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Periodic and quasi-periodic orbits in nearly integrable Hamiltonian systems

MAT/07

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Introduction

The study of periodic and quasi-periodic orbits in nearly integrable Hamiltonian systems is a long standing and challenging problem, that dates back to Poincaré. Quoting Poincaré, they represent *the only opening through which we can try to enter a place which, up to now, was deemed inaccessible.* The aim of this thesis is to find effective and constructive algorithms for constructing both periodic and quasi-periodic solutions via a modification of the normal form methods related to Kolmogorov's theorem. The thesis is divided in two parts. The first part concerns the classical problem of the continuation of periodic orbits surviving to the breaking of invariant maximal or lower dimensional completely resonant tori in nearly integrable Hamiltonian systems: we here propose a new scheme which allows to deal with the problem of degeneracy at any order of perturbation. The second part regards the development of a variation of the Kolmogorov's normalization algorithm, by avoiding the so-called translation step at the price of fixing only the *final* frequency, while the initial one can only be determined *a posteriori*.

Consider the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi) \ ,$$

where $I \in \mathcal{U} \subset \mathbb{R}^n$, $\varphi \in \mathbb{T}^n$ are action-angle variables and ε is a small perturbation parameter. The unperturbed system, H_0, is clearly integrable and the bounded orbits, lying on invariant tori, are generically quasi-periodic. Besides, if the unperturbed frequencies satisfy resonance relations, one has periodic orbits on a dense set of resonant tori. The Kolmogorov's theorem ensures the persistence of a set of large measure of quasi-periodic orbits, lying on (strongly) non-resonant tori, for the perturbed system, if ε is small enough and a suitable non-degeneracy condition for H_0 is satisfied. Instead, considering a completely resonant torus foliated by periodic orbits, when a small perturbation is added such a torus is generically destroyed and only a finite number of the periodic orbits carried by the torus are expected to survive.

In the first part of the thesis, after an initial preparation of the Hamiltonian which exhibits one fast rotating angle and the remaining slow angles, we develop a constructive normal form scheme that allows to identify and approximate the periodic orbits which continue to exist after the breaking of the resonant torus of maximal or low dimension (the results have been collected in [74], [85]). In particular, it enables to treat the continuation of those periodic orbits which are at leading order degenerate, hence not covered by classical averaging methods (see [77], [78]). Degeneracy (the so-called Poincaré degeneracy) may arise for instance when the approximate periodic orbits, which correspond to critical points of time averaged perturbation

$$\langle H_1 \rangle_T = \frac{1}{T} \int_0^T H_1 \, dt$$

evaluated at the unperturbed periodic orbits, are not isolated and appear as families depending on one or more parameters.

The method, inspired to classical Kolmogorov's schemes, consists of a finite number of normalization steps, each of which includes also averaging of leading terms and translation of the actions of the torus. The novelty of this construction lies in the "parametric" dependence of the translation on the slow angles that are candidates for the continuation and are determined a posteriori, at the end of the procedure. Given a finite number of normal form steps, possible candidates for the continuation are first identified as critical points of a functional (which is a

ε-perturbation of the average $\langle H_1 \rangle_T$) on the sub-torus of slow angles: these are relative equilibria of the truncated Hamiltonian. Then, the existence of a unique continuation of the so obtained approximate periodic orbits is ensured by an iterative fixed point method, provided appropriate spectral properties of the period map are fulfilled. The method is completely constructive, so suitable for implementation of possible applications with an algebraic manipulator. Besides, it allows to deal with any degree of degeneracy, including the completely degenerate case considered in [61] and extended in [73], when $\langle H_1 \rangle_T \equiv \text{const}$.

Moreover, a further development in the normal form algorithm enables to study the approximate linear stability of the periodic orbit, by obtaining a high-order approximation of its Floquet exponents. If additional requirements are asked on this spectrum, we can obtain information about the effective linear stability of the continued periodic orbits; here arguments from Krein's signature and resolvent theory are necessary. In Chapter 1 we deal with the case of full dimensional tori, while Chapter 2 concerns the extension to lower dimensional elliptic tori and the study of linear stability.

The original motivation for this kind of investigation is the mathematical study of spatially localized time-periodic solutions in Hamiltonian lattices, such as chains of weakly coupled anharmonic oscillators (like Klein-Gordon models). This class of periodic orbits, sometimes called multibreathers (as generalization of their forerunners breathers), can be looked at as periodic orbits which survive to the breaking of a lower dimensional completely resonant torus. Their existence and stability have been studied since the late 90's with an averaging procedure named effective Hamiltonian method (introduced already in [58], [5] and developed for example in [1], [55]) which fails in those cases where approximate solutions are degenerate.

In this context, our scheme enables to face degenerate scenarios which naturally emerge when studying discrete solitons in one-dimensional discrete non-linear Schrödinger lattices (standard dNLS, coupled dNLS, Zig-Zag dNLS,...): in these models, one-parameter families of solutions of the averaged Hamiltonian appear when non consecutive excited sites are considered or when in the model long range interactions are added (like next-to-nearest neighbourhood). Otherwise, one-parameter families of approximate solutions appear when studying vortexes in 2D square lattices. In these problems, the only approach till now explored was based on an ansatz of the solution and then implemented with bifurcation methods suitably combined with a perturbation scheme (see, e.g., [70], [75]).

Up to our knowledge, the existing results for chains of weakly coupled oscillators are valid for specific configurations (e.g. restricting to consecutive oscillators) and degenerate solutions can be hardly explored. On the contrary, our approach allows us to face every kind of degeneracy and to tackle the problem of studying discrete solitons in models with several resonant modules, not only the standard $(1 : \ldots : 1)$ resonance (the results will be collected in [22]). Moreover, the already available methods for non degenerate solutions can be recovered by a single step of our normal form scheme.

Applications of our algorithm to several degenerate models are collected in Chapter 3.

The second part of the thesis (Chapter 4) focuses on the Kolmogorov's normalization algorithm with a variation on the handling of the frequencies. The motivation behind the development of this approach has strong connections with the problem of persistence of lower dimensional elliptic invariant tori under sufficiently small perturbations.

Indeed, in [38] the authors gave a constructive proof of the existence of lower dimensional elliptic tori for planetary systems, adapting the classical Kolmogorov's normalization algorithm and a result of Pöschel (see [79]) that allows to estimate the measure of a suitable set of non-resonant frequencies. The key point is that both the *internal* frequencies of the torus and the *transversal* ones vary at each normalization step, and cannot be kept fixed as in Kolmogorov's algorithm. This makes the accumulation of small divisors much more tricky to control and, more important, the result is only valid *in measure* and therefore one cannot know *a priori* if a specific torus exists or not.

In order to overcome this problem, as a first step, we decide to adapt the classical Kolmogorov's normalization algorithm so as to avoid the translation that keeps the frequencies fixed and to introduce a *detuning* between the fixed *final* frequencies and the corresponding initial ones, to be determined *a posteriori*.

This approach, in principle, also allows to start from a resonant torus carrying frequencies ω_0 that *by construction* falls into a strongly non-resonant one. The results have been collected in [84].

The next goal will be the extension of the scheme developed to the elliptic lower dimensional case.

Part I

Continuation of degenerate periodic orbits

Chapter 1

Continuation of degenerate periodic orbits: full dimensional tori

We consider a canonical system of differential equations with Hamiltonian

$$H(I,\varphi,\varepsilon) = H_0(I) + \varepsilon H_1(I,\varphi) + \varepsilon^2 H_2(I,\varphi) + \dots \ , \tag{1.1}$$

where $I \in \mathcal{U} \subset \mathbb{R}^{n_1}$, $\varphi \in \mathbb{T}^{n_1}$ are action-angle variables and ε is a small perturbation parameter. The orbits of the integrable unperturbed system, $H_0(I)$ lie on invariant tori and are generically quasi-periodic. Moreover, if the unperturbed frequencies satisfy resonance relations, one has periodic orbits on a dense set of resonant tori. As we have recalled in the Introduction, the KAM theorem ensures the persistence of a set of large measure of quasi-periodic orbits, lying on strongly non-resonant tori, if ε is small enough and a suitable non-degeneracy condition for H_0 is satisfied (the so-called Kolmogorov non-degeneracy or twist condition). Instead, for arbitrary small perturbations, a resonant torus is generically destroyed and only a finite number of periodic orbits are expected to survive. The location and stability of the continued periodic orbits are determined by a theorem of Poincaré (see [77,78]), who approached the problem locally: with an averaging method, he was able to select those isolated unperturbed solutions which, under a suitable non-degeneracy condition (nowadays called Poincaré non-degeneracy), can be continued by means of an implicit function theorem. In particular, introducing the $n_1 - 1$ resonant angles $q_j = k_j\varphi_1 - k_1\varphi_j$, one can consider the time averaged perturbation

$$\langle H_1 \rangle_T (q) = \frac{1}{T}\int_0^T H_1 \, dt \ ,$$

evaluated at the unperturbed periodic orbits, which only depends on the particular periodic orbit taken into account and not on the initial point on it. Hence, it is defined as a function of the resonant angles and, according to Poincaré, periodic orbits for which it holds that

$$\nabla_q \langle H_1 \rangle = 0 \ , \qquad |D^2_q \langle H_1 \rangle| \neq 0$$

can be continued also for $\varepsilon \neq 0$.

A modern approach aiming at studying continuation of periodic orbits has been developed in the seventies by Weinstein [89] and Moser [65] using bifurcation techniques, turning the problem to the investigation of critical points of a functional on a compact manifold, whose number can be estimated from below with geometrical methods, like Morse theory (see, e.g., [3, 10]). The latter approach allows to extend the Poincaré-Birkhoff Theorem for two-dimensional twist maps, obtaining a global result which claims that a minimal number of periodic solutions surely survive.

The drawback lies in the fact that the method is not at all constructive, thus it does not permit the localization of the periodic orbits on the torus. In the same spirit, variational methods which make use of the mountain pass theorem were developed some years later by Fadell and Rabinowitz, under different hypotheses (see Chapter 1 in [11] for a simplified exposition of this result).

More recently, the problem of continuation of degenerate periodic orbits in nearly integrable Hamiltonian systems, using perturbation techniques, has been studied in [88] and [61], respectively in the sense of Kolmogorov and Poincaré degeneracy. On the other hand, from the early nineties great attention has been devoted to the generalization of Poincaré's result to partially resonant tori, where the unperturbed torus is foliated by quasi-periodic orbits, since the number of resonances is strictly less than $n_1 - 1$. In this case, the starting point still consists in looking for non-degenerate critical points of the perturbation averaged over the unperturbed quasi-periodic solution. However, the presence of more than a single frequency requires the assumption of additional hypotheses, which allow to implement suitable versions of the KAM scheme. Along this line, first results were due to Treshchev [87], Cheng [15], Cheng and Wang [16], Li and Yi [56]. Recently, these results have been successfully extended to multiscale nearly integrable Hamiltonian systems, where the integrable part of the Hamiltonian $H_0(I, \varepsilon)$, properly involves several time scales, see, e.g., [90,91]. All the quoted works deal with the case where the unperturbed invariant torus is degenerate due to resonances among its frequencies. Instead, we remark that the problems of existence of invariant tori of dimension less than the number of degrees of freedom in weakly perturbed Hamiltonian system, i.e., the extension to lower dimensional tori of the classical KAM theory, has been widely investigated by many authors, see, e.g., [26,44,47,62,63,66,93,94] in a general abstract framework, and [12,20,21,38,46,86] for more recent problems mainly emerging in Celestial Mechanics.

In this first part of the thesis we follow the line traced by Poincaré and deal with those cases when the Poincaré non-degeneracy condition is not fulfilled. In particular, under a twist-like condition of the form (1.7) (see, e.g., [9]) and analytic estimates of the perturbation, we develop an original normal form scheme, inspired by a recent completely constructive proof of the classical Lyapunov's theorem on periodic orbits (see [33]), and which allows to investigate the continuation of degenerate periodic orbits for completely resonant tori of maximal or low dimension. Precisely, first we identify possible candidates for the continuation via normal form, then we prove the existence of a unique solution by using the Newton-Kantorovich method.

Generically, the estimates of our normal form procedure do not give a convergent normal form. Actually, looking for a convergent normal form which is valid for all possible periodic orbits could be too much to ask. The idea is that a suitably truncated normal form allows to produce the approximated periodic orbits and the continuation can be performed via a contraction theorem or with a further convergent normal form around a selected periodic orbit.

The strength of the present perturbation algorithm is at least twofold. First, it provides a way to construct approximate periodic solutions at any desired order in ε, thus going beyond the average approximation mostly used in the literature. One of the few results which represents an improvement with respect to the usual average method is the one claimed in [61], where a criterion for the existence of periodic orbits on completely degenerate resonant tori is proved. In that work the authors, by means of a standard Lindstedt expansion as the original works of Poincaré, are able to push the perturbation scheme at second order in the small parameter ε. However, the possibility to provide a criterion for the continuation, although remarkable, is a consequence of the restriction to completely degenerate cases, like when the Fourier expansion of H_1 with respect to the angle variables does not include a certain resonance class. In this way, all the partial degeneracies are excluded. Such a limitation is overcome by the normal form that is here proposed: indeed, by being able to deal with any degree of degeneracy, it results more general (also in terms of order of accuracy), thus including also the above mentioned result. The formal scheme itself has also a second relevant aspect. Since this approximation is given by a recursive explicit algorithm, it can be much useful for numerical applications (see, e.g., [41]) and it is independent of the possibility to conclude the proof with a contraction theorem.

Furthermore, this approach provides a constructive normal form that might be applied to a sufficiently general class of models, in particular as regards the extension to lower dimensional completely resonant tori in Chapter 2. It allows to study spatially localized time-periodic solutions (often called multibreathers) in chains of weakly coupled oscillators, provided that it is possible to explicitly perform a transformation to action-angle variables for the oscillators which are excited in the unperturbed system. Indeed, these solutions can be interpreted as periodic orbits that survive to the breaking of a lower dimensional completely resonant torus. For instance, our procedure would also include non-linear Hamiltonian lattices with next-to-nearest neighbor interactions, such as

$$H = \sum_{j\in\mathcal{J}} \frac{y_j^2}{2} + \sum_{j\in\mathcal{J}} V(x_j) + \varepsilon \sum_{l=1}^{i} \sum_{j\in\mathcal{J}} W(x_{j+l} - x_j) \ ;$$

where $V(x)$ is the potential of an anharmonic oscillator which allows for action variables (at least locally, like the Morse potential), and $W(x)$ represents a generic also next-to-nearest neighbour (possibly linear) interaction, with i the maximal range of the interaction. In the class of nearly integrable Hamiltonian lattices, the possibility to generalize the formal scheme to lower dimensional tori represents a breakthrough in the investigation of degenerate multibreathers and vortexes in one and two-dimensional lattices (see, e.g., [54,70,72,75,76]). In this context, degenerate scenarios which stem from the study of discrete solitons in dNlS lattices will be investigated in Chapter 3, thanks also to the possibility of easily introducing the suitable setting with action-angle variables.

Existence and stability of multibreathers have been studied since the 90's with an averaging procedure, named effective Hamiltonian method (introduced in [5,58] and developed e.g. in [1,55]), which however fails in those cases where approximate solutions are degenerate, for example not isolated. The method is an extension of the Poincaré's result to the case of lower dimensional completely resonant tori in chains of weakly coupled oscillators. Moreover, it considers an effective Hamiltonian which, in the lowest order of approximation, is exactly the time averaged perturbation $\langle H_1 \rangle$ evaluated at the unperturbed periodic orbit. Hence, critical points of this effective Hamiltonian at leading order are the candidates for continuation which results in the application of the implicit function theorem under the same hypotheses of non-degeneracy as before (Kolmogorov and Poincaré non-degeneracy) and a non-resonance condition (first Melnikov condition) between the internal oscillators of the torus and the external ones.

This first Chapter is devoted to the normal form algorithm which allows to treat continuation of degenerate periodic orbits in the case of completely resonant tori of maximal dimension. Before entering the details of the normalization procedure, I will introduce the generic analytic setting which will be used also in Chapter 2.

In Section 1.5 the special degenerate case of one-parameter families of periodic solutions is analyzed and a simplified version of the continuation theorem is presented, exploiting perturbation theory of matrices. Indeed, the hypotheses of the continuation theorem are not always easy to be verified and the just mentioned simplification provides a more applicable formulation which can be applied in several applications. In Chapter 2, by means of the improvement of the normal form algorithm, we will exploit the properties of the monodromy matrix in order to get a generic strategy which enables to more easily check the validity of the assumptions and obtain continuation of periodic orbits.

1.1 Analytic setting

In this section we detail the analytic setting which is helpful to go into the details of the normal form procedure. For the sake of simplicity, we introduce the setting for the generic case of lower dimensional tori. This enables us to better justify some choices of notation due to the extension of the normal form algorithm in the next Chapter.

Consider a canonical system of differential equations with $n = n_1 + n_2$ d.o.f. and Hamiltonian

$$H(I, \varphi, \xi, \eta, \varepsilon) = H_0(I, \xi, \eta) + \varepsilon H_1(I, \varphi, \xi, \eta; \varepsilon) \ , \tag{1.2}$$

where $(I, \varphi) \in \mathcal{U}(I^*) \times \mathbb{T}^{n_1}$ are action-angle variables defined in a neighbourhood $\mathcal{U}(I^*) \subset \mathbb{R}^{n_1}$ of the action I^*, $(\xi, \eta) \in \mathcal{V}(0) \subset \mathbb{C}^{2n_2}$ are Cartesian variables defined in a neighbourhood $\mathcal{V}(0)$

of the origin. The Hamiltonian (1.2) is assumed to be analytic in all variables and in the small parameter ε.

Introduce the distinguished classes of functions $\widehat{\mathcal{P}}_{l,m}$, with integers l and m, which can be written as a Taylor-Fourier expansion

$$g(I,\varphi,\xi,\eta) = \sum_{\substack{i\in\mathbb{N}^{n_1}\\ |i|=l}} \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2}\\ |m_1|+|m_2|=m}} \sum_{k\in\mathbb{Z}^{n_1}} g_{i,m_1,m_2,k}\, I^i \exp(\mathbf{i}\langle k,\,\varphi\rangle)\xi^{m_1}\eta^{m_2} \ , \tag{1.3}$$

with coefficients $g_{i,m_1,m_2,k}\in\mathbb{C}$. We say that $g\in\mathcal{P}_\ell$ in case

$$g\in \bigcup_{\substack{l\geq 0,m\geq 0\\ 2l+m=\ell}} \widehat{\mathcal{P}}_{l,m} \ .$$

We also set $\mathcal{P}_{-4}=\mathcal{P}_{-3}=\mathcal{P}_{-2}=\mathcal{P}_{-1}=\{0\}$.

Consider the Hamiltonian (1.2) and select a completely resonant elliptic lower dimensional torus for the unperturbed Hamiltonian setting $I=I^*$ and $\xi=\eta=0$ such that

$$\hat\omega(I^*) = \nabla_I H_0(I^*) = \omega k \ , \quad \text{with } \omega\in\mathbb{R} \ , \ k\in\mathbb{Z}^{n_1} \ . \tag{1.4}$$

Expanding the Hamiltonian in Taylor series of the translated actions $J=I-I^*$ and the Cartesian coordinates (ξ,η), and in Fourier series of the angles φ we get

$$H^{(0)} = \langle\hat\omega, J\rangle + \sum_{j=1}^{n_2}\mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(J,\xi,\eta) + \sum_{s>0}\sum_{\ell\geq 0} f_\ell^{(0,s)}(\varphi,J,\xi,\eta) \ , \tag{1.5}$$

where $f_\ell^{(0,s)}\in\mathcal{P}_\ell$ is of order $\mathcal{O}(\varepsilon^s)$ and the first superscript stands for the normalization step.

We also define the (n_1-1)-dimensional resonant module associated to the resonant frequency $\hat\omega(I^*)$ as

$$\mathcal{M}_\omega = \Big\{h\in\mathbb{Z}^{n_1} : \langle\hat\omega(I^*),h\rangle = 0\Big\} \ .$$

In a neighborhood of the resonant torus, it is useful to introduce the resonant variables $(\hat q,\hat p)$ in place of (φ,J), in order to better describe the periodic dynamics. Without affecting the generality of the result, we will assume $k_1=1$; this choice simplifies the interpretation of the new variables. The canonical change of coordinates is built with an unimodular matrix which defines the *slow* angles $\hat q_j = k_j\varphi_1-\varphi_j$, for $j=2,\ldots,n_1$, as the phase differences with respect to the *fast* angle $\hat q_1$ of the periodic orbit; the momenta are defined so as to complement the canonical change of coordinates, in particular $\hat p_1=\langle k,J\rangle$.

In order to underline the dependence on fast and slow angles in the normal form scheme, we introduce the notations $\hat p=(p_1,p)$, $\hat q=(q_1,q)$ with $p_1=\hat p_1$, $p=(\hat p_2,\ldots,\hat p_n)$ and correspondingly for q_1 and q. The Hamiltonian (1.5) then reads

$$H^{(0)} = \omega p_1 + \sum_{j=1}^{n_2}\mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(\hat p,\xi,\eta) + \sum_{s>0}\sum_{\ell\geq 0} f_\ell^{(0,s)}(\hat q,\hat p,\xi,\eta) \ ,$$

where $f_\ell^{(0,s)}\in\mathcal{P}_\ell$ and it is a function of order $\mathcal{O}(\varepsilon^s)$. Indeed, the linear change of coordinates does not affect the belonging to the classes of functions $\mathcal{P}_\ell$.

Besides, we introduce the extended complex domains $\mathcal{D}_{\rho,\sigma,R}=\mathcal{G}_\rho\times\mathbb{T}^{n_1}_\sigma\times\mathcal{B}_R$, namely

$$\begin{aligned}
\mathcal{G}_\rho &= \big\{\hat p\in\mathbb{C}^{n_1} : \max_{1\leq j\leq n_1}|\hat p_j|<\rho\big\} \ ,\\
\mathbb{T}^{n_1}_\sigma &= \big\{\hat q\in\mathbb{C}^{n_1} : \operatorname{Re}\hat q_j\in\mathbb{T},\ \max_{1\leq j\leq n_1}|\operatorname{Im}\hat q_j|<\sigma\big\} \ ,\\
\mathcal{B}_R &= \big\{(\xi,\eta)\in\mathbb{C}^{2n_2} : \max_{1\leq j\leq n_2}(|\xi_j|+|\eta_j|)<R\big\} \ .
\end{aligned}$$

Given a generic analytic function $g : \mathcal{D}_{\rho,\sigma,R} \to \mathbb{C}$, we define the weighted Fourier norm

$$\|g\|_{\rho,\sigma,R} = \sum_{i\in\mathbb{N}^{n_1}} \sum_{(m_1,m_2)\in\mathbb{N}^{2n_2}} \sum_{k\in\mathbb{Z}^{n_1}} |g_{i,m_1,m_2,k}|\rho^{|i|}R^{|m_1|+|m_2|}e^{|k|\sigma} .$$

Hereafter, we are going to use the shorthand notation $\|\cdot\|_\alpha$ for $\|\cdot\|_{\alpha(\rho,\sigma,R)}$.

Let us remark that, for the case of full dimensional tori, the transversal variables ξ and η are simply absent.

1.2 Main results

Consider a completely resonant maximal torus of H_0 with unperturbed frequencies $\hat{\omega}(I^*)$ as in (1.4). This corresponds to a suitable choice of the actions $I = I^*$ with non-vanishing components.

Expanding (1.1) in power series of the translated actions $J = I - I^*$, one has

$$\begin{aligned} H^{(0)} = \langle\hat{\omega}, J\rangle &+ f_4^{(0,0)}(J) + \sum_{l>2} f_{2l}^{(0,0)}(J) \\ &+ f_0^{(0,1)}(\varphi) + f_2^{(0,1)}(\varphi, J) \\ &+ \sum_{s>1} f_0^{(0,s)}(\varphi) + \sum_{s>1} f_2^{(0,s)}(\varphi, J) \\ &+ \sum_{s>0}\sum_{l>1} f_{2l}^{(0,s)}(\varphi, J) , \end{aligned}$$

where $f_{2l}^{(0,s)}$ is a homogeneous polynomial of degree l in J and it is a function of order $\mathcal{O}(\varepsilon^s)$.

Introducing the convenient notations $\hat{p} = (p_1, p)$, $\hat{q} = (q_1, q)$, the Hamiltonian can be written in the form

$$\begin{aligned} H^{(0)} = \omega p_1 &+ f_4^{(0,0)}(p_1, p) + \sum_{l>2} f_{2l}^{(0,0)}(p_1, p) \\ &+ f_0^{(0,1)}(q_1, q) + f_2^{(0,1)}(q_1, q, p_1, p) \\ &+ \sum_{s>1} f_0^{(0,s)}(q_1, q) + \sum_{s>1} f_2^{(0,s)}(q_1, q, p_1, p) \\ &+ \sum_{s>0}\sum_{l>1} f_{2l}^{(0,s)}(q_1, q, p_1, p) \end{aligned} \tag{1.6}$$

where $f_{2l}^{(0,s)}$ is a homogeneous polynomial of degree l in $\hat{p}$ and it is a function of order $\mathcal{O}(\varepsilon^s)$.

Since we aim to continue a generic unperturbed periodic orbit $q_1 = q_1(0) + \omega t$, $q = q^*$, $p_1 = 0$, $p = 0$ with fixed frequency ω, we look for a normal form which is able to select those phase shifts, q^*, which represent good candidates for continuation. The Hamiltonian is said to be in normal form up to order r if the constant and linear terms in the actions are averaged (up to order r) with respect to the fast angle, q_1, and if, for a fixed but arbitrary q^*, the linear terms in the action, evaluated at $q = q^*$, vanish identically.

We state here our main result concerning the normal form.

Proposition 1.2.1 *Consider a Hamiltonian $H^{(0)}$ expanded as in* (1.6) *that is analytic in a domain $\mathcal{D}_{\rho,\sigma}$. Let us assume that*

(H1) there exists a positive constant m such that for every $v \in \mathbb{R}^{n_1}$ one has

$$m\sum_{i=1}^{n_1}|v_i| \le \sum_{i=1}^{n_1}\Big|\sum_{j=1}^{n_1} C_{0,ij}v_j\Big| , \quad \textit{where} \quad C_0 = D_{\hat{p}}^2 f_4^{(0,0)} ; \tag{1.7}$$

(H2) the terms appearing in the expansion of the Hamiltonian satisfy

$$\|f_{2l}^{(0,s)}\|_1 \le \frac{E}{2^{2l}}\varepsilon^s , \quad \textit{with} \quad E > 0. \tag{1.8}$$

Then, for every positive integer r there is a positive ε_r^* such that for $0 \le \varepsilon < \varepsilon_r^*$ there exists an analytic canonical transformation $\Phi^{(r)}$ satisfying

$$\mathcal{D}_{\frac{1}{4}(\rho,\sigma)} \subset \Phi^{(r)}\left(\mathcal{D}_{\frac{1}{2}(\rho,\sigma)}\right) \subset \mathcal{D}_{\frac{3}{4}(\rho,\sigma)}$$

such that the Hamiltonian $H^{(r)} = H^{(0)} \circ \Phi^{(r)}$ has the following expansion

$$\begin{aligned} H^{(r)}(p_1, p, q_1, q; q^*) = \omega p_1 + f_4^{(r,0)}(p_1, p) + \sum_{l>2} f_{2l}^{(r,0)}(p_1, p) \\ + \sum_{s=1}^{r} f_0^{(r,s)}(q; q^*) + \sum_{s=1}^{r} f_2^{(r,s)}(q, p_1, p; q^*) \\ + \sum_{s>r} f_0^{(r,s)}(q_1, q; q^*) + \sum_{s>r} f_2^{(r,s)}(q_1, q, p_1, p; q^*) \\ + \sum_{s>0}\sum_{l>1} f_{2l}^{(r,s)}(q_1, q, p_1, p; q^*) \ , \end{aligned} \tag{1.9}$$

where q^* is a fixed but arbitrary parameter and $f_{2l}^{(r,s)} \in \mathcal{P}_{2l}$ is a function of order $\mathcal{O}(\varepsilon^s)$. The Hamiltonian (1.9) *is said to be in normal form up to order r since for $s \le r$ satisfies:*

1. *$f_0^{(r,s)}(q; q^*)$ do not depend on the fast angle q_1;*
2. *$f_2^{(r,s)}(q, \hat{p}; q^*)$ do not depend on q_1 and, evaluated at $q = q^*$, satisfy*

$$f_2^{(r,s)}(q^*, \hat{p}; q^*) = 0 \ .$$

The Hamilton equations associated to the truncated normal form, i.e., neglecting terms of order $\mathcal{O}(\varepsilon^{r+1})$, once evaluated at $x^* = (q = q^*, \hat{p} = 0)$, read

$$\dot{q}_1 = \omega \ , \qquad \dot{q} = 0 \ , \qquad \dot{p}_1 = 0 \ , \qquad \dot{p} = -\sum_{s=1}^{r} \nabla_q f_0^{(r,s)} \ .$$

Hence, if

$$\sum_{s=1}^{r} \nabla_q f_0^{(r,s)}\big|_{q=q^*} = 0 \ , \tag{1.10}$$

then $q_1 = q_1(0)$, $q = q^*$, $p_1 = 0$, $p = 0$ is the initial datum of a periodic orbit with frequency ω for the truncated normal form. Considering the whole system given by $H^{(r)}$, the initial datum provides an *approximate* periodic orbit with frequency ω, which turns out to be a relative equilibrium of the truncated Hamiltonian. In order to provide a precise definition of *approximate periodic orbit* we introduce the variation over the T-period map $\Upsilon : \mathcal{U}(x^*) \subset \mathbb{R}^{2n_1-1} \to \mathcal{V}(x^*) \subset \mathbb{R}^{2n_1-1}$, a smooth function of the $2n_1 - 1$ variables $x = (q, \hat{p})$, parameterized by the initial phase $q_1(0)$ and the small parameter ε, precisely[1]

$$\Upsilon(x(0); \varepsilon, q_1(0)) = \begin{pmatrix} \mathfrak{F}(x(0); \varepsilon, q_1(0)) \\ \mathfrak{G}(x(0); \varepsilon, q_1(0)) \end{pmatrix} = \begin{pmatrix} \hat{q}(T) - \hat{q}(0) - \Lambda T \\ \frac{1}{\varepsilon}(p(T) - p(0)) \end{pmatrix} \ , \tag{1.11}$$

with $\Lambda = (\omega, 0) \in \mathbb{R}^{n_1}$.

Let us stress that $(q_1 = \omega t + q_1(0), x^*)$ corresponds to a periodic orbit for the truncated normal form, thus $\Upsilon(x^*; \varepsilon, q_1(0))$ is of order[2] $\mathcal{O}(\varepsilon^r)$, as we will prove in Lemma1.4.1. Hence, a true periodic orbit, close to the approximate one, is identified by an initial datum $x^*_{\text{p.o.}} = (q^*_{\text{p.o.}}, \hat{p}_{\text{p.o.}}) \in \mathcal{U}(x^*)$ such that

$$\Upsilon(x^*_{\text{p.o.}}; \varepsilon, q_1(0)) = 0 \ .$$

[1]Let us remark that the actions p have been scaled by ε in Υ, in order to reveal degeneracy of periodic orbits.

[2]The actions p have been scaled by ε in Υ, hence only $\mathfrak{G}$ is of order $\mathcal{O}(\varepsilon^{r+1})$, while $\mathfrak{F}$ is of order $\mathcal{O}(\varepsilon^r)$.

In order to prove the existence of a unique solution $q_1 = q_1(0)$, $q^* = q^*_{\text{p.o.}}$, $\hat{p} = \hat{p}_{\text{p.o.}}$, close enough to the approximate one, we will apply the Newton-Kantorovich algorithm. Therefore we need to ensure that the Jacobian matrix (with respect to the initial datum)

$$M(\varepsilon) = D_{x(0)}\Upsilon(x^*;\varepsilon, q_1(0)) \tag{1.12}$$

is invertible and its eigenvalues are not too small with respect to ε^r.

We now state the main result concerning continuation of periodic orbits

Theorem 1.2.1 *Consider the map Υ defined in (1.11) in a neighbourhood of the torus $\hat{p} = 0$ and let $x^*(\varepsilon) = (q^*(\varepsilon), 0)$, with $q^*(\varepsilon)$ satisfying (1.10), an approximate zero of Υ, namely*

$$\|\Upsilon(x^*(\varepsilon);\varepsilon, q_1(0))\| \le c_1\varepsilon^r \ ,$$

where c_1 is a positive constant depending on $\mathcal{U}$ and r. Assume that the matrix $M(\varepsilon)$ defined in (1.12) is invertible and its eigenvalues satisfy

$$|\lambda| \gtrsim \varepsilon^\alpha \ , \qquad \text{for} \quad \lambda \in \operatorname{spec}(M(\varepsilon)) \quad \text{with} \quad 2\alpha < r \ . \tag{1.13}$$

Then, there exist $c_0 > 0$ and $\varepsilon^* > 0$ such that for any $0 \le \varepsilon < \varepsilon^*$ there exists a unique $x^*_{\text{p.o.}}(\varepsilon) = (q^*_{\text{p.o.}}(\varepsilon), \hat{p}_{\text{p.o.}}(\varepsilon)) \in \mathcal{U}$ which solves

$$\Upsilon(x^*_{\text{p.o.}};\varepsilon, q_1(0)) = 0 \ , \qquad \left\|x^*_{\text{p.o.}} - x^*\right\| \le c_0\varepsilon^{r-\alpha} \ . \tag{1.14}$$

We will see in the next Section that the above Theorem generalizes the classical result by Poincaré, which corresponds to the construction of the first order normal form together with a non-degeneracy assumption on the ε-independent version of (1.10), precisely

$$\nabla_q f_0^{(1,1)} = 0 \ , \qquad |D^2_q f_0^{(1,1)}| \neq 0 \ . \tag{1.15}$$

In such a case, due to the simplified form of Υ, the solution $(q^*_{\text{p.o.}}, \hat{p}_{\text{p.o.}})$ can be obtained via implicit function theorem in a neighborhood of the approximate initial datum x^*, q^* being a solution of the first of (1.15), independent of ε. Hence, our high-order normal form construction becomes a necessary way in order to deal with degenerate cases, where for example solutions of (1.15) are not isolated and appear as d-parameter families, thus leading to $|D^2_q f_0^{(1,1)}| = 0$. Let us also remark that our scheme provides a refined averaged Hamiltonian which allows to treat the totally degenerate case, i.e., $\nabla_q f_0^{(1,1)} \equiv 0$. In particular, the results presented in [61] by means of Lindstedt perturbation scheme can be obtained as special cases.

1.3 Normal form algorithm

This Section is dedicated to detail the first step of the normal form algorithm and the generic one that takes the Hamiltonian (1.6) and brings it into normal form up to an arbitrary and finite order r. We will use the formalism of Lie series (see, e.g., [43] and [32]).

The transformation at step r is generated via composition of two Lie series of the form

$$\exp(L_{\chi_2^{(r)}}) \circ \exp(L_{\chi_0^{(r)}}) \ ,$$

where

$$\chi_0^{(r)} = X_0^{(r)} + \langle \zeta^{(r)}, \hat{q}\rangle \ ,$$

with $\zeta^{(r)} \in \mathbb{R}^{n_1}$ and $X_0^{(r)} \in \mathcal{P}_0$, $\chi_2^{(r)} \in \mathcal{P}_2$ of order $\mathcal{O}(\varepsilon^r)$. The generating functions $\chi_0^{(r)}$ and $\chi_2^{(r)}$ are unknowns to be determined so that the transformed Hamiltonian is in normal form up to order r. We also denote by $L_g\cdot$ the Poisson bracket $\{\cdot, g\}$.

We now state an algebraic property of the $\mathcal{P}_\ell$ classes of functions:

Lemma 1.3.1 *Let $f \in \mathcal{P}_{\ell_1}$ and $g \in \mathcal{P}_{\ell_2}$, then $\{f, g\} \in \mathcal{P}_{\ell_1+\ell_2-2}$.*

Proof. We set $\ell_i = 2l_i + m_i$ with $i = 1, 2$ and, by considering the Poisson bracket between f and g

$$\{f, g\} = \sum_{j=1}^{n_1} \left(\frac{\partial f}{\partial \hat{q}_j} \frac{\partial g}{\partial \hat{p}_j} - \frac{\partial f}{\partial \hat{p}_j} \frac{\partial g}{\partial \hat{q}_j} \right) + \sum_{j=1}^{n_2} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} - \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \right) ,$$

we can deduce that each partial derivative with respect to $\hat{p}_j$ turns the degree $2l_i$ into $2(l_i - 1) = 2l_i - 2$ and, similarly, each partial derivative with respect to ξ_j or η_j transforms the degree m_i into $m_i - 1$. This concludes the proof.

□

1.3.1 First normalization step

Consider the starting Hamiltonian (1.6).

First stage of the first normalization step

Our aim is to put the term $f_0^{(0,1)}$ in normal form, by averaging it with respect to the fast angle q_1. Furthermore, we want to perform a translation of the linear terms in the actions, in order to keep the frequencies of the selected resonant torus fixed, when we are considering $q = q^*$. It means that we want to keep the frequency ω unchanged and not to introduce transversal frequencies on the torus. We determine the generating function

$$\chi_0^{(1)}(\hat{q}) = X_0^{(1)}(\hat{q}) + \langle \zeta^{(1)}, \hat{q} \rangle \qquad \text{with} \qquad \zeta^{(1)} \in \mathbb{R}^{n_1} ,$$

belonging to $\mathcal{P}_0$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equations

$$\begin{aligned} &L_{X_0^{(1)}} \omega p_1 + f_0^{(0,1)} = \langle f_0^{(0,1)} \rangle_{q_1} , \\ &L_{\langle \zeta^{(1)}, \hat{q} \rangle} f_4^{(0,0)} + \left\langle f_2^{(0,1)} \Big|_{q=q^*} \right\rangle_{q_1} = 0 . \end{aligned}$$

where $\langle \cdot \rangle_{q_1}$ denotes the average with respect to the fast angle q_1. Considering the Taylor-Fourier expansion

$$f_0^{(0,1)}(\hat{q}) = \sum_k c_{0,k}^{(0,1)} \exp(\mathbf{i}\langle k, \hat{q} \rangle) ,$$

we get

$$X_0^{(1)}(\hat{q}) = \sum_{k_1 \neq 0} \frac{c_{0,k}^{(0,1)}}{\mathbf{i} k_1 \omega} \exp(\mathbf{i}\langle k, \hat{q} \rangle) .$$

The translating vector $\zeta^{(1)}$ is the solution of the following linear system

$$\sum_j C_{0,ij} \zeta_j^{(1)} = \frac{\partial}{\partial \hat{p}_i} \left\langle f_2^{(0,1)} \Big|_{q=q^*} \right\rangle_{q_1} .$$

Remark 1.3.1 *This translation does not involve the linear term in the actions $L_{X_0^{(1)}(\hat{q})} f_4^{(0,0)}(p_1, p)$, since one has*

$$\left\langle L_{X_0^{(1)}(\hat{q})} f_4^{(0,0)} \Big|_{q=q^*} \right\rangle_{q_1} = 0 .$$

The transformed Hamiltonian is calculated as

$$
\begin{aligned}
H^{(\mathrm{I};0)} &= \exp\Big(L_{\chi_0^{(1)}}\Big) H^{(0)} = \\
&= \omega p_1 \\
&\quad + f_0^{(\mathrm{I};0,1)} + f_2^{(\mathrm{I};0,1)} \\
&\quad + \sum_{s>1} f_0^{(\mathrm{I};0,s)} + \sum_{s>1} f_2^{(\mathrm{I};0,s)} \\
&\quad + \sum_{s\geq 0} \sum_{l>1} f_{2l}^{(\mathrm{I};0,s)} \ .
\end{aligned}
$$

The functions $f_{2l}^{(\mathrm{I};0,s)}$ are recursively defined as

$$
\begin{aligned}
f_0^{(\mathrm{I};0,1)} &= \langle f_0^{(0,1)} \rangle_{q_1} \ , \\
f_2^{(\mathrm{I};0,1)} &= f_2^{(0,1)} - \big\langle f_2^{(0,1)}(q^*) \big\rangle_{q_1} + L_{X_0^{(1)}} f_4^{(0,0)} \ , \\
f_{2l}^{(\mathrm{I};0,s)} &= \sum_{j=0}^{s} \frac{1}{j!} L_{\chi_0^{(1)}}^{j} f_{2l+2j}^{(0,s-j)} \ , \qquad && \text{for } l = 0, 1,\ s \neq 1 \ , \\
& && \text{or } l \geq 2,\ s \geq 0 \ ,
\end{aligned}
$$

with $f_{2l}^{(\mathrm{I};0,s)} \in \mathcal{P}_{2l}$.

Second stage of the first normalization step

Our goal is to put in normal form the term $f_2^{(\mathrm{I};0,1)}$, by averaging it with respect to the fast angle q_1. We determine the generating function $\chi_2^{(1)}$, belonging to $\mathcal{P}_2$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equation

$$
L_{\chi_2^{(1)}} \omega p_1 + f_2^{(\mathrm{I};0,1)} = \langle f_2^{(\mathrm{I};0,1)} \rangle_{q_1} \ . \tag{1.16}
$$

Hence, considering the Taylor-Fourier expansion

$$
f_2^{(\mathrm{I};0,1)}(\hat{p}, \hat{q}) = \sum_{\substack{|l|=1 \\ k}} c_{l,k}^{(\mathrm{I};0,1)} \hat{p}^l \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ ,
$$

we get

$$
\chi_2^{(1)}(\hat{p}, \hat{q}) = \sum_{\substack{|l|=1 \\ k_1 \neq 0}} \frac{c_{l,k}^{(\mathrm{I};0,1)} \hat{p}^l \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)}{\mathbf{i} k_1 \omega} \ .
$$

The transformed Hamiltonian is computed as

$$
H^{(1)} = \exp\left(L_{\chi_2^{(1)}}\right) H^{(\mathrm{I};0)}
$$

and is given by

$$
\begin{aligned}
H^{(1)} &= \omega p_1 \\
&\quad + f_0^{(1,1)} + f_2^{(1,1)} \\
&\quad + \sum_{s>1} f_0^{(1,s)} + \sum_{s>1} f_2^{(1,s)} \\
&\quad + \sum_{s\geq 0} \sum_{l>1} f_{2l}^{(1,s)} \ ,
\end{aligned}
$$

with

$$\begin{aligned}
f_2^{(1,1)} &= \langle f_2^{(\mathrm{I};0,1)}\rangle_{q_1} \ ,\\
f_2^{(1,s)} &= \frac{1}{(s-1)!}L^{s-1}_{\chi_2^{(1)}}\left(f_2^{(\mathrm{I};0,1)}+\frac{1}{s}L_{\chi_2^{(1)}}f_2^{(\mathrm{I};0,0)}\right)\\
&\quad+\sum_{j=0}^{s-2}\frac{1}{j!}L^{j}_{\chi_2^{(1)}}f_2^{(\mathrm{I};0,s-j)} =\\
&= \frac{1}{(s-1)!}L^{s-1}_{\chi_2^{(1)}}\left(\frac{1}{s}\langle f_2^{(\mathrm{I};0,1)}\rangle_{q_1}+\frac{s-1}{s}f_2^{(\mathrm{I};0,1)}\right)\\
&\quad+\sum_{j=0}^{s-2}\frac{1}{j!}L^{j}_{\chi_2^{(1)}}f_2^{(\mathrm{I};0,s-j)} \ , \qquad\qquad \text{for } s\neq 1 \ ,\\
f_{2l}^{(1,s)} &= \sum_{j=0}^{s}\frac{1}{j!}L^{j}_{\chi_2^{(1)}}f_{2l}^{(\mathrm{I};0,s-j)} \ , \qquad\qquad \text{for } l\neq 1,\ s\geq 0 \ ,
\end{aligned}$$

where we have exploited the homological equation (1.16).

We can stress that, of course, this stage does not give rise to terms which change the function $f_0^{(\mathrm{I};0,1)}$.

Let us consider the Hamilton equations of $H^{(1)}$

$$\begin{aligned}
\dot q_1 &= \omega+\nabla_{p_1}\left[f_4^{(1,0)}+f_2^{(1,1)}\right]+\mathcal{O}(|\hat p|^2)+\mathcal{O}(\varepsilon|\hat p|)+\mathcal{O}(\varepsilon^2)\\
\dot q &= \nabla_{p}\left[f_4^{(1,0)}+f_2^{(1,1)}\right]+\mathcal{O}(|\hat p|^2)+\mathcal{O}(\varepsilon|\hat p|)+\mathcal{O}(\varepsilon^2)\\
\dot p_1 &= \mathcal{O}(\varepsilon|\hat p|^2)+\mathcal{O}(\varepsilon^2)\\
\dot p &= -\nabla_q f_0^{(1,1)}-\nabla_q f_2^{(1,1)}+\mathcal{O}(\varepsilon|\hat p|^2)+\mathcal{O}(\varepsilon^2) \ ,
\end{aligned}$$

When we neglect terms of order $\mathcal{O}(\varepsilon^2)$ and we evaluate the equations at $x^*=(q=q^*,\hat p=0)$, we get

$$\dot q_1=\omega \ , \quad \dot q=0 \ , \quad \dot p_1=0 \ , \quad \dot p=-\nabla_q f_0^{(1,1)}\big|_{q=q^*} \ .$$

Let us remark that, for $q=q^*$, one has $f_2^{(1,1)}\big|_{q=q^*}=0$, because of the translation performed in the first stage of the normalization step.

Therefore, if

$$\nabla_q f_0^{(1,1)}\big|_{q=q^*}=0 \ , \tag{1.17}$$

then $(q_1=q_1(0)+\omega t, x^*)$ represents a relative equilibrium of the truncated Hamiltonian which we aim to continue. We remark that the equation (1.17) allows to select the candidates q^* for the continuation, which are independent of ε. The periodicity of an orbit for the Hamiltonian $H^{(1)}$ is given by the following condition:

$$\begin{aligned}
\hat q(T)-\hat q(0)-\Lambda T &= \int_0^T\nabla_p\Big[f_4^{(1,0)}+f_2^{(1,1)}\Big]\,ds+\mathcal{O}(|\hat p|^2)+\mathcal{O}(\varepsilon|\hat p|)+\mathcal{O}(\varepsilon^2)=0 \ ,\\
p_1(T)-p_1(0) &= \mathcal{O}(\varepsilon|\hat p|^2)+\mathcal{O}(\varepsilon^2)=0 \ ,\\
p(T)-p(0) &= -\int_0^T\nabla_q\Big[f_0^{(1,1)}+f_2^{(1,1)}\Big]\,ds+\mathcal{O}(\varepsilon|\hat p|^2)+\mathcal{O}(\varepsilon^2)=0 \ ,
\end{aligned}$$

Due to the conservation of the energy, we can neglect the equation for p_1. In addition, we can divide the n_1-1 actions p by ε. Hence, we get a system of $2n_1-1$ equations in $2n_1-1$ unknowns $x(0)=(q(0),p_1(0),p(0))$. We can now define the map Υ as in (1.11). Therefore, the approximate periodic solution

$$q_1(t)=\omega t+q_1(0) \ , \qquad q(t)=q^* \ , \qquad \hat p(t)=0$$

coincides with an approximate zero of the map Υ, thus, for $\varepsilon = 0$, one has $\Upsilon(x^*; 0, q_1(0)) = 0$. In order to apply the implicit function theorem and obtain the continuation of the unperturbed periodic orbit, we have to verify the condition on the determinant of the Jacobian matrix $M(0)$.

To this end, we expand[3] the solution $x = (\hat{q}, \hat{p})$ w.r.t ε, getting

$$x(t, \varepsilon) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \mathcal{O}(\varepsilon^2) \ .$$

Since we consider an initial datum x_0 which does not depend on ε, we have

$$x^{(0)}(0) = x_0 \ , \qquad x^{(k)}(0) = 0 \ , \qquad k \geq 1 \ .$$

In particular, we have the following expansions

$$\begin{aligned} &\hat{p}(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \hat{p}_0 + \mathcal{O}(\varepsilon) \\ &\hat{q}(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \hat{q}^{(0)}(t) + \mathcal{O}(\varepsilon) \ , \quad \text{with} \quad q_1^{(0)}(t) = \omega t + q_1^{(0)}(0), \ q^{(0)}(t) = q^* \ , \end{aligned} \tag{1.18}$$

where the dependence on $q_1(0)$ is implied.

Let us compute the differential of the functions $\mathfrak{F}$ and $\mathfrak{G}$. By inserting (1.18) in $\mathfrak{F}$ and $\mathfrak{G}$, we get

$$\begin{aligned} \mathfrak{F} &= \int_0^T C_0 \hat{p}_0 \, ds + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon) \ , \\ \mathfrak{G} &= -\frac{1}{\varepsilon} \int_0^T \left[\nabla_q f_0^{(1,1)}(q^{(0)}(t)) + \nabla_q f_2^{(1,1)}(q^{(0)}(t), \hat{p}_0) \right] ds + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon) \ . \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &D_{q_0} \mathfrak{F} \big|_{(x^*;0)} = O \ , \qquad D_{\hat{p}_0} \mathfrak{F} \big|_{(x^*;0)} = C_0 T \ , \\ &D_{q_0} \mathfrak{G} \big|_{(x^*;0)} = -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) \ , \qquad D_{\hat{p}_0} \mathfrak{G} \big|_{(x^*;0)} = -\frac{T}{\varepsilon} D_{\hat{p}q}^2 f_2^{(1,1)}(q^*) \end{aligned}$$

and the matrix

$$M(0) = \begin{pmatrix} O & C_0 T \\ & \\ -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) & -\frac{T}{\varepsilon} D_{\hat{p}q}^2 f_2^{(1,1)}(q^*) \end{pmatrix} .$$

Due to the twist condition (1.7), in order to apply the implicit function theorem, we only need that

$$\left| -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) \right| \neq 0 \ .$$

To conclude, under the above hypothesis of non-degeneracy, we can infer the continuation of the unperturbed approximate periodic orbit to a true periodic orbit, with the same frequency ω obtaining the following Theorem

Theorem 1.3.1 *Consider the starting Hamiltonian* (1.6) *and assume the twist condition* (1.7). *The unperturbed approximate periodic orbits for which it holds that*

$$\nabla_q f_0^{(1,1)}(q^*) = 0 \ , \qquad \left| D_q^2 f_0^{(1,1)}(q^*) \right| \neq 0 \ ,$$

namely non-degenerate periodic orbits, are analytically continued at fixed period, i.e. there exists a value ε^* such that for $|\varepsilon| < \varepsilon^*$ we get continuation.

The result is exactly the continuation statement of Poincaré's Theorem.

[3] Let us stress that the solution is analytic w.r.t. the parameter ε, in view of the analyticity of the Hamiltonian.

1.3.2 Generic r-th normalization step

We now describe the generic r-th normalization step, starting from the Hamiltonian in normal form up to order $r-1$, $H^{(r-1)}$, namely

$$\begin{aligned} H^{(r-1)} = \omega p_1 &+ \sum_{s<r} f_0^{(r-1,s)} + \sum_{s<r} f_2^{(r-1,s)} \\ &+ f_0^{(r-1,r)} + f_2^{(r-1,r)} \\ &+ \sum_{s>r} f_0^{(r-1,s)} + \sum_{s>r} f_2^{(r-1,s)} \\ &+ \sum_{s\geq 0}\sum_{l>1} f_{2l}^{(r-1,s)} \ , \end{aligned} \tag{1.19}$$

where $f_{2l}^{(r-1,s)} \in \mathcal{P}_{2l}$ is of order $\mathcal{O}(\varepsilon^s)$; $f_0^{(r-1,s)}$ and $f_2^{(r-1,s)}$ for $1 \leq s < r$ are in normal form.

First stage of the normalization step

Our aim is to put the term $f_0^{(r-1,r)}$ in normal form and to keep the harmonic frequencies of the selected resonant torus fixed. We determine the generating function $\chi_0^{(r)} = X_0^{(r)} + \langle \zeta^{(r)}, \hat{q} \rangle$ by solving the homological equations

$$\begin{aligned} L_{X_0^{(r)}} \omega p_1 + f_0^{(r-1,r)} &= \langle f_0^{(r-1,r)} \rangle_{q_1} \ , \\ L_{\langle \zeta^{(r)}, \hat{q} \rangle} f_4^{(0,0)} + \left\langle f_2^{(r-1,r)} \Big|_{q=q^*} \right\rangle_{q_1} &= 0 \ . \end{aligned}$$

Considering the Taylor-Fourier expansion

$$f_0^{(r-1,r)}(\hat{q}) = \sum_k c_{0,k}^{(r-1,r)} \exp(\mathbf{i}\langle k, \hat{q} \rangle) \ ,$$

we readily get

$$X_0^{(r)}(\hat{q}) = \sum_{k_1 \neq 0} \frac{c_{0,k}^{(r-1,r)}}{\mathbf{i} k_1 \omega} \exp(\mathbf{i}\langle k, \hat{q} \rangle) \ .$$

The translation vector, $\zeta^{(r)}$, is determined by solving the linear system

$$\sum_j C_{0,ij} \zeta_j^{(r)} = \frac{\partial}{\partial \hat{p}_i} \left\langle f_2^{(r-1,r)} \Big|_{q=q^*} \right\rangle_{q_1} \ .$$

This translation, which involves the linear term in the actions $f_2^{(r-1,r)}$, allows to keep the frequency ω fixed and kills the small transversal frequencies in the angles q.

The transformed Hamiltonian is computed as

$$H^{(\mathrm{I};r-1)} = \exp\left(L_{\chi_0^{(r)}} \right) H^{(r-1)}$$

and has a form similar to (1.19), precisely

$$\begin{aligned} H^{(\mathrm{I};r-1)} &= \exp\Big(L_{\chi_0^{(r)}} \Big) H^{(r-1)} = \\ &= \omega p_1 + \sum_{s<r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{I};r-1,s)} \\ &\quad + f_0^{(\mathrm{I};r-1,r)} + f_2^{(\mathrm{I};r-1,r)} \\ &\quad + \sum_{s>r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{I};r-1,s)} \\ &\quad + \sum_{s\geq 0}\sum_{l>1} f_{2l}^{(\mathrm{I};r-1,s)} \ . \end{aligned}$$

The functions $f_{2l}^{(\mathrm{I};r-1,s)}$ are recursively defined as

$$
\begin{aligned}
f_0^{(\mathrm{I};r-1,r)} &= \left\langle f_0^{(r-1,r)} \right\rangle_{q_1} \ , \\
f_2^{(\mathrm{I};r-1,r)} &= f_2^{(r-1,r)} - \left\langle f_2^{(r-1,r)}(q^*) \right\rangle_{q_1} + L_{X_0^{(r)}} f_4^{(0,0)} \ , \\
f_{2l}^{(\mathrm{I};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_0^{(r)}} f_{2l+2j}^{(r-1,s-jr)} \ , \qquad \text{for } l=0,1,\ s \neq r \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } l \geq 2,\ s \geq 0 \ ,
\end{aligned}
\tag{1.20}
$$

with $f_{2l}^{(\mathrm{I};r-1,s)} \in \mathcal{P}_{2l}$.

Second stage of the normalization step

We now put $f_2^{(\mathrm{I};r-1,r)}$ in normal form, by averaging with respect to the fast angle q_1. This is necessary in order to avoid small oscillations of q around q^*. We determine the generating function $\chi_2^{(r)}$ by solving the homological equation

$$
L_{\chi_2^{(r)}} \omega p_1 + f_2^{(\mathrm{I};r-1,r)} = \left\langle f_2^{(\mathrm{I};r-1,r)} \right\rangle_{q_1} \ .
$$

Considering again the Taylor-Fourier expansion

$$
f_2^{(\mathrm{I};r-1,r)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=1 \\ k}} c_{l,k}^{(\mathrm{I};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\hat{q}\rangle)
$$

we get

$$
\chi_2^{(r)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=1 \\ k_1 \neq 0}} \frac{c_{l,k}^{(\mathrm{I};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\hat{q}\rangle)}{\mathbf{i} k_1 \omega} \ .
$$

The transformed Hamiltonian is computed as

$$
H^{(r)} = \exp\left(L_{\chi_2^{(r)}} \right) H^{(\mathrm{I};r-1)}
$$

and is given the form (1.19), replacing the upper index $r-1$ by r , with

$$
\begin{aligned}
f_2^{(r,r)} &= \langle f_2^{(\mathrm{I};r-1,r)} \rangle_{q_1} \ , \\
f_2^{(r,jr)} &= \frac{1}{(j-1)!} L^{j-1}_{\chi_2^{(r)}} \left(\frac{1}{j} \langle f_2^{(\mathrm{I};r-1,r)} \rangle_{q_1} + \frac{j-1}{j} f_2^{(\mathrm{I};r-1,r)} \right) \\
&\quad + \sum_{j=0}^{\lfloor s/r \rfloor - 2} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_2^{(\mathrm{I};r-1,s-jr)} \ , \\
f_{2l}^{(r,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_{2l}^{(\mathrm{I};r-1,s-jr)} \qquad \text{for } l=0,\ s \geq 0 \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } l=1,\ s \neq jr \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } l \geq 2,\ s \geq 0 \ .
\end{aligned}
\tag{1.21}
$$

1.4 Proof of Theorem 1.2.1

By means of the normal form construction, it is possible to give the original Hamiltonian the form (1.9).

The Hamilton equations associated to the Hamiltonian $H^{(r)}$ read

$$\begin{aligned}
\dot{q}_1 &= \omega + \nabla_{p_1}\left[f_4^{(r,0)} + \sum_{s=1}^{r} f_2^{(r,s)}\right] + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon|\hat{p}|) + \mathcal{O}(\varepsilon^{r+1})\\
\dot{q} &= \nabla_{p}\left[f_4^{(r,0)} + \sum_{s=1}^{r} f_2^{(r,s)}\right] + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon|\hat{p}|) + \mathcal{O}(\varepsilon^{r+1})\\
\dot{p}_1 &= \mathcal{O}(\varepsilon|\hat{p}|^2) + \mathcal{O}(\varepsilon^{r+1})\\
\dot{p} &= -\sum_{s=1}^{r}\nabla_q f_0^{(r,s)} - \sum_{s=1}^{r}\nabla_q f_2^{(r,s)} + \mathcal{O}(\varepsilon|\hat{p}|^2) + \mathcal{O}(\varepsilon^{r+1})
\end{aligned}$$

Evaluating at x^* and neglecting terms of order $\mathcal{O}(\varepsilon^{r+1})$, the Hamilton equations provide a periodic orbit of frequency ω once q^* fulfills the equation (1.10). Generically, for $r \geq 2$, the value q^* would depend analytically on ε, precisely $q^*(\varepsilon) = q_0^* + \mathcal{O}(\varepsilon)$, with q_0^* solution of the ε-independent equation (1.15). The periodicity of an orbit for the Hamiltonian $H^{(r)}$ is given by

$$\begin{aligned}
\hat{q}(T) - \hat{q}(0) - \Lambda T &= \int_0^T \nabla_p\left[f_4^{(r,0)} + \sum_{s=1}^{r} f_2^{(r,s)}\right] ds + \mathcal{O}(|p|^2) + \mathcal{O}(\varepsilon|p|) + \mathcal{O}\big(\varepsilon^{r+1}\big) = 0\ ,\\
p_1(T) - p_1(0) &= \mathcal{O}(\varepsilon|p|^2) + \mathcal{O}\big(\varepsilon^{r+1}\big) = 0\ ,\\
p(T) - p(0) &= -\int_0^T \sum_{s=1}^{r}\nabla_q\left[f_0^{(r,s)} + f_2^{(r,s)}\right] ds + \mathcal{O}(\varepsilon|p|^2) + \mathcal{O}\big(\varepsilon^{r+1}\big) = 0\ ,
\end{aligned}$$

Due to conservation of the energy, we can eliminate the equation for p_1, divide the n_1-1 actions p by ε and look at $q_1(0)$ as a parameter (the phase along the orbit). The system of $2n_1-1$ equations in $2n_1-1$ unknowns $x(0)$

$$\begin{aligned}
\hat{q}(T) - \hat{q}(0) - \Lambda T &= \int_0^T \nabla_p\left[f_4^{(r,0)} + \sum_{s=1}^{r} f_2^{(r,s)}\right] ds + \mathcal{O}(|p|^2) + \mathcal{O}(\varepsilon|p|) + \mathcal{O}\big(\varepsilon^{r+1}\big) = 0\ ,\\
\frac{p(T) - p(0)}{\varepsilon} &= -\frac{1}{\varepsilon}\int_0^T \sum_{s=1}^{r}\nabla_q\left[f_0^{(r,s)} + f_2^{(r,s)}\right] ds + \mathcal{O}(|p|^2) + \mathcal{O}(\varepsilon^{r}) = 0\ ,
\end{aligned}$$

takes the form (1.11). The approximate periodic solution

$$q_1(t) = \omega t + q_1(0)\ , \qquad q(t) = q^*\ , \qquad \hat{p}(t) = 0\ ,$$

corresponds to an approximate zero x^* for the map Υ.

Introduce the quantities

$$\Xi_r = \max\left(\frac{eE}{\omega\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\omega\delta_r\rho\sigma}, \frac{2eE}{\omega\delta_r^2\rho\sigma}\right)\ ,$$

that will be useful for the estimates in Section 1.6, with ρ, σ, δ_r the constants and the restrictions of the domain due to Cauchy's estimates (see Lemma 1.6.1). Now, we can state the following Lemma

Lemma 1.4.1 *Let $x^* = (q^*, 0)$ be a relative equilibrium for the truncated normal form, i.e. an approximate periodic orbit for the Hamiltonian $H^{(r)}$, then $\Upsilon(x^*; \varepsilon, q_1(0))$ is of order $\mathcal{O}(\varepsilon^r)$.*

Proof. Considering the remainder of the Hamiltonian $H^{(r)}$, namely $\sum_{s>r}\sum_{2l\geq 0} f_{2l}^{(r,s)}$, we obtain the following estimate, if ε is small enough (i.e. take $\varepsilon < \frac{1}{100\Xi_r^3}$):

$$\begin{aligned}
\sum_{s>r}\sum_{l\geq 0}\|f_{2l}^{(r,s)}\| &\leq \sum_{s>r}\sum_{l\geq 0}\frac{100^s}{20}\Xi_r^{3s}\frac{E}{2^{2l}}\varepsilon^s \leq \frac{E}{15}\left(\sum_{s\geq 0}\big(100\Xi_r^3\varepsilon\big)^s - \sum_{s=0}^{r}\big(100\Xi_r^3\varepsilon\big)^s\right)\ ,\\
&= \frac{E}{15}\frac{\big(100\Xi_r^3\varepsilon\big)^{r+1}}{1 - 100\Xi_r^3\varepsilon}
\end{aligned}$$

where we have used the estimates contained in Lemma 1.6.4. By considering the estimates for the symplectic gradient in Corollary A.2.1 and integrating over the period T, we can deduce that there exist a domain $\mathcal{U}$ and a constant $c_1(r)$, dependent on the domain, such that

$$\|\Upsilon(x^*;\varepsilon,q_1(0))\| \leq c_1(r)\varepsilon^r \ ,$$

where we have taken into account the scaling of the actions with respect to ε.

□

We come now to the proof of the Theorem 1.2.1.

Proof. The proof of the Theorem consists in the application of the Proposition A.1.1 reported in section A.1 of the Appendix. Since we are seeking for a true periodic solution close to the approximate one, we take x in a small ball centered in x^*. Thus both the variables can be interpreted locally as Cartesian variables in $\mathbb{R}^{2n_1-1}$. Consider the differential of the map Υ evaluated at x^*, namely the matrix $M(\varepsilon)$ defined in (1.12). Extracting from $M(\varepsilon)$ its leading order in ε, we get

$$M(\varepsilon) = M_0 + \mathcal{O}(\varepsilon) \ , \qquad M_0 := M(0) = \begin{pmatrix} 0 & C_0 T \\ -B_1 T & -D_1 T \end{pmatrix} \ ,$$

where

$$B_1 = \frac{1}{\varepsilon} D_q^2 f_0^{(r,1)}\big|_{q=q_0^*}$$

and C_0 is the twist matrix defined in (1.7). The first hypothesis (A.1) in Proposition A.1.1 is satisfied with $\beta = r$, due to Lemma 1.4.1. The third assumption (A.3) on Lipschitz continuity is satisfied in view of the analyticity of the flow at time T w.r.t. the initial datum (it keeps the same smoothness as its vector field). The core of the statement is then the requirement on the invertibility of $M(\varepsilon)$. If B_1 is invertible, then the same holds true for M_0 (the twist C_0 being invertible); thus $M(\varepsilon)$ is also invertible and the second hypothesis (A.2) is satisfied with $\alpha = 0$, M_0 being independent of ε. This is actually the non-degenerate case, namely the Poincaré's theorem. If instead B_1 has a nontrivial Kernel, namely we have degeneracy, then the same holds also for M_0 (typically with a greater dimension). The required invertibility of $M(\varepsilon)$, asked by Theorem 1.2.1, is necessarily due to the ε-corrections, which are responsible for the bifurcations of the zero eigenvalues of the matrix M_0. Hence, in order to fulfill (A.2) for a generic step r, we need the smallest eigenvalues of $M(\varepsilon) = N(\varepsilon) + \mathcal{O}(\varepsilon^r)$ to bifurcate from zero as $\lambda_j(\varepsilon) \sim \varepsilon^\alpha$, with $\alpha < r$, which is guaranteed by the condition $2\alpha < r$ needed in Proposition A.1.1. Thus, we get the hypothesis (1.13). Finally, estimates (1.14) are of the same type as the one in Proposition A.1.1, even after back-transforming the solutions to the original canonical variables with $\Phi^{(r)}$. Indeed, the normalizing transformation $\Phi^{(r)}$ is a near the identity transformation (see the Appendix for the proof of Proposition 2.1.1, which is just a generalization of Proposition 1.2.1).

