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EQUILIBRIUM, SYSTEMIC RISK MEASURES AND OPTIMAL TRANSPORT: A CONVEX DUALITY APPROACH

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Abstract

This Thesis focuses on two main topics. Firstly, we introduce and analyze the novel concept of Systemic Optimal Risk Transfer Equilibrium (SORTE), and we progressively generalize it (i) to a multivariate setup and (ii) to a dynamic (conditional) setting. Additionally we investigate its relation to a recently introduced concept of Systemic Risk Measures. We present Conditional Systemic Risk Measures and study their properties, dual representation and possible interpretations of the associated allocations as equilibria in the sense of SORTE. On a parallel line of work, we develop a duality for the Entropy Martingale Optimal Transport problem and provide applications to problems of nonlinear pricing-hedging. The mathematical techniques we exploit are mainly borrowed from functional and convex analysis, as well as probability theory. More specifically, apart from a wide range of classical results from functional analysis, we extensively rely on Fenchel-Moreau-Rockafellar type conjugacy results, Minimax Theorems, theory of Orlicz spaces, compactness results in the spirit of Komlós Theorem. At the same time, mathematical results concerning utility maximization theory (existence of optima for primal and dual problems, just to mention an example) and optimal transport theory are widely exploited.

Contents

Introduction

This Thesis takes its beginnings from a quite natural question: given the notion of Systemic Risk Measures (SRM) of Biagini et al. (2020) [20], what can we say from the point of view of equilibrium? In [20] the authors study the following problem defining SRM via scenario-dependent allocations:

$$X \mapsto \rho(X) := \inf\left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathcal{C}, \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] \geq B\right\}$$

where $\mathcal{C}$ stands for a set of feasible random allocations such that for every $Y \in \mathcal{C}$ it holds that $\sum_{j=1}^{N} Y^j$ is deterministic, even though each Y^j is a random variable. One of the key findings in [20] is that for every initial datum X there exists a vector of probability measures $\mathbb{Q}_X$ such that

$$\rho(X) = \rho^{\mathbb{Q}_X}(X) := \inf\left\{\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}_X^j}[Y^j] \mid Y \text{ satisfies } \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] \geq B\right\}.$$

We might infer that each feasible allocation Y has a natural, initial-time deterministic counterpart $[\mathbb{E}_{\mathbb{Q}_X^j}[Y^j]]_{j=1}^{N}$ one might use to build the capital $\rho(X)$. Consequently, SRM in [20] are set in fact in a uniperiodal framework in which exchanges take place both at initial and at terminal time. Inspired by the setup of [20] we consider a system of agents, each with an initial risky endowment, interacting in a uniperiodal environment and having at disposal some exogenously assigned amount A. We design an equilibrium concept for the following game: the agents share deterministic amounts at initial time (similarly to the procedure of constructing the total amount securing the system in [20]) in such a way that the total amount exchanged in the system is equal to the given A, and at terminal time a second, scenario dependent reallocation takes place. In order to properly define an equilibrium, we clearly need a criterion for optimality. We adopt the utility maximization approach. Each set of initial time deterministic allocations (namely a vector $a \in \mathbb{R}^N$ with $\sum_{j=1}^{N} a^j = A$) acts as set of budget constraints, given "pricing functionals" $[p^1, \dots, p^N]$, for the utility maximization of each agent (who optimizes over terminal time random allocations, at first ignoring the actions of other players), that is we consider the problems $U_j^{p^j}(a^j) = \sup\left\{\mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right] \mid p^j(Y^j) \leq a^j\right\}$. This produces indirect utilities for each player, depending on the particular initial exchanges. Aggregating (i.e. summing, in a utilitarian approach) these indirect utilities, we get a value function for a second optimization problem over deterministic exchanges: $a \mapsto \sum_{j=1}^{N} U_j^{p^j}(a^j)$. An equilibrium for such a game will consist then of three ingredients: initial time allocations $[a_X^1, \dots, a_X^N]$ maximizing the aggregate indirect utility $\sum_{j=1}^N U_j^{p_X^j}(a^j)$, terminal time allocations $[Y_X^1, \dots, Y_X^N]$ each maximizing the indirect utility of single agents $U_j^{p_X^j}(a_X^j)$, "pricing functionals"$[p_X^1, \dots, p_X^N]$ linking them via $a_X^j = p_X^j(Y_X^j)$. This essentially explains the genesis of the concept of Systemic Optimal Risk Transfer Equilibrium (SORTE), to be introduced more in detail in Section I.1.

Given this initial step, our research branches out into two main directions. Firstly, we consider a more explicit interaction between agents from the point of view preferences, allowing for indirect utility of single agents to depend on the actions of other agents. Using multivariate utility functions, we generalize the concept of SORTE to the one of Multivariate SORTE and study the consequences of the new interactions from the point of view of Nash equilibria. An introductory motivation for this step can be found in Section I.2. Secondly (see Section I.3), dynamics are introduced both for Systemic Risk Measures and (Multivariate) SORTE using as customary in the literature conditional expectations. We also explicitly link optimal allocations for Systemic Risk Measures to (Multivariate) SORTE.

Convex duality methods and functional analysis results used in the research described above are also applied in the last part of the Thesis, where we turn our attention to a different topic. We study Entropy Martingale Optimal Transport, a relaxation of the classical Martingale Optimal Transport problem of Beiglböck et al. [14], in which the marginal constraints are replaced by general penalization terms. The key point in our analysis is inspired by the work of Liero et al. [108]: the authors consider a generalization of the classical transport problem in which the usual infimum, over probability measures with given marginals $\mathbb{P}_1, \mathbb{P}_2$, of a cost functional in integral form, namely $\inf\left\{\int_{K_1\times K_2} c\,\mathrm{d}\mu \mid \mu_1 = \mathbb{P}_1, \mu_2 = \mathbb{P}_2\right\}$ is replaced by the more general problem

$$\inf_{\mu\in\mathrm{Meas}(K_1\times K_2)}\left(\int_{K_1\times K_2} c\,\mathrm{d}\mu + \mathcal{D}_1(\mu_1) + \mathcal{D}_2(\mu_2)\right).$$

The infimum is taken over all measures, but penalization terms for the marginals (i.e. $\mathcal{D}_1, \mathcal{D}_2$) appear. The classical optimal transport problem admits under suitable assumptions the duality

$$\inf\left\{\int_{K_1\times K_2} c\,\mathrm{d}\mu \mid \mu_1 = \mathbb{P}_1, \mu_2 = \mathbb{P}_2\right\} = \sup_{\varphi+\psi\le c}\left(S_1(\varphi) + S_2(\psi)\right)$$

where $S_1(\cdot) = \mathbb{E}_{\mathbb{P}_1}[\cdot], S_2(\cdot) = \mathbb{E}_{\mathbb{P}_2}[\cdot]$. The duality we develop can be applied to obtain a nonlinear pricing-hedging duality, in the spirit of [14]: as one might intuitively guess, the expectations under the given marginals in the classical dual problem will be replaced by more general (and possibly nonlinear) valuation functionals S_1, S_2. Further explanations are deferred to Section I.4.

I.1 Systemic Optimal Risk Transfer Equilibrium

In Chapter 1 we introduce the concept of Systemic Optimal Risk Transfer Equilibrium, denoted by SORTE, that conjugates the classical Bühlmann's notion of an equilibrium risk exchange with capital allocation based on systemic expected utility optimization.

The capital allocation and risk sharing equilibrium that we consider can be applied to many contexts, such as: equilibrium among financial institutions, agents, or countries; insurance and reinsurance markets; capital allocation among business units of a single firm; wealth allocation among investors.
To fix terminology, we will refer to a participant in these problems (either financial institution or firms or countries) as an **agent**; the class consisting of these N agents as the **system**; the individual risk of the agents (or the random endowment or future profit and loss) as the **risk vector** $X := [X^1, ..., X^N]$; the amount $Y := [Y^1, ..., Y^N]$ that can be exchanged among the agents as random **allocation**. We will generically refer to a central regulator authority, or CCP, or executive manager as a **central bank** (CB). In a one period framework, we consider N agents, each one characterized by a concave, strictly monotone utility function $u_j : \mathbb{R} \to \mathbb{R}$ and by the original risk $X^j \in L^0(\Omega, \mathcal{F}, \mathbb{P})$, for $j = 1, ..., N$. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is the vector space of real valued $\mathcal{F}$-measurable random variables. The sigma algebra $\mathcal{F}$ represents all possible measurable events at the final time T. $\mathbb{E}_{\mathbb{P}}[\cdot]$ denotes the expectation under $\mathbb{P}$. Given another probability measure $\mathbb{Q}$, $\mathbb{E}_{\mathbb{Q}}[\cdot]$ denotes the expectation under $\mathbb{Q}$. For the sake of simplicity we are assuming zero interest rate.
We present now the main concepts of our approach and leave the details and the mathematical rigorous presentation to Chapter 1. In order to adequately present the core problem, we need to shortly elaborate on Bühlmann's risk exchange equilibrium and Systemic Optimal (deterministic) Allocation.

1. **Bühlmann's risk exchange equilibrium**

   We recall Bühlmann's definition of a risk exchange equilibrium in a pure exchange economy (or in a reinsurance market). The initial wealth of agent n is denoted by $x^j \in \mathbb{R}$ and the variable X^j represents the original risk of this agent. In this economy each agent is allowed to exchange risk with the other agents. Each agent has to agree to receive (if positive) or to provide (if negative) the amount $\widetilde{Y}^j(\omega)$ at the final time in exchange of the amount $\mathbb{E}_{\mathbb{Q}}[\widetilde{Y}^j]$ paid (if positive) or received (if negative) at the initial time, where $\mathbb{Q}$ is some pricing probability measure. Hence $\widetilde{Y}^j$ is a time T measurable random variable. In order that at the final time this risk sharing procedure is indeed possible, the exchange variables $\widetilde{Y}^j$ have to satisfy the *clearing condition*

$$\sum_{j=1}^{N} \widetilde{Y}^j = 0 \quad \mathbb{P} - \text{a.s.}$$

   As in Bühlmann (1980) [32] and (1984) [33], we say that a pair $(\widetilde{Y}_X, \mathbb{Q}_X)$ is a **risk exchange equilibrium** if:

   (a) for each j, $\widetilde{Y}^j_X$ maximizes: $\mathbb{E}_{\mathbb{P}}\left[u_j(x^j + X^j + \widetilde{Y}^j - \mathbb{E}_{\mathbb{Q}_X}[\widetilde{Y}^j])\right]$ among all variables $\widetilde{Y}^j$,
   (b) $\sum_{j=1}^N \widetilde{Y}^j_X = 0$ $\mathbb{P}$−a.s.
   It is clear that only for some particular choice of the equilibrium pricing measure $\mathbb{Q}_X$, the optimal solutions $\widetilde{Y}^j_X$ to the problems in (a) will also satisfy the condition in (b).

In addition it is evident that the clearing condition in (b) requires that all agents accept to exchange the amount $\widetilde{Y}^j_X(\omega)$ at the final time T.

Define

$$\mathcal{C}_{\mathbb{R}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N \mid \sum_{j=1}^N Y^j \in \mathbb{R} \right\} \tag{I.1}$$

that is, $\mathcal{C}_{\mathbb{R}}$ is the set of random vectors such that the sum of the components is $\mathbb{P}$-a.s. a deterministic number.

Observe that with the change of notations $Y^j := x^j + \widetilde{Y}^j - \mathbb{E}_{\mathbb{Q}_X}[\widetilde{Y}^j]$, we obtain variables with $\mathbb{E}_{\mathbb{Q}_X}[Y^j] = x^j$ for each n, and an optimal solution Y^j_X still belonging to $\mathcal{C}_{\mathbb{R}}$ and satisfying

$$\sum_{j=1}^N Y^j_X = \sum_{j=1}^N x^j \quad \mathbb{P} - \text{a.s.} \tag{I.2}$$

As can be easily checked

$$\sup_{\widetilde{Y}^j} \mathbb{E}_{\mathbb{P}} \left[u_j(x^j + X^j + \widetilde{Y}^j - \mathbb{E}_{\mathbb{Q}_X}[\widetilde{Y}^j]) \right] = \sup_{Y^j} \left\{ \mathbb{E}_{\mathbb{P}} \left[u_j(X^j + Y^j) \right] \mid \mathbb{E}_{\mathbb{Q}_X}[Y^j] \leq x^j \right\}.$$

Hence the two above conditions in the definition of a risk exchange equilibrium may be equivalently reformulated as

(a') for each j, Y^j_X maximizes $\mathbb{E}_{\mathbb{P}} [u_j(X^j + Y^j)]$ among all variables satisfying $\mathbb{E}_{\mathbb{Q}_X}[Y^j] \leq x^j$,

(b') $Y_X \in \mathcal{C}_{\mathbb{R}}$ and $\sum_{j=1}^N Y^j_X = \sum_{j=1}^N x^j$ $\mathbb{P}$-a.s.

