative correction $S^{(1)}$ is evaluated in terms of the dynamical fields e_μ^{a+} as follows: from (19) and (18) we find that $F_{\mu\nu}^1 = F_{\mu\nu}^5 = 0$. Thus $S^{(1)}$ is simply given by

$$S^{(1)} = -i\int d^4x \epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}F_{\mu\nu}^{ab(1)}F_{\rho\sigma}^{cd} \quad (45)$$

Inserting the constraints

$$\begin{aligned} a_\mu &= b_\mu = 0 \\ e_\mu^{a-} &= \beta e_\mu^{a+} & e_\mu^{a-(1)} &= -\beta e_\mu^{a+(1)} \end{aligned} \quad (46)$$

in the expression (18) for $\tilde{F}_{\mu\nu}^{ab}$, we obtain

$$\begin{aligned} F_{\mu\nu}^{ab} &= \partial_\mu\omega_\nu^{ab} + \omega_{\mu c}^a\omega_\nu^{cb} - 8\beta e_\mu^{a+}e_\nu^{b+} - \mu \leftrightarrow \nu \\ F_{\mu\nu}^{ab(1)} &= \partial_\mu\omega_\nu^{ab(1)} + \omega_{\mu c}^{a(1)}\omega_\nu^{cb} + \omega_{\mu c}^a\omega_\nu^{cb(1)} - \mu \leftrightarrow \nu \end{aligned} \quad (47)$$

Now solving the constraints $F_{\mu\nu}^{a+} = 0$ in ω_μ^{ab} and $F_{\mu\nu}^{a+(1)} = 0$ in $\omega_\mu^{ab(1)}$ we determine the spin connections in terms of the vierbein e_μ^{a+} .

At 0th-order in θ we have

$$F_{\mu\nu}^{a+} = \partial_\mu e_\nu^{a+} - \partial_\nu e_\mu^{a+} + \omega_\mu^{ab} e_{\nu b}^+ - \omega_\nu^{ab} e_{\mu b}^+ = 0 \quad (48)$$

It is solved by

$$\omega_\mu^{ab} = -\frac{1}{2}\epsilon^{\rho a+}\epsilon^{\nu b+}(\partial_\mu e_\nu^{k+}e_{\rho k}^+ - \partial_\nu e_\rho^{k+}e_{\mu k}^+ + \partial_\rho e_\mu^{k+}e_{\nu k}^+) \quad (49)$$

where $\epsilon_a^{\mu+}$ is the inverse vierbein

$$e_\nu^{a+}\epsilon_b^{\nu+} = \delta_b^a \quad e_\mu^{a+}\epsilon_a^{\nu+} = \delta_\mu^\nu \quad (50)$$

In the same way at 1st-order we have

$$F_{\mu\nu}^{a+(1)} = \partial_\mu e_\nu^{a+(1)} - 2b_\mu^{(1)}e_\nu^{a+} + \omega_\mu^{ab(1)}e_{\nu b}^+ + \omega_\mu^{ab}e_{\nu b}^{+(1)} - \frac{1}{4}\theta^{\alpha\beta}\epsilon^a{}_{bcd}\partial_\alpha e_\mu^{b+}\partial_\beta\omega_\nu^{cd} - \mu \leftrightarrow \nu = 0 \quad (51)$$

With the definition

$$A_{\mu\nu}^{a(1)} \equiv \partial_\mu e_\nu^{a+(1)} - 2b_\mu^{(1)}e_\nu^{a+} + \omega_\mu^{ab}e_{\nu b}^{+(1)} - \frac{1}{4}\theta^{\alpha\beta}\epsilon^a{}_{bcd}\partial_\alpha e_\mu^{b+}\partial_\beta\omega_\nu^{cd} \quad (52)$$

we rewrite (51) as

$$\omega_\mu^{ab(1)}e_{\nu b}^+ - \omega_\nu^{ab(1)}e_{\mu b}^+ = -A_{\mu\nu}^{a(1)} + A_{\nu\mu}^{a(1)} \quad (53)$$

Therefore we obtain

$$\omega_\mu^{ab(1)} = -\frac{1}{2}\epsilon^{\rho a+}\epsilon^{\nu b+}(A_{\mu\nu}^{k(1)}e_{\rho k}^+ - A_{\nu\rho}^{k(1)}e_{\mu k}^+ + A_{\rho\mu}^{k(1)}e_{\nu k}^+) \quad (54)$$

At this stage we have $\omega_\mu^{ab}(e_\mu^{a+})$ from (49) and $\omega_\mu^{ab(1)}(e_\mu^{a+}, e_\mu^{a+(1)}, b_\mu^{(1)})$ from (54).