□

In case of absence of the scaling w.r.t. ε in the map Υ, the result about the continuation of periodic orbits may be stated as follows

Theorem 1.4.1 *Consider the map Υ defined in* (1.11) *(without the scaling of the actions) in a neighbourhood of the torus $\hat{p} = 0$ and let $x^*(\varepsilon) = (q^*(\varepsilon), 0)$, with $q^*(\varepsilon)$ satisfying* (1.10)*, an approximate zero of Υ, namely*

$$\|\Upsilon(x^*(\varepsilon);\varepsilon,q_1(0))\| \leq c_1\varepsilon^{r+1} \ ,$$

where c_1 is a positive constant depending on $\mathcal{U}$ and r. Assume that the matrix $M(\varepsilon)$ defined in (1.12) *is invertible and its eigenvalues satisfy*

$$|\lambda| \gtrsim \varepsilon^\alpha \ , \qquad \textit{for} \quad \lambda \in \mathrm{spec}(M(\varepsilon)) \quad \textit{with} \quad 2\alpha < r+1 \ .$$

Then, there exist $c_0 > 0$ and $\varepsilon^* > 0$ such that for any $0 \leq \varepsilon < \varepsilon^*$ there exists a unique $x^*_{\rm p.o.}(\varepsilon) = (q^*_{\rm p.o.}(\varepsilon), \hat{p}_{\rm p.o.}(\varepsilon)) \in \mathcal{U}$ which solves

$$\Upsilon(x^*_{\rm p.o.};\varepsilon,q_1(0)) = 0 \ , \qquad \left\|x^*_{\rm p.o.} - x^*\right\| \leq c_0\varepsilon^{r+1-\alpha} \ .$$

1.5 One-parameter families

Generically we expect that, in most cases, two normal form steps should be enough to get a clear insight into the degeneracy. In particular, with a second order approximation one can investigate whether one-parameter families which are solutions of (1.15), are destroyed or not. In the first case, the isolated solutions which survive to the breaking of the family are natural candidates for continuation, once (1.13) has been verified. In the second case, at least a third step of normalization is necessary, unless there are good reasons to believe that the whole family survives, due to the effect of some hidden symmetry of the model.

What we are going to develop is exactly the case when the first of (1.15) admits one-parameter families of solutions on the torus $\mathbb{T}^{n_1-1}$, which means that $\dim(\mathrm{Ker}(B_1)) = 1$. In this easier case (which represents the weakest degeneracy for B_1), under suitable conditions on the matrix M_0, it is possible to apply some results of perturbation theory of matrices to $M(\varepsilon)$ (see [92], Chap. IV, par. 1.4) in order to replace assumption (1.13) with a more accessible criterion. This allows to get a more applicable formulation of Theorem 1.2.1, which will be used in applications.

1.5.1 Some few facts on matrix perturbation theory

The degeneracy we are considering implies that $0 \in \mathrm{Spec}(-B_1T)$, with the geometric multiplicity equal to one ($m_g(0, -B_1T) = 1$). Let a_1 be the $(n_1 - 1)$-dimensional vector generating $\mathrm{Ker}(B_1)$. Let us introduce also f_1 as the embedding of a_1 into $\mathbb{R}^{2n_1-1}$, namely the $(2n_1 - 1)$ vector

$$f_1 = \begin{pmatrix} a_1 \\ \underline{0} \end{pmatrix} .$$

It is the vector generating $\ker(M_0)$. Indeed, in order to study the $\mathrm{Ker}(M_0)$, we have to solve

$$\begin{pmatrix} O & C_0T \\ -B_1T & -D_1T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_0Ty \\ -B_1Tx - D_1Ty \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which, due to the invertibility of C_0, gives $y = 0$, and thus $x \in \mathrm{Ker}(B_1)$. We have the following Lemma

Lemma 1.5.1 *Consider the matrix M_0, with $\mathrm{Ker}(M_0) = \mathrm{Span}(f_1)$. If the orthogonality condition*

$$\left\langle C_0^{-1} D_1^\top a_1, \begin{pmatrix} a_1 \\ 0 \end{pmatrix} \right\rangle = 0 \ , \tag{1.22}$$

is fulfilled (with $\begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ a n_1-dimensional vector), then the algebraic multiplicity of the zero eigenvalue is greater than two ($m_a(0, M_0) \geq 2$).

Proof. The statement can be derived investigating the Kernel of the adjoint matrix $M_0^\top$. It is easy to see that

$$\mathrm{Ker}(M_0^\top) = \mathrm{Span}\,(g) \ , \qquad g = \begin{pmatrix} -C_0^{-1} D_1^\top a_1 \\ a_1 \end{pmatrix}$$

and to deduce that the assumption (1.22) is equivalent to $\langle f_1, g \rangle = 0$, where the right hand vector in (1.22) is the n_1-dimensional vector built by complementing a_1 with one 0. The last, according to Lemma III, Chapter 1.16 of [92], is not compatible with $m_a(0, M_0) = 1$. Precisely, we can observe that the orthogonality condition between the two vectors allows to find a second generalized eigenvector f_2 for $\mathrm{Ker}(M_0)$, as a solution of $M_0 f_2 = f_1$. Indeed, the Fredholm alternative theorem guarantees the existence of f_2 under exactly the condition $\langle f_1, g \rangle = 0$. □

In order to determine the asymptotic behavior of the eigenvalues $\lambda(\varepsilon) \in \mathrm{spec}(M(\varepsilon))$, we make use of the fact that $dim(\mathrm{Ker}(M_0)) = 1$ and we consider the following expansion

$$M(\varepsilon) = M_0 + \varepsilon M_1 + \mathcal{O}(\varepsilon^2) = \begin{pmatrix} \varepsilon A_1 T & C_0T + \varepsilon C_1 T \\ -B_1T - \varepsilon B_2 T & -D_1T - \varepsilon D_2 T \end{pmatrix} + \mathcal{O}(\varepsilon^2) \ .$$

Then the following Lemma holds true (see [92], Chapter IV, § 1)

Lemma 1.5.2 *Let λ_0 be an eigenvalue of M_0 with $m_g(\lambda_0, M_0) = 1$ and $m_a(\lambda_0, M_0) = h \geq 2$ and let $f_1, \ldots, f_h$ be the generalized eigenvectors relative to λ_0, defined by the recursive scheme*

$$M_0 f_1 = \lambda_0 f_1, \quad M_0 f_2 = \lambda_0 f_2 + f_1, \ldots, M_0 f_h = \lambda_0 f_h + f_{h-1}.$$

Moreover, let $g_1, \ldots, g_h$ be the generalized eigenvectors for $M_0^\top$ relative to λ_0, such that

$$\langle f_j, g_i \rangle = \delta_{ji}, \quad \textit{with} \quad j, i = 1, \ldots, h$$

and define

$$\gamma = \langle M_1 f_1, g_h \rangle \ .$$

If $\gamma \neq 0$, then the h solutions $\lambda_j(\varepsilon)$ of the characteristic equation

$$\det(M(\varepsilon) - \lambda I) = 0$$

are given by

$$\lambda_j(\varepsilon) = \lambda_0 - (\varepsilon\gamma)_j^{1/h} + \mathcal{O}(\varepsilon^{2/h}) \ ,$$

where $(\varepsilon\gamma)_j^{1/h}$ are the h distinct roots of $\sqrt[h]{\varepsilon\gamma}$.

1.5.2 The special case of $m_a(0, M_0) = 2$.

We have to bound the inverse matrix $M^{-1}(\varepsilon)$, hence we are interested in the bifurcations of the zero eigenvalue, thus in the previous Lemma 1.5.2 we can take $\lambda_0 = 0$ and f_1 as the eigenvector generating $\mathrm{Ker}(M_0)$. Moreover, since

$$\begin{pmatrix} A_1 T & C_1 T \\ -B_2 T & -D_2 T \end{pmatrix} f_1 = \begin{pmatrix} A_1 T a_1 \\ -B_2 T a_1 \end{pmatrix} \ ,$$

the value of γ does not depend on the whole matrix M_1, but only on the blocks A_1 and B_2. The problem is further simplified when $m_a(0, M_0) = 2$: in this case g_2 coincides with g and γ reduces to

$$\gamma = \langle M_1 f_1, g_2 \rangle = \left\langle \begin{pmatrix} A_1 T a_1 & -B_2 T a_1 \end{pmatrix}, \begin{pmatrix} -C_0^{-1} D_1^\top a_1 \\ a_1 \end{pmatrix} \right\rangle = \left\langle -T\big(B_2 + D_1 C_0^{-1} A_1\big) a_1, a_1 \right\rangle \ .$$

Thus, under the easier condition

$$\gamma = \langle \big(B_2 + D_1 C_0^{-1} A_1\big) a_1, a_1 \rangle \neq 0 \ ,$$

Theorem 1.2.1 can be formulated as

Theorem 1.5.1 *Consider $\Upsilon = (\mathfrak{F}, \mathfrak{G})$ defined by (1.11) in a neighborhood of the point x^*, with $q^*(\varepsilon)$ defined by (1.10) and $r = 2$. Let $\dim(\mathrm{Ker}(B_1)) = 1$, a_1 being its generator. Assume also that $m_a(0, M_0) = 2$ and that*

$$\gamma \neq 0 \ . \tag{1.23}$$

Then, there exist positive constants c_0 and ε^* such that, for $|\varepsilon| < \varepsilon^*$ there exists a point $x^*_{\mathrm{p.o.}}(\varepsilon) \in \mathcal{U} \times \mathbb{T}^{n_1 - 1}$ which solves

$$\Upsilon(x^*_{\mathrm{p.o.}}; \varepsilon, q_1(0)) = 0 \ , \qquad \left\| x^*_{\mathrm{p.o.}} - x^* \right\| \leq c_0 \varepsilon^{3/2} \ .$$

In order to verify condition (1.23), the block matrices A_1 and B_2 are needed; as a consequence, the first order corrections to the generic Cauchy problem, $\hat{q}^{(1)}(t)$ and $\hat{p}^{(1)}(t)$ have to be derived. With a standard approach, as the one performed in [61], and after expanding in ε both the map Υ and the solution $q^*(\varepsilon) = q_0^* + \mathcal{O}(\varepsilon)$, one gets

$$
\begin{aligned}
\varepsilon A_1 T &= -\frac{T^2}{2} C_0 D_{q\hat{q}} f_0^{(2,1)}(q_0^*) + T D_{q\hat{p}} f_2^{(2,1)}(q_0^*) \\
-\varepsilon B_2 T &= -T D_q^3 f_0^{(2,1)}(q_0^*) q_1^* - \frac{T}{\varepsilon} D_q^2 f_0^{(2,2)}(q_0^*) \\
&\quad + \frac{T^2}{2\varepsilon} \left[D_{qp}^2 f_2^{(2,1)}(q_0^*) D_q^2 f_0^{(2,1)}(q_0^*) - D_q^2 f_0^{(2,1)}(q_0^*) D_{qp}^2 f_2^{(2,1)}(q_0^*) \right] \\
&\quad + \frac{T^3}{6\varepsilon} \left[D_q^2 f_0^{(2,1)}(q_0^*) C_0 D_q^2 f_0^{(2,1)}(q_0^*) \right] \ .
\end{aligned}
$$

Despite the formulation of Theorem 1.5.1 is simplified with respect to the abstract result stated in Theorem 1.2.1, it is evident from the above formulas that it may be a hard task to verify condition (1.23). However, if the original Hamiltonian is even in the angle variables, as often happens in models of weakly coupled anharmonic oscillators, then condition (1.23) can be further simplified if the solutions to be investigated are the in/out-of-phase solutions $q^* = 0, \pi$, as shown in the pedagogical example 3.2.1 in Chapter 3.

Moreover, in Chapter 2 a different approach, which exploits the connection of the matrix (1.12) with the monodromy matrix, will be described. It will allow to get a more accessible condition to be verified in applications for all degenerate scenarios.

1.6 Analytic estimates

The formal algorithm we have described results in a recursive scheme of estimates on the norms of the various functions. Prior to state the main results, we introduce some helpful technical tools.

1.6.1 Estimates for Poisson brackets and Lie series

Lemma 1.6.1 *Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $g \in \mathcal{P}_{2l}$ be an analytic function with bounded norm $\|g\|_1$. Then one has*

$$
\left\| \frac{\partial g}{\partial \hat{p}_j} \right\|_{1-d} \leq \frac{\|g\|_1}{d\rho} \ , \qquad \left\| \frac{\partial g}{\partial \hat{q}_j} \right\|_{1-d} \leq \frac{\|g\|_1}{ed\sigma} \ .
$$

Lemma 1.6.2 *Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $j \geq 1$. Then one has*

$$
\begin{aligned}
\left\| L^j_{\chi_0^{(r)}} f \right\|_{1-d-d'} &\leq \frac{j!}{e} \left(\frac{e\|X_0^{(r)}\|_{1-d'}}{d^2\rho\sigma} + \frac{e|\zeta^{(r)}|}{d\rho} \right)^j \|f\|_{1-d'} \ , \\
\left\| L^j_{\chi_2^{(r)}} f \right\|_{1-d-d'} &\leq \frac{j!}{e^2} \left(\frac{2e\|\chi_2^{(r)}\|_{1-d'}}{d^2\rho\sigma} \right)^j \|f\|_{1-d'} \ .
\end{aligned}
$$

1.6.2 Recursive scheme of estimates

We need to introduce a sequence of restrictions of the domain so as to apply Cauchy's estimate. Having fixed $d \in \mathbb{R}$, $0 < d \leq 1/4$, we consider a sequence $\delta_{r\geq 1}$ of positive real numbers satisfying

$$
\delta_{r+1} \leq \delta_r \ , \quad \sum_{r\geq 1} \delta_r \leq \frac{d}{2} \ ;
$$

thus the sequence δ_r has to satisfy the inequality $\delta_r < C/r$ for some $r > \overline{r}$ and $C \in \mathbb{R}$. Moreover, we introduce a further sequence $d_{r\geq 0}$ of real numbers recursively defined as

$$d_0 = 0 \ , \quad d_r = d_{r-1} + 2\delta_r \ .$$

In order to precisely state the iterative Lemma, we need to introduce the quantities Ξ_r, parameterized by the index r, as

$$\Xi_r = \max\left(\frac{eE}{\omega\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\omega\delta_r\rho\sigma}, \frac{2eE}{\omega\delta_r^2\rho\sigma}\right) \ .$$

Following the approach described in [35], the number of terms generated recursively by formulæ (1.20) and (1.21) is controlled by the two sequences $\{\nu_{r,s}\}_{r\geq 0\,,\, s\geq 0}$ and $\{\nu_{r,s}^{(\mathrm{I})}\}_{r\geq 1\,,\, s\geq 0}$ of integer numbers that are recursively defined as

$$\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \geq 0\,,\\
\nu_{r,s}^{(\mathrm{I})} &= \sum_{j=0}^{\lfloor s/r\rfloor} \nu_{r-1,r}^{j}\nu_{r-1,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,,\\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor} (3\nu_{r-1,r})^{j}\nu_{r,s-jr}^{(\mathrm{I})} && \text{for } r \geq 1\,,\ s \geq 0\,.
\end{aligned} \tag{1.24}$$

Let us stress that when $s < r$, the above (1.24) simplify as

$$\nu_{r,s}^{(\mathrm{I})} = \nu_{r-1,s} \ , \qquad \nu_{r,s} = \nu_{r,s}^{(\mathrm{I})} \ ,$$

namely

$$\nu_{r,s} = \nu_{r-1,s} = \ldots = \nu_{s,s} \ .$$

Lemma 1.6.3 *The sequence of positive integers $\{\nu_{r,s}\}_{r\geq 0\,,\, s\geq 0}$ defined in (1.24) is bounded by the exponential growth*

$$\nu_{r,s} \leq \nu_{s,s} \leq \frac{100^s}{20} \qquad \textit{for} \quad r \geq 0\,,\ s \geq 0 \ .$$

Let us introduce the quantities $b(\mathrm{I}; r, s, 2l)$ and $b(r, s, 2l)$ (r being a positive integer, while s and l are non-negative ones) that will be useful to control the exponents of the Ξ_r in the normalization procedure,

$$b(\mathrm{I}; r, s, 2l) = \begin{cases}
s & \text{if } r = 1\ ,\\
0 & \text{if } r \geq 2, s = 0\ ,\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 2 & \text{if } r \geq 2, 0 < s \leq r, l = 0\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 1 & \text{if } r \geq 2, r < s \leq 2r, l = 0\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 1 & \text{if } r \geq 2, 0 < s \leq r, l = 1\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor & \text{in the other cases}
\end{cases}$$

and

$$b(r, s, 2l) = \begin{cases}
0 & \text{if } r > 0, s = 0\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - w_{2l} & \text{if } r = 1, s > 0\ ,\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 2 & \text{if } r \geq 2, 0 < s \leq r, l = 0\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 1 & \text{if } r \geq 2, r < s \leq 2r, l = 0\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor - 1 & \text{if } r \geq 2, 0 < s \leq r, l = 1\\
3s - \lfloor\frac{s+r-1}{r}\rfloor - \lfloor\frac{s+r-2}{r}\rfloor & \text{in the other cases}
\end{cases}$$

with $w_0 = 2$, $w_2 = 1$ and $w_{2l} = 0$ for $l \geq 2$.

We are now ready to state the main Lemma collecting the estimates for the generic r-th normalization step of the normal form algorithm.

Lemma 1.6.4 *Consider a Hamiltonian $H^{(r-1)}$ expanded as in (1.19). Let $\chi_0^{(r)} = X_0^{(r)} + \langle\zeta^{(r)},\varphi\rangle$ and $\chi_2^{(r)}$ be the generating functions used to put the Hamiltonian in normal form at order r, then one has*

$$\|X_0^{(r)}\|_{1-d_{r-1}} \le \frac{1}{\omega}\nu_{r-1,r}\Xi_r^{3r-4}E\varepsilon^r \ ,$$

$$|\zeta^{(r)}| \le \frac{1}{4m\rho}\nu_{r-1,r}\Xi_r^{3r-3}E\varepsilon^r \ ,$$

$$\|\chi_2^{(r)}\|_{1-d_{r-1}-\delta_r} \le \frac{1}{\omega}3\nu_{r-1,r}\Xi_r^{3r-3}\frac{E}{4}\varepsilon^r \ .$$

The terms appearing in the expansion of $H^{(\mathrm{I};r-1)}$ in (1.20) are bounded as

$$\|f_{2l}^{(\mathrm{I;r-1,s})}\|_{1-d_{r-1}-\delta_r} \le \nu_{r,s}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r,s,2l)}\frac{E}{2^{2l}}\varepsilon^s \ .$$

The terms appearing in the expansion of $H^{(r)}$ in (1.21) are bounded as

$$\|f_{2l}^{(r,s)}\|_{1-d_r} \le \nu_{r,s}\Xi_r^{b(r,s,2l)}\frac{E}{2^{2l}}\varepsilon^s \ .$$

Remark 1.6.1 *Our estimates do not provide the convergence of the normal form algorithm. We are considering a completely resonant normal form, thus, if $\omega \neq 0$, the divisors $k_1\omega$ introduced in the solution of the homological equations cannot become arbitrarily small. In particular, we do not need a strong non-resonance condition on the frequencies. However, the restrictions of the domains, due to the Cauchy's estimates for derivatives, introduce the small denominators δ_r that actually accumulate to zero. We think it is reasonable that the normal form diverges, since looking for a convergent normal form which is valid for all possible periodic orbits could be too much to ask. It would mean to have a local normal form which gives a global result.*

All the Lemmas of this Section and Proposition 1.2.1 are just a particular case of those regarding the lower dimensional torus. Hence, refer to the Appendix (Section A.2) for the proofs, *mutatis mutandis*.

Chapter 2

Continuation of degenerate periodic orbits: lower dimensional tori

This Chapter is devoted to the extension to lower dimensional resonant tori of the algorithm developed in the first Chapter. Hence, consider a canonical system of differential equations with $n = n_1 + n_2$ d.o.f. and the analytic Hamiltonian

$$H(I, \varphi, x, y, \varepsilon) = H_0(I, x, y) + \varepsilon H_1(I, \varphi, x, y) \ , \tag{2.1}$$

where $I \in \mathcal{U}(I^*) \subset \mathbb{R}^{n_1}$, $\varphi \in \mathbb{T}^{n_1}$ are action-angle variables, $(x, y) \in \mathcal{V}(0) \subset \mathbb{R}^{2n_2}$ and ε is a small perturbation parameter. This Hamiltonian is a generalization of the Hamiltonian (1.1) considered for the full dimensional case. It is also assumed to be the perturbation of an integrable Hamiltonian $H_0 = \tilde{H}_0(I) + \hat{H}_0(x, y)$, where $\hat{H}_0(x, y)$ is at least quadratic. Furthermore, we assume that the Hamiltonian $\hat{H}_0$ has an elliptic equilibrium at the origin, namely

$$\hat{H}_0 = \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \Omega_j \frac{x_j^2 + y_j^2}{2} + h.o.t. \ .$$

More generally, we can consider the Hamiltonian

$$H(I, \varphi, x, y, \varepsilon) = \tilde{H}_0(I) + \hat{H}_0(x, y) + \varepsilon H_1(I, \varphi, x, y) + \varepsilon^2 H_2(I, \varphi, x, y) + \ldots \ .$$

Introduce a set of indexes $\mathcal{J}$ and a subset of indexes $\mathcal{I} \subset \mathcal{J}$, with $|\mathcal{I}| = n_1$ and $|\mathcal{J} \setminus \mathcal{I}| = n_2$. Select now a completely resonant elliptic lower dimensional torus of H_0, setting $I_j = I_j^* \neq 0$ for $j \in \mathcal{I}$ and $x_j = y_j = 0$ for $j \in \mathcal{J} \setminus \mathcal{I}$. We also assume the same analytical setting of the Chapter 1.

We want to investigate the problem of the continuation of periodic orbits which survive to the breaking of a completely resonant n_1-dimensional torus I^* of (2.1). A typical example is provided by physical models described by a Hamiltonian (2.1) made by identical and weakly coupled nonlinear oscillators (see references in Chapter 1 and the editorial review [68] on Hamiltonian Lattices), with n_1 ones that have been excited and oscillate periodically with the same frequencies $\hat{\omega}(I^*)$ and n_2 ones are at rest. In this context, spatially localized time-periodic solutions, the so-called multibreathers, are relevant objects in the investigation of phenomena of confinement and transfer of the energy along the chain. Their continuation for the perturbed system has been tackled in the non-degenerate case by means of the effective Hamiltonian method cited in Chapter 1.

The normal form procedure that will be presented in this Chapter allows to extend the above mentioned method and enables to face degenerate scenarios which require a different and more powerful approach. In addition, it permits to investigate also the linear stability of continued periodic orbits. Indeed, the normal form step described will include three stages which suffice to achieve an abstract result on continuation of periodic orbits. However, the algorithm will be refined in order to add a fourth stage which enables to get easier conditions to be verified in applications, and which, together with a fifth one, is necessary to study the approximate linear stability of periodic orbits. Then, effective linear stability can be inferred with further assumptions on the spectrum of the approximate periodic orbits.

2.1 Main results

We now perform the canonical change of coordinates

$$x_j = \frac{1}{\sqrt{2}}(\xi_j + \mathbf{i}\eta_j), \quad y_j = \frac{\mathbf{i}}{\sqrt{2}}(\xi_j - \mathbf{i}\eta_j), \qquad j \in \mathcal{J} \setminus \mathcal{I}\ ,$$

so that

$$\sum_{j\in\mathcal{J}\setminus\mathcal{I}} \Omega_j \frac{x_j^2 + y_j^2}{2} = \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j \xi_j \eta_j\ .$$

In addition, expanding (2.1) in power series of the translated actions $J = I - I^*$, one has

$$\begin{aligned} H^{(0)} = {}& \langle \hat{\omega}, J\rangle + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(J,\xi,\eta) \\ & + f_0^{(0,1)}(\varphi) + f_1^{(0,1)}(\varphi,\xi,\eta) + f_2^{(0,1)}(\varphi,J,\xi,\eta) + f_3^{(0,1)}(\varphi,J,\xi,\eta) + f_4^{(0,1)}(\varphi,J,\xi,\eta) \\ & + \sum_{s>1} f_0^{(0,s)}(\varphi) + \sum_{s>1} f_1^{(0,s)}(\varphi,\xi,\eta) + \sum_{s>1} f_2^{(0,s)}(\varphi,J,\xi,\eta) + \sum_{s>1} f_3^{(0,s)}(\varphi,J,\xi,\eta) \\ & + \sum_{s>1} f_4^{(0,s)}(\varphi,J,\xi,\eta) \\ & + \sum_{s>0}\sum_{\ell>4} f_\ell^{(0,s)}(\varphi,J,\xi,\eta)\ , \end{aligned}$$

where $f_\ell^{(0,s)} \in \mathcal{P}_\ell$ is a function of order $\mathcal{O}(\varepsilon^s)$.

Introducing the resonant variables $(\hat{q},\hat{p})$ in place of (φ, J), the Hamiltonian can be written in the form

$$\begin{aligned} H^{(0)} = {}& \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(\hat{p},\xi,\eta) \\ & + f_0^{(0,1)}(\hat{q}) + f_1^{(0,1)}(\hat{q},\xi,\eta) + f_2^{(0,1)}(\hat{q},\hat{p},\xi,\eta) + f_3^{(0,1)}(\hat{q},\hat{p},\xi,\eta) + f_4^{(0,1)}(\hat{q},\hat{p},\xi,\eta) \\ & + \sum_{s>1} f_0^{(0,s)}(\hat{q}) + \sum_{s>1} f_1^{(0,s)}(\hat{q},\xi,\eta) + \sum_{s>1} f_2^{(0,s)}(\hat{q},\hat{p},\xi,\eta) + \sum_{s>1} f_3^{(0,s)}(\hat{q},\hat{p},\xi,\eta) \\ & + \sum_{s>1} f_4^{(0,s)}(\hat{q},\hat{p},\xi,\eta) \\ & + \sum_{s>0}\sum_{\ell>4} f_\ell^{(0,s)}(\hat{q},\hat{p},\xi,\eta) \end{aligned} \tag{2.2}$$

where $f_\ell^{(0,s)} \in \mathcal{P}_\ell$ and it is a function of order $\mathcal{O}(\varepsilon^s)$.

In addition, let us assume that

(H1) there exists a positive constant m such that for every $v \in \mathbb{R}^{n_1}$ one has

$$m\sum_{i=1}^{n_1} |v_i| \le \sum_{i=1}^{n_1} |\sum_{j=1}^{n_1} C_{0,ij} v_j|\ , \quad \text{where} \quad C_0 = D^2_{\hat{p}} f_4^{(0,0)}\ ; \tag{2.3}$$

(H2) the terms appearing in the expansion of the Hamiltonian satisfy

$$\|f_\ell^{(0,s)}\|_1 \le \frac{E}{2^\ell}\varepsilon^s \ , \quad \text{with} \quad E > 0. \tag{2.4}$$

(H3) the frequencies satisfy the conditions of non-resonance

$$k_1\omega \pm \Omega_j \neq 0, \qquad k_1 \in \mathbb{Z} \ , \tag{2.5}$$
$$k_1\omega \pm \Omega_l \pm \Omega_k \neq 0, \qquad k_1 \in \mathbb{Z}\setminus\{0\} \ , \tag{2.6}$$

known as first and second Melnikov conditions.

Let us remark that the hypotheses (H1) and (H2) are the same as the maximal dimension case. The third assumption (H3) ensures absence of resonances between the periodic motion and the transverse linear oscillations. The first Melnikov condition is enough to obtain continuation of periodic orbits, the second one is needed to study the linear stability of the orbits.

We now state our main result concerning the normal form algorithm.

Proposition 2.1.1 *Consider a Hamiltonian $H^{(0)}$ expanded as in (2.2) that is analytic in a domain $\mathcal{D}_{\rho,\sigma}$. Let us assume the hypotheses (H1), (H2), (H3). Then, for every positive integer r there is a positive ε_r^* such that for $0 \le \varepsilon < \varepsilon_r^*$ there exists an analytic canonical transformation $\Phi^{(r)}$ satisfying*

$$\mathcal{D}_{\frac{1}{4}(\rho,\sigma,R)} \subset \Phi^{(r)}\Big(\mathcal{D}_{\frac{1}{2}(\rho,\sigma,R)}\Big) \subset \mathcal{D}_{\frac{3}{4}(\rho,\sigma,R)} \tag{2.7}$$

such that the Hamiltonian $H^{(r)} = H^{(0)} \circ \Phi^{(r)}$ has the following expansion

$$\begin{aligned} H^{(r)}(\hat{q},\hat{p},\xi,\eta;q^*) &= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(r,0)}(\hat{p},\xi,\eta) \\ &+ \sum_{s=1}^{r}\Big(f_0^{(r,s)}(q;q^*) + f_2^{(r,s)}(q,\hat{p},\xi,\eta;q^*) + f_3^{(r,s)}(\hat{q},\xi,\eta;q^*) + f_4^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)\Big) \\ &+ \sum_{s>r}\sum_{\ell=0}^{4} f_\ell^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*) + \sum_{s>0}\sum_{\ell>4} f_\ell^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*) \ , \end{aligned} \tag{2.8}$$

where q^* is a fixed but arbitrary parameter and $f_\ell^{(r,s)} \in \mathcal{P}_\ell$ is a function of order $\mathcal{O}(\varepsilon^s)$. The Hamiltonian (2.8) is said to be in normal form up to order r since for $s \le r$ satisfies:

1. *$f_0^{(r,s)}(q;q^*)$ do not depend on the fast angle q_1;*
2. *$f_1^{(r,s)}(\hat{q},\xi,\eta;q^*)$ have been completely removed from (2.2);*
3. *$f_2^{(r,s)}(q,\hat{p},\xi,\eta;q^*)$ do not depend on q_1 and, evaluated at $\xi=\eta=0$ and $q=q^*$, satisfy*

$$f_2^{(r,s)}(q^*,\hat{p},0,0;q^*) = 0 \ ;$$

4. *$f_3^{(r,s)}(\hat{q},\xi,\eta;q^*)$ do not depend on the actions $\hat{p}$;*
5. *$f_4^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)$, evaluated at $\xi=\eta=0$, do not depend on the fast angle q_1.*

The Hamilton equations associated to the truncated normal form, i.e., neglecting terms of order $\mathcal{O}(\varepsilon^{r+1})$, once evaluated at $x^* = (q=q^*, \hat{p}=0, \xi=0, \eta=0)$, read

$$\dot{q}_1 = \omega \ , \qquad \dot{q} = 0 \ , \qquad \dot{p}_1 = 0 \ , \qquad \dot{p} = -\sum_{s=1}^{r}\nabla_q f_0^{(r,s)} \ , \qquad \dot{\xi} = 0 \ , \qquad \dot{\eta} = 0 \ .$$

Hence, if
$$\sum_{s=1}^{r} \nabla_q f_0^{(r,s)}\big|_{q=q^*} = 0 \ , \tag{2.9}$$
then $q_1 = q_1(0)$, $q = q^*$, $p_1 = 0$, $p = 0$, $\xi = 0$, $\eta = 0$ is the initial datum of a periodic orbit with frequency ω for the truncated normal form[1].

In order to investigate the continuation of the approximate periodic orbit, we introduce, once again, the variation over the T-period map $\Upsilon : \mathcal{U}(x^*) \subset \mathbb{R}^{2n-1} \to \mathcal{V}(x^*) \subset \mathbb{R}^{2n-1}$, a smooth function of the variables $x = (q, \hat{p}, \xi, \eta)$, parameterized by the initial phase $q_1(0)$ and the parameter ε, namely[2]
$$\Upsilon(x(0); \varepsilon, q_1(0)) = \begin{pmatrix} \mathfrak{F}(x(0); \varepsilon, q_1(0)) \\ \mathfrak{G}(x(0); \varepsilon, q_1(0)) \\ \mathfrak{R}(x(0); \varepsilon, q_1(0)) \\ \mathfrak{S}(x(0); \varepsilon, q_1(0)) \end{pmatrix} := \begin{pmatrix} \hat{q}(T) - \hat{q}(0) - \Lambda T \\ p(T) - p(0) \\ \xi(T) - \xi(0) \\ \eta(T) - \eta(0) \end{pmatrix} \ , \tag{2.10}$$
with $\Lambda = (\omega, 0) \in \mathbb{R}^{n_1}$ and where we have neglected the equation for p_1, due to conservation of the energy.

A true periodic orbit, close to the approximate one, is identified by an initial datum $x^*_{\rm p.o.} = (q^*_{\rm p.o.}, \hat{p}_{\rm p.o.}, \xi_{\rm p.o.}, \eta_{\rm p.o.}) \in \mathcal{U}(x^*)$ such that
$$\Upsilon(x^*_{\rm p.o.}; \varepsilon, q_1(0)) = 0 \ .$$
Therefore, in order to prove its existence, we apply the Newton-Kantorovich method, under the assumption that the Jacobian matrix
$$M(\varepsilon) = D_{x(0)}\Upsilon(x^*; \varepsilon, q_1(0)) = N(\varepsilon) + \mathcal{O}(\varepsilon^{r+1}) \tag{2.11}$$
is invertible and its eigenvalues are not too small w.r.t. ε^{r+1}. The $(2n-1)$-dimensional square matrix $N(\varepsilon)$ has a block diagonal structure, which will be revealed in Section 2.3, with the $(2n_1-1)$-dimensional square matrix $\tilde{N}(\varepsilon)$ as first block.

As a consequence, we can state the following result

Theorem 2.1.1 *Consider the map Υ defined in (2.10) in a neighbourhood of the torus $\hat{p} = 0, \xi = 0, \eta = 0$ and let $x^*(\varepsilon) = (q^*(\varepsilon), 0, 0, 0)$, with $q^*(\varepsilon)$ satisfying (2.9), an approximate zero of Υ, namely*
$$\|\Upsilon(x^*(\varepsilon); \varepsilon, q_1(0))\| \leq c_1 \varepsilon^{r+1} \ ,$$
where c_1 is a positive constant depending on $\mathcal{U}$ and r. Assume that the matrix $N(\varepsilon)$ defined in (2.11) is invertible and its eigenvalues satisfy
$$|\lambda| \gtrsim \varepsilon^{\alpha} \ , \qquad \text{for} \quad \lambda \in \sigma(\tilde{N}(\varepsilon)) \quad \text{with} \quad 2\alpha < r+1 \ , \tag{2.12}$$
where $\tilde{N}(\varepsilon)$ stands for the first block of the matrix $N(\varepsilon)$. Then, there exist $c_0 > 0$ and $\varepsilon^* > 0$ such that for any $0 \leq \varepsilon < \varepsilon^*$ there exists a unique $x^*_{\rm p.o.}(\varepsilon) = (q^*_{\rm p.o.}(\varepsilon), \hat{p}_{\rm p.o.}(\varepsilon), \xi_{\rm p.o.}(\varepsilon), \eta_{\rm p.o.}(\varepsilon)) \in \mathcal{U}$ which solves
$$\Upsilon(x^*_{\rm p.o.}; \varepsilon, q_1(0)) = 0 \ ,$$
$$\left\|x^*_{\rm p.o.} - x^*\right\| \leq c_0 \varepsilon^{r+1-\alpha} \ .$$

This Theorem generalizes the result obtained by means of the effective Hamiltonian method for chains of weakly coupled anharmonic oscillators. In particular, in the next Section we will prove that the first step of our normal form procedure allows to reproduce this non-degenerate

[1] Let us remark that $q^*(\varepsilon)$ is analytic in ε. Indeed, $(\omega t + q_1(0), x^*)$ is a periodic solution of an analytic Hamiltonian system. Hence, it has to be analytic in ε.

[2] Let us stress that, differently from the first Chapter, the actions p have not been scaled by ε in Υ. The scaling will be necessary only in the first normalization step in order to obtain continuation of non-degenerate periodic orbits by means of the implicit function theorem.

result. Moreover, comparisons with the literature are also made; more precisely, we will discuss differences with respect to the result in [55].

In order to study the linear stability of continued periodic orbits we have to consider the quadratic Hamiltonian of the normal form that gives the linear approximation of the dynamics close to the approximate periodic orbit. Its canonical linear vector field can be represented by a block diagonal matrix $L(\varepsilon)$, whose spectrum provides information about the approximate linear stability. The effective linear stability of the true periodic orbit can be easily inferred from the approximate one in the generic case of distinct eigenvalues, under suitable assumptions on the spectrum of the first $(2n_1 - 1)$-dimensional block $L_{11}(\varepsilon)$ of the matrix $L(\varepsilon)$. One of the main ingredients which allows to get the result about effective linear stability will be the monodromy matrix of the periodic orbit. As we will see in Section 2.3, it also corresponds to the differential of the flow at the period T w.r.t. the initial datum of the periodic orbits. This results in a strong connection with the matrices $M(\varepsilon)$ and $L(\varepsilon)$ for the approximate periodic orbit. By considering the true periodic orbit given by $x^*_{p.o.}$ and the Hamiltonian $H^{(r)}$, the associated monodromy matrix turns out to be equal to $\exp(L(\varepsilon)T) + \mathcal{O}(\varepsilon^{r+1-\alpha})$ with α as in Theorem 2.1.1. Its spectrum is the union of two different components, one of which, Σ_{11}, is close to $\Sigma(\exp(L_{11}(\varepsilon)T))$. Hence, the following Theorem and its Corollary can be stated:

Theorem 2.1.2 *Assume that $L_{11}(\varepsilon)$ has $2n_1 - 2$ distinct non-zero eigenvalues and let $\tilde{c} > 0$ and $\beta < r + 1 - \alpha$, with $2\alpha < r + 1$ as in Theorem 2.1.1, be such that*

$$|\lambda_j - \lambda_k| > \tilde{c}\varepsilon^{\beta} \ , \qquad \text{for all} \quad \lambda_j, \lambda_k \in \Sigma(L_{11}(\varepsilon)) \setminus \{0\} \ . \tag{2.13}$$

Then there exists $\varepsilon^* > 0$ such that if $|\varepsilon| < \varepsilon^*$ and $\mu = e^{\lambda T} \in \Sigma(\exp(L_{11}(\varepsilon)T))$, there exists one eigenvalue $\nu \in \Sigma_{11}$ inside the complex disk $D_\varepsilon(\mu) = \left\{z \in \mathbb{C} \ : \ |z - \mu| < c\varepsilon^{r+1-\alpha}\right\}$, with $c > 0$ a suitable constant independent of μ.

Corollary 2.1.1 *Under the assumptions of Theorem 2.1.2 the periodic orbit $x^*_{\rm p.o.}$ is linearly stable if and only if the same holds for the approximate periodic orbit x^*. In the unstable case, the number of hyperbolic directions of the periodic orbit $x^*_{\rm p.o.}$ is the same as for x^*.*

The stability results are achieved in Section 2.4.

Let me stress that the structure of the monodromy matrix also enables to more easily verify the condition (2.12), giving a more applicable criterion for the continuation with respect to the one in the first Chapter.

2.2 Normal form algorithm

In this section, by using the formalism of Lie series, we detail the first step of the normal form algorithm that takes the Hamiltonian (2.2) and brings it into normal form up to order 1. Afterward, we will describe the generic r-th normalization step.

The transformation at step r is generated via composition of four Lie series of the form

$$\exp(L_{\chi_4^{(r)}}) \circ \exp(L_{\chi_3^{(r)}}) \circ \exp(L_{\chi_2^{(r)}}) \circ \exp(L_{\chi_1^{(r)}}) \circ \exp(L_{\chi_0^{(r)}}) \ ,$$

where $\chi_0^{(r)} = X_0^{(r)} + \langle \zeta^{(r)}, \hat{q} \rangle$, with $\zeta^{(r)} \in \mathbb{R}^{n_1}$ and $X_0^{(r)} \in \mathcal{P}_0$, $\chi_1^{(r)} \in \mathcal{P}_1$, $\chi_2^{(r)} \in \mathcal{P}_2$, $\chi_3^{(r)} \in \mathcal{P}_3$, $\chi_4^{(r)} \in \mathcal{P}_4$ of order $\mathcal{O}(\varepsilon^r)$. The generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$, $\chi_3^{(r)}$ and $\chi_4^{(r)}$ are unknowns to be determined so that the transformed Hamiltonian is in normal form up to order r. Once again, we denote by $L_g \cdot$ the Poisson bracket $\{\cdot, g\}$.

2.2.1 First normalization step

Consider the starting Hamiltonian (2.2). We now describe the five stages of the first normalization step. We remark that the first three stages are sufficient to study the continuation of periodic orbits. The fourth stage allows to simplify the conditions that have to be verified to obtain continuation. Moreover, part of the third stage and the last two stages are needed in order to investigate linear stability of periodic orbits.

First stage of the first normalization step

As in the maximal dimension case, we put the term $f_0^{(0,1)}$ in normal form. We determine the generating function

$$\chi_0^{(1)}(\hat{q}) = X_0^{(1)}(\hat{q}) + \langle\zeta^{(1)}, \hat{q}\rangle \qquad \text{with} \qquad \zeta^{(1)} \in \mathbb{R}^{n_1} \ ,$$

belonging to $\mathcal{P}_0$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equations

$$\begin{aligned}
&L_{X_0^{(1)}}\omega p_1 + f_0^{(0,1)} = \langle f_0^{(0,1)}\rangle_{q_1} \ , \\
&L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)}\Big|_{\xi=\eta=0} + \Big\langle f_2^{(0,1)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\Big\rangle_{q_1} = 0 \ .
\end{aligned}$$

Considering the Taylor-Fourier expansion

$$f_0^{(0,1)}(\hat{q}) = \sum_k c_{0,0,0,k}^{(0,1)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ ,$$

we get

$$X_0^{(1)}(\hat{q}) = \sum_{k_1\neq 0} \frac{c_{0,0,0,k}^{(0,1)}}{\mathbf{i}k_1\omega} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ .$$

The translating vector $\zeta^{(1)}$ is the solution of the following linear system

$$\sum_j C_{0,ij}\zeta_j^{(1)} = \frac{\partial}{\partial\hat{p}_i}\Big\langle f_2^{(0,1)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\Big\rangle_{q_1} \ .$$

The transformed Hamiltonian is calculated as

$$\begin{aligned}
H^{(\mathrm{I};0)} &= \exp\Big(L_{\chi_0^{(1)}}\Big) H^{(0)} = \\
&= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\
&\quad + f_0^{(\mathrm{I};0,1)} + f_1^{(\mathrm{I};0,1)} + f_2^{(\mathrm{I};0,1)} + f_3^{(\mathrm{I};0,1)} + f_4^{(\mathrm{I};0,1)} \\
&\quad + \sum_{s>1} f_0^{(\mathrm{I};0,s)} + \sum_{s>1} f_1^{(\mathrm{I};0,s)} + \sum_{s>1} f_2^{(\mathrm{I};0,s)} + \sum_{s>1} f_3^{(\mathrm{I};0,s)} + \sum_{s>1} f_4^{(\mathrm{I};0,s)} \\
&\quad + \sum_{s\geq 0}\sum_{\ell>2} f_\ell^{(\mathrm{I};0,s)} \ .
\end{aligned}$$

The functions $f_\ell^{(\mathrm{I};0,s)}$ are recursively defined as

$$\begin{aligned}
f_0^{(\mathrm{I};0,1)} &= \langle f_0^{(0,1)}\rangle_{q_1} \ , \\
f_\ell^{(\mathrm{I};0,s)} &= \sum_{j=0}^{s} \frac{1}{j!} L^j_{\chi_0^{(1)}} f_{\ell+2j}^{(0,s-j)} \ , \qquad && \text{for } \ell = 0,\ s \neq 1 \ , \\
& && \text{or } \ell \geq 1,\ s \geq 0 \ ,
\end{aligned}$$

with $f_\ell^{(\mathrm{I};0,s)} \in \mathcal{P}_\ell$.

Second stage of the first normalization step

We now put in normal form the term $f_1^{(\mathrm{I};0,1)}$, by removing the linear terms in the transversal variables ξ, η from the Hamiltonian. We determine the generating function $\chi_1^{(1)}$, belonging to $\mathcal{P}_1$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equation

$$L_{\chi_1^{(1)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_1^{(\mathrm{I};0,1)} = 0 \ . \tag{2.14}$$

Considering again the Taylor-Fourier expansion

$$f_1^{(\mathrm{I};0,1)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{I};0,1)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\xi^{m_1}\eta^{m_2} \ ,$$

we obtain

$$\chi_1^{(1)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{I};0,1)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\,\Omega\rangle\big]} \ .$$

with $\Omega\in\mathbb{R}^{n_2}$.

Remark 2.2.1 *Due to the first Melnikov condition* (2.5) *and to the constraint* $|m_1|+|m_2|=1$, *the denominator cannot vanish. Indeed,* $\sum_i (m_{1_i}-m_{2_i}) = \pm 1$, *from which one obtains*

$$k_1\omega + \langle m_1-m_2,\Omega\rangle = k_1\omega + \sum_i \Omega_i\,(m_{1_i}-m_{2_i}) = k_1\omega \pm \Omega_j \neq 0 \quad \textit{for some } j\in\mathcal{J}\setminus\mathcal{I} \ .$$

The transformed Hamiltonian is computed as

$$\begin{aligned} H^{(\mathrm{II};0)} &= \exp\Big(L_{\chi_1^{(1)}}\Big)H^{(\mathrm{I};0)} = \\ &= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\ &\quad + f_0^{(\mathrm{II};0,1)} + f_2^{(\mathrm{II};0,1)} + f_3^{(\mathrm{II};0,1)} + f_4^{(\mathrm{II};0,1)} \\ &\quad + \sum_{s>1} f_0^{(\mathrm{II};0,s)} + \sum_{s>1} f_1^{(\mathrm{II};0,s)} + \sum_{s>1} f_2^{(\mathrm{II};0,s)} + \sum_{s>1} f_3^{(\mathrm{II};0,s)} + \sum_{s>1} f_4^{(\mathrm{II};0,s)} \\ &\quad + \sum_{s\geq 0}\sum_{\ell>2} f_\ell^{(\mathrm{II};0,s)} \ , \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} f_1^{(\mathrm{II};0,1)} &= 0 \ , \\ f_0^{(\mathrm{II};0,2)} &= f_0^{(\mathrm{I};0,2)} + L_{\chi_1^{(1)}} f_1^{(\mathrm{I};0,1)} + \frac{1}{2} L_{\chi_1^{(1)}}\Big(L_{\chi_1^{(1)}} f_2^{(\mathrm{I};0,0)}\Big) = \\ &= f_0^{(\mathrm{I};0,2)} + \frac{1}{2} L_{\chi_1^{(1)}} f_1^{(\mathrm{I};0,1)} \ , \\ f_\ell^{(\mathrm{II};0,s)} &= \sum_{j=0}^{s} \frac{1}{j!} L_{\chi_1^{(1)}}^j f_{\ell+j}^{(\mathrm{I};0,s-j)}, \qquad \text{for } \ell=0,\ s\neq 2 \ , \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell=1,\ s\neq 1 \ , \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell\geq 2,\ s\geq 0 \ . \end{aligned}$$

where we have used (2.14).

Third stage of the first normalization step

Our goal is to put in normal form the term $f_2^{(\mathrm{II};0,1)}$, by averaging it with respect to the fast angle q_1. We determine the generating function $\chi_2^{(1)}$, belonging to $\mathcal{P}_2$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equation

$$L_{\chi_2^{(1)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_2^{(\mathrm{II};0,1)} = \langle f_2^{(\mathrm{II};0,1)}\rangle_{q_1} \ . \tag{2.16}$$

Hence, considering the Taylor-Fourier expansion

$$f_2^{(\mathrm{II};0,1)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k}} c_{l,0,0,k}^{(\mathrm{II};0,1)}\hat{p}^l\exp(\mathbf{i}\langle k,\,\hat{q}\rangle) + \sum_{\substack{|m_1|+|m_2|=2\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{II};0,1)}\exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\xi^{m_1}\eta^{m_2} \ ,$$

we get

$$\chi_2^{(1)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{II};0,1)}\hat{p}^l\exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega} + \sum_{\substack{|m_1|+|m_2|=2\\ k_1\neq 0}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{II};0,1)}\exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\,\Omega\rangle\big]} \ .$$

Remark 2.2.2

1. *Notice that the term* $f_2^{(\mathrm{II};0,1)}$*, being either linear in the actions or quadratic in the transversal variables, can be rewritten as* $\tilde{f}_2^{(\mathrm{II};0,1)}(\hat{p},\hat{q}) + \hat{f}_2^{(\mathrm{II};0,1)}(\hat{q},\xi,\eta)$*. As a consequence,* $\chi_2^{(1)}$ *can be split in two terms in the same way.*
2. *We can observe that, in order to obtain the continuation of periodic orbits, it suffices to average the term* $\tilde{f}_2^{(\mathrm{II};0,1)}(\hat{p},\hat{q})$ *w.r.t. the fast angle* q_1*. We also added an average for the term* $\hat{f}_2^{(\mathrm{II};0,1)}(\hat{q},\xi,\eta)$ *so as to study the linear stability of periodic orbits.*
3. *Due to the second Melnikov condition* (2.6) *and to the constraint* $|m_1|+|m_2|=2$*, the denominator cannot vanish. Indeed,*

$$k_1\omega+\langle m_1-m_2,\Omega\rangle = k_1\omega+\sum_i \Omega_i\,(m_{1_i}-m_{2_i}) = \begin{cases} k_1\omega\pm 2\Omega_l \\ k_1\omega\pm\Omega_l\pm\Omega_k \end{cases} \neq 0 \quad \textit{for some } l,k\in\mathcal{J}\setminus\mathcal{I} \ .$$

The transformed Hamiltonian is computed as

$$H^{(\mathrm{III},0)} = \exp\Big(L_{\chi_2^{(1)}}\Big)\,H^{(\mathrm{II};0)}$$

and is given in the form (2.15), replacing the upper index II by III, with

$$\begin{aligned}
f_2^{(\mathrm{III};0,1)} &= \langle f_2^{(\mathrm{II};0,1)}\rangle_{q_1} \ ,\\
f_2^{(\mathrm{III};0,s)} &= \frac{1}{(s-1)!}L_{\chi_2^{(1)}}^{s-1}\Big(f_2^{(\mathrm{II};0,1)} + \frac{1}{s}L_{\chi_2^{(1)}}f_2^{(\mathrm{II};0,0)}\Big)\\
&\quad + \sum_{j=0}^{s-2}\frac{1}{j!}L_{\chi_2^{(1)}}^{j}f_2^{(\mathrm{II};0,s-j)} =\\
&= \frac{1}{(s-1)!}L_{\chi_2^{(1)}}^{s-1}\Big(\frac{1}{s}\langle f_2^{(\mathrm{II};0,1)}\rangle_{q_1} + \frac{s-1}{s}f_2^{(\mathrm{II};0,1)}\Big)\\
&\quad + \sum_{j=0}^{s-2}\frac{1}{j!}L_{\chi_2^{(1)}}^{j}f_2^{(\mathrm{II};0,s-j)} \ , \qquad \text{for } s\neq 1 \ ,\\
f_\ell^{(\mathrm{III};0,s)} &= \sum_{j=0}^{s}\frac{1}{j!}L_{\chi_2^{(1)}}^{j}f_\ell^{(\mathrm{II};0,s-j)} \ , \qquad \text{for } \ell\neq 2,\ s\geq 0 \ ,
\end{aligned}$$

where we have exploited the homological equation (2.16).

Fourth stage of the first normalization step

We now put in normal form the term $\tilde{f}_3^{(\mathrm{III};0,1)} = f_3^{(\mathrm{III};0,1)} - f_3^{(\mathrm{III};0,1)}\Big|_{\hat{p}=0}$, by removing the cubic terms which depend both on the actions and on the transversal variables ξ, η. We determine the generating function $\chi_3^{(1)}$, belonging to $\mathcal{P}_3$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equation

$$L_{\chi_3^{(1)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + \tilde{f}_3^{(\mathrm{III};0,1)} = 0 \ . \tag{2.17}$$

Considering again the Taylor-Fourier expansion

$$\tilde{f}_3^{(\mathrm{III};0,1)}(\hat{q},\hat{p},\xi,\eta) = \sum_{\substack{|l|=1\\ |m_1|+|m_2|=1\\ k}} c_{l,m_1,m_2,k}^{(\mathrm{III};0,1)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\hat{p}^l\xi^{m_1}\eta^{m_2} \ ,$$

we obtain

$$\chi_3^{(1)}(\hat{q},\hat{p},\xi,\eta) = \sum_{\substack{|l|=1\\ |m_1|+|m_2|=1\\ k}} \frac{c_{l,m_1,m_2,k}^{(\mathrm{III};0,1)}\ \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\, \hat{p}^l\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1 - m_2,\, \Omega\rangle\big]} \ .$$

with $\Omega \in \mathbb{R}^{n_2}$.

Remark 2.2.3 *Let us stress that we do not need to put the term* $f_3^{(\mathrm{III};0,1)}\Big|_{\hat{p}=0} = \hat{f}_3^{(\mathrm{III};0,1)}(\hat{q},\xi,\eta)$ *in normal form in order to study the linear stability of periodic orbits. Indeed, it does not affect the linearization of the system in normal form.*

The transformed Hamiltonian is calculated as

$$H^{(\mathrm{IV};0)} = \exp\Big(L_{\chi_3^{(1)}}\Big)\, H^{(\mathrm{III};0)} \ ,$$

and is given in the form (2.15), replacing the upper index II by IV, with

$$\begin{aligned}
f_3^{(\mathrm{IV};0,1)} &= f_3^{(\mathrm{III};0,1)}\Big|_{\hat{p}=0} \ ,\\
f_4^{(\mathrm{IV};0,2)} &= f_4^{(\mathrm{III};0,2)} + L_{\chi_3^{(1)}} f_3^{(\mathrm{III};0,1)} + \frac{1}{2}L^2_{\chi_3^{(1)}} f_2^{(\mathrm{III};0,0)} =\\
&= f_4^{(\mathrm{III};0,2)} + \frac{1}{2}L_{\chi_3^{(1)}} f_3^{(\mathrm{III};0,1)} + \frac{1}{2}L_{\chi_3^{(1)}} f_3^{(\mathrm{III};0,1)}\Big|_{\hat{p}=0} \ ,\\
f_\ell^{(\mathrm{IV};0,s)} &= \sum_{j=0}^{s}\frac{1}{j!}L^j_{\chi_3^{(1)}} f_{\ell-j}^{(\mathrm{III};0,s-j)}, \qquad \text{for } \ell=3,\ s\neq 1 \ ,\\
&\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell=4,\ s\neq 2 \ ,\\
&\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell\neq 3,4,\ s\geq 0 \ .
\end{aligned}$$

where we have used (2.17).

Fifth stage of the first normalization step

Our aim is to put in normal form the term $f_4^{(\mathrm{IV};0,1)}\Big|_{\xi=\eta=0}$, by averaging it with respect to the fast angle q_1. We determine the generating function $\chi_4^{(1)}$, belonging to $\mathcal{P}_4$ and of order $\mathcal{O}(\varepsilon)$, by solving the homological equation

$$L_{\chi_4^{(1)}}\omega p_1 + f_4^{(\mathrm{IV};0,1)}\Big|_{\xi=\eta=0} = \langle f_4^{(\mathrm{IV};0,1)}\Big|_{\xi=\eta=0}\rangle_{q_1} \ .$$

Therefore, considering the Taylor-Fourier expansion

$$f_4^{(\mathrm{IV};0,1)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=2\\ k}} c_{l,0,0,k}^{(\mathrm{IV};0,1)}\hat{p}^l \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ ,$$

we get

$$\chi_4^{(1)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=2\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{IV};0,1)}\hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega} \ .$$

The transformed Hamiltonian is calculated as

$$\begin{aligned} H^{(1)} &= \exp\left(L_{\chi_4^{(1)}}\right) H^{(\mathrm{IV};0)} = \\ &= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\ &\quad + f_0^{(1,1)} + f_2^{(1,1)} + f_3^{(1,1)} + f_4^{(1,1)} \\ &\quad + \sum_{s>1} f_0^{(1,s)} + \sum_{s>1} f_1^{(1,s)} + \sum_{s>1} f_2^{(1,s)} + \sum_{s>1} f_3^{(1,s)} + \sum_{s>1} f_4^{(1,s)} \\ &\quad + \sum_{s\geq 0}\sum_{\ell>2} f_\ell^{(1,s)} \ , \end{aligned} \tag{2.18}$$

with

$$\begin{aligned} f_4^{(1,1)} &= \langle f_4^{(\mathrm{IV};0,1)}\Big|_{\xi=\eta=0}\rangle_{q_1} + \left(f_4^{(\mathrm{IV};0,1)} - f_4^{(\mathrm{IV};0,1)}\Big|_{\xi=\eta=0}\right) \ , \\ f_\ell^{(1,s)} &= \sum_{j=0}^{s} \frac{1}{j!} L^j_{\chi_4^{(1)}} f_{\ell-2j}^{(\mathrm{IV};0,s-j)} \ . \qquad \text{for } \ell=4,\ s\neq 1 \ , \\ &\qquad\qquad\qquad\qquad\qquad \text{or } \ell\neq 4,\ s\geq 0 \ . \end{aligned}$$

Remark 2.2.4 *Considering the function*

$$f_2^{(1,1)} = f_2^{(0,1)} + L_{\chi_0^{(1)}} f_4^{(0,0)} + L_{\chi_1^{(1)}} f_3^{(0,0)} \ ,$$

we can observe that the term $L_{\chi_1^{(1)}} f_3^{(0,0)}$ *depends only on the transversal variables and the angles, due to the separability of the unperturbed initial Hamiltonian. Hence, it keeps unchanged the frequencies of the resonant torus.*

Let us consider the Hamiltonian $H^{(1)}$ in (2.18)

$$\begin{aligned} H^{(1)} &= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j + \hat{f}_3^{(1,0)}(\xi,\eta) + \tilde{f}_4^{(1,0)}(\hat{p}) + \hat{f}_4^{(1,0)}(\eta,\xi) \\ &\quad + f_0^{(1,1)}(q) + \tilde{f}_2^{(1,1)}(q,\hat{p}) + \hat{f}_2^{(1,1)}(q,\xi,\eta) + \hat{f}_3^{(1,1)}(\hat{q},\xi,\eta) \\ &\quad + \tilde{f}_4^{(1,1)}(q,\hat{p}) + \overline{f}_4^{(1,1)}(\hat{q},\hat{p},\xi,\eta) + \hat{f}_4^{(1,1)}(\hat{q},\xi,\eta) \\ &\quad + \sum_{s=0}^{1}\sum_{\ell>4} f_\ell^{(1,s)} + \mathcal{O}(\varepsilon^2) \\ &= K^{(1)} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

and its Hamilton equations

$$\begin{aligned} \dot{q}_1 &= \omega + \nabla_{p_1}\left[\tilde{f}_4^{(1,0)} + \tilde{f}_2^{(1,1)}\right] + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon|\hat{p}|^{a-1}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) \\ \dot{q} &= \nabla_p\left[\tilde{f}_4^{(1,0)} + \tilde{f}_2^{(1,1)}\right] + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon|\hat{p}|^{a-1}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) \\ \dot{p}_1 &= -\nabla_{q_1}\hat{f}_3^{(1,1)} + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) \\ \dot{p} &= -\nabla_q f_0^{(1,1)} - \nabla_q\left[f_2^{(1,1)} + \hat{f}_3^{(1,1)}\right] + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) \\ \dot{\xi} &= \mathbf{i}\tilde{\Omega}\xi + \nabla_\eta\left[\hat{f}_2^{(1,1)} + \hat{f}_3^{(1,1)}\right] + \mathcal{O}(|\xi|^m|\eta|^{n-1}) + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^{c-1}) + \mathcal{O}(\varepsilon^2) \\ \dot{\eta} &= -\mathbf{i}\tilde{\Omega}\eta - \nabla_\xi\left[\hat{f}_2^{(1,1)} + \hat{f}_3^{(1,1)}\right] + \mathcal{O}(|\xi|^{m-1}|\eta|^n) + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^{b-1}|\eta|^c) + \mathcal{O}(\varepsilon^2) \ , \end{aligned}$$

where $a, b, c, m, n \in \mathbb{N}$ such that $2a+b+c=4$ and $m+n=3$, and $\tilde{\Omega}$ is the diagonal matrix with frequencies Ω_j on the diagonal. We stress that, for $q=q^*$, one has

$$f_2^{(1,1)}\big|_{\substack{\xi=\eta=0\\ q=q^*}} = \tilde{f}_2^{(1,1)}\big|_{q=q^*} = 0 \ ,$$

because of the translation performed in the first stage of the normalization step. Hence, by neglecting terms of order $\mathcal{O}(\varepsilon^2)$ and evaluating the equations at $x^* = (q^*, 0, 0, 0)$, we get

$$\dot{q}_1 = \omega \ , \quad \dot{q} = 0 \ , \quad \dot{p}_1 = 0 \ , \quad \dot{p} = -\nabla_q f_0^{(1,1)}\big|_{q=q^*} \ , \quad \dot{\xi} = 0 \ , \quad \dot{\eta} = 0 \ .$$

Therefore, if $\nabla_q f_0^{(1,1)}\big|_{q=q^*} = 0$, then $(q_1 = q_1(0) + \omega t, x^*)$ represents a relative equilibrium of the truncated Hamiltonian. The periodicity condition is the following:

$$\begin{aligned}
\hat{q}(T) - \hat{q}(0) - \Lambda T &= \int_0^T \nabla_p \Big[\tilde{f}_4^{(1,0)} + \tilde{f}_2^{(1,1)}\Big]\, ds + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon|\hat{p}|^{a-1}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) = 0 \ , \\
p_1(T) - p_1(0) &= -\int_0^T \nabla_{q_1} \hat{f}_3^{(1,1)}\, ds + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) = 0 \ , \\
p(T) - p(0) &= -\int_0^T \nabla_q \Big[f_0^{(1,1)} + f_2^{(1,1)} + \hat{f}_3^{(1,1)}\Big]\, ds + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^2) = 0 \ , \\
\xi(T) - \xi(0) &= \int_0^T \mathbf{i}\tilde{\Omega}\xi + \nabla_\eta \left[\hat{f}_2^{(1,1)} + \hat{f}_3^{(1,1)}\right]\, ds + \mathcal{O}(|\xi|^m|\eta|^{n-1}) + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^b|\eta|^{c-1}) + \\
&\quad + \mathcal{O}(\varepsilon^2) = 0 \ , \\
\eta(T) - \eta(0) &= -\int_0^T \mathbf{i}\tilde{\Omega}\eta + \nabla_\xi \left[\hat{f}_2^{(1,1)} + \hat{f}_3^{(1,1)}\right]\, ds + \mathcal{O}(|\xi|^{m-1}|\eta|^n) + \mathcal{O}(\varepsilon|\hat{p}|^a|\xi|^{b-1}|\eta|^c) + \\
&\quad + \mathcal{O}(\varepsilon^2) = 0 \ ,
\end{aligned}$$

Once again, by neglecting the equation for p_1 and dividing the $n_1 - 1$ actions p by ε, we get a system of $2n-1$ equations in $2n-1$ unknowns $x(0) = (q(0), p_1(0), p(0), \xi(0), \eta(0))$.

With the aim of applying the implicit function theorem in the non-degenerate case, in this first normalization step we define the map Υ in (2.10) with the scaling of the actions by ε. Hence, the approximate periodic solution

$$q_1(t) = \omega t + q_1(0) \ , \qquad q(t) = q^* \ , \qquad \hat{p}(t) = 0 \ , \qquad \xi(t) = 0 \ , \qquad \eta(t) = 0$$

coincides with an approximate zero of the map. Thus, for $\varepsilon = 0$, one has $\Upsilon(x^*; 0, q_1(0)) = 0$. In order to apply the implicit function theorem, it remains to verify the condition on the determinant of the Jacobian matrix $M(0)$.

For this purpose, we follow the same procedure of the maximal dimension case, considering the expansion of the solution $x = (\hat{q}, \hat{p}, \xi, \eta)$ w.r.t ε

$$x(t, \varepsilon) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \mathcal{O}(\varepsilon^2) \ .$$

with

$$x^{(0)}(0) = x_0 \ , \qquad x^{(k)}(0) = 0 \ , \qquad k \geq 1 \ .$$

In particular, we have the following expansions

$$\begin{aligned}
&\hat{p}(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \hat{p}_0 + \mathcal{O}(\varepsilon) \\
&\hat{q}(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \hat{q}^{(0)}(t) + \mathcal{O}(\varepsilon) \ , \quad \text{with} \quad q_1^{(0)}(t) = \omega t + q_1^{(0)}(0),\ q^{(0)}(t) = q^* \ , \\
&\xi(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \xi^{(0)}(t) + \mathcal{O}(\varepsilon) \ , \quad \text{with} \quad \xi^{(0)}(t) = \xi_0 e^{i\tilde{\Omega}t} + \mathcal{O}(|\xi_0|^m|\eta_0|^{n-1}) \ , \\
&\eta(t, q_0, \hat{p}_0, \xi_0, \eta_0, \varepsilon) = \eta^{(0)}(t) + \mathcal{O}(\varepsilon) \ , \quad \text{with} \quad \eta^{(0)}(t) = \eta_0 e^{-i\tilde{\Omega}t} + \mathcal{O}(|\xi_0|^{m-1}|\eta_0|^n) \ ,
\end{aligned} \tag{2.19}$$

where the dependence on $q_1(0)$ is implied.