We remark that here the quantity $x^j \in \mathbb{R}$ is preassigned to each agent.

2. **Systemic Optimal (deterministic) Allocation**

To simplify the presentation, we now suppose that the initial wealth of each agent is already absorbed in the notation X^j, so that X^j represents the initial wealth plus the original risk of agent j. We assume that the system has at disposal a total amount of capital $A \in \mathbb{R}$ to be used at a later time in case of necessity. This amount could have been assigned by the Central Bank, or could have been the result of previous trading in the system, or could have been collected ad hoc by the agents. The amount A could represent an insurance pot or a fund collected (as guarantee for future investments) in a community of homeowners. For further interpretation of A, see also the related discussion in Section 5.2 of Biagini et al. (2020) [20]. In any case, we consider the quantity A as exogenously determined. This amount is allocated among the agents in order to optimize the overall systemic satisfaction. If we denote with $a^j \in \mathbb{R}$ the cash received (if positive) or provided (if negative) by agent j, then the terminal time wealth at disposal of agent j will be $(X^j + a^j)$. The optimal vector $a_X \in \mathbb{R}^N$ could be determined according to the following aggregate time-T criterion

$$\sup \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[u_j(X^j + a^j) \right] \mid a \in \mathbb{R}^N \text{ s.t. } \sum_{j=1}^N a^j = A \right\}. \tag{I.3}$$

Note that each agent is not optimizing his own utility function. As the vector $a \in\mathbb{R}^N$ is deterministic, it is known at time $t = 0$ and therefore the agents have to agree to provide or receive money only at such initial time.

However, under the assumption that also at the final time the agents have confidence in the overall reliability of the other agents, one can combine the two approaches outlined in Items 1 and 2 above to further increase the optimal total expected systemic utility and simultaneously guarantee that each agent will optimize his/her own single expected utility, taking into consideration an aggregated budget constraint assigned by the system. Of course an alternative assumption to trustworthiness could be that the rules are enforced by the CB.
We denote with $\mathcal{L}^j \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ a space of admissible random variables and assume that $\mathcal{L}^j + \mathbb{R} = \mathcal{L}^j$. We will consider maps $p^j : \mathcal{L}^j \to \mathbb{R}$ that represent the pricing or cost functionals, one for each agent j. As we shall see, in some relevant cases, all agents will adopt the same functional $p^1 = ... = p^N$, which will then be interpreted as the equilibrium pricing functional, as in Bühlmann's setting above, where $p^j(\cdot) := \mathbb{E}_{\mathbb{Q}}[\cdot]$ for all j. However, we do not have to assume this a priori. Instead we require that the maps p^j satisfy for all $j = 1, ..., N$:
i) p^j is monotone increasing,
ii) $p^j(0) = 0$,
iii) $p^j(Y + c) = p^j(Y) + c$ for all $c \in \mathbb{R}$ and $Y \in \mathcal{L}^j$.
Such assumptions in particular imply $p^j(c) = c$ for all constants $c \in \mathbb{R}$. A relevant example of such functionals are

$$p^j(\cdot) := \mathbb{E}_{\mathbb{Q}^j}[\cdot]\,, \tag{I.4}$$

where $\mathbb{Q}^j$ are probability measures for $j = 1, ..., N$. Another example could be $p^j = -\rho^j$, for convex risk measures ρ^j.
Now we will apply both approaches, outlined in Items 1 and 2 above, to describe the concept of a Systemic Optimal Risk Transfer Equilibrium.

3. **Systemic Optimal Risk Transfer Equilibrium.**

   As explained in Item 1, given some amount a^j assigned to agent j, this agent may buy $\widetilde{Y}^j$ at the price $p^j(\widetilde{Y}^j)$ in order to optimize

$$\mathbb{E}_{\mathbb{P}}\left[u_j(a^j + X^j + \widetilde{Y}^j - p^j(\widetilde{Y}^j))\right].$$

   The pricing functionals p^j, $j = 1, ..., N$ have to be selected in such a way that the optimal solution verifies the clearing condition

$$\sum_{j=1}^N \widetilde{Y}^j = 0 \quad \mathbb{P} - \text{a.s.}$$

   However, as in Item 2, a^j is not exogenously assigned to each agent, but only the total amount A is at disposal of the whole system. Thus the optimal way to allocate A among the agents is given by the solution $(\widetilde{Y}_X, p_X, a_X)$ of the following problem:

$$\sup_{a\in\mathbb{R}^N}\left\{\sum_{j=1}^{N}\sup_{\widetilde{Y}^j}\left\{\mathbb{E}_{\mathbb{P}}\left[u_j(a^j+X^j+\widetilde{Y}^j-p_X^j(\widetilde{Y}^j))\right]\right\}\mid\sum_{j=1}^{N}a^j=A\right\}, \tag{I.5}$$

$$\sum_{j=1}^{N}\widetilde{Y}_X^j=0\ \ \mathbb{P}-\text{a.s.} \tag{I.6}$$

From (I.5) and (I.6) it easily follows that an optimal solution $(\widetilde{Y}_X^j, p_X^j, a_X^j)$ fulfills

$$\sum_{j=1}^{N}p_X^j(\widetilde{Y}_X^j)=0. \tag{I.7}$$

Furthermore, letting $Y^j := a^j + \widetilde{Y}^j - p_X^j(\widetilde{Y}^j)$, from the Cash Additivity of p_X^j we deduce $p_X^j(Y^j) = a^j + p_X^j(\widetilde{Y}^j) - p_X^j(\widetilde{Y}^j) = a^j$ and $\sum_{j=1}^N Y_X^j = \sum_{j=1}^N a^j + \sum_{j=1}^N \widetilde{Y}_X^j - \sum_{j=1}^N p_X^j(\widetilde{Y}_X^j) = \sum_{j=1}^N a^j$ and, as before, the above optimization problem can be reformulated as

$$\sup_{a\in\mathbb{R}^N}\left\{\sum_{j=1}^{N}\sup_{Y^j}\left\{\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Y^j)\right]\mid p_X^j(Y^j)\le a^j\right\}\mid\sum_{j=1}^{N}a^j=A\right\}, \tag{I.8}$$

$$\sum_{j=1}^{N}Y_X^j=A\ \ \mathbb{P}-\text{a.s.} \tag{I.9}$$

Analogously to (I.7) we have that a solution (Y_X^j, p_X^j, a_X^j) satisfies

$$\sum_{j=1}^{N}p_X^j(Y_X^j)=A$$

by (I.8) and (I.9).

The two optimal values in (I.5) and (I.8) coincide. We see that while each agent is behaving optimally according to his preferences, the budget constraint $p_X^j(Y^j) \le a^j$ are not a priori assigned, but are endogenously determined through an aggregate optimization problem. The optimal value a_X^j determines the optimal risk allocation of each agent. It will turn out that $a_X^j = p_X^j(Y_X^j)$. Obviously, the optimal value in (I.5) is greater than (or equal to) the optimal value in (I.3), which can be economically translated into the statement that *allowing for exchanges also at terminal time increases the systemic performance.*

In addition to the condition in (I.9), we introduce further possible constraints on the optimal solution, by requiring that

$$Y_X \in \mathcal{B}, \tag{I.10}$$

where $\mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$.

In Chapter 1, see Section 1.2.4, we formalize the above discussion and show the existence of the solution (Y^j_X, p^j_X, a^j_X) to (I.8), (I.9) and (I.10), which we call Systemic Optimal Risk Transfer Equilibrium (SORTE). We show that p^j_X can be chosen to be of the particular form $p^j_X(\cdot) := \mathbb{E}_{\mathbb{Q}^j_X}[\cdot]$, for a probability vector $\mathbb{Q}_X = [\mathbb{Q}^1_X, ..., \mathbb{Q}^N_X]$. The crucial step, Theorem 1.3.5, is the proof of the dual representation and the existence of the optimizer of the associated problem (1.14). The optimizer of the dual formulation provides the optimal probability vector $\mathbb{Q}_X$ that determines the functional $p^j_X(\cdot) := \mathbb{E}_{\mathbb{Q}^j_X}[\cdot]$. The characteristics of the optimal $\mathbb{Q}_X$ depend on the feasible allocation set $\mathcal{B}$. When no constraints are enforced, i.e., when $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$, then all the components of $\mathbb{Q}_X$ turn out to be equal. Hence we find that the implicit assumption of one single equilibrium pricing measure, made in the Bühlmann's framework, is in our theory a consequence of the particular selection $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$, but for general $\mathcal{B}$ this in not always the case. We emphasize that the existence of multiple equilibrium pricing measures $\mathbb{Q}_X = [\mathbb{Q}^1_X, ..., \mathbb{Q}^N_X]$ is a natural consequence of the presence of the - non trivial - constraints set $\mathcal{B}$, see Reamrk 1.2.9 and Example 1.2.10 for further details.

Bühlmann's equilibrium (Y_X) satisfies two relevant properties: *Pareto optimality* (there are no feasible allocation Y such that all agents are equal or better off - compared with Y_X - and at least one of them is better off) and *Individual Rationality* (each agent is better off with Y^j_X than without it). Any feasible allocation satisfying these two properties is called an *optimal risk sharing rule*, see Barrieu and El Karoui (2005) [9] or Jouini et al. (2007) [99].
We show that a SORTE is unique (once the class of pricing functionals is restricted to those in the form $p^j(\cdot) = \mathbb{E}_{\mathbb{Q}^j}[\cdot]$). We also prove Pareto optimality, see the Definition 1.2.1 and the exact formulation in Theorem 1.3.17.
However, a SORTE lacks Individual Rationality. This is shown in the toy example of Section 1.4.2, but it is also evident from the expression in equation (I.8). As already mentioned, each agent is performing rationally, maximizing her expected utility, but under a budget constraint $p^j_X(Y^j) \leq a^j_X$ that is determined globally via an additional systemic maximization problem ($\sup_{a\in\mathbb{R}^N}\{... \mid \sum_{j=1}^N a^j = A\}$) that assigns priority to the systemic performance, rather than to each individual agent. In the SORTE we replace individual rationality with such a *systemic induced individual rationality*, which also shows the difference between the concepts of SORTE and of an optimal risk sharing rule. We also point out that the participation in the risk sharing mechanism may be appropriately mitigated or enforced by the use of adequate sets $\mathcal{B}$, see e.g. Example 1.3.20 for risk sharing restricted to subsystems. From the technical point of view, we will not rely on any of the methods and results related to the notion of inf-convolution, which is a common tool to prove existence of optimal risk sharing rules (see for example [9] or [99]) in the case of monetary utility functions, as we do not require the utility functions to be cash additive. Our proofs are based on the dual approach to (systemic) utility maximization. This is summarized in Section 1.3.1. Furthermore, the exponential case is treated in detail in Section 1.4.

Remark I.1.1. We stress the fact that in (I.8) we might add positive weights $\gamma = [\gamma_1, ..., \gamma_N] \in \mathbb{R}^N$, substituting $u_j(\cdot)$ in (I.8) with $\gamma_j u_j(\cdot)$, $j = 1, \dots, N$. Section 1.4.3 is devoted to a detailed discussion of the non-utilitarian setup ($\gamma \neq 1$), with stability results in the case of exponential utility functions.