We still have to evaluate $b_\mu^{(1)}$ and $e_\mu^{a+(1)}$ in terms of e_μ^{a+} . We do this via the Seiberg-Witten map: in addition to the relations in (36) and (37) we have correspondingly

$$\begin{aligned} \delta_\lambda e_\mu^+ &= \lambda e_\mu^+ - e_\mu^+ \lambda \\ \delta_{\hat{\lambda}} \hat{e}_\mu^+ &= \hat{\lambda} \star \hat{e}_\mu^+ - \hat{e}_\mu^+ \star \hat{\lambda} \end{aligned} \quad (55)$$

and [17]

$$\delta \hat{e}_\mu^+(\theta) = -\frac{1}{2}\delta\theta^{\alpha\beta}(\{\hat{A}_\alpha, \partial_\beta \hat{e}_\mu^+\}_\star + \frac{1}{2}\{[\hat{e}_\mu^+, \hat{A}_\alpha]_\star, \hat{A}_\beta\}_\star) \quad (56)$$

From (37) we have to 1st-order

$$A_\mu^{(1)} = -\frac{i}{4}\theta^{\alpha\beta}\{A_\alpha, (\partial_\beta A_\mu + F_{\beta\mu})\} \quad (57)$$

Substituting in the above equation $A_\mu = \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$, $\lambda = \lambda^{ab}\gamma_{ab}$, $F_{\mu\nu} = \frac{1}{4}F_{\mu\nu}^{ab}\gamma_{ab}$, with ω_μ^{ab} as given in (49) we obtain

$$\begin{aligned}
A_\mu^{(1)} &= -\frac{i}{16}\theta^{\alpha\beta}\omega_\alpha^{ab}(\partial_\beta\omega_\mu^{cd} + F_{\beta\mu}^{cd})\{\gamma_{ab}, \gamma_{cd}\} \\
&= -\frac{i}{16}\theta^{\alpha\beta}\omega_\alpha^{ab}(\partial_\beta\omega_\mu^{cd} + F_{\beta\mu}^{cd})(-2i\eta_{ac}\eta_{bd}i\mathbb{1} - 2i\epsilon_{abcd}\gamma_5) \\
&= a_\mu^{(1)}i\mathbb{1} + b_\mu^{(1)}\gamma_5
\end{aligned} \tag{58}$$

In this way

$$\begin{aligned}
a_\mu^{(1)} &= -\frac{1}{8}\theta^{\alpha\beta}\omega_\alpha^{ab}(\partial_\beta\omega_{\mu ab} + F_{\beta\mu ab}) \\
b_\mu^{(1)} &= -\frac{1}{8}\theta^{\alpha\beta}\epsilon_{abcd}\omega_\alpha^{ab}(\partial_\beta\omega_\mu^{cd} + F_{\beta\mu}^{cd})
\end{aligned} \tag{59}$$

are determined as functions of e_μ^{a+} .

In the same way from (56) we obtain

$$\begin{aligned}
e_\mu^{+(1)}(e, A) &= -\frac{i}{2}\theta^{\alpha\beta}\left(\{A_\alpha, \partial_\beta e_\mu^{a+}\} + \frac{1}{2}\{[e_\mu^{a+}, A_\alpha], A_\beta\}\right) \\
&= -\frac{i}{2}\theta^{\alpha\beta}\left(\frac{1}{4}\omega_\alpha^{bc}\partial_\beta e_\mu^{a+}\{i\gamma_a^+, \gamma_{bc}\} + \frac{1}{2}\{e_\mu^{a+}\frac{1}{4}\omega_\alpha^{bc}[i\gamma_a^+, \gamma_{bc}], \frac{1}{4}\omega_\beta^{ef}\gamma_{ef}\}\right)
\end{aligned} \tag{60}$$

Using

$$\begin{aligned}
[i\gamma_a^+, \gamma_{bc}] &= 2\eta_{ab}i\gamma_c^+ - 2\eta_{ac}i\gamma_b^+ \\
\{i\gamma_a^+, \gamma_{bc}\} &= 2i\epsilon_{abcd}(i\gamma_d^+)
\end{aligned} \tag{61}$$

finally we obtain:

$$e_\mu^{a+(1)} = \frac{1}{4}\theta^{\alpha\beta}\epsilon_{bcd}^a(\partial_\beta e_\mu^{b+}\omega_\alpha^{cd} - \frac{1}{4}\omega_{\alpha k}^b e_\mu^{k+}\omega_\beta^{cd}). \tag{62}$$

Using the expressions given in (62) and in (59) we can reconstruct the first order correction of the spin-connection (54) which finally allows to obtain $F_{\mu\nu}^{ab(1)}$ in (47). In this way one can reexpress the first order correction in (45) explicitly in terms of the vierbeins e_μ^{a+} . The complete result requires a straightforward but quite lengthy algebra. It would be interesting to proceed further and write the action explicitly in terms of the metric. Then one could evaluate the corrected propagator and investigate how these θ -dependent terms affect the renormalization properties of the theory.

Acknowledgements

We thank S. Cacciatori, L. Martucci and M. Picariello for very helpful comments and suggestions.

This work has been partially supported by INFN, MURST, and the European Commission RTN program HPRN-CT-2000-00113 in which the authors are associated to the University of Torino.

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