Let us start computing the differential of the functions $\mathfrak{F}$ and $\mathfrak{G}$. By inserting (2.19) in $\mathfrak{F}$ and $\mathfrak{G}$, we get

$$\begin{aligned}
\mathfrak{F} &= \int_0^T C_0 \hat{p}_0 \, ds + \mathcal{O}(|\hat{p}|^2) + \mathcal{O}(\varepsilon) \ , \\
\mathfrak{G} &= -\frac{1}{\varepsilon} \int_0^T \Big[\nabla_q f_0^{(1,1)}(q^{(0)}(t)) + \nabla_q f_2^{(1,1)}(q^{(0)}(t), \hat{p}_0, \xi^{(0)}(t), \eta^{(0)}(t)) + \\
&\qquad + \nabla_q \hat{f}_3^{(1,1)}(\hat{q}^{(0)}(t), \xi^{(0)}(t), \eta^{(0)}(t)) \Big] \, ds + \mathcal{O}(|\hat{p}|^a |\xi|^b |\eta|^c) + \mathcal{O}(\varepsilon) \ .
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&D_{q_0}\mathfrak{F}\big|_{(x^*;0)} = O \ , \qquad D_{\hat{p}_0}\mathfrak{F}\big|_{(x^*;0)} = C_0 T \ , \\
&D_{\xi_0}\mathfrak{F}\big|_{(x^*;0)} = O \ , \qquad D_{\eta_0}\mathfrak{F}\big|_{(x^*;0)} = O \ , \\
&D_{q_0}\mathfrak{G}\big|_{(x^*;0)} = -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) \ , \qquad D_{\hat{p}_0}\mathfrak{G}\big|_{(x^*;0)} = -\frac{T}{\varepsilon} D_{\hat{p}q}^2 \tilde{f}_2^{(1,1)}(q^*) \ , \\
&D_{\xi_0}\mathfrak{G}\big|_{(x^*;0)} = O \ , \qquad D_{\eta_0}\mathfrak{G}\big|_{(x^*;0)} = O \ .
\end{aligned}$$

As regards the functions $\mathfrak{R}$ and $\mathfrak{S}$, by inserting (2.19) in their expressions, we get, for $\varepsilon = 0$,

$$\begin{aligned}
\mathfrak{R} &= \int_0^T \mathbf{i}\tilde{\Omega}\xi^{(0)} \, ds + \mathcal{O}(|\xi^{(0)}|^m |\eta^{(0)}|^{n-1}) \ , \\
\mathfrak{S} &= -\int_0^T \mathbf{i}\tilde{\Omega}\eta^{(0)} \, ds + \mathcal{O}(|\xi^{(0)}|^{m-1} |\eta^{(0)}|^n) \ .
\end{aligned}$$

It means that $\xi^{(0)}, \eta^{(0)}$ are the solutions of a system of the following type:

$$\frac{d}{dt} z = Az + \mathcal{P}(z) \ , \qquad \text{with} \quad \mathcal{P}(z) = \mathcal{O}(|z|^2) \ ,$$

with $z = (\xi^{(0)}, \eta^{(0)})$. Therefore, using the variation of constants method, the solutions can be rewritten as

$$z(t, z_0) = e^{At} z_0 + \int_0^t e^{(t-s)A} \, \mathcal{P}(z(s, z_0)) \, ds \ ,$$

from which follows

$$D_{z_0} z(t, z_0) = e^{At} + \int_0^t e^{(t-s)A} \, \mathcal{P}'(z(s, z_0)) D_{z_0} z(s, z_0) \, ds \ .$$

Evaluating the differential in $z_0 = 0$, one has

$$z(s, 0) = 0 \qquad \text{and} \qquad \mathcal{P}'(z(s, 0)) = 0 \ ,$$

and, consequently,

$$D_{z_0} z(t, z_0)\big|_{z_0=0} = e^{At} \ .$$

We can deduce that

$$D_{z_0}(z(T, z_0) - z_0)\big|_{z_0=0} = e^{AT} - I \ ,$$

with, in our case, the matrix

$$A := \begin{pmatrix} \mathbf{i}\tilde{\Omega} & O \\ O & -\mathbf{i}\tilde{\Omega} \end{pmatrix} .$$

Finally, we have

$$\begin{aligned} &D_{q_0}\mathfrak{R}\big|_{(x^*;0)} = O \ , \qquad D_{\hat{p}_0}\mathfrak{R}\big|_{(x^*;0)} = O \ , \\ &D_{\xi_0}\mathfrak{R}\big|_{(x^*;0)} = e^{\mathbf{i}\tilde{\Omega}T} - I \ , \qquad D_{\eta_0}\mathfrak{R}\big|_{(x^*;0)} = O \ , \\ &D_{q_0}\mathfrak{S}\big|_{(x^*;0)} = O \ , \qquad D_{\hat{p}_0}\mathfrak{S}\big|_{(x^*;0)} = O \ , \\ &D_{\xi_0}\mathfrak{S}\big|_{(x^*;0)} = O \ , \qquad D_{\eta_0}\mathfrak{S}\big|_{(x^*;0)} = e^{-\mathbf{i}\tilde{\Omega}T} - I \ , \end{aligned}$$

hence, we get

$$\left(\begin{array}{cc|cc} O & C_0 T & O & O \\ -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) & -\frac{T}{\varepsilon} D_{\hat{p}q}^2 f_2^{(1,1)}(q^*) & O & O \\ \hline O & O & e^{2\pi\mathbf{i}\frac{\tilde{\Omega}}{\omega}} - I & O \\ O & O & O & e^{-2\pi\mathbf{i}\frac{\tilde{\Omega}}{\omega}} - I \end{array}\right) .$$

In order to apply the implicit function theorem, we only need that

$$\left| -\frac{T}{\varepsilon} D_q^2 f_0^{(1,1)}(q^*) \right| \neq 0 \ ,$$

because of the twist condition of the form (2.3) and the first Melnikov condition (2.5). Indeed, it is necessary that

$$\left| e^{\pm 2\pi\mathbf{i}\frac{\tilde{\Omega}}{\omega}} - I \right| \neq 0$$

and the above condition reads

$$e^{\pm 2\pi\mathbf{i}\frac{\Omega_j}{\omega}} - 1 \neq 0 \ , \quad \forall\, j \ ,$$

which is equivalent to $k\omega \pm \Omega_j \neq 0$. In conclusion, the applicability of the implicit function theorem results in the continuation of the unperturbed periodic orbit for $\varepsilon \neq 0$. Namely there exist an open interval of ε values around zero and a neighborhood $\mathcal{U}(x^*)$ such that the system with Hamiltonian $H^{(1)}$ admits a unique periodic orbit with frequencies ω and initial condition $x^*_{p.o.} = (q^*_{p.o.}, \hat{p}_{p.o.}, \xi_{p.o.}, \eta_{p.o.}) = (q^*_{p.o.}(\varepsilon), \hat{p}_{p.o.}(\varepsilon), \xi_{p.o.}(\varepsilon), \eta_{p.o.}(\varepsilon)) = (q^*, 0, 0, 0) + \mathcal{O}(\varepsilon)$.

We now want to investigate the linear stability of the unperturbed periodic orbit. Let us consider the Hamiltonian $H^{(1)}$ after a normalizing step; it has the structure

$$H^{(1)} = K^{(1)} + \mathcal{O}(\varepsilon^2) \ ,$$

For the truncated normal form $K^{(1)}$, the periodic orbit $q_1 = \omega t + q_1(0), q = q^*, \hat{p} = 0, \xi = 0, \eta = 0$ is a relative equilibrium. In order to study its linear stability, we introduce small displacement variables $(\hat{Q}, \hat{P})$ around the relative equilibrium

$$\begin{aligned} &Q_1 = q_1 - \omega t - q_1(0) \ , \qquad Q = q - q^* \ , \\ &P_1 = p_1 \ , \qquad P = p \ . \end{aligned}$$

The linearized equations for the variables $(\hat{Q}, \hat{P}, \xi, \eta)$ are Hamiltonian, and correspond to the Hamiltonian field given by the quadratic term in the Taylor expansion of $K^{(1)}$

$$
\begin{aligned}
K^{(1)} = {} & K^{(1)}(\omega t + q_1(0), x^*) + \\
& + D_q K^{(1)}(\omega t + q_1(0), x^*) Q + D_{\hat{p}} K^{(1)}(\omega t + q_1(0), x^*) \hat{P} + \\
& + D_\xi K^{(1)}(\omega t + q_1(0), x^*) \xi + D_\eta K^{(1)}(\omega t + q_1(0), x^*) \eta + \\
& + \frac{1}{2} Q^\top D_q^2 K^{(1)}(\omega t + q_1(0), x^*) Q + \frac{1}{2} \hat{P}^\top D_{\hat{p}}^2 K^{(1)}(\omega t + q_1(0), x^*) \hat{P} + \\
& + Q^\top D_{q\hat{p}}^2 K^{(1)}(\omega t + q_1(0), x^*) \hat{P} + \\
& + \frac{1}{2} \xi^\top D_\xi^2 K^{(1)}(\omega t + q_1(0), x^*) \xi + \frac{1}{2} \eta^\top D_\eta^2 K^{(1)}(\omega t + q_1(0), x^*) \eta + \\
& + \xi^\top D_{\xi\eta}^2 K^{(1)}(\omega t + q_1(0), x^*) \eta + \dots \ ,
\end{aligned}
$$

where respectively

$$
\begin{aligned}
B(\varepsilon) &= D_q^2 K^{(1)}(\omega t + q_1(0), x^*) = D_q^2 \left[f_0^{(1,1)} \right](x^*) = \varepsilon B_1 \\
D(\varepsilon) &= D_{q\hat{p}}^2 K^{(1)}(\omega t + q_1(0), x^*) = D_{q\hat{p}}^2 \left[\tilde{f}_2^{(1,1)} \right](x^*) = \varepsilon D_1 \\
C(\varepsilon) &= D_{\hat{p}}^2 K^{(1)}(\omega t + q_1(0), x^*) = D_{\hat{p}}^2 \left[\tilde{f}_4^{(1,0)} + \tilde{f}_4^{(1,1)} \right](x^*) = C_0 + \varepsilon C_1 \\
G(\varepsilon) &= D_\xi^2 K^{(1)}(\omega t + q_1(0), x^*) = D_\xi^2 \left[\hat{f}_2^{(1,1)} \right](x^*) = \varepsilon G_1 \\
F(\varepsilon) &= D_\eta^2 K^{(1)}(\omega t + q_1(0), x^*) = D_\eta^2 \left[\hat{f}_2^{(1,1)} \right](x^*) = \varepsilon F_1 \\
E(\varepsilon) &= D_{\xi\eta}^2 K^{(1)}(\omega t + q_1(0), x^*) = D_{\xi\eta}^2 \left[\hat{f}_2^{(1,0)} + \hat{f}_2^{(1,1)} \right](x^*) = E_0 + \varepsilon E_1
\end{aligned}
$$

Remark 2.2.5 *We observe that we do not have the terms $D_{\hat{p}\eta}^2 K^{(1)}(\omega t + q_1(0), x^*)$ and $D_{\hat{p}\xi}^2 K^{(1)}(\omega t + q_1(0), x^*)$, due to the absence of the term $f_3^{(1,1)}(\hat{q}, \hat{p}, \xi, \eta)$ in the Hamiltonian $H^{(1)}$ and to the evaluation at the relative equilibrium. For the latter reason, also the terms $D_{q\eta}^2 K^{(1)}(\omega t + q_1(0), x^*)$ and $D_{q\xi}^2 K^{(1)}(\omega t + q_1(0), x^*)$ are absent.*

In order to write the linear Hamiltonian field, we extend the matrices $B(\varepsilon)$ and $D(\varepsilon)$ (the last one being $(n_1 - 1) \times n_1$ rectangular) in order to include the Q_1 dependence. Hence, we denote by $\tilde{B}(\varepsilon)$ the square-matrix obtained adding a zero row at first position and a zero column at first position to $B(\varepsilon)$. Similarly, we denote by $\tilde{D}(\varepsilon)$ the square-matrix obtained adding a zero row at first position to $D(\varepsilon)$. So doing, the quadratic Hamiltonian which represents the linear motion around the approximate periodic orbit is

$$
K_2^{(1)} = \frac{1}{2} \hat{Q}^\top \tilde{B}(\varepsilon) \hat{Q} + \hat{Q}^\top \tilde{D}(\varepsilon) \hat{P} + \frac{1}{2} \hat{P}^\top C(\varepsilon) \hat{P} + \frac{1}{2} \xi^\top G(\varepsilon) \xi + \xi^\top E(\varepsilon) \eta + \frac{1}{2} \eta^\top F(\varepsilon) \eta \ ,
$$

and the field is

$$
\begin{pmatrix} \dot{\hat{Q}} \\ \dot{\hat{P}} \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \left(\begin{array}{cc|cc} O & I & O & O \\ -I & O & O & O \\ \hline O & O & O & I \\ O & O & -I & O \end{array}\right) \left(\begin{array}{cc|cc} \tilde{B}(\varepsilon) & \tilde{D}(\varepsilon) & O & O \\ \tilde{D}(\varepsilon)^\top & C(\varepsilon) & O & O \\ \hline O & O & G(\varepsilon) & E(\varepsilon) \\ O & O & E(\varepsilon)^\top & F(\varepsilon) \end{array}\right) \begin{pmatrix} \hat{Q} \\ \hat{P} \\ \xi \\ \eta \end{pmatrix} =
$$

$$
= \left(\begin{array}{cc|cc} \tilde{D}(\varepsilon)^\top & C(\varepsilon) & O & O \\ -\tilde{B}(\varepsilon) & -\tilde{D}(\varepsilon) & O & O \\ \hline O & O & E(\varepsilon)^\top & F(\varepsilon) \\ O & O & -G(\varepsilon) & -E(\varepsilon) \end{array}\right) \begin{pmatrix} \hat{Q} \\ \hat{P} \\ \xi \\ \eta \end{pmatrix} = L(\varepsilon) \begin{pmatrix} \hat{Q} \\ \hat{P} \\ \xi \\ \eta \end{pmatrix} .
$$

Since $L(\varepsilon)$ is constant in time, the stability of the periodic orbit reduces to the study of the spectrum of $L(\varepsilon)$, hence to the zeros of

$$
\det(L(\varepsilon) - \lambda \mathbb{I}) = 0 .
$$

With reference to $L(\varepsilon) - \lambda\mathbb{I}$, if we "lift" the (n_1+1)-th row at second position and the (n_1+1)-th column at second position, then the determinant does not change (we make $2n_1$ changes of the sign), and we can factor out a λ^2 dependence

$$
\det(L(\varepsilon) - \lambda \mathbb{I}) = \lambda^2 \det(V(\varepsilon) - \lambda\mathbb{I}) = 0 ,
$$

where $V(\varepsilon) - \lambda\mathbb{I}$ is a $2n-2$ square-matrix with the following block form

$$
V(\varepsilon) - \lambda\mathbb{I} = \left(\begin{array}{cc|cc} (\lfloor D(\varepsilon) - \lambda\mathbb{I})^\top & \lfloor\overline{C}(\varepsilon) & O & O \\ -B(\varepsilon) & -\lfloor D(\varepsilon) - \lambda\mathbb{I} & O & O \\ \hline O & O & (E(\varepsilon) - \lambda\mathbb{I})^\top & F(\varepsilon) \\ O & O & -G(\varepsilon) & -E(\varepsilon) - \lambda\mathbb{I} \end{array}\right) ,
$$

where $\lfloor D(\varepsilon)$ is the n_1-1 square-matrix $D(\varepsilon)$ without the first columns, and $\lfloor\overline{C}(\varepsilon)$ is the n_1-1 square-matrix $C(\varepsilon)$ without both the first column and row, i.e. any dependence on p_1. The determinant $\det(V(\varepsilon) - \lambda\mathbb{I})$ is the product of the determinants of the two squared blocks on the diagonal. As regards the first block, in order to compute the eigenvalues, we make an exchange of rows so as to obtain

$$
\begin{vmatrix} -B(\varepsilon) & -\lfloor D(\varepsilon) - \lambda\mathbb{I} \\ (\lfloor D(\varepsilon) - \lambda\mathbb{I})^\top & \lfloor\overline{C}(\varepsilon) \end{vmatrix} = 0 . \tag{2.20}
$$

If the periodic orbit is non-degenerate, i.e. $|B(\varepsilon)| \neq 0$, then we can calculate (2.20) as

$$
\det(-B(\varepsilon)) \cdot \det\left(\lfloor\overline{C}(\varepsilon) - (\lfloor D(\varepsilon) - \lambda\mathbb{I})^\top (-B(\varepsilon))^{-1} (-\lfloor D(\varepsilon) - \lambda\mathbb{I})\right) = 0 ,
$$

namely

$$
\det\left(\lfloor\overline{C}_0 + \varepsilon\lfloor\overline{C}_1 - (\varepsilon\lfloor D_1 - \lambda\mathbb{I})^\top (-\varepsilon B_1)^{-1} (-\varepsilon\lfloor D_1 - \lambda\mathbb{I})\right) = 0 .
$$

Since the relation must hold in the limit $\varepsilon \to 0$, under the assumption of invertibility[3] for the matrix $\lfloor\overline{C}_0$, it is necessary that the leading order of approximation of λ is $\mathcal{O}(\sqrt{\varepsilon})$. So, we can

[3]This condition is always guaranteed if the the symmetric matrix C_0 is positive definite. Indeed, there exists a matrix S such that $S^\top S = C_0$. Moreover, considering the matrix K which is the $n_1 \times (n_1 - 1)$ submatrix of the unimodular matrix used to introduced the resonant variables, we get $\lfloor\overline{C}_0 = K^\top C_0 K$ and

$$
rank(K^\top C_0 K) = rank((SK)^\top SK) = rank(SK) = rank(K) = n_1 - 1 .
$$

rewrite the eigenvalues as $\lambda = \sqrt{\varepsilon}\mu = \sqrt{\varepsilon}(\mu_0 + \mathcal{O}(\varepsilon^{\frac{1}{l}}))$, with $\mu_0 \neq 0$ and $l \in \mathbb{N}$, μ being the solution of an algebraic equation. Hence, we get

$$\det\left(\lfloor\overline{C}_0 + \varepsilon\lfloor\overline{C}_1 - (\sqrt{\varepsilon}\lfloor D_1 - \mu\mathbb{I})^\top(-B_1)^{-1}(-\sqrt{\varepsilon}\lfloor D_1 - \mu\mathbb{I})\right) = 0 \ ,$$

which, in the limit $\varepsilon \to 0$, reads

$$\det\left(-B_1\lfloor\overline{C}_0 - \mu_0^2\mathbb{I}\right) = 0 \ .$$

Therefore, up to terms of order $\mathcal{O}(\sqrt{\varepsilon})$, the eigenvalues of the first block of $V(\varepsilon)$ are determined by B_1 and $\lfloor\overline{C}_0$.

Regarding the second block of $V(\varepsilon)$, since $E_0 = \mathbf{i}\tilde{\Omega}$, for $\varepsilon = 0$ it corresponds to the Hamiltonian $H = \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \Omega_j \frac{x_j^2+y_j^2}{2}$. If we assume $\Omega_j > 0$ (or $\Omega_j < 0$), the latter Hamiltonian is positive definite (or negative definite). Hence, for $\varepsilon \neq 0$, it cannot lose definiteness, so, due to the Dirichlet's criterion[4], it cannot lose stability and its eigenvalues remain imaginary for all orders of approximation w.r.t ε.

Remark 2.2.6 *Let us observe that, concerning the transversal variables, one can obtain the approximate (and actually also the effective) stability of periodic orbits by means of the Krein's theory[5]. If the Floquet multipliers lie on the unit circle and are definite, then periodic orbits are stable. The assumption on the approximate multipliers may be translated in terms of approximate characteristic exponents (which are the eigenvalues of $L(\varepsilon)$), asking that all of them be purely imaginary. The hypothesis on definiteness results in the condition $\Omega_j > 0$ (or $\Omega_j < 0$), provided that we assume the second Melnikov condition. Taking advantage of the normal form construction, we have reduced existence and stability of periodic orbits to the ones of relative equilibrium points. By doing so, it is not necessary to invoke Krein's signature: the definiteness assumption is enough for an equilibrium point not to lose stability. However, we remark that the hypothesis on second Melnikov condition has been already assumed during the construction of the normal form algorithm, which allows to turn the study of continuation and stability of periodic orbits into the ones of equilibrium points.*

The result can be stated as follows

Proposition 2.2.1 *Consider the starting Hamiltonian* (2.2) *and assume the twist condition* (2.3) *and the Melnikov conditions* (2.5), (2.6). *The unperturbed approximate periodic orbits for which it holds that*

$$\nabla_q f_0^{(1,1)}(q^*) = 0 \ , \qquad \left|D_q^2 f_0^{(1,1)}(q^*)\right| \neq 0 \ ,$$

namely non-degenerate periodic orbits, are analytically continued at fixed period, i.e. there exists a value ε^* such that for $|\varepsilon| < \varepsilon^*$ we get continuation. Moreover, as regards the approximate linear stability, the characteristic exponents λ of the internal variables of the approximate periodic orbits can be expanded as $\sqrt{\varepsilon}(\mu_0 + \mathcal{O}(\varepsilon^{\frac{1}{l}}))$, with $l \in \mathbb{N}$, and μ_0^2 the eigenvalues of the matrix $-B_1\lfloor\overline{C}_0$; while the characteristic exponents $\mathbf{i}\Omega_j$ of the transversal variables remain purely imaginary under perturbation, if we assume $\Omega_j > 0$ (or $\Omega_j < 0$) for all $j \in \mathcal{J}\setminus\mathcal{I}$.

The result may also be formulated and proved in terms of effective linear stability of continued periodic orbits. For the effective stability result in the generic case with r normal form steps, see the Section 2.4. Instead, for a proof in the non-degenerate case with the particular scaling of the internal characteristic exponents, refer to [73], where we get an extension of the Poincaré's result to lower dimensional tori by means of a standard perturbation expansion of the solutions w.r.t. the small parameter.

Furthermore, the result we have just achieved is a generalization of the extension to lower dimensional tori of the Poincaré's result obtained by means of the effective Hamiltonian method

[4] See [57].
[5] See [4, 25, 59, 60, 92].

for chains of weakly coupled oscillators. So it also includes the result given by V. Koukouloyannis and P. G. Kevrekidis, in [55], for chains of weakly coupled oscillators. They consider a countable set of oscillators with a nearest-neighbor coupling, with Hamiltonian

$$H = H_0 + \varepsilon H_1 = \sum_{i=-\infty}^{+\infty} \left(\frac{p_i^2}{2} + V(x_i) \right) + \frac{\varepsilon}{2} \sum_{i=-\infty}^{+\infty} (x_{i+1} - x_i)^2$$

where $V(x_i)$ is the potential function, x_i the displacement from the equilibrium and p_i the momentum of the i-th oscillator. Assume that, in the "anticontinuous" limit $\varepsilon = 0$, $n+1$ adjacent "central" oscillators move in periodic orbits with frequency ω and arbitrary phases, while the "non-central" oscillators lie at rest. They seek conditions under which these orbits can be continued for $\varepsilon \neq 0$, giving rise to multibreathers with the same frequency. After performing an action-angle canonical transformation for the central oscillators, they consider the averaged Hamiltonian resulting from the effective Hamiltonian method at the lowest order of approximation

$$H^{eff} = H_0(I_i) + \varepsilon \langle H_1 \rangle (\phi_i, I_i) \ , \qquad\qquad i = 1, \ldots, n \ ,$$

where ϕ_i are the slow angles and $\langle H_1 \rangle$ is the average value of H_1 over the fast angle. The critical points of this effective Hamiltonian which satisfy the conditions

$$\begin{aligned}
&\left| D_I^2 H_0 \right| \neq 0 && \text{(twist condition)} \ , \\
&\left| D_\phi^2 \langle H_1 \rangle \right| \neq 0 && \text{(Poincaré non-degeneracy)} \ , \\
&\omega_p \neq k\omega && \text{(first Melnikov condition)} \ ,
\end{aligned}$$

where $\omega_p = V''(0)$, can be continued for $\varepsilon \neq 0$.

In addition, they determine the linear stability of the multibreathers investigating the linear stability of the critical points of H^{eff}. Similarly to the above discussion, this can be reduced to the study of the non-zero characteristic exponents of the central oscillators, for which their approach provides an $\mathcal{O}(\sqrt{\varepsilon})$ estimate.

Let us remark that, in order to apply the Krein's signature theory for the stability of the external oscillators, also in their procedure it would be needed an additional hypothesis of non resonance, represented by the second Melnikov condition. Moreover, due to equivalence of potential functions, it would be reduced to the simplified second Melnikov condition $2\omega_p \neq k\omega$.

Furthermore, since they consider solutions with $\phi_i = 0, \pi$ (which are the only solutions in this kind of systems with consecutive excited oscillators, see [54]), they obtain $\lfloor D(\varepsilon) = 0$, so the first correction to the characteristic exponents is of order $\mathcal{O}(\varepsilon^{3/2})$. We also remark that, by explicitly computing the matrix $C(\varepsilon)$, one gets a tridiagonal invertible matrix, hence the matrix $\lfloor \overline{C}(\varepsilon)$ is automatically invertible in this system.

2.2.2 Generic r-th normalization step

We summarize the five stages of a generic r-th normalizing step. The starting Hamiltonian has the form

$$\begin{aligned}
H^{(r-1)} = \omega p_1 &+ \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \mathbf{i} \Omega_j \xi_j \eta_j \\
&+ \sum_{s<r} f_0^{(r-1,s)} + \sum_{s<r} f_2^{(r-1,s)} + \sum_{s<r} f_3^{(r-1,s)} + \sum_{s<r} f_4^{(r-1,s)} \\
&+ f_0^{(r-1,r)} + f_1^{(r-1,r)} + f_2^{(r-1,r)} + f_3^{(r-1,r)} + f_4^{(r-1,r)} \\
&+ \sum_{s>r} f_0^{(r-1,s)} + \sum_{s>r} f_1^{(r-1,s)} + \sum_{s>r} f_2^{(r-1,s)} + \sum_{s>r} f_3^{(r-1,s)} + \sum_{s>r} f_4^{(r-1,s)} \\
&+ \sum_{s \geq 0} \sum_{\ell > 2} f_\ell^{(r-1,s)} \ .
\end{aligned} \tag{2.21}$$

where $f_0^{(r-1,s)}$, $f_2^{(r-1,s)}$, $f_3^{(r-1,s)}$ and $f_4^{(r-1,s)}$, for $1 \leq s < r$, are in normal form.

First stage of the r-th normalization step

We average the term $f_0^{(r-1,r)}$ with respect to the fast angle q_1, determining the generating function

$$\chi_0^{(r)}(\hat{q}) = X_0^{(r)}(\hat{q}) + \langle\zeta^{(r)}, \hat{q}\rangle \qquad \text{with} \qquad \zeta^{(r)} \in \mathbb{R}^{n_1} \ ,$$

belonging to $\mathcal{P}_0$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equations

$$\begin{aligned} &L_{X_0^{(r)}}\omega p_1 + f_0^{(r-1,r)} = \langle f_0^{(r-1,r)}\rangle_{q_1} \ , \\ &L_{\langle\zeta^{(r)},\hat{q}\rangle} f_4^{(0,0)}\Big|_{\xi=\eta=0} + \left\langle f_2^{(r-1,r)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\right\rangle_{q_1} = 0 \ . \end{aligned}$$

By considering the Taylor-Fourier expansion

$$f_0^{(r-1,r)}(\hat{q}) = \sum_k c_{0,0,0,k}^{(r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ ,$$

we obtain

$$X_0^{(r)}(\hat{q}) = \sum_{k_1\neq 0} \frac{c_{0,0,0,k}^{(r-1,r)}}{\mathbf{i}k_1\omega} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ .$$

The vector $\zeta^{(r)}$ is determined by solving the linear system

$$\sum_j C_{0,ij}\zeta_j^{(r)} = \frac{\partial}{\partial\hat{p}_i}\left\langle f_2^{(r-1,r)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\right\rangle_{q_1} \ .$$

The transformed Hamiltonian is computed as

$$\begin{aligned} H^{(\mathrm{I};r-1)} &= \exp\Big(L_{\chi_0^{(r)}}\Big) H^{(r-1)} = \\ &= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\ &\quad + \sum_{s<r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{I};r-1,s)} \\ &\quad + f_0^{(\mathrm{I};r-1,r)} + f_1^{(\mathrm{I};r-1,r)} + f_2^{(\mathrm{I};r-1,r)} + f_3^{(\mathrm{I};r-1,r)} + f_4^{(\mathrm{I};r-1,r)} \\ &\quad + \sum_{s>r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{I};r-1,s)} \\ &\quad + \sum_{s\geq 0}\sum_{\ell>2} f_\ell^{(\mathrm{I};r-1,s)} \ . \end{aligned}$$

The functions $f_\ell^{(\mathrm{I};r-1,s)}$ are recursively defined as

$$\begin{aligned} f_0^{(\mathrm{I};r-1,r)} &= \langle f_0^{(r-1,r)}\rangle_{q_1} \ , \\ f_\ell^{(\mathrm{I};r-1,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L_{\chi_0^{(r)}}^j f_{\ell+2j}^{(r-1,s-jr)} \ , \qquad \text{for } \ell=0,\ s\neq r \ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell\neq 0\ s\geq 0 \ \ , \end{aligned} \tag{2.22}$$

with $f_\ell^{(\mathrm{I};r-1,s)} \in \mathcal{P}_\ell$.

Second stage of the r-th normalization step

We now remove the term $f_1^{(\mathrm{I};r-1,r)}$ by means of the generating function $\chi_1^{(r)}$, belonging to $\mathcal{P}_1$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_1^{(r)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_1^{(\mathrm{I};r-1,r)} = 0 \ . \tag{2.23}$$

Considering again the Taylor-Fourier expansion

$$f_1^{(\mathrm{I};r-1,r)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\xi^{m_1}\eta^{m_2}\ ,$$

we get

$$\chi_1^{(r)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\,\Omega\rangle\big]}\ .$$

with $\Omega\in\mathbb{R}^{n_2}$.

The transformed Hamiltonian is calculated as

$$\begin{aligned}
H^{(\mathrm{II};r-1)} &= \exp\left(L_{\chi_1^{(r)}}\right) H^{(\mathrm{I};r-1)} = \\
&= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\
&\quad + \sum_{s<r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + f_0^{(\mathrm{II};r-1,r)} + f_2^{(\mathrm{II};r-1,r)} + f_3^{(\mathrm{II};r-1,r)} + f_4^{(\mathrm{II};r-1,r)} \\
&\quad + \sum_{s>r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + \sum_{s\ge 0}\sum_{\ell>2} f_\ell^{(\mathrm{II};r-1,s)}\ ,
\end{aligned} \tag{2.24}$$

with

$$\begin{aligned}
f_1^{(\mathrm{II};r-1,r)} &= 0\ , \\
f_0^{(\mathrm{II};r-1,2r)} &= f_0^{(\mathrm{I};r-1,2r)} + L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)} + \frac{1}{2} L_{\chi_1^{(r)}}\left(L_{\chi_1^{(r)}} f_2^{(\mathrm{I};r-1,0)}\right) = \\
&= f_0^{(\mathrm{I};r-1,2r)} + \frac{1}{2} L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)}\ , \\
f_\ell^{(\mathrm{II};r-1,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L_{\chi_1^{(r)}}^j f_{\ell+j}^{(\mathrm{I};r-1,s-jr)}, \qquad \text{for } \ell=0,\ s\neq 2r\ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell=1\ s\neq r\ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell\geq 2\ s\geq 0\ ,
\end{aligned} \tag{2.25}$$

where we have exploited (2.23).

Third stage of the r-th normalization step

We now average the term $f_2^{(\mathrm{II};r-1,r)}$ with respect to the fast angle q_1, determining the generating function $\chi_2^{(r)}$, belonging to $\mathcal{P}_2$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_2^{(r)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_2^{(\mathrm{II};r-1,r)} = \langle f_2^{(\mathrm{II};r-1,r)}\rangle_{q_1}\ . \tag{2.26}$$

Therefore, considering the Taylor-Fourier expansion

$$f_2^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k}} c_{l,0,0,k}^{(\mathrm{II};r-1,r)}\hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle) + \sum_{\substack{|m_1|+|m_2|=2\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{II};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\xi^{m_1}\eta^{m_2}\ ,$$

we obtain

$$\chi_2^{(r)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{II};r-1,r)}\hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega} + \sum_{\substack{|m_1|+|m_2|=2\\ k_1\neq 0}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{II};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\,\Omega\rangle\big]}\ .$$

The transformed Hamiltonian is computed as

$$H^{(\mathrm{III};r-1)} = \exp\left(L_{\chi_2^{(r)}}\right) H^{(\mathrm{II};r-1)}$$

and is in the form (2.24), replacing the upper index II by III, with $f_\ell^{(\mathrm{III};r-1,s)} \in \mathcal{P}_\ell$ given by

$$\begin{aligned}
f_2^{(\mathrm{III};r-1,r)} &= \langle f_2^{(\mathrm{II};r-1,r)}\rangle_{q_1} \ , \\
f_2^{(\mathrm{III};r-1,ri)} &= \frac{1}{(i-1)!} L^{i-1}_{\chi_2^{(r)}} \left(f_2^{(\mathrm{II};r-1,r)} + \frac{1}{i} L_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,0)} \right) \\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,ri-rj)} = \\
&= \frac{1}{(i-1)!} L^{i-1}_{\chi_2^{(r)}} \left(\frac{1}{i} \langle f_2^{(\mathrm{II};r-1,r)}\rangle_{q_1} + \frac{i-1}{i} f_2^{(\mathrm{II};r-1,r)} \right) \\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,ri-rj)} \ , \\
f_\ell^{(\mathrm{III};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_\ell^{(\mathrm{II};r-1,s-jr)} \ , \qquad \begin{array}{l} \text{for } \ell = 2,\ s \neq ri \ , \\ \text{or } \ell \neq 2,\ s \geq 0 \ \ , \end{array}
\end{aligned} \tag{2.27}$$

where we have used the homological equation (2.26).

Fourth stage of the r-th normalization step

We now remove the term $\tilde{f}_3^{(\mathrm{III};r-1,r)} = f_3^{(\mathrm{III};r-1,r)} - f_3^{(\mathrm{III};r-1,r)}\Big|_{\hat{p}=0}$, which depends both on the actions and on the transversal variables ξ, η. We determine the generating function $\chi_3^{(r)}$, belonging to $\mathcal{P}_3$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_3^{(r)}} \Big(\omega p_1 + \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \mathbf{i} \Omega_j \xi_j \eta_j \Big) + \tilde{f}_3^{(\mathrm{III};r-1,r)} = 0 \ . \tag{2.28}$$

Hence, considering the Taylor-Fourier expansion

$$\tilde{f}_3^{(\mathrm{III};r-1,r)}(\hat{q}, \hat{p}, \xi, \eta) = \sum_{\substack{|l|=1 \\ |m_1|+|m_2|=1 \\ k}} c_{l,m_1,m_2,k}^{(\mathrm{III};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \hat{p}^l \xi^{m_1} \eta^{m_2} \ ,$$

we get

$$\chi_3^{(r)}(\hat{q}, \hat{p}, \xi, \eta) = \sum_{\substack{|l|=1 \\ |m_1|+|m_2|=1 \\ k}} \frac{c_{l,m_1,m_2,k}^{(\mathrm{III};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\, \hat{p}^l \xi^{m_1} \eta^{m_2}}{\mathbf{i}\big[k_1 \omega + \langle m_1 - m_2,\, \Omega \rangle\big]} \ .$$

with $\Omega \in \mathbb{R}^{n_2}$.

The transformed Hamiltonian is computed as

$$H^{(\mathrm{IV};r-1)} = \exp\left(L_{\chi_3^{(r)}}\right) H^{(\mathrm{III};r-1)}$$

and is given in the form (2.24), replacing the upper index II by IV, with

$$
\begin{aligned}
f_3^{(\mathrm{IV};r-1,r)} &= f_3^{(\mathrm{III};r-1,r)}\Big|_{\hat p=0} \ , \\
f_4^{(\mathrm{IV};r-1,2r)} &= f_4^{(\mathrm{III};r-1,2r)} + L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,r)} + \frac{1}{2} L^2_{\chi_3^{(r)}} f_2^{(\mathrm{III};r-1,0)} = \\
&= f_4^{(\mathrm{III};r-1,2r)} + \frac{1}{2} L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,r)} + \frac{1}{2} L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,2)}\Big|_{\hat p=0} \ , \\
f_\ell^{(\mathrm{IV};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_3^{(r)}} f_{\ell-j}^{(\mathrm{III};r-1,s-jr)}, \qquad \begin{aligned}&\text{for } \ell=3,\ s\neq r\ ,\\ &\text{or } \ell=4,\ s\neq 2r\ ,\\ &\text{or } \ell\neq 3,4,\ s\geq 0\ .\end{aligned}
\end{aligned}
\tag{2.29}
$$

where we have exploited (2.28).

Fifth stage of the r-th normalization step

We average the term $f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0}$ with respect to the fast angle q_1. We determine the generating function $\chi_4^{(r)}$, belonging to $\mathcal{P}_4$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$
L_{\chi_4^{(r)}} \omega p_1 + f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0} = \langle f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0} \rangle_{q_1} \ .
$$

By considering the Taylor-Fourier expansion

$$
f_4^{(\mathrm{IV};r-1,r)}(\hat p,\hat q) = \sum_{\substack{|l|=2\\ k}} c_{l,0,0,k}^{(\mathrm{IV};r-1,r)} \hat p^l \exp(\mathbf{i}\langle k,\,\hat q\rangle) \ ,
$$

we obtain

$$
\chi_4^{(r)}(\hat p,\hat q) = \sum_{\substack{|l|=2\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{IV};r-1,r)} \hat p^l \exp(\mathbf{i}\langle k,\,\hat q\rangle)}{\mathbf{i} k_1 \omega} \ .
$$

The transformed Hamiltonian is calculated as

$$
H^{(r)} = \exp\left(L_{\chi_4^{(r)}}\right) H^{(\mathrm{IV};r-1)}
$$

and is given in the form (2.21), replacing the upper index $r-1$ by r, with

$$
\begin{aligned}
f_4^{(r,r)} &= \langle f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0} \rangle_{q_1} + \left(f_4^{(\mathrm{IV};r-1,r)} - f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0} \right) \ , \\
f_\ell^{(r,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_4^{(r)}} f_{\ell-2j}^{(\mathrm{IV};r-1,s-jr)} \ . \qquad \begin{aligned}&\text{for } \ell=4,\ s\neq r\ ,\\ &\text{or } \ell\neq 4,\ s\geq 0\ .\end{aligned}
\end{aligned}
\tag{2.30}
$$

2.3 Proof of Theorem 2.1.1

The Hamilton equations associated to the Hamiltonian in normal form up to order r read

$$
\begin{aligned}
\dot q_1 &= \omega + \nabla_{p_1}\left[\tilde f_4^{(r,0)} + \sum_{s=1}^{r}\tilde f_2^{(r,s)}\right] + \mathcal{O}(|\hat p|^2) + \mathcal{O}(\varepsilon|\hat p|^{a-1}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^{r+1})\\
\dot q &= \nabla_{p}\left[\tilde f_4^{(r,0)} + \sum_{s=1}^{r}\tilde f_2^{(r,s)}\right] + \mathcal{O}(|\hat p|^2) + \mathcal{O}(\varepsilon|\hat p|^{a-1}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^{r+1})\\
\dot p_1 &= -\sum_{s=1}^{r}\nabla_{q_1}\hat f_3^{(r,s)} + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^{r+1})\\
\dot p &= -\sum_{s=1}^{r}\nabla_{q} f_0^{(r,s)} - \sum_{s=1}^{r}\nabla_{q}\left[f_2^{(r,s)} + \hat f_3^{(r,s)}\right] + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^{r+1})\\
\dot\xi &= \mathbf{i}\tilde\Omega\xi + \sum_{s=1}^{r}\nabla_{\eta}\left[\hat f_2^{(r,s)} + \hat f_3^{(r,s)}\right] + \mathcal{O}(|\xi|^m|\eta|^{n-1}) + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^{c-1}) + \mathcal{O}(\varepsilon^{r+1})\\
\dot\eta &= -\mathbf{i}\tilde\Omega\eta - \sum_{s=1}^{r}\nabla_{\xi}\left[\hat f_2^{(r,s)} + \hat f_3^{(r,s)}\right] + \mathcal{O}(|\xi|^{m-1}|\eta|^{n}) + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^{b-1}|\eta|^{c}) + \mathcal{O}(\varepsilon^{r+1})\ ,
\end{aligned}
$$

where $a, b, c, m, n \in \mathbb{N}$ such that $2a + b + c = 4$ and $m + n = 3$, and $\tilde\Omega$ is the diagonal matrix with frequencies Ω_j on the diagonal.

Neglecting terms of order $\mathcal{O}(\varepsilon^{r+1})$ and evaluating at $x^* = (q = q^*, \hat p = 0, \xi = 0, \eta = 0)$, the Hamilton equations of the truncated normal form read

$$
\dot q_1 = \omega\ ,\quad \dot q = 0\ ,\quad \dot p_1 = 0\ ,\quad \dot p = -\sum_{s=1}^{r}\nabla_q f_0^{(r,s)}\big|_{q=q^*}\ ,\quad \dot\xi = 0\ ,\quad \dot\eta = 0\ .
$$

Thus, if q^* fulfills the equation (2.9) then $(q_1 = q_1(0) + \omega t, x^*)$ represents a relative equilibrium of the truncated Hamiltonian, i.e. it is the initial datum of an approximate periodic orbit for the whole system[6]. Moreover, the periodicity condition for an orbit of the Hamiltonian $H^{(r)}$ can be rewritten as follows

$$
\begin{aligned}
\hat q(T) - \hat q(0) - \Lambda T &= \int_0^T \nabla_{\hat p}\left[\tilde f_4^{(r,0)} + \sum_{s=1}^{r}\tilde f_2^{(r,s)}\right] ds + \mathcal{O}(|\hat p|^2) + \mathcal{O}(\varepsilon|\hat p|^{a-1}|\xi|^b|\eta|^c) +\\
&\quad + \mathcal{O}(\varepsilon^{r+1}) = 0\\
p_1(T) - p_1(0) &= -\int_0^T \sum_{s=1}^{r}\nabla_{q_1}\hat f_3^{(r,s)}\, ds + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^c) + \mathcal{O}(\varepsilon^{r+1}) = 0\\
p(T) - p(0) &= -\int_0^T \left(\sum_{s=1}^{r}\nabla_q f_0^{(r,s)} + \sum_{s=1}^{r}\nabla_q\left[f_2^{(r,s)} + \hat f_3^{(r,s)}\right]\right) ds + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^c) +\\
&\quad + \mathcal{O}(\varepsilon^{r+1}) = 0\\
\xi(T) - \xi(0) &= \int_0^T \left(\mathbf{i}\tilde\Omega\xi + \sum_{s=1}^{r}\nabla_\eta\left[\hat f_2^{(r,s)} + \hat f_3^{(r,s)}\right]\right) ds + \mathcal{O}(|\xi|^m|\eta|^{n-1}) +\\
&\quad + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^b|\eta|^{c-1}) + \mathcal{O}(\varepsilon^{r+1}) = 0\\
\eta(T) - \eta(0) &= -\int_0^T \left(\mathbf{i}\tilde\Omega\eta - \sum_{s=1}^{r}\nabla_\xi\left[\hat f_2^{(r,s)} + \hat f_3^{(r,s)}\right]\right) ds + \mathcal{O}(|\xi|^{m-1}|\eta|^{n}) +\\
&\quad + \mathcal{O}(\varepsilon|\hat p|^{a}|\xi|^{b-1}|\eta|^{c}) + \mathcal{O}(\varepsilon^{r+1}) = 0\ ,
\end{aligned}
$$

[6] Let us observe that the candidate q^* for the continuation are now dependent on ε.

and defines the map Υ as in (2.10).

As in the first Chapter, we introduce some quantities useful for the estimates in Section 2.5:

$$\Xi_r = \max\left(\frac{eE}{\alpha\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\alpha\delta_r^2\rho\sigma}, \frac{E}{\alpha\delta_r^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right) ,$$

with

$$\alpha = \min_{k_1,j,l,k}\left(|\omega|, |k_1\omega \pm \Omega_j|, |k_1\omega \pm \Omega_l \pm \Omega_k|\right) ,$$

and ρ, σ, δ_r which are the constants and the restrictions of the domain due to Cauchy's estimates (see Lemma 2.5.1). Since the approximate periodic orbit is a periodic orbit for the truncated normal form, we get

Lemma 2.3.1 *Let $x^* = (q^*, 0, 0, 0)$ be a relative equilibrium for the truncated normal form $K^{(r)}$, i.e. an approximate periodic orbit for the Hamiltonian $H^{(r)}$, then $\Upsilon(x^*; \varepsilon, q_1(0))$ is of order $\mathcal{O}(\varepsilon^{r+1})$.*

Proof. Consider the remainder of the Hamiltonian $H^{(r)}$, namely $\sum_{s>r}\sum_{\ell\geq 0} f_\ell^{(r,s)}$. For $\varepsilon < \frac{1}{2^{12}\Xi_r^5}$, we get the estimate:

$$\sum_{s>r}\sum_{\ell\geq 0} \|f_\ell^{(r,s)}\| \leq 2E\left(2^{14}\Xi_r^5\varepsilon\right)^{r+1} .$$

Hence, following the same procedure of Lemma 1.4.1, we obtain the estimate for the map Υ.

□

The proof of Theorem 2.1.1 simply consists in the application of Proposition A.1.1 [7]. In particular, the main assumption concerns the invertibility of the matrix $M(\varepsilon)$ and its eigenvalues. In contrast to the non-degenerate case, now it is really difficult to directly verify condition (A.2) starting from the definition (2.11). Indeed, it requires to insert (2.19) in the new functions $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{R}$ and $\mathfrak{S}$, to expand it w.r.t. ε and, finally, to compute the differential w.r.t. the initial datum, evaluated at x^*. However, we can take advantage of the normal form construction and of the connection with the monodromy matrix, which allow the matrix $M(\varepsilon)$ to be more easily calculated, without computing all the expansions w.r.t. ε just mentioned.

The monodromy matrix of a system is the fundamental matrix $\Phi(t)$ evaluated at the period T, under the hypothesis $\Phi(0) = I$. It also corresponds to the differential of the flow at the period T w.r.t. the initial datum of the periodic orbit. Indeed, the flow satisfies the equation

$$\frac{d\phi^t(x_0)}{dt} = X_{H^{(r)}}(\phi^t(x_0)),$$

where $X_{H^{(r)}}$ is the Hamiltonian vector field of $H^{(r)}$. By computing the differential w.r.t. the initial datum, one gets the variational equation

$$\begin{cases} \dfrac{d}{dt} d_{x_0}\phi^t(x_0) = dX_{H^{(r)}}(\phi^t(x_0)) d_{x_0}\phi^t(x_0) \\ d_{x_0}\phi^0(x_0) = \mathbb{I} \end{cases} ,$$

As a result, $\Phi(T; x_0) = d_{x_0}\phi^T(x_0)$. Moreover, we observe that the matrix $M(\varepsilon)$ corresponds to $d_{x_0}\phi^T(x^*) - \mathbb{I}$, in which we have neglected the first column and the equation for p_1. As a consequence, from the structure of the Jacobian matrix $dX_{H^{(r)}}(\phi^t(x^*))$, we can deduce the structure of the monodromy matrix and, consequently, of $M(\varepsilon)$. Indeed, similarly to the first normalization step, if we consider the linearized equation of the Hamiltonian $K^{(r)}$ in normal form up to order r (i.e. the approximate variational equation), around the approximate equilibrium, we get a constant matrix with null block on the anti-diagonal. Taking into account the exponential of

[7] See section A.1 in the Appendix.

this matrix multiplied by the period T, we obtain the approximate monodromy matrix evaluated at x^*, with null block on the anti-diagonal. Neglecting the first column and the equation for p_1 and subtracting the identity matrix, we get the structure for $M(\varepsilon)$ claimed by the following Lemma. Prior to state the Lemma, we introduce a convenient notation. Let M be a $2n$-dimensional square matrix. We denote by M_{red} the reduced matrix, namely the $(2n-1)$-dimensional square matrix obtained from M by removing the first column (related to the fast angle q_1) and the (n_1+1)-th row (related to the momentum p_1).

Lemma 2.3.2 *The differential $M(\varepsilon)$ defined in* (2.11) *is the reduction of $\Phi(T;H^{(r)},x^*)-\mathbb{I}$, namely*

$$M(\varepsilon) = \Big(\Phi(T;H^{(r)},x^*)-\mathbb{I}\Big)_{\mathsf{red}} \ .$$

Moreover, $M(\varepsilon)$ has the following decomposition

$$M(\varepsilon) = N(\varepsilon) + \mathcal{O}(\varepsilon^{r+1}) \ , \qquad \textit{with} \quad N(\varepsilon) = \left(\begin{array}{c|c} \tilde{N}(\varepsilon) & O \\ \hline O & \hat{N}(\varepsilon) \end{array}\right) \ ,$$

where the leading term reads

$$N(\varepsilon) = \Big(\Phi(T;K^{(r)},x^*)-\mathbb{I}\Big)_{\mathsf{red}} \ , \qquad \textit{with} \quad \Phi(T;K^{(r)},x^*) = \exp\left(dX_{K^{(r)}}(x^*)T\right) \ .$$

We can now prove the Theorem 2.1.1:

Proof. Let us stress that we have proved (A.1) in Lemma 2.3.1, with $\beta = r+1$. Besides, the condition (A.3), which is a Lipschitz continuity requirement, is also satisfied, in view of the analyticity of the Hamiltonian and its vector field. As regards the second hypothesis (A.2) on the invertibility of the Jacobian matrix and on the smallness of its eigenvalues, in order to investigate them, we have to exploit the Lemma 2.3.2, obtaining the matrix

$$M(\varepsilon) = N(\varepsilon) + \mathcal{O}(\varepsilon^{r+1}) \qquad \text{with} \quad N(\varepsilon) = \left(\begin{array}{c|c} \tilde{N}(\varepsilon) & O \\ \hline O & \hat{N}(\varepsilon) \end{array}\right) \ ,$$

and

$$\hat{N}(0) = \begin{pmatrix} e^{2\pi\mathbf{i}\frac{\bar{\Omega}}{\omega}} - I & O \\ O & e^{-2\pi\mathbf{i}\frac{\bar{\Omega}}{\omega}} - I \end{pmatrix} \ .$$

If the matrix $N(\varepsilon)$ is invertible, then the same holds true for $M(\varepsilon)$, by continuity. One can also prove[8] that if $|\lambda| \gtrsim \varepsilon^\alpha$ (with $\alpha < r$, which is guaranteed by the hypothesis $2\alpha < r+1$), with $\lambda \in \sigma(N)$, then $|\nu| \gtrsim \varepsilon^\alpha$, with $\nu \in \sigma(M)$. Moreover, from the structure of the matrix $N(\varepsilon)$, we can deduce that its spectrum is the union of the spectrum of the two blocks $\tilde{N}$ and $\hat{N}$. We also remark that the block $\hat{N}$ is always invertible, because of the invertibility of its leading order, due to the first Melnikov condition (2.5). Furthermore, the smallest eigenvalue of the matrix N is the smallest eigenvalue of the matrix $\tilde{N}$. Therefore, we only need the requirement on the spectrum of the block $\tilde{N}$.

□

[8]See Proposition A.3.1 in the Appendix.

It is well worth noting how the statement of Theorem 2.1.1 may be formulated, according to the structure of the normal form algorithm. In order to state a theorem about the continuation of periodic orbits, one can perform only three stages of the normalization step, the third one consisting of the average of the term $\tilde{f}_2^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q})$ only. By so doing, the abstract result requires an assumption on the eigenvalues of the matrix $N(\varepsilon)$, not only on the block $\tilde{N}(\varepsilon)$. Indeed, three stages do not suffice to obtain the two block on the anti-diagonal equal to zero, in general. This does not allow to split the spectrum of the matrix $N(\varepsilon)$ in the spectrum of its diagonal blocks.

As a consequence, with the purpose of getting a more accessible criterion for applications, it is necessary to perform a fourth stage in the normalization step. It permits to remove the term $f_3^{(\mathrm{III};r-1,r)} - f_3^{(\mathrm{III};r-1,r)}\Big|_{\hat{p}=0}$, achieving the desired structure with null blocks on the anti-diagonal. Let us stress that the fourth stage does not need a second Melnikov condition. We also remark that the matrix of the linearized system is not independent of time in this case, due to lack of averaging of the terms $\hat{f}_2^{(\mathrm{II};r-1,r)}(\xi,\eta)$ and $\tilde{f}_4^{(\mathrm{II};r-1,r)}(\hat{q},\hat{p})$, namely of the second half of the third stage and of the fifth stage. Therefore, we cannot easily deduce the structure of the matrix $M(\varepsilon)$, simply considering the exponential of the linearized matrix. However, it allows to simplify the statement, giving a criterion on the eigenvalues of the block $\tilde{N}(\varepsilon)$, in all models in which the Hamiltonian does not depend on the fast angle q_1 (because of the effect of some symmetries of the systems). So, the fourth stage actually allows to get an easier condition to be verified for applications.

2.4 Approximate and effective linear stability

Coming back to the complete normal form scheme with five stages, in order to investigate the approximate linear stability of the approximate periodic orbit we have to compute the eigenvalues of the matrix of the linearization obtained from the quadratic Hamiltonian $K_2^{(r)}$ of the normal form. Hence, as in the first normalization step, we need to study the spectrum of the constant matrix

$$L(\varepsilon) = \begin{pmatrix} L_{11}(\varepsilon) & O \\ O & L_{22}(\varepsilon) \end{pmatrix} ,$$

with

$$L_{11}(\varepsilon) = \begin{pmatrix} \tilde{D}(\varepsilon)^\top & C(\varepsilon) \\ -\tilde{B}(\varepsilon) & -\tilde{D}(\varepsilon) \end{pmatrix} , \qquad \text{and} \qquad L_{22}(\varepsilon) = \begin{pmatrix} E(\varepsilon)^\top & F(\varepsilon) \\ -G(\varepsilon) & -E(\varepsilon) \end{pmatrix} ,$$

where

$$B(\varepsilon) = D^2_q K^{(r)}(\omega t + q_1(0), x^*) = D^2_q \left[\sum_{s=1}^{r} f_0^{(r,s)} \right] (x^*) = \varepsilon B_1 + \ldots + \varepsilon^r B_r \ ,$$

$$C(\varepsilon) = D^2_{\hat{p}} K^{(r)}(\omega t + q_1(0), x^*) = D^2_{\hat{p}} \left[\sum_{s=0}^{r} \tilde{f}_4^{(r,s)} \right] (x^*) = C_0 + \ldots + \varepsilon^r C_r \ ,$$

$$D(\varepsilon) = D^2_{q\hat{p}} K^{(r)}(\omega t + q_1(0), x^*) = D^2_{q\hat{p}} \left[\sum_{s=1}^{r} \tilde{f}_2^{(r,s)} \right] (x^*) = \varepsilon D_1 + \ldots + \varepsilon^r D_r \ ,$$

$$E(\varepsilon) = D^2_{\xi\eta} K^{(r)}(\omega t + q_1(0), x^*) = D^2_{\xi\eta} \left[\sum_{s=0}^{r} \hat{f}_2^{(r,s)} \right] (x^*) = E_0 + \ldots + \varepsilon^r E_r \ ,$$

$$F(\varepsilon) = D^2_{\eta} K^{(r)}(\omega t + q_1(0), x^*) = D^2_{\eta} \left[\sum_{s=1}^{r} \hat{f}_2^{(r,s)} \right] (x^*) = \varepsilon F_1 + \ldots + \varepsilon^r F_r \ ,$$

$$G(\varepsilon) = D^2_{\xi} K^{(r)}(\omega t + q_1(0), x^*) = D^2_{\xi} \left[\sum_{s=1}^{r} \hat{f}_2^{(r,s)} \right] (x^*) = \varepsilon G_1 + \ldots + \varepsilon^r G_r \ ,$$

$\tilde{B}(\varepsilon)$ is the square-matrix obtained adding a zero row at first position and a zero column at first position to $B(\varepsilon)$ and $\tilde{D}(\varepsilon)$ is the square-matrix obtained adding a zero row at first position to $D(\varepsilon)$. By factoring out a λ^2 dependence in the equation $\det(L(\varepsilon) - \lambda\mathbb{I})$, similarly to the first normalization step, we are reduced to compute the eigenvalues of

$$V(\varepsilon) = \begin{pmatrix} V_{11}(\varepsilon) & O \\ O & V_{22}(\varepsilon) \end{pmatrix} ,$$

with

$$V_{11}(\varepsilon) = \begin{pmatrix} (\lfloor D(\varepsilon))^\top & \lfloor \overline{C}(\varepsilon) \\ -B(\varepsilon) & -\lfloor D(\varepsilon) \end{pmatrix} \qquad \text{and} \qquad V_{22}(\varepsilon) = L_{22}(\varepsilon) \tag{2.31}$$

where $\lfloor D(\varepsilon)$ is the $n_1 - 1$ square-matrix $D(\varepsilon)$ without the first columns, and $\lfloor \overline{C}(\varepsilon)$ is the $n_1 - 1$ square-matrix $C(\varepsilon)$ without both the first column and row. Observe that the spectrum of the approximate matrix $V(\varepsilon)$ is the union of the spectrum of two diagonal blocks. The first one is $\Sigma(V_{11}(\varepsilon))$, made of $2n_1 - 2$ eigenvalues which vanish as $\varepsilon \to 0$. The second one is $\Sigma(V_{22}(\varepsilon))$. As in the first normalization step, we assume $\Omega_j > 0$ (or $\Omega_j < 0$) so that the matrix V_{22} is positive (negative) definite and the elliptic equilibrium persists for ε small enough. Let us stress that the same claim may be inferred invoking the Krein's signature, as observed in Remark 2.2.6.

As a consequence, the approximate linear stability of the periodic orbits only depends on the vanishing part of the spectrum $\Sigma(V_{11}(\varepsilon))$. Moreover, in order to get linear stability, it is necessary to demand that all its eigenvalues be purely imaginary.

We now investigate to what extent the stability of the true periodic orbit can be inferred by the stability of the approximate one, under suitable assumptions on $\Sigma(V_{11}(\varepsilon))$. Let us stress that the above matrix is an approximation of the matrix one has to consider in order to investigate the linear stability of true periodic orbits. To this end, we have to add $\mathcal{O}(\varepsilon^{r+1})$, due to the perturbation of the normal form $K^{(r)}$, and a second perturbation $\mathcal{O}(\varepsilon^{r+1-\alpha})$, because of the error $\left\| x^*_{\text{p.o.}} - x^* \right\| \leq c_0 \varepsilon^{r+1-\alpha}$.

We can now state the following Theorem

Theorem 2.4.1 *Consider the monodromy matrix $\Phi(T; H^{(r)}, x^*_{\text{p.o.}})$ and its approximation given by $\exp(L(\varepsilon)T)$, with $L_{22}(\varepsilon)$ positive definite. Then for $|\varepsilon|$ small enough the following holds true:*

1. *there exists a positive constant c_A such that one has*

$$\Phi(T; H^{(r)}, x^*_{\text{p.o.}}) = \exp\left(L(\varepsilon)T\right) + A \ , \qquad \textit{with} \quad \|A\|_{op} \le c_A |\varepsilon|^{r+1-\alpha} \ , \tag{2.32}$$

where α is the same as in Theorem 2.1.1;

2. *$\Sigma(\Phi(T; H^{(r)}, x^*_{\text{p.o.}})) = \Sigma_{11} \cup \Sigma_{22}$, where Σ_{11} is close to $\Sigma(\exp(L_{11}(\varepsilon)T))$ and includes at least two elements equal to 1, while Σ_{22} is close to $\Sigma(\exp(L_{22}(\varepsilon)T))$ and all its elements lie on the unit circle.*

Proof. In view of continuity and separation of the two spectra $\Sigma(L_{11}(\varepsilon))$ and $\Sigma(L_{22}(\varepsilon))$, the spectrum of the monodromy matrix splits into two different components. Moreover, Krein's signature theory ensures that Σ_{22}, which is a deformation of $\Sigma(\exp(L_{22}(\varepsilon)T))$, lies on the unit circle.

In order to obtain the estimate of the error in (2.32), we exploit the fact that the monodromy matrix is the differential of the flow with respect to the initial datum. Considering the matrix $\Phi(T; K^{(r)}, x^*_{\text{p.o.}})$, we take into account two different sources of approximation: the one of the Hamiltonian $H^{(r)}$ with its normal form $K^{(r)}$ and the one due to the approximation of the initial datum of the periodic orbit. Hence, the error term consists of the normal form remainder $\mathcal{O}(\varepsilon^{r+1})$ and of the error of the periodic orbit, which is of order $\mathcal{O}(\varepsilon^{r+1-\alpha})$ (with $2\alpha < r+1$), as it follows from Theorem 2.1.1. The latter is the dominant one and this concludes the proof.

□

Let us stress that the matrix $N(\varepsilon)$ in Theorem 2.1.1 can now be rewritten as

$$N(\varepsilon) = (\exp(L(\varepsilon)T) - \mathbb{I})_{red}$$

with $\tilde{N}(\varepsilon) = (\exp(L_{11}(\varepsilon)T) - \mathbb{I})_{red}$.

We are now ready to prove the Theorem 2.1.2 on the localization of the eigenvalues[9] of Σ_{11} by exploiting the spectrum of the matrix $L_{11}(\varepsilon)$ in the generic case of distinct eigenvalues.

Proof. [Theorem 2.1.2] The proof follows from Proposition A.3.2 in the Appendix, by exploiting (2.32) and the fact that the difference between the Floquet multipliers close to 1, $e^{\lambda_j T} - e^{\lambda_k T}$, is, at leading order, the same as the exponents $\lambda_j - \lambda_k$.

□

Remark 2.4.1 *If the eigenvalues in Theorem 2.1.2 are not distinct or β does not satisfy the condition $\beta < r + 1 - \alpha$, one can also take advantage of the normal form algorithm and perform further normalization steps, in order to increase the accuracy of the approximation and try to apply the Theorem. An example of this procedure will be given in the railway model in Chapter 3.*

2.5 Analytic estimates

Now, our aim is to turn the formal algorithm into a recursive scheme of estimates. We here report only the statements needed to describe the analytic estimates. The detailed proofs are given in Section A.2 of the Appendix for three stages of the normalization step, which contain all the key aspects of the procedure. The case of five stages only requires further calculations. Before detailing the main results, we must anticipate some useful technical tools.

2.5.1 Estimates for Poisson brackets and Lie series

We report some basic Cauchy's estimates which will be needed to bound the transformed Hamiltonian.

[9] See also Lemma 2 in [1].

Lemma 2.5.1 *Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $g \in \mathcal{P}_\ell$ be an analytic function with bounded norm $\|g\|_1$. Then one has*

$$\left\|\frac{\partial g}{\partial \hat{p}_j}\right\|_{1-d} \le \frac{\|g\|_1}{d\rho}\ , \qquad \left\|\frac{\partial g}{\partial \hat{q}_j}\right\|_{1-d} \le \frac{\|g\|_1}{ed\sigma}\ , \qquad \left\|\frac{\partial g}{\partial \xi_j}\right\|_{1-d} \le \frac{\|g\|_1}{dR}\ , \qquad \left\|\frac{\partial g}{\partial \eta_j}\right\|_{1-d} \le \frac{\|g\|_1}{dR}\ ,$$

Lemma 2.5.2 *Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $j \ge 1$. Then one has*

$$\begin{aligned}
\left\|L^j_{\chi_0^{(r)}} f\right\|_{1-d-d'} &\le \frac{j!}{e}\left(\frac{e\|X_0^{(r)}\|_{1-d'}}{d^2\rho\sigma} + \frac{e|\zeta^{(r)}|}{d\rho}\right)^j \|f\|_{1-d'}\ ,\\
\left\|L^j_{\chi_1^{(r)}} f\right\|_{1-d-d'} &\le \frac{j!}{e^2}\left(\frac{\|\chi_1^{(r)}\|_{1-d'}}{d^2}\left(\frac{e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right)^j \|f\|_{1-d'}\ ,\\
\left\|L^j_{\chi_2^{(r)}} f\right\|_{1-d-d'} &\le \frac{j!}{e^2}\left(\frac{\|\chi_2^{(r)}\|_{1-d'}}{d^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right)^j \|f\|_{1-d'}\ ,\\
\left\|L^j_{\chi_3^{(r)}} f\right\|_{1-d-d'} &\le \frac{j!}{e^2}\left(\frac{\|\chi_3^{(r)}\|_{1-d'}}{d^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right)^j \|f\|_{1-d'}\ ,\\
\left\|L^j_{\chi_4^{(r)}} f\right\|_{1-d-d'} &\le \frac{j!}{e^2}\left(\frac{2e\|\chi_4^{(r)}\|_{1-d'}}{d^2\rho\sigma}\right)^j \|f\|_{1-d'}\ .
\end{aligned}$$

2.5.2 Recursive scheme of estimates

Having fixed $d \in \mathbb{R}$, $0 < d \le 1/4$, we consider a sequence $\delta_{r\ge1}$ of positive real numbers satisfying

$$\delta_{r+1} \le \delta_r\ , \quad \sum_{r\ge1}\delta_r \le \frac{d}{5}\ ,$$

and a further sequence $d_{r\ge0}$ defined as

$$d_0 = 0\ , \quad d_r = d_{r-1} + 5\delta_r\ .$$

This sequence allows to control the restrictions of the domain due to the Cauchy's estimate.

The factors entered by the estimate of the norm of the Poisson brackets are bounded by

$$\Xi_r = \max\left(\frac{eE}{\alpha\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\alpha\delta_r^2\rho\sigma}, \frac{E}{\alpha\delta_r^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right)\ ,$$

with

$$\alpha = \min_{k_1,j,l,k}\left(|\omega|, |k_1\omega \pm \Omega_j|\,, |k_1\omega \pm \Omega_l \pm \Omega_k|\right)\ ,$$

that is strictly greater than zero in view of the the Melnikov conditions.

The number of terms in (2.22), (2.25), (2.27), (2.29) and (2.30) is controlled by the five sequences

$$\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \geq 0\,, \\
\nu^{(\mathrm{I})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} \nu^{j}_{r-1,r}\nu_{r-1,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu^{(\mathrm{II})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{I})}_{r,r})^{j}\nu^{(\mathrm{I})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu^{(\mathrm{III})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (2\nu^{(\mathrm{II})}_{r,r})^{j}\nu^{(\mathrm{II})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,. \\
\nu^{(\mathrm{IV})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{III})}_{r,r})^{j}\nu^{(\mathrm{III})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,. \\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{IV})}_{r,r})^{j}\nu^{(\mathrm{IV})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,.
\end{aligned}$$

We can now state the following Lemma:

Lemma 2.5.3 *The sequence of positive integers* $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ *defined in* (A.13) *is bounded by*

$$\nu_{r,s} \leq \nu_{s,s} \leq \frac{2^{14s}}{2^{8}}\ .$$

The following Lemma collects all the key estimates concerning the generating functions and the transformed Hamiltonians.