The work presented in Chapter 1 originates from the systemic risk approach developed in Biagini et al. (2019) [19] and (2020) [20]. The notion of a SORTE is inspired by the following utility maximization problem, associated to the risk minimization problem

$$\sup_{Y\in\mathcal{B}\subset\mathcal{C}_{\mathbb{R}}}\left\{\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}u_j(X^j+Y^j)\right]\mid\sum_{j=1}^{N}Y^j\leq A\right\},\ A\in\mathbb{R}, \tag{I.11}$$

that was also introduced in [20] and linked to the systemic risk approach adopted there. Related papers on Systemic Risk Measures are Acharya et al. (2016) [3], Armenti et al. (2018) [7], Chen et al. (2013) [42], Feinstein et al. (2017) [69], Kromer et al. (2016) [105]. For an exhaustive overview on the literature on systemic risk, see Fouque and Langsam (2013) [78] and Hurd (2016) [98].
For a review on Arrow-Debreu Equilibrium (see Debreu (1959) [50]; Mas Colell and Zame (1991) [109] for the infinite dimensional case) we refer to Section 3.6 of Föllmer and Schied (2016) [77], which is close to our setup. In the spirit of the Arrow-Debreu Equilibrium, Bühlmann (1980) [32] and (1984) [33] proved the existence of risk exchange equilibria in a pure exchange economy. Such risk sharing equilibria had been studied in different forms starting from the seminal papers of Borch (1962) [26], where Pareto-optimal allocations were proved to be comonotonic for concave utility functions, and Bühlmann and Jewell (1979) [34]. The differences with Bühlmann's setup and our approach have been highlighted before in detail.
In Barrieu and El Karoui (2005) [9] inf-convolution of convex Risk Measures has been introduced as a fundamental tool for studying risk sharing. Existence of optimal risk sharing for law-determined monetary utility functions is obtained in Jouini et al. (2008) [99] and then generalized to the case of non-monotone Risk Measures by Acciaio (2007) [1] and Filipović and Svindland (2008) [73], to multivariate risks by Carlier and Dana (2013) [40] and Carlier et al. (2012)[41], to cash-subadditive and quasi-convex Risk Measures by Mastrogiacomo and Rosazza Gianin (2015) [110]. Further works on risk sharing are also Dana and Le Van (2010) [48], Heath and Ku (2004) [87], Tsanakas (2009) [127], Weber (2018) [128]. Risk sharing problems with quantile-based Risk Measures are studied in Embrechts et al. (2018) [68] by explicit construction, and in (2020) [67] for heterogeneous beliefs. In Filipović and Kupper (2008) [70] Capital and Risk Transfer is modeled as (deterministically determined) redistribution of capital and risk by means of a finite set of non deterministic financial instruments. Existence issues are studied and related concepts of equilibrium are introduced. Recent further extensions have been obtained in Liebrich and Svindland (2018) [107].

The SORTE concept was analyzed, jointly with F. Biagini, J.-P. Fouque, M. Frittelli and T. Meyer-Brandis, in [18] (2021).

I.2 Multivariate Systemic Optimal Risk Transfer Equilibrium

We proceed in extending the notion of SORTE to the case when the value function to be optimized has two components, one being the sum of the single agents' utility functions, as in the aforementioned case of SORTE, the other consisting of a truly systemic component. This marks the progress from SORTE to Multivariate Systemic Optimal Risk Transfer Equilibrium (mSORTE). In Chapter 2 we will consider multivariate utility functions $U : \mathbb{R}^N \to \mathbb{R}$ of the form

$$U(x) := \sum_{j=1}^{N} u_j(x^j) + \Lambda(x) \tag{I.12}$$

where $u_1, \dots, u_j : \mathbb{R} \to \mathbb{R}$ are (univariate) utility functions and

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

is a (not necessarily strictly) concave, increasing function on $\mathbb{R}^N$ that is bounded from above. Using the additional aggregative term Λ we can model the fact that the choices of each single agent in the system depend not only on his/her individual preferences, but also on others agents' behavior.
Before moving to the discussion of the generalization of SORTE to the multivariate case, we believe it is instructive to look back at the arguments that motivated the introduction of SORTE and see how these could be generalized under multivariate utility. More specifically, we consider possible multivariate generalizations for Bühlmann's risk exchange equilibrium and for the Systemic Optimal Allocation problem.

1. **Bühlmann's risk exchange equilibrium** We find it more comfortable to look at the formulation in (a') and (b'). Condition (b') does not depend on the particular choice of utilities, hence we will focus on (a'). We observe that the condition (a') can be rewritten as follows:

   (a'_1) the $j-$th allocation Y^j_X, *given all other agents' positions*

   $$Y^{[-j]}_X = [Y^1_X, \dots, Y^{j-1}_X, Y^{j+1}_X, \dots, Y^N_X]$$

   maximizes the function (see Equation (2.15))

   $$Z \mapsto \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Z)\right] + \mathbb{E}_{\mathbb{P}}\left[\Lambda(X + [Y^{[-j]}_X; Z])\right]$$

   where $[Y^{[-j]}; Z] := \left[Y^1, \dots, Y^{j-1}, Z, Y^{j+1}, \dots, Y^N\right]$, for $\Lambda \equiv 0$, among all scalar variables Z satisfying $\mathbb{E}_{\mathbb{Q}_X}[Z] \le x^j$;

   (a'_2) Y_X maximizes $Z \mapsto \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Z^j)\right] + \mathbb{E}_{\mathbb{P}}\left[\Lambda(X + Z)\right]$, for $\Lambda \equiv 0$, among all vector variables Z with $\mathbb{E}_{\mathbb{Q}_X}[Z^j] \le x^j$ for every $j = 1, \dots, N$.

   It is now natural to "turn on"the aggregation term Λ in (a'_1) and (a'_2) to get a multivariate extension, observing that whenever we drop the assumption $\Lambda \equiv 0$ the two conditions (a'_1) and (a'_2) are not equivalent anymore. We then say that a pair $(Y_X, \mathbb{Q}_X)$ is a **weak multivariate risk exchange equilibrium** (resp. **multivariate risk exchange equilibrium**) if conditions (a'_1) (resp. (a'_2)) and (b') are met, with the suppression of the condition $\Lambda \equiv 0$.

2. **Systemic Optimal (deterministic) Allocation**

   The generalization to the multivariate case is very natural: the optimal vector $a_X \in \mathbb{R}^N$ can be determined according to the following aggregate terminal time criterion

$$\sup\left\{\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+a^j)\right]+\mathbb{E}_{\mathbb{P}}\left[\Lambda(X+a)\right] \mid a\in\mathbb{R}^N \text{ s.t. } \sum_{j=1}^{N}a^j=A\right\}. \quad \text{(I.13)}$$

We may now proceed following conceptually Section I.1 in introducing the natural generalization of SORTE to the multivariate setup. As explained in detail in Section 2.4.1, and due to the presence of the alternative conditions (a_1') and (a_2'), there are two (a priori non equivalent) paths to follow.

The first approach considers the most natural counterpart of the definition of SORTE (Definition 1.2.7) in the multivariate setup, which leads to the definition of Weak Multivariate SORTE (Definition 2.4.1), and is conceptually related to the weak multivariate risk exchange equilibrium.

The second one is motivated by the formulation (I.8) of the SORTE, and yields the concept of Multivariate SORTE. As can be easily verified, a Multivariate SORTE turns out to be in particular a Weak Multivariate SORTE and we will mostly focus our attention on the stronger concept.

To be more precise, we will define (Section 2.4.1) a Multivariate Systemic Optimal Risk Transfer Equilibrium (mSORTE) as a triple $(Y_X, \mathbb{Q}_X, a_X)$ such that $Y_X = [Y_X^1, ..., Y_X^N]$ satisfies (I.9) and (I.10), and (Y_X, a_X) is an optimum for

$$\sup_{a\in\mathbb{R}^N}\left\{\sup_{Y}\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j\right]\leq a^j \ \forall j=1,\dots,N\right\} \mid \sum_{n=1}^{N}a^n=A\right\} \quad \text{(I.14)}$$

where $U(\cdot)$ is defined in (I.12). We emphasize the fact that the setup and results for SORTE in Chapter 1 can be recovered from the ones we are to present in Chapter 2 by setting $\Lambda \equiv 0$. As explained in Section 2.4.3, we prove existence, uniqueness, Pareto optimality of an mSORTE under three different setups of assumptions. A detailed study of these assumptions is collected in Section 2.4.5. In Section 2.4.6 we also compare such assumptions with the one considered in Chapter 1. We stress here that such assumptions are reasonably weak and weaker than those assumed in Chapter 1. Just to mention a few examples, any of the following multivariate utility functions satisfies our assumptions:

$$U(x) := \sum_{j=1}^{N}u_j(x^j)+u\left(\sum_{j=1}^{N}\beta_j x^j\right), \quad \text{with } \beta_j\geq 0, \text{ for all } j, \quad \text{(I.15)}$$

where $u:\mathbb{R}\to\mathbb{R}$, for some $p>1$, is any one of the following functions:

$$u_{\exp}(x) := 1-\exp(-px); \quad u_p(x) = \begin{cases} p\,\frac{x}{x+1} & x\geq 0 \\ 1-|x-1|^p & x<0 \end{cases};$$

$$u_{\mathrm{atan}}(x) = \begin{cases} p\,\arctan(x) & x\geq 0 \\ 1-|x-1|^p & x<0 \end{cases};$$

and $u_1,\dots,u_N$ are exponential utility functions ($u_j(x^j)=1-\exp(-\alpha_j x^j)$, $\alpha>0$) for any choice of u as above, or $u_j(x^j)=u_{p_j}(x^j)$, $p_j>p$ for $u=u_p$ or $u=u_{\text{atan}}$. The function Λ could also be constructed as follows. Let $G:\mathbb{R}^N\to\mathbb{R}$ be convex, monotone decreasing and bounded from below, and $F:\mathbb{R}\to\mathbb{R}$ be concave and monotone decreasing on $range(G)$. Then $\Lambda:\mathbb{R}^N\to\mathbb{R}$ defined by

$$\Lambda(x)=F(G(x)) \tag{I.16}$$

is concave, monotone increasing and bounded above by $F(\inf G)$. Notice that, as detailed in Section 2.3, we will require differentiability only in few circumstances. We here provide an example in which our assumptions are met, covering the non differentiable case. Take $\gamma_j\geq 0$, $j=1,\dots,N$, $G(x):=\sum_{j=1}^N\gamma_j(x^j-k^j)^-$ and take $F:\mathbb{R}\to\mathbb{R}$ defined by $F(x):=-x^\alpha$, $\alpha\geq 1$, which is concave and monotone decreasing on $range(G)=[0,\infty)$. Then

$$\Lambda(x):=-\left(\sum_{j=1}^N\gamma_j(x^j-k^j)^-\right)^\alpha \tag{I.17}$$

is concave, monotone increasing and bounded above by 0, and $U(x):=\sum_{j=1}^N u_j(x^j)+\Lambda(x)$, with $u_1,\dots,u_N$ exponential utility functions and Λ assigned in (I.17), satisfies our assumptions.

Quite remarkably, this generalization of a SORTE allows us to introduce and to study a Nash Equilibrium property for an mSORTE, as shown in Section 2.4.3. We prove that, in addition to being Pareto optimal, the component Y_X of an mSORTE is a Nash Equilibrium (see Theorem 2.4.11 and Theorem 2.4.12). We point out that, in interpreting the component Y_X as Nash Equilibrium, we are considering that each agent's value function is not simply given by its expected (univariate) utility. In fact, we require that the $j-$th agent, *given all other agents' positions* $Y^{[-j]}=[Y^1,\dots,Y^{j-1},Y^{j+1},\dots,Y^N]$, optimizes the function (see Equation (2.15))

$$Z\mapsto U_j^{Y^{[-j]}}(Z):=\mathbb{E}\left[u_j(X^j+Z)\right]+\mathbb{E}\left[\Lambda(X+[Y^{[-j]};Z])\right]$$

where $[Y^{[-j]};Z]:=\left[Y^1,\dots,Y^{j-1},Z,Y^{j+1},\dots,Y^N\right]$.

From a technical perspective, our results can be considered as consequences of Theorem 2.4.9 and Theorem 2.4.10. The proof of Theorem 2.4.9, which is the most lengthy and complex, is split according to the Setups we work in. The proofs for Setup A and B (collected in Theorem 2.5.16) use a Komlós- type argument. This allows us to obtain existence of optimizers for both the primal and the dual problems without requiring differentiability of $U(\cdot)$, which is a rather unusual result in the literature. The one for Setup C instead (see Theorem 2.5.17) is somehow inspired by Theorem 1.3.5 in Chapter 1, and is based on a minimax argument. A duality result links the content of Theorems 2.4.9 and 2.4.10 yielding the existence result in Theorem 2.4.11. The uniqueness argument in Theorem 2.4.12 is inspired by the corresponding uniqueness result in Chapter 1 (Theorem 1.3.17). We also remark that, differently from the case of SORTE, we need to construct the dual system (M^Φ,K_Φ), where M^Φ is a multivariate Orlicz Heart having as topological dual space the Köthe dual K_Φ. Here, we denote with $\Phi : (\mathbb{R}_+)^N \to \mathbb{R}$ the multivariate Orlicz function $\Phi(y) := U(0) - U(-|y|)$ associated to the multivariate utility function. Details of this construction are provided in Sections 2.1 and 2.2.

As already mentioned, our study of mSORTE is a somehow natural prosecution of the one of SORTE. Thus, as far as the conceptual aspects are concerned, we refer to the all the literature reviewed in Section I.1 for extended comments. Here, we limit multivariate utility functions have been widely exploited in the study of optimal investment under transaction costs, and we cite Campi and Owen (2011) [38], Deelestra et al. (2001) [51], Kamizono (2004) [100], Bouchard and Pham (2005) [28] and references therein for more details on these functions and their study.

The results regarding mSORTE and its relation to SORTE are collected in Doldi and Frittelli (2019) [58].

I.3 Conditional Systemic Risk Measures

As mentioned before, the notions of SORTE and mSORTE were inspired by recent results on Systemic Risk Measures. It is then natural to investigate further the link between these concepts. In Chapter 3 we provide a natural extension of static Systemic Risk Measures to a dynamic, conditional setting. We then study related concepts of time consistency and link the SORTE and mSORTE to Conditional Shortfall Systemic Risk Measures.
To put our principal findings into perspective, we briefly review the literature pertaining to Systemic Risk Measures. We let $X = [X^1, \dots, X^N] \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ be a vector of N random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, representing a configuration of risky (financial) factors at a future time T associated to a system of N financial institutions/banks.
A traditional approach to evaluate the risk of each institution $j \in \{1, ...N\}$ is to apply a *univariate Monetary Risk Measure* η^j to the single financial position X^j, yielding $\eta^j(X^j)$. Let L be a subspace of $L^0(\Omega, \mathcal{F}, \mathbb{P})$. A Monetary Risk Measure (see [77]) is a map $\eta : L \to \mathbb{R}$ that can be interpreted as the minimal capital needed to secure a financial position with payoff $Z \in L$, i.e., the minimal amount $m \in \mathbb{R}$ that must be added to Z in order to make the resulting (discounted) payoff at time T acceptable

$$\eta(Z) := \inf\{m \in \mathbb{R} \mid Z + m \in \mathbb{A}\}, \tag{I.18}$$

where the acceptance set $\mathbb{A} \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to be monotone, i.e., $Z \geq Y \in \mathbb{A}$ implies $Z \in \mathbb{A}$. Then η is monotone decreasing and satisfies the Cash Additivity property

$$\eta(Z + m) = \eta(Z) - m, \text{ for all } m \in \mathbb{R} \text{ and } Z \in L. \tag{I.19}$$

Under the assumption that the set $\mathbb{A}$ is convex (resp. is a convex cone) the maps in (I.18) are convex (resp. convex and positively homogeneous) and are called *Convex* (resp. Coherent) *Risk Measures*, see Artzner et al. (1999) [8], Föllmer and Schied (2002) [76], Frittelli and Rosazza Gianin (2002) [83]. Once the risk $\eta^j(X^j)$ of each institution $j \in \{1, ...N\}$ has been determined, the quantity

$$\rho(X) := \sum_{j=1}^{N} \eta^j(X^j)$$

could be used as a very preliminary and naïf assessment of the risk of the entire system.

I.3.1 Static Systemic Risk Measures

The approach sketched above does not clearly capture systemic risk of an interconnected system, and the design of more adequate Risk Measures for financial systems is the topic of a vast literature on systemic risk. Let $L_{\mathcal{F}}$ be a vector subspace of $(L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$. A Systemic Risk Measure is a map $\rho : L_{\mathcal{F}} \to \mathbb{R}$ that evaluates the risk $\rho(X)$ of the complete system $X \in L_{\mathcal{F}}$ and satisfies additionally financially reasonable properties.

First aggregate then allocate. In Chen et al. (2013) [42] the authors investigated under which conditions a Systemic Risk Measure could be written in the form

$$\rho(X) = \eta(U(X)) = \inf\{m \in \mathbb{R} \mid U(X) + m \in \mathbb{A}\}, \tag{I.20}$$

for some univariate Monetary Risk Measure η and some aggregation rule

$$U : \mathbb{R}^N \to \mathbb{R}$$

that aggregates the N-dimensional risk factors into a univariate risk factor. We also refer to Kromer et al. (2016) [105] for extension to general probability space.
Such systemic risk might again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $U(X)$, given that $U(X)$ allows for a monetary loss interpretation. Note, however, that in (I.20) systemic risk is the minimal capital added to secure the system *after aggregating individual risks.*

First allocate then aggregate. A second approach consisted in measuring systemic risk as the minimal cash that secures the aggregated system by adding the capital into the single institutions *before aggregating their individual risks.* This way of measuring systemic risk can be expressed by

$$\rho(X) := \inf \left\{ \sum_{j=1}^{N} m^j \mid m = [m^1, \cdots, m^N] \in \mathbb{R}^N, \, U(X + m) \in \mathbb{A} \right\}. \tag{I.21}$$

Here, the amount m^j is added to the financial position X^j of institution $j \in \{1, \cdots, N\}$ before the corresponding total loss $U(X + m)$ is computed. We refer to Armenti et al. (2018) [7] and Biagini et al. (2019) [19] for a detailed study of Systemic Risk Measures in the form (I.21) and to Feinstein et al. (2017) [69] for a similar approach for set valued Risk Measures. Dual representations of Systemic Risk Measures based on acceptance sets have recently been studied in Arduca et al. (2019) [6].

Scenario dependent allocations. The "*first allocate and then aggregate*" approach was then extended in Biagini et al. (2019) [19] and (2020) [20] by adding to X not merely a vector $m = [m_1, \cdots, m_N] \in \mathbb{R}^N$ of deterministic amounts but, more generally, a random vector $Y \in \mathcal{C}$, for some given class $\mathcal{C}$. In particular, one main example considered in [20] is given by the class $\mathcal{C}$ such that

$$\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}, \text{ where } \mathcal{C}_{\mathbb{R}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N \mid \sum_{j=1}^{N} Y^j \in \mathbb{R} \right\}, \tag{I.22}$$

and $\mathcal{L}$ is a subspace of $(L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ representing possible additional integrability or boundedness requirements. The set $\mathcal{C}$ represents the class of feasible allocations, and it is possible to model additional constraints on the allocation $Y \in \mathcal{C}_{\mathbb{R}}$ by requiring $Y \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}$, with strict inclusion. It is assumed that $\mathbb{R}^N \subseteq \mathcal{C}$.
Under these premises the Systemic Risk Measure considered in [20] takes the form

$$\rho(X) := \inf \left\{ \sum_{j=1}^{N} Y^j \mid Y \in \mathcal{C},\ U(X+Y) \in \mathbb{A} \right\} \tag{I.23}$$

and can still be interpreted, since $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, as the minimal total cash amount $\sum_{j=1}^{N} Y^j \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector X. However, while the total capital requirement $\sum_{j=1}^{N} Y^j$ is determined today, contrary to (I.21) the individual allocation $Y^j(\omega)$ to institution j does not need to be decided today but in general depends on the scenario ω realized at time T. As explained in detail in [20], this total cash amount $\rho(X)$ can be composed today through the formula

$$\sum_{j=1}^{N} a^j(X) = \rho(X), \tag{I.24}$$

where each cash amount $a^j(X) \in \mathbb{R}$ can be interpreted as a *risk allocation* of bank j. The exact formula for the risk allocation $a^j(X)$ will be introduced later in (I.27).
We remark that by selecting $\mathcal{C} = \mathbb{R}^N$ in (I.23), one recovers the deterministic case (I.21); while when $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ no further requirements are imposed on the set of feasible allocations.

Shortfall Systemic Risk Measures. A special, relevant case of Systemic Risk Measures of the form (I.23) "*first allocate and then aggregate, with scenario dependent allocation*" is given by the class of Shortfall Systemic Risk Measures, where the acceptance set has the form $\mathbb{A} = \{Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathbb{E}_{\mathbb{P}}[Z] \geq B\}$ for a given constant $B \in \mathbb{R}$, namely:

$$\rho(X) := \inf \left\{ \sum_{j=1}^{N} Y^j \mid Y \in \mathcal{C}, \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \geq B \right\}. \tag{I.25}$$

For the financial motivation behind these choices and for a detailed study of this class of measures, we refer to [19] and [20] when $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, and to Armenti et al. (2018) [7]

for the analysis of such Risk Measures in the special case $\mathcal{C} = \mathbb{R}^N$, i.e. when only deterministic allocations are allowed.
The choice of the aggregation functions $U : \mathbb{R}^N \to \mathbb{R}$ is also a key ingredient in the construction of ρ and we refer to Acharia et al. (2016) [3], Adrian and Brunnermeier (2016) [4], Huang et al. (2009) [97], Lehar (2005) [106], Brunnermeier and Cheridito (2019) [31], Biagini et al. (2019) and (2020) [19], [20] for the many examples of aggregators adopted in literature. In order to obtain more specific and significant properties of ρ, [20] selected the aggregator

$$U(x) = \sum_{j=1}^{N} u_j(x^j), \quad x \in \mathbb{R}^N, \tag{I.26}$$

for strictly concave increasing utility functions $u_j : \mathbb{R} \to \mathbb{R}$, for each $j = 1, \cdots, N$.
Systemic Risk Measures can be applied not only to determine the overall risk $\rho(X)$ of the system, but also to establish the riskiness of each individual financial institution. As explained in [20] it is possible to determine the risk allocations $a^j(X) \in \mathbb{R}$ of each bank j that satisfy (I.24) and additional meaningful properties. It was there shown that, with the choice (I.26), a *fair risk allocation* of bank j is given by:

$$a^j(X) := \mathbb{E}_{\mathbb{Q}^j(X)}\left[Y^j(X)\right], \quad j = 1, \cdots, N, \tag{I.27}$$

where the vector $Y(X)$ is the optimizer in (I.25) and $\mathbb{Q}(X)$ is the vector of probability measures that optimizes the dual problem associated to $\rho(X)$.
We will adopt the generalization of the aggregation function (I.26) defined by (I.12), namely

$$U(x) = \sum_{j=1}^{N} u_j(x_j) + \Lambda(x), \quad x \in \mathbb{R}^N,$$

where the multivariate aggregator $\Lambda : \mathbb{R}^N \to \mathbb{R}$ is concave and increasing (not necessarily in a strict sense). Thus the selection $\Lambda \equiv 0$ is possible and hence, in this case, (I.12) reduces to (I.26). Similarly to the extension from SORTE to mSORTE, the addition of the term Λ allows for modeling interdependence among agents from the point of view of preferences.
As shown in Chapter 3, a fairness property for the risk allocation of each bank can be established also in a conditional setting and for the aggregator expressed by (I.12).