Lemma 2.5.4 *Consider a Hamiltonian* $H^{(r-1)}$ *expanded as in* (2.2)*. Let* $\chi^{(r)}_0$, $\chi^{(r)}_1$, $\chi^{(r)}_2$, $\chi^{(r)}_3$ *and* $\chi^{(r)}_4$ *be the generating functions used to put the Hamiltonian in normal form at order* r*, then one has*

$$\begin{aligned}
\|X^{(r)}_0\|_{1-d_{r-1}} &\leq \frac{1}{\alpha}\nu_{r-1,r}\Xi_r^{5r-5}E\varepsilon^{r}\ , \\
|\zeta^{(r)}| &\leq \frac{1}{4m\rho}\nu_{r-1,r}\Xi^{5r-3}E\varepsilon^{r}\ , \\
\|\chi^{(r)}_1\|_{1-d_{r-1}-\delta_r} &\leq \frac{1}{\alpha}\nu^{(\mathrm{I})}_{r,r}\Xi_r^{5r-4}\frac{E}{2}\varepsilon^{r}\ , \\
\|\chi^{(r)}_2\|_{1-d_{r-1}-2\delta_r} &\leq \frac{1}{\alpha}2\nu^{(\mathrm{II})}_{r,r}\Xi_r^{5r-3}\frac{E}{2^2}\varepsilon^{r}\ , \\
\|\chi^{(r)}_3\|_{1-d_{r-1}-3\delta_r} &\leq \frac{1}{\alpha}\nu^{(\mathrm{III})}_{r,r}\Xi_r^{5r-2}\frac{E}{2^3}\varepsilon^{r}\ , \\
\|\chi^{(r)}_4\|_{1-d_{r-1}-4\delta_r} &\leq \frac{1}{\alpha}\nu^{(\mathrm{IV})}_{r,r}\Xi_r^{5r-1}\frac{E}{2^4}\varepsilon^{r}\ .
\end{aligned}$$

The terms appearing in the expansion of $H^{(r)}$*, i.e. in* (2.8)*, are bounded as*

$$\|f^{(r,s)}_{\ell}\|_{1-d_r} \leq \nu_{r,s}\Xi_r^{5s}\frac{E}{2^{\ell}}\varepsilon^{s}\ . \tag{2.33}$$

Let us stress that the proof of Lemma 2.5.4 actually requires stricter estimates in (2.33) both for the lower order terms (as it is evident from the bounds on the generating functions) and for the intermediate stages of the r-th normalization step. The detailed proof with the stricter estimates for the exponents is given for three stages of the normalization step in the Appendix.

Chapter 3

Applications

The aim of this Chapter is to present applications of the abstract results gained in previous Chapters. The normal form scheme developed allows to investigate different kinds of degeneracy with the help of a symbolic manipulator, thus confirming the practical applicability of the abstract results. In terms of possible applications, one can consider the problem of the existence of degenerate discrete solitons or multibreathers in one-dimensional discrete non-linear Schrödinger or Klein-Gordon lattices, as well as discrete vortexes in two-dimensional lattices.

Considering a chain of weakly coupled anharmonic oscillators, the simplest case of degeneracy arises when in the multibreather configuration there are holes between oscillators which are large in comparison with the interaction range, thus leading to the lack of some terms in the averaged perturbation. A more subtle form of degeneracy is related to internal symmetries generated by beyond nearest-neighbor interactions and their relative strength, even for consecutive sites configurations.

Several degenerate scenarios, in particular in the dNLS class of Hamiltonian lattices or also in Klein-Gordon models, can also be treated with a suitable rotating frame ansatz[1] and the combination of a Lyapunov-Schmidt decomposition with perturbative techniques (see, e.g., [50, 69, 75, 76]). As regards the Klein-Gordon model, it can be proved that for low energy and small coupling parameter the dNLS model represents a normal form for the Klein-Gordon, so the latter can be approximated with the former in this regime and the standard approach based on the ansatz can be applied (see for instance [67, 71]). However, the normal form approach enables to face every kinds of degeneracy and any resonant module in the same way, so to include, differently from the literature, cases in which the resonances among the excited frequencies differ from the usual $(1 : \ldots : 1)$, since different amplitudes have been chosen for the selected sites. In this latter case the investigation of periodic orbits can be extended to the analysis of two-dimensional subtori foliated by periodic orbits.

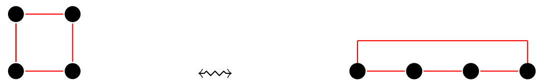
3.1 Main results

In the following Sections we will investigate continuation and linear stability of degenerate periodic orbits in different dNLS models. In particular, all the forthcoming applications are chains of weakly coupled anharmonic oscillators where the coupling parameter ε has to be considered small enough (we are considering the anti-continuum limit $\varepsilon \to 0$) and which turn out to be dNLS models. We will investigate different forms of degeneracy; the first two examples, which contain all the details and calculations in order to clarify how the normal form algorithm and the abstract results work, immediately describe the two main mechanisms of degeneracy mentioned above.

I here summarize the main results obtained in each application:

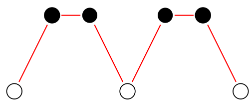
[1] See also Section A.4 for the standard approach for the dNLS model.

- **Square dNLS cell**



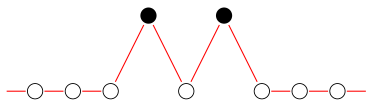
Candidates for continuation at first order: two non-degenerate configurations and three one-parameter families which all intersect in the square vortex configurations (phase differences $\pm\pi/2$). Two families break down at second order, apart from the in/out-of-phase configurations (phase differences 0 or π) which can be continued; the third one and also the two square vortex configurations which are contained in it, reveal to be true solutions of the model. Only the solution with all the phase differences equal to π turns out to be linearly stable.

- **The seagull**



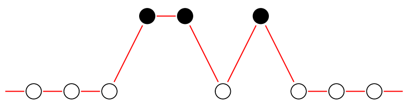
Candidates for continuation at first order: four one-parameter families which do not intersect. At second order the four families break down and only the continuation of the eight in/out-of-phase configurations is feasible. One of the latter is also linearly stable. The effect of focusing/defocusing non-linearity (γ positive or negative) on the linear stability is also studied.

- **Multi-pulse solutions (2 excited sites)**



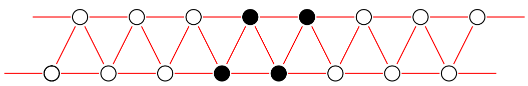
It is a completely degenerate case: at first order every phase difference between the two oscillators is a candidate for the continuation. At second order, only the two in/out-of-phase configurations survive and can be continued, one of which is also linearly stable.

- **Multi-pulse solutions (3 excited sites)**



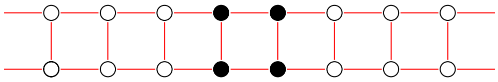
Candidates for continuation at first order: two disjoint one-parameter families. At second order the two families break down and only the four in/out-of-phase configurations still persist and can be continued. One of them is also linearly stable. The role of the non-linear parameter γ is stressed.

- **ZigZag model**



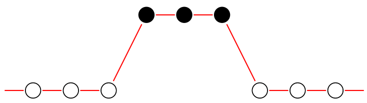
Candidates for continuation at first order: two one-parameter families, intersecting in the vortex-like solutions, and four isolated solutions, three of which non-degenerate and the fourth one degenerate. At second order, the two families break down and only the in/out-of-phase solutions survive. All these latter configurations and the isolated one can be continued. This proves the non-existence of four-sites vortex-like structures in ZigZag models, confirming the result in [76]. Only the solution with all the phase differences equal to 0 turns out to be linearly stable.

- **Railway model**



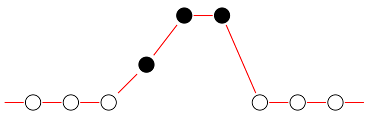
Candidates for continuation at first order: three one-parameter families, which all intersect in the vortex configurations, and four isolated non-degenerate solutions. At second order, we only get four in/out-of-phase solutions and two square vortex configurations. The standard in/out-of-phase ones can be continued, while for the vortexes we need a third normal form step, which is conclusive for their non-existence. Once again, only the solution with all the phase differences equal to 0 is linearly stable. The advantages of the normal form procedure in the study of the effective linear stability are also stressed.

- **Multi-pulse solutions (3 excited sites) with purely non-linear coupling**



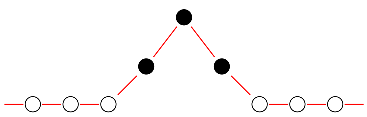
Candidates for continuation at first order: four isolated solutions, three of which degenerate and the fourth one non-degenerate. The degeneracy can be removed only at order three, where we get that all the in/out-of-phase solutions survive and can be continued. One of them is also linearly stable.

- **Different resonances: non-degenerate case**



Due to the resonances different from the standard $(1 : \ldots : 1)$ and to the action of the symmetry in the dNLS, the first normalization order results in existence and continuation of non-degenerate two-dimensional subtori foliated by periodic orbits, for phase differences between the third and the second excited oscillator equal to $0, \pi$.

- **Different resonances: degenerate case**



This is a completely degenerate case, where the degeneracy is removed at second order which provides the continuation of two-dimensional subtori foliated by periodic orbits, for phase differences between the third and the first excited oscillator equal to $0, \pi$.

The normal form algorithm and all the other computations have been performed by means of Mathematica.

Before entering the details of the applications, we introduce the dNLS model.

The dNLS model

We consider the dNLS equation in the general form

$$\mathbf{i}\dot{\psi}_j = \psi_j - \varepsilon(L\psi)_j + \gamma\psi_j|\psi_j|^2 \ , \qquad \text{with} \quad j \in \mathcal{J} \ , \tag{3.1}$$

where $\psi_j \in \mathbb{C}$, $\gamma \neq 0$ is a parameter tuning the non-linearity and L is a linear operator which can include beyond nearest-neighbors interactions. In this Chapter we will make different choices for the linear operator L. The boundary conditions will be either periodic or of Dirichlet type in the case of $\mathcal{J}$ finite[2]. The equations can be written in Hamiltonian form $\mathbf{i}\dot{\psi}_j = \frac{\partial H}{\partial \bar{\psi}_j}$ with

$$H = H_0 + \varepsilon H_1 \ , \qquad\qquad H_0 = \sum_{j\in\mathcal{J}} |\psi_j|^2 + \frac{\gamma}{2}\sum_{j\in\mathcal{J}} |\psi_j|^4 \ .$$

We introduce the set of excited sites $\mathcal{I} = \{j_1, \dots, j_{n_1}\} \subset \mathcal{J}$, not necessarily consecutive, in order to include also configurations where the localization of the amplitude (hence of the energy), is clustered, with holes separating the different clusters along the lattice.

In the limit of $\varepsilon = 0$, we consider unperturbed excited oscillators $\left\{\psi_j^{(0)}\right\}_{j\in\mathcal{I}}$ with resonant frequencies in order to get a periodic flow on the resonant torus. The typical choice is given by the $(1 : \dots : 1)$ resonance, obtained by choosing a common frequency ω for all the $\left\{\psi_j^{(0)}\right\}_{j\in\mathcal{I}}$. All the unperturbed periodic orbits foliate a n-dimensional torus of the phase space: the torus corresponds to $|\psi_j|^2 = R^2$ for $j \in \mathcal{I}$ and $\psi_j = 0$ for the remaining $j \notin \mathcal{I}$. Hence, this problem for the dNLS model can be recast in the investigation of the breaking of a completely resonant torus, namely we want to determine via our normal form procedure which solutions, degenerate or not, are going to survive as $\varepsilon \neq 0$, at fixed period.

For the applications of the normal form approach, we start from the "real" Hamiltonian formulation of systems of weakly coupled anharmonic oscillators, which can be obtained from the original Hamiltonian by using real and imaginary parts of the complex amplitudes ψ_j as canonical configurations-momenta: more precisely

$$x_j = \frac{\mathbf{i}}{\sqrt{2}}\big(\psi_j - \bar{\psi}_j\big) \ , \qquad y_j = \frac{1}{\sqrt{2}}\big(\bar{\psi}_j + \psi_j\big) \quad \Rightarrow \quad \frac{1}{2}\big(x_j^2 + y_j^2\big) = |\psi_j|^2 \ . \tag{3.2}$$

Let me stress that in all the following applications the twist condition (1.7) or (2.3) will be satisfied thanks to the anharmonicity of the oscillators. Instead, as regards the first and the second Melnikov conditions, it is necessary to choose suitable values for the parameters γ and I_j^* with $j \in \mathcal{I}$, the latter being the actions which define the torus.

3.2 Applications: full dimensional tori

3.2.1 Square dNLS cell with nearest-neighbour interaction

The following example is a model of weakly coupled oscillators which reveals to be a dNLS model (3.1) of the following kind

$$\mathbf{i}\dot{\psi}_j = \psi_j - \varepsilon(L\psi)_j + \gamma\psi_j|\psi_j|^2 \ , \qquad -(L\psi)_j = \psi_{j+1} + \psi_{j-1} \ , \tag{3.3}$$

with $j \in \mathcal{J} = \{1, \dots, 4\}$, $\gamma = 2$, and periodic boundary conditions. The equations can be written in the Hamiltonian form $\mathbf{i}\dot{\psi}_j = \frac{\partial H}{\partial \bar{\psi}_j}$ from

$$H = \sum_{j\in\mathcal{J}} \big(|\psi_j|^2 + |\psi_j|^4 + \varepsilon\big(\psi_{j+1}\overline{\psi}_j + \overline{\psi}_{j+1}\psi_j\big)\big) \ , \tag{3.4}$$

[2] Or vanishing at infinity as $\psi \in \ell^2(\mathbb{C})$ in the case of infinite $\mathcal{J}$: this case is not properly covered by our normal form technique, since the analytical estimates should be extended; however the formal algorithm works as well.

with $\psi_5 = \psi_1$. This is a model where extrema of the averaged Hamiltonian are not isolated. In particular, degenerate solutions appear with the weakest possible degeneracy, namely they are one-parameter families of approximate solutions. This allows to get a more applicable formulation of Theorem 1.2.1, as proved in Section 1.5. Otherwise, the structure of the monodromy matrix can be exploited and the Theorem 2.1.1 can be applied.

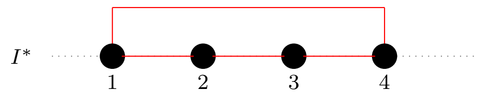
Let us rewrite the Hamiltonian system in real coordinates with the canonical transformation (3.2), obtaining

$$H = H_0 + \varepsilon H_1 = \sum_{j=1}^{4} \left(\frac{x_j^2 + y_j^2}{2} + \left(\frac{x_j^2 + y_j^2}{2} \right)^2 + \varepsilon (x_{j+1} x_j + y_{j+1} y_j) \right) ,$$

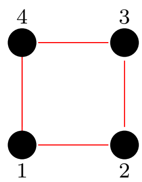
with periodic boundary conditions, i.e., $x_5 \equiv x_1$ and $y_5 \equiv y_1$. Introducing the action-angle variables $(x_j, y_j) = (\sqrt{2I_j}\cos\varphi_j, -\sqrt{2I_j}\sin\varphi_j)$, the Hamiltonian reads

$$H = \sum_{j=1}^{4} \left(I_j + I_j^2 + 2\varepsilon \sqrt{I_{j+1} I_j} \cos(\varphi_{j+1} - \varphi_j) \right) .$$

Let us now fix the completely resonant torus $I^* = (I^*, I^*, I^*, I^*)$. The model is a one-dimensional discrete Hamiltonian lattice with an interaction beyond nearest neighbors, as represented in the following picture.



Otherwise, the configuration can be seen as a two-dimensional discrete Hamiltonian lattice with nearest neighbor interactions, which explains the name square cell.



Making a Taylor expansion around I^*, i.e. setting $I_j = J_j + I^*$ for $j = 1, \ldots, 4$, the unperturbed part H_0 reads

$$H_0(J) = 4I^* + 4(I^*)^2 + (1 + 2I^*)(J_1 + J_2 + J_3 + J_4) + J_1^2 + J_2^2 + J_3^2 + J_4^2 ,$$

while the perturbation H_1 takes the form

$$\begin{aligned} H_1(J, \varphi) = {} & 2I^*(\cos(\varphi_2 - \varphi_1) + \cos(\varphi_3 - \varphi_2) + \cos(\varphi_4 - \varphi_3) + \cos(\varphi_4 - \varphi_1)) \\ & + (J_1 + J_2)\cos(\varphi_2 - \varphi_1) + (J_3 + J_2)\cos(\varphi_3 - \varphi_2) \\ & + (J_4 + J_3)\cos(\varphi_4 - \varphi_3) + (J_1 + J_4)\cos(\varphi_4 - \varphi_1) \\ & - \frac{(J_1 - J_2)^2\cos(\varphi_2 - \varphi_1)}{4I^*} - \frac{(J_2 - J_3)^2\cos(\varphi_3 - \varphi_2)}{4I^*} \\ & - \frac{(J_3 - J_4)^2\cos(\varphi_4 - \varphi_3)}{4I^*} - \frac{(J_1 - J_4)^2\cos(\varphi_4 - \varphi_1)}{4I^*} + \mathcal{O}(|J|^3) . \end{aligned}$$

We introduce[3] the resonant angles $\hat{q} = (q_1, q)$ and their conjugate actions $\hat{p} = (p_1, p)$

$$\begin{cases} q_1 = \varphi_1 \\ q_2 = \varphi_2 - \varphi_1 \\ q_3 = \varphi_3 - \varphi_2 \\ q_4 = \varphi_4 - \varphi_3 \end{cases} , \qquad \begin{cases} p_1 = J_1 + J_2 + J_3 + J_4 \\ p_2 = J_2 + J_3 + J_4 \\ p_3 = J_3 + J_4 \\ p_4 = J_4 \end{cases} .$$

[3] In this example, we have preferred the angles to be the relative phase differences among consecutive angles, rather than the phase differences with respect to the first angle φ_1, as in the previous Chapters.

Thus, ignoring the constant terms, we can rewrite H as

$$\begin{aligned}
H &= \omega p_1 + \Big((p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_4)^2+p_4^2\Big)\\
&\quad + \varepsilon\Big[\Big(2I^*\cos(q_2)+2I^*\cos(q_3)+2I^*\cos(q_4)+2I^*\cos(q_2+q_3+q_4)\Big)\\
&\quad + (p_1-p_3)\cos(q_2)+(p_2-p_4)\cos(q_3)+p_3\cos(q_4)\\
&\quad + (p_1-p_2+p_4)\cos(q_2+q_3+q_4)\\
&\quad - \frac{\cos(q_2)(p_1-2p_2+p_3)^2}{4I^*} - \frac{\cos(q_3)(p_2-2p_3+p_4)^2}{4I^*} - \frac{\cos(q_4)(p_3-2p_4)^2}{4I^*}\\
&\quad - \frac{\cos(q_2+q_3+q_4)(p_1-p_2-p_4)^2}{4I^*}\Big] + \mathcal{O}(\varepsilon|\hat{p}|^3)\\
&= \omega p_1 + f_4^{(0,0)}(p_1,p_2,p_3,p_4) + f_0^{(0,1)}(q_2,q_3,q_4)\\
&\quad + f_2^{(0,1)}(p_1,p_2,p_3,p_4,q_2,q_3,q_4) + f_4^{(0,1)}(p_1,p_2,p_3,p_4,q_2,q_3,q_4) + \mathcal{O}(\varepsilon|\hat{p}|^3)\ ,
\end{aligned}$$

where $\omega = 1+2I^*$.

We now proceed to polish the Hamiltonian in order to study continuation and stability of periodic orbits.

We observe that the Hamiltonian does not depend on the fast angle q_1. This is due to the effect of the Gauge symmetry of the model, as visible in the complex form (3.4). As a consequence, $f_0^{(0,1)}(q_2,q_3,q_4)$ is already in normal form and the first stage only consists in the translation of the actions, which allows to keep ω fixed. Since $f_2^{(0,1)}$ is automatically averaged w.r.t. q_1, the homological equation defining $\zeta^{(1)}$ is equivalent to the following linear system

$$\langle \nabla_{\hat{p}} f_4^{(0,0)}, \zeta^{(1)}\rangle = f_2^{(0,1)}\Big|_{q=q^*}\ ,$$

whose solution is given by

$$\begin{cases}
\zeta_1^{(1)} = \varepsilon\left(\cos(q_2^*)+\cos(q_3^*)+\cos(q_4^*)+\cos(q_2^*+q_3^*+q_4^*)\right)\\
\zeta_2^{(1)} = \varepsilon\left(\dfrac{\cos(q_2^*)}{2}+\cos(q_3^*)+\cos(q_4^*)+\dfrac{\cos(q_2^*+q_3^*+q_4^*)}{2}\right)\\
\zeta_3^{(1)} = \varepsilon\left(\dfrac{\cos(q_3^*)}{2}+\cos(q_4^*)+\dfrac{\cos(q_2^*+q_3^*+q_4^*)}{2}\right)\\
\zeta_4^{(1)} = \varepsilon\left(\dfrac{\cos(q_4^*)}{2}+\dfrac{\cos(q_2^*+q_3^*+q_4^*)}{2}\right)
\end{cases}.$$

Since the normal form preserves the symmetry, the newly generated term $f_2^{(\mathrm{I};0,1)}$ is again independent of q_1. This also applies to the term $f_4^{(\mathrm{I};0,1)}$, so no further average is required. The values q^*, which define the approximate periodic orbit at leading order, are given by the solutions of the trigonometric system $\nabla_q f_0^{(1,1)} = 0$ (depending only on sines, due to the parity of the Hamiltonian), which reads

$$\begin{cases}
-2I^*\sin(q_2) - 2I^*\sin(q_2+q_3+q_4) = 0\\
-2I^*\sin(q_3) - 2I^*\sin(q_2+q_3+q_4) = 0\\
-2I^*\sin(q_4) - 2I^*\sin(q_2+q_3+q_4) = 0
\end{cases}.$$

Such solutions are given by the two isolated configurations $(0,0,0)$, (π,π,π), and the three one-parameter families $Q_1 = (\vartheta,\vartheta,\pi-\vartheta)$, $Q_2 = (\vartheta,\pi-\vartheta,\vartheta)$, $Q_3 = (\vartheta,\pi-\vartheta,\pi-\vartheta)$, with $\vartheta\in S^1$, which all intersect in the two opposite configurations $\pm(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. Since the twist condition (2.3) is verified, we only need the invertibility of the matrix B_1 in (1.4) in order to apply the implicit function theorem (which reduces to the classical result of Poincaré). Factoring out $-2I^*$, the non-degeneracy condition reads

$$\begin{vmatrix} \cos(q_2)+\cos(q_2+q_3+q_4) & \cos(q_2+q_3+q_4) & \cos(q_2+q_3+q_4) \\ \cos(q_2+q_3+q_4) & \cos(q_3)+\cos(q_2+q_3+q_4) & \cos(q_2+q_3+q_4) \\ \cos(q_2+q_3+q_4) & \cos(q_2+q_3+q_4) & \cos(q_4)+\cos(q_2+q_3+q_4) \end{vmatrix} \neq 0 \ .$$

If we evaluate the determinant in the two isolated configurations, we get $\det(B_1) = \pm 4 \neq 0$, hence the corresponding solutions can be continued for small enough ε. In the three families we obviously get a degeneracy, since the tangent direction to each family represents a Kernel direction, hence $\det\big(B_1\big|_{Q_j}\big) = 0$. Furthermore, in the intersections $\pm(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ the matrices are identically zero. For all these families a second normalization step is thus needed.

The first stage of the second normalization step deals with

$$f_0^{(1,2)} = f_0^{(\mathrm{I};0,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_2^{(0,1)} + \frac{1}{2} L^2_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)} \ ,$$

which is already averaged over q_1, due to the preservation of the symmetry. The same holds also for the linear term in the action variables $f_2^{(1,2)}$, given by

$$f_2^{(1,2)} = f_2^{(\mathrm{I};0,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,1)} \ .$$

Hence, the homological equation providing the new translation $\zeta^{(2)}$ reads

$$L_{\langle\zeta^{(2)},\hat{q}\rangle} f_4^{(0,0)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,1)} \Big|_{q=q^*} = 0 \ .$$

The new linear term in the action

$$f_2^{(\mathrm{I};1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,1)} + L_{\langle\zeta^{(2)},\hat{q}\rangle} f_4^{(0,0)} \ ,$$

is again already averaged over q_1 and, similarly, no further average is needed for the term

$$f_4^{(\mathrm{I};1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_6^{(0,1)} \ .$$

Hence, the second step is concluded, and the transformed Hamiltonian reads

$$\begin{aligned} H^{(2)} = \omega p_1 &+ f_4^{(2,0)}(\hat{p}) \\ &+ f_0^{(2,1)}(q) + f_2^{(2,1)}(\hat{p},q) + f_4^{(2,1)}(\hat{p},q) \\ &+ f_0^{(2,2)}(q) + f_2^{(2,2)}(\hat{p},q) + f_4^{(2,2)}(\hat{p},q) \\ &+ \mathcal{O}(\varepsilon|\hat{p}|^3) + \mathcal{O}\big(\varepsilon^3\big) \ . \end{aligned}$$

The approximate periodic orbits correspond to the q^* for which

$$\nabla_q \left(f_0^{(2,1)}(q) + f_0^{(2,2)}(q) \right) = \nabla_q f_0^{(2,1)}(q) + \nabla_q \left\langle \nabla_{\hat{p}} f_2^{(0,1)}(q),\, \zeta^{(1)} \right\rangle_{q_1} = 0 \ ,$$

where in the correction due to $f_0^{(2,2)}$, only the term $L_{\langle\zeta^{(1)},\hat{q}\rangle} f_2^{(0,1)}$ really matters, having a nontrivial dependence on the slow angles q. By exploiting the explicit expression for $\zeta^{(1)}$ previously derived, and replacing q^* with q in it, we explicitly get the system

$$\begin{cases} -8\left(\sin(q_2)+\sin(q_2+q_3+q_4)\right)+\varepsilon\Big(2\sin(2q_2)+\sin(q_2-q_3)+2\sin(q_2+q_3) \\ \qquad +2\sin(2q_2+2q_3+2q_4)+2\sin(2q_2+q_3+q_4) \\ \qquad +\sin(q_2+q_3+2q_4)\Big)=0 \\ -8\left(\sin(q_3)+\sin(q_2+q_3+q_4)\right)+\varepsilon\Big(2\sin(2q_3)+\sin(q_3-q_2)+2\sin(q_2+q_3) \\ \qquad +\sin(q_3-q_4)+2\sin(q_3+q_4) \\ \qquad +2\sin(2q_2+2q_3+2q_4)+\sin(2q_2+q_3+q_4) \\ \qquad +\sin(q_2+q_3+2q_4)\Big)=0 \\ -8\left(\sin(q_4)+\sin(q_2+q_3+q_4)\right)+\varepsilon\Big(2\sin(2q_4)+\sin(q_4-q_3)+2\sin(q_3+q_4) \\ \qquad +2\sin(2q_2+2q_3+2q_4)+\sin(2q_2+q_3+q_4) \\ \qquad +2\sin(q_2+q_3+2q_4)\Big)=0 \end{cases},$$

depending on the effective small parameter $\tilde{\varepsilon}=\frac{\varepsilon}{I^*}$. The above system has the structure

$$F(q,\varepsilon)=F_0(q)+\varepsilon F_1(q)=0\ , \tag{3.5}$$

where $F:\mathbb{T}^3\times\mathcal{U}(0)\to\mathbb{R}^3$. Moreover, we have already found at first normalization step that

$$F(Q_j(\vartheta),0)=F_0(Q_j(\vartheta))=0\ .$$

Suppose that there exists a solution $q(\varepsilon)=(q_2(\varepsilon),q_3(\varepsilon),q_4(\varepsilon))$ which is at least continuous in the small parameter, i.e. $\mathcal{C}^0(\mathcal{U}(0),\mathbb{T}^3)$. Hence, by continuity, we must have

$$\lim_{\varepsilon\to 0}F(q_2(\varepsilon),q_3(\varepsilon),q_4(\varepsilon),\varepsilon)=F_0(q_2(0),q_3(0),q_4(0))=0\ ,$$

which means that $q(0)\in Q_j$. Let us introduce the matrices $\tilde{B}_{1,j}(\vartheta)=\frac{\partial F_0(Q_j(\vartheta))}{\partial q}$ and observe that the tangent directions to the three families

$$\partial_\vartheta Q_1=\begin{pmatrix}1\\1\\-1\end{pmatrix},\qquad \partial_\vartheta Q_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}\qquad\text{and}\qquad \partial_\vartheta Q_3=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

represent the Kernel directions of $\tilde{B}_{1,j}$, for $j=1,2,3$, respectively. A standard proposition of bifurcation theory provides a necessary condition for the existence of a solution $Q_j(\vartheta,\varepsilon)$ which is a continuation of $Q_j(\vartheta)$.

Proposition 3.2.1 *Necessary condition for the existence of a solution $q(\varepsilon)=Q_j(\vartheta,\varepsilon)$ of (3.5) is that*

$$F_1(Q_j(\vartheta,0))\in\mathrm{Range}(\tilde{B}_{1,j}(\vartheta))\ .$$

If $\tilde{B}_{1,j}(\vartheta)$ is symmetric, the above condition simplifies

$$F_1(Q_j(\vartheta,0))\perp\mathrm{Ker}(\tilde{B}_{1,j}(\vartheta)). \tag{3.6}$$

Let us apply the above Proposition to show that the families Q_1 and Q_3 break down. Precisely, all their points, except for those corresponding to $\vartheta=\{0,\pi/2,\pi\}$, do not represent true candidates for the continuation. We compute $\langle F_1(Q_j(\vartheta,0)),\partial_\vartheta Q_j\rangle$ for $j=1,3$

$$\langle F_1(Q_1(\vartheta)),\partial_\vartheta Q_1\rangle=8\sin(2\vartheta)=\langle F_1(Q_3(\vartheta)),\partial_\vartheta Q_3\rangle\ ,$$

which shows that the necessary condition is generically violated for the two families $Q_{1,3}$, apart from the in/out-of-phase configurations $(0,0,\pi)$, $(\pi,\pi,0)$, $(0,\pi,\pi)$, $(\pi,0,0)$ and the symmetric vortex configurations $\pm\left(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}\right)$, the last being also points of $Q_2(\vartheta)$.

A way to conclude that the above mentioned in/out-of-phase configurations can be continued to periodic solutions is to apply Theorem 1.5.1. Indeed, the main and first fact to notice is that if $q_0^* = 0,\pi$ then $D_1 = 0$, since it depends only on sines; then by Lemma 1.5.1 we get $m_a(0, M_0) \geq 2$. Moreover, a direct computation shows that the algebraic multiplicity of the zero eigenvalue of M_0 is exactly two, so that we can apply Theorem 1.5.1. In order to verify the main condition (1.23), since $D_1 = 0$, we can restrict to compute only B_2. In the configurations $(0,0,\pi)$ and $(\pi,\pi,0)$, we get

$$B_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 16\,(I^*)^2 & 0 & 16\,(I^*)^2 \\ 0 & 32\,(I^*)^2 & 32\,(I^*)^2 \\ 16\,(I^*)^2 & 32\,(I^*)^2 & 48\,(I^*)^2 \end{pmatrix} \frac{T^2}{6} \ ,$$

while, in $(0,\pi,\pi)$ and $(\pi,0,0)$, we have

$$B_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 48\,(I^*)^2 & 32\,(I^*)^2 & 16\,(I^*)^2 \\ 32\,(I^*)^2 & 32\,(I^*)^2 & 0 \\ 16\,(I^*)^2 & 0 & 16\,(I^*)^2 \end{pmatrix} \frac{T^2}{6} \ .$$

Anyway, we immediately obtain in all the four cases

$$\gamma = \big\langle \langle B_2, a_1 \rangle, a_1 \big\rangle = 4 \neq 0 \ ,$$

with $a_1 = \partial_\vartheta Q_1$ for the first matrix B_2, and $a_1 = \partial_\vartheta Q_3$ for the second one. Hence, the above in/out-of-phase configurations can be continued with $|\lambda| \gtrsim \varepsilon^{1/2}$, λ being an eigenvalue of the matrix $M(\varepsilon)$.

Otherwise, we can infer the continuation of these configurations, taking advantage of the structure of the monodromy matrix, as stated in Lemma 2.3.2. In such a way, we get $|\lambda| \gtrsim \varepsilon$ for all the points, then Theorem 2.1.1 can be applied with $\alpha = 1$ and $r = 2$. Let us remark that the discrepancy in the scaling of the eigenvalues for the two different procedures is only due to the scaling of the actions in the first method.

It remains to investigate the second family Q_2, which satisfies the necessary condition (3.6) because it represents a solution for (3.5), namely $F(Q_2(\vartheta)) \equiv 0$.

We explicitly constructed the normal form up to order three by using Mathematica and checked that this family still persists. This led us to conjecture that it represents a true solution of the problem. Indeed, using the complex coordinates as in (3.4) and the usual ansatz (A.42), we obtain the stationary equation for the amplitudes ϕ_j

$$\lambda\phi_j = 2\phi_j|\phi_j|^2 - \varepsilon(L\phi)_j \ , \qquad \lambda = \omega - 1 \ , \qquad -(L\phi)_j = \phi_{j+1} + \phi_{j-1} \ .$$

If we further assume that the continued solutions have the same amplitude at all the sites, $|\phi_j| = R$, and the phase-shifts belong to the second family Q_2

$$\phi_j = Re^{\mathbf{i}\varphi_j} \ , \qquad \varphi = (\varphi_1, \varphi_1 + \theta, \varphi_1 + \pi, \varphi_1 + \theta + \pi) \ ,$$

then we realize that for any $\theta \in S^1$ one has

$$Le^{\mathbf{i}\varphi(\theta)} = 0 \ .$$

Hence the stationary equation becomes

$$\lambda = 2R^2 = 2I^* \ ,$$

which implies that a two-dimensional resonant torus, embedded in the original unperturbed four-dimensional torus, survives for any given ε.

To study the approximate linear stability of the approximate periodic orbits we have to consider the $2n_1 - 2$ square-matrix $V_{11}(\varepsilon)$ as in (2.31) with the block form

$$V_{11}(\varepsilon) = \begin{pmatrix} (\lfloor D)^\top & \lfloor \overline{C} \\ -B & \lfloor D \end{pmatrix} .$$

In this first example, I will explicitly report the matrix $V_{11}(\varepsilon)$ and its eigenvalues for all the configurations that can be continued, degenerate or not, in order to show the structure of the matrix for different values of q^* and better work out the details for studying the stability properties.

Since by definition the matrix D carries one derivative only w.r.t. q, then it depends only on $\sin(\langle k, q\rangle)$ and then it vanishes at $q^* = 0, \pi$. In other terms, we are reduced to study the spectrum of the simplified $V_{11}(\varepsilon)$ matrix

$$V_{11}(\varepsilon) = \begin{pmatrix} 0 & \lfloor \overline{C} \\ -B & 0 \end{pmatrix} .$$

We notice that in this example the matrices

$$\lfloor \overline{C} = D_p^2 \left(f_4^{(2,0)} + f_4^{(2,1)} + f_4^{(2,2)} \right)(q^*) \qquad \text{and} \qquad -B = -D_q^2 \left(f_0^{(2,1)} + f_0^{(2,2)} \right)(q^*)$$

reduce to

$$D_p^2 \left(f_4^{(0,0)} + f_4^{(0,1)} + L_{\langle \zeta^{(1)}, \hat{q}\rangle} f_6^{(0,1)} \right)(q^*) \qquad \text{and} \qquad -D_q^2 \left(f_0^{(0,1)} + L_{\langle \zeta^{(1)}, \hat{q}\rangle} f_2^{(0,1)} \right)(q^*) .$$

Hence, for the in/out-of-phase configurations $(0,0,\pi)$, $(\pi,\pi,0)$, $(0,\pi,\pi)$, $(\pi,0,0)$, we get $V(\varepsilon)$ respectively equal to

$$\begin{pmatrix}
0 & 0 & 0 & 4 - \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} & -2 + \frac{2\varepsilon}{I^*} + \frac{2\varepsilon^2}{(I^*)^2} & -\frac{\varepsilon^2}{2(I^*)^2} \\
0 & 0 & 0 & -2 + \frac{2\varepsilon}{I^*} + \frac{2\varepsilon^2}{(I^*)^2} & 4 - \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} & -2 \\
0 & 0 & 0 & -\frac{\varepsilon^2}{2(I^*)^2} & -2 & 4 + \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} \\
-2\varepsilon^2 & -2I^*\varepsilon - \varepsilon^2 & -2I^*\varepsilon - \varepsilon^2 & 0 & 0 & 0 \\
-2I^*\varepsilon - \varepsilon^2 & -2\varepsilon^2 & -2I^*\varepsilon - \varepsilon^2 & 0 & 0 & 0 \\
-2I^*\varepsilon - \varepsilon^2 & -2I^*\varepsilon - \varepsilon^2 & -4I^*\varepsilon - 2\varepsilon^2 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 4 + \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} & -2 - \frac{2\varepsilon}{I^*} + \frac{2\varepsilon^2}{(I^*)^2} & -\frac{\varepsilon^2}{2(I^*)^2} \\
0 & 0 & 0 & -2 - \frac{2\varepsilon}{I^*} + \frac{2\varepsilon^2}{(I^*)^2} & 4 + \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} & -2 \\
0 & 0 & 0 & -\frac{\varepsilon^2}{2(I^*)^2} & -2 & 4 - \frac{2\varepsilon}{I^*} - \frac{3\varepsilon^2}{2(I^*)^2} \\
-2\varepsilon^2 & 2I^*\varepsilon - \varepsilon^2 & 2I^*\varepsilon - \varepsilon^2 & 0 & 0 & 0 \\
2I^*\varepsilon - \varepsilon^2 & -2\varepsilon^2 & 2I^*\varepsilon - \varepsilon^2 & 0 & 0 & 0 \\
2I^*\varepsilon - \varepsilon^2 & 2I^*\varepsilon - \varepsilon^2 & 4I^*\varepsilon - 2\varepsilon^2 & 0 & 0 & 0
\end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 4-\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} & -2 & -\frac{\varepsilon^2}{2(I^*)^2} \\ 0 & 0 & 0 & -2 & 4+\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & -\frac{\varepsilon^2}{2(I^*)^2} & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4+\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} \\ 4I^*\varepsilon-2\varepsilon^2 & 2I^*\varepsilon-\varepsilon^2 & 2I^*\varepsilon-\varepsilon^2 & 0 & 0 & 0 \\ 2I^*\varepsilon-\varepsilon^2 & -2\varepsilon^2 & 2I^*\varepsilon-\varepsilon^2 & 0 & 0 & 0 \\ 2I^*\varepsilon-\varepsilon^2 & 2I^*\varepsilon-\varepsilon^2 & -2\varepsilon^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 4+\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} & -2 & -\frac{\varepsilon^2}{2(I^*)^2} \\ 0 & 0 & 0 & -2 & 4-\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & -\frac{\varepsilon^2}{2(I^*)^2} & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4-\frac{2\varepsilon}{I^*}-\frac{3\varepsilon^2}{2(I^*)^2} \\ -4I^*\varepsilon-2\varepsilon^2 & -2I^*\varepsilon-\varepsilon^2 & -2I^*\varepsilon-\varepsilon^2 & 0 & 0 & 0 \\ -2I^*\varepsilon-\varepsilon^2 & -2\varepsilon^2 & -2I^*\varepsilon-\varepsilon^2 & 0 & 0 & 0 \\ -2I^*\varepsilon-\varepsilon^2 & -2I^*\varepsilon-\varepsilon^2 & -2\varepsilon^2 & 0 & 0 & 0 \end{pmatrix}.$$

By computing the eigenvalues of the above matrices, we obtain

$$\begin{aligned} \lambda_{1,2}(\varepsilon) &= \pm\mathbf{i}\left(2\varepsilon+\frac{\varepsilon^3}{4\,(I^*)^2}+h.o.t.\right) , \\ \lambda_{3,4}(\varepsilon) &= \pm\left(2^{7/4}\sqrt{I^*}\sqrt{\varepsilon}-\frac{7\varepsilon^{3/2}}{2^{7/4}\sqrt{I^*}}+h.o.t.\right) , \\ \lambda_{5,6}(\varepsilon) &= \pm\mathbf{i}\left(2^{7/4}\sqrt{I^*}\sqrt{\varepsilon}+\frac{7\varepsilon^{3/2}}{2^{7/4}\sqrt{I^*}}+h.o.t.\right) . \end{aligned}$$

Therefore, all the above degenerate in/out-of-phase configurations are unstable. Let me stress that the additional factor $\sqrt{\varepsilon}$ in the eigenvalues $\lambda_{1,2}$ stems from the degenerate tangential direction to the families.

We now study the linear stability of the non-degenerate in/out-of-phase configurations $(0, 0, 0)$

and (π, π, π). We get the $V_{11}(\varepsilon)$ matrices

$$\begin{pmatrix} 0 & 0 & 0 & 4-\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & -\frac{\varepsilon}{I^*}-\frac{\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4-\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & -\frac{\varepsilon}{I^*}-\frac{\varepsilon^2}{(I^*)^2} & -2+\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4-\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} \\ 4I^*\varepsilon-4\varepsilon^2 & 2I^*\varepsilon-2\varepsilon^2 & 2I^*\varepsilon-2\varepsilon^2 & 0 & 0 & 0 \\ 2I^*\varepsilon-2\varepsilon^2 & 4I^*\varepsilon-4\varepsilon^2 & 2I^*\varepsilon-2\varepsilon^2 & 0 & 0 & 0 \\ 2I^*\varepsilon-2\varepsilon^2 & 2I^*\varepsilon-2\varepsilon^2 & 4I^*\varepsilon-4\varepsilon^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 4+\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & \frac{\varepsilon}{I^*}-\frac{\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4+\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} \\ 0 & 0 & 0 & \frac{\varepsilon}{I^*}-\frac{\varepsilon^2}{(I^*)^2} & -2-\frac{2\varepsilon}{I^*}+\frac{2\varepsilon^2}{(I^*)^2} & 4+\frac{3\varepsilon}{I^*}-\frac{3\varepsilon^2}{(I^*)^2} \\ -4I^*\varepsilon-4\varepsilon^2 & -2I^*\varepsilon-2\varepsilon^2 & -2I^*\varepsilon-2\varepsilon^2 & 0 & 0 & 0 \\ -2I^*\varepsilon-2\varepsilon^2 & -4I^*\varepsilon-4\varepsilon^2 & -2I^*\varepsilon-2\varepsilon^2 & 0 & 0 & 0 \\ -2I^*\varepsilon-2\varepsilon^2 & -2I^*\varepsilon-2\varepsilon^2 & -4I^*\varepsilon-4\varepsilon^2 & 0 & 0 & 0 \end{pmatrix},$$

with eigenvalues respectively equal to

$$\begin{aligned} \lambda_{1,2}(\varepsilon) &= -2\sqrt{2I^*}\sqrt{\varepsilon} + \frac{3\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t.\ , \\ \lambda_{3,4}(\varepsilon) &= 2\sqrt{2I^*}\sqrt{\varepsilon} - \frac{3\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t.\ , \\ \lambda_{5,6}(\varepsilon) &= \pm\left(4\sqrt{I^*}\sqrt{\varepsilon} - \frac{4\varepsilon^{3/2}}{\sqrt{I^*}} + h.o.t.\right)\ , \end{aligned}$$

and

$$\begin{aligned} \lambda_{1,2}(\varepsilon) &= \mathbf{i}\left(-2\sqrt{2I^*}\sqrt{\varepsilon} - \frac{3\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t.\right)\ , \\ \lambda_{3,4}(\varepsilon) &= \mathbf{i}\left(2\sqrt{2I^*}\sqrt{\varepsilon} + \frac{3\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t.\right)\ , \\ \lambda_{5,6}(\varepsilon) &= \pm\mathbf{i}\left(4\sqrt{I^*}\sqrt{\varepsilon} + \frac{4\varepsilon^{3/2}}{\sqrt{I^*}} + h.o.t.\right)\ . \end{aligned}$$

We can deduce that only the configuration (π, π, π) is stable, the eigenvalues being purely imaginary. We can observe that all the eigenvalues of this non-degenerate configuration are of order $\mathcal{O}(\sqrt{\varepsilon})$, as proved in Theorem 2.2.1 in Chapter 2. In order to infer the effective linear stability, we cannot apply the Theorem 2.1.2, since the eigenvalues are not distinct. However, it can be derived by using the definiteness of $J^{-1}L_{11}(\varepsilon)$ which results in the definiteness of the matrices $\lfloor\overline{C}(\varepsilon)$ and

$B(\varepsilon)$. These latter being positive definite at leading order of approximation, they remain so for all orders and the eigenvalues are always purely imaginary.

Instead, if we want to investigate the linear stability of the symmetric vortex configurations $\pm(\pi/2, \pi/2, \pi/2)$, we have to study the spectrum of the simplified $V_{11}(\varepsilon)$ matrix

$$V_{11}(\varepsilon) = \begin{pmatrix} \lfloor D^\top & \lfloor \overline{C} \\ 0 & \lfloor D \end{pmatrix} .$$

Indeed, the matrix B depends only on $\cos(\langle k, q\rangle)$, so it vanishes at $q^* = \pm\pi/2$, whereas the matrix $\lfloor D$ is different from zero. Furthermore, in this example, the matrix

$$\lfloor D = D^2_{q\hat{p}} \left(f_2^{(2,1)} + f_2^{(2,2)} \right) (q^*)$$

reduces to

$$D^2_{q\hat{p}} \left(f_2^{(0,1)} + L_{\langle \zeta^{(1)}, \hat{q}\rangle} f_4^{(0,1)} \right) (q^*) .$$

Since the translating vector $\zeta^{(1)}$ and the functions $f_4^{(0,1)}$ and $f_6^{(0,1)}$ depend only on $\cos(k \cdot q)$, we can observe that also the matrices D_2, C_1, and C_2 vanish at $q^* = \pm\pi/2$. Hence, we obtain the matrices

$$\begin{pmatrix} -\varepsilon & \varepsilon & \varepsilon & 4 & -2 & 0 \\ -2\varepsilon & 0 & 2\varepsilon & -2 & 4 & -2 \\ -\varepsilon & -\varepsilon & \varepsilon & 0 & -2 & 4 \\ 0 & 0 & 0 & \varepsilon & 2\varepsilon & \varepsilon \\ 0 & 0 & 0 & -\varepsilon & 0 & \varepsilon \\ 0 & 0 & 0 & -\varepsilon & -2\varepsilon & -\varepsilon \end{pmatrix} ,$$

$$\begin{pmatrix} \varepsilon & -\varepsilon & -\varepsilon & 4 & -2 & 0 \\ 2\varepsilon & 0 & -2\varepsilon & -2 & 4 & -2 \\ \varepsilon & \varepsilon & -\varepsilon & 0 & -2 & 4 \\ 0 & 0 & 0 & -\varepsilon & -2\varepsilon & -\varepsilon \\ 0 & 0 & 0 & \varepsilon & 0 & -\varepsilon \\ 0 & 0 & 0 & \varepsilon & 2\varepsilon & \varepsilon \end{pmatrix} ,$$

with the following eigenvalues

$$\begin{aligned} \lambda_{1,2} &= 0 , \\ \lambda_{3,4}(\varepsilon) &= -2\mathbf{i}\varepsilon , \\ \lambda_{5,6}(\varepsilon) &= 2\mathbf{i}\varepsilon . \end{aligned}$$

The two vortex configurations are not stable: they belong to the family of true solutions Q_2 and the zero eigenvalues always persist at every order because of the surviving of a two-dimensional torus of periodic orbits. Hence, periodic orbits are not isolated and they cannot be stable in strict sense.

We can observe that the additional factor $\sqrt{\varepsilon}$ in the eigenvalues $\lambda_{3,4}$ and $\lambda_{5,6}$ results from the degenerate tangential directions to the families Q_1 and Q_3, while the zero eigenvalue comes from the persistence of the second family Q_2.

Consider now the two configurations with $q^* = 0, \pi$, belonging to the family of true solutions Q_2, namely $(0, \pi, 0)$ and $(\pi, 0, \pi)$. We get the $V_{11}(\varepsilon)$ matrices

$$\begin{pmatrix} 0 & 0 & 0 & 4 - \frac{\varepsilon}{I^*} & -2 & \frac{\varepsilon}{I^*} \\ 0 & 0 & 0 & -2 & 4 + \frac{\varepsilon}{I^*} & -2 \\ 0 & 0 & 0 & \frac{\varepsilon}{I^*} & -2 & 4 - \frac{\varepsilon}{I^*} \\ 0 & -2I^*\varepsilon & -2I^*\varepsilon & 0 & 0 & 0 \\ -2I^*\varepsilon & -4I^*\varepsilon & -2I^*\varepsilon & 0 & 0 & 0 \\ -2I^*\varepsilon & -2I^*\varepsilon & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 4 + \frac{\varepsilon}{I^*} & -2 & -\frac{\varepsilon}{I^*} \\ 0 & 0 & 0 & -2 & 4 - \frac{\varepsilon}{I^*} & -2 \\ 0 & 0 & 0 & -\frac{\varepsilon}{I^*} & -2 & 4 + \frac{\varepsilon}{I^*} \\ 0 & 2I^*\varepsilon & 2I^*\varepsilon & 0 & 0 & 0 \\ 2I^*\varepsilon & 4I^*\varepsilon & 2I^*\varepsilon & 0 & 0 & 0 \\ 2I^*\varepsilon & 2I^*\varepsilon & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the following eigenvalues

$$\begin{aligned} &\lambda_{1,2} = 0 \ , \\ &\lambda_{3.4}(\varepsilon) = \pm \mathbf{i} \left(2\sqrt{2I^*}\sqrt{\varepsilon} + \frac{\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t. \right) \ , \\ &\lambda_{5,6}(\varepsilon) = \pm \left(2\sqrt{2I^*}\sqrt{\varepsilon} - \frac{\varepsilon^{3/2}}{\sqrt{2I^*}} + h.o.t. \right) \ . \end{aligned}$$

We can conclude that both configurations are not stable and we remark, once again, that the null eigenvalues are due to the persistence of the family Q_2. Moreover, also the family Q_2 cannot be stable, because the eigenvalues different from zero are not all purely imaginary.

3.3 Applications: lower dimensional tori

3.3.1 The seagull

In the following lower dimensional example we consider the same dNLS model of the previous example, namely (3.3), but with fixed boundary conditions and a generic value of γ. Although it does not represent a dNLS lattice in the proper sense due to the limited number of sites, it is nevertheless suitable to see the advantages of the normal form construction. Furthermore, it sheds some light onto the role of the non-linearity in the linear stability of multi-peaked discrete solitons in dNLS lattices. Indeed, at variance with models considered in literature, we notice that a change in the sign of the non-linear parameter γ does not influence the nature of the degenerate eigenspaces. On the contrary, considering consecutive excited sites as in [69], a change in the sign of γ (at fixed linear interaction ε) produces an exchange of stable and unstable directions around the periodic solutions.

Let us remark that, in order to apply Theorem 2.1.1, we have to control the smallest eigenvalue of the matrix $M(\varepsilon)$. This is a delicate point, particularly in actual applications. Indeed, to numerically verify this assumption one has to investigate the spectrum of the matrix $(\exp(L_{11}(\varepsilon)T) - \mathbb{I})_{red}$, by interpolating the decay of the smallest eigenvalue with respect to ε.

In some specific cases like the example here considered, it might not be the easiest way to verify the condition. However, one can further decompose the quadratic Hamiltonian $K_2^{(r)}$ in order to decouple the fast variables (Q_1, P_1) from the slow variables (Q, P) with a linear canonical change of coordinates (see [87]).

Precisely, we decompose the matrix C so as to put in evidence the first row and column vectors, namely

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where C_{11} is the first element, $C_{12} = C_{21}^\top$ is the (n_1-1)-dimensional row vector and C_{22} is the (n_1-1)-dimensional square matrix.

Assume now that C_{22} is invertible, then we can introduce the canonical change of coordinates

$$u = Q \ , \quad u_1 = Q_1 - C_{12}C_{22}^{-1}Q \ , \quad P = v - v_1 C_{12}C_{22}^{-1} \ , \quad P_1 = v_1 \ .$$

The transformed quadratic Hamiltonian $K_2^{(r)}$ now reads

$$K_2^{(r)} = \frac{1}{2}c_{11}v_1^2 + \frac{1}{2}\left[u^\top Bu + v^\top C_{22}v\right] + u^\top Dv + \frac{1}{2}\xi^\top G\xi + \xi^\top E\eta + \frac{1}{2}\eta^\top F\eta \ ,$$

where $c_{11} = C_{11} - C_{12}C_{22}^{-1}C_{21}$ and the term $u^\top Dv$ contains mixed terms in action-angles variables. The main advantage is that, if $D = 0$, then the fast dynamics and the slow one turn out to be decoupled, hence it suffices to investigate the eigenvalues of the matrix

$$\begin{pmatrix} 0 & C_{22} \\ -B & 0 \end{pmatrix}$$

which represents the linear vector fields of the new slow variables (Q, P). Hence (2.12) can be easily checked, possibly without the needs of numerical interpolation.

Let us consider a system of coupled anharmonic oscillators with Hamiltonian

$$H = H_0 + \varepsilon H_1 = \sum_{j=-3}^{3} \left(\frac{x_j^2 + y_j^2}{2} + \gamma \left(\frac{x_j^2 + y_j^2}{2} \right)^2 \right) + \varepsilon \sum_{j=-3}^{2} (x_{j+1}x_j + y_{j+1}y_j) \ ,$$

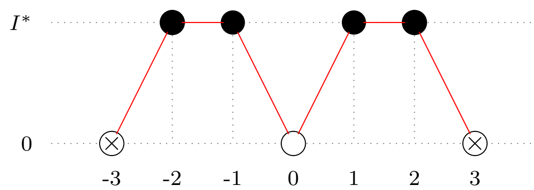
with fixed boundary conditions $x_{-3} \equiv y_{-3} \equiv x_3 \equiv y_3 \equiv 0$. Introducing the action-angle variables $(x_j, y_j) = (\sqrt{2I_j}\cos\varphi_j, -\sqrt{2I_j}\sin\varphi_j)$, for the set of indices $\mathcal{I} = \{-2, -1, 1, 2\}$, and the complex canonical coordinates for the central oscillator

$$x_0 = \frac{1}{\sqrt{2}}(\xi_0 + \mathbf{i}\eta_0), \quad y_0 = \frac{\mathbf{i}}{\sqrt{2}}(\xi_0 - \mathbf{i}\eta_0) \ ,$$

the Hamiltonian reads

$$\begin{aligned} H = &\sum_{j\in\mathcal{I}} \left(I_j + \gamma I_j^2\right) + \mathbf{i}\xi_0\eta_0 - \gamma\xi_0^2\eta_0^2 \\ &+ \varepsilon \Bigg(2\sqrt{I_{-1}I_{-2}}\cos(\varphi_{-1} - \varphi_{-2}) + 2\sqrt{I_2 I_1}\cos(\varphi_2 - \varphi_1) \\ &+ (\xi_0 + \mathbf{i}\eta_0)\left(\sqrt{I_{-1}}\cos(\varphi_{-1}) + \sqrt{I_1}\cos(\varphi_1)\right) + \mathbf{i}\,(\xi_0 - \mathbf{i}\eta_0)\left(\sqrt{I_{-1}}\sin(\varphi_{-1}) + \sqrt{I_1}\sin(\varphi_1)\right)\Bigg) \ . \end{aligned}$$

Let us now fix the lower dimensional resonant torus $I_j^* = I^*$, for $j \in \mathcal{I}$, and $\xi_0 = \eta_0 = 0$ and make a Taylor expansion around I^*.



The unperturbed Hamiltonian H_0 reads

$$H_0(J,\xi_0,\eta_0) = 4I^*+4\gamma(I^*)^2+(1+2\gamma I^*)(J_{-2}+J_{-1}+J_1+J_2)+\gamma\left(J_{-2}^2+J_{-1}^2+J_1^2+J_2^2\right)+\mathbf{i}\xi_0\eta_0-\gamma\xi_0^2\eta_0^2\,,$$

while the perturbation H_1 takes the form

$$\begin{aligned}
H_1(J,\varphi,\xi_0,\eta_0) &= 2I^*\cos(\varphi_{-1}-\varphi_{-2}) + 2I^*\cos(\varphi_2-\varphi_1)\\
&\quad + (\xi_0+\mathbf{i}\eta_0)\sqrt{I^*}\cos(\varphi_{-1}) + (\xi_0+\mathbf{i}\eta_0)\sqrt{I^*}\cos(\varphi_1)\\
&\quad + \mathbf{i}(\xi_0-\mathbf{i}\eta_0)\sqrt{I^*}\sin(\varphi_{-1}) + \mathbf{i}(\xi_0-\mathbf{i}\eta_0)\sqrt{I^*}\sin(\varphi_1)+\\
&\quad + (J_{-2}+J_{-1})\cos(\varphi_{-1}-\varphi_{-2}) + (J_1+J_2)\cos(\varphi_2-\varphi_1)\\
&\quad + \frac{J_{-1}(\xi_0+\mathbf{i}\eta_0)\cos(\varphi_{-1})}{2\sqrt{I^*}} + \frac{\mathbf{i}J_{-1}(\xi_0-\mathbf{i}\eta_0)\sin(\varphi_{-1})}{2\sqrt{I^*}}\\
&\quad + \frac{J_1(\xi_0+\mathbf{i}\eta_0)\cos(\varphi_1)}{2\sqrt{I^*}} + \frac{\mathbf{i}J_1(\xi_0-\mathbf{i}\eta_0)\sin(\varphi_1)}{2\sqrt{I^*}}\\
&\quad - \frac{(J_{-2}-J_{-1})^2\cos(\varphi_{-1}-\varphi_{-2})}{4I^*}\\
&\quad - \frac{(J_1-J_2)^2\cos(\varphi_2-\varphi_1)}{4I^*} + \mathcal{O}(|\xi_0|^a|\eta_0|^b|J|^3)\,,
\end{aligned}$$

where $a,b\in\mathbb{N}$ such that $a+b=1$. We now introduce the angles $\hat{q}=(q_1,q)$ and their conjugate actions $\hat{p}=(p_1,p)$

$$\begin{cases} q_1=\varphi_{-2}\\ q_2=\varphi_{-1}-\varphi_{-2}\\ q_3=\varphi_1-\varphi_{-1}\\ q_4=\varphi_2-\varphi_1 \end{cases},\qquad \begin{cases} p_1=J_{-2}+J_{-1}+J_1+J_2\\ p_2=J_{-1}+J_1+J_2\\ p_3=J_1+J_2\\ p_4=J_2 \end{cases}.$$

Hence, the Hamiltonian can be rewritten as

$$\begin{aligned}
H &= \omega p_1 + \mathbf{i}\xi_0\eta_0 + \gamma\left((p_1-p_2)^2+(p_2-p_3)^2+(p_3-p_4)^2+p_4^2\right) - \gamma\xi_0^2\eta_0^2\\
&\quad + \varepsilon\Bigg[2I^*\cos(q_2)+2I^*\cos(q_4)\\
&\qquad + (\xi_0+\mathbf{i}\eta_0)\sqrt{I^*}\cos(q_1+q_2) + (\xi_0+\mathbf{i}\eta_0)\sqrt{I^*}\cos(q_1+q_2+q_3)\\
&\qquad + \mathbf{i}(\xi_0-\mathbf{i}\eta_0)\sqrt{I^*}\sin(q_1+q_2) + \mathbf{i}(\xi_0-\mathbf{i}\eta_0)\sqrt{I^*}\sin(q_1+q_2+q_3)\\
&\qquad + (p_1-p_3)\cos(q_2) + p_3\cos(q_4)\\
&\qquad + \frac{(p_2-p_3)(\xi_0+\mathbf{i}\eta_0)\cos(q_1+q_2)}{2\sqrt{I^*}} + \frac{\mathbf{i}(p_2-p_3)(\xi_0-\mathbf{i}\eta_0)\sin(q_1+q_2)}{2\sqrt{I^*}}\\
&\qquad + \frac{(p_3-p_4)(\xi_0+\mathbf{i}\eta_0)\cos(q_1+q_2+q_3)}{2\sqrt{I^*}} + \frac{\mathbf{i}(p_3-p_4)(\xi_0-\mathbf{i}\eta_0)\sin(q_1+q_2+q_3)}{2\sqrt{I^*}}\\
&\qquad - \frac{(p_1-2p_2+p_3)^2\cos(q_2)}{4I^*} - \frac{(p_3-2p_4)^2\cos(q_4)}{4I^*}\Bigg] + \mathcal{O}(\varepsilon|\xi_0|^a|\eta_0|^b|\hat{p}|^3)\\
&= \omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(0,0)}(\hat{p},\xi_0,\eta_0) + f_0^{(0,1)}(q_2,q_4) + f_1^{(0,1)}(\hat{q},\xi_0,\eta_0)\\
&\quad + f_2^{(0,1)}(\hat{p},q) + f_3^{(0,1)}(p,\hat{q},\xi_0,\eta_0) + f_4^{(0,1)}(p,q) + \mathcal{O}(\varepsilon|\xi_0|^a|\eta_0|^b|\hat{p}|^3)\,,
\end{aligned}$$

where $\omega = 1+2\gamma I^*$.

Remark 3.3.1 *Let us observe that this is the same kind of model of the square dNLS cell in the first example. Hence, the Hamiltonian turns out to be a dNLS model with a Gauge symmetry. As regards terms that depend only on the internal variables of the torus, the symmetry once again* *reveals itself as an independence from the fast angle q_1. On the contrary, the terms that also depend on the transversal variables contain the angle q_1, since they are not expressed in a system of coordinates which allows the symmetry to appear by an independence from a particular variable.*

Since $f_0^{(0,1)}$ is automatically averaged w.r.t. q_1, the first stage of the first normalization step only consists in the translation correcting the frequencies. Moreover, also $f_2^{(0,1)}$ does not depend on the fast angle q_1, hence the homological equation defining $\zeta^{(1)}$ is equivalent to the following linear system

$$\sum_j C_{0,i,j}\zeta_j^{(1)} = \frac{\partial}{\partial \hat{p}_i} f_2^{(0,1)}\Big|_{q=q^*} ,$$

with

$$C_0 = \gamma \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix} .$$

The solution is given by

$$\begin{cases} \zeta_1^{(1)} = \dfrac{\varepsilon}{\gamma}\left(\cos(q_2^*) + \cos(q_4^*)\right) \\ \zeta_2^{(1)} = \dfrac{\varepsilon}{\gamma}\left(\dfrac{\cos(q_2^*)}{2} + \cos(q_4^*)\right) \\ \zeta_3^{(1)} = \dfrac{\varepsilon}{\gamma}\cos(q_4^*) \\ \zeta_4^{(1)} = \dfrac{\varepsilon}{2\gamma}\cos(q_4^*) \end{cases} . \tag{3.7}$$

We remove the term $f_1^{(\mathrm{I};0,1)} = f_1^{(0,1)}$ from the Hamiltonian by means of the generating function

$$\chi_1^{(1)} = \varepsilon\left(\mathbf{i}\frac{\sqrt{I^*}\left(e^{-\mathbf{i}(q_1+q_2)} + e^{-\mathbf{i}(q_1+q_2+q_3)}\right)\xi_0}{\omega - 1} + \frac{\sqrt{I^*}\left(e^{\mathbf{i}(q_1+q_2)} + e^{\mathbf{i}(q_1+q_2+q_3)}\right)\eta_0}{\omega - 1}\right) .$$

The new term $f_2^{(\mathrm{II};0,1)} = f_2^{(0,1)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)}$ is again independent of q_1, so no further average is required. We now remove the cubic terms which depend both on the actions and on the transversal variables from $f_3^{(\mathrm{III},0,1)} = f_3^{(0,1)} + L_{\chi_1^{(1)}} f_4^{(0,0)}$, by means of the generating function

$$\begin{aligned} \chi_3^{(1)} &= -\varepsilon e^{-\mathbf{i}(q_1+q_2+q_3)}\frac{\left(-1+e^{\mathbf{i}q_3}\right)p_3\left(e^{\mathbf{i}(2q_1+2q_2+q_3)}\eta_0 - \mathbf{i}\xi_0\right) + e^{\mathbf{i}q_3}p_2\left(e^{2\mathbf{i}(q_1+q_2)}\eta_0 + \mathbf{i}\xi_0\right)}{2\sqrt{I^*}\,(\omega-1)} \\ &\quad + \varepsilon e^{-\mathbf{i}(q_1+q_2+q_3)}\frac{p_4\left(e^{2\mathbf{i}(q_1+q_2+q_3)}\eta_0 + \mathbf{i}\xi_0\right)}{2\sqrt{I^*}\,(\omega-1)} . \end{aligned}$$

The new term $f_4^{(\mathrm{IV},0,1)} = f_4^{(0,1)}$ turns out to be independent of q_1, thus the first step is concluded.

The values q^*, which determine the approximate periodic orbits at leading order, are the solutions of the following system

$$\begin{cases} -2I^*\sin(q_2) = 0 \\ -2I^*\sin(q_4) = 0 \end{cases} .$$

Such solutions are given by the four one-parameter families $Q_1 = (0, \vartheta, 0)$, $Q_2 = (0, \vartheta, \pi)$, $Q_3 = (\pi, \vartheta, 0)$, $Q_4 = (\pi, \vartheta, \pi)$, with $\vartheta \in S^1$. We observe that the non-degeneracy condition is not fulfilled, indeed

$$\begin{vmatrix} -2I^*\cos(q_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2I^*\cos(q_4) \end{vmatrix} = 0 .$$

Thus, a second normalization step is needed in order to investigate the continuation of all the one-parameter families.

The first stage of the second normalization step consists, once again, in the translation which keeps the frequencies fixed, since the term

$$f_0^{(1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_2^{(0,1)} + \frac{1}{2} L^2_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)} + \frac{1}{2} L_{\chi_1^{(1)}} f_1^{(0,1)}$$

is already averaged w.r.t q_1. The same occurs also for the term

$$f_2^{(1,2)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,1)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}} + L_{\chi_1^{(1)}} f_3^{(0,1)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}} + \frac{1}{2} L^2_{\chi_1^{(1)}} f_4^{(0,0)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}} .$$

Therefore, the homological equation is given by

$$L_{\langle\zeta^{(2)},\hat{q}\rangle} f_4^{(0,0)}\Big|_{\xi=\eta=0} + f_2^{(1,2)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}} = 0 .$$

The second stage deals with the term

$$f_1^{(\mathrm{I};1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_3^{(0,1)} + L_{\chi_1^{(1)}} \left(f_2^{(0,1)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)} \right) + L_{\chi_3^{(1)}} f_0^{(0,1)} ,$$

while the new term

$$f_2^{(\mathrm{II};1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,1)} + L_{\chi_1^{(1)}} f_3^{(0,1)} + \frac{1}{2} L^2_{\chi_1^{(1)}} f_4^{(0,0)} + L_{\langle\zeta^{(2)},\hat{q}\rangle} f_4^{(0,0)}$$

is again already averaged over q_1.

The fourth stage copes with

$$f_3^{(\mathrm{III};1,2)} = L_{\langle\zeta^{(1)},\hat{q}\rangle} f_5^{(0,1)} + L_{\chi_1^{(1)}} f_4^{(0,1)} + L_{\chi_3^{(1)}} \left(f_2^{(0,1)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)} \right) + L_{\chi_1^{(2)}} f_4^{(0,0)} ,$$

and the term

$$\begin{aligned} f_4^{(\mathrm{IV};1,2)}\Big|_{\xi=\eta=0} = \Bigg(& L_{\langle\zeta^{(1)},\hat{q}\rangle} f_6^{(0,1)} + L_{\chi_1^{(1)}} f_5^{(0,1)} + \frac{1}{2} L_{\chi_3^{(1)}} \left(f_3^{(0,1)} + f_3^{(0,1)}\Big|_{\hat{p}=0} \right) \\ & + \frac{1}{2} L_{\chi_3^{(1)}} \left(L_{\chi_1^{(1)}} \left(f_4^{(0,0)} + f_4^{(0,0)}\Big|_{\hat{p}=0} \right) \right) \Bigg)\Bigg|_{\xi=\eta=0} \end{aligned}$$

is, once again, averaged w.r.t. q_1, hence the second step is concluded. The transformed Hamiltonian reads

$$\begin{aligned} H^{(2)} = {} & \omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(2,0)}(\hat{p},\xi_0,\eta_0) \\ & + f_0^{(2,1)}(q) + f_2^{(2,1)}(\hat{p},q,\xi_0,\eta_0) + f_3^{(2,1)}(\hat{q},\xi_0,\eta_0) + f_4^{(2,1)}(\hat{p},q,\xi_0,\eta_0) + \mathcal{O}(\varepsilon|\xi_0|^a|\eta_0|^b|\hat{p}|^c) \\ & + f_0^{(2,2)}(q) + f_2^{(2,2)}(\hat{p},q,\xi_0,\eta_0) + f_3^{(2,2)}(\hat{q},\xi_0,\eta_0) + f_4^{(2,2)}(\hat{p},q,\xi_0,\eta_0) + \mathcal{O}(\varepsilon^2|\xi_0|^a|\eta_0|^b|\hat{p}|^c) \\ & + \mathcal{O}\big(\varepsilon^3\big) , \end{aligned}$$

with $a,b,c \in \mathbb{N}$ such that $a+b+2c=5$. The approximate periodic orbits are the solutions q^* of the system

$$\nabla_q \left(f_0^{(2,1)}(q) + f_0^{(2,2)}(q) \right) = 0 .$$

Since in the correction $f_0^{(2,2)}(q)$ only the terms $L_{\langle\zeta^{(1)},\hat{q}\rangle} f_2^{(0,1)}$ and $\frac{1}{2} L_{\chi_1^{(1)}} f_1^{(0,1)}$ have a non trivial dependence on the angles q, using (3.7), we get the system

$$\begin{cases} -2I^* \sin(q_2) + \dfrac{\varepsilon}{\gamma} \sin(q_2)\cos(q_2) = 0 \\ \dfrac{\varepsilon}{\gamma} \sin(q_3) = 0 \\ -2I^* \sin(q_4) + \dfrac{\varepsilon}{\gamma} \sin(q_4)\cos(q_4) = 0 \end{cases} .$$

The above system has the structure

$$F(q,\varepsilon) = F_0(q) + \varepsilon F_1(q) = 0 \tag{3.8}$$

where $F : \mathbb{T}^3 \times \mathcal{U}(0) \to \mathbb{R}^3$. In addition, we know from the first normalization step that

$$F(Q_j(\vartheta), 0) = F_0(Q_j(\vartheta)) = 0 \ .$$

Suppose that there exists a solution $q(\varepsilon) = (q_2(\varepsilon), q_3(\varepsilon), q_4(\varepsilon))$ which is at least continuous in the small parameter. Thus, we must have

$$\lim_{\varepsilon\to 0} F(q_2(\varepsilon), q_3(\varepsilon), q_4(\varepsilon), \varepsilon) = F_0(q_2(0), q_3(0), q_4(0)) = 0 \ .$$

We introduce the matrices $\tilde{B}_{1,j}(\vartheta) = \frac{\partial F_0(Q_j(\vartheta))}{\partial q}$ and observe that the tangent direction to the four families Q_j

$$\partial_\vartheta Q_j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is the Kernel direction of $\tilde{B}_{1,j}(\vartheta)$, for $j = 1, \ldots, 4$. By computing

$$\langle F_1(Q_j(\vartheta, 0)), \partial_\vartheta Q_j \rangle = \frac{\sin(\vartheta)}{\gamma} \qquad j = 1, \ldots, 4 \ ,$$

from Proposition 3.2.1 we obtain that, apart from the in/out-of-phase configurations $(0,0,0)$, $(0,\pi,0)$, $(0,0,\pi)$, $(0,\pi,\pi)$, $(\pi,0,0)$, $(\pi,\pi,0)$, $(\pi,0,\pi)$ and (π,π,π), the four families break down. We observe that the eight in/out-of-phase configurations are solutions of the equation (3.8).

In order to ensure the continuation of these configurations, we have to verify the condition (2.12), with x^* approximate zero of Υ, namely

$$\|\Upsilon(x^*; \varepsilon, q_1(0))\| \le c_1 \varepsilon^3 \ .$$

In order to verify (2.12) we have two options: (i) examine the spectrum of $(\exp(L_{11}(\varepsilon)T) - \mathbb{I})_{red}$ and numerically interpolate the smallest eigenvalue, getting $|\lambda| \gtrsim \varepsilon$ for each of the eight configuration; (ii) since $q_j = q^* = \{0, \pi\}$, the mixed terms in action-angle variables are missing and we can compute $e^{T\sigma_l} - 1$, with σ_l eigenvalues of the matrix

$$\begin{pmatrix} 0 & C_{22} \\ -B & 0 \end{pmatrix} \ ,$$

with

$$B = \begin{pmatrix} -2\varepsilon I^* \cos(q_2) + \frac{\varepsilon^2}{\gamma}\cos^2(q_2) & 0 & 0 \\ 0 & -\frac{\varepsilon^2}{\gamma}\cos(q_3) & 0 \\ 0 & 0 & -2\varepsilon I^* \cos(q_4) + \frac{\varepsilon^2}{\gamma}\cos^2(q_4) \end{pmatrix} \ .$$

Hence, C_{22} being definite and of order $\mathcal{O}(1)$ in the limit of small ε, we obtain that condition (2.12) is verified with $\alpha = 1$ and $r = 2$. Applying the Theorem 2.1.1, we can infer the existence of a unique $x^*_{\text{p.o.}}(\varepsilon) = (q^*_{\text{p.o.}}(\varepsilon), \hat{p}_{\text{p.o.}}(\varepsilon), \xi_{\text{p.o.}}(\varepsilon), \eta_{\text{p.o.}}(\varepsilon))$, with $q^*_{\text{p.o.}}(\varepsilon) = q^* = \{0, \pi\}$ such that $\left\|x^*_{\text{p.o.}} - x^*\right\| \le c_0 \varepsilon^2$, for each candidate for the continuation.

Coming to the linear stability, first we exploit the structure of the matrix $V(\varepsilon)$, with $D \equiv 0$, in order to get the approximate linear stability of the continued periodic orbits. It turns out that the stable and unstable directions correspond to the positive or negative eigenvalues of $C_{22}B$, where the prefactor γ in C_{22} accounts for the positive or negative signature of C_{22} (which is the same of C_0). Hence, the stability depends on the signature of γB, given by the elements on its diagonal. The degenerate direction depends only on ε^2, thus it is always a saddle at $q_3 = 0$ and always a centre at $q_3 = \pi$. Instead, the non-degenerate directions depend on the sign of the product $\gamma\varepsilon$, which converts hyperbolic subspace into centre subspace at fixed $q_{2,4} \in \{0, \pi\}$. In particular, by studying the spectrum in the in/out-of-phase configurations and for attractive interactions ($\varepsilon > 0$), we find that the only stable approximate periodic orbit corresponds to the configuration (π, π, π) when $\gamma > 0$ and to $(0, \pi, 0)$ when $\gamma < 0$. In order to derive the effective linear stability, we have to verify (2.13). Symbolic calculations implemented in Mathematica give

$$\begin{aligned}
\lambda_{1,2}(\varepsilon) &= \pm\mathbf{i}\left(2\sqrt{2}\sqrt{I^*|\gamma|}\sqrt{\varepsilon} + \frac{\sqrt{2}\varepsilon^{3/2}}{\sqrt{I^*|\gamma|}} + h.o.t.\right) \ , \\
\lambda_{3,4}(\varepsilon) &= \pm\mathbf{i}\left(2\sqrt{2}\sqrt{I^*|\gamma|}\sqrt{\varepsilon} + \frac{5\varepsilon^{3/2}}{2\sqrt{2}\sqrt{I^*|\gamma|}} + h.o.t.\right) \ , \\
\lambda_{5,6}(\varepsilon) &= \pm\mathbf{i}\left(\sqrt{2}\varepsilon - \frac{\varepsilon^2}{4\sqrt{2}I^*|\gamma|} + h.o.t.\right) \ .
\end{aligned}$$

Thus condition (2.13) holds true with $r + 1 - \alpha = 2$ and $\beta = \frac{3}{2}$.

3.3.2 Standard dNLS models

We now start to consider a dNLS model in the proper sense, with a large number of sites. We consider the dNLS equation in the general form

$$\mathbf{i}\dot{\psi}_j = \psi_j - \varepsilon(L\psi)_j + \gamma\psi_j|\psi_j|^2 \ ,$$

where the linear term L includes beyond nearest-neighbors terms

$$L\psi = \sum_{l=1}^{i} \kappa_l(\Delta_l\psi) \ , \qquad (\Delta_l\psi)_j = \psi_{j+l} - 2\psi_j + \psi_{j-l} \qquad \forall j \in \mathcal{J} \ ,$$

i is the biggest length of interaction, and the boundary conditions are periodic. The equations can be written in Hamiltonian form $\mathbf{i}\dot{\psi}_j = \frac{\partial H}{\partial \overline{\psi}_j}$ with

$$H = H_0 + \varepsilon H_1 \ , \qquad \begin{aligned} H_0 &= \sum_{j\in\mathcal{J}} |\psi_j|^2 + \frac{\gamma}{2}\sum_{j\in\mathcal{J}} |\psi_j|^4 \ , \\ H_1 &= \sum_{l=1}^{i} \kappa_l \sum_{j\in\mathcal{J}} |\psi_{j+l} - \psi_j|^2 \ . \end{aligned} \tag{3.9}$$

With the transformation (3.2), the original Hamiltonian is thus turned into a system of weakly coupled anharmonic oscillators with Hamiltonian

$$\begin{aligned} H = H_0 + \varepsilon H_1 &= \sum_{j\in\mathcal{J}} \left(\frac{1}{2}\left(x_j^2 + y_j^2\right) + \frac{\gamma}{8}\left(x_j^2 + y_j^2\right)^2\right) \\ &\quad + \varepsilon \sum_{l=1}^{i} \kappa_l \sum_{j\in\mathcal{J}} \left(x_j^2 + y_j^2\right) - \varepsilon \sum_{l=1}^{i} \kappa_l \sum_{j\in\mathcal{J}} (x_{j+l}x_j + y_{j+l}y_j) \ . \end{aligned} \tag{3.10}$$

We now introduce action-angle variables $(x_j, y_j) = \left(\sqrt{2I_j}\cos\varphi_j, -\sqrt{2I_j}\sin\varphi_j\right)$ for the set of indices $j \in \mathcal{I}$, and the complex canonical coordinates

$$x_j = \frac{1}{\sqrt{2}}(\xi_j + \mathbf{i}\eta_j) \ , \qquad\qquad y_j = \frac{\mathbf{i}}{\sqrt{2}}(\xi_j - \mathbf{i}\eta_j) \ ,$$

for the remaining ones $\mathcal{J}\setminus\mathcal{I}$, so that the Hamiltonian reads

$$H(I,\varphi,\xi,\eta,\varepsilon)=H_0(I,\xi,\eta)+\varepsilon H_1(I,\varphi,\xi,\eta)\ ,\quad H_0=\tilde{H}_0(I)+\hat{H}_0(\xi,\eta)\ ,$$

with

$$\begin{aligned}
\tilde{H}_0&=\sum_{j\in\mathcal{I}}\Big(I_j+\frac{\gamma}{2}I_j^2\Big)\ ,\\
\hat{H}_0&=\sum_{j\in\mathcal{J}\setminus\mathcal{I}}\Big(\mathbf{i}\Omega\xi_j\eta_j-\frac{\gamma}{2}\xi_j^2\eta_j^2\Big)\ ,\qquad \Omega=1\ ,\\
H_1&=2\sum_{l=1}^{i}\kappa_l\sum_{j\in\mathcal{I}}I_j+2\sum_{l=1}^{i}\kappa_l\sum_{j\in\mathcal{J}\setminus\mathcal{I}}\mathbf{i}\xi_j\eta_j-\text{effective coupling terms}\ ,
\end{aligned}$$

where the effective coupling terms come from the products $x_{j+l}x_j+y_{j+l}y_j$ in (3.10).

Multi-pulse solutions

We now consider the Hamiltonian in (3.9), with only nearest-neighbors interactions ($\kappa_1=1$) and periodic boundary conditions. We are going to consider two different kind of sets $\mathcal{I}$, both dealing with problem of degeneracy due to non consecutive excited sites, the periodic boundary conditions being irrelevant. In the first case we take only two non consecutive sites $\mathcal{I}=\{-l,l\}$, with $l\geq 1$: the larger the distance $2l$ among the sites is, the greater the number of normal form steps needed to remove the degeneracy is, r being equal to $2l$. In the second case we take 3 sites, giving an asymmetric configuration $\mathcal{I}=\{-2,-1,1\}$: this is the easiest (and shortest) asymmetric example witch exhibits degeneracy, due to the lack of the interaction at order ε between the second and the fourth sites. In both the cases it will be shown, accordingly to the already existing literature (see for example [50, 69]), that only standard in/out-of-phase solutions do exist. Linear stability analysis provides a scaling of Floquet exponents coherent with the literature and Theorem 2.1.2 can be always applied in these examples. Moreover, the normal form remarkably shows the effect of switching from focusing to defocusing dNLS, obtained by changing the sign of γ: non-degenerate saddle and center eigenspaces exchange their stability, while degenerate eigenspaces keep unchanged their stability whenever the order of degeneracy is odd, as with $\mathcal{I}=\{-2,-1,1\}$.

First case: $\mathcal{I}=\{-l,l\}$, $\mathcal{J}=\{-7,\ldots,7\}$. In the first case, the perturbation H_1, given by the nearest neighbours interactions, reads

$$H_1=2\sum_{j=\pm l}I_j+2\sum_{j\neq\pm l}\mathbf{i}\xi_j\eta_j-\sum_{j\in\mathcal{J}}(x_{j+1}x_j+y_{j+1}y_j)\ ,$$

where the products $x_{j+1}x_j+y_{j+1}y_j$ are of the following three types:

$$\begin{aligned}
x_{j+1}x_j+y_{j+1}y_j&=\mathbf{i}(\xi_{j+1}\eta_j+\xi_j\eta_{j+1})\\
x_jx_l+y_jy_l&=\sqrt{I_l}[\cos(\varphi_l)(\xi_j+\mathbf{i}\eta_j)-\mathbf{i}\sin(\varphi_l)(\xi_j-\mathbf{i}\eta_j)]\qquad j=l\pm1\\
x_jx_{-l}+y_jy_{-l}&=\sqrt{I_{-l}}[\cos(\varphi_{-l})(\xi_j+\mathbf{i}\eta_j)-\mathbf{i}\sin(\varphi_{-l})(\xi_j-\mathbf{i}\eta_j)]\qquad j=-l\pm1
\end{aligned}\ ,$$

while no term of the form $\cos(\varphi_l-\varphi_{-l})$ appears at order $\mathcal{O}(\varepsilon)$. We expand H_0 and H_1 in Taylor series of the actions around I^*, forget constant terms and introducing the resonant angles $\hat{q}=(q_1,q)$ and their conjugated actions $\hat{p}=(p_1,p)$

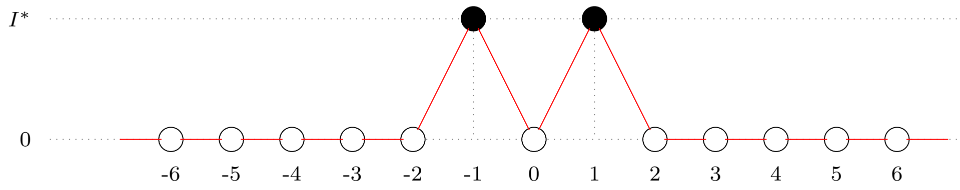
$$\begin{cases}q_1=\varphi_{-l}\\ q_2=\varphi_l-\varphi_{-l}\end{cases}\ ,\qquad\begin{cases}p_1=J_{-l}+J_l\\ p_2=J_l\end{cases}\ ,$$

we can rewrite, as in the previous examples, the initial Hamiltonian in the form

$$\begin{aligned} H = \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\xi_j\eta_j + f_4^{(0,0)}(\hat{p},\xi,\eta) + f_1^{(0,1)}(\hat{q},\xi,\eta) + f_2^{(0,1)}(\hat{p},\xi,\eta) \\ + f_3^{(0,1)}(\hat{q},\hat{p},\xi,\eta) + \sum_{l\geq 5} f_l^{(0,1)}(\hat{q},\hat{p},\xi,\eta) \ , \end{aligned}$$

with $\omega = 1+\gamma I^* = 1+I^*$. Notice in particular that $f_4^{(0,1)}$ and $f_0^{(0,1)}$ are missing, this latter being a constant term. Moreover, $f_2^{(0,1)}$ does not depend on $\hat{q}$: this is due to the lack of coupling terms $x_jx_{j+1}+y_jy_{j+1}$ with both j and $j+1$ belonging to $\mathcal{I}$.

We start considering $l=1$, hence $\mathcal{I}=\{-1,1\}$.



The first stage of the normalization step consists only in the translation correcting the frequencies, given by a constant vector. The homological equation defining $\zeta^{(1)}$ is equivalent to the system

$$\sum_j C_{0,i,j}\zeta_j^{(1)} = \frac{\partial}{\partial \hat{p}_i} f_2^{(0,1)}\Big|_{q=q^*,\xi=\eta=0} \ ,$$

with

$$C_0 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \ ,$$

and $f_2^{(0,1)} = f_2^{(0,1)}(p_1) + f_2^{(0,1)}(\xi,\eta)$. Hence $f_2^{(0,1)}\Big|_{q=q^*,\xi=\eta=0}$ is independent of q^* and the solution of the system is $\zeta^{(1)} = 2\varepsilon(2,1)$.

The remaining stages are needed to remove $f_1^{(\mathrm{I},0,1)} = f_1^{(0,1)}$ and the part of $f_3^{(\mathrm{III},0,1)} = f_3^{(0,1)} + L_{\chi_1^{(1)}} f_4^{(0,0)}$ depending of both $(\hat{q},\hat{p})$ and (ξ,η). Indeed, the term $f_2^{(\mathrm{II},0,1)} = f_2^{(0,1)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)}$ is independent of the angles and the term $f_4^{(\mathrm{IV},0,1)}(\hat{q},\hat{p},\xi,\eta)\Big|_{\xi=\eta=0} = f_4^{(0,1)}(\hat{q},\hat{p},\xi,\eta)\Big|_{\xi=\eta=0} \equiv 0$, as for $f_0^{(1,0)}$; hence the first step in concluded. Being $f_0^{(0,1)}(\hat{q}) \equiv 0$, any q_2 is a critical point and the problem is trivially degenerate. A second normalization step is needed.

The second step can be performed in a similar way to the seagull example, but this time the only stage that is absent is the fourth one. Once computed, we get from the gradient of $f_0^{(2,2)}$

$$-2\varepsilon^2\sin(q_2) = 0 \ ,$$

which provides only standard solutions $q_2^* \in \{0,\pi\}$. In order to conclude the existence of the two in/out-of-phase configurations above, we need to check condition (2.12) with $\alpha < \frac{3}{2}$. Indeed, explicit symbolic calculations made with Mathematica clearly show that (2.12) holds with $\alpha = 1$. The stability analysis shows that $q_2^*=0$ is the unstable configuration, while $q_2^*=\pi$ is the stable one, with Floquet exponents

$$\lambda_{1,2}(\varepsilon) = \pm\mathbf{i}\left(2\varepsilon + \frac{\varepsilon^3}{(I^*)^2} + h.o.t.\right) \ .$$

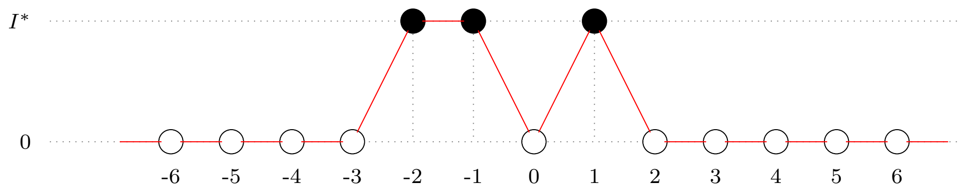
Theorem 2.1.2 applies with $\beta = 1 < 2 = r+1-\alpha$, hence Floquet multipliers are ε^2-close to the approximate ones $e^{\lambda T}$ (where T is the period).

If $l > 1$, the procedure for the continuation is clearly the same: it turns out that degeneracy persists up to order $r = 2l - 1$, namely $f_0^{(r,r)} \equiv 0$ for $r \leq 2l - 1$. At order $r = 2l$ one gets from $f_0^{(r,r)}$ an equation of the form

$$c(I^*)\varepsilon^r \sin(q_2) = 0 \ ,$$

with $c(I^*)$ a constant depending on I^*, which again provides only standard solutions $q_2 = \{0, \pi\}$. Existence of the two in/out-of-phase configurations above is ensured by condition (2.12) with $\alpha = r/2 < (r+1)/2$. Linear stability is clearly affected by the increased (odd) order of the degeneracy; stable and unstable configurations are expected to be respectively $q_2 = \pi$ and $q_2 = 0$, with approximate Floquet exponents of order $\mathcal{O}(\varepsilon^l)$.

Second case: $\mathcal{I} = \{-2, -1, 1\}$, $\mathcal{J} = \{-6, \ldots, 7\}$.



The perturbation H_1, given by the nearest neighbors interactions, reads

$$H_1 = 2\sum_{j\in\mathcal{I}} I_j + 2\sum_{j\neq -2,-1,1} \mathbf{i}\xi_j\eta_j - \sum_{j\in\mathcal{J}} (x_{j+1}x_j + y_{j+1}y_j) \ ,$$

where the products $x_l x_j + y_l y_j$ are of the following three types:

$$\begin{aligned} x_{j+1}x_j + y_{j+1}y_j &= \mathbf{i}(\xi_{j+1}\eta_j + \xi_j\eta_{j+1}) \\ x_j x_l + y_j y_l &= \sqrt{I_l}[\cos(\varphi_l)(\xi_j + \mathbf{i}\eta_j) - \mathbf{i}\sin(\varphi_l)(\xi_j - \mathbf{i}\eta_j)] \qquad l = j \pm 1 \ . \\ x_{-1}x_{-2} + y_{-1}y_{-2} &= 2\sqrt{I_{-2}I_{-1}}\cos(\varphi_{-1} - \varphi_{-2}) \end{aligned}$$

By expanding H_0 and H_1 in Taylor series of the actions around I^*, forgetting constant terms and introducing the resonant angles $\hat{q} = (q_1, q)$ and their conjugated actions $\hat{p} = (p_1, p)$

$$\begin{cases} q_1 = \varphi_{-2} \\ q_2 = \varphi_{-1} - \varphi_{-2} \\ q_3 = \varphi_1 - \varphi_{-1} \end{cases} , \qquad \begin{cases} p_1 = J_{-2} + J_{-1} + J_1 \\ p_2 = J_{-1} + J_1 \\ p_3 = J_1 \end{cases} ,$$

we can rewrite the initial Hamiltonian in the form

$$\begin{aligned} H = \omega p_1 &+ \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\xi_j\eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(q_2) + f_1^{(0,1)}(\hat{q}, \xi, \eta) \\ &+ f_2^{(0,1)}(q, \hat{p}, \xi, \eta) + f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(q, \hat{p}) + \sum_{l\geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) \ , \end{aligned}$$

with $\omega = 1 + \gamma I^* = 1 + I^*$. The first stage of the normalization step consists only in the translation $\zeta^{(1)}$ correcting the frequencies

$$\begin{cases} \zeta_1^{(1)} = 2\varepsilon\,(3 - \cos(q_2^*)) \\ \zeta_2^{(1)} = \varepsilon\,(4 - \cos(q_2^*)) \\ \zeta_3^{(1)} = 2\varepsilon \end{cases} ,$$

since $f_0^{(0,1)}(q_2)$, due to the effect of the symmetry, is already independent of q_1 and no average is required. We remark that the normal form construction keeps the symmetry, thus the terms depending only on $(q, \hat{p})$ remain independent of q_1 and no averages are required, as for the original

Hamiltonian H. In particular the terms $f_2^{(0,1)}(q,\hat{p},\xi,\eta)$ and $f_4^{(0,1)}(q,\hat{p})$ are already independent of q_1; in this example they explicitly read

$$\begin{aligned} f_2^{(0,1)}(q,\hat{p},\xi,\eta) = -\varepsilon\Bigg(& (-2+\cos(q_2))\,p_1 - \cos(q_2)\,p_3 - 2\mathbf{i}\sum_{j\in\mathcal{J}\setminus\mathcal{I}}\xi_j\eta_j + \mathbf{i}\Big(\eta_{11}\xi_1+\eta_3\xi_2+\eta_4\xi_3+\eta_2\xi_1 \\ & +\eta_2\xi_3+\eta_3\xi_4+\eta_5\xi_4+\eta_4\xi_5+\eta_6\xi_5+\eta_5\xi_6+\eta_9\xi_8+\eta_8\xi_9+\eta_{10}\xi_9+\eta_9\xi_{10} \\ & +\eta_{11}\xi_{10}+\eta_{10}\xi_{11}+\eta_1\xi_2+\eta_1\xi_{11}\Big)\Bigg)\ , \\ f_4^{(0,1)}(q,\hat{p}) = \frac{\varepsilon\cos(q_2)}{4I^*} & (p_1-2p_2+p_3)^2\ . \end{aligned}$$

The remaining stages consist in removing $f_1^{(\mathrm{I},0,1)}$, which connects the torus with the transversal variables, and the part of $f_3^{(\mathrm{III},0,1)}(\hat{q},\hat{p},\xi,\eta)$ which depends on both the sets of variables. This concludes the first normalization step. The critical points q^* of $f_0^{(1,1)} = -2\varepsilon I^*\cos(q_2)$ are solutions of the following trigonometric equation

$$2\varepsilon I^*\sin(q_2) = 0\ ,$$

hence we get two disjoint one-parameter families $Q_1(\vartheta) = (0,\vartheta)$ and $Q_2(\vartheta) = (\pi,\vartheta)$ on the torus $\mathbb{T}^2$, where $\vartheta = q_3$. In order to remove the degeneracy, a second normalization step is needed, where in this case the first and the fourth stages are absent.

Proceeding according to the normal form algorithm, we are led to consider the trigonometric system

$$\begin{cases} 2\varepsilon I^*\sin(q_2) - 2\varepsilon^2(2-\cos(q_2))\sin(q_2) = 0 \\ -\,2\varepsilon^2\sin(q_3) = 0 \end{cases}\ ,$$

which provides the critical points of $f_0^{(2,1)} + f_0^{(2,2)}$. The system only admits the four in/out-of-phase solutions $(q_2^*, q_3^*) \in \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$. In order to conclude the existence of the four in/out-of-phase configurations above, we need to check condition (2.12), again with $\alpha < \frac{3}{2}$. In all the considered cases, explicit symbolic calculations made with Mathematica show that (2.12) holds with $\alpha = 1$. Approximate linear stability analysis provides $(0,\pi)$ as the only stable configuration with Floquet exponents

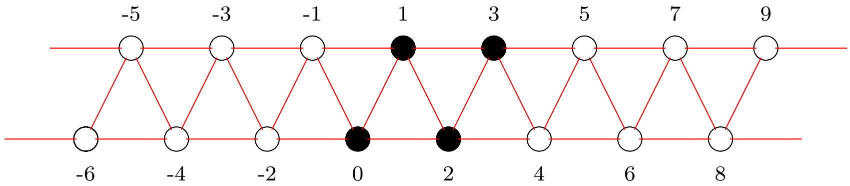
$$\begin{aligned} \lambda_{1,2}(\varepsilon) &= \pm\mathbf{i}\left(2\sqrt{I^*}\sqrt{\varepsilon} + \frac{\varepsilon^{3/2}}{4\sqrt{I^*}} + h.o.t.\right)\ , \\ \lambda_{3,4}(\varepsilon) &= \pm\mathbf{i}\sqrt{3}\left(\varepsilon - \frac{\varepsilon^2}{8I^*} + h.o.t.\right)\ , \end{aligned}$$

while the other configurations are unstable. Approximate linear stability corresponds to effective linear stability, since also in this case Theorem 2.1.2 applies with $\beta = 1 < 2 = r+1-\alpha$, hence Floquet multipliers are located ε^2-close to the approximate ones, and fulfill the usual symmetries of the spectrum of a symplectic matrix.

Remark 3.3.2 *We here remark what happens to the Floquet exponents once the sign of the nonlinear coefficient γ is changed. It turns out, as already stressed in the literature, that eigenvalues of order $\mathcal{O}(\sqrt{\varepsilon})$ switch from real to imaginary and viceversa, hence stable and unstable eigenspaces are exchanged. However, eigenvalues of order $\mathcal{O}(\varepsilon)$ keep their nature. This is the effect of a cancellation of γ in front of the equation $-2\varepsilon^2\sin(q_3)$, as already stressed in the seagull example. The new stable configuration would be (π,π).*

ZigZag model

Let us consider the Hamiltonian system (3.9) with $\kappa_1 = \kappa_2 = 1$, namely the so-called ZigZag model. This is a particular case of two coupled one-dimensional dNLS models, where the ZigZag coupling provides a one-dimensional Hamiltonian system with nearest and next-to-nearest neighbor interactions. We want to investigate the continuation of vortex-like localized structures given by four consecutive excited sites; hence the lower dimensional resonant torus is $I^* = (I^*, I^*, I^*, I^*)$, $\xi = \eta = 0$ and $\mathcal{I} = \{0, 1, 2, 3\} \subset \mathcal{J} = \{-6, \ldots, 9\}$.



These configurations have been the object of investigation of [76]: there, methods of bifurcation theory (namely a Lyapunov-Schmidt reduction) have been used to show non-existence of four-sites solutions with phase differences q_l different from $\{0, \pi\}$, first in the dNLS model and then in the Klein-Gordon lattice. Here we want to obtain the same results by means of our normal form algorithm, and correct a minor statement on non-degeneracy of the isolated configurations.

In this case the perturbation, given by the nearest and next-to-nearest neighbors interactions, reads

$$H_1 = 4\sum_{j=0}^{3} I_j + 4\sum_{j<0\vee j>3} \mathbf{i}\xi_j\eta_j - 2\sum_{j\in\mathcal{J}} \left((x_{j+1}x_j + y_{j+1}y_j) + (x_{j+2}x_j + y_{j+2}y_j)\right)$$

where the products $x_{j+1}x_j + y_{j+1}y_j$ are of the following three types:

$$\begin{aligned}
x_{j+1}x_j + y_{j+1}y_j &= \mathbf{i}(\xi_{j+1}\eta_j + \xi_j\eta_{j+1}) \\
x_{j+1}x_j + y_{j+1}y_j &= \sqrt{I_{j+1}}\left[\cos(\varphi_{j+1})(\xi_j + \mathbf{i}\eta_j) - \mathbf{i}\sin(\varphi_{j+1})(\xi_j - \mathbf{i}\eta_j)\right] . \\
x_{j+1}x_j + y_{j+1}y_j &= 2\sqrt{I_{j+1}I_j}\cos(\varphi_{j+1} - \varphi_j)
\end{aligned}$$

By expanding H_0 and H_1 in Taylor series of the actions around I^*, forgetting constant terms and introducing the resonant angles $\hat{q} = (q_1, q)$ and their conjugated actions $\hat{p} = (p_1, p)$

$$\begin{cases} q_1 = \varphi_0 \\ q_2 = \varphi_1 - \varphi_0 \\ q_3 = \varphi_2 - \varphi_1 \\ q_4 = \varphi_3 - \varphi_2 \end{cases}, \qquad \begin{cases} p_1 = J_0 + J_1 + J_2 + J_3 \\ p_2 = J_1 + J_2 + J_3 \\ p_3 = J_2 + J_3 \\ p_4 = J_3 \end{cases},$$

we can rewrite the initial Hamiltonian in the form

$$\begin{aligned}
H = \omega p_1 &+ \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\xi_j\eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(q_2, q_3, q_4) + f_1^{(0,1)}(\hat{q}, \xi, \eta) \\
&+ f_2^{(0,1)}(q, \hat{p}, \xi, \eta) + f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(q, \hat{p}, \xi, \eta) + \sum_{l\geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) ,
\end{aligned}$$

with $\omega = 1 + \gamma I^* = 1 + I^*$. The first stage of the normalization step consists only in the translation correcting the frequencies. The homological equation defining $\zeta^{(1)}$ gives the solution

$$\begin{cases}
\zeta_1^{(1)} = 2\varepsilon\left(4 - \cos(q_2^*) - \cos(q_3^*) - \cos(q_4^*) - \cos(q_2^* + q_3^*) - \cos(q_3^* + q_4^*)\right) \\
\zeta_2^{(1)} = \varepsilon\left(6 - \cos(q_2^*) - 2\cos(q_3^*) - 2\cos(q_4^*) - \cos(q_2^* + q_3^*) - 2\cos(q_3^* + q_4^*)\right) \\
\zeta_3^{(1)} = \varepsilon\left(4 - \cos(q_3^*) - 2\cos(q_4^*) - \cos(q_2^* + q_3^*) - \cos(q_3^* + q_4^*)\right) \\
\zeta_4^{(1)} = \varepsilon\left(2 - \cos(q_4^*) - \cos(q_3^* + q_4^*)\right)
\end{cases}.$$

The remaining steps are the same as for the previous example $\mathcal{I} = \{-2, -1, 1\}$ in the focusing dNLS model. Let us determine the q^*-values, critical points of $f_0^{(1,1)}$, that solve the trigonometric system

$$\begin{cases} 2I^* \sin(q_2) + 2I^* \sin(q_2 + q_3) = 0 \\ 2I^* \sin(q_3) + 2I^* \sin(q_2 + q_3) + 2I^* \sin(q_3 + q_4) = 0 \\ 2I^* \sin(q_4) + 2I^* \sin(q_3 + q_4) = 0 \end{cases} ,$$

obtaining only the following solutions: we have four isolated solutions $(0, 0, 0)$, $(0, 0, \pi)$, $(\pi, 0, 0)$, $(\pi, 0, \pi)$, and two one-parameter families $(\vartheta, \pi, \vartheta - \pi)$ and $(\vartheta, \pi, -\vartheta)$. In order to apply an implicit function theorem we have to verify the non-degeneracy condition, which factoring out $2I^*$, reads

$$\begin{vmatrix} \cos(q_2) + \cos(q_2 + q_3) & \cos(q_2 + q_3) & 0 \\ \cos(q_2 + q_3) & \cos(q_3) + \cos(q_2 + q_3) + \cos(q_3 + q_4) & \cos(q_3 + q_4) \\ 0 & \cos(q_3 + q_4) & \cos(q_4) + \cos(q_3 + q_4) \end{vmatrix} \neq 0 .$$

By calculating the determinant in correspondence of the q^*-values determined above, we see that non-degeneracy is fulfilled in three of the four isolated solutions $(0, 0, 0)$, $(0, 0, \pi)$, $(\pi, 0, 0)$, while for the fourth isolated configuration $(\pi, 0, \pi)$ and the two families we have degeneracy. The topologically isolated configuration $(\pi, 0, \pi)$ is a degenerate minimizer of $f_0^{(1,1)}$, since along the tangent direction $(\pi + t, -2t, \pi + t)$ it is possible to observe a growth as $\mathcal{O}(t^4)$; this represents an example of degenerate isolated configuration.

For all these configurations we have to perform a second normalization step. In this example only the fourth stage is missing.

The equation that allows us to have periodic orbits for the approximate dynamic is the following

$$\nabla_q \left(f_0^{(2,1)} + f_0^{(2,2)} \right) = 0 ,$$

and can be rewritten as

$$F(q_2, q_3, q_4, \varepsilon) = F_0(q_2, q_3, q_4) + \varepsilon F_1(q_2, q_3, q_4) = 0,$$

with $F : \mathbb{T}^3 \times \mathcal{U}(0) \to \mathbb{R}^3$. We know that, for $\varepsilon = 0$, F admits as solutions four points and two families $Q_j(\vartheta)$ with

$$\begin{aligned} Q_1(\vartheta) &= (\vartheta, \pi, \vartheta - \pi) \\ Q_2(\vartheta) &= (\vartheta, \pi, -\vartheta). \end{aligned}$$

We wonder under which conditions it is possible to continue w.r.t. ε the degenerate solutions. Observe that the vectors

$$\partial_\vartheta Q_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \qquad \text{and} \qquad \partial_\vartheta Q_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

generate the Kernel of $\frac{\partial F_0(Q_j(\vartheta))}{\partial q}$ with $j = 1, 2$. Applying Proposition 3.2.1, the necessary condition to obtain the continuation is $F_1(Q_j(\vartheta)) \perp \partial_\vartheta Q_j(\vartheta)$, so from

$$\begin{aligned} \langle F_1(Q_1(\vartheta)), \partial_\vartheta Q_1 \rangle &= 4 \sin(2\vartheta) \\ \langle F_1(Q_2(\vartheta)), \partial_\vartheta Q_2 \rangle &= -4 \sin(\vartheta), \end{aligned}$$

we can deduce that the two families $Q_j(\vartheta)$ break down and only four solutions

$$(0, \pi, \pi),\ (\pi, \pi, 0),\ (0, \pi, 0),\ (\pi, \pi, \pi)$$

are allowed for continuation. Indeed, the two vortex-like solutions $\left(\frac{\pi}{2}, \pi, -\frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, \pi, \frac{\pi}{2}\right)$, belonging to the intersection of the two families, solve only the necessary condition for Q_1 and not the one for Q_2.

In order to conclude the existence of the three non-degenerate in/out-of-phase configurations above, we need to check condition (2.12) with $\alpha < \frac{3}{2}$. Explicit symbolic calculations made with Mathematica clearly show that (2.12) holds with $\alpha = 1/2$ in the three isolated and non-degenerate configurations[4] while $\alpha = 1$ for the remaining four configurations belonging to $Q_{1,2}(\vartheta)$ and for the fourth isolated solution $(\pi, 0, \pi)$. The additional $\sqrt{\varepsilon}$ factor comes from the degenerate tangential directions; hence existence is proved.

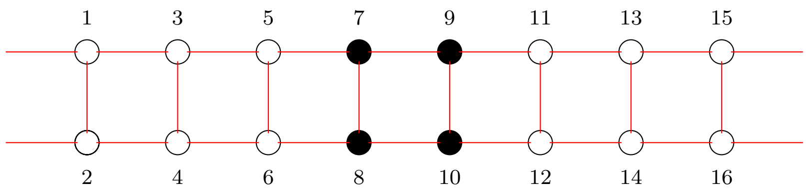
Concerning the approximate linear stability analysis, $(0,0,0)$ is the unique stable configuration; indeed

$$\lambda_{1,2}(\varepsilon) = \pm 2\sqrt{2}\mathbf{i}\left(\sqrt{I^*}\sqrt{\varepsilon} + \frac{\varepsilon^{3/2}}{\sqrt{I^*}} + h.o.t.\right) ,$$
$$\lambda_{3,4}(\varepsilon) = \pm\mathbf{i}\left(2\sqrt{2}\sqrt{I^*}\sqrt{\varepsilon} + \frac{5\varepsilon^{3/2}}{\sqrt{2}\sqrt{I^*}} + h.o.t.\right) ,$$
$$\lambda_{5,6}(\varepsilon) = \pm\mathbf{i}\left(2\sqrt{I^*}\sqrt{\varepsilon} + \frac{\varepsilon^{3/2}}{\sqrt{I^*}} + h.o.t.\right) .$$

However it has two pairs of exponents which coincide at order $\sqrt{\varepsilon}$, but they are different at order $\varepsilon^{3/2}$; this leads to $\beta = 3/2$ in the assumption of Theorem 2.1.2. Since $r + 1 - \alpha = 3 - \frac{1}{2} = \frac{5}{2}$, the statement ensures existence of two couples of distinct Floquet multipliers which are $\varepsilon^{5/2}$-close to $e^{\lambda T}$ on the unitary circle, which means effective linear stability of the solution.

Railway model

We here consider the so-called railway-model: it simply consists of two coupled dNLS models, in each of which only nearest neighbors interactions are active.



The model, labeling the sites of the two one-dimensional lattices according to the index set $\mathcal{J} = \{1, \dots, N\}$, with $N = 16$, is described by the Hamiltonian

$$H = \sum_{j\in\mathcal{J}} \left(\frac{1}{2}(x_j^2 + y_j^2) + \frac{\gamma}{8}(x_j^2 + y_j^2)^2\right)$$
$$+ 3\varepsilon \sum_{j\in\mathcal{J}} \frac{1}{2}(x_j^2 + y_j^2) - \varepsilon \sum_{j\in\mathcal{J}}(x_{j+1}x_{j-1} + y_{j+1}y_{j-1}) - \varepsilon \sum_{j=1}^{N/2}(x_{2j}x_{2j-1} + y_{2j}y_{2j-1}) ,$$

which is a minor variation of the Hamiltonian system (3.9). Indeed, we are considering $k_1 = 1/2$ and $k_2 = 1$ and we have rewrite the second part of the perturbation in terms of odd and even terms. We want to investigate the continuation of the minimal vortex configuration, namely the localized structures given by four consecutive excited sites, that we here take as $\mathcal{I} = \{7, 8, 9, 10\}$, with phase differences between the neighboring ones all equal to $\pi/2$. The existence of such rotating structures has been shown in proper two-dimensional lattices in [70], by expanding at very high perturbation orders the Kernel equation obtained with a Lyapunov-Schmidt reduction. On the other hand, in [75] similar structures have been proved not to exists in the one-dimensional dNLS lattice (3.10) with $\kappa_1 = \kappa_3 = 1$, which at first order in the perturbation parameter ε exhibits the same averaged term $f_0^{(1,1)}(q)$ as the two-dimensional problem, hence the same critical points.

[4] Indeed for these non-degenerate solutions one normal form step would have been enough, being $\alpha < (r+1)/2 = 1$.

The present railway-model represents a natural hybrid setting between one-dimensional and two-dimensional square lattices: here, by means of our normal form algorithm, we are going to show non-existence of the minimal vortex, thus enforcing the proper two-dimensional nature of these kind of localized solutions.

As in the previous examples, we introduce action-angle variables (I, φ) and complex coordinates (ξ, η) and we expand H_0 and H_1 in Taylor series of the actions around I^*; by forgetting constant terms and introducing the resonant angles $\hat{q} = (q_1, q)$ and their conjugated actions $\hat{p} = (p_1, p)$

$$\begin{cases} q_1 = \varphi_7 \\ q_2 = \varphi_8 - \varphi_7 \\ q_3 = \varphi_9 - \varphi_8 \\ q_4 = \varphi_{10} - \varphi_9 \end{cases} , \qquad \begin{cases} p_1 = J_7 + J_8 + J_9 + J_{10} \\ p_2 = J_8 + J_9 + J_{10} \\ p_3 = J_9 + J_{10} \\ p_4 = J_{10} \end{cases} ,$$

we can rewrite the starting Hamiltonian again in the form

$$\begin{aligned} H = \omega p_1 + \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \mathbf{i} \xi_j \eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(q_2, q_3, q_4) + f_1^{(0,1)}(\hat{q}, \xi, \eta) \\ + f_2^{(0,1)}(q, \hat{p}, \xi, \eta) + f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(q, \hat{p}, \xi, \eta) + \sum_{l \geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) , \end{aligned}$$

with $\omega = 1 + \gamma I^* = 1 + I^*$. The homological equation defining $\zeta^{(1)}$ gives the solution

$$\begin{cases} \zeta_1^{(1)} = \varepsilon \left(12 - 2\cos(q_2^*) - 2\cos(q_4^*) - 2\cos(q_2^* + q_3^*) - 2\cos(q_3^* + q_4^*)\right) \\ \zeta_2^{(1)} = \varepsilon \left(9 - \cos(q_2^*) - 2\cos(q_4^*) - \cos(q_2^* + q_3^*) - 2\cos(q_3^* + q_4^*)\right) \\ \zeta_3^{(1)} = \varepsilon \left(6 - 2\cos(q_4^*) - \cos(q_2^* + q_3^*) - \cos(q_3^* + q_4^*)\right) \\ \zeta_4^{(1)} = \varepsilon \left(3 - \cos(q_4^*) - \cos(q_3^* + q_4^*)\right) \end{cases} ,$$

while the remaining stages are the same of the previous example (the second and the fourth stages are absent). The q^*-values, critical points of $f_0^{(1,1)}$, have to solve the trigonometric system

$$\begin{cases} 2I^* \sin(q_2) + 2I^* \sin(q_2 + q_3) = 0 \\ 2I^* \sin(q_2 + q_3) + 2I^* \sin(q_3 + q_4) = 0 \\ 2I^* \sin(q_4) + 2I^* \sin(q_3 + q_4) = 0 \end{cases} .$$

We obtain the following solutions: we have two isolated solutions $(0, 0, 0)$, $(\pi, 0, \pi)$, and three one-parameter families $Q_j(\vartheta)$

$$\begin{aligned} Q_1(\vartheta) &= (\vartheta, \pi, -\vartheta) \\ Q_2(\vartheta) &= (\vartheta, \pi, \vartheta + \pi). \\ Q_3(\vartheta) &= (\vartheta, -2\vartheta, \vartheta + \pi). \end{aligned} .$$

Moreover we notice that, as in the previous ZigZag model, the three families all intersect in the two vortex configurations $\pm\left(\frac{\pi}{2}, \pi, -\frac{\pi}{2}\right)$. These are completely degenerate configurations, since the Kernel admits three independent directions $\partial_\vartheta Q_j$ on the tangent space to the torus $\mathbb{T}^3$; hence $D_q^2 f_0^{(1,1)}\left(\frac{\pi}{2}, \pi, -\frac{\pi}{2}\right) \equiv 0$ and the following non-degeneracy condition is not fulfilled:

$$\begin{vmatrix} \cos(q_2) + \cos(q_2 + q_3) & \cos(q_2 + q_3) & 0 \\ \cos(q_2 + q_3) & \cos(q_2 + q_3) + \cos(q_3 + q_4) & \cos(q_3 + q_4) \\ 0 & \cos(q_3 + q_4) & \cos(q_4) + \cos(q_3 + q_4) \end{vmatrix} \neq 0 ,$$

where we have factored out $2I^*$. It is immediate to verify that the two isolated configurations are non-degenerate, hence also a strategy based on the implicit function theorem works out. For all the remaining solutions on Q_j we have to perform a second normalization step. As in the previous example, the fourth stage is missing. The equation that allows us to have periodic orbits for the approximate dynamic is the following

$$\nabla_q \left(f_0^{(2,1)} + f_0^{(2,2)} \right) = 0 \ ,$$

which can be rewritten again as

$$F(q_2, q_3, q_4, \varepsilon) = F_0(q_2, q_3, q_4) + \varepsilon F_1(q_2, q_3, q_4) = 0,$$

with $F : \mathbb{T}^3 \times \mathcal{U}(0) \to \mathbb{R}^3$. In this case there are three vectors

$$\partial_\vartheta Q_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \partial_\vartheta Q_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \partial_\vartheta Q_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

that generate the Kernel of $\partial_\vartheta F_0(Q_j(\vartheta))$ with $j = 1, 2, 3$. Once again, applying Proposition 3.2.1, the necessary condition to obtain the continuation is $F_1(Q_j(\vartheta)) \perp \partial_\vartheta Q_j(\vartheta)$; it turns out that $F_1(Q_1(\vartheta)) \equiv 0$, hence nothing can be concluded on Q_1, similarly to what already observed also in the square dNLS cell. For the other two families we get

$$\begin{aligned} \langle F_1(Q_2(\vartheta)), \partial_\vartheta Q_2 \rangle &= 4\sin(2\vartheta) \\ \langle F_1(Q_3(\vartheta)), \partial_\vartheta Q_3 \rangle &= 4\sin(2\vartheta), \end{aligned}$$

and we can deduce that they break down and only the four solutions

$$(0, \pi, \pi),\ (\pi, \pi, 0),\ (0, 0, \pi),\ (\pi, 0, 0)$$

or the two vortexes $\left(\frac{\pi}{2}, \pi, -\frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, \pi, \frac{\pi}{2}\right)$ are allowed for continuation. The existence of the four in/out-of-phase configurations above is ensured by condition (2.12); indeed, explicit symbolic calculations made with Mathematica clearly show that (2.12) holds with $\alpha = 1 < \frac{3}{2}$. In the two vortexes, instead, the decay of the smallest eigenvalue is too fast to fulfill (2.12), being $\alpha \approx 3 > \frac{3}{2}$. Hence, a third normal form step is needed to study the continuation of the configurations in $Q_1(\vartheta)$, vortexes included. And indeed, by applying at order $\mathcal{O}(\varepsilon^3)$ the previous bifurcation argument, we get

$$\langle F_2(Q_1(\vartheta)), \partial_\vartheta Q_1 \rangle = \frac{4}{I^*} \sin(\vartheta) \ ,$$

where clearly $\varepsilon^3 F_2(q) = \nabla_q f_0^{(3,3)}$. The third normalization step is thus conclusive for the non-existence of the two vortex configurations: only the two in/out-of-phase solutions (π, π, π) and $(0, \pi, 0)$ can be continued. Continuation is derived from (2.12), which holds true with $\alpha = 3/2 < 2 = (r + 1)/2$ with $r = 3$.

Concerning the approximate linear stability analysis, $(0, 0, 0)$ is the unique stable configuration.

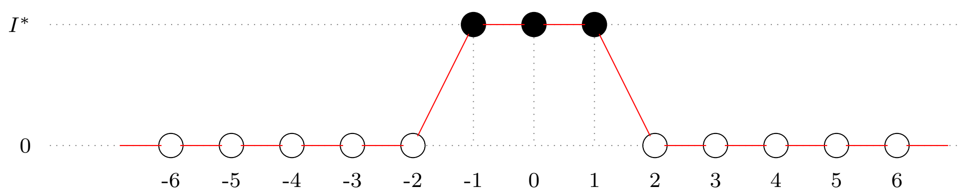
$$\begin{aligned} \lambda_{1,2}(\varepsilon) &= \pm \mathbf{i} \left(2\sqrt{I^*}\sqrt{\varepsilon} + \frac{6}{(I^*)^{5/2}} \varepsilon^{7/2} + h.o.t. \right) \ , \\ \lambda_{3,4}(\varepsilon) &= \pm \mathbf{i} \left(2\sqrt{I^*}\sqrt{\varepsilon} - \frac{\varepsilon^{5/2}}{(I^*)^{3/2}} + h.o.t. \right) \ , \\ \lambda_{5,6}(\varepsilon) &= \pm \mathbf{i} \left(2\sqrt{2}\sqrt{I^*}\sqrt{\varepsilon} + \frac{\sqrt{2}\varepsilon^{3/2}}{\sqrt{I^*}} + h.o.t. \right) \ . \end{aligned}$$

They split only at order $\varepsilon^{5/2}$. This leads to $\beta = 5/2$ in the assumption of Theorem 2.1.2. Since $r + 1 - \alpha = 4 - \frac{1}{2} = \frac{7}{2}$, the statement ensures existence of two couples of distinct Floquet multipliers which are $\varepsilon^{7/2}$-close to $e^{\lambda T}$ on the unitary circle, which means effective linear stability of the solution.

Remark 3.3.3 *It is remarkable the effect of having studied the non-degenerate configurations with the more accurate normal form at order $r = 3$, although useless for the continuation purpose. Indeed, while stopping at order $r = 1, 2$, Theorem 2.1.2 could not have applied to conclude effective stability of the solution and the localization of its Floquet exponents; indeed the two couples of Floquet exponents would have been equal. On the contrary, for $r = 3$ the eigenvalues split at order $5/2$ and the Theorem can be applied. This shows the power of the normal form, which allows to increase the accuracy of the approximation beyond the minimal order needed to ensure existence of the continuation.*

3.3.3 Multi-pulse solutions in the dNLS model with purely non-linear coupling

We consider here a dNLS model with purely non-linear coupling, hence not covered by (3.9); as the simplest choice, only nearest-neighbors interactions are active. It is well known that in this model single-site discrete solitons (as breathers in Klein-Gordon models) are more compactly supported, with tails decaying more than exponentially fast (for this kind of models see for example [28, 81]). In terms of normal form, due to this weaker interaction among the sites of the chain, we need $r = 3$ to remove the degeneracy, even if there are no holes.



To be precise, we consider a perturbation H_1 of the form

$$H_1 = \sum_j |\psi_{j+1} - \psi_j|^4 \ ,$$

$\mathcal{I} = \{-1, 0, 1\}$, $\mathcal{J} = \{-7, \dots, 7\}$ and, as before, periodic boundary conditions. The perturbation H_1 is given by the quartic nearest neighbors interaction, which in real coordinates reads

$$\begin{aligned} H_1 = &\frac{1}{2}\sum_{j\in\mathcal{J}}(x_j^2 + y_j^2)^2 + \sum_{j\in\mathcal{J}}\left(x_{j+1}x_j + y_{j+1}y_j\right)^2 \\ &- \sum_{j\in\mathcal{J}}(x_j^2 + y_j^2)(x_{j+1}x_j + y_{j+1}y_j) - \sum_{j\in\mathcal{J}}(x_j^2 + y_j^2)(x_{j-1}x_j + y_{j-1}y_j) \\ &+ \frac{1}{2}\sum_{j\in\mathcal{J}}(x_{j+1}^2 + y_{j+1}^2)(x_j^2 + y_j^2) \ . \end{aligned}$$

By expanding H_0 and H_1 in Taylor series of the actions around I^*, forgetting constant terms and introducing the resonant angles and their conjugated actions

$$\begin{cases} q_1 = \varphi_{-1} \\ q_2 = \varphi_0 - \varphi_{-1} \\ q_3 = \varphi_1 - \varphi_0 \end{cases} , \qquad \begin{cases} p_1 = J_{-1} + J_0 + J_1 \\ p_2 = J_0 + J_1 \\ p_3 = J_1 \end{cases} ,$$

we can rewrite the initial Hamiltonian in the usual form

$$\begin{aligned} H = \omega p_1 + &\sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\xi_j\eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(q) + f_1^{(0,1)}(\hat{q}, \xi, \eta) + f_2^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) \\ &+ f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + \sum_{l\geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) \ , \end{aligned}$$

where $\omega = 1 + \gamma I^* = 1 + I^*$. This problem is degenerate at order $\mathcal{O}(\varepsilon)$, indeed

$$f_0^{(1,1)} = 8\,(I^*)^2\,\varepsilon\,(\cos(2q_2) - \cos(q_2) + \cos(2q_3) - \cos(q_3))$$

(after removing constant terms after the expansion around the torus), hence the solutions of the system

$$\begin{cases} 8\,(I^*)^2\sin(q_2) - 4\,(I^*)^2\sin(2q_2) = 0 \\ 8\,(I^*)^2\sin(q_2) - 4\,(I^*)^2\sin(2q_2) = 0 \end{cases} .$$

are critical points of $f_0^{(1,1)}$. We obtain the four isolated solutions $(0, \pi)$, $(\pi, 0)$, $(0, \pi)$ and (π, π). The non- degeneracy condition is fulfilled only in the last configuration. For the remaining solutions, the degeneracy persists also at order $\mathcal{O}(\varepsilon^2)$. This requires a third normal form step by means of we can conclude the existence of all the four configurations above. Indeed, we check condition (2.12) with $\alpha < 2$ and explicit symbolic calculations made with Mathematica clearly show that (2.12) holds with $\alpha = 3/2$.

The approximate stability analysis easily shows that $(0, 0)$ is the only stable configuration, with Floquet exponents

$$\lambda_{1,2}(\varepsilon) = \pm 2\mathbf{i}I^*\left(\varepsilon^{3/2} + \varepsilon^{5/2} + h.o.t.\right) ,$$
$$\lambda_{3,4}(\varepsilon) = \pm 2\mathbf{i}I^*\left(\sqrt{3}\varepsilon^{3/2} + \frac{1}{\sqrt{3}}\varepsilon^{5/2} + h.o.t.\right) .$$

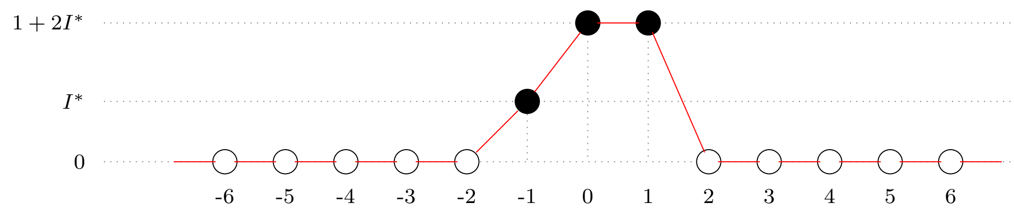
Theorem 2.1.2 applies with $\beta = 3/2 < 5/2 = r + 1 - \alpha$, hence Floquet multipliers are $\mathcal{O}(\varepsilon^{5/2})$-close to the approximate ones.

3.3.4 Other resonances and persistence of two-dimensional tori.

Let us consider again the standard dNLS model (3.9) with $\kappa_1 = 1$ and $\kappa_l = 0$ for any $2 \le l \le i$, and periodic boundary conditions. Differently from most of the literature on localized solutions, we now consider a resonant torus with resonance relationships different from the classical $(1 : \ldots : 1)$. In this case, the action of the symmetry group is transversal to the action of the periodic flow on the unperturbed torus; hence, the objects which have to survive in this model with one additional conserved quantity (A.43) are two-dimensional resonant subtori of the given initial resonant torus. Our normal form allows to approximate at any perturbation order the subtori surviving to the breaking of the original resonant torus; such a good approximation can be used to prove the persistence of the considered subtorus. The persistence of these objects in Hamiltonian systems (and more generic dynamical systems) with symmetries is known in the literature, as well as applications of this theory due to Nekhoroshev to dNLS lattices (see [6–8]). However, in contrast to the literature, the continuation is here made at fixed period and not at fixed values of the independent conserved quantities, and, more important, the normal form allows to treat both non-degenerate and degenerate subtori (while in the mentioned results only non-degenerate objects are covered).

We show how to construct the leading order approximation of these subtori in both a non-degenerate and a degenerate case, in the easiest case of three consecutive excited sites $\mathcal{I} = \{-1, 0, 1\}$, and always assuming $\gamma = 1$. By restricting to these considered examples, we also explain how to modify the proof of Theorem 2.1.1 in terms of the map Υ, so to prove the persistence of these families of localized and time-periodic structures in this class of dNLS models.

Non-degenerate case: as a first case, we consider the sets $\mathcal{I} = \{-1, 0, 1\}$ and $\mathcal{J} = \{-7, \ldots, 7\}$, with the following excited actions $\{I^*, 1 + 2I^*, 1 + 2I^*\}$, so that at $\varepsilon = 0$ the flow lies on a resonant torus with frequencies $\hat{\omega} = \omega(1, 2, 2)$, with $\omega = 1 + I^*$.



After expanding H_0 and H_1 in Taylor series of the actions around I_l^*, with $l \in \{-1, 0, 1\}$, we introduce the resonant angles $\hat{q} = (q_1, q)$ and their conjugated actions $\hat{p} = (p_1, p)$ as follows

$$\begin{cases} q_1 = \varphi_{-1} \\ q_2 = \varphi_0 - 2\varphi_{-1} \\ q_3 = \varphi_1 - \varphi_0 \end{cases} , \qquad \begin{cases} p_1 = J_{-1} + 2J_0 + 2J_1 \\ p_2 = J_0 + J_1 \\ p_3 = J_1 \end{cases} ,$$

so that we can rewrite the initial Hamiltonian in the form

$$\begin{aligned} H = \omega p_1 + \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \mathbf{i} \xi_j \eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(\hat{q}) + f_1^{(0,1)}(\hat{q}, \xi, \eta) \\ + f_2^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + \sum_{l \geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) \ . \end{aligned}$$

Of course, we could have performed a different canonical linear transformation to resonant angles $q_{2,3}$, since the basis of the resonant modulus $\mathcal{M}_\omega$ is not unique; for example, we could have performed the admissible choice $q_3 = \varphi_3 - 2\varphi_1$. However, with this latter choice, in the term $f_0^{(1,1)}$ we would have had a dependence on a suitable combination of the angles q_2 and q_3, which means that it is possible to choose the coordinates so that to have a dependence on a proper angle. Hence, the choice we have made is more convenient and better reveals the action of the symmetry.

Hence at any order r the normal form terms $f_0^{(r,s)}$ will be independent both of q_1 (because of averaging) and of q_2. These variables $q_{1,2}$ are going to parameterize the persisting subtori. At first order, the averaging of $f_0^{(0,1)}$ gives the normal form

$$f_0^{(1,1)} = -2\varepsilon(1 + 2I^*) \cos(q_3) \ ,$$

whose critical points are only $q_3^* = 0, \pi$. As already stressed, the absence of the resonant angle q_2 has not to be interpreted in this case as the effect of a proper degeneracy, since we expect a finite numbers of two-dimensional subtori to be continued. Thus the two subtori are clearly non-degenerate, satisfying $D^2_{q_3} f_0^{(1,1)}(q^*) \neq 0$.

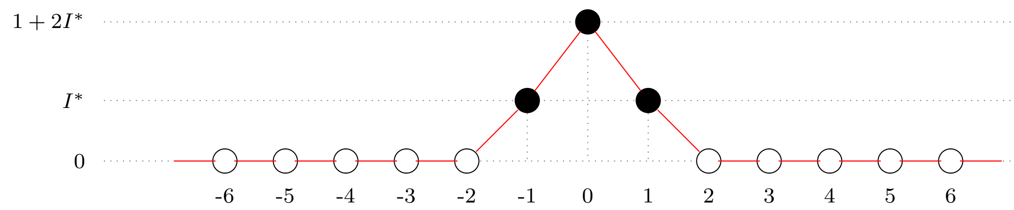
In order to prove the persistence of the obtained subtori, we keep both $q_1(0)$ and $q_2(0)$ as parameters in the map Υ introduced in (2.10), and we also forget the variation of the second action p_2, since in this case we have two independent constant of motion; hence $\Upsilon : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$. Then, coherently with the previous modification, the reduced monodromy matrix $(\exp(dX_{K^{(1)}}(x^*)T) - \mathbb{I})_{\mathsf{red}}$, where now $x^* = (q_3 \in \{0, \pi\}, \hat{p} = 0, \xi = \eta = 0)$, is constructed removing the two columns related to $q_{1,2}$ and the two rows related to $p_{1,2}$. Then, under the same assumptions of Theorem 2.1.1 on the spectrum of $\hat{N}$, we get existence and approximation of the considered subtori. In our case, explicit calculations with Mathematica, provide the typical non-degenerate values of $\alpha = \frac{1}{2} < \frac{r+1}{2} = 1$, which allows to apply the new version of the Theorem.

The first subtorus $q_3^* = \pi$ is linearly unstable, while $q_3^* = 0$ is linearly stable, since its Floquet exponents are

$$\lambda_{1,2}(\varepsilon) = \pm \mathbf{i} \left(2\sqrt{1 + 2I^*}\sqrt{\varepsilon} + \frac{\varepsilon^{3/2}}{\sqrt{1 + 2I^*}} + h.o.t. \right) \ .$$

The Theorem 2.1.2 can be applied with $\beta = 1/2 < r + 1 - \alpha = 1$, also obtaining the effective linear stability.

Degenerate case: as a second case, we still consider the sets $\mathcal{I} = \{-1, 0, 1\}$ and $\mathcal{J} = \{-7, \dots, 7\}$, with the new excited actions $\{I^*, 1 + 2I^*, I^*\}$, so that at $\varepsilon = 0$ the flow lies on a resonant torus with frequencies $\hat{\omega} = \omega(1, 2, 1)$, with again $\omega = 1 + I^*$.