I.3.2 Conditional Systemic Risk Measures

The temporal setting in the approaches described above is static, meaning that the Risk Measures do not allow for possible dynamic elements, such as additional information, or the possibility of risk monitoring in continuous time, or the possibility of intermediate payoffs and valuation. In order to model the conditional setting we then assume that $\mathcal{G} \subseteq \mathcal{F}$ is a sub sigma algebra of $\mathcal{F}$ and we consider Risk Measures $\rho_{\mathcal{G}}$ with range in $L^0(\Omega, \mathcal{G}, \mathbb{P})$ and interpret $\rho_{\mathcal{G}}(X)$ as the risk of the whole system X given the information $\mathcal{G}$.
Conditional Risk Measures have mostly been studied in the framework of *univariate* Dynamic Risk Measures, where one adjusts the risk measurement in response to

the flow of information that is revealed when time elapses. Detlefsen and Scandolo (2005) [57] was one of the first contribution in the study of Conditional Convex Risk Measures and since then a vast literature appeared. We refer the reader to Barrieu and El Karoui (2005) [10] for a good overview on univariate Dynamic Risk Measures. We observe that such a conditional and dynamic framework generated a florilegium of interesting ramification in different fields, including the relationships with BSDEs (Barrieu and El Karoui (2005) [10], Rosazza Gianin (2006) [120], Bion-Nadal (2008) [25], Delbaen et al. (2010) [53]) and Non Linear Expectation (Peng (2004) [113]). Several results have also been obtained for the case of quasi-convex conditional maps and Risk Measures, see Frittelli and Maggis (2011) [80], (2014) [81] and (2014) [82]. A vast literature has focused on conditional counterparts to classical static results regarding dual representation and separation properties, using L^0-modules. Among the many contributions in this stream of research we mention Filipović et al. (2009) [71] and (2012) [72], Drapeau et al. (2016) [63], Drapeau et al. (2019) [64], Guo (2010) [86] and references therein. We will prefer here a more direct approach. We do not really need to rely on this kind of very abstract results, nor to extend them in a multidimensional framework required for our systemic perspective. Overall, the fact that natural conditional counterparts hold for static results is not so surprising. The two are intrinsically related by a Boolean Logic principle. As seen in Carl and Jamneshan (2018) [39] traditional theorems carry over to the conditional setup assuming that suitable concatenation properties hold.

A Conditional Systemic Risk Measure is a map $\rho_{\mathcal{G}} : L_{\mathcal{F}} \to L^0(\Omega, \mathcal{G}, \mathbb{P})$ that associates to a N-dimensional risk factor $X \in L_{\mathcal{F}} \subseteq (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N$ a $\mathcal{G}$-measurable random variable. A Conditional Systemic Risk Measure thus models the risk of a system as new information arises in the course of time. The study of Conditional Systemic (multivariate) Risk Measure was initiated by Hoffmann et al. (2016) [94] and (2018) [95]. However, as pointed out in [95], in the context of Systemic Risk Measures, a second interesting and important dimension of conditioning arises, besides dynamic conditioning: risk measurement conditional on information in space in order to identify systemic relevant structures. In that case $\mathcal{G}$ represents for example information on the state of a subsystem, see Föllmer (2014) [74] or Föllmer and Kluppelberg (2014) [75]. In order to allow for both interpretations, in Chapter 3 a general sigma algebra $\mathcal{G} \subseteq \mathcal{F}$ will be considered.
We finally observe that the papers [94] and [95] consider exclusively the conditional extension of (static) Systemic Risk Measures of the "first aggregate and then allocate" form expressed by (I.20).
Our interest in Chapter 3 is the study of general Conditional (convex) Systemic Risk Measures and the detailed analysis of Conditional Shortfall Systemic Risk Measures. Our findings show that most properties of Shortfall Systemic Risk Measures carry over to the conditional setting, even if the proofs become more technical, and that a new vector type consistency, with respect to sub sigma algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, replaces the scalar recursiveness property of univariate Risk Measures.

More precisely, we define axiomatically a *Conditional Systemic Risk Measure* (CSRM) on $L_{\mathcal{F}}$ as a map $\rho_{\mathcal{G}} : L_{\mathcal{F}} \to L^1(\Omega, \mathcal{G}, \mathbb{P})$ satisfying Monotonicity, Conditional Convexity and the Conditional Monetary Property (see Definition 3.2.5). Our first results

(Theorem 3.2.9 and Corollary 3.2.10) show, under fairly general assumptions, that: (i) $\rho_{\mathcal{G}}$ admits the conditional dual representation

$$\rho_{\mathcal{G}}(X) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}_{\mathcal{G}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \alpha(\mathbb{Q}) \right), \; X \in L_{\mathcal{F}}, \tag{I.28}$$

where $\mathscr{Q}_{\mathcal{G}}$, defined in Equation (3.11), is a set of vectors of probability measures and the penalty $\alpha(\mathbb{Q}) \in L^0(\Omega, \mathcal{G}, \mathbb{P})$ is defined in Equation (3.12); (ii) the supremum in (I.28) is attained.
We then specialize our analysis by considering the *Conditional Shortfall Systemic Risk Measure, associated to multivariate utility functions* U of the form (I.12), defined by

$$\rho_{\mathcal{G}}(X) := \operatorname*{ess\,inf} \left\{ \sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathcal{G}}, \; \mathbb{E}_{\mathbb{P}}\left[U(X+Y)|\mathcal{G}\right] \geq B \right\}, \tag{I.29}$$

where B is now a random variable in $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ and the set of $\mathcal{G}$-admissible allocations is

$$\mathscr{C}_{\mathcal{G}} \subseteq \left\{ Y \in (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N \text{ such that } \sum_{j=1}^{N} Y^j \in L^\infty(\Omega, \mathcal{G}, \mathbb{P}) \right\}.$$

Thus, with these definitions that mimic those in (I.22) and in (I.25), the same motivations explained in the unconditional setting, mutatis mutandis, remain true in the conditional one.
Observe that for the trivial selection $\mathcal{G} = \{\varnothing, \Omega\}$, for which Conditional Risk Measures reduce to static ones, our work extends the results in [20] to the more general aggregator U in the form (I.12).

In Theorem 3.4.4 we prove the main properties of the Conditional Shortfall Systemic Risk Measure $\rho_{\mathcal{G}}$ and, in particular, we show that (i) $\rho_{\mathcal{G}}$ is continuous from above and from below; (ii) the essential infimum in (I.29) is attained by a vector $Y(\mathcal{G}, X) = [Y^1(\mathcal{G}, X), ..., Y^N(\mathcal{G}, X)] \in \mathscr{C}_{\mathcal{G}}$; (iii) $\rho_{\mathcal{G}}$ admits the dual representation described in (3.23); (iv) the supremum in the dual formulation (3.23) of $\rho_{\mathcal{G}}$ is attained by a vector $\mathbb{Q}(\mathcal{G}, X) = [\mathbb{Q}^1(\mathcal{G}, X), ..., \mathbb{Q}^N(\mathcal{G}, X)]$ of probability measures satisfying:

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j(\mathcal{G},X)}\left[Y^j(\mathcal{G}, X) \middle| \mathcal{G}\right] = \sum_{j=1}^{N} Y^j(\mathcal{G}, X) = \rho_{\mathcal{G}}(X) \quad \mathbb{P}-\text{a.s.}$$

In the same spirit of [20], we will then interpret the quantity

$$a^j(\mathcal{G}, X) := \mathbb{E}_{\mathbb{Q}^j(\mathcal{G},X)}\left[Y^j(\mathcal{G}, X) \middle| \mathcal{G}\right]$$

as a *fair risk allocation of institution* j, *given* $\mathcal{G}$.

Section 3.5 is then devoted to the particular case of exponential utility functions

$$u_j(x^j) := -e^{-\alpha_j x^j}, \;\; \alpha_j > 0, \; j = 1, ..., N,$$

and with $\Lambda \equiv 0$. As in the static case (see [20]), also in the conditional case it is possible to find the explicit formulas for: (i) the value of the Conditional Shortfall Systemic Risk Measure $\rho_{\mathcal{G}}(X)$; (ii) the optimizer $Y(\mathcal{G}, X)$ in (I.29) of $\rho_{\mathcal{G}}(X)$; (iii) the vector $\mathbb{Q}(\mathcal{G}, X)$ of probability measures that attains the supremum in the dual formulation. We prove these properties first for a finitely generated sigma algebra $\mathcal{G}$, where we rely on the results in [20], and then we extend them to a general $\mathcal{G}$.
Finally, for sub sigma algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ we prove a particular consistency property, which does not have a counterpart in the univariate case. Indeed, a recursive property of the type $\rho_{\mathcal{H}}(-\rho_{\mathcal{G}}(X)) = \rho_{\mathcal{H}}(X)$ is not even well defined in the systemic setting, as $\rho_{\mathcal{G}}(X)$ is a random variable but the argument of $\rho_{\mathcal{H}}$ is a vector of random variables. However, we explain that consistency properties may be well defined for: (i) the vector optimizers $Y(\mathcal{G}, X)$ of $\rho_{\mathcal{G}}(X)$ and $Y(\mathcal{H}, Y(\mathcal{G}, X))$ of $\rho_{\mathcal{H}}(Y(\mathcal{G}, X))$; (ii) the fair risk allocations vectors $[a(\mathcal{G}, X)]_k := [E_{\mathbb{Q}^k(\mathcal{G},X)}[Y^k(\mathcal{G}, X)|\mathcal{G}]]_k$ of $\rho_{\mathcal{G}}(X)$ and $a(\mathcal{H}, a(\mathcal{G}, X))$ of $\rho_{\mathcal{H}}(a(\mathcal{G}, X))$. The consistency properties are shown in (3.56) and (3.58) and proven in Theorem 3.5.8 for the entropic Conditional Systemic Risk Measure.

In a final Section we elaborate on the concept of Systemic Optimal Risk Transfer Equilibrium, a notion anticipated above and formalized in Chapter 1. Based on the results on the Conditional Shortfall Systemic Risk Measure developed in Section 3.6, we are able to provide in Theorem 3.6.3 a direct extension of this equilibrium in the conditional setting. At the same time, we show that the optimal allocations for Shortfall Systemic Risk Measures, in both the static and dynamic cases, admit an interpretation in the sense of a suitably defined equilibrium. By the choice of the trivial $\mathcal{G} = \{\emptyset, \Omega\}$ our findings cover the static setup, and provide an explicit link between the theory in Chapter 1 and [20].

The analysis of Conditional Systemic Risk Measures of Chapter 3 can be found in Doldi and Frittelli (2020) [59].

I.4 Entropy Martingale Optimal Transport

Chapter 4 is conceptually separate from all the previous ones, even though duality theory plays a major role here too. We exploit Optimal Transport theory to develop the duality

$$A := \inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c] + \mathcal{D}_U(\mathbb{Q})\right) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_{\Delta}(c)} S^U(\varphi) := B. \tag{I.30}$$

Problem (A) is inspired by the Entropy Optimal Transport primal problem (Liero et al. (2018) [108]) with the additional constraint that the infimum of the cost functional c is taken over *martingale probability* measures. This is mirrored in the additional supremum over the integrands $\Delta \in \mathcal{H}$ in problem (B), and in the Cash Additivity of the functional S^U. The functional S^U is associated to a typically non linear utility functional U and represents the pricing rule over continuous bounded functions φ defined on Ω. In order to provide a context for our arguments, we believe it is instructive to look back at some classical concepts in financial mathematics. Since we aim at a nonlinear pricing-hedging duality, we will briefly summarize the highlights of the classical theory and of the robust one. We will also recall the key points regarding

Coherent Risk Measures, and the passage to the more general convex ones, since our relaxation-by-penalization procedure follows a somehow parallel path. We will resume the discussion of our work on page 28.

I.4.1 Pricing-hedging Duality in Financial Mathematics

The notion of subhedging price is one of the most analyzed concepts in financial mathematics. In this introduction we will take the point of view of the subhedging price, but obviously an analogous theory for the superhedging price can be developed as well. We are assuming a discrete time market model with zero interest rate.