After expanding H_0 and H_1 in Taylor series of the actions around I_l^*, with $l \in \{-1, 0, 1\}$, we introduce the resonant angles $\hat{q} = (q_1, q)$ and their conjugated actions $\hat{p} = (p_1, p)$ as follows

$$\begin{cases} q_1 = \varphi_{-1} \\ q_2 = \varphi_0 - 2\varphi_{-1} \\ q_3 = \varphi_1 - \varphi_{-1} \end{cases}, \qquad \begin{cases} p_1 = J_{-1} + 2J_0 + J_1 \\ p_2 = J_0 \\ p_3 = J_1 \end{cases},$$

getting the starting Hamiltonian

$$\begin{aligned} H = \omega p_1 &+ \sum_{j \in \mathcal{J} \setminus \mathcal{I}} \mathbf{i} \xi_j \eta_j + f_4^{(0,0)}(\hat{p}, \xi, \eta) + f_0^{(0,1)}(\hat{q}) + f_1^{(0,1)}(\hat{q}, \xi, \eta) \\ &+ f_2^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_3^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + f_4^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) + \sum_{l \geq 5} f_l^{(0,1)}(\hat{q}, \hat{p}, \xi, \eta) \ . \end{aligned}$$

Also in this case a different canonical linear transformation to resonant angles $q_{2,3}$ might have been performed, but, as stressed before, this one is more convenient since at any order r the normal form terms $f_0^{(r,s)}$ will be independent both of q_1 (because of averaging) and of q_2. Differently from the previous examples, at first order, the averaging of $f_0^{(0,1)}$ gives a trivial normal form term $f_0^{(1,1)} \equiv 0$, because the two resonant oscillators at sites $\{-1, 1\}$ are not interacting at order $\mathcal{O}(\varepsilon)$; hence a second normal form step is required. At order $r = 2$ the degeneracy is removed, since we get

$$f_0^{(2,2)} = \frac{2(I^*)^2}{(1 + I^*)^2} \varepsilon^2 \cos(q_3) \ ,$$

whose critical points are again only $q_3^* = 0, \pi$. Explicit calculations with Mathematica, provide the expected value of $\alpha = 1 < \frac{r+1}{2} = \frac{3}{2}$, which allows to apply the same strategy used before. The first subtorus $q_3^* = 0$ is linearly unstable, while $q_3^* = \pi$ is linearly stable, since the Floquet exponents are

$$\lambda_{1,2}(\varepsilon) = \pm \mathbf{i} \left(\frac{2I^*}{1 + I^*} \varepsilon + \frac{2I^* \left(5 + 22I^* + 24(I^*)^2\right) \varepsilon^3}{(1 + I^*)^5} + h.o.t. \right) \ .$$

The Theorem 2.1.2 can be applied with $\beta = 1 < r + 1 - \alpha = 2$, also getting the effective linear stability.

3.4 Conclusions

In this Chapter we have applied our abstract results on continuation and stability of periodic orbits in nearly integrable Hamiltonian system, to revisit the existence of time-periodic and spatially localized solutions in dNLS lattices, such as discrete solitons or multi-pulse solutions. It has been shown in several different dNLS models, starting from the standard one, moving to coupled dNLS chains or models with a purely non-linear interaction, that in the limit of small coupling parameter ε this class of solutions are frequently degenerate at leading order, so that a first step of averaging is not enough. The abstract normal form scheme and the main theorems on existence and linear stability, allow to investigate with the help of a symbolic manipulator different kinds of degenerate configurations in dNLS lattices, thus confirming the practical applicability of the abstract algorithm. At the same time, they allow to shed some light on a wider class of localized periodic solutions, leading to the existence of two-dimensional resonant tori, thanks for example to the effect of the rotation symmetry of dNLS models. However, it is necessary to stress that the possibility to apply the present approach to multibreathers in weakly coupled chains of anharmonic oscillators as the Klein-Gordon models collides with the need to explicitly transform the excited oscillators to action-angle variables. This is a problem which might be overcome with special choices of the non-linear potential, like the Morse potential, or with a preliminary dNLS normal form approximation of the non-linear lattice, in the limit of small coupling parameter and energy. As a different direction of future development, the scheme might be extended so to study the existence of degenerate quasi-periodic solutions (degenerate KAM-subtori), both from an abstract point of view and in terms of applications to physical models. Furthermore, this normal form construction could be a preliminary unavoidable step in order to perform further investigations, for instance the study of long-time stability of periodic orbits.

Part II

Variation on the Kolmogorov's theorem

Chapter 4

KAM with knobs

The aim of this Chapter is to reconsider the proof of Kolmogorov's theorem with a variation on the handling of the frequencies. Particular attention is paid to the constructive aspect: we want to produce an algorithm that can be explicitly applied, e.g., with the aid of an algebraic manipulator.

The motivation that gives rise to the development of this approach is the problem of persistence of lower dimensional elliptic invariant tori under sufficiently small perturbations. Indeed, in [38] the authors gave a constructive proof of the existence of lower dimensional elliptic tori for planetary systems, adapting the classical Kolmogorov's normalization algorithm (see also [86]) and a result of Pöschel [79], that allows to estimate the measure of a suitable set of non-resonant frequencies. The key point is that both the *internal* frequencies of the torus and the *transversal* ones vary at each normalization step, and cannot be kept fixed as in Kolmogorov's algorithm. This makes the accumulation of small divisors much more tricky to control and, more important, the result is only valid *in measure* and therefore one cannot know *a priori* if a specific torus exists or not.

A different approach based on Lindstedt's series, that allows to control the frequencies, has been proposed in [18, 19] in the context of FPU problem. However, the algorithm has been so far introduced and used, up to our knowledge, only in a formal way and the literature lacks of rigorous convergence estimates.

The idea is to overcome the issue of having a result that is valid only in measure, playing with the frequency like one do with a control *knob*. The Chapter focuses on full dimensional invariant tori, thus representing a first step in this direction.

Of course, considering full dimensional invariant tori, the original Kolmogorov's normalization algorithm allows to have a complete control of the frequencies, that are fixed along the whole normalization procedure. However, considering lower dimensional elliptic tori, as explained in detail by Pöschel [79], we cannot fix the frequencies and we have to let them vary. Thus, as a first result, we decide to adapt the classical Kolmogorov's normalization algorithm so as to avoid the translation that keep the frequencies fixed and introducing a *detuning*[1] between the fixed *final* frequencies and the corresponding initial ones, to be determined *a posteriori*. This approach, in principle, also allows to start from a resonant torus carrying frequencies $\omega^{(0)}$ that *by construction* falls into a strongly non-resonant one.

In order to better illustrate the point of view of our variation, I briefly recall some classical results on KAM theory. Consider the so-called fundamental problem of dynamics as stated by Poincaré, i.e., a canonical system of differential equations with Hamiltonian

$$H(p, q) = H_0(p) + \varepsilon H_1(p, q; \varepsilon) \ , \tag{4.1}$$

where $(p, q) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables and ε is a small parameter. The functions H_0 and H_1 are assumed to be analytic in all the variables and in the small parameter. Komogorov, in his seminal paper [52] that, together with the works of Arnold [2] and Moser [64], gave birth to

[1]The detuning can be figured as the action of turning a control *knob*.

the KAM theory, proved the existence of quasi-periodic solutions for this Hamiltonian, with given strongly non-resonant frequencies.

The original idea of Kolmogorov is to select the actions p^* such that the frequency vector $\omega = \nabla_p H_0(p^*)$ satisfies a Diophantine condition

$$|\langle k, \omega \rangle| > \gamma |k|^{-\tau} \quad \text{for all } k \in \mathbb{Z}^n \ , \ k \neq 0 \ ,$$

for some positive γ and $\tau \geq n-1$. Hence, the Hamiltonian can be expanded around p^* and, denoting again by p the translated actions $p - p^*$, we rewrite the Hamiltonian as

$$H(p, q) = \langle \omega, p \rangle + \mathcal{O}(p^2) + \varepsilon H_1(p, q; \varepsilon) \ . \tag{4.2}$$

In rough words, Kolmogorov theorem ensures the persistence of the torus $p = 0$ ($p = p^*$ in the original variables) carrying quasi-periodic solutions with frequencies ω, if ε is small enough and $H_0(p)$ is non-degenerate (the so-called Kolmogorov non-degeneracy or twist condition).

Let me stress that the role of the non-degeneracy assumption on $H_0(p)$ is twofold: (i) it allows to select the desired frequencies, parameterized by the actions; (ii) it allows to perform the *translation step* that keeps the frequency fixed along the normalization algorithm.

However, if the Hamiltonian is already in the form (4.2) or satisfies the so-called *twistless* property, i.e., it consists of a sum of a kinetic term, quadratic in p, and of a potential energy, depending only on the angles, it turns out that the non-degeneracy assumption can be removed (see, e.g., [29, 34]).

Nowadays, the literature about KAM theory is so vast that an exhaustive list would fill several pages. Different proofs have been given by many authors. Here, we just mention some contributions in the fields (for full dimensional and lower dimensional tori), adopting different methods, i.e. [24, 26, 27, 29–31, 34–38, 45, 46, 79, 80, 83].

A final remark is about the so called *quadratic* (or superconvergent or Newton-like) method, originally adopted by Kolmogorov and considered crucial until Russmann (see [82,83]) pointed out that a careful analysis of the accumulation of the small divisors allows to sharpen some estimates and get rid of it. Eventually, a proof of Kolmogorov theorem via classical expansions in a small parameter has been obtained by Giorgilli and Locatelli (see [34–36]).

Between the different approaches, a proof of the Kolmogorov's theorem based on an a posteriori format has been given e.g. in [23,24] and applied to dissipative systems for instance in [14]. Instead, for a proof which uses the Lindstedt's series method, see for example [17, 29, 30].

Let me stress that the approach based on classical expansions on some parameter gives absolutely convergent series and allows to unveil the mechanism of the accumulation of the small divisors, leading in a natural way to introduce a more relaxed non-resonant condition for the frequency vector ω, introduced in [40] and adopted in [38, 39]. Precisely

Condition $\boldsymbol{\tau}$: The sequence $\{\alpha_r\}_{r\geq 0}$ satisfies

$$-\sum_{r\geq 1} \frac{\ln \alpha_r}{r(r+1)} = \Gamma < \infty \ , \qquad \text{with} \quad \min_{0<|k|\leq rK} |\langle k, \omega \rangle| \geq \alpha_r \ , \tag{4.3}$$

where K and Γ are two positive constants. Furthermore, the classical approach is the only way to directly implement KAM theory in practical applications and it proved advantageous in different contexts, e.g., the construction of lower-dimensional elliptic tori in [38, 86] or the continuation of periodic orbits as we have seen in previous Chapters. In the present Chapter too, we adopt the classical approach, which also turns out to be better suited in order to devise a normal form algorithm that introduce a detuning of the initial frequencies that will be determined, step by step, along the normalization procedure.

4.1 Main results

Consider the Hamiltonian (4.1) and assume that $H_0(p)$ and $H_1(p, q; \varepsilon)$, for ε small enough, are real analytic bounded functions in the domain $\mathcal{G} \subseteq \mathbb{R}^n \times \mathbb{T}^n$.

Given a point $p_0 \in \mathcal{G}$, denote by $\omega^{(0)}(p_0) \in \mathbb{R}^n$ the corresponding frequency vector and expand the Hamiltonian in a neighbourhood of p_0, denoting again by p the translated actions $p - p_0$, precisely

$$H(p,q) = \langle \omega^{(0)}, p \rangle + \mathcal{O}(p^2) + \varepsilon H_1(p,q;\varepsilon) \ . \tag{4.4}$$

As remarked before, one can assume a non-degeneracy condition on $H_0(p)$ so as to ensure that the frequency vector is parameterized by the actions. However, if the Hamiltonian is already in this form, no non-degeneracy assumption is required.

We can now state our main theorem

Theorem 4.1.1 *Consider the Hamiltonian* (4.4) *with unknown frequency vector* $\omega^{(0)}$. *Pick a strongly non-resonant frequency vector* $\omega \in \mathbb{R}^n$ *satisfying the condition* $\boldsymbol{\tau}$ *in* (4.3). *Then there exists a positive* ε^* *such that for* $|\varepsilon| < \varepsilon^*$ *the following statement holds true: there exists a real analytic near to the identity canonical transformation* $(p,q) = \mathcal{C}^{(\infty)}(p^{(\infty)}, q^{(\infty)})$ *leading the Hamiltonian* (4.4) *in normal form, i.e.,*

$$H^{(\infty)} = \langle \omega, p^{(\infty)} \rangle + \mathcal{O}({p^{(\infty)}}^2) \ . \tag{4.5}$$

The initial frequency vector $\omega^{(0)}$ *is determined a posteriori and the detuning* $\omega - \omega^{(0)}$ *is of order* $\mathcal{O}(\varepsilon)$.

A more quantitative statement, including a detailed definition of the threshold on the smallness of the perturbation, is given in Section 4.4.

At difference with respect to the original Kolmogorov's theorem, we do not keep the frequencies fixed along the normalization procedure. The idea, that will be fully detailed in the next Section, is to replace the classical *translation step* with a change of the frequencies. Thus, once selected the *end* invariant torus, the theorem ensures the existence of the *starting* one that, by construction, falls into the wanted invariant torus.

Finally, the normalization algorithm is based on classical expansions, thus we have a sequence of detuning $\{\omega^{(s)}\}_{s\geq 1}$ between the frequencies ω and $\omega^{(0)}$ satisfying

$$\omega = \omega^{(0)} + \sum_{s\geq 1} \omega^{(s)} \ . \tag{4.6}$$

The sequence $\{\omega^{(s)}\}_{s\geq 1}$ will be determined step by step by the normalization procedure and at any given finite normal form order r we get an approximation of the initial frequency vector given by $\omega^{(0)} = \omega - \sum_{s=1}^{r} \omega^{(s)}$. This can be useful in practical applications, e.g., constructing invariant KAM tori in planetary systems, where only a finite number of explicit normal form steps can be actually performed.

4.2 Analytic setting and expansion of the Hamiltonian

In this section we detail the analytic setting which will be useful in the following.

The Hamiltonian (4.1) is assumed to be real analytic for sufficiently small values of ε and real holomorphic function of the (p,q) variables in the complex domain $\mathcal{D}_{\rho_0,\sigma_0} = \mathcal{G}_{\rho_0} \times \mathbb{T}^n_{\sigma_0}$ where ρ_0 and σ_0 are positive parameters, $\mathcal{G}_{\rho_0} = \bigcup_{p\in\mathcal{G}} \Delta_{\rho_0}(p)$, with

$$\Delta_{\rho_0}(p) = \{z \in \mathbb{C}^n \colon |p_j - z_j| < \rho_0\} \ ,$$

$$\mathbb{T}^n_{\sigma_0} = \{q \in \mathbb{C}^n \colon |\mathrm{Im}(q_j)| < \sigma_0\}$$

are the usual complex extensions of the real domains.

We now define the norms that we are going to use. For real vectors $x \in \mathbb{R}^n$ we use

$$|x| = \sum_{j=1}^{n} |x_j| \ .$$

while for an analytic function $f(p,q)$ with $q \in \mathbb{T}^n$ we use the weighted Fourier norm

$$\|f\|_{\rho,\sigma} = \sum_{k\in\mathbb{Z}^n} |f_k|_\rho e^{|k|\sigma} \ ,$$

with

$$|f_k|_\rho = \sup_p |f_k(p)| \ .$$

Hereafter, we are going to use the shorthand notation $\|\cdot\|_\alpha$ for $\|\cdot\|_{\alpha(\rho,\sigma)}$ and $|\cdot|_\alpha$ for $|\cdot|_{\alpha\rho}$, α being any positive real number.

We here describe how to expand the Hamiltonian (4.4) in power series of a small parameter. Indeed, we split the Hamiltonian in a sum of trigonometric polynomials and we introduce an artificial parameter μ which will represent the expansion parameter, instead of ε. The idea (see e.g. [35]) is to exploit the exponential decay of the Fourier coefficients for analytic functions: pick an arbitrary positive integer K and split the Fourier expansion of $f_\ell(p,q,\varepsilon)$ as

$$f_\ell^{(1)} = \sum_{0\le|k|\le K} c_{\ell,k}(p,\varepsilon)\exp(\mathbf{i}\langle k,q\rangle) \ ,$$

$$f_\ell^{(s)} = \sum_{(s-1)K<|k|\le sK} c_{\ell,k}(p,\varepsilon)\exp(\mathbf{i}\langle k,q\rangle) \ , \quad \text{for } s>1 \ .$$

This splitting allows to simplify the control of the small divisors that show up in the normalization procedure. Let me also remark that, in contrast to the first part of the thesis, the subscript ℓ directly stands for the degree in the actions. Thus we can expand the Hamiltonian (4.4) as

$$H(p,q) = \langle\omega^{(0)},p\rangle + \sum_{\ell\ge2} f_\ell + \sum_{s\ge1}\sum_{\ell\ge0} f_\ell^{(s)} \tag{4.7}$$

where the terms are bounded as

$$\|f_\ell\|_1 \le \frac{E}{2^\ell} \ , \qquad \|f_\ell^{(s)}\|_1 \le \frac{\varepsilon_0 E}{2^\ell}\mu^s \ ,$$

with

$$\rho = \frac{\rho_0}{4} \ , \quad E = 2^{n-1}E_0 \ , \quad \mu = e^{-\frac{K\sigma_0}{8}} \ , \quad \varepsilon_0 = \varepsilon e^{\frac{K\sigma_0}{8}}\left(\frac{1+e^{-\frac{\sigma_0}{8}}}{1-e^{-\frac{\sigma_0}{8}}}\right)^n \frac{F_0}{E_0} \ , \quad \sigma = \frac{\sigma_0}{4} \ .$$

and

$$\sup_p |H_0(p)| \le E_0 \ , \qquad \sup_{p,q,\varepsilon} |H_1(p,q;\varepsilon)| \le F_0 \ . \tag{4.8}$$

The definitions of ρ and E result from the Cauchy's estimate of the Taylor expansion, while for μ, ε_0 and σ the exponential decay of the coefficients in Fourier expansion has been used. These latter estimates can be found in Lemma 8 in [42].

4.3 Normal form algorithm

In this section we present the algorithm leading the Hamiltonian (4.7) in normal form. The procedure is here described from a purely formal point of view, while the study of the convergence is postponed to the next sections.

First, let us exploit the detuning (4.6), that we report here for convenience

$$\omega = \omega^{(0)} + \sum_{s\ge1}\omega^{(s)} \ ,$$

and rewrite the Hamiltonian in the form

$$H^{(0)}(p,q) = \langle \omega, p\rangle + \sum_{\ell\geq 2} f_\ell^{(0,0)}(p,q) + \sum_{s\geq 1}\sum_{\ell\geq 0} f_\ell^{(0,s)}(p,q) - \sum_{s\geq 1}\langle \omega^{(0,s)}, p\rangle \ , \tag{4.9}$$

where we have modified the notation, so as to add a superscript 0 to the Hamiltonian, which will keep track of the normalization order. We stress again that the quantities $\omega^{(0,s)}$, with $s \geq 1$, are unknowns that will be determined along the normalization procedure.

As in the original Kolmogorov's proof scheme, starting from $H^{(0)}$, we construct an infinite sequence of Hamiltonians $\{H^{(r)}\}_{r\geq 0}$, where $H^{(r)}$ is in normal form up to order r, namely

$$H^{(r)}(p,q) = \langle \omega, p\rangle + \sum_{s>r}\left(f_0^{(r,s)}(q) + f_1^{(r,s)}(p,q) - \langle \omega^{(r,s)}, p\rangle\right) + \sum_{s\geq 0}\sum_{\ell\geq 2} f_\ell^{(r,s)} \ . \tag{4.10}$$

Assume that $r-1$ steps have been performed, so that the Hamiltonian (4.10) has the wanted form with $r-1$ in place of r. The transformation that brings the Hamiltonian in normal form up to order r is computed by the composition of two Lie series,

$$\exp(L_{\chi_1^{(r)}}) \circ \exp(L_{\chi_0^{(r)}}) \ ,$$

with generating functions $\chi_0^{(r)}$ and $\chi_1^{(r)}$ that are determined in order to kill the unwanted terms $f_0^{(r-1,r)}(q)$ and $f_1^{(r-1,r)}(p,q) - \langle \omega^{(r-1,r)}, p\rangle$, hence the Hamiltonian is in Kolmogorov's normal form up to order r. At difference with respect to the original approach designed by Kolmogorov we do not introduce a translation of the actions p, since we do not keep the *initial* frequency $\omega^{(0,0)}$ fixed. Indeed, in our algorithm the role of the translation is played by the detuning of the frequency of order r, i.e., $\omega^{(r-1,r)}$.

4.3.1 Generic r-th normalization step

First stage of the normalization step

Our aim is to remove the term $f_0^{(r-1,r)}(q)$, determining the generating function $\chi_0^{(r)}(q)$ which is the solution of the homological equation

$$L_{\chi_0^{(r)}}\langle \omega, p\rangle + f_0^{(r-1,r)} = 0 \ . \tag{4.11}$$

Considering the Fourier expansion

$$f_0^{(r-1,r)}(q) = \sum_{0<|k|\leq rK} c_{0,k}^{(r-1,r)} \exp(\mathbf{i}\langle k, q\rangle) \ ,$$

one can easily check that the solution of (4.11) is given by

$$\chi_0^{(r)}(q) = \sum_{0<|k|\leq rK} \frac{c_{0,k}^{(r-1,r)}}{\mathbf{i}\langle k, \omega\rangle} \exp(\mathbf{i}\langle k, q\rangle) \ .$$

Let me remark that the homological equation (4.11) can be solved only provided the function $f_0^{(r-1,r)}$ has null average with respect to the angles q. In this case, the average is a constant that can be neglected. The intermediate Hamiltonian $H^{(\mathrm{I};r-1)} = \exp(L_{\chi_0^{(r)}})H^{(r-1)}$ reads

$$\begin{aligned} H^{(\mathrm{I};r-1)}(p,q) =& \langle \omega, p\rangle \\ &+ f_1^{(\mathrm{I};r-1,r)}(p,q) - \langle \omega^{(r-1,r)}, p\rangle \\ &+ \sum_{s>r}\left(f_0^{(\mathrm{I};r-1,s)}(q) + f_1^{(\mathrm{I};r-1,s)}(p,q) - \langle \omega^{(r-1,s)}, p\rangle\right) + \sum_{s\geq 0}\sum_{\ell\geq 2} f_\ell^{(\mathrm{I};r-1,s)} \ , \end{aligned}$$

with

$$
f_0^{(\mathrm{I};r-1,s)} = \begin{cases} 0 \ , & s \le r \ ; \\ f_0^{(r-1,s)} \ , & r < s < 2r \ ; \\ f_0^{(r-1,s)} + L_{\chi_0^{(r)}} \big(f_1^{(r-1,s-r)} - \langle \omega^{(r-1,s-r)}, p \rangle \big) + \sum_{j=2}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_0^{(r)}} f_j^{(r-1,s-jr)} \ , & s \ge 2r \ . \end{cases}
$$

$$
f_1^{(\mathrm{I};r-1,s)} = \begin{cases} 0 \ , & s < r \ ; \\ \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_0^{(r)}} f_{1+j}^{(r-1,s-jr)} \ , & s \ge r \ . \end{cases}
$$

$$
f_\ell^{(\mathrm{I};r-1,s)} = \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_0^{(r)}} f_{\ell+j}^{(r-1,s-jr)} \ , \qquad \ell \ge 2 \ . \tag{4.12}
$$

Second stage of the normalization step

We now remove the term $f_1^{(\mathrm{I};r-1,r)}(p,q) - \langle \omega^{(r-1,r)}, p \rangle$, by determining the generating function $\chi_1^{(r)}(p,q)$ which solves the homological equation

$$
L_{\chi_1^{(r)}} \langle \omega, p \rangle + f_1^{(\mathrm{I};r-1,r)} - \langle \omega^{(r-1,r)}, p \rangle = 0 \ , \tag{4.13}
$$

with

$$
f_1^{(\mathrm{I};r-1,r)} = f_1^{(r-1,r)} + L_{\chi_0^{(r)}} f_2^{(0,0)} \ .
$$

Considering the Taylor-Fourier expansion

$$
f_1^{(\mathrm{I};r-1,r)}(p,q) = \sum_{|k| \le rK} c_{1,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k, q \rangle) \ ,
$$

one can easily check that the solution of (4.13) is given by

$$
\chi_1^{(r)}(p,q) = \sum_{0<|k|\le rK} \frac{c_{1,k}^{(\mathrm{I};r-1,r)}}{\mathbf{i}\langle k, \omega \rangle} \exp(\mathbf{i}\langle k, q \rangle) \quad \text{and} \quad \langle \omega^{(r-1,r)}, p \rangle = \langle f_1^{(r-1,r)} \rangle_q \ ,
$$

where $\langle \cdot \rangle_q$ denotes the average with respect to the angles q. Let me stress that, in order to solve the homological equation, we should have imposed $\langle \omega^{(r-1,r)}, p \rangle = \langle f_1^{(\mathrm{I};r-1,r)} \rangle_q$, but the term $L_{\chi_0^{(r)}} f_2^{(0,0)}$ has null average.

We complete the normalization step by computing the Hamiltonian $H^{(r)} = \exp(L_{\chi_1^{(r)}}) H^{(\mathrm{I};r-1)}$

that takes the form (4.10) and setting $k = \lfloor s/r \rfloor$, $m = s \mod r$, $s = kr + m$ we have

$$f_0^{(r,s)} = \sum_{j=0}^{k-1} \frac{1}{j!} L^j_{\chi_1^{(r)}} f_0^{(\mathrm{I};r-1,s-jr)} \ ,$$

$$f_1^{(r,s)} = \begin{cases} 0 \ , & s \le r \ ; \\ f_1^{(\mathrm{I};r-1,s)} + \displaystyle\sum_{j=1}^{k-1} \frac{1}{j!} L^j_{\chi_1^{(r)}} \Big(f_1^{(\mathrm{I};r-1,s-jr)} - \langle \omega^{(r-1,s-jr)}, p \rangle \Big) \ , & s > r \ , \ m \neq 0 \ ; \\ f_1^{(\mathrm{I};r-1,s)} + \displaystyle\sum_{j=1}^{k-2} \frac{1}{j!} L^j_{\chi_1^{(r)}} \Big(f_1^{(\mathrm{I};r-1,s-jr)} - \langle \omega^{(r-1,s-jr)}, p \rangle \Big) & \\ \quad + \dfrac{k-1}{k!} L^{k-1}_{\chi_1^{(r)}} \Big(f_1^{(\mathrm{I};r-1,r)} - \langle \omega^{(r-1,r)}, p \rangle \Big) \ , & s > r \ , \ m = 0 \ ; \end{cases} \tag{4.14}$$

$$f_\ell^{(r,s)} = \sum_{j=0}^{k} \frac{1}{j!} L^j_{\chi_1^{(r)}} f_\ell^{(\mathrm{I};r-1,s-jr)} \ , \qquad \ell \ge 2 \ ,$$

$$\langle \omega^{(r,s)}, p \rangle = \langle \omega^{(r-1,s)}, p \rangle \ .$$

The justification of the formulas (4.12) and (4.14) is just a matter of straightforward computations, exploiting (4.11) and (4.13) in order to kill the unwanted terms and to simplify the expression of $f_1^{(r,kr)}$.

Let us remark that in the Hamiltonian in normal form up to order r, the quantities $\{\omega^{(r,s)}\}_{s=1}^r$ have been explicitly determined (and will not be modified by the normalization algorithm), while $\{\omega^{(r,s)}\}_{s>r}$ are still unknowns to be determined. Finally the redefinition of the sequence of detunings at each normalization step, by updating the corresponding label, is a mere technical detail that turns out to be useful when dealing with the quantitative estimates.

4.4 Analytic estimates

In this section, we translate our formal algorithm into a recursive scheme of estimates on the norms of the functions. This essentially requires to bound the norm of the Lie series. The useful estimates are collected in the following statements.

Lemma 4.4.1 *Let f be analytic in $\mathcal{D}_{\rho,\sigma}$ with finite norm $\|f\|_1$ Then for $0 < d < 1$ and for $1 \le j \le n$ we have*

$$\left\| \frac{\partial f}{\partial p_j} \right\|_{(1-d)} \le \frac{\|f\|_1}{d\rho} \ , \quad \left\| \frac{\partial f}{\partial q_j} \right\|_{(1-d)} \le \frac{\|f\|_1}{ed\sigma} \ ;$$

Lemma 4.4.2 *Let d and d' be real numbers such that $d > 0$, $d' \ge 0$ and $d + d' < 1$; let χ and f be two analytic functions on $\mathcal{D}_{(1-d')(\rho,\sigma)}$ having finite norms $\|\chi\|_{1-d'}$ and $\|f\|_{1-d'}$, respectively. Then, for $j \ge 1$, we have*

$$\left\| L^j_\chi f \right\|_{1-d-d'} \le \frac{j!}{e^2} \left(\frac{2e\|\chi\|_{1-d'}}{d^2\rho\sigma} \right)^j \|f\|_{1-d'} \ . \tag{4.15}$$

The proof of these Lemmas are just a minor variation of the proofs of Lemmas A.2.3 and A.2.4.

The estimates for the Lie derivatives require to restrict the analytic domain. Hence, at each normalization step r, we need to introduce a restriction of the domain, d_r. Of course, the series taking into account all these restrictions must converge; thus, the divisors d_r^2 appearing in (4.15) must shrink to zero as $r \to \infty$. To this end, we define the sequences $\{d_r\}_{r\geq 0}$ and $\{\delta_r\}_{r\geq 1}$ as

$$d_0 = 0 \ , \quad d_r = d_{r-1} + 2\delta_r \ , \quad \delta_r = \frac{3}{4\pi^2} \cdot \frac{1}{r^2} \ . \tag{4.16}$$

At the r-th step of the normalization algorithm, we make a restriction δ_r of the domain for both the canonical transformations, thus the Hamiltonian $H^{(r)}$ is analytic in $\mathcal{D}_{(1-d_r)}$. Our definition ensures that the sequences of domains converge to a compact set whose interior is not empty, since $\lim_{r\to\infty} d_r = 1/4$.

We also introduce the real and non increasing sequence $\{\alpha_r\}_{r\geq 0}$ defined[2] as

$$\alpha_0 = 1 \ , \quad \alpha_r = \min\Big(1, \min_{0<|k|\leq rK} \big|\langle k, \omega\rangle\big|\Big) \ . \tag{4.17}$$

This means that α_r represents the smallest divisor that may occur in the solutions of the homological equations at step r.

As we will see in the estimates of the functions, the small divisors α_r and δ_r always appear as products $\beta_r = \alpha_r \delta_r^2$, thus we can control all the divisors with the same method. For convenience, we also define $\beta_0 = 1$.

By using (4.3), one can check that the generating functions, $\chi_0^{(r)}$ and $\chi_1^{(r)}$, and the detuning $\omega^{(r-1,r)}$, are bounded as

$$\|\chi_0^{(r)}\|_{1-d_{r-1}} \leq \frac{\|f_0^{(r-1,r)}\|_{1-d_{r-1}}}{\alpha_r} \ , \tag{4.18}$$

$$\|\langle\omega^{(r-1,r)}, p\rangle\|_{1-d_{r-1}} \leq \|\langle f_1^{(r-1,r)}\rangle_q\|_{1-d_{r-1}} \ , \tag{4.19}$$

$$\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \leq \frac{\|f_1^{(\mathrm{I};r-1,r)}\|_{1-d_{r-1}-\delta_r}}{\alpha_r} \ . \tag{4.20}$$

We are now ready to estimate the terms appearing in the intermediate Hamiltonian and in the Hamiltonian in normal form up to order r, namely (4.12) and (4.14), respectively.

Setting

$$G_{r,0} = \frac{2e}{\rho\sigma}\|f_0^{(r-1,r)}\|_{1-d_{r-1}}$$

one has

$$\begin{aligned}
&\|f_0^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} \leq \begin{cases} \|f_0^{(r-1,s)}\|_{1-d_{r-1}} \ , & r < s < 2r \ ; \\[2ex] \|f_0^{(r-1,s)}\|_{1-d_{r-1}} + \dfrac{G_{r,0}}{\delta_r^2\alpha_r}\|f_1^{(r-1,s-r)} - \langle\omega^{(r-1,s-r)}, p\rangle\|_{1-d_{r-1}} & \\ \qquad + \displaystyle\sum_{j=2}^{\lfloor s/r\rfloor}\left(\frac{G_{r,0}}{\delta_r^2\alpha_r}\right)^j \|f_j^{(r-1,s-jr)}\|_{1-d_{r-1}} \ , & s \geq 2r \ . \end{cases} \\[2ex]
&\|f_1^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} \leq \sum_{j=0}^{\lfloor s/r\rfloor}\left(\frac{G_{r,0}}{\delta_r^2\alpha_r}\right)^j \|f_{1+j}^{(r-1,s-jr)}\|_{1-d_{r-1}} \ , \qquad s \geq r \ . \\[2ex]
&\|f_\ell^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} \leq \sum_{j=0}^{\lfloor s/r\rfloor}\left(\frac{G_{r,0}}{\delta_r^2\alpha_r}\right)^j \|f_{\ell+j}^{(r-1,s-jr)}\|_{1-d_{r-1}} \ , \qquad \ell \geq 2 \ .
\end{aligned} \tag{4.21}$$

Setting

$$G_{r,1} = \frac{2e}{\rho\sigma}\|f_1^{(\mathrm{I};r-1,r)}\|_{1-d_{r-1}-\delta_r}$$

[2]For consistency reasons we slightly modify the definition in (4.3).

and recalling that $k = \lfloor s/r \rfloor$, $m = s \mod r$, $s = kr + m$, $\ell \geq 2$, one has

$$\begin{aligned}
&\|f_0^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k-1} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|f_0^{(\mathrm{I};r-1,s-jr)}\|_{1-d_{r-1}-\delta_r} \ , \\
&\|f_1^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k-1} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|f_1^{(\mathrm{I};r-1,s-jr)} - \langle \omega^{(r-1,s-jr)}, p\rangle\|_{1-d_{r-1}-\delta_r} \ , \\
&\|f_\ell^{(r,s)}\|_{1-d_r} \leq \sum_{j=0}^{k} \left(\frac{G_{r,1}}{\delta_r^2 \alpha_r}\right)^j \|f_\ell^{(\mathrm{I};r-1,s-jr)}\|_{1-d_{r-1}-\delta_r} \ , \\
&\|\langle \omega^{(r,s)}, p\rangle\| \leq \|\langle \omega^{(r-1,s)}, p\rangle\| \ .
\end{aligned} \tag{4.22}$$

As already remarked, a crucial difference with respect to the classical Kolmogorov's normal form algorithm is that the corrections $\{\omega^{(r,s)}\}_{s>r}$ are still unknowns to be determined. Therefore, strictly speaking, we cannot bound any term containing these quantities. However, this is not a true issue since, once we are able to estimate a term, it remains unchanged by the subsequent normalization steps. Moreover, the terms $\langle \omega^{(r,s)}, p\rangle$ are always paired with the terms $f_1^{(r,s)}$ and by construction they simply cancel out the zero average part, thus they do not play any role in the normal form estimates.

To be more formal, we will assume that the norms of the detunings $\{\omega^{(r,s)}\}_{s>r}$ decay according to a prescribed geometrical law given *a priori*, and then we prove by induction that the assumption is satisfied.

Let us remark that all the estimates exhibit a common structure: they are sums of different contributions obtained by multiplying a factor $G_{r,0}/(\delta_r^2 \alpha_r)$ or $G_{r,1}/(\delta_r^2 \alpha_r)$ at some power by the known norms of some functions. Thus, we are naturally led to consider the quantities

$$\beta_0 = 1 \ , \quad \beta_r = \delta_r^2 \alpha_r \ .$$

These are the *small divisors* that we are going to carefully analyze in the next subsection. Indeed, it is well known that the accumulation of the small divisors can prevent the convergence of any normalization procedure.

4.4.1 Accumulation of small divisors

The accumulation of small divisors can be analyzed by just focusing on indices of the succession $\{\beta_r\}_{r\geq 0}$. Indeed, the mechanism of accumulation is rather matter of indices: the key point is not the actual values of the divisors, but which divisors can appear.

We call *list of indices* a collection $\{j_1, \ldots, j_s\}$ of non negative integers, with length $s \geq 0$. The empty list $\{\}$ of length 0 is allowed, as well as repeated indices. The index 0 is allowed, too, and will be used in order to pad a short list to the wanted length, when needed. As we said before, the lists of indices provide a full characterization of the products of small divisors: to the list $\{j_1, \ldots, j_s\}$ we associate the product of divisors $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$. Adding any number of zeros to a list of indices is harmless, for we have set $\beta_0 = 1$.

We say that *a function f owns a list of indices $I = \{j_1, \ldots, j_k\}$ if its estimate presents a divisor $\beta_{j_1} \cdots \beta_{j_k}$.*

In order to identify the worst possible product of divisors in every function, we need to look for:

(i) the number of divisors β_j;

(ii) a selection rule which identifies which lists can really arise.

Hence, let $I = \{j_1, \ldots, j_s\}$ and $I' = \{j'_1, \ldots, j'_s\}$ be two sets of indices with the same number s of elements. Let us introduce the following relation of partial ordering on those sets: we say that $I \lhd I'$ in case there is a permutation of the indices such that the relation $j_m \leq j'_m$ holds true for $m = 1, \ldots, s$. If two sets of indices contain a different number of elements, first, we pad the shorter one with zeros and, then, we use the same method to compare them. With the symbol $\cup$, we mean concatenation of lists.

I_1^*	{}
I_2^*	{1}
I_3^*	{1, 1}
I_4^*	{1, 1, 2}
I_5^*	{1, 1, 1, 2}
I_6^*	{1, 1, 1, 2, 3}
I_7^*	{1, 1, 1, 1, 2, 3}
I_8^*	{1, 1, 1, 1, 2, 2, 4}
I_9^*	{1, 1, 1, 1, 1, 2, 3, 4}
I_{10}^*	{1, 1, 1, 1, 1, 2, 2, 3, 5}

Table 4.1: The special lists I_s^* for $1 \leq s \leq 10$.

So, we can state the following definition

Definition 4.4.1 *For all integers $r \geq 0$ and $s > 0$, let us introduce the family of indices sets*

$$\mathcal{J}_{r,s} = \left\{ I = \{j_1, \ldots, j_{s-1}\} \,:\, 0 \leq j_m \leq \min\{r, \lfloor s/2 \rfloor\} \,,\, I \lhd I_s^* \right\} \,, \tag{4.23}$$

where the special lists of indices I_s^* are defined as

$$I_s^* = \left\{ \left\lfloor \frac{s}{s} \right\rfloor, \left\lfloor \frac{s}{s-1} \right\rfloor, \ldots, \left\lfloor \frac{s}{2} \right\rfloor \right\} \;.$$

The condition $I \lhd I_s^*$ represents the *selection rule* **S** which enables to select the divisors that can appear along the normalization procedure. In table 4.1 we give examples of the special lists just defined.

We are now ready to claim technical Lemmas which will be useful in the following.

Lemma 4.4.3 *For the sets of indices $I_s^* = \{j_1, \ldots, j_s\}$ the following statements hold true:*
(i) the maximal index is $j_{\max} = \left\lfloor \frac{s}{2} \right\rfloor$;
(ii) for every $k \in \{1, \ldots, j_{\max}\}$ the index k appears exactly $\left\lfloor \frac{s}{k} \right\rfloor - \left\lfloor \frac{s}{k+1} \right\rfloor$ times;
(iii) for $0 < r \leq s$ one has

$$\left(\{r\} \cup I_r^* \cup I_s^*\right) \lhd I_{r+s}^* \;.$$

For the proof of this Lemma see [40] and [38].

We now come to exploit the relation between lists of indices and products of small divisors. Consider the sequence $\{\alpha_k\}_{k \geq 0}$. To a list I we associate the quantity $Q(I) = \prod_{j \in I} \frac{1}{\alpha_j}$, getting also the following property

$$\text{if} \quad I \lhd I' \quad \text{then} \quad Q(I) > Q(I') \;.$$

Let us consider the special sequence

$$Q_s^* = \prod_{j \in I_s^*} \frac{1}{\alpha_j} \;.$$

We look for a sufficient condition assuring that the sequence has a finite limit. In view of Lemma 4.4.3, we evaluate

$$\ln Q_s^* = \ln \prod_{j \in I_s^*} \frac{1}{\alpha_j} \leq -\sum_{k=1}^{s} \left(\left\lfloor \frac{s}{k} \right\rfloor - \left\lfloor \frac{s}{k+1} \right\rfloor \right) \ln \alpha_k \leq -s \sum_{k \geq 1} \frac{\ln \alpha_k}{k(k+1)} \;.$$

Hence, this justifies the introduction of the condition $\boldsymbol{\tau}$ in (4.3) for the sequence $\{\alpha_k\}_{k\geq 0}$, and allows to get $Q_s^* < e^{s\Gamma}$, which means that it grows not faster than geometrically.

Remark 4.4.1 *The condition $\boldsymbol{\tau}$ is weaker than the diophantine one and equivalent to the Bruno's condition*

$$-\sum_{r\geq 1} \frac{\ln \alpha_{2^{r-1}}}{2^r} = \text{Б} < \infty$$

introduced by Alexander Bruno. Indeed, it can be proved that $\Gamma < \text{Б} < 2\Gamma$. Moreover, Bruno's condition is the optimal one for the problem of Schröder–Siegel, as proved by J.C. Yoccoz, while for the problem of Kolmogorov the question is still open.

We now return to the sets $\mathcal{J}_{r,s}$ and the small divisors β_r.

Lemma 4.4.4 *For the sets of indices $\mathcal{J}_{r,s}$ the following statements hold true:*

(i) $\mathcal{J}_{r,s} = \mathcal{J}_{\{\min r, \lfloor s/2 \rfloor\},s}$;

(ii) $\mathcal{J}_{r-1,s} \subseteq \mathcal{J}_{r,s}$;

(iii) if $I \in \mathcal{J}_{r-1,r}$ *and* $I' \in \mathcal{J}_{r,s}$, *then* $\big(\{\min\{r,s\}\} \cup I \cup I'\big) \in \mathcal{J}_{r,s+r}$;

We now associate to the collections of lists $\mathcal{J}_{r,s}$ introduced in (4.23) the sequence of positive numbers

$$T_{0,s} = T_{s,0} = 1 \qquad \text{for} \quad s \geq 0 \ , \qquad T_{r,s} = \max_{I\in\mathcal{J}_{r,s}} \prod_{j\in I, j\geq 1} \frac{1}{\beta_j} \ , \qquad \text{for} \quad r,s \geq 1 \ . \tag{4.24}$$

Lemma 4.4.5 *The sequence $T_{r,s}$ satisfies the following properties for all $r,s \geq 1$*

(i) $T_{r-1,s} \leq T_{r,s}$ *and* $T_{r',s} = T_{s,s}$ *for* $r' > s$;

(ii) $\dfrac{1}{\beta_m} T_{r-1,r} T_{r,s} \leq T_{r,r+s}$ *where* $m = \min\{r,s\}$.

The proofs of the above two Lemmas can be found in [38].

In the following table we summarize the relevant information concerning the set of indices appearing in the denominators, namely the accumulation of the small divisors.

Function	conditions	set of indices	bounded by
$f_0^{(r,s)}$	$0 \leq r < s$	$(\mathcal{J}_{r,s})^2$	$T_{r,s}^2$
$f_1^{(r,s)}$	$0 \leq r < s$	$\{r\} \cup (\mathcal{J}_{r,s})^2$	$\frac{1}{\beta_r} T_{r,s}^2$
$f_{\ell\geq 2}^{(r,s)}$	$r \geq 0$, $s \geq 1$	$(\{\min\{r,s\}\} \cup \mathcal{J}_{r,s})^2$	$\frac{1}{\beta_{\min\{r,s\}}^2} T_{r,s}^2$
$\chi_0^{(r)}$	$r \geq 1$	$\{r\} \cup (\mathcal{J}_{r-1,r})^2$	$\frac{1}{\beta_r} T_{r-1,r}^2$
$\chi_1^{(r)}$	$r \geq 1$	$(\{r\} \cup \mathcal{J}_{r-1,r})^2$	$\frac{1}{\beta_r^2} T_{r-1,r}^2$

Table 4.2: The number of indices and the selection rules for the functions $f_\ell^{(r,s)}$.

With a little abuse of notation, in the table above we have introduced a sort of power of a set so that, for instance, $(\mathcal{J}_{r,s})^2 = \mathcal{J}_{r,s} \cup \mathcal{J}_{r,s}$. The selection rules for the indices appearing at the denominators are really the keystone of the whole proof. The proof of these selection rules is just an application of Lemma 4.4.4 and is reported in Section A.5.

It basically requires to unfold all the recursive inequalities (4.18)–(4.22).

Now, we can state a Lemma which provides a geometrical bounds for the accumulation of small divisors.

Lemma 4.4.6 *Let the sequence* $\{\alpha_r\}_{r\geq 1}$*, introduced by* (4.17)*, satisfy condition* $\boldsymbol{\tau}$ *and the sequence* $\{\delta_r\}_{r\geq 1}$ *be defined as in* (4.16)*. Then, the sequence* $\{T_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ *defined by* (4.24) *is bounded by*

$$T_{r,s} \leq \frac{1}{\alpha_s \delta_s^2} T_{r,s} \leq \left(2^{13} e^{\Gamma}\right)^s \qquad \textit{for } r \geq 1\,,\ s \geq 1\ .$$

The proof of this Lemma is deferred to the Appendix, Section A.5.

4.4.2 Recurrent estimates

In the previous subsection, we provided the tools needed to control the accumulation of the small divisors. We now collect the estimates of all the different contributions and prove the convergence of the normal form algorithm.

First, it is convenient to introduce the constant

$$M = \max\left\{1, \frac{4eE}{\rho\sigma}\right\}\ , \tag{4.25}$$

that allows to get rid of many contributions in an uniform way. Furthermore, in order to translate the normalization scheme into recursive estimates, we need to control also the number of sums involved in the recursive formulas (4.21) and (4.22). Hence, we introduce the sequences $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$, $\{\nu^{(\mathrm{I})}_{r,s}\}_{r\geq 1\,,\,s\geq 0}$:

$$\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \geq 0\,,\\
\nu^{(\mathrm{I})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} \nu^{j}_{r-1,r}\nu_{r-1,s-jr} && \text{for } r\geq 1\,,\ s\geq 0\,,\\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{I})}_{r,r})^j \nu^{(\mathrm{I})}_{r,s-jr} && \text{for } r\geq 1\,,\ s\geq 0\,.
\end{aligned} \tag{4.26}$$

Here too, we can provide a geometric bound for this sequences, by means of the following Lemma

Lemma 4.4.7 *The sequence of positive integers* $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ *defined in* (4.26) *is bounded by the exponential growth*

$$\nu_{r,s} \leq \nu_{s,s} \leq \frac{2^{6s}}{2^4} \qquad \textit{for} \quad r\geq 0\,,\ s\geq 0\ .$$

The proof is deferred to the Appendix, Section A.5, and is only a simplified version of Lemma A.2.2.

In order to complete the proof, we need an estimate also for the transformed Hamiltonian and the generating functions.

Lemma 4.4.8 *Consider the Hamiltonian* $H^{(0)}$ *expanded as in* (4.9) *with*

$$\|f_\ell^{(0,0)}\|_1 \leq \frac{E}{2^\ell} \qquad \textit{and} \qquad \|f_\ell^{(0,s)}\|_1 \leq \frac{\varepsilon_0 E}{2^\ell}\mu^s \quad \textit{for } s>0\ .$$

Assume that on $H^{(0)}$ we have performed $r \geq 1$ normalization steps of the formal algorithm described in Section 4.3. Then the following estimates hold true

$$
\begin{aligned}
\frac{1}{\delta_r^2}\|\chi_0^{(r)}\|_{1-d_{r-1}} &\leq \varepsilon_0 E \mu^r M^{2r-2} \frac{T_{r-1,r}^2}{\beta_r} \nu_{r-1,r} \ , \\
\frac{1}{\delta_r^2}\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} &\leq \frac{\varepsilon_0 E}{2} \mu^r M^{2r-1} \frac{T_{r-1,r}^2}{\beta_r^2} \nu_{r,r}^{(\mathrm{I})} \ . \\
\|\langle \omega^{(r-1,r)}, p\rangle\|_{1-d_{r-1}} &\leq \frac{\varepsilon_0 E}{2} \mu^r M^{2r-1} \frac{T_{r-1,r}^2}{\beta_{r-1}} \nu_{r-1,r} \ .
\end{aligned}
\tag{4.27}
$$

Moreover, the terms appearing in the expansion of the transformed Hamiltonian $H^{(r)}$ in (4.10) *are bounded by*

$$
\begin{aligned}
\|f_\ell^{(r,s)}\|_{1-d_r} &\leq \frac{\varepsilon_0 E \mu^s M^{2s-2+\ell}}{2^\ell} \frac{T_{r,s}^2}{\beta_r^\ell} \nu_{r,s} & &\text{for} \quad 0 \leq \ell \leq 1 \ , \ s > r \ , \\
\|f_\ell^{(r,s)}\|_{1-d_r} &\leq \frac{\varepsilon_0 E \mu^s M^{2s}}{2^\ell} \frac{T_{r,s}^2}{\beta_{\min\{r,s\}}^2} \nu_{r,s} \ , & &\text{for} \quad \ell \geq 2 \ , \ s > 1 \ ,
\end{aligned}
$$

The proof of this Lemma is just a simplified version of Lemma 1.6.4: instead of the quantities Ξ_r we deal with the constant M and the sequence $T_{r,s}$, the latter being bounded as in Table 4.2. In Section A.5 we report the proof for the first normalization step and the sketch for the r-th step.

To summarize, we have bounded geometrically the accumulation of small divisors, the number of sums involved in the recursive formulas and the generating functions. This allows to use the general result on convergence of Lie series and to obtain an infinite sequence of canonical transformations which produces an Hamiltonian in normal form of Kolmogorov.

We can now provide a more quantitative statement of Theorem 4.1.1.

Proposition 4.4.1 *Consider the Hamiltonian $H^{(0)}$ expanded as in* (4.9) *with*

$$
\|f_\ell^{(0,0)}\|_1 \leq \frac{E}{2^\ell} \quad \text{and} \quad \|f_\ell^{(0,s)}\|_1 \leq \frac{\varepsilon_0 E \mu^s}{2^\ell} \quad \text{for } s > 0 \ .
$$

Assume that the frequency vector ω satisfy the τ-condition (4.3). *Then if*

$$
\mu \leq \bar{\mu} = \frac{1}{2^6 (2^{13} M e^{\Gamma})^2} \quad \text{and} \quad \varepsilon < \mu \left(\frac{1 - e^{-\frac{\sigma_0}{8}}}{1 + e^{-\frac{\sigma_0}{8}}} \right)^n \frac{E_0}{F_0} \ ,
$$

where M, Γ, σ_0, E_0 and F_0 are defined in (4.25), (4.3) *and* (4.8), *respectively. Then there exists an analytic near to the identity canonical transformation $\mathcal{C}^{(\infty)} : \mathcal{D}_{1/2} \to \mathcal{D}_{3/4}$ such that the transformed Hamiltonian $H^{(\infty)} = H^{(0)} \circ \mathcal{C}^{(\infty)}$ is in normal form, i.e.,*

$$
H^{(\infty)}(q,p) = \langle \omega, p \rangle + \sum_{s \geq 0} \sum_{\ell \geq 2} f_\ell^{(\infty,s)}(q,p) \ .
$$

The functions $f_\ell^{(\infty,s)}(q,p)$ are bounded by

$$
\|f_\ell^{(\infty,0)}\|_1 \leq \frac{E}{2^\ell} \quad \text{and} \quad \|f_\ell^{(\infty,s)}\|_1 \leq \frac{\varepsilon_0 E \bar{\mu}^s}{2^\ell} \quad \text{for } s > 0 \ .
$$

and the detuning is bounded as $|\omega - \omega^{(0)}| < \mathcal{O}(\varepsilon)$.

Proof. Having bounded geometrically the accumulation of small divisors, the number of sums involved in the recursive formulas and hence the generating functions, it is now straightforward to prove this Proposition. Thus we only sketch here the key points.

We denote by $(p^{(0)}, q^{(0)})$ the original coordinates and by $\hat{\mathcal{C}}^{(r)}$ the canonical transformation which maps $(p^{(r)}, q^{(r)})$ to $(p^{(r-1)}, q^{(r-1)})$. The canonical transformation can be written as

$$\begin{aligned} p^{(r-1)} &= \exp(L_{\chi_0^{(r)}})p^{(\mathrm{I},r-1)} = p^{(\mathrm{I},r-1)} - \frac{\partial \chi_0^{(r)}}{\partial q^{(\mathrm{I},r-1)}} \ , \\ p^{(\mathrm{I},r-1)} &= \exp(L_{\chi_1^{(r)}})p^{(r)} = p^{(r)} - \sum_{s\geq 1} \frac{1}{s!} L^{s-1}_{\chi_1^{(r)}} \frac{\partial \chi_1^{(r)}}{\partial q^{(r)}} \ , \\ q^{(r-1)} &= \exp(L_{\chi_1^{(r)}})q^{(r)} = q^{(r)} + \sum_{s\geq 1} \frac{1}{s!} L^{s-1}_{\chi_1^{(r)}} \frac{\partial \chi_1^{(r)}}{\partial p^{(r)}} \ . \end{aligned}$$

Hence, considering a domain $\mathcal{D}_{(3d-d_r)(\rho,\sigma)}$ and using Lemma 4.4.8, we get

$$\begin{aligned} \left|p^{(r-1)} - p^{(\mathrm{I},r-1)}\right| &< \frac{\|\chi_0^{(r)}\|_{1-d_{r-1}}}{e\delta_r\sigma} \leq M^{2r}\left(2^{13}e^{\Gamma}\right)^{2r} 2^{6r}\varepsilon_0 E\mu^r \ , \\ \left|p^{(\mathrm{I},r-1)} - p^{(r)}\right| &< \frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{e\delta_r\sigma} \sum_{s\geq 1} \frac{1}{e^2} \left(\frac{2e\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r^2\rho\sigma}\right)^{s-1} \leq \\ &\leq \delta_r\rho M^{2r}\left(2^{13}e^{\Gamma}\right)^{2r} 2^{6r}\frac{\varepsilon_0 E}{2}\mu^r \sum_{s\geq 1}\left(M^{2r}\left(2^{13}e^{\Gamma}\right)^{2r} 2^{6r}\frac{\varepsilon_0 E}{2}\mu^r\right)^{s-1} \ , \qquad (4.28) \\ \left|q^{(r-1)} - q^{(r)}\right| &< \frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r\rho} \sum_{s\geq 1} \frac{1}{e^2} \left(\frac{2e\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r^2\rho\sigma}\right)^{s-1} \leq \\ &\leq \delta_r\sigma M^{2r}\left(2^{13}e^{\Gamma}\right)^{2r} 2^{6r}\frac{\varepsilon_0 E}{2}\mu^r \sum_{s\geq 1}\left(M^{2r}\left(2^{13}e^{\Gamma}\right)^{2r} 2^{6r}\frac{\varepsilon_0 E}{2}\mu^r\right)^{s-1} \ . \end{aligned}$$

If $\varepsilon_0 < 1$ and $\mu < \overline{\mu}$ with

$$\overline{\mu} = \frac{1}{2^6(2^{13}e^{\Gamma}M)^2} \ ,$$

then the series in the estimates (4.28) converges. So, we have an absolutely convergent series which defines the canonical transformation which turns out to be analytic. Let me stress that $\varepsilon_0 < 1$ implies that the smaller μ is, the smaller ε is. Thus, we have a threshold for the parameters and

$$\left|p^{(r-1)} - p^{(r)}\right| < \varepsilon_0\delta_r\rho \ , \qquad \left|q^{(r-1)} - q^{(r)}\right| < \varepsilon_0\delta_r\sigma \ .$$

A similar argument applies to the inverse transformation, thus we obtain

$$\mathcal{D}_{(3d-d_r)(\rho,\sigma)} \subset \hat{\mathcal{C}}^{(r)}(\mathcal{D}_{(3d-d_{r-1}-\delta_r)(\rho,\sigma)}) \subset \mathcal{D}_{(3d-d_{r-1})(\rho,\sigma)} \ .$$

Consider now the sequence of transformations $\mathcal{C}^{(r)} = \hat{\mathcal{C}}^{(1)} \circ \ldots \circ \hat{\mathcal{C}}^{(r)}$.

For $(p^{(r-1)}, q^{(r-1)}) \in \mathcal{D}_{(3d-d_{r-1})(\rho,\sigma)}$ the transformation is clearly analytic. Setting $d = \frac{1}{4}$ and using (4.16), one has $\sum_{j\geq 1}\delta_j \leq \frac{d}{2} = \frac{1}{8}$, and $\mathcal{C}^{(r)}$ converges to an analytic canonical transformation $\mathcal{C}^{(\infty)}$ which satisfies

$$\mathcal{D}_{\frac{1}{4}(\rho,\sigma)} \subset \mathcal{C}^{(\infty)}\left(\mathcal{D}_{\frac{1}{2}(\rho,\sigma)}\right) \subset \mathcal{D}_{\frac{3}{4}(\rho,\sigma)} \ .$$

Due to the condition $\mu < \overline{\mu}$, it can be also proved that the sequence of functions $H^{(r)}$ defines an analytic Hamiltonian $H^{(\infty)} = H \circ \mathcal{C}^{(\infty)}$ in normal form of Kolmogorov, as in (4.5).

From the third of (4.27), we get also that $|\omega - \omega^{(0)}| = \mathcal{O}(\varepsilon)$, so this concludes the proof of the Proposition.

□

4.5 Conclusions

In this Chapter we have presented a variation of the Kolmogorov's normal form scheme, in order to provide an algorithm which enables to completely control the frequencies, avoiding the so-called translation step. Although in the case of full dimensional tori the Kolmogorov's scheme allows to get a complete handling of the frequencies, our approach can be useful to face the problem in the generic case of lower dimensional tori, where the frequencies cannot be fixed.

Hence, in the light of the results just discussed, the natural future development will be the construction of a normal form procedure for lower dimensional tori, with possible applications to planetary systems. In particular, this extension will allow to select a specific frequency of a torus and to start also from a resonant torus, showing its existence and effectively construct it.

Appendix A

A.1 Newton-Kantorovich method

Proposition A.1.1 (Newton-Kantorovich method) *Consider* $\Upsilon \in \mathcal{C}^1\left(\mathcal{U}(x^*) \times \mathcal{U}(0), V\right)$. *Assume that there exist three constants* $c_{1,2,3} > 0$ *dependent, for* ε *small enough, on* $\mathcal{U}(x^*) \subset V$ *only, and two parameters* $0 \leq 2\alpha < \beta$ *such that*

$$\|\Upsilon(x^*, \varepsilon)\| \leq c_1 |\varepsilon|^\beta \ , \tag{A.1}$$

$$\|[\Upsilon'(x^*, \varepsilon)]^{-1}\|_{\mathcal{L}(V)} \leq c_2 |\varepsilon|^{-\alpha} \ , \tag{A.2}$$

$$\|\Upsilon'(z, \varepsilon) - \Upsilon'(x^*, \varepsilon)\|_{\mathcal{L}(V)} \leq c_3 \|z - x^*\| \ . \tag{A.3}$$

Then there exist positive c_0 *and* ε^* *such that, for* $|\varepsilon| < \varepsilon^*$, *there exists a unique* $x_0(\varepsilon) \in \mathcal{U}(x^*)$ *which fulfills*

$$\Upsilon(x_0, \varepsilon) = 0 \ , \qquad\qquad \|x_0 - x^*\| \leq c_0 |\varepsilon|^{\beta - \alpha} \ .$$

Furthermore, Newton's algorithm converges to x_0.

We recall that $\|\cdot\|_{\mathcal{L}(V)}$ represents the usual norm for a linear operator from V to$V$.

Proof. The result is a direct consequence of the Contraction Principle applied to a suitable closed ball centered in x^*. Indeed, by following a standard procedure (see, i.e., [53]), let us formulate the original problem as a fixed point problem, namely $\Upsilon(x, \varepsilon) = 0$ if and only if $A(x, \varepsilon) = x$, where

$$A(x, \varepsilon) = x - [\Upsilon'(x^*, \varepsilon)]^{-1} \Upsilon(x, \varepsilon) \ .$$

We first of all show that A is a contraction of a sufficiently small ball centered in x^*. We first rewrite our assumptions in a more general form

$$\|\Upsilon(x^*, \varepsilon)\| \leq \mu \ , \qquad \|[\Upsilon'(x^*, \varepsilon)]^{-1}\|_{\mathcal{L}(V)} \leq M \ ,$$

and we introduce the auxiliary quantities

$$\eta = M\mu = c_1 c_2 |\varepsilon|^{\beta - \alpha} \ , \qquad\qquad h = M c_3 \eta = c_1 c_2^2 c_3 |\varepsilon|^{\beta - 2\alpha} \ .$$

Notice that the condition $\beta > 2\alpha$ is necessary in order to have

$$\lim_{\varepsilon \to 0} h = 0 \ .$$

The main ingredient is the continuity of Υ', since $\Upsilon \in \mathcal{C}^1$ locally around x^* (independently from ε). From finite increment formula we get, for $x, y \in B(x^*, r) \subset \mathcal{U}(x^*)$

$$\|A(x, \varepsilon) - A(y, \varepsilon)\| \leq \left(\sup_{z \in B(x^*, r)} \|A'(z, \varepsilon)\|_{\mathcal{L}(V)} \right) \|x - y\| \ ;$$

thus, we aim at showing that, with a suitable choice of the radius r, we have

$$\sup_{z \in B(x^*, r)} \|A'(z, \varepsilon)\|_{\mathcal{L}(V)} < 1 \ .$$

Since

$$A'(z,\varepsilon) = \mathbb{I} - [\Upsilon'(x^*,\varepsilon)]^{-1}\Upsilon'(z,\varepsilon) = [\Upsilon'(x^*,\varepsilon)]^{-1}[\Upsilon'(x^*,\varepsilon) - \Upsilon'(z,\varepsilon)]$$

we get

$$\begin{aligned}\|A'(z,\varepsilon)\|_{\mathcal{L}(V)} &\leq \left\|[\Upsilon'(x^*,\varepsilon)]^{-1}\right\|_{\mathcal{L}(V)} \|\Upsilon'(x^*,\varepsilon) - \Upsilon'(z,\varepsilon)\|_{\mathcal{L}(V)} \leq \\ &\leq M \|\Upsilon'(x^*,\varepsilon) - \Upsilon'(z,\varepsilon)\|_{\mathcal{L}(V)} \ .\end{aligned}$$

From the continuity of Υ' it follows that, provided $\|z - x^*\|$ is small enough, it is possible to make $\Upsilon'(x^*,\varepsilon) - \Upsilon'(z,\varepsilon)$ arbitrary small. The Lipschitz-continuity estimate[1] in the hypotheses of the Proposition allows to explicitly deal with this issue. Indeed, from

$$\|\Upsilon'(x^*,\varepsilon) - \Upsilon'(z,\varepsilon)\|_{\mathcal{L}(V)} \leq c_3 \|z - x^*\| \ ,$$

we get

$$\|A'(z,\varepsilon)\|_{\mathcal{L}(V)} \leq Mc_3 \|z - x^*\| \leq Mc_3 r =: q \ , \qquad \forall z \in B(x^*,r) \ ,$$

and also

$$\sup_{z\in B(x^*,r)} \|A'(z,\varepsilon)\|_{\mathcal{L}(V)} \leq q \ .$$

In order to show that $\Upsilon(B(x^*,r)) \subset B(x^*,r)$, namely that $\|z - x^*\| \leq r$ implies $\|A(z,\varepsilon) - x^*\| \leq r$, we start splitting

$$\|A(z,\varepsilon) - x^*\| \leq \|A(z,\varepsilon) - A(x^*,\varepsilon)\| + \|A(x^*,\varepsilon) - x^*\| \ .$$

We will separately estimate the two r.h.t.. From the bound on $A'(z,\varepsilon)$ we get

$$\|A(z,\varepsilon) - A(x^*,\varepsilon)\| \leq \sup_{z\in B(x^*,r)} \|A'(z,\varepsilon)\|_{\mathcal{L}(V)} \|z - x^*\| \leq qr \ .$$

on the other hand, by exploiting the initial definition of $A(x,\varepsilon)$, one has

$$\begin{aligned}\|A(x^*,\varepsilon) - x^*\| &= \left\|x^* - [\Upsilon'(x^*,\varepsilon)]^{-1}\Upsilon(x^*,\varepsilon) - x^*\right\| = \left\|[\Upsilon'(x^*,\varepsilon)]^{-1}\Upsilon(x^*,\varepsilon)\right\| \leq \\ &\leq \left\|[\Upsilon'(x^*,\varepsilon)]^{-1}\right\|_{\mathcal{L}(V)} \|\Upsilon(x^*,\varepsilon)\| \leq M\mu \ .\end{aligned}$$

Hence, in order to have $\Upsilon(B(x^*,r)) \subset B(x^*,r)$, it must happen

$$M\mu + qr \leq r \ .$$

Thus, two independent conditions have to be satisfied:

$$Mc_3 r < 1 \ , \qquad \eta + Mc_3 r^2 \leq r \ .$$

The second is equivalent to

$$Mc_3 r^2 - r + \eta \leq 0 \ ,$$

which can be re-scaled to

$$r = \eta\rho \ , \qquad h\rho^2 - \rho + 1 \leq 0 \ .$$

The corresponding equation, under the condition $h < \frac{1}{4}$, has the two zeros

$$t_\pm = \frac{1}{2h}\left(1 \pm \sqrt{1 - 4h}\right) \ .$$

Moreover one has $t_- < 2$, since $1 - 4h < \sqrt{1 - 4h}$, and for $h \sim 0$ we get $t_-(h) \sim 1$. Collecting the above information, the radius r has to fulfill

$$\eta t_- \leq r \leq t_+ \eta \ .$$

[1]Actually Holder-continuity would be sufficient, modifying the conditions on α and β.