The classical setup In the classical setup of stochastic securities market models, one considers an adapted stochastic process $X = (X_t)_t,\ t = 0, ..., T$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, representing the price of some underlying asset. Let $\mathcal{P}(\mathbb{P})$ be the set of all probability measures on Ω that are absolutely continuous with respect to $\mathbb{P}$, $\mathrm{Mart}(\Omega)$ be the set of all probability measures on Ω under which X is a martingale and $\mathcal{M}(\mathbb{P}) = \mathcal{P}(\mathbb{P}) \cap \mathrm{Mart}(\Omega)$. We also let $\mathcal{H}$ be the class of admissible integrands and $I^{\Delta} := I^{\Delta}(X)$ be the stochastic integral of X with respect to $\Delta \in \mathcal{H}$. Under reasonable assumptions on $\mathcal{H}$, the equality

$$\mathbb{E}_{\mathbb{Q}}\left[I^{\Delta}(X)\right] = 0 \tag{I.31}$$

holds for all $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$ and, as well known, all linear pricing functionals compatible with no arbitrage are expectations $\mathbb{E}_{\mathbb{Q}}[\cdot]$ under some probability $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$ such that $\mathbb{Q} \sim \mathbb{P}$.
We denote with p the **subhedging price** of a contingent claim $c : \mathbb{R} \to \mathbb{R}$ written on the payoff X_T of the underlying asset. If we let $\mathcal{L}(\mathbb{P}) \subseteq L^0((\Omega, \mathcal{F}_T, \mathbb{P}))$ be the space of random payoff and let $Z := c(X_T) \in \mathcal{L}(\mathbb{P})$, then $p : \mathcal{L}(\mathbb{P}) \to \mathbb{R}$ is defined by

$$p(Z) := \sup\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H} \text{ s.t. } m + I^{\Delta}(X) \leq Z,\ \mathbb{P} - \text{a.s.}\right\}. \tag{I.32}$$

The subhedging price is independent from the preferences of the agents, but it depends on the reference probability measure via the class of $\mathbb{P}$-null events. It satisfies the following two key properties:

(CA) Cash Additivity on $\mathcal{L}(\mathbb{P})$: $p(Z + k) = p(Z) + k$, for all $k \in \mathbb{R}$, $Z \in \mathcal{L}(\mathbb{P})$.

(IA) Integral Additivity on $\mathcal{L}(\mathbb{P})$: $p(Z + I^{\Delta}) = p(Z)$, for all $\Delta \in \mathcal{H}$, $Z \in \mathcal{L}(\mathbb{P})$.

When a functional p satisfies (CA), then Z, k and $p(Z)$ must be expressed in the same monetary unit and this allows for the *monetary* interpretation of p, as the price of the contingent claim. This will be one of the key features that we will require also in the novel definition of the nonlinear subhedging value. The (IA) property and $p(0) = 0$ imply that the p price of any stochastic integral $I^{\Delta}(X)$ is equal to zero, as in (I.31).
Since the seminal works of El Karoui and Quenez (1995) [66], Karatzas (1997) [101], Delbaen and Schachermayer (1994) [54], it was discovered that, under the no arbitrage assumption, the dual representation of the subhedging price p is

$$p(Z) = \inf_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[Z]. \tag{I.33}$$

More or less in the same period, the concept of **Coherent Risk Measure** was introduced in the pioneering work by Artzner et al. (1999) [8]. A Coherent Risk Measure $\rho : \mathcal{L}(\mathbb{P}) \to \mathbb{R}$ determines the minimal capital required to make a financial position acceptable and its dual formulation is assigned by

$$-\rho(Y) = \inf_{\mathbb{Q} \in \mathcal{Q} \subseteq \mathcal{P}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[Y], \tag{I.34}$$

where Y is a random variable representing future profit-and-loss and $\mathcal{Q} \subseteq \mathcal{P}(\mathbb{P})$. Coherent Risk Measures ρ are convex, cash additive, monotone and positively homogeneous. We take the liberty to label both the representations in (I.33) and in (I.34) as the "*sublinear case*".

In the study of incomplete markets the concept of the (buyer) **indifference price** p^b, originally introduced by Hodges and Neuberger (1989) [93], received, in the early 2000, increasing consideration (see Frittelli (2000) [79], Rouge and El Karoui (2000) [121], Delbaen et al. (2002) [52], Bellini and Frittelli (2002) [15]) as a tool to assess, *consistently with the no arbitrage principle*, the value of non replicable contingent claims, and not just to determine an upper bound (the superhedging price) or a lower bound (the subhedging price) for the price of the claim. Differently from the notion of subhedging, p^b is based on some concave increasing utility function $u : \mathbb{R} \to [-\infty, +\infty)$ of the agent. By defining the indirect utility function

$$U(w_0) := \sup_{\Delta \in \mathcal{H}} \mathbb{E}_{\mathbb{P}}[u(w_0 + I^{\Delta}(X))],$$

where $w_0 \in \mathbb{R}$ is the initial wealth, the indifference price p^b is defined as

$$p^b(Z) := \sup \{ m \in \mathbb{R} \mid U(Z - m) \geq U(0) \}.$$

Under suitable assumptions, the dual formulation of p^b is

$$p^b(Z) = \inf_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} \{ \mathbb{E}_{\mathbb{Q}}[Z] + \alpha_u(\mathbb{Q}) \}, \tag{I.35}$$

and the penalty term $\alpha_u : \mathcal{M}(\mathbb{P}) \to [0, +\infty]$ is associated to the particular utility function u appearing in the definition of p^b via the Fenchel conjugate of u. Observe that the functional p^b is concave, monotone increasing and satisfies both properties (CA) and (IA), but it is not necessarily linear on the space of all contingent claims. As recalled in the conclusion of Frittelli (2000) [79], "there is no reason why a price functional defined on the whole space of bundles and consistent with no arbitrage should be linear also outside the space of marketed bundles".
It was exactly the particular form (I.35) of the indifference price that suggested to Frittelli and Rosazza Gianin (2002) [83] to introduce the concept of **Convex Risk Measure** (also independently introduced by Föllmer and Schied (2002) [76]), as a map $\rho : \mathcal{L}(\mathbb{P}) \to \mathbb{R}$ that is convex, cash additive and monotone decreasing. Under good continuity properties, the Fenchel-Moreau Theorem shows that any Convex Risk Measure admits the following representation

$$-\rho(Y) = \inf_{\mathbb{Q} \in \mathcal{P}(\mathbb{P})} \{ \mathbb{E}_{\mathbb{Q}}[Y] + \alpha(\mathbb{Q}) \} \tag{I.36}$$

for some penalty α. We will then label functional in the form (I.35) or (I.36) as the "*convex case*". As a consequence of the Cash Additivity property, in the dual representations (I.35) or (I.36) the infimum is taken with respect to *probability measures*, namely with respect to normalized non negative elements in the dual space, which in this case can be taken as $L^1(\mathbb{P})$. Differently from the indifference price p^b, Convex Risk Measures do not necessarily take into account the presence of the stochastic security market, as reflected by the absence of any reference to martingale measures in the dual formulation (I.34) and (I.36), in contrast to (I.33) and (I.35).

Pathwise finance As a consequence of the financial crises in 2008, the uncertainty in the selection of a reference probability $\mathbb{P}$ gained increasing attention and led to the investigation of the notions of arbitrage and pricing-hedging duality in different settings. On the one hand, the single reference probability $\mathbb{P}$ was replaced with a family of - a priori non dominated - probability measures, leading to the theory of Quasi-Sure Stochastic Analysis (see Bayraktar and Zhang (2016) [12], Bayraktarand Zhou (2017) [13], Bouchard and Nutz (2015) [27], Cohen (2012) [43], Denis and Martini (2006) [56], Peng (2019) [114], Soner et al. (2011) [125]). On the other hand, taking an even more radical approach, a probability free, pathwise, theory of financial markets was developed, as in Acciaio et al. (2016) [2], Burzoni et al. (2016) [36], Burzoni et al. (2017) [37], Burzoni et al. (2019) [35], Riedel (2015) [117]. In such a framework, Optimal Transport theory became a very powerful tool to prove pathwise pricing-hedging duality results with very relevant contributions by many authors (Beiglböck et al. (2013) [14], Davis et al. (2014) [49], Dolinksi and Soner (2014) [61], Dolinsky and Soner (2015) [62]; Galichon et al. (2014) [85], Henry-Labordère (2013) [88], Henry-Labordère et al. (2016) [89]; Hou and Obłój (2018) [96], Tan and Touzi (2013) [126]). These contributions mainly deal with what we labeled above as the sublinear case, while our main interest in Chapter 4 is to develop the convex case theory, as explained below.

We will now abandon the classical setup described above and work without a reference probability measure. We consider $T \in \mathbb{N}$, $T \geq 1$, and

$$\Omega := K_0 \times \cdots \times K_T$$

for $K_0, \ldots, K_T$ subsets of $\mathbb{R}$ and denote with $X_0, \ldots, X_T$ the canonical projections $X_t : \Omega \to K_t$, for $t = 0, 1, ..., T$. We set

$$\mathrm{Mart}(\Omega) := \{\text{Martingale probability measures for the canonical process of } \Omega\},$$

and when μ is a measure defined on the Borel sigma algebra of $(K_0 \times \cdots \times K_T)$, its marginals will be denoted with $\mu_0, \ldots, \mu_T$. We consider a contingent claim $c : \Omega \to (-\infty, +\infty]$ which is now allowed to depend on the whole path and we will adopt semistatic trading strategies for hedging. This means that in addition to dynamic trading in X via the admissible integrands $\Delta \in \mathcal{H}$, we may invest in "vanilla" options $\varphi_t : K_t \to \mathbb{R}$. For modeling purposes we take vector subspaces $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ for $t = 0, \ldots, T$, where $\mathcal{C}_b(K_t)$ is the space of real-valued, continuous, bounded functions on K_t. For each t, $\mathcal{E}_t$ is the set of static options that can be used for hedging, say

affine combinations of options with different strikes and same maturity t. The key assumption in the robust, Optimal Transport based, formulation is that the marginals $(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ..., \widehat{\mathbb{Q}}_T)$ of the underlying price process X are known. This assumption can be justified (see the seminal papers by Breeden and Litzenberger (1978) [29] and Hobson (1998) [92], as well as the many contributions by Hobson (2011) [90], Cox and Obłój (2011) [44], [45], Cox and Wang (2013) [46], Henry-Labordère et al. (2016) [89], Brown et al. (2001) [30], Hobson and Klimmerk (2013) [91]) by assuming the knowledge of prices of a sufficiently large number of plain vanilla options maturing at each intermediate date.
Thus the class of arbitrage-free pricing measures that are compatible with the observed prices of the options is given by

$$\mathcal{M}(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ...\widehat{\mathbb{Q}}_T) := \left\{ \mathbb{Q} \in \mathrm{Mart}(\Omega) \mid X_t \sim_{\mathbb{Q}} \widehat{\mathbb{Q}}_t \text{ for each } t = 0, \dots, T \right\}.$$

In this framework,

$$\mathcal{H} := \{\Delta = [\Delta_0, \dots, \Delta_{T-1}] \mid \Delta_t \in \mathcal{C}_b(K_0 \times \cdots \times K_t; \mathbb{R})\} \ , \tag{I.37}$$

$$\mathcal{I} := \left\{ I^{\Delta}(x) = \sum_{t=0}^{T-1} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) \mid \Delta \in \mathcal{H} \right\} \ , \tag{I.38}$$

and the subhedging duality, obtained in [14] Theorem 1.1, takes the form:

$$\begin{aligned} &\inf_{\mathbb{Q} \in \mathcal{M}(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ...\widehat{\mathbb{Q}}_T)} \mathbb{E}_{\mathbb{Q}}[c] \\ &= \sup \left\{ \sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leq c(x) \ \forall x \in \Omega \right\} \end{aligned} \tag{I.39}$$

where the RHS of (I.39) is known as the **robust subhedging price** of c. Comparing (I.39) with the duality between (I.32) and (I.33), we observe that: (i) the $\mathbb{P}-$a.s. inequality in (I.32) has been replaced by an inequality that holds for all $x \in \Omega$; (ii) in (I.39) the infimum of the price of the contingent claim c is taken under all martingale measures compatible with the option prices, with no reference to the probability $\mathbb{P}$; (iii) static hedging with options is allowed.
As can be seen from the LHS of (I.39), this case falls into the category labeled above as the *sublinear case*, and the purpose of our work is to investigate the *convex case*, in the robust setting, using the tools from Entropy Optimal Transport (EOT) recently developed in Liero et al. (2018) [108].
Let us first describe the financial interpretation of the problems that we are going to study.