If we make the more restrictive choice

$$\eta t_- \le r \le 2\eta \ ,$$

then, from $h < \frac{1}{4}$, it follows that Υ is an $\frac{1}{2}$-contraction map

$$Mc_3 r < 2Mc_3\eta = 2h < \frac{1}{2} \ .$$

In our case, $h < \frac{1}{4}$ comes directly from being $h(\varepsilon)$ infinitesimal w.r.t. ε; thus for ε small enough the condition is satisfied. Moreover, from $h(\varepsilon) \approx 1$, one deduces that the optimal choice for the radius is

$$r(\varepsilon) = \eta t_- \approx c_1 c_2 |\varepsilon|^{\beta-\alpha} \ .$$

□

A.2 Analytic estimates: lower dimensional tori

In this Section we report the analytic estimates for the generic r-th normalization step with three stages, which are enough in order to obtain continuation of periodic orbits. This simplified version of the normalization algorithm, with the third stage which only consists of an average of the linear term in the actions $f_2^{(\mathrm{II};r-1,r)}(\hat{q},\hat{p})$, enables us to show all the key aspects of the estimates that can be extended to the case of five stages with further calculations, and include obviously the maximal dimension case.

A.2.1 Generic r-th normalization step with three stages

We here summarize the three stages of a generic r-th normalizing step needed in order to obtain continuation of periodic orbits. The starting Hamiltonian has the form

$$\begin{aligned} H^{(r-1)} = \omega p_1 &+ \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j \xi_j \eta_j \\ &+ \sum_{s<r} f_0^{(r-1,s)} + \sum_{s<r} f_2^{(r-1,s)} \\ &+ f_0^{(r-1,r)} + f_1^{(r-1,r)} + f_2^{(r-1,r)} \\ &+ \sum_{s>r} f_0^{(r-1,s)} + \sum_{s>r} f_1^{(r-1,s)} + \sum_{s>r} f_2^{(r-1,s)} \\ &+ \sum_{s\ge 0}\sum_{l>2} f_l^{(r-1,s)} \ . \end{aligned} \tag{A.4}$$

where $f_0^{(r-1,s)}$, $f_2^{(r-1,s)}$ for $1 \le s < r$, are in normal form.

First stage of the r-th normalization step

We average the term $f_0^{(r-1,r)}$ with respect to the fast angle q_1, determining the generating function

$$\chi_0^{(r)}(\hat{q}) = X_0^{(r)}(\hat{q}) + \langle \zeta^{(r)}, \hat{q}\rangle \qquad \text{with} \qquad \zeta^{(r)} \in \mathbb{R}^{n_1} \ ,$$

by solving the homological equations

$$\begin{aligned} &L_{X_0^{(r)}}\omega p_1 + f_0^{(r-1,r)} = \langle f_0^{(r-1,r)}\rangle_{q_1} \ , \\ &L_{\langle\zeta^{(r)},\hat{q}\rangle} f_4^{(0,0)}\Big|_{\xi=\eta=0} + \Big\langle f_2^{(r-1,r)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\Big\rangle_{q_1} = 0 \ . \end{aligned}$$

By considering the Taylor-Fourier expansion

$$f_0^{(r-1,r)}(\hat{q}) = \sum_{k} c_{0,0,0,k}^{(r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ ,$$

we obtain

$$X_0^{(r)}(\hat{q}) = \sum_{k_1\neq 0} \frac{c_{0,0,0,k}^{(r-1,r)}}{\mathbf{i}k_1\omega} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \ .$$

The vector $\zeta^{(r)}$ is determined by solving the linear system

$$\sum_{j} C_{ij}\zeta_j^{(r)} = \frac{\partial}{\partial \hat{p}_i}\Big\langle f_2^{(r-1,r)}\Big|_{\substack{\xi=\eta=0\\ q=q^*}}\Big\rangle_{q_1} \ . \tag{A.5}$$

The transformed Hamiltonian is computed as

$$\begin{aligned}
H^{(\mathrm{I};r-1)} &= \exp\Big(L_{\chi_0^{(r)}}\Big)H^{(r-1)} = \\
&= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\
&\quad + \sum_{s<r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{I};r-1,s)} \\
&\quad + f_0^{(\mathrm{I};r-1,r)} + f_1^{(\mathrm{I};r-1,r)} + f_2^{(\mathrm{I};r-1,r)} + f_3^{(\mathrm{I};r-1,r)} + f_4^{(\mathrm{I};r-1,r)} \\
&\quad + \sum_{s>r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{I};r-1,s)} \\
&\quad + \sum_{s\geq 0}\sum_{l>2} f_l^{(\mathrm{I};r-1,s)} \ .
\end{aligned}$$

The functions $f_\ell^{(\mathrm{I};r-1,s)}$ are recursively defined as

$$\begin{aligned}
f_0^{(\mathrm{I};r-1,r)} &= \langle f_0^{(r-1,r)}\rangle_{q_1} \ , \\
f_\ell^{(\mathrm{I};r-1,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L^j_{\chi_0^{(r)}} f_{\ell+2j}^{(r-1,s-jr)} \ , \qquad \text{for } \ell=0,\ s\neq r \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell\neq 0 \ s\geq 0 \ ,
\end{aligned} \tag{A.6}$$

with $f_\ell^{(\mathrm{I};r-1,s)} \in \mathcal{P}_\ell$.

Second stage of the r-th normalization step

We remove the term $f_1^{(\mathrm{I};r-1,r)}$ by solving the homological equation

$$L_{\chi_1^{(r)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_1^{(\mathrm{I};r-1,r)} = 0 \ . \tag{A.7}$$

Considering again the Taylor-Fourier expansion

$$f_1^{(\mathrm{I};r-1,r)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\xi^{m_1}\eta^{m_2} \ ,$$

we get

$$\chi_1^{(r)}(\hat{q},\xi,\eta) = \sum_{\substack{|m_1|+|m_2|=1\\ k}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\, \Omega\rangle\big]} \ .$$

with $\Omega \in \mathbb{R}^{n_2}$.

The transformed Hamiltonian is calculated as

$$
\begin{aligned}
H^{(\mathrm{II};r-1)} &= \exp\left(L_{\chi_1^{(r)}}\right) H^{(\mathrm{I};r-1)} = \\
&= \omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j \\
&\quad + \sum_{s<r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + f_0^{(\mathrm{II};r-1,r)} + f_2^{(\mathrm{II};r-1,r)} + f_3^{(\mathrm{II};r-1,r)} + f_4^{(\mathrm{II};r-1,r)} \\
&\quad + \sum_{s>r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + \sum_{s\geq 0}\sum_{l>2} f_l^{(\mathrm{II};r-1,s)} \ ,
\end{aligned}
$$

with

$$
\begin{aligned}
f_1^{(\mathrm{II};r-1,r)} &= 0 \ , \\
f_0^{(\mathrm{II};r-1,2r)} &= f_0^{(\mathrm{I};r-1,2r)} + L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)} + \frac{1}{2} L_{\chi_1^{(r)}}\left(L_{\chi_1^{(r)}} f_2^{(\mathrm{I};r-1,0)}\right) = \\
&= f_0^{(\mathrm{I};r-1,2r)} + \frac{1}{2} L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)} \ , \\
f_\ell^{(\mathrm{II};r-1,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L^j_{\chi_1^{(r)}} f_{\ell+j}^{(\mathrm{I};r-1,s-jr)}, \qquad \begin{array}{l} \text{for } \ell=0,\ s\neq 2r \ , \\ \text{or } \ell=1\ s\neq r \ \ , \\ \text{or } \ell\geq 2\ s\geq 0 \ \ , \end{array}
\end{aligned}
\tag{A.8}
$$

where we have exploited (A.7).

Third stage of the r-th normalization step

We average the term $f_2^{(\mathrm{II};r-1,r)}\Big|_{\xi=\eta=0}$ with respect to the fast angle q_1, by solving the homological equation

$$
L_{\chi_2^{(r)}}\Big(\omega p_1 + \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \mathbf{i}\Omega_j\xi_j\eta_j\Big) + f_2^{(\mathrm{II};r-1,r)}\Big|_{\xi=\eta=0} = \langle f_2^{(\mathrm{II};r-1,r)}\Big|_{\xi=\eta=0}\rangle_{q_1} \ . \tag{A.9}
$$

Therefore, considering the Taylor-Fourier expansion

$$
f_2^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q},0,0) = \sum_{\substack{|l|=1\\ k}} c_{l,0,0,k}^{(\mathrm{II};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle) \ ,
$$

we obtain

$$
\chi_2^{(r)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=1\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{II};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega} \ .
$$

The transformed Hamiltonian is computed as

$$
H^{(r)} = \exp\left(L_{\chi_2^{(r)}}\right) H^{(\mathrm{II};r-1)}
$$

and is in the form (A.4), replacing the upper index $r-1$ by r, with $f_\ell^{(r,s)} \in \mathcal{P}_\ell$ given by

$$
\begin{aligned}
f_2^{(r,r)} &= \langle f_2^{(\mathrm{II};r-1,r)} \rangle_{q_1} \ , \\
f_2^{(r,ri)} &= \frac{1}{(i-1)!} L^{i-1}_{\chi_2^{(r)}} \left(f_2^{(\mathrm{II};r-1,r)} + \frac{1}{i} L_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,0)} \right) \\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,ri-rj)} = \\
&= \frac{1}{(i-1)!} L^{i-1}_{\chi_2^{(r)}} \left(\frac{1}{i} \langle f_2^{(\mathrm{II};r-1,r)} \rangle_{q_1} + \frac{i-1}{i} f_2^{(\mathrm{II};r-1,r)} \right) \\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,ri-rj)} \ , \\
f_\ell^{(r,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L^j_{\chi_2^{(r)}} f_\ell^{(\mathrm{II};r-1,s-jr)} \ , \qquad \begin{array}{l} \text{for } \ell = 2,\ s \neq ri \ , \\ \text{or } \ell \neq 2,\ s \geq 0 \ , \end{array}
\end{aligned}
\tag{A.10}
$$

where we have used the homological equation (A.9).

A.2.2 Recursive scheme of estimates

In order to estimate the norms of the functions involved in the normalization algorithm, we need to introduce a sequence of restrictions of the domain so as to apply Cauchy's estimate. Having fixed $d \in \mathbb{R}$, $0 < d \leq 1/4$, we consider a sequence $\delta_{r\geq 1}$ of positive real numbers satisfying

$$
\delta_{r+1} \leq \delta_r \ , \quad \sum_{r\geq 1} \delta_r \leq \frac{d}{3} \ , \tag{A.11}
$$

thus the sequence δ_r has to satisfy the inequality $\delta_r < C/r$ for some $r > \overline{r}$ and $C \in \mathbb{R}$. Moreover, we introduce a further sequence $d_{r\geq 0}$ of real numbers recursively defined as

$$
d_0 = 0 \ , \quad d_r = d_{r-1} + 3\delta_r \ .
$$

We also need to introduce the quantities Ξ_r, parametrized by the index r, as

$$
\Xi_r = \max\left(\frac{eE}{\alpha\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\alpha\delta_r^2\rho\sigma}, \frac{E}{\alpha\delta_r^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right) \ , \tag{A.12}
$$

with

$$
\alpha = \min_{k_1,j} \{ |k_1\omega \pm \Omega_j|, |\omega| \} > 0 \ ,
$$

in view of the first Melnikov condition.

The number of terms in formulæ (A.6), (A.8), and (A.10) is controlled by the three sequences $\{\nu_{r,s}\}_{r\geq 0\,,\, s\geq 0}$, $\{\nu^{(\mathrm{I})}_{r,s}\}_{r\geq 1\,,\, s\geq 0}$ and $\{\nu^{(\mathrm{II})}_{r,s}\}_{r\geq 1\,,\, s\geq 0}$:

$$
\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \geq 0\,, \\
\nu^{(\mathrm{I})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} \nu^j_{r-1,r}\nu_{r-1,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu^{(\mathrm{II})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{I})}_{r,r})^j \nu^{(\mathrm{I})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (2\nu^{(\mathrm{II})}_{r,r})^j \nu^{(\mathrm{II})}_{r,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,.
\end{aligned}
\tag{A.13}
$$

Let us stress that when $s < r$, the above simplify as

$$\nu^{(\mathrm{I})}_{r,s} = \nu_{r-1,s} \ , \qquad \nu^{(\mathrm{II})}_{r,s} = \nu^{(\mathrm{I})}_{r,s} \ , \qquad \nu_{r,s} = \nu^{(\mathrm{II})}_{r,s} \ ,$$

namely

$$\nu_{r,s} = \nu_{r-1,s} = \ldots = \nu_{s,s} \ .$$

Let us introduce the quantities $b(\mathrm{I};r,s,\ell)$, $b(\mathrm{II};r,s,\ell)$, $b(r,s,\ell)$ in order to control the exponents of Ξ_r in the normalization procedure:

$$b(r,s,\ell) = \begin{cases} 0 & \text{if } r=0 \\ 0 & \text{if } r>0, \ell\neq 1, s=0 \\ 0 & \text{if } r>0, \ell=1, s\leq r \\ 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor - w_\ell & \text{if } r>0, \ell\neq 1, s>0 \\ & \quad \text{or if } r>0, \ell=1, s>r \end{cases}$$

$$b(\mathrm{I};r,s,\ell) = \begin{cases} 0 & \text{if } r=0, \ell=0, s\leq 1 \\ s & \text{if } r=0, \ell=0, s\neq 1 \\ 0 & \text{if } r=0, \ell>0, s=0 \\ s & \text{if } r=0, \ell>0, s>0 \\ 0 & \text{if } r>0, \ell=0, s=0 \\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor - w_\ell & \text{if } r>0, \ell=0, s>0 \\ 0 & \text{if } r>0, \ell=1, s<r \\ 5s-2-2\left\lfloor\frac{s-2}{r}\right\rfloor - w_\ell & \text{if } r>0, \ell=1, s\geq r \\ 0 & \text{if } r>0, \ell=2, s=0 \\ 2 & \text{if } r>0, \ell=2, s=1 \\ 5s-2-\left\lfloor\frac{s-1}{r}\right\rfloor - \left\lfloor\frac{s-2}{r}\right\rfloor - w_\ell & \text{if } r>0, \ell=2, s>1 \\ 0 & \text{if } r>0, \ell>2, s=0 \\ 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor - w_\ell & \text{if } r>0, \ell>2, s>0 \end{cases}$$

$$b(\mathrm{II};r,s,\ell)=\begin{cases}0 & \text{if } r=0,\ell=0,s\le 1\\ 3 & \text{if } r=0,\ell=0,s=2\\ 2s & \text{if } r=0,\ell=0,s>2\\ 0 & \text{if } r=0,\ell=1,s\le 1\\ 2s & \text{if } r=0,\ell=1,s>1\\ 2s & \text{if } r=0,\ell\ge 2\\ 0 & \text{if } r=1,\ell=0,s=0\\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor-w_\ell & \text{if } r=1,\ell=0,s>0\\ 0 & \text{if } r=1,\ell=1,s\le 2\\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor-w_\ell & \text{if } r=1,\ell=1,s>2\\ 0 & \text{if } r=1,\ell=2,s<1\\ 5s-2-\left\lfloor\frac{s-1}{r}\right\rfloor-\left\lfloor\frac{s-2}{r}\right\rfloor+\left\lfloor\frac{s}{r+1}\right\rfloor-w_\ell & \text{if } r=1,\ell=2,s\ge 1\\ 0 & \text{if } r=1,\ell>2,s=0\\ 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor+\left\lfloor\frac{s}{r+1}\right\rfloor-w_\ell & \text{if } r=1,\ell>2,s>0\\ 0 & \text{if } r>1,\ell=0,s=0\\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor-w_\ell & \text{if } r>1,\ell=0,s>0\\ 0 & \text{if } r>1,\ell=1,s\le r+1\\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor-w_\ell & \text{if } r>1,\ell=1,s>r+1\\ 0 & \text{if } r>1,\ell=2,s=0\\ 5s-2-2\left\lfloor\frac{s-1}{r+1}\right\rfloor-w_\ell & \text{if } r>1,\ell=2,s>0\\ 0 & \text{if } r>1,\ell>2,s=0\\ 5s-2-w_\ell & \text{if } r>1,\ell>2,s=1\\ 5s-2-\left\lfloor\frac{s-1}{r}\right\rfloor-\left\lfloor\frac{s-2}{r}\right\rfloor-w_\ell & \text{if } r>1,\ell>2,s>1\end{cases}$$

with $w_\ell=\max(0,3-\ell)$.

We are now ready to state the main Lemma collecting the estimates for the generic r-th normalization step of the normal form algorithm.

Lemma A.2.1 *Consider a Hamiltonian $H^{(r-1)}$ expanded as in (A.4). Let $\chi_0^{(r)}=X_0^{(r)}+\langle\zeta^{(r)},\varphi\rangle$, $\chi_1^{(r)}$ and $\chi_2^{(r)}$ be the generating functions used to put the Hamiltonian in normal form at order r, then one has*

$$\begin{aligned}\|X_0^{(r)}\|_{1-d_{r-1}} &\le \frac{1}{\alpha}\nu_{r-1,r}\Xi_r^{5r-7}E\varepsilon^r\ ,\\ |\zeta^{(r)}| &\le \frac{1}{4m\rho}\nu_{r-1,r}\Xi_r^{5r-5}E\varepsilon^r\ ,\\ \|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} &\le \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{I})}\Xi_r^{5r-4}\frac{E}{2}\varepsilon^r\\ \|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} &\le \frac{1}{\alpha}2\nu_{r,r}^{(\mathrm{II})}\Xi_r^{5r-3}\frac{E}{4}\varepsilon^r\ .\end{aligned}$$

The terms appearing in the expansion of $H^{(\mathrm{I};r-1)}$ in (A.6) *are bounded as*

$$\|f_\ell^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r}\le\nu_{r,s}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r-1,s,\ell)}\frac{E}{2^\ell}\varepsilon^s\ .$$

The terms appearing in the expansion of $H^{(\mathrm{II};r-1)}$ in (A.8) *are bounded as*

$$\|f_\ell^{(\mathrm{II};r-1,s)}\|_{1-d_{r-1}-2\delta_r}\le\nu_{r,s}^{(\mathrm{II})}\Xi_r^{b(\mathrm{II};r-1,s,\ell)}\frac{E}{2^\ell}\varepsilon^s\ .$$

The terms appearing in the expansion of $H^{(r)}$ in (A.10) *are bounded as*

$$\|f_\ell^{(r,s)}\|_{1-d_r}\le\nu_{r,s}\Xi_r^{b(r,s,\ell)}\frac{E}{2^\ell}\varepsilon^s\ .$$

A.2.3 Estimates for the $\nu_{r,s}$ sequence

Lemma A.2.2 *The sequence of positive integers* $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ *defined in* (A.13) *is bounded by the exponential growth*

$$\nu_{r,s}\leq\nu_{s,s}\leq\frac{2^{10s}}{2^6}\qquad\textit{for}\quad r\geq 0\,,\ s\geq 0\ .$$

Proof. First of all, we rewrite the definition of $\nu_{r,s}$, by eliminating the two sequence $\{\nu_{r,s}^{(\mathrm{I})}\}_{r\geq 1\,,\,s\geq 0}$ and $\{\nu_{r,s}^{(\mathrm{II})}\}_{r\geq 1\,,\,s\geq 0}$. In particular, let us remove the symbol $\nu^{(\mathrm{II})}$, by writing

$$\begin{aligned}\nu_{r,s}&=\sum_{j=0}^{\lfloor s/r\rfloor}2^j\big(\nu_{r,r}^{(\mathrm{I})}+\nu_{r,r}^{(\mathrm{I})}\nu_{r,0}^{(\mathrm{I})}\big)^j\sum_{i=0}^{\lfloor s/r\rfloor-j}(\nu_{r,r}^{(\mathrm{I})})^i\nu_{r,s-(i+j)r}^{(\mathrm{I})}=\sum_{j=0}^{\lfloor s/r\rfloor}(4\nu_{r,r}^{(\mathrm{I})})^j\sum_{i=j}^{\lfloor s/r\rfloor}(\nu_{r,r}^{(\mathrm{I})})^{i-j}\nu_{r,s-ir}^{(\mathrm{I})}\\&=\sum_{i=0}^{\lfloor s/r\rfloor}(\nu_{r,r}^{(\mathrm{I})})^i\nu_{r,s-ir}^{(\mathrm{I})}\sum_{j=0}^{i}4^j=\frac{1}{3}\sum_{i=0}^{\lfloor s/r\rfloor}\big(4^{i+1}-1\big)\,(\nu_{r,r}^{(\mathrm{I})})^i\nu_{r,s-ir}^{(\mathrm{I})}\ ,\end{aligned}$$

where in the second equality we have exploited $\nu_{r,0}^{(\mathrm{I})}=1$. Similarly, we eliminate $\nu^{(\mathrm{I})}$, by writing

$$\begin{aligned}&\frac{1}{3}\sum_{i=0}^{\lfloor s/r\rfloor}\big(4^{i+1}-1\big)\,(\nu_{r,r}^{(\mathrm{I})})^i\nu_{r,s-ir}^{(\mathrm{I})}=\\&=\frac{1}{3}\sum_{i=0}^{\lfloor s/r\rfloor}\big(4^{i+1}-1\big)\,\big(\nu_{r-1,r}+\nu_{r-1,r}\nu_{r-1,0}\big)^i\sum_{j=0}^{\lfloor s/r\rfloor-i}\nu_{r-1,r}^j\nu_{r-1,s-(i+j)r}\ ,\end{aligned}$$

from which, using $\nu_{r-1,0}=0$, we get

$$\begin{aligned}\nu_{r,s}&=\frac{1}{3}\sum_{i=0}^{\lfloor s/r\rfloor}2^i\,\big(4^{i+1}-1\big)\,\nu_{r-1,r}^i\sum_{j=i}^{\lfloor s/r\rfloor}\nu_{r-1,r}^{j-i}\nu_{r-1,s-jr}\\&=\frac{1}{3}\sum_{j=0}^{\lfloor s/r\rfloor}\nu_{r-1,r}^j\nu_{r-1,s-jr}\sum_{i=0}^{j}\big(2^{3i+2}-2^i\big)=\sum_{j=0}^{\lfloor s/r\rfloor}\theta_j\nu_{r-1,r}^j\nu_{r-1,s-jr}\ ,\end{aligned}$$

with

$$\theta_j=\frac{1}{3}\sum_{i=0}^{j}\big(2^{3i+2}-2^i\big)=\frac{1}{21}\big(2^{3j+5}-7\cdot 2^{j+1}+3\big)\ .$$

From the definition above, we get

$$\theta_0=1\ ,\qquad\theta_1=11\ ,$$

and we can derive

$$\theta_{j+1}\leq 11\theta_j\qquad\text{for } j\geq 0.\tag{A.14}$$

Thus we can rewrite the sequence as

$$\nu_{0,s}=1\quad\text{for } s\geq 0\ ,\qquad\nu_{r,s}=\sum_{j=0}^{\lfloor s/r\rfloor}\theta_j\nu_{r-1,r}^j\nu_{r-1,s-jr}\quad\text{for } r\geq 1\,,\ s\geq 0\ .$$

As a result, we remark that

$$\nu_{0,s}\leq\nu_{1,s}\leq\ldots\leq\nu_{s,s}=\nu_{s+1,s}=\ldots\tag{A.15}$$

and, observing that $\nu_{r,r}=\theta_0\nu_{r-1,r}+\theta_1\nu_{r-1,r}$, we get

$$\nu_{r,r}=12\nu_{r-1,r}\qquad\text{for } r\geq 1\ .\tag{A.16}$$

From the definition of $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$, we can obtain the following: for $r\geq 2$ and $s>2r$ we have

$$\begin{aligned}\nu_{r,s} &= \nu_{r-1,s}+\nu_{r-1,r}\sum_{j=0}^{\lfloor s/r\rfloor -1}\theta_{j+1}\nu_{r-1,r}^{j}\nu_{r-1,s-r-jr}\\ &\leq \nu_{r-1,s}+11\nu_{r-1,r}\sum_{j=0}^{\lfloor s/r\rfloor -1}\theta_{j}\nu_{r-1,r}^{j}\nu_{r-1,s-r-jr}\\ &\leq \nu_{r-1,s}+11\nu_{r-1,r}\nu_{r,s-r}\leq \nu_{r-1,s}+\nu_{r,r}\nu_{s-r,s-r}\ ,\end{aligned}$$

where we have used (A.14) and (A.16); for $r=1$ we have

$$\begin{aligned}\nu_{1,s} &= \nu_{0,s}+\nu_{0,1}\sum_{j=0}^{s-1}\theta_{j+1}\nu_{0,1}^{j}\nu_{0,s-1-j}\leq\\ &\leq (1+\theta_1)\nu_{0,s-1}+11\sum_{j=1}^{s-1}\theta_{j}\nu_{0,1}^{j}\nu_{0,s-1-j}\leq 12\nu_{1,s-1}\leq 12\nu_{s-2,s-1}=\nu_{s-1,s-1}\ ,\end{aligned}$$

where (A.14), (A.15), (A.16) have been used, together with $\theta_1=11$ and $\nu_{0,s}=1$ for $s\geq 0$. Due to the above properties, we can estimate $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ by means of its diagonal elements $\nu_{r,r}$. Indeed, $\nu_{1,1}=12$ and for $r\geq 2$ one has

$$\begin{aligned}\nu_{r,r} &= 12\nu_{r-1,r}\leq 12\nu_{r-2,r}+12\nu_{r-1,r-1}\nu_{1,1}\leq\ldots\\ &\leq 12\nu_{1,r}+12\left(\nu_{2,2}\nu_{r-2,r-2}+\ldots+\nu_{r-1,r-1}\nu_{1,1}\right)\leq 12\sum_{j=1}^{r-1}\nu_{j,j}\nu_{r-j,r-j}\ .\end{aligned}$$

From this upper bound, one can easily verify that

$$\nu_{r,r}\leq 2^{8r-4}\lambda_r\qquad \text{for } r\geq 1,$$

with $\{\lambda_r\}_{r\geq 1}$ being the Catalan sequence, which satisfies $\lambda_r\leq 4^{r-1}$. Therefore, we can conclude that

$$\nu_{r,s}\leq \nu_{s,s}\leq \frac{2^{10s}}{2^6}\qquad \text{for}\quad r\geq 0\,,\ s\geq 0\ .$$

□

A.2.4 Estimates for multiple Poisson brackets

We report some Cauchy's estimates on the derivatives in the restricted domains which will be useful in the estimates for the generating functions.

Lemma A.2.3 *Let $d\in\mathbb{R}$ such that $0<d<1$ and $g\in\mathcal{P}_\ell$ be an analytic function with bounded norm $\|g\|_1$. Then one has*

$$\left\|\frac{\partial g}{\partial \hat p_j}\right\|_{1-d}\leq \frac{\|g\|_1}{d\rho}\ ,\qquad \left\|\frac{\partial g}{\partial \hat q_j}\right\|_{1-d}\leq \frac{\|g\|_1}{ed\sigma}\ ,\qquad \left\|\frac{\partial g}{\partial \xi_j}\right\|_{1-d}\leq \frac{\|g\|_1}{dR}\ ,\qquad \left\|\frac{\partial g}{\partial \eta_j}\right\|_{1-d}\leq \frac{\|g\|_1}{dR}\ ,$$

Proof. Given g as in (1.3), one has

$$\begin{aligned}\left\|\frac{\partial g}{\partial \hat p_j}\right\|_{1-d} &\leq \sum_{\substack{i\in\mathbb{N}^n\\ |i|=l}}\ \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2}\\ |m_1|+|m_2|=m}}\ \sum_{k\in\mathbb{Z}^n}\frac{i_j}{\rho}|g_{i,m_1,m_2,k}|(1-d)^{l-1}\rho^l e^{|k|(1-d)\sigma}R^m(1-d)^m\\ &\leq \frac{1}{d\rho}\sum_{\substack{i\in\mathbb{N}^n\\ |i|=l}}\ \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2}\\ |m_1|+|m_2|=m}}\ \sum_{k\in\mathbb{Z}^n}|g_{i,m_1,m_2,k}|\rho^l e^{|k|\sigma}R^m=\frac{\|g\|_1}{d\rho}\ ,\end{aligned}$$

where we have used the elementary inequality $m(\lambda - x)^{m-1} \le \lambda^m/x$, for $0 < x < \lambda$ and $m \ge 1$. Similarly, we can deduce the estimates for the partial derivatives with respect to ξ_j and η_j.

Besides,

$$\left\| \frac{\partial g}{\partial \hat{q}_j} \right\|_{1-d} \le \sum_{\substack{i\in\mathbb{N}^n \\ |i|=l}} \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2} \\ |m_1|+|m_2|=m}} \sum_{k\in\mathbb{Z}^n} |k_j|\, |g_{i,m_1,m_2,k}|(1-d)^{l+m}\rho^l R^m e^{|k|(1-d)\sigma}$$
$$\le \frac{1}{ed\sigma} \sum_{\substack{i\in\mathbb{N}^n \\ |i|=l}} \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2} \\ |m_1|+|m_2|=m}} \sum_{k\in\mathbb{Z}^n} |g_{i,m_1,m_2,k}|\rho^l R^m e^{|k|\sigma} = \frac{\|g\|_1}{ed\sigma} \ ,$$

where we have used the elementary inequality $x^\alpha e^{-\delta x} \le (\alpha/(e\delta))^\alpha$, for positive α, x and δ.

□

The previous Lemma enables us to estimate gradients of functions and, consequently, the Hamiltonian vector fields. Therefore, we can state the following Corollary

Corollary A.2.1 *Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $g \in \mathcal{P}_\ell$ be an analytic function with bounded norm $\|g\|_1$. Then one gets*

$$\|\nabla_{\hat{p}}\, g\|_{1-d} \le n_1 \frac{\|g\|_1}{d\rho} \ , \qquad \|\nabla_{\hat{q}}\, g\|_{1-d} \le n_1 \frac{\|g\|_1}{ed\sigma} \ ,$$
$$\|\nabla_{\xi}\, g\|_{1-d} \le n_2 \frac{\|g\|_1}{dR} \ , \qquad \|\nabla_{\eta}\, g\|_{1-d} \le n_2 \frac{\|g\|_1}{dR} \ .$$

Proof. The proof simply follows from the estimates in Lemma A.2.3, applied on each component of the gradient.

□

Lemma A.2.4 *Let $d, d' \in \mathbb{R}$ such that $d > 0$, $d' \ge 0$ and $d + d' < 1$. Then one has*

$$\left\| \frac{\partial \chi_0^{(r)}}{\partial \hat{q}_j} \right\|_{1-d} \le \frac{\|X_0^{(r)}\|_1}{ed\sigma} + |\zeta^{(r)}| \ , \tag{A.17}$$

$$\left\| \frac{\partial \chi_1^{(r)}}{\partial \hat{q}_j} \right\|_{1-d} \le \frac{\|\chi_1^{(r)}\|_1}{ed\sigma} \ , \tag{A.18}$$

$$\left\| \frac{\partial \chi_1^{(r)}}{\partial \xi_j} \right\|_{1-d} \le \frac{\|\chi_1^{(r)}\|_1}{dR} \ , \qquad \left\| \frac{\partial \chi_1^{(r)}}{\partial \eta_j} \right\|_{1-d} \le \frac{\|\chi_1^{(r)}\|_1}{dR} \ , \tag{A.19}$$

$$\left\| \frac{\partial \chi_2^{(r)}}{\partial \hat{q}_j} \right\|_{1-d} \le \frac{\|\chi_2^{(r)}\|_1}{ed\sigma} \ , \tag{A.20}$$

$$\left\| \frac{\partial \chi_2^{(r)}}{\partial \hat{p}_j} \right\|_{1-d} \le \frac{\|\chi_2^{(r)}\|_1}{d\rho} \ ; \tag{A.21}$$

moreover, for $j \ge 1$,

$$\left\| L^j_{\chi_0^{(r)}} f \right\|_{1-d-d'} \le \frac{j!}{e} \left(\frac{e\|X_0^{(r)}\|_{1-d'}}{d^2\rho\sigma} + \frac{e|\zeta^{(r)}|}{d\rho} \right)^j \|f\|_{1-d'} \ , \tag{A.22}$$

$$\left\| L^j_{\chi_1^{(r)}} f \right\|_{1-d-d'} \le \frac{j!}{e^2} \left(\frac{\|\chi_1^{(r)}\|_{1-d'}}{d^2} \left(\frac{e}{\rho\sigma} + \frac{e^2}{R^2} \right) \right)^j \|f\|_{1-d'} \ , \tag{A.23}$$

$$\left\| L^j_{\chi_2^{(r)}} f \right\|_{1-d-d'} \le \frac{j!}{e^2} \left(\frac{2e\|\chi_2^{(r)}\|_{1-d'}}{d^2\rho\sigma} \right)^j \|f\|_{1-d'} \ , \tag{A.24}$$

Proof. The proofs of (A.17)–(A.21) are just minor modifications of Lemma A.2.3, thus are left to the reader.

Coming to (A.22), let $\delta = d/j$ with $j \geq 1$. Proceeding iteratively we get

$$\begin{aligned}\left\| L^j_{\chi_0^{(r)}} f \right\|_{1-d-d'} &\leq \left(\frac{\|X_0^{(r)}\|_{1-d'}}{j\delta^2 e\rho\sigma} + \frac{|\zeta^{(r)}|}{\delta\rho} \right) \left\| L^{j-1}_{\chi_0^{(r)}} f \right\|_{1-d'-(j-1)\delta} \\ &\leq \ldots \\ &\leq \frac{j!}{e} \left(\frac{e\|X_0^{(r)}\|_{1-d'}}{d^2\rho\sigma} + \frac{e|\zeta^{(r)}|}{d\rho} \right)^j \|f\|_{1-d'} \ ,\end{aligned}$$

where we have used the trivial inequality $j^j \leq j!\, e^{j-1}$, holding true for $j \geq 1$. Finally, the proofs of (A.23) and (A.24) are the same, mutatis mutandis.

□

A.2.5 Estimates for the generating functions

Lemma A.2.5 *Let $d \in \mathbb{R}$ such that $0 < d < 1$. The generating function $X_0^{(r)}$ and the vector $\zeta^{(r)}$ are bounded by*

$$\|X_0^{(r)}\|_{1-d} \leq \frac{\|f_0^{(r-1,r)}\|_{1-d}}{\alpha} \ , \qquad |\zeta^{(r)}| \leq \frac{\|f_2^{(r-1,r)}\|_{1-d}}{m\rho} \ . \tag{A.25}$$

The generating functions $\chi_1^{(r)}$ and $\chi_2^{(r)}$ are instead bounded by

$$\|\chi_1^{(r)}\|_{1-d} \leq \frac{\|f_1^{(\mathrm{I},r-1,r)}\|_{1-d}}{\alpha} \ , \tag{A.26}$$

$$\begin{aligned}\|\chi_2^{(r)}\|_{1-d} \leq \frac{1}{\alpha} \Bigg(& 2\|f_2^{(r-1,r)}\|_{1-d} + \frac{1}{\delta_r^2\rho\sigma} \frac{\|f_0^{(r-1,r)}\|_1}{\alpha} \|f_4^{(0,0)}\|_1 + \\ & + \frac{1}{e\delta_r^2} \left(\frac{1}{\rho\sigma} + \frac{e}{R^2} \right) \frac{\|f_1^{(\mathrm{I};r-1,r)}\|_{1-d}}{\alpha} \|f_3^{(0,0)}\|_1 \Bigg) \ ,\end{aligned} \tag{A.27}$$

Proof. The estimates for $X_0^{(r)}$ and $\chi_1^{(r)}$ are trivial. The estimate for $\chi_2^{(r)}$, that is controlled by $f_2^{(\mathrm{I};r-1,r)}$, is a little bit tricky. Indeed, one has to explicitly exploit the fact that

$$f_2^{(\mathrm{II};r-1,r)} = f_2^{(\mathrm{I};r-1,r)} + L_{\chi_1^{(r)}} f_3^{(0,0)} = f_2^{(r-1,r)} - \left\langle f_2^{(r-1,r)} \Big|_{\substack{\xi=\eta=0 \\ q=q^*}} \right\rangle_{q_1} + L_{X_0^{(r)}} f_4^{(0,0)} + L_{\chi_1^{(r)}} f_3^{(0,0)} \ ,$$

together with the trivial estimate

$$\|f - \langle f\big|_{\substack{\xi=\eta=0 \\ q=q^*}} \rangle_{q_1} \|_{1-d} \leq 2\|f\|_{1-d} \ .$$

Concerning the second of (A.25), as C satisfies (2.3), there exists a solution $\zeta^{(r)}$ of (A.5) which satisfies

$$\left\| \nabla_{\hat{p}} \langle f_2^{(r-1,r)} \big|_{\substack{\xi=\eta=0 \\ q=q^*}} \rangle_{q_1} \right\|_{1-d_{r-1}} = \left| \sum_j C_{ij}\zeta_j^{(r)} \right| \geq m|\zeta^{(r)}| \ .$$

Moreover, by the definition of the norm one has

$$\left\| \nabla_{\hat{p}} \langle f_2^{(r-1,r)} \big|_{\substack{\xi=\eta=0 \\ q=q^*}} \rangle_{q_1} \right\|_{1-d_{r-1}} = \frac{\left\| \langle f_2^{(r-1,r)} \big|_{\substack{\xi=\eta=0 \\ q=q^*}} \rangle_{q_1} \right\|_{1-d_{r-1}}}{\rho} \leq \frac{\left\| f_2^{(r-1,r)} \right\|_{1-d_{r-1}}}{\rho} \ .$$

Combining the latter inequalities one gets (A.25).

□

A.2.6 Estimates for the first step

The following Lemma collects the estimates concerning the first step of the normal form algorithm previously described.

Lemma A.2.6 *Consider a Hamiltonian $H^{(0)}$ expanded as in* (2.2)*. Let $\chi_0^{(1)}$, $\chi_1^{(1)}$ and $\chi_2^{(1)}$ be the generating functions used to put the Hamiltonian in normal form at order one, then one has*

$$
\begin{aligned}
\|X_0^{(1)}\|_1 &\le \frac{1}{\alpha}\nu_{0,1}E\varepsilon \ , \\
|\zeta^{(1)}| &\le \frac{1}{4m\rho}\nu_{0,1}E\varepsilon \ , \\
\|\chi_1^{(1)}\|_{1-\delta_1} &\le \frac{1}{\alpha}\nu_{1,1}^{(\mathrm{I})}\Xi_1\frac{E}{2}\varepsilon \ , \\
\|\chi_2^{(1)}\|_{1-2\delta_1} &\le \frac{1}{\alpha}2\nu_{1,1}^{(\mathrm{II})}\Xi_1^2\frac{E}{4}\varepsilon \ .
\end{aligned}
$$

The terms appearing in the expansion of $H^{(\mathrm{I};0)}$, i.e. in (A.6) *with $r = 1$, are bounded as*

$$
\begin{aligned}
\|f_0^{(\mathrm{I};0,1)}\|_{1-\delta_1} &\le E\varepsilon \ , \\
\|f_\ell^{(\mathrm{I};0,s)}\|_{1-\delta_1} &\le \nu_{1,s}^{(\mathrm{I})}\Xi_1^s\frac{E}{2^\ell}\varepsilon^s \ .
\end{aligned}
$$

The terms appearing in the expansion of $H^{(\mathrm{II};0)}$, i.e. in (A.8) *with $r = 1$, are bounded as*

$$
\begin{aligned}
\|f_0^{(\mathrm{II};0,2)}\|_{1-2\delta_1} &\le \nu_{1,2}^{(\mathrm{II})}\Xi_1^3E\varepsilon^2 \ , \\
\|f_\ell^{(\mathrm{II};0,s)}\|_{1-2\delta_1} &\le \nu_{1,s}^{(\mathrm{II})}\Xi_1^{2s}\frac{E}{2^\ell}\varepsilon^s \ .
\end{aligned}
$$

The terms appearing in the expansion of $H^{(1)}$, i.e. in (A.10) *with $r = 1$, are bounded as*

$$
\begin{aligned}
\|f_0^{(1,s)}\|_{1-d_1} &\le \nu_{1,s}\Xi_1^{3s-3}E\varepsilon^s \ , \\
\|f_1^{(1,s)}\|_{1-d_1} &\le \nu_{1,s}\Xi_1^{3s-2}\frac{E}{2}\varepsilon^s \ , \\
\|f_2^{(1,s)}\|_{1-d_1} &\le \nu_{1,s}\Xi_1^{3s-1}\frac{E}{2^2}\varepsilon^s \ , \\
\|f_\ell^{(1,s)}\|_{1-d_1} &\le \nu_{1,s}\Xi_1^{3s}\frac{E}{2^\ell}\varepsilon^s \ .
\end{aligned}
$$

Proof. Using Lemma A.2.5, we immediately get the bounds

$$
\|X_0^{(1)}\|_1 \le \frac{1}{\alpha}\|f_0^{(0,1)}\|_1 \le \frac{1}{\alpha}E\varepsilon \ , \qquad |\zeta^{(1)}| \le \frac{1}{m\rho}\|f_2^{(0,1)}\|_1 \le \frac{E\varepsilon}{4m\rho} \ ,
$$

thus, from (A.17) with $r = 1$ we get

$$
\left\|\frac{\partial\chi_0^{(1)}}{\partial\hat{q}_j}\right\|_{1-\delta_1} \le \frac{E\varepsilon}{\alpha e\delta_1\sigma} + \frac{E\varepsilon}{4m\rho} \le \left(\frac{1}{\alpha e\delta_1\sigma} + \frac{1}{4m\rho}\right)E\varepsilon \ .
$$

The terms $f_\ell^{(\mathrm{I};0,s)}$ appearing in the expansion of the Hamiltonian $H^{(\mathrm{I};0)}$ are bounded as follows. For $\ell = 0$ and $s = 1$ one has

$$
\|f_0^{(\mathrm{I};0,1)}\|_{1-\delta_1} \le \|f_0^{(0,1)}\|_{1-\delta_1} \le E\varepsilon \ , \tag{A.28}
$$

while for the remaining terms one has

$$
\begin{aligned}
\|f_\ell^{(\mathrm{I};0,s)}\|_{1-\delta_1} &\le \sum_{j=0}^{s} \frac{1}{j!} \|L^j_{\chi_0^{(1)}} f_{\ell+2j}^{(0,s-j)}\|_{1-\delta_1} \\
&\le \sum_{j=0}^{s} \frac{1}{e} \left(\frac{e\|X_0^{(1)}\|_1}{\delta_1^2 \rho\sigma} + \frac{e|\zeta^{(1)}|}{\delta_1 \rho} \right)^j \|f_{\ell+2j}^{(0,s-j)}\|_1 \\
&\le \sum_{j=0}^{s} \frac{1}{e} \left(\frac{e}{\alpha\delta_1^2 \rho\sigma} + \frac{e}{4m\delta_1 \rho^2} \right)^j E^j \varepsilon^j \frac{E}{2^{\ell+2j}} \varepsilon^{s-j} \\
&\le \frac{E\varepsilon^s}{2^\ell} \sum_{j=0}^{s} \frac{1}{e} \left(\frac{eE}{4\alpha\delta_1^2 \rho\sigma} + \frac{eE}{16m\delta_1 \rho^2} \right)^j \\
&< (s+1)\Xi_1^s \frac{E}{2^\ell}\varepsilon^s = \nu_{1,s}^{(\mathrm{I})} \Xi_1^s \frac{E}{2^\ell}\varepsilon^s ,
\end{aligned}
$$

where we used the definition of the constant Ξ_1 in (A.12) and Lemma A.2.4.

As regards the second stage of the normalization step, the generating function $\chi_1^{(1)}$ is bounded, as in (A.26), by

$$
\|\chi_1^{(1)}\|_{1-\delta_1} \le \frac{1}{\alpha} \|f_1^{(\mathrm{I};0,1)}\|_{1-\delta_1} \le \frac{1}{\alpha} \nu_{1,1}^{(\mathrm{I})} \Xi_1 \frac{E}{2} \varepsilon \ .
$$

For the term $f_0^{(\mathrm{II};0,2)}$ one has

$$
\begin{aligned}
\|f_0^{(\mathrm{II};0,2)}\|_{1-2\delta_1} &\le \|f_0^{(\mathrm{I};0,2)}\|_{1-\delta_1} + \frac{1}{2} \|L_{\chi_1^{(1)}} f_1^{(\mathrm{I};0,1)}\|_{1-2\delta_1} \\
&\le \nu_{1,2}^{(\mathrm{I})} \Xi_1^2 E\varepsilon^2 + \frac{1}{2} \frac{\|\chi_1^{(1)}\|_{1-\delta_1}}{\delta_1^2 R^2} \|f_1^{(\mathrm{I};0,1)}\|_{1-\delta_1} \\
&\le \nu_{1,2}^{(\mathrm{I})} \Xi_1^2 E\varepsilon^2 + \frac{1}{2} \frac{\nu_{1,1}^{(\mathrm{I})} \Xi_1 E\varepsilon}{2\alpha\delta_1^2 R^2} \nu_{1,1}^{(\mathrm{I})} \Xi_1 \frac{E}{2}\varepsilon \\
&\le \nu_{1,2}^{(\mathrm{II})} \Xi_1^3 E\varepsilon^2 \ ,
\end{aligned}
$$

while the terms $f_\ell^{(\mathrm{II};0,s)}$ are bounded by

$$
\begin{aligned}
\|f_\ell^{(\mathrm{II};0,s)}\|_{1-2\delta_1} &\le \sum_{j=0}^{s} \frac{1}{j!} \|L^j_{\chi_1^{(1)}} f_{\ell+j}^{(\mathrm{I};0,s-j)}\|_{1-2\delta_1} \\
&\le \sum_{j=0}^{s} \frac{1}{e^2} \left(\frac{\|\chi_1^{(1)}\|_{1-\delta_1}}{\delta_1^2} \left(\frac{e}{\rho\sigma} + \frac{e^2}{R^2} \right) \right)^j \|f_{\ell+j}^{(\mathrm{I};0,s-j)}\|_{1-\delta_1} \\
&\le \sum_{j=0}^{s} \frac{1}{e^2} \left(\frac{\nu_{1,1}^{(\mathrm{I})} \Xi_1 E\varepsilon}{2\alpha\delta_1^2} \left(\frac{e}{\rho\sigma} + \frac{e^2}{R^2} \right) \right)^j \nu_{1,s-j}^{(\mathrm{I})} \Xi_1^{s-j} \frac{E}{2^{\ell+j}} \varepsilon^{s-j} \\
&\le \nu_{1,s}^{(\mathrm{II})} \Xi_1^{2s} \frac{E}{2^\ell} \varepsilon^s \ .
\end{aligned}
$$

Coming to the third stage of the normalization step, the generating function $\chi_2^{(1)}$ is bounded, as in (A.27), by

$$\begin{aligned}
\|\chi_2^{(1)}\|_{1-2\delta_1} &\le \frac{1}{\alpha}\left(2\|f_2^{(0,1)}\|_1 + \frac{1}{\delta_1^2\rho\sigma}\frac{\|f_0^{(0,1)}\|_1}{\alpha}\|f_4^{(0,0)}\|_1 + \frac{1}{e\delta_1^2}\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)\frac{\|f_1^{(\mathrm{I};0,1)}\|_{1-\delta_1}}{\alpha}\|f_3^{(0,0)}\|_1\right)\\
&\le \frac{1}{\alpha}\left(\frac{2E}{4}\varepsilon + \frac{1}{\delta_1^2\rho\sigma}\frac{E\varepsilon}{\alpha}\frac{E}{2^4} + \frac{1}{e\delta_1^2}\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)\nu_{1,1}^{(\mathrm{I})}\Xi_1\frac{E}{2}\varepsilon\frac{1}{\alpha}\frac{E}{8}\right)\\
&\le \frac{1}{\alpha}\left(2 + \frac{E}{4\alpha\delta_1^2\rho\sigma} + \frac{1}{e\delta_1^2}\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)\nu_{1,1}^{(\mathrm{I})}\Xi_1\frac{E}{4}\frac{1}{\alpha}\right)\frac{E}{4}\varepsilon\\
&\le \frac{1}{\alpha}\left(\Xi_1 + \nu_{1,1}^{(\mathrm{I})}\Xi_1^2\right)\frac{E}{4}\varepsilon\\
&\le \frac{1}{\alpha}2\nu_{1,1}^{(\mathrm{II})}\Xi_1^2\frac{E}{4}\varepsilon\ .
\end{aligned}$$

The terms $f_\ell^{(1,s)}$ appearing in the expansion of the Hamiltonian $H^{(1)}$ are bounded as follows. The term $f_0^{(1,1)}$ is unchanged, while for $\ell = 0$ and $s = 2$ one has

$$\begin{aligned}
\|f_0^{(1,2)}\|_{1-d_1} &\le \|f_0^{(\mathrm{II};0,2)}\|_{1-2\delta_1} + \frac{1}{e}\frac{2}{\delta_1^2\rho\sigma}\|\chi_2^{(1)}\|_{1-2\delta_1}\|f_0^{(\mathrm{II};0,1)}\|_{1-2\delta_1}\\
&\le \nu_{1,2}^{(\mathrm{II})}\Xi_1^3E\varepsilon^2 + \frac{1}{e}\frac{2}{\delta_1^2\rho\sigma}\frac{1}{\alpha}2\nu_{1,1}^{(\mathrm{II})}\Xi_1^2\frac{E}{4}\varepsilon E\varepsilon\\
&\le \nu_{1,2}\Xi_1^3E\varepsilon^2\ .
\end{aligned}$$

For $\ell = 0$ and $s \ne 1, 2$, using (A.28) for the estimate of the last term in the sum, one has

$$\begin{aligned}
\|f_0^{(1,s)}\|_{1-d_1} \le{}& \sum_{j=0}^{s-3}\frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j\|\chi_2^{(1)}\|_{1-2\delta_1}^j\|f_0^{(\mathrm{II};0,s-j)}\|_{1-2\delta_1}\\
&+ \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-2}\|\chi_2^{(1)}\|_{1-2\delta_1}^{s-2}\|f_0^{(\mathrm{II};0,2)}\|_{1-2\delta_1}\\
&+ \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-1}\|\chi_2^{(1)}\|_{1-2\delta_1}^{s-1}\|f_0^{(\mathrm{II};0,1)}\|_{1-2\delta_1}\\
\le{}& \sum_{j=0}^{s-3}\frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j\frac{1}{\alpha^j}(2\nu_{1,1}^{(\mathrm{II})})^j\Xi_1^{2j}\frac{E^j}{4^j}\varepsilon^j\nu_{1,s-j}^{(\mathrm{II})}\Xi_1^{2(s-j)}E\varepsilon^{s-j}\\
&+ \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-2}\frac{1}{\alpha^{s-2}}(2\nu_{1,1}^{(\mathrm{II})})^{s-2}\Xi_1^{2s-4}\frac{E^{s-2}}{4^{s-2}}\varepsilon^{s-2}\nu_{1,2}^{(\mathrm{II})}\Xi_1^3E\varepsilon^2\\
&+ \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-1}\frac{1}{\alpha^{s-1}}(2\nu_{1,1}^{(\mathrm{II})})^{s-1}\Xi_1^{2s-2}\frac{E^{s-1}}{4^{s-1}}\varepsilon^{s-1}E\varepsilon\\
\le{}& \nu_{1,s}\Xi_1^{3s-3}E\varepsilon^s\ .
\end{aligned}$$

The term $f_1^{(1,1)}$ is unchanged, while for $\ell = 1$ and $s \ne 1$ one has

$$\begin{aligned}
\|f_1^{(1,s)}\|_{1-d_1} &\le \sum_{j=0}^{s-2}\frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j\|\chi_2^{(1)}\|_{1-2\delta_1}^j\|f_1^{(\mathrm{II};0,s-j)}\|_{1-2\delta_1}\\
&\le \sum_{j=0}^{s-2}\frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j\frac{1}{\alpha^j}(2\nu_{1,1}^{(\mathrm{II})})^j\Xi_1^{2j}\frac{E^j}{4^j}\varepsilon^j\nu_{1,s-j}^{(\mathrm{II})}\Xi_1^{2(s-j)}\frac{E}{2}\varepsilon^{s-j}\\
&\le \nu_{1,s}\Xi_1^{3s-2}\frac{E}{2}\varepsilon^s\ .
\end{aligned}$$

The term $f_2^{(1,1)}$ is unchanged, while for $\ell = 2$ and $s \neq 1$ one has

$$\begin{aligned}
\|f_2^{(1,s)}\|_{1-d_1} \leq & \sum_{j=0}^{s-2} \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j \|\chi_2^{(1)}\|_{1-2\delta_1}^j \|f_2^{(\mathrm{II};0,s-j)}\|_{1-2\delta_1} \\
& + \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-1} \|\chi_2^{(1)}\|_{1-2\delta_1}^{s-1} \|f_2^{(\mathrm{II};0,1)}\|_{1-2\delta_1} + \\
\leq & \sum_{j=0}^{s-2} \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j \frac{1}{\alpha^j}(2\nu_{1,1}^{(\mathrm{II})})^j \Xi_1^{2j} \frac{E^j}{4^j}\varepsilon^j \nu_{1,s-j}^{(\mathrm{II})} \Xi_1^{2(s-j)} \frac{E}{4}\varepsilon^{s-j} \\
& + \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^{s-1} \frac{1}{\alpha^{s-1}}(2\nu_{1,1}^{(\mathrm{II})})^{s-1} \Xi_1^{2s-2} \frac{E^{s-1}}{4^{s-1}}\varepsilon^{s-1} \nu_{1,1}^{(\mathrm{II})} \Xi_1^2 \frac{E}{4}\varepsilon \\
\leq & \, \nu_{1,s} \Xi_1^{3s-1} \frac{E}{2^2}\varepsilon^s \ .
\end{aligned}$$

Finally, for $\ell > 2$ one has

$$\begin{aligned}
\|f_\ell^{(1,s)}\|_{1-d_1} \leq & \sum_{j=0}^{s} \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j \|\chi_2^{(1)}\|_{1-2\delta_1}^j \|f_\ell^{(\mathrm{II};0,s-j)}\|_{1-2\delta_1} \\
\leq & \sum_{j=0}^{s} \frac{1}{e^2}\left(\frac{2e}{\delta_1^2\rho\sigma}\right)^j \frac{1}{\alpha^j}(2\nu_{1,1}^{(\mathrm{II})})^j \Xi_1^{2j} \frac{E^j}{4^j}\varepsilon^j \nu_{1,s-j}^{(\mathrm{II})} \Xi_1^{2(s-j)} \frac{E}{2^\ell}\varepsilon^{s-j} \\
\leq & \, \nu_{1,s} \Xi_1^{3s} \frac{E}{2^\ell}\varepsilon^s \ .
\end{aligned}$$

This concludes the proof of the Lemma.

□

A.2.7 Proof of Lemma A.2.1

For $r = 0$ we do not have any generating functions and by assumption the Hamiltonian satisfies

$$\|f_\ell^{(0,s)}\|_1 \leq \frac{E}{2^\ell}\varepsilon^s \ .$$

We proceed by induction. For $r = 1$ use Lemma A.2.6. For $r > 1$, we now complete the proof performing the step from $r-1$ to r. The estimates for the generating function $\chi_0^{(r)}$ result from Lemma A.2.5, observing that

$$b(r-1, r, 2) = 5r - 2 - 2\left\lfloor \frac{r-1}{r-1} \right\rfloor - 1 = 5r - 5$$

and

$$b(r-1, r, 0) = 5r - 2 - 2\left\lfloor \frac{r-1}{r-1} \right\rfloor - 3 = 5r - 7 \ .$$

Indeed,

$$\|X_0^{(r)}\|_{1-d_{r-1}} \leq \frac{1}{\alpha}\nu_{r-1,r}\Xi_r^{b(r-1,r,0)} E\varepsilon^r \leq \frac{1}{\alpha}\nu_{r-1,r}\Xi_r^{5r-7} E\varepsilon^r \ ,$$

$$|\zeta^{(r)}| \leq \frac{1}{4m\rho}\nu_{r-1,r}\Xi_r^{b(r-1,r,2)} E\varepsilon^r \leq \frac{1}{4m\rho}\nu_{r-1,r}\Xi_r^{5r-5} E\varepsilon^r \ .$$

We now come to the functions appearing in the expansion of the Hamiltonian $H^{(\mathrm{I};r-1)}$, where one has

$$
\begin{aligned}
\|f_\ell^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} &\le \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!}\|L^j_{\chi_0^{(r)}} f_{\ell+2j}^{(r-1,s-jr)}\|_{1-d_{r-1}-\delta_r}\\
&\le \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{e}\left(\frac{e\|X_0^{(r)}\|_{1-d_{r-1}}}{\delta_r^2\rho\sigma}+\frac{e|\zeta^{(r)}|}{\delta_r\rho}\right)^j \nu_{r-1,s-jr}\Xi_r^{b(r-1,s-jr,\ell+2j)}\frac{E}{2^{\ell+2j}}\varepsilon^{s-jr}\\
&\le \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{e}\left(\frac{eE}{\alpha\delta_r^2\rho\sigma}+\frac{eE}{4m\delta_r\rho^2}\right)^j \nu_{r-1,r}^j\Xi_r^{5rj-5j}\varepsilon^{rj}\\
&\qquad\times \nu_{r-1,s-jr}\Xi_r^{b(r-1,s-jr,\ell+2j)}\frac{E}{2^{\ell+2j}}\varepsilon^{s-jr}\\
&\le \nu_{r,s}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r-1,s,\ell)}\frac{E}{2^\ell}\varepsilon^s\ .
\end{aligned}
$$

The main difference with respect to the case $r=1$ is due to the power of the constant Ξ_r, that is controlled by the quantity $b(\mathrm{I};r-1,s,\ell)$.

For $\ell\ge 0$ and $s=0$ we have
$$f_\ell^{(\mathrm{I};r-1,0)}=f_\ell^{(r-1,0)}\ ,$$
thus
$$b(\mathrm{I};r-1,0,\ell)=b(r-1,0,\ell)=0\ .$$

For $r\ge 2$, $\ell>2$, one has

$$
\begin{aligned}
5rj-4j+b(r-1,s-jr,\ell+2j) &= 5rj-4j+5(s-jr)-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-w_{\ell+2j}\\
&\le 5s-2-2\left\lfloor\frac{s-1}{r-1}\right\rfloor-w_\ell=b(\mathrm{I};r-1,s,\ell)\ ,
\end{aligned}
$$

where we have exploited $w_{\ell+2j}=0$ for $\ell+2j\ge 3$.

Moreover, for $r\ge 2$, $\ell=2$ and $s=1$ we have
$$f_2^{(\mathrm{I};r-1,1)}=f_2^{(r-1,1)}\ ,$$
thus
$$b(\mathrm{I};r-1,1,2)=b(r-1,1,2)=3-w_2=2\ .$$

For $\ell=2$, $s>1$, $j=0$ one has
$$5s-2-2\left\lfloor\frac{s-1}{r-1}\right\rfloor-w_2\le b(\mathrm{I};r-1,s,2)\ .$$

For $\ell=2$, $s>1$, $j\ge 1$, just notice that
$$\left\lfloor\frac{s-1+j(r-2)}{r-1}\right\rfloor\ge\left\lfloor\frac{s-2}{r-1}\right\rfloor+1\ .$$

Thus for $j\ge 1$ we have

$$
\begin{aligned}
5rj-4j+b(r-1,s-jr,2+2j) &= 5rj-4j+5(s-jr)-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-w_{2+2j}\\
&= 5s-2-2\left\lfloor\frac{s-1+j(r-2)}{r-1}\right\rfloor\\
&\le 5s-2-\left\lfloor\frac{s-1}{r-1}\right\rfloor-\left\lfloor\frac{s-2}{r-1}\right\rfloor-w_2=b(\mathrm{I};r-1,s,2)\ .
\end{aligned}
$$

Similarly, for $r \geq 2$, $\ell = 1$, $s \geq r$ and $j = 0$ we have

$$5s - 2 - 2\left\lfloor \frac{s-1}{r-1} \right\rfloor - w_1 \leq b(\mathrm{I}; r-1, s, 1) \ ,$$

while, for $j \geq 1$, we get

$$\begin{aligned} 5rj - 4j + b(r-1, s-jr, 1+2j) &= 5rj - 4j + 5(s-jr) - 2 - 2\left\lfloor \frac{s-jr-1}{r-1} \right\rfloor - w_{1+2j} \\ &\leq 5s - 2 - 2\left\lfloor \frac{s-2}{r-1} \right\rfloor - w_1 = b(\mathrm{I}; r-1, s, 1) \ . \end{aligned}$$

We remark that for $r \geq 2$, $\ell = 1$ and $s < r$ one has

$$f_1^{(\mathrm{I}; r-1, s)} = f_1^{(r-1,s)} = 0 \ ,$$

thus

$$b(\mathrm{I}; r-1, s, 1) = b(r-1, s, 1) = 0 \ .$$

For $r \geq 2$, $\ell = 0$ and $s > 0$, our goal is to prove that

$$5rj - 4j + b(r-1, s-jr, 2j) = 5s - 2 - 2\left(\left\lfloor \frac{s-jr-1}{r-1} \right\rfloor + 2j\right) - w_{2j} \leq 5s - 2 - 2\left\lfloor \frac{s-1}{r} \right\rfloor - w_0 \ .$$

First notice that for $j = 0$ is trivial, due to the definition of w_0 and to the estimate $\left\lfloor \frac{s-1}{r-1} \right\rfloor \geq \left\lfloor \frac{s-1}{r} \right\rfloor$.

For $j = 1$ and $s = r$, the estimate is true because of the definition of $f_0^{(\mathrm{I}; r-1, r)}$. For $j = 1$, which implies $s > r$ ($s - jr$ being a non negative integer), just notice that one has

$$\left\lfloor \frac{s-r-1}{r-1} \right\rfloor + 2 \geq \left\lfloor \frac{s-1}{r} \right\rfloor + 1 \ .$$

Indeed, letting $s = s' + r$, we have

$$\left\lfloor \frac{s-r-1}{r-1} \right\rfloor + 2 = \left\lfloor \frac{s'-1}{r-1} \right\rfloor + 2$$

and

$$\left\lfloor \frac{s-1}{r} \right\rfloor + 1 = \left\lfloor \frac{s'+r-1}{r} \right\rfloor + 1 = \left\lfloor \frac{s'-1}{r} \right\rfloor + 2 \ .$$

While for $j > 1$, $2\left\lfloor \frac{s-jr-1}{r-1} \right\rfloor + 4j$ being increasing in j, one has

$$2\left\lfloor \frac{s-jr-1}{r-1} \right\rfloor + 4j \geq 2\left\lfloor \frac{s-2r-1}{r-1} \right\rfloor + 8$$

and in view of $s > 2r$ one has

$$2\left\lfloor \frac{s-2r-1}{r-1} \right\rfloor + 8 \geq 2\left\lfloor \frac{s-1}{r} \right\rfloor + 3 \ .$$

Thus we have

$$\begin{aligned} 5rj - 4j + b(r-1, s-jr, 2j) &= 5rj - 4j + 5(s-jr) - 2 - 2\left\lfloor \frac{s-jr-1}{r-1} \right\rfloor - w_{2j} \\ &\leq b(\mathrm{I}; r-1, s, 0) \ . \end{aligned}$$

The estimate for the generating function $\chi_1^{(r)}$ follows from Lemma A.2.5, observing that

$$b(\mathrm{I}; r-1, r, 1) = 5r - 2 - 2\left\lfloor \frac{r-2}{r-1} \right\rfloor - 2 = 5r - 4 \ .$$

Indeed,

$$\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \le \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r-1,r,1)}\frac{E}{2}\varepsilon^r \le \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{I})}\Xi_r^{5r-4}\frac{E}{2}\varepsilon^r \ .$$

We now come to the functions appearing in the expansion of the Hamiltonian $H^{(\mathrm{II};r-1)}$, where one has

$$\begin{aligned}
\|f_\ell^{(\mathrm{II};r-1,s)}\|_{1-d_{r-1}-2\delta_r} &\le \sum_{j=0}^{\lfloor s/r\rfloor}\frac{1}{j!}\|L^j_{\chi_1^{(r)}}f_{\ell+j}^{(\mathrm{I};r-1,s-jr)}\|_{1-d_{r-1}-2\delta_r}\\
&\le \sum_{j=0}^{\lfloor s/r\rfloor}\frac{1}{e^2}\left(\frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r^2}\left(\frac{e}{\rho\sigma}+\frac{e^2}{R^2}\right)\right)^j\\
&\qquad\times\nu_{r-1,s-jr}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r-1,s-jr,\ell+j)}\frac{E}{2^{\ell+j}}\varepsilon^{s-jr}\\
&\le \sum_{j=0}^{\lfloor s/r\rfloor}\frac{1}{e^2}\left(\frac{E}{2\alpha\delta_r^2}\left(\frac{e}{\rho\sigma}+\frac{e^2}{R^2}\right)\right)^j\left(\nu_{r,r}^{(\mathrm{I})}\right)^j\Xi_r^{5rj-4j}\varepsilon^{rj}\\
&\qquad\times\nu_{r-1,s-jr}^{(\mathrm{I})}\Xi_r^{b(\mathrm{I};r-1,s-jr,\ell+j)}\frac{E}{2^{\ell+j}}\varepsilon^{s-jr}\\
&\le \nu_{r,s}^{(\mathrm{II})}\Xi_r^{b(\mathrm{II};r-1,s,\ell)}\frac{E}{2^\ell}\varepsilon^s \ .
\end{aligned}$$

The main difference with respect to the case $r=1$ is due to the power of the constant Ξ_r, that is controlled by the quantity $b(\mathrm{II};r-1,s,\ell)$.

For $\ell\ge 0$ and $s=0$, we get

$$f_\ell^{(\mathrm{II};r-1,0)} = f_\ell^{(r-1,0)} \ ,$$

thus

$$b(\mathrm{II};r-1,0,\ell) = b(r-1,0,\ell) = 0 \ .$$

For $r=2$, $\ell>2$, $s>0$, exploiting $w_{\ell+j}=0$ for $\ell+j\ge 3$, one has

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j \le 5s-2-2\left\lfloor\frac{s-1}{r-1}\right\rfloor+\left\lfloor\frac{s}{r}\right\rfloor-w_\ell = b(\mathrm{II};r-1,s,\ell) \ .$$

For $r=2$, $\ell=2$, $s=1$ we have

$$f_2^{(\mathrm{II};1,1)} = \langle f_2^{(\mathrm{II};0,1)}\rangle_{q_1} \ ,$$

then

$$b(\mathrm{II};1,1,2) = b(\mathrm{II};0,1,2) = 2 \ .$$

For $r=2$, $\ell=2$, $s>1$, similarly to the case of $b(\mathrm{I};r-1,s,2)$, one has

$$\begin{aligned}
5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_{2+j} &\le 5s-2-\left\lfloor\frac{s-1}{r-1}\right\rfloor-\left\lfloor\frac{s-2}{r-1}\right\rfloor+\left\lfloor\frac{s}{r}\right\rfloor-w_2\\
&\le b(\mathrm{II};1,s,2) \ .
\end{aligned}$$

For $r=2$, $\ell=1$, $s<2$, we obtain

$$f_1^{(\mathrm{II};1,s)} = f_1^{(\mathrm{I};1,s)} = f_1^{(1,s)} = 0 \ ,$$

thus

$$b(\mathrm{II};1,s,1) = b(1,s,1) = 0 \ .$$

For $r=2$, $\ell=1$, $s=2$, we get

$$f_1^{(\mathrm{II};1,2)} = 0 \ ,$$

hence

$$b(\mathrm{II};1,s,1)=0\ ,$$

while, if $s=3$ we have $j<2$ thus

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_{1+j}\le 9=b(\mathrm{II},1,3,1)\ .$$

For $r=2$, $\ell=1$, $s>3$ one has

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_{1+j}\le 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor-w_1=b(\mathrm{II};1,s,1)\ .$$

For $r=2$, $\ell=0$, $s=1$ one has

$$f_0^{(\mathrm{II};1,1)}=f_0^{(\mathrm{I};1,1)}=f_0^{(1,1)}=\langle f_0^{(0,1)}\rangle_{q_1}\ ,$$

thus

$$b(\mathrm{II};1,1,0)=b(1,1,0)=0\ ,$$

while for $s=2$ we obtain

$$f_0^{(\mathrm{II};1,2)}=\langle f_0^{(1,2)}\rangle_{q_1}\ ,$$

thus

$$b(\mathrm{II};1,2,0)=b(1,2,0)=3\ .$$

For $r=2$, $\ell=0$, $s\ge 3$, one has

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_j\le 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor-w_0\le b(\mathrm{II};1,s,0)\ .$$

For $r>2$, $\ell>2$ and $s>1$ one has

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_{\ell+j}\le 5s-2-\left\lfloor\frac{s-1}{r-1}\right\rfloor-\left\lfloor\frac{s-2}{r-1}\right\rfloor-w_\ell=b(\mathrm{II};r-1,s,l)\ ,$$

where we have exploited $w_{\ell+j}=0$ for $\ell+j\ge 3$. Indeed, remarking that $2\lfloor(s-jr-1)/(r-1)\rfloor+3j$ is a non-decreasing function, one has

$$2\left\lfloor\frac{s-1}{r-1}\right\rfloor\ge\left\lfloor\frac{s-1}{r-1}\right\rfloor+\left\lfloor\frac{s-2}{r-1}\right\rfloor\ .$$

For $r>2$, $\ell>2$ and $s=1$ we get

$$f_\ell^{(\mathrm{II};r-1,1)}=f_\ell^{(r-1,1)}\ ,$$

thus

$$b(r-1,1,\ell)=b(\mathrm{II};r-1,1,\ell)\ .$$

In the same way, for $r>2$, $\ell=2$, $s=1$, we get

$$f_2^{(\mathrm{II};r-1,1)}=f_2^{(r-1,1)}\ ,$$

thus

$$b(r-1,1,2)=2=b(\mathrm{II};r-1,1,2)\ .$$

For $r>2$, $\ell=2$, $s>1$ one has

$$5s-2-2\left\lfloor\frac{s-jr-1}{r-1}\right\rfloor-3j-w_{2+j}\le 5s-2-2\left\lfloor\frac{s-1}{r}\right\rfloor-w_2\le b(\mathrm{II};r-1,s,2)\ ,$$

which can be proved distinguishing the cases for $j = 0$ and $j = 1$.

Similarly to the case $r = 2$, for $r > 2$, $\ell = 1$, $s < r$, we obtain

$$f_1^{(\mathrm{II};r-1,s)} = f_1^{(\mathrm{I};r-1,s)} = f_1^{(r-1,s)} = 0 \ ,$$

thus

$$b(\mathrm{II}; r-1, s, 1) = b(r-1, s, 1) = 0 \ .$$

For $r > 2$, $\ell = 1$, $s = r$, we get

$$f_1^{(\mathrm{II};r-1,r)} = 0 \ ,$$

hence

$$b(\mathrm{II}; r-1, s, 1) = 0 \ .$$

For $r > 2$, $\ell = 1$, $s \geq r+1$ one has

$$5s - 2 - 2\left\lfloor \frac{s - jr - 1}{r-1} \right\rfloor - 3j - w_{1+j} \leq 5s - 2 - 2\left\lfloor \frac{s-1}{r} \right\rfloor - w_1 \leq b(\mathrm{II}; r-1, s, 1) \ .$$

Similarly to the previous case, one has to consider the different values of j up to $j = 2$.

For $r > 2$, $\ell = 0$, $s > 0$ one has

$$5s - 2 - 2\left\lfloor \frac{s - jr - 1}{r-1} \right\rfloor - 3j - w_j \leq 5s - 2 - 2\left\lfloor \frac{s-1}{r} \right\rfloor - w_0 \leq b(\mathrm{II}; r-1, s, 0) \ .$$

Once again, one has to consider the different values of j up to $j = 3$.

The estimate for the generating function $\chi_2^{(r)}$ follows from Lemma A.2.5, observing that

$$b(\mathrm{II}; r-1, r, 2) = 5r - 2 - 2\left\lfloor \frac{r-1}{r} \right\rfloor - 1 = 5r - 3 \ .$$

Indeed,

$$\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} \leq \frac{1}{\alpha} 2\nu_{r,r}^{(\mathrm{II})} \Xi_r^{b(\mathrm{II};r-1,r,2)} \frac{E}{4} \varepsilon^r \leq \frac{1}{\alpha} 2\nu_{r,r}^{(\mathrm{II})} \Xi_r^{5r-3} \frac{E}{4} \varepsilon^r \ .$$

We now come to the functions appearing in the expansion of the Hamiltonian $H^{(r)}$, where one has

$$\begin{aligned}
\|f_\ell^{(r,s)}\|_{1-d_r} &\leq \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} \|L_{\chi_2^{(r)}}^j f_\ell^{(\mathrm{II};r-1,s-jr)}\|_{1-d_r} \\
&\leq \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{e^2} \left(\frac{2e\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r}}{\delta_r^2 \rho\sigma} \right)^j \\
&\qquad \times \nu_{r-1,s-jr}^{(\mathrm{II})} \Xi_r^{b(\mathrm{II};r-1,s-jr,\ell)} \frac{E}{2^\ell} \varepsilon^{s-jr} \\
&\leq \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{e^2} \left(\frac{2eE}{4\alpha\delta_r^2\rho\sigma} \right)^j \left(2\nu_{r,r}^{(\mathrm{II})} \right)^j \Xi_r^{5rj-3j} \varepsilon^{rj} \\
&\qquad \times \nu_{r-1,s-jr}^{(\mathrm{II})} \Xi_r^{b(\mathrm{II};r-1,s-jr,\ell)} \frac{E}{2^\ell} \varepsilon^{s-jr} \\
&\leq \nu_{r,s} \Xi_r^{b(r,s,\ell)} \frac{E}{2^\ell} \varepsilon^s \ .
\end{aligned}$$

For $\ell \geq 0$ and $s = 0$ the claim is true because

$$f_\ell^{(r,0)} = f_\ell^{(\mathrm{II};r,0)} = f_\ell^{(\mathrm{I};r,0)} = f_\ell^{(r-1,0)} \ ,$$

which implies

$$b(r, 0, \ell) = 0 \ .$$

For $r = 2$, $\ell \leq 1$ and $s > 0$ (with $s > r$ if $\ell = 1$)

$$5s - 2 - 2\left\lfloor \frac{s - jr - 1}{r} \right\rfloor - 2j - w_\ell = 5s - 2 - 2\left\lfloor \frac{s-1}{r} \right\rfloor - w_\ell = b(r, s, l) \ .$$

For $r = 2$, $\ell = 1$ and $s = r$, we remark that $f_1^{(2,2)} = 0$, thus $b(2, 2, 1) = 0$. For $r = 2$, $\ell = 1$ and $s < r$, we obtain $b(2, s, 1) = b(\mathrm{II}; 2, s, 1) = 0$.

For $r = 2$, $\ell = 2$, if $s = 2\lfloor \frac{s}{2} \rfloor$, the homological equation allows the summation to run only up to $j = \lfloor \frac{s}{2} \rfloor - 1$. Similarly, if $s = 2\lfloor \frac{s}{2} \rfloor + 1$, we get $s - 2j \geq 1$, since the summation runs up to $j = \lfloor \frac{s}{2} \rfloor$. The case $s - 2j = 1$, which implies $j = \lfloor \frac{s}{2} \rfloor$, is trivial, because

$$f_2^{(\mathrm{II};1,1)} = \langle f_2^{(\mathrm{II};0,1)} \rangle_{q_1} \ .$$

For $r > 2$, $\ell > 2$ one has

$$5rj - 2j + b(\mathrm{II}; r - 1, s - jr, \ell) \leq b(r, s, \ell) \ .$$

Indeed, if $s = \lfloor \frac{s}{r} \rfloor r$, the homological equation allows the summation to run only up to $j = \lfloor \frac{s}{r} \rfloor - 1$. Moreover, if $s = \lfloor \frac{s}{r} \rfloor r + 1$, we get $s - jr \geq 1$, since the summation runs up to $j = \lfloor \frac{s}{r} \rfloor$. If instead $s = \lfloor \frac{s}{r} \rfloor r + m$ with $2 \leq m \leq r - 1$, we have $s - jr \geq 2$, since the summation runs up to $j = \lfloor \frac{s}{r} \rfloor$.

For $r > 2$, $\ell \leq 2$ and $s > 0$ (with $s > r$ if $\ell = 1$) one has

$$5s - 2 - 2\left\lfloor \frac{s - jr - 1}{r} \right\rfloor - 2j - w_\ell \leq 5s - 2 - 2\left\lfloor \frac{s-1}{r} \right\rfloor - w_2 \leq b(r, s, l) \ .$$

For $r > 2$, $\ell = 1$ and $s \leq r$, we get $f_1^{(r,s)} = f_1^{(\mathrm{II};r-1,s)} = 0$, then $b(r, s, 1) = b(\mathrm{II}; r - 1, s, 1) = 0$. This concludes the proof of the Lemma.

A.2.8 Proof of Proposition 2.1.1

We prove the Proposition 2.1.1 in its simplified version with three stages in a normalization step. We give an estimate for the canonical transformation. We denote by $(\hat{p}^{(0)}, \hat{q}^{(0)}, \xi^{(0)}, \eta^{(0)})$ the original coordinates, and by $(\hat{p}^{(r)}, \hat{q}^{(r)}, \xi^{(r)}, \eta^{(r)})$ the coordinates at step r. We also denote by $\phi^{(r)}$ the canonical transformation mapping $(\hat{p}^{(r)}, \hat{q}^{(r)}, \xi^{(r)}, \eta^{(r)})$ to $(\hat{p}^{(r-1)}, \hat{q}^{(r-1)}, \xi^{(r-1)}, \eta^{(r-1)})$. Precisely, we can give an explicit expression of the canonical flow, writing

$$\begin{aligned}
\hat{p}^{(r-1)} &= \exp(L_{\chi_0^{(r)}})\hat{p}^{(\mathrm{I},r-1)} = \hat{p}^{(\mathrm{I},r-1)} - \sum_{s\geq 1} \frac{1}{s!} L_{\chi_0^{(r)}}^{s-1} \frac{\partial \chi_0^{(r)}}{\partial \hat{q}^{(\mathrm{I},r-1)}} = \hat{p}^{(\mathrm{I},r-1)} - \frac{\partial \chi_0^{(r)}}{\partial \hat{q}^{(\mathrm{I},r-1)}} \ , \\
\hat{p}^{(\mathrm{I},r-1)} &= \exp(L_{\chi_1^{(r)}})\hat{p}^{(\mathrm{II},r-1)} = \hat{p}^{(\mathrm{II},r-1)} - \sum_{s\geq 1} \frac{1}{s!} L_{\chi_1^{(r)}}^{s-1} \frac{\partial \chi_1^{(r)}}{\partial \hat{q}^{(\mathrm{II},r-1)}} = \\
&= \hat{p}^{(\mathrm{II},r-1)} - \frac{\partial \chi_1^{(r)}}{\partial \hat{q}^{(\mathrm{II},r-1)}} - \frac{1}{2} L_{\chi_1^{(r)}} \frac{\partial \chi_1^{(r)}}{\partial \hat{q}^{(\mathrm{II},r-1)}} \ , \\
\xi^{(r-1)} &= \exp(L_{\chi_1^{(r)}})\xi^{(r)} = \xi^{(r)} + \frac{\partial \chi_1^{(r)}}{\partial \eta^{(r)}} \ , \\
\eta^{(r-1)} &= \exp(L_{\chi_1^{(r)}})\eta^{(r)} = \eta^{(r)} - \frac{\partial \chi_1^{(r)}}{\partial \xi^{(r)}} \ , \\
\hat{p}^{(\mathrm{II},r-1)} &= \exp(L_{\chi_2^{(r)}})\hat{p}^{(r)} = \hat{p}^{(r)} - \sum_{s\geq 1} \frac{1}{s!} L_{\chi_2^{(r)}}^{s-1} \frac{\partial \chi_2^{(r)}}{\partial \hat{q}^{(r)}} \ , \\
\hat{q}^{(r-1)} &= \exp(L_{\chi_2^{(r)}})\hat{q}^{(r)} = \hat{q}^{(r)} + \sum_{s\geq 1} \frac{1}{s!} L_{\chi_2^{(r)}}^{s-1} \frac{\partial \chi_2^{(r)}}{\partial \hat{p}^{(r)}} \ .
\end{aligned}$$

where we have taken into account that the generating function $\chi_0^{(r)}$ depends only on $\hat{q}^{(\mathrm{I},r-1)}$, so it leaves the angles and the transversal variables unchanged. Similarly, the generating function $\chi_1^{(r)}$ does not modify the angles, and the function $\chi_2^{(r)}$ leaves the transversal variables unchanged.

Consider now a sequence of domains $\mathcal{D}_{(3d-d_r)(\rho,\sigma,R)}$, and, using Lemma A.2.1, we get

$$
\begin{aligned}
\left|\hat{p}^{(r-1)}-\hat{p}^{(\mathrm{I},r-1)}\right| &< \frac{\|X_0^{(r)}\|_{1-d_{r-1}}}{e\delta_r\sigma}+|\zeta^{(r)}| \le \left(\frac{1}{\alpha e\delta_r\sigma}+\frac{1}{4m\rho}\right)\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\ ,\\
\left|\hat{p}^{(\mathrm{I},r-1)}-\hat{p}^{(\mathrm{II},r-1)}\right| &< \frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{e\delta_r\sigma}\left(1+\frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{2e\delta_r^2}\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)\right)\le\\
&\le \frac{1}{2\alpha e\delta_r\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\left(1+\frac{1}{4\alpha e\delta_r^2}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)\right)\ ,\\
\left|\xi^{(r-1)}-\xi^{(r)}\right| &< \frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r R}\le\frac{1}{2\alpha\delta_r R}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\ ,\\
\left|\eta^{(r-1)}-\eta^{(r)}\right| &< \frac{\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r}}{\delta_r R}\le\frac{1}{2\alpha\delta_r R}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\ ,\\
\left|\hat{p}^{(\mathrm{II},r-1)}-\hat{p}^{(r)}\right| &< \frac{\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r}}{e\delta_r\sigma}\sum_{s\ge1}\frac{1}{e^2}\left(\frac{2e\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r}}{\delta_r^2\rho\sigma}\right)^{s-1}\le\\
&\le \frac{1}{2\alpha e\delta_r\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\sum_{s\ge1}\left(\frac{2e}{2\alpha\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\right)^{s-1}\ ,\\
\left|\hat{q}^{(r-1)}-\hat{q}^{(r)}\right| &< \frac{\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r}}{\delta_r\rho}\sum_{s\ge1}\frac{1}{e^2}\left(\frac{2e\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r}}{\delta_r^2\rho\sigma}\right)^{s-1}\le\\
&\le \frac{1}{2\alpha\delta_r\rho}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\sum_{s\ge1}\left(\frac{2e}{2\alpha\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\right)^{s-1}\ .
\end{aligned}
\tag{A.29}
$$

Thus if

$$
\left(\frac{1}{\alpha e\delta_r^2\rho\sigma}+\frac{1}{4m\delta_r\rho^2}\right)\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r+\frac{3}{4\alpha e\delta_r^2}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\left(\frac{1}{\rho\sigma}+\frac{e}{R^2}\right)+\frac{e}{\alpha\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\le\frac{1}{2}\ ,
\tag{A.30}
$$

then $\frac{e}{\alpha\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\le\frac{1}{2}$, so the series (A.29) defining the canonical transformation are absolutely convergent in the domain $\mathcal{D}_{(3d-d_{r-1}-\delta_r)(\rho,\sigma,R)}$. The absolute convergence implies the uniform convergence in any compact subset of the domain $\mathcal{D}_{(3d-d_{r-1}-\delta_r)(\rho,\sigma,R)}$, and so, by Weierstrass theorem, the canonical transformation is also analytic. Furthermore, due to (A.30), one also has the estimates

$$
|\hat{p}^{(r-1)}-\hat{p}^{(r)}|<\delta_r\rho\ ,\qquad |\hat{q}^{(r-1)}-\hat{q}^{(r)}|<\delta_r\sigma\ ,\qquad |\xi^{(r-1)}-\xi^{(r)}|<\delta_r R\ ,\qquad |\eta^{(r-1)}-\eta^{(r)}|<\delta_r R\ .
$$

Indeed,

$$
\begin{aligned}
|\hat{p}^{(r-1)}-\hat{p}^{(r)}| &\le |\hat{p}^{(r-1)}-\hat{p}^{(\mathrm{I},r-1)}|+|\hat{p}^{(\mathrm{I},r-1)}-\hat{p}^{(\mathrm{II},r-1)}|+|\hat{p}^{(\mathrm{II},r-1)}-\hat{p}^{(r)}|\le\\
&\le\left(\left(\frac{1}{\alpha e\delta_r^2\rho\sigma}+\frac{1}{4m\delta_r\rho^2}\right)\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r+\frac{3}{4\alpha e\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r+\right.\\
&\qquad\left.+\frac{e}{\alpha\delta_r^2\rho\sigma}\Xi_r^{5r}\frac{2^{10r}}{2^6}E\varepsilon^r\right)\delta_r\rho<\\
&<\delta_r\rho\ .
\end{aligned}
$$

In the same way, we can deduce the other estimates.

A similar argument applies to the inverse of $\phi^{(r)}$, which is defined as a composition of Lie series generated by $-\chi_2^{(r)}$, $-\chi_1^{(r)}$ and $-\chi_0^{(r)}$, thus we get

$$\mathcal{D}_{(3d-d_r)(\rho,\sigma,R)} \subset \phi^{(r)}(\mathcal{D}_{(3d-d_{r-1}-\delta_r)(\rho,\sigma,R)}) \subset \mathcal{D}_{(3d-d_{r-1})(\rho,\sigma,R)} \ .$$

Consider now the sequence of transformations $\Phi^{(\bar{r})} = \phi^{(1)} \circ \ldots \circ \phi^{(\bar{r})}$.

For $(\hat{p}^{(r-1)}, \hat{q}^{(r-1)}, \xi^{(r-1)}, \eta^{(r-1)}) \in \mathcal{D}_{(3d-d_{r-1})(\rho,\sigma,R)}$ the transformation is clearly analytic and one has

$$|\hat{p}^{(0)} - \hat{p}^{(\bar{r})}| < \rho \sum_{j=1}^{\bar{r}} \delta_j \ , \qquad |\hat{q}^{(0)} - \hat{q}^{(\bar{r})}| < \sigma \sum_{j=1}^{\bar{r}} \delta_j \ ,$$
$$|\xi^{(0)} - \xi^{(\bar{r})}| < R \sum_{j=1}^{\bar{r}} \delta_j \ , \qquad |\eta^{(0)} - \eta^{(\bar{r})}| < R \sum_{j=1}^{\bar{r}} \delta_j \ .$$

Setting $d = \frac{1}{4}$ and using (A.11), one has $\sum_{j\geq 1} \delta_j \leq \frac{d}{3} = \frac{1}{12}$, thus (2.7) immediately follows. Finally, the estimates for the Hamiltonian in normal form had been already gathered in Lemma A.2.1. This concludes the proof of Proposition 2.1.1.

Remark A.2.1 *Since the non convergence of the normalization algorithm represents one of the main points, let us stress that in view of the definition of* Ξ_r *in* (A.12) *and of* $\delta_r < C/r$*, one immediately get* $\Xi_r > Cr$*,* C *being a suitable positive constant. Thus* $\sum_{r>0} \Xi_r^{5r} \varepsilon^r$ *cannot converge for any positive* ε*.*

A.3 Spectrum deformation under matrix perturbations

In this section we collect some useful results concerning the deformation of the spectrum of a matrix under small perturbations. The results rely on resolvent theory, for which we refer to [49] for a detailed treatment of the subject and to [1,59] for the study of the linear stability of breathers and multibreathers.

We first set some notations. Given a matrix M defined on a vector space X, we denote by $\Sigma(M)$ its spectrum and by $\rho(M) = \max_{\lambda\in\Sigma(M)} |\lambda_j|$ its spectral radius. For any $z \notin \Sigma(M)$ it is well defined $R(z) = (M - z)^{-1}$, which is the *resolvent* of M. The inverse of the spectral radius of $R(z)$ measures the distance between $z \in \mathbb{C}$ and the spectrum of M

$$\mathrm{dist}\,(z, \Sigma(M)) = \frac{1}{\rho(R(z))} \ . \tag{A.31}$$

Let us recall that $\rho(M)$ and the operatorial norm $\|M\|_{\mathsf{op}} = \sup_{x\neq 0} \frac{\|Mx\|}{\|x\|}$ satisfy

$$\rho(M) \leq \|M\|_{\mathsf{op}} \ . \tag{A.32}$$

Moreover, a converse inequality is given by the following

Lemma A.3.1 *Let* $M : X \to X$*, then there exists* $c_{op} > 1$*, depending on the dimension of* X *only, such that*

$$\|M\|_{op} \leq c_{op} \max\{\rho(M), 1\} \ . \tag{A.33}$$

If M *is diagonable, or if* $\rho(M) \geq 1$*, then the above simplifies to*

$$\|M\|_{op} \leq c_{op}\rho(M) \ . \tag{A.34}$$

A.3.1 On the minimum eigenvalue

Let us now consider a given matrix N and its perturbation $M(\mu) = N + \mu P$ depending analytically on a small parameter μ. The resolvent $R(z, \mu)$ of $M(\mu)$ is still well defined and holomorphic in the two variables, provided $z \notin \Sigma(M(\mu))$; moreover, it is possible to relate the perturbed and unperturbed resolvents via the following series expansion

$$R(z, \mu) = R_0(z)[\mathbb{I} + A(z, \mu)R_0(z)]^{-1} \ , \quad R_0(z) = R(z, 0) \ , \tag{A.35}$$

where $R_0(z)$ is the resolvent of the leading term $N = M(0)$, while $A(z, \mu) = R(z, \mu) - R_0(z)$ represents the *deformation* due to the small perturbation. In what follows we collect some useful results relating the *unperturbed spectrum* $\Sigma(N)$ to the *perturbed spectrum* $\Sigma(M(\mu))$.

We collect the results relating the minimum eigenvalue of N and M in the following

Proposition A.3.1 *Let us consider a matrix $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon)P(\varepsilon)$, depending on the small parameter $\varepsilon \in \mathcal{U}(0)$. Let us assume that for any $\varepsilon \in \mathcal{U}$ it holds true:*

(1) $N(\varepsilon)$ is invertible and there exist $c_1 > 0$ and $\alpha > 0$ independent of ε such that

$$\left\|N^{-1}\right\|_{\mathsf{op}} \leq c_1|\varepsilon|^{-\alpha} \ ;$$

(2) $P(\varepsilon) = P(0) + \mathcal{O}(\varepsilon)$ and there exist $c_2 > 0$ and $\beta > \alpha$ such that

$$|\mu(\varepsilon)| \leq c_2|\varepsilon|^{\beta} \ .$$

Then for ε small enough $M(\varepsilon)$ is invertible and there exists $c_3 > 0$ independent of ε such that

$$|\nu| \geq c_3|\varepsilon|^{\alpha} \ , \qquad \textit{for all} \quad \nu \in \Sigma(M).$$

Moreover, the same result holds true if we replace the first assumption with

$$\min_{\lambda \in \Sigma(N(\varepsilon))} \{|\lambda|\} \geq c_1|\varepsilon|^{\alpha} \ . \tag{A.36}$$

Proof. Due to the invertibility of N, we can rewrite M as $M = N\,(\mathbb{I} + \mu T)$, with $T = N^{-1}P$. We now show that $(\mathbb{I} + \mu T)^{-1}$ is well defined, thus the inverse of M is given by $M^{-1} = (\mathbb{I} + \mu T)^{-1}\,N^{-1}$. The operator $(\mathbb{I} + \mu T)$ being a small perturbation of the identity $\mathbb{I}$, its inverse exists provided $|\mu|\,\|T\|_{\mathsf{op}} < 1$, hence hypothesis *(1)* implies

$$|\mu|\,\|T\|_{\mathsf{op}} \leq |\mu|\,\left\|N^{-1}\right\|_{\mathsf{op}} \|P\|_{\mathsf{op}} \leq c|\varepsilon|^{\beta-\alpha} < 1 \ , \tag{A.37}$$

where $c \approx c_2 c_1 \|P(0)\|_{\mathsf{op}}$ for sufficiently small ε and in such a regime is independent of ε; as a consequence, since $\beta - \alpha > 0$, also $M(\varepsilon)$ is invertible for ε small enough. From the estimate

$$\left\|(\mathbb{I} + \mu T)^{-1}\right\|_{\mathsf{op}} \leq \sum_{n \geq 0} |\mu|^n \|T\|^n_{\mathsf{op}} \leq \frac{1}{1 - |\mu|\,\|T\|_{\mathsf{op}}} \ ,$$

and recalling (A.32) we get

$$\rho(M^{-1}) \leq \left\|M^{-1}\right\|_{\mathsf{op}} \leq \frac{\left\|N^{-1}\right\|_{\mathsf{op}}}{1 - |\mu|\,\|T\|_{\mathsf{op}}} \ ,$$

where the spectral radius $\rho(M^{-1}) = \frac{1}{\min\{|\nu_k|\}}$, with $\nu_k \in \Sigma(M)$. Let ν_1 be the minimum, then

$$|\nu_1| \geq \frac{1 - |\mu|\,\|T\|_{\mathsf{op}}}{\|N^{-1}\|_{\mathsf{op}}} \ .$$

For ε sufficiently small, condition (A.37) holds and we have

$$|\mu| \, \|T\|_{\mathsf{op}} \leq \frac{1}{2} \, , \qquad \text{that implies} \quad |\nu_1| \geq c_3 |\varepsilon|^{\alpha} \, , \quad \text{with} \quad c_3 = \frac{1}{2c_1} \, .$$

It is clear that the main point in the proof is the bound of $\left\|N^{-1}\right\|_{\mathsf{op}}$. In fact, the estimate (A.37) can be obtained also replacing the assumption on $\|N\|_{\mathsf{op}}$ with (A.36). Indeed by means of (A.33) one can obtain the following estimate

$$\left\|N^{-1}\right\|_{\mathsf{op}} \leq c_{\mathsf{op}} \max\left\{1, \rho(N^{-1})\right\} = c_{\mathsf{op}} \max\left\{1, \frac{1}{\min_{\lambda\in\Sigma(N(\varepsilon))}\{|\lambda|\}}\right\} \leq c_{\mathsf{op}} c_1^{-1} |\varepsilon|^{-\alpha} \, ,$$

since $c_1^{-1}|\varepsilon|^{-\alpha} \gg 1$, which allows to conclude

$$|\nu_1| \geq c_3 |\varepsilon|^{\alpha} \, , \qquad \text{with} \quad c_3 = \frac{1}{2} c_{\mathsf{op}}^{-1} c_1 \, .$$

□

A.3.2 Deformation of eigenvalues.

We aim at localizing the eigenvalues of $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon)P(\varepsilon)$, when the spectrum of its leading part $N(\varepsilon)$ is known, provided ε is taken small enough in a small neighbourhood of the origin $\mathcal{U}(0)$. We start with a preliminary Lemma

Lemma A.3.2 *Let $z \notin \Sigma(N)$ be a complex number satisfying*

$$\mathrm{dist}(z, \Sigma(N)) \geq 4\mu c_{op} \|P\|_{op} \, . \tag{A.38}$$

Then the following inequality holds true

$$\frac{1}{c_{op}} \mathrm{dist}(z, \Sigma(N)) - \mu \|P\|_{op} \leq \mathrm{dist}(z, \Sigma(M)) \leq c_{op} \, \mathrm{dist}(z, \Sigma(N)) + \mu c_{op} \|P\|_{op} \, . \tag{A.39}$$

Proof. To shorten the proof, let us introduce the notations

$$\delta_N(z) = \mathrm{dist}(z, \Sigma(N)) \, , \qquad \delta_M(z) = \mathrm{dist}(z, \Sigma(M)) \, .$$

By setting $R_0(z)$ the resolvent of N, from (A.31) we have

$$\rho(R_0(z)) = \frac{1}{\delta_N} \, .$$

In the rest of the proof we aim at deriving bounds for δ_M by exploiting the perturbation of R_0 given by μP. Thus, let us set R the resolvent of M and recall that

$$\Sigma(R(z)) = \left\{\frac{1}{\lambda_j - z}\right\}_{\lambda_j \in \Sigma(M)} \, ;$$

from (A.32) and (A.34) we have

$$\frac{1}{\delta_N} = \rho(R_0(z)) \leq \|R_0(z)\|_{\mathsf{op}} \leq c_{\mathsf{op}} \rho(R_0(z)) = \frac{c_{\mathsf{op}}}{\delta_N} \, .$$

From the second Neumann series (A.35) we can write

$$R(z) = R_0(z)[\mathbb{I} + \mu P R_0]^{-1} \, , \tag{A.40}$$

where, due to (A.38), the product $\mu P R_0$ represents a perturbation of the identity

$$\|\mu P R_0\|_{\mathsf{op}} \leq \mu \|P\|_{\mathsf{op}} \|R_0\|_{\mathsf{op}} \leq \frac{c_{\mathsf{op}}\mu \|P\|_{\mathsf{op}}}{\delta_N} \leq \frac{1}{4} \ , \tag{A.41}$$

so that $[\mathbb{I} + \mu P R_0]^{-1}$ is well defined in terms of power series of $\mu P R_0$. As a consequence, from the estimate of $\left\|\sum_{k\geq 0}(-1)^k \mu^k (P R_0)^k\right\|_{\mathsf{op}}$, we get

$$\delta_M(z) = \frac{1}{\rho(R(z))} \geq \frac{1}{\|R\|_{\mathsf{op}}} \geq \frac{1}{\|R_0\|_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} \geq \frac{\delta_N(z)}{c_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} \ ,$$

which gives the lower bound in (A.39). Let us consider again (A.35) but reversing the roles of R and R_0

$$R_0(z) = R(z)[\mathbb{I} - \mu P R]^{-1} \ ;$$

from (A.40) and (A.41) we have

$$\|R\|_{\mathsf{op}} \leq \frac{4}{3} \|R_0\|_{\mathsf{op}} \qquad \text{that implies} \quad \|\mu P R\|_{\mathsf{op}} \leq \frac{1}{3} \ ,$$

which provides the upper bound in (A.39)

$$\delta_N(z) = \frac{1}{\rho(R_0(z))} \geq \frac{1}{\|R_0\|_{\mathsf{op}}} \geq \frac{3/2}{\|R\|_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} \geq \frac{3}{2c_{\mathsf{op}}}\delta_M(z) - \mu \|P\|_{\mathsf{op}} \ .$$

□

The localization result is collected in the following

Proposition A.3.2 *Let $M(\varepsilon) \in \mathsf{Mat}(n)$ be decomposed into $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon)P(\varepsilon)$. Assume that:*

(1) $P(\varepsilon) = P(0) + \mathcal{O}(\varepsilon)$ with $\|P(0)\|_{op} \leq c_P$ and there exists $\beta_1 > 0$ such that

$$|\mu(\varepsilon)| \leq |\varepsilon|^{\beta_1} \ ;$$

(2) there exists a $c_N > 0$ such that for any couple of distinct eigenvalues $\lambda_i \neq \lambda_j \in \Sigma(N)$

$$|\lambda_i - \lambda_j| \geq c_N \varepsilon^{\beta_2} \ , \qquad \textit{with} \quad \beta_2 < \beta_1 \ .$$

Then there exists $\varepsilon^* > 0$ (depending on $\Sigma(N)$) such that, given $|\varepsilon| < \varepsilon^*$, for any $\lambda \in \Sigma(N)$ there exist one eigenvalue $\nu \in \Sigma(M)$ inside the disk $D_\varepsilon(\lambda) = \left\{z \in \mathbb{C} \ : \ |z - \lambda| < c_M|\varepsilon|^{\beta_1}\right\}$, with $c_M > 0$ a suitable constant independent of λ.

Proof. Take an arbitrary eigenvalue $\lambda \in \Sigma(N)$ and consider a complex number $z \in \mathbb{C}$ at distance $\tilde{\delta} = c|\varepsilon|^{\beta_1}$, with c independent of ε to be determined along the proof. In view of *(2)* and for ε small enough, one has that $c_N|\varepsilon|^{\beta_2} \gg c_P|\varepsilon|^{\beta_1}$; hence by defining $\delta_N(z) = \mathrm{dist}(z, \Sigma(N))$ it turns out

$$\delta_N(z) = \tilde{\delta} = c|\varepsilon|^{\beta_1} \ .$$

We want to use upper bound of (A.39) to control $\delta_M(z) = \mathrm{dist}\,(z, \Sigma(M))$; to fulfill the requirements of Lemma A.3.2 we take $c \geq 4c_{\mathsf{op}}c_P$ and we exploit *(1)*, so that

$$\delta_M(z) \leq c_{\mathsf{op}}(c + c_P)|\varepsilon|^{\beta_1} \ .$$

This ensures the existence of an eigenvalue $\nu \in \Sigma(M)$, which depends on the choice of z, whose distance from the initially chosen z is of order $\mathcal{O}(\varepsilon^{\beta_1})$

$$\exists \nu \in \Sigma(M) \ : \ |\nu - z| \leq c_{\mathsf{op}}(c + c_P)|\varepsilon|^{\beta_1} \ ,$$

which provides the final estimate

$$|\nu - \lambda| \leq |\nu - z| + |z - \lambda| \leq c_M|\varepsilon|^{\beta_1} \ , \qquad \text{with} \quad c_M = c + c_{\mathsf{op}}(c + c_P) \ .$$

□

A.4 dNLS model: standard approach

Consider the dNLS equation (3.1). Introducing the set of excited sites $\mathcal{I} = \{j_1, \ldots, j_{n_1}\} \subset \mathcal{J}$, not necessarily consecutive, in the limit of $\varepsilon = 0$ we consider unperturbed excited oscillators $\left\{\psi_j^{(0)}\right\}_{j\in\mathcal{I}}$ with resonant frequencies in order to get a periodic flow on the resonant torus. In this way, at $\varepsilon = 0$ we can assume the usual ansatz

$$\psi^{(0)}(t) = e^{-\mathbf{i}\omega t}\phi^{(0)} , \qquad t \in [0, T = 2\pi/\omega] , \tag{A.42}$$

where the unperturbed spatial profile $\phi^{(0)}$ reads

$$\phi_j^{(0)} = \begin{cases} Re^{\mathbf{i}\varphi_j} , & j \in \mathcal{I} \\ 0 , & j \in \mathcal{J} \setminus \mathcal{I} \end{cases} .$$

The choice of the amplitude R uniquely defines the detuning of the frequency ω from the harmonic limit of the uncoupled oscillators

$$\omega(R) = 1 + \gamma R^2 .$$

Hence $\psi^{(0)}(t)$ coincides with the orbit of the Gauge symmetry $e^{\mathbf{i}\sigma}$ acting on the phase space. In other terms, in any point of the orbit $\Gamma_0 = \left\{\psi^{(0)}(t) : t \in [0, T]\right\}$, the Hamiltonian flow is parallel to the flow of the symmetry

$$X_H(\psi^{(0)}(t)) = \omega X_P(\psi^{(0)}(t)) , \qquad \forall t \in [0, T] ,$$

where P is the additionally conserved ℓ^2-norm

$$P = \sum_{j\in\mathcal{J}} |\psi_j|^2 , \tag{A.43}$$

whose Hamiltonian field X_P generates the symmetry. This identification of the "group-orbit" with the "flow-orbit" does not hold anymore if a different resonance relationship is chosen among the unperturbed oscillators. All these orbits are uniquely defined except for a phase shift σ, due to the action of the symmetry $e^{\mathbf{i}\sigma}$ along the orbit, which corresponds to a change of the initial configuration in the ansatz (A.42). In order to study the continuation of solutions (A.42) at $\varepsilon \neq 0$, the usual approach is then to assume the same ansatz for the continued solution $\psi(t, \varepsilon) = e^{-\mathbf{i}\omega t}\phi(\varepsilon)$ and insert it into the dNLS equation (3.1), thus obtaining a time-independent stationary equation of the form $F(\phi, \varepsilon) = 0$; the latter then is studied by methods of bifurcation theory, namely a Lyapunov-Schmidt reduction which exploits the variational formulation of $F(\phi, \varepsilon) = 0$ (see [48,51,69,70,75]). Continuation is easily achieved for non-degenerate critical points of the averaged perturbation $\langle H_1\rangle_T$ by means of the following Proposition

Proposition A.4.1 (Kapitula's criterium) *Let $\phi^{(0)}(\varphi)$ be the profile of an unperturbed periodic solution $\psi^{(0)}$ given by* (A.42)*. A necessary condition for $\phi^{(0)}$ to be continued at $\varepsilon \neq 0$ is that $\phi^{(0)}$ is a critical point of the functional*

$$S_1(\phi^{(0)}) = \frac{1}{2\pi}\int_0^{2\pi} H_1(e^{-\mathbf{i}\tau}\phi^{(0)}(\varphi))d\tau .$$

If the critical point $\phi^{(0)}$ is not degenerate (modulo phase shift), then for ε small enough there exists a continuation $\phi(\varepsilon)$, solution of $F(\phi(\varepsilon), \varepsilon) = 0$ which is analytic in ε.

A.5 Analytic estimates: KAM with knobs

In this Section we report the proofs of some Lemmas of Chapter 4.

Proof. [**Lemma 4.4.6**] From the definitions (4.17) and (4.16), we get $\alpha_s\delta_s^2 < 1$. So, it remains to prove the second part of the inequality. Starting from the definition (4.24), using the selection rule in (4.23) and the decreasing character of the sequence $\{\alpha_s\delta_s^2\}_{s\ge1}$, we obtain

$$\frac{T_{r,s}}{\alpha_s\delta_s^2} = \frac{1}{\alpha_s\delta_s^2}\max_{I\in\mathcal{J}_{s,s}}\prod_{j\in I\,,\,j\ge1}\frac{1}{\alpha_j\delta_j^2} \le \prod_{j\in\{s\}\cup I_s^*\,,\,j\ge1}\frac{1}{\alpha_j\delta_j^2}\ .$$

Hence, we get

$$\begin{aligned}\log\frac{T_{r,s}}{\alpha_s\delta_s^2} &\le -\log(\alpha_s\delta_s^2) - \sum_{k=1}^{\lfloor s/2\rfloor}\left(\left\lfloor\frac{s}{k}\right\rfloor - \left\lfloor\frac{s}{k+1}\right\rfloor\right)\log(\alpha_k\delta_k^2)\\ &\le -\sum_{k=1}^{s}\left(\left\lfloor\frac{s}{k}\right\rfloor - \left\lfloor\frac{s}{k+1}\right\rfloor\right)(\log\alpha_k + 2\log\delta_k)\\ &\le -s\sum_{k\ge1}\frac{\log\alpha_k + 2\log\delta_k}{k(k+1)} = s\left(\Gamma - \sum_{k\ge1}\frac{2\log\delta_k}{k(k+1)}\right)\ ,\end{aligned} \tag{A.44}$$

where we used properties (i) and (ii) of Lemma 4.4.3, the fact that the sequence $\{\alpha_s\delta_s^2\}_{s\ge1}$ is decreasing and the condition $\boldsymbol{\tau}$ in (4.3). Recalling the definition of δ_r we get

$$\begin{aligned}-\sum_{k\ge1}\frac{2\log\delta_k}{k(k+1)} &= 2\sum_{k\ge1}\frac{\log\left(\frac{4\pi^2}{3}\right) + 2\log k}{k(k+1)}\\ &= 2\log\left(\frac{4\pi^2}{3}\right) + 4\sum_{k\ge1}\frac{\log k}{k(k+1)}\ .\end{aligned}$$

where the relation $\sum_{k\ge1} 1/[k(k+1)] = 1$ is used. As regards the second series, we exploit the relation

$$\sum_{k\ge2} f(k) \le f(2) + \int_2^\infty f(x)\,dx\ , \qquad \text{with} \quad f(k) = \frac{\log k}{k^2}\ ,$$

so that

$$4\sum_{k\ge1}\frac{\log k}{k(k+1)} < 4\left(\frac{\log 2}{6} + \int_2^\infty\frac{\log x\,dx}{x^2}\right) = 4\left(\frac{\log 2}{6} + \frac{1}{2}(\log 2 + 1)\right)\ .$$

To conclude, we obtain

$$-\sum_{k\ge1}\frac{2\log\delta_k}{k(k+1)} < 13\log 2$$

Putting the estimate above into (A.44), we conclude the proof.

□

Proof. [**Selection rules in Table 4.2**]

For $r = 0$ there are no divisors, therefore every function in $H^{(0)}$ owns an empty list $\{\}$. Hence, by padding every list with an appropriate number of zeros, the table 4.2 is correct, because $\mathcal{J}_{0,s}$ is a list of $s-1$ zeros. For $r > 0$ we proceed by induction, supposing that the table applies up to step $r-1$. To this end, we follow the formal algorithm defined in (4.12) and (4.14), and the estimates of the norms reported in (4.21) and (4.22).

We now exploit the definitions of generating functions and factors $G_{r,0}$, $G_{r,1}$ and, for $f_1^{(\mathrm{I};r-1,r)}$, we select the worst divisors that can appear in the sum. So, we obtain

$$
\begin{aligned}
f_0^{(r-1,r)} &\quad \text{owns} \quad \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ ; \\
G_{r,0} &\quad \text{owns} \quad \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ ; \\
\chi_0^{(r)} &\quad \text{owns} \quad \{r\} \cup \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ ; \\
f_1^{(\mathrm{I};r-1,r)} &\quad \text{owns} \quad \{r\} \cup \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ ; \\
G_{r,1} &\quad \text{owns} \quad \{r\} \cup \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ ; \\
\chi_1^{(r)} &\quad \text{owns} \quad \{r\} \cup \{r\} \cup \mathcal{J}_{r-1,r} \cup \mathcal{J}_{r-1,r} \ .
\end{aligned}
$$

By using the selection rule for $\chi_0^{(r)}$ for all allowed values of j in the sums, we get

$$
\begin{aligned}
f_0^{(\mathrm{I};r-1,s)} &\quad \text{owns} \quad \mathcal{J}_{r-1,s}^2 + \{r\} \cup \mathcal{J}_{r-1,r}^2 \cup \{r-1\} \cup \mathcal{J}_{r-1,s-r}^2 + \\
&\qquad\qquad + \left(\{r\} \cup \mathcal{J}_{r-1,r}^2\right)^j \cup \left(\min\{r-1, s-jr\} \cup \mathcal{J}_{r-1,s-jr}\right)^2 \subset \ \mathcal{J}_{r,s}^2 \ , \\
f_1^{(\mathrm{I};r-1,s)} &\quad \text{owns} \quad \{r-1\} \cup \mathcal{J}_{r-1,s}^2 + \left(\{r\} \cup \mathcal{J}_{r-1,r}^2\right)^j \cup \left(\min\{r-1, s-jr\} \cup \mathcal{J}_{r-1,s-jr}\right)^2 \\
&\qquad\qquad \subset \ \{r\} \cup \mathcal{J}_{r,s}^2 \ , \\
f_{\ell\geq 2}^{(\mathrm{I};r-1,s)} &\quad \text{owns} \quad \left(\{r\} \cup \mathcal{J}_{r-1,r}^2\right)^j \cup \left(\min\{r-1, s-jr\} \cup \mathcal{J}_{r-1,s-jr}\right)^2 \ \subset \ \left(\{\min\{r,s\}\} \cup \mathcal{J}_{r,s}\right)^2
\end{aligned}
$$

where we recall that the notation $\mathcal{J}_{r,s}^2$ stands for $\mathcal{J}_{r,s} \cup \mathcal{J}_{r,s}$. The inclusion relations follow from Lemma 4.4.4.

Using the information on $\chi_1^{(r)}$ for all allowed values of j in the sums, we obtain

$$
\begin{aligned}
f_0^{(r,s)} &\quad \text{owns} \quad \left(\{r\} \cup \mathcal{J}_{r-1,r}\right)^{2j} \cup \mathcal{J}_{r,s-jr}^2 \ \subset \ \mathcal{J}_{r,s}^2 \ ; \\
f_1^{(r,s)} &\quad \text{owns} \quad \left(\{r\} \cup \mathcal{J}_{r-1,r}\right)^{2j} \cup \{r\} \cup \mathcal{J}_{r,s-jr}^2 \ \subset \ \{r\} \cup \mathcal{J}_{r,s}^2 \ ; \\
f_{\ell\geq 2}^{(r,s)} &\quad \text{owns} \quad \left(\{r\} \cup \mathcal{J}_{r-1,r}\right)^{2j} \cup \left(\{\min\{r, s-jr\}\} \cup \mathcal{J}_{r,s-jr}\right)^2 \ \subset \ \left(\{\min\{r,s\}\} \cup \mathcal{J}_{r,s}\right)^2 \ .
\end{aligned}
$$

Once again, the inclusion relations follow from Lemma 4.4.4 and hold true for every term in the sums. This completes the proof.

□

Proof. [**Lemma 4.4.7**]
The definition of $\nu_{r,s}$ can be rewrite by eliminating the sequence $\{\nu_{r,s}^{(\mathrm{I})}\}_{r\geq 1\,,\,s\geq 0}$.

$$
\begin{aligned}
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} \left(2\nu_{r-1,r}\right)^j \sum_{i=0}^{\lfloor s/r \rfloor - j} \nu_{r-1,r}^i \nu_{r-1,s-(i+j)r} = \sum_{j=0}^{\lfloor s/r \rfloor} \left(2\nu_{r-1,r}\right)^j \sum_{i=j}^{\lfloor s/r \rfloor} \nu_{r-1,r}^{i-j} \nu_{r-1,s-ir} \\
&= \sum_{i=0}^{\lfloor s/r \rfloor} \left(2^{i+1} - 1\right) \nu_{r-1,r}^i \nu_{r-1,s-ir} = \sum_{i=0}^{\lfloor s/r \rfloor} \theta_i \nu_{r-1,r}^i \nu_{r-1,s-ir} \ .
\end{aligned}
$$

Hence, we get

$$
\theta_i = 2^{i+1} - 1 \qquad \theta_0 = 1 \ , \qquad \theta_1 = 3 \ .
$$

So, we can derive

$$
\theta_{i+1} \leq 4\theta_i \qquad \text{for } i \geq 0.
$$

and $\nu_{r,r} = \theta_0 \nu_{r-1,r} + \theta_1 \nu_{r-1,r} = 4\nu_{r-1,r}$. Following the same path of the proof of Lemma A.2.2, we obtain $\nu_{1,1} = 4$ and for $r \geq 2$ one has

$$
\begin{aligned}
\nu_{r,r} &= 4\nu_{r-1,r} \leq 4\nu_{r-2,r} + 4\nu_{r-1,r-1}\nu_{1,1} \leq \ldots \\
&\leq 4\nu_{1,r} + 4\left(\nu_{2,2}\nu_{r-2,r-2} + \ldots + \nu_{r-1,r-1}\nu_{1,1}\right) \leq 4\sum_{j=1}^{r-1} \nu_{j,j}\nu_{r-j,r-j} \ .
\end{aligned}
$$

From this upper bound, one can easily verify that

$$\nu_{r,r} \le 2^{4r-2}\lambda_r \qquad \text{for} \quad r \ge 1,$$

with $\{\lambda_r\}_{r\ge1}$ being the Catalan sequence, which satisfies $\lambda_r \le 4^{r-1}$. Therefore, we can conclude that

$$\nu_{r,s} \le \nu_{s,s} \le \frac{2^{6s}}{2^4} \qquad \text{for} \quad r \ge 0\,,\; s \ge 0\,.$$

□

A.5.1 Proof of Lemma 4.4.8

We proceed by induction. Let $r = 1$, the generating function $\chi_0^{(1)}$ is bounded as

$$\|\chi_0^{(1)}\|_1 \le \frac{\|f_0^{(0,1)}\|_1}{\alpha_1} \le \varepsilon_0 E\mu \frac{1}{\alpha_1} \le \varepsilon_0 E\mu \frac{T_{0,1}^2}{\alpha_1}\nu_{0,1}\ ,$$

this verifies the first of (4.27).

We can now bound the terms appearing in $H^{(\mathrm{I};0)} = \exp(L_{\chi_0^{(1)}})H^{(0)}$. For $\ell > 0$ we have

$$\begin{aligned}
\|f_\ell^{(\mathrm{I};0,s)}\|_{1-\delta_1} &\le \sum_{j=0}^{s-1}\left(\frac{2e}{\rho\sigma}\right)^j\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)^j\frac{\varepsilon_0 E\mu^{s-j}}{2^{\ell+j}} + \left(\frac{2e}{\rho\sigma}\right)^s\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)^s\frac{E}{2^{\ell+s}}\\
&\le (s+1)\left(\frac{2e}{\rho\sigma}E\right)^s\frac{\varepsilon_0 E\mu^s}{2^\ell}\frac{1}{\beta_1^s}\\
&\le \frac{\varepsilon_0 E\mu^s M^s}{2^\ell}T_{1,s}^2\nu_{1,s}^{(\mathrm{I})}\ .
\end{aligned}$$

For $\ell = 0$ and $s \ge 2$, we have

$$\begin{aligned}
\|f_0^{(\mathrm{I};0,s)}\|_{1-\delta_1} &\le \sum_{j=0}^{s-1}\left(\frac{2e}{\rho\sigma}\right)^j\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)^j\frac{\varepsilon_0 E\mu^{s-j}}{2^j}\\
&\qquad + \left(\frac{2e}{\rho\sigma}\right)^s\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)^s\frac{E}{2^s} + \left(\frac{2e}{\rho\sigma}\right)\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)\|\langle\omega^{(0,s-1)},p\rangle\|_1\\
&\le (s+1)\left(\frac{2e}{\rho\sigma}E\right)^s\varepsilon_0 E\mu^s\frac{1}{\beta_1^s} + \left(\frac{2e}{\rho\sigma}\right)\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)\|\langle\omega^{(0,s-1)},p\rangle\|_1\ .
\end{aligned}$$

Let us stress that the last addend is still unknown.

The norm of the generating function $\chi_1^{(1)}$ can be bounded as

$$\|\chi_1^{(1)}\|_1 \le \frac{\|f_1^{(\mathrm{I};0,1)}\|_{1-\delta_1}}{\alpha_1} \le 2\left(\frac{2e}{\rho\sigma}E\right)\frac{\varepsilon_0 E\mu}{2}\frac{1}{\alpha_1\beta_1} \le \frac{\varepsilon_0 E\mu M}{2}\frac{T_{0,1}^2}{\alpha_1\beta_1}\nu_{1,1}^{(\mathrm{I})}\ ,$$

and the one of the quantity $\langle\omega^{(0,1)},p\rangle$ as

$$\|\langle\omega^{(0,1)},p\rangle\|_1 \le \|f_1^{(0,1)}\|_1 \le \frac{\varepsilon_0 E\mu}{2}\ ,$$

thus (4.27) holds true.

Having just determined $\|\langle\omega^{(0,1)},p\rangle\|_1$, we can now estimate the norm of $f_0^{(\mathrm{I};0,2)}$ as

$$\begin{aligned}
\|f_0^{(\mathrm{I};0,2)}\|_{1-\delta_1} &\le 3\left(\frac{2e}{\rho\sigma}E\right)^2 \varepsilon_0 E\mu^2\frac{1}{\beta_1^2} + \left(\frac{2e}{\rho\sigma}\right)\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)\|\langle\omega^{(0,1)},p\rangle\|_1 \\
&\le 3\left(\frac{2e}{\rho\sigma}E\right)^2 \varepsilon_0 E\mu^2\frac{1}{\beta_1^2} + \left(\frac{2e}{\rho\sigma}\right)\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)\frac{\varepsilon_0 E\mu}{2} \\
&\le 3\left(\frac{4e}{\rho\sigma}E\right)^2 \varepsilon_0 E\mu^2\frac{1}{\beta_1^2} \\
&\le \varepsilon_0 E\mu^2 M^2 T_{1,2}^2 \nu_{1,2}^{(\mathrm{I})}\ .
\end{aligned}$$

Besides, the higher order terms, $f_0^{(\mathrm{I};0,s)}$ with $s>2$, will be explicitly bounded to subsequent normalization orders.

To conclude the inductive basis, we now bound the terms appearing in $H^{(1)}=\exp(L_{\chi_1^{(1)}})H^{(\mathrm{I};0)}$. First notice that $\|f_0^{(1,2)}\|_{1-d_1}=\|f_0^{(\mathrm{I};0,2)}\|_{1-d_1}$. The higher order terms with $\ell=0$ are bounded as

$$\begin{aligned}
\|f_0^{(1,s)}\|_{1-d_1} \le{}& \sum_{j=0}^{s-2}\left(\frac{2e}{\rho\sigma}\right)^j\left(2\left(\frac{2e}{\rho\sigma}E\right)\frac{\varepsilon_0 E\mu}{2}\frac{1}{\beta_1^2}\right)^j \\
&\times\left((s-j+1)\left(\frac{2e}{\rho\sigma}E\right)^{s-j}\varepsilon_0 E\mu^{s-j}\frac{1}{\beta_1^{s-j}} + \left(\frac{2e}{\rho\sigma}\right)\left(\varepsilon_0 E\mu\frac{1}{\beta_1}\right)\|\langle\omega^{(0,s-j-1)},p\rangle\|_1\right) \\
\le{}& \left(\frac{2e}{\rho\sigma}E\right)^{2s-2}\varepsilon_0 E\mu^s\frac{1}{\beta_1^{2s-2}}\left(\sum_{j=0}^{s-2}(s-j+1)\right) \\
&+\sum_{j=0}^{s-2}\left(\frac{2e}{\rho\sigma}E\right)^{j+1}\varepsilon_0 E\mu\frac{1}{\beta_1^{2j+1}}\|\langle\omega^{(0,s-j-1)},p\rangle\|_1
\end{aligned}$$

Again, the last addend is still unknown, but this is harmless for the inductive step.

Considering $\ell=1$, we immediately get

$$\begin{aligned}
\|f_1^{(1,s)}\|_{1-d_1} \le \|f_1^{(\mathrm{I};0,s)}\|_1 &+ \sum_{j=1}^{s-1}\left(\frac{2e}{\rho\sigma}\right)^j\left(2\left(\frac{2e}{\rho\sigma}E\right)\frac{\varepsilon_0 E\mu}{2}\frac{1}{\beta_1^2}\right)^j \\
&\times\left((s-j+1)\left(\frac{2e}{\rho\sigma}E\right)^{s-j}\left(\frac{\varepsilon_0 E\mu^{s-j}}{2}\frac{1}{\beta_1^{s-j}}+\|\langle\omega^{(0,s-j)},p\rangle\|_1\right)\right)\ .
\end{aligned}$$

Here also, for $s>2$ the last addend is still unknown, while for $s=2$ we have

$$\begin{aligned}
\|f_1^{(1,2)}\|_{1-d_1} \le{}& 3\left(\frac{2e}{\rho\sigma}E\right)^2\frac{\varepsilon_0 E\mu^2}{2}\frac{1}{\beta_1^2} \\
&+\left(\frac{2e}{\rho\sigma}\right)\left(2\left(\frac{2e}{\rho\sigma}E\right)\right)\left(\frac{\varepsilon_0 E\mu}{2}\frac{1}{\beta_1^2}\right)\left(\|f_1^{(\mathrm{I};0,1)}\|_{1-\delta_1}+\|\langle\omega^{(0,1)},p\rangle\|_1\right) \\
\le{}& 5\left(\frac{4e}{\rho\sigma}E\right)^3\frac{\varepsilon_0 E\mu^2}{2}\frac{1}{\beta_1^3} \\
\le{}& \frac{\varepsilon_0 E\mu^2 M^3}{2}\frac{T_{1,2}^2}{\beta_1}\nu_{1,2}\ .
\end{aligned}$$

Coming to $\ell \geq 2$, we have

$$\begin{aligned}
\|f_\ell^{(1,s)}\|_{1-d_1} &\leq \sum_{j=0}^{s} \left(\frac{2e}{\rho\sigma}\right)^j \left(2\left(\frac{2e}{\rho\sigma}E\right)\frac{\varepsilon_0 E\mu}{2}\frac{1}{\beta_1^2}\right)^j \\
&\qquad \times (s-j+1)\left(\frac{2e}{\rho\sigma}E\right)^{s-j}\frac{\varepsilon_0 E\mu^{s-j}}{2^\ell}\frac{1}{\beta_1^{s-j}} \\
&\leq \left(\frac{2e}{\rho\sigma}E\right)^{2s}\frac{\varepsilon_0 E\mu^s}{2^\ell}\frac{1}{\beta_1^{2s}}\sum_{j=0}^{s}(s-j+1) \\
&\leq \frac{\varepsilon_0 E\mu^s M^{2s}}{2^\ell}\frac{T_{1,s}^2}{\beta_1^2}\nu_{1,s}\ .
\end{aligned}$$

This complete the inductive basis, we can now consider the inductive step from $r-1$ to r.

The bound for the generating function $\chi_0^{(r)}$ is trivial,

$$\|\chi_0^{(r)}\|_{1-d_{r-1}} \leq \frac{\|f_0^{(r-1,r)}\|_{1-d_{r-1}}}{\alpha_r} \leq \frac{\varepsilon_0 E\mu^r M^{2r-2}T_{r-1,r}^2\nu_{r-1,r}}{\alpha_r}\ ,$$

this proves the first of (4.27).

For $\ell = 0$ and $s \geq 2$, we have

$$\begin{aligned}
\|f_0^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} &\leq \sum_{j=0}^{\lfloor s/r\rfloor}\left(\frac{2e}{\rho\sigma}\right)^j\left(\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\right)^j\|f_j^{(r-1,s-jr)}\|_{1-d_{r-1}} \\
&\qquad + \frac{2e}{\rho\sigma}\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\|\langle\omega^{(r-1,s-r)},p\rangle\|_{1-d_{r-1}} \\
&\leq \|f_0^{(r-1,s)}\|_{1-d_{r-1}} + \left(\frac{2e}{\rho\sigma}\right)\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\|f_1^{(r-1,s-r)}\|_{1-d_{r-1}} \\
&\qquad + \sum_{j=2}^{\lfloor s/r\rfloor}\left(\frac{2e}{\rho\sigma}\right)^j\left(\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\right)^j\|f_j^{(r-1,s-jr)}\|_{1-d_{r-1}} \\
&\qquad + \frac{2e}{\rho\sigma}\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\|\langle\omega^{(r-1,s-r)},p\rangle\|_{1-d_{r-1}}\ ,
\end{aligned}$$

Here, for $s > 2r$ the last addend is still unknown and will be determined in the following normalization steps. Instead, for $s < 2r$ the last inequality is bounded by

$$\begin{aligned}
&\|f_0^{(r-1,s)}\|_{1-d_{r-1}} + \left(\frac{2e}{\rho\sigma}\right)\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\|f_1^{(r-1,s-r)}\|_{1-d_{r-1}} \\
&\qquad + \sum_{j=2}^{\lfloor s/r\rfloor}\left(\frac{2e}{\rho\sigma}\right)^j\left(\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\right)^j\|f_j^{(r-1,s-jr)}\|_{1-d_{r-1}} \\
&\qquad + \left(\frac{2e}{\rho\sigma}\right)\varepsilon_0 E\mu^r M^{2r-2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r-1,r}\frac{\varepsilon_0 E\mu^{s-r}M^{2(s-r)-1}}{2}\frac{T_{r-1,s-r}^2}{\beta_{\min\{r-1,s-r\}}}\nu_{r-1,s-r} \\
&\leq \varepsilon_0 E\mu^s M^{2s-2}T_{r,s}^2\nu_{r,s}^{(\mathrm{I})}\ ,
\end{aligned}$$

The bound for the norms of $\|f_\ell^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r}$ with $\ell \geq 1$ only requires minor modifications, thus we only report hereafter the corresponding estimates, i.e.,

$$\begin{aligned}
\|f_1^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} &\leq \frac{\varepsilon_0 E\mu^s M^{2s-1}}{2}\frac{T_{r,s}^2}{\beta_r}\nu_{r,s}^{(\mathrm{I})}\ , \\
\|f_{\ell\geq 2}^{(\mathrm{I};r-1,s)}\|_{1-d_{r-1}-\delta_r} &\leq \frac{\varepsilon_0 E\mu^s M^{2s}}{2^\ell}\frac{T_{r,s}^2}{\beta_{\min\{r,s\}}^2}\nu_{r,s}^{(\mathrm{I})}\ ,
\end{aligned}$$

where the first bound is valid for $s \le r$, since the higher order detunings are still unknown.

The norm of the generating function $\chi_1^{(r)}$ can be bounded as

$$\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} \le \frac{\|f_1^{(\mathrm{I};r-1,r)}\|_{1-\delta_{r-1}}}{\alpha_r} \le \frac{\varepsilon_0 E\mu^r M^{2r-1}}{2}\frac{T_{r-1,r}^2}{\alpha_r\beta_r}\nu_{r,r}^{(\mathrm{I})} ,$$

and the one of the quantity $\langle\omega^{(r-1,r)}, p\rangle$ as

$$\|\langle\omega^{(r-1,r)}, p\rangle\|_{1-d_{r-1}} \le \|f_1^{(r-1,r)}\|_{1-d_{r-1}} \le \frac{\varepsilon_0 E\mu^r M^{2r-1}}{2}\frac{T_{r-1,r}^2}{\beta_{r-1}}\nu_{r-1,r} ,$$

this proves (4.27).

Having determined the norm of $\langle\omega^{(r-1,r)}, p\rangle$, we are now able to bound also $\|f_0^{(\mathrm{I};r-1,2r)}\|_{1-d_{r-1}-\delta_r}$.

We can now complete the proof, bounding the terms appearing in $H^{(r)} = \exp(L_{\chi_1^{(r)}})H^{(\mathrm{I};r-1)}$. For $\ell = 0$ and $s \le 2r$, we have

$$\begin{aligned}
\|f_0^{(r,s)}\|_{1-d_r} &\le \sum_{j=0}^{\lfloor s/r\rfloor}\left(\frac{2e}{\rho\sigma}\right)^j\left(\frac{\varepsilon_0 E\mu^r M^{2r-1}}{2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r,r}^{(\mathrm{I})}\right)^j \varepsilon_0 E\mu^{s-jr}M^{2(s-jr)-2}T_{r,s-jr}^2\nu_{r,s-jr}^{(\mathrm{I})}\\
&\le \varepsilon_0 E\mu^s M^{2s-2}T_{r,s}^2\nu_{r,s} .
\end{aligned}$$

where we make use of Lemma 4.4.5 to get rid of the β_r.

For $\ell = 1$ and $s \le 2r$, we have

$$\begin{aligned}
\|f_1^{(r,s)}\|_{1-d_r} \le &\sum_{j=0}^{\lfloor s/r\rfloor}\left(\frac{2e}{\rho\sigma}\right)^j\left(\frac{\varepsilon_0 E\mu^r M^{2r-1}}{2}\frac{T_{r-1,r}^2}{\beta_r}\nu_{r,r}^{(\mathrm{I})}\right)^j\\
&\times\left(\frac{\varepsilon_0 E\mu^{s-jr}M^{2(s-jr)-1}}{2^\ell}\frac{T_{r,s-jr}^2}{\beta_r}\nu_{r,s-jr}^{(\mathrm{I})} + \frac{\varepsilon_0 E\mu^r M^{2(s-jr)-1}}{2}\frac{T_{r-1,s-jr}^2}{\beta_{\min\{r-1,s-jr\}}}\nu_{r-1,s-jr}\right)\\
\le &\frac{\varepsilon_0 E\mu^s M^{2s-1}}{2}\frac{T_{r,s}^2}{\beta_r}\nu_{r,s} .
\end{aligned}$$

where again we make use of Lemma 4.4.5 to get rid of the β_r and the definition of M in (4.25) so as to take into account the contribute of the detunings.

The remaining cases only require minor modifications.

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