The dual problem
The LHS of (I.39), namely $\inf_{\mathbb{Q} \in \mathcal{M}(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ...\widehat{\mathbb{Q}}_T)} \mathbb{E}_{\mathbb{Q}}[c]$, represents the dual problem in the financial application, but is typically the primal problem in **Martingale Optimal Transport** (MOT). We label this case as the *sublinear case* of MOT. In [108], the primal **Entropy Optimal Transport** (EOT) problem takes the form

$$\inf_{\mu \in \mathrm{Meas}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c] + \sum_{t=0}^{T} \mathcal{D}_{F_t, \widehat{\mathbb{Q}}_t}(\mu_t) \right) , \tag{I.40}$$

where Meas(Ω) is the set of all positive finite measures μ on Ω, and $\mathcal{D}_{F_t,\widehat{\mathbb{Q}}_t}(\mu_t)$ is a divergence in the form:

$$\mathcal{D}_{F_t,\widehat{\mathbb{Q}}_t}(\mu_t) := \int_{K_t} F_t\left(\frac{\mathrm{d}\mu_t}{\mathrm{d}\widehat{\mathbb{Q}}_t}\right) \mathrm{d}\widehat{\mathbb{Q}}_t, \text{ if } \mu_t \ll \widehat{\mathbb{Q}}_t, \tag{I.41}$$

otherwise $\mathcal{D}_{F_t,\widehat{\mathbb{Q}}_t}(\mu_t) := +\infty$. Problem (I.40) represents the *convex case* of Optimal Transport (OT) theory. Notice that in the EOT primal problem (I.40) the typical constraint that μ has prescribed marginals $(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ...\widehat{\mathbb{Q}}_T)$ is relaxed (as the infimum is taken with respect to all positive finite measures) by introducing the divergence functional $\mathcal{D}_{F_t,\widehat{\mathbb{Q}}_t}(\mu_t)$, which penalizes those measures μ that are "far" from some reference marginals $(\widehat{\mathbb{Q}}_0, \widehat{\mathbb{Q}}_1, ...\widehat{\mathbb{Q}}_T)$. Observing that we could potentially push our smoothing argument above even further, we might consider more general marginal penalizations, not necessarily in the divergence form (I.41), that is we could take functionals $\mathcal{D}_t$ in place of $\mathcal{D}_{F_t,\widehat{\mathbb{Q}}_t}(\mu_t)$, $t = 0, ..., T$, in (I.40). We are then naturally let to the study of the convex case of MOT, i.e. to the Entropy Martingale Optimal Transport (EMOT) problem:

$$\mathfrak{D}(c) := \inf_{\mathbb{Q}\in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c\right] + \sum_{t=0}^{T} \mathcal{D}_t(\mathbb{Q}_t)\right). \tag{I.42}$$

These penalizations $\mathcal{D}_0, \dots, \mathcal{D}_T$ will be better specified later.

The primal problem: the Nonlinear Subhedging Value

We provide the financial interpretation of the primal problem which will yield the EMOT problem $\mathfrak{D}$ as its dual. It is convenient to reformulate the robust subhedging price in the RHS of (I.39) in a more general setting.

Definition I.4.1. *Consider a measurable function $c : \Omega \to \mathbb{R}$ representing a (possibly path dependent) option, the set $\mathcal{V}$ of hedging instruments and a suitable pricing functional $\pi : \mathcal{V} \to \mathbb{R}$. Then the robust Subhedging Value of c is defined by*

$$\Pi_{\pi,\mathcal{V}}(c) = \sup\left\{\pi(v) \mid v \in \mathcal{V} \text{ s.t. } v \le c\right\}.$$

In the classical setting, functionals of this form (and even with a more general formulation) are known as general capital requirement, see for example Frittelli and Scandolo (2006) [84]. We stress however that in Definition I.4.1 the inequality $v \le c$ holds for all elements in Ω with no reference to a probability measure whatsoever. The novelty in this definition is that a priori π may not be linear and it is crucial to understand which evaluating functional π we may use. For our discussion, we assume that the vector subspaces $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ satisfies $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$, for $t = 0, \dots, T$. We let $\mathcal{E} := \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$, and $\mathcal{V} := \mathcal{E}_0 + \cdots + \mathcal{E}_T + \mathcal{I}$. Suppose we took a linear pricing rule $\pi : \mathcal{V} \to \mathbb{R}$ defined via a $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ by

$$\pi(v) := \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\sum_{t=0}^{T} \varphi_t + I^{\Delta}\right] \overset{(i)}{=} \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\sum_{t=0}^{T} \varphi_t\right] \overset{(ii)}{=} \sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t], \tag{I.43}$$

where we used (I.31) and the fact that $\widehat{\mathbb{Q}}_t$ is the marginal of $\widehat{\mathbb{Q}}$. In this case, we would trivially obtain for the robust subhedging value of c

$$\begin{aligned}
\Pi_{\pi,\mathcal{V}}(c) &= \sup\left\{\pi(v) \mid v \in \mathcal{V} \text{ s.t. } v \leq c\right\} \qquad\qquad (\text{I.44})\\
&= \sup\left\{\sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leq c(x)\ \forall x \in \Omega\right\}\\
&= \sup\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H},\ \varphi \in \mathcal{E},\ \text{s.t. } m - \sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] + \sum_{t=0}^{T} \varphi_t + I^{\Delta} \leq c\right\}\\
&= \sup\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H},\ \varphi \in \mathcal{E},\ \text{with } \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] = 0 \text{ s.t. } m + \sum_{t=0}^{T} \varphi_t + I^{\Delta} \leq c\right\}, \qquad (\text{I.45})
\end{aligned}$$

where in the last equality we replaced φ_t with $(\mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] - \varphi_t) \in \mathcal{E}_t$, which satisfies:

$$\mathbb{E}_{\widehat{\mathbb{Q}}_t}\left[\mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] - \varphi_t\right] = 0. \qquad (\text{I.46})$$

Interpretation: $\Pi_{\pi,\mathcal{V}}(c)$ is the supremum amount $m \in \mathbb{R}$ for which we may buy options φ_t and dynamic strategies $\Delta \in \mathcal{H}$ such that $m + \sum_{t=0}^{T} \varphi_t + I^{\Delta} \leq c$, where the value of both the options and the stochastic integrals are computed as the expectation under the same martingale measure ($\widehat{\mathbb{Q}}$ for the integral I^{Δ}; its marginals $\widehat{\mathbb{Q}}_t$ for each option φ_t).

However, as mentioned above when presenting the indifferent price p^b, there is a priori no reason why one has to allow only linear functional in the evaluation of $v \in \mathcal{V}$.
We thus generalize the expression for $\Pi_{\pi,\mathcal{V}}(c)$ *by considering valuation functionals* $S : \mathcal{V} \to \mathbb{R}$ and $S_t : \mathcal{E}_t \to \mathbb{R}$ *more general than* $\mathbb{E}_{\widehat{\mathbb{Q}}}[\cdot]$ *and* $\mathbb{E}_{\widehat{\mathbb{Q}}_t}[\cdot]$.
Nonetheless, in order to be able to repeat the same key steps we used in (I.44)-(I.45) and therefore to keep the same interpretation, we shall impose that such functionals S and S_t satisfy the property in (I.46) and the two properties (i) and (ii) in equation (I.43), that is:

(a) $S_t[\varphi_t + k] = S_t[\varphi_t] + k$ and $S_t[0] = 0$, for all $\varphi_t \in \mathcal{C}_b(K_t)$, $k \in \mathbb{R}$, $t = 0, \ldots, T$,

(b) $S\left[\left(\sum_{t=0}^{T} \varphi_t\right) + I^{\Delta}(x)\right] = S\left[\sum_{t=0}^{T} \varphi_t\right]$ for all $\Delta \in \mathcal{H}$ and $\varphi \in \mathcal{E}$,

(c) $S\left[\sum_{t=0}^{T} \varphi_t\right] = \sum_{t=0}^{T} S_t[\varphi_t]$ for all $\varphi \in \mathcal{E}$.

We immediately recognize that (a) is the Cash Additivity (CA) property on $\mathcal{C}_b(K_t)$ of the functional S_t and (b) implies the Integral Additivity (IA) property on $\mathcal{V}$. As a consequence, repeating the same steps in (I.44)-(I.45), we will obtain as primal problem the **nonlinear subhedging value** of c :

$$\begin{aligned}\mathfrak{P}(c) &= \sup\{S(v) \mid v \in \mathcal{V} : v \le c\} \\ &= \sup\left\{\sum_{t=0}^{T} S_t(\varphi_t) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T}\varphi_t(x_t) + I^{\Delta}(x) \le c(x)\ \forall x \in \Omega\right\} \qquad \text{(I.47)} \\ &= \sup\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H},\, \varphi \in \mathcal{E},\ \text{with } S_t(\varphi_t) = 0 \text{ s.t. } m + \sum_{t=0}^{T}\varphi_t + I^{\Delta} \le c\right\}, \qquad \text{(I.48)}\end{aligned}$$

to be compared with (I.45).
Interpretation: $\mathfrak{P}(c)$ is the supremum amount $m \in \mathbb{R}$ for which we may buy *zero value* options φ_t and dynamic strategies $\Delta \in \mathcal{H}$ such that $m + \sum_{t=0}^{T}\varphi_t + I^{\Delta} \le c$, where the value of both the options and the stochastic integrals are computed with the same functional S.

Before further elaborating on these issues, let us introduce the concept of Stock Additivity, which is the natural counterpart of properties (IA) and (CA) when we are evaluating hedging instruments depending solely on the value of the underlying stock X at some fixed date $t \in \{0, \dots, T\}$. Let Id_t be the identity function $x_t \mapsto x_t$ on K_t. As before, the set of hedging instruments is denoted by $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ and we will suppose that $\mathrm{Id}_t \in \mathcal{E}_t$ (that is, we can use units of stock at time t for hedging) and that $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ (that is, deterministic amounts of cash can be used for hedging as well).

Definition I.4.2. *A functional* $p_t : \mathcal{E}_t \to \mathbb{R}$ *is stock additive on* $\mathcal{E}_t$ *if* $p_t(0) = 0$ *and*

$$p_t(\varphi_t + \alpha_t \mathrm{Id}_t + \lambda_t) = p_t(\varphi_t) + \alpha_t x_0 + \lambda_t \quad \forall \varphi_t \in \mathcal{E}_t, \lambda_t \in \mathbb{R}, \alpha_t \in \mathbb{R}\,.$$

We now clarify the role of stock additive functionals in our setup. Suppose that $S_t : \mathcal{E}_t \to \mathbb{R}$ are stock additive on $\mathcal{E}_t$, $t = 0, \dots, T$. It can be shown (see Lemma 4.4.7) that if there exist $\varphi, \psi \in \mathcal{E}_0 \times ... \times \mathcal{E}_T$ and $\Delta \in \mathcal{H}$ such that $\sum_{t=0}^{T}\varphi_t = \sum_{t=0}^{T}\psi_t + I^{\Delta}$ then

$$\sum_{t=0}^{T} S_t(\varphi_t) = \sum_{t=0}^{T} S_t(\psi_t).$$

This allows us to define a functional $S : \mathcal{V} = \mathcal{E}_0 + \dots + \mathcal{E}_T + \mathcal{I} \to \mathbb{R}$ by

$$S(v) := \sum_{t=0}^{T} S_t(\varphi_t), \quad \text{for } v = \sum_{t=0}^{T}\varphi_t + I^{\Delta}. \qquad \text{(I.49)}$$

Then S is a well defined, integral additive functional on $\mathcal{V}$, and $S, S_0, \dots, S_T$ satisfy the properties (a), (b), (c). There is a natural way to produce a variety of stock additive functionals, as explained in Example I.4.3 below.

Example I.4.3. Consider a Martingale measure $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ and a concave non decreasing utility function $u_t : \mathbb{R} \to [-\infty, +\infty)$, satisfying $u(0) = 0$ and $u_t(x_t) \le x_t\ \forall x_t \in \mathbb{R}$. We can then take

$$S_t(\varphi_t) = U_{\widehat{\mathbb{Q}}_t}(\varphi_t) := \sup_{\alpha \in \mathbb{R},\, \lambda \in \mathbb{R}} \left(\int_{\Omega} u_t\left(\varphi_t(x_t) + \alpha x_t + \lambda\right) \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - (\alpha x_0 + \lambda)\right).$$

As shown in Lemma 4.3.2 the Stock Additivity property is then satisfied for these functionals.

When we consider stock additive functionals $S_0, \dots, S_T$ that induce the functional S as explained in (I.49), we can focus our attention to the optimization problem (I.47) or (I.48), that will be referred to as our primal problem.

The Duality As a consequence of our main results we prove the following duality (see Theorem 4.2.3). If

$$\mathcal{D}_t(\mathbb{Q}_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left(S_t(\varphi_t) - \int_{K_t} \varphi_t \, \mathrm{d}\mathbb{Q}_t \right) \quad \text{for } \mathbb{Q}_t \in \mathrm{Prob}(K_t), \quad t = 0, \dots, T,$$

and $\mathfrak{D}(c)$ and $\mathfrak{P}(c)$ are defined respectively in (I.42) and (I.47), then

$$\mathfrak{D}(c) = \mathfrak{P}(c).$$

We also treat the particular case of $S_0, \dots, S_T$ induced by utility functions, as explained in Example I.4.3. In the special case of linear utility functions $u_t(x_t) = x_t$, we recover the sublinear MOT theory.

I.4.2 Entropy Martingale Optimal Transport Duality

We described and provided the financial interpretation of the new duality $\mathfrak{D}(c) = \mathfrak{P}(c)$. This will be a particular case of a more general duality established in Theorem 4.1.3 and Theorem 4.1.4.
In our main result (Theorem 4.1.4) we start by introducing two general functionals, $U : \mathcal{E} \to [-\infty, +\infty)$ and $\mathcal{D}_U : \mathrm{ca}(\Omega) \to (-\infty, +\infty]$, that are associated through a Fenchel-Moreau type relation (see (4.1)). The vector space $\mathcal{E} \subseteq \mathcal{C}_b(\Omega; \mathbb{R}^{T+1})$ consists of continuous and bounded functions defined on some Polish space Ω and with values in $\mathbb{R}^{T+1}$. The map U needs not be either cash additive or lower semicontinuous. We then rely on the notion of the Optimized Certainty Equivalent (OCE), that was introduced in Ben Tal and Teboulle (1986) [16] and further analyzed in Ben Tal and Teboulle (2007) [17]. As it is easily recognized, any OCE is, except for the sign, a particular Convex Risk Measure and so it is cash additive. We introduce the Generalized Optimized Certainty Equivalent (Generalized OCE) associated to U as the functional $S^U : \mathcal{E} \to [-\infty, +\infty]$ defined by

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \quad \varphi \in \mathcal{E}. \tag{I.50}$$

Thus we obtain a cash additive map $S^U(\varphi + \xi) = S^U(\varphi) + \sum_{t=0}^{T} \xi_t$, which will guarantee that in the problem (I.40) the elements $\mu \in \mathrm{Meas}(\Omega)$ are normalized, i.e. are probability measures. Then the duality will take the form

$$\inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c] + \mathcal{D}_U(\mathbb{Q}) \right) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} S^U(\varphi), \tag{I.51}$$

where

$$\mathbf{\Phi}_\Delta(c) := \left\{ \varphi \in \mathrm{dom}(U),\ \sum_{t=0}^{T} \varphi_t(x_t) + I^\Delta(x) \leq c(x)\ \ \forall\, x \in \Omega \right\},$$

and we also prove the existence of the optimizer for the problem in the LHS of (I.51), see Proposition 4.1.6.
The penalization term $\mathcal{D}$:=$\mathcal{D}_U$ associated to U does not necessarily have an additive structure ($\mathcal{D}(\mathbb{Q}) = \sum_{t=0}^{T} \mathcal{D}_t(\mathbb{Q}_t)$) nor needs to have the divergence formulation, as described in (I.41), and so it does not necessarily depend on a given martingale measure $\widehat{\mathbb{Q}}$. As explained in Sections 4.3.1 and 4.3.2 this additional flexibility in choosing $\mathcal{D}$ allows several different interpretations and constitutes one generalization of the Entropy Optimal Transport theory of [108]. Of course, the other additional difference with EOT is the presence in (I.51) of the additional supremum with respect to admissible integrand $\Delta \in \mathcal{H}$. As a consequence, in the LHS of (I.51) the infimum is now taken with respect to martingale measures. We also point out that in [108], the cost functional c is required to be lower semicontinuous and nonnegative and that the theory is developed only for the bivariate case ($t = 0, 1$), while in Chapter 4 we take c lower semicontinuous and with compact level sets, hence bounded from below, and consider the multivariate case ($t = 0, ..., T$).
In [108] the authors work with a Hausdorf topological space Ω, while we request Ω to be a Polish space and for some of the results even a compact subset of $\mathbb{R}^N$. This stronger assumption however is totally reasonable for the applications we deal with (see Remark 4.3.4) and, in case of the Polish space assumption, it would be also compatible with a theory for stochastic processes X in continuous time, a topic left for future investigation. We stress the fact that our work has some points in common with Pennanen and Perkkiö (2019) [115], in particular regarding our additive setup in Section 4.2. The authors in this paper consider a more general underlying space (unbounded claims) and a more general cost function in place of the classical integral of a cost function, but only with additive (in time) penalizations and valuation functionals. [115] Section 3.3 contains some considerations on possible applications of their generalized Optimal Transport duality to robust superhedging, again covering the additive case, but working with martingale nonnegative measures instead of martingale **probability** measures. This is mirrored by the fact that the authors do not consider cash additive valuation functionals. On the contrary, Cash Additivity is one of the key properties we consider here together with the new concepts of Stock and Integral Additivity, and causes Generalized Optimized Certainty Equivalents to appear in our duality. Finally, we do not require the strong assumption of any (semi)continuity for pricing functionals, and allow for static parts of semistatic superhedging strategies in a strict (and possibly not norm closed) subset $\mathcal{E}_t$ of $\mathcal{C}_b(K_0 \times \cdots \times K_t)$.

The analysis of EMOT, described in Chapter 4, can be found in Doldi and Frittelli (2020) [60].

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Chapter 1

Systemic Optimal Risk Transfer Equilibrium

In Chapter 1 we propose a novel concept of a Systemic Optimal Risk Transfer Equilibrium (SORTE), which is inspired by the Bühlmann's classical notion of an Equilibrium Risk Exchange. In both the Bühlmann and the SORTE definition, each agent is behaving rationally by maximizing his/her expected utility given a budget constraint. The two approaches differ by the budget constraints. In Bühlmann's definition the vector that assigns the budget constraint is given a priori. On the contrary, in the SORTE approach, the vector that assigns the budget constraint is endogenously determined by solving a systemic utility maximization. SORTE gives priority to the systemic aspects of the problem, in order to optimize the overall systemic performance, rather than to individual rationality. We provide sufficient general assumptions that guarantee existence, uniqueness, and Pareto optimality of a SORTE with budget $A \in \mathbb{R}$ and set of admissible allocations $\mathcal{B}$, namely a triple $(Y_X, \mathbb{Q}_X, a_X)$ consisting respectively of a vector of random allocations, a vector of probability measures and a deterministic vector such that

- for each j, Y_X^j is optimal for

$$U_j^{\mathbb{Q}_X^j}(a^j) := \sup \left\{ \mathbb{E}_{\mathbb{P}} \left[u_j(X^j + Y) \right] \mid Y \in \mathcal{L}^j, \, \mathbb{E}_{\mathbb{Q}_X^j}[Y] \leq a_X^j \right\} ,$$

- a_X is optimal for

$$\sup \left\{ \sum_{j=1}^N U_j^{\mathbb{Q}_X^j}(a^j) \mid a \in \mathbb{R}^N \text{ s.t. } \sum_{j=1}^N a^j \leq A \right\} ,$$

- $Y_X \in \mathcal{B}$ and $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}-$a.s.

Chapter 1 is structured as follows: after fixing notation and setup, in Section 1.2 we formalize mathematically the several notions of equilibrium we already mentioned in Section I.1 and state our main results, that is existence (Theorem 1.2.14) and uniqueness (Theorem 1.2.15) of SORTE. Section 1.3 is devoted to their proofs. Section 1.4 collects examples and explicit formulas for the specific case of exponential utility functions. Finally, Section 1.5 collects all auxiliary notions and results we need.

1.1 Notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider the following set of probability vectors on $(\Omega, \mathcal{F})$

$$\mathcal{P}^N := \left\{ \mathbb{Q} = [\mathbb{Q}^1, ..., \mathbb{Q}^N] \mid \text{ such that } \mathbb{Q}^j \ll \mathbb{P} \text{ for all } j = 1, ..., N \right\}.$$

For a vector of probability measures $\mathbb{Q}$ we write $\mathbb{Q} \ll \mathbb{P}$ to denote $\mathbb{Q}^1 \ll \mathbb{P}, \dots, \mathbb{Q}^N \ll \mathbb{P}$. Similarly for $\mathbb{Q} \sim \mathbb{P}$. For $\mathbb{Q} \in \mathcal{P}^1$ let

$$L^0(\mathbb{Q}) := L^0(\Omega, \mathcal{F}, \mathbb{Q}; \mathbb{R}) \quad L^1(\mathbb{Q}) := L^1(\Omega, \mathcal{F}, \mathbb{Q}; \mathbb{R}) \quad L^\infty(\mathbb{Q}) := L^\infty(\Omega, \mathcal{F}, \mathbb{Q}; \mathbb{R})$$

be the vector spaces of $\mathbb{Q}-$ a.s. finite, $\mathbb{Q}-$ integrable and $\mathbb{Q}-$ essentially bounded random variables respectively, and set $L^p_+(\mathbb{Q}) = \{Z \in L^p(\mathbb{Q}) | Z \geq 0 \, \mathbb{Q} - \text{a.s.}\}$ and $L^p(\Omega, \mathcal{F}, \mathbb{Q}; \mathbb{R}^N) = (L^p(\mathbb{Q}))^N$, for $p \in \{0, 1, \infty\}$. For $\mathbb{Q} = [\mathbb{P}^1, \dots, \mathbb{Q}^N] \in \mathcal{P}^N$ and $p \in \{0, 1, \infty\}$ define

$$L^p(\mathbb{Q}) := L^p(\mathbb{Q}^1) \times ... \times L^p(\mathbb{Q}^N), \qquad L^p_+(\mathbb{Q}) := L^p_+(\mathbb{Q}^1) \times ... \times L^p_+(\mathbb{Q}^N).$$

We finally set $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_{++} = (0, +\infty) = \mathbb{R}_+ \setminus \{0\}$.
For each $j = 1, ..., N$ consider a vector subspace $\mathcal{L}^j$ with $\mathbb{R} \subseteq \mathcal{L}^j \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and set

$$\mathcal{L} := \mathcal{L}^1 \times ... \times \mathcal{L}^N \subseteq (L^0(\mathbb{P}))^N.$$

Consider now a subset $\mathbb{Q} \subseteq \mathcal{P}^N$ and assume that the pair $(\mathcal{L}, \mathscr{Q})$ satisfies that for every $\mathbb{Q} \in \mathscr{Q}$

$$\mathcal{L} \subseteq L^1(\mathbb{Q}).$$

One could take as $\mathcal{L}^j$, for example, L^∞ or some Orlicz space. Our optimization problems will be defined on the vector space $\mathcal{L}$ to be specified later.
For each $j = 1, ..., N$, let $u_j : \mathbb{R} \to \mathbb{R}$ be concave and strictly increasing. Fix $X = (X^1, ..., X^N) \in \mathcal{L}$.
For $(\mathbb{Q}, a, A) \in \mathscr{Q} \times \mathbb{R}^N \times \mathbb{R}$ define

$$U_j^{\mathbb{Q}^j}(a^j) := \sup \left\{ \mathbb{E}_\mathbb{P} \left[u_j(X^j + Y) \right] \mid Y \in \mathcal{L}^j, \, \mathbb{E}_{\mathbb{Q}^j}[Y] \leq a^j \right\}, \tag{1.1}$$

$$S^\mathbb{Q}(A) := \sup \left\{ \sum_{j=1}^N U_j^{\mathbb{Q}^j}(a^j) \mid a \in \mathbb{R}^N \text{ s.t. } \sum_{j=1}^N a^j \leq A \right\}, \tag{1.2}$$

$$\Pi^\mathbb{Q}(A) := \sup \left\{ \mathbb{E}_\mathbb{P} \left[\sum_{j=1}^N u_j(X^j + Y^j) \right] \mid Y \in \mathcal{L}, \, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] \leq A \right\}. \tag{1.3}$$

Obviously, such quantities depend also on X, but as X will be kept fixed throughout most of the analysis, we may avoid to explicitly specify this dependence in the notations. As u_j is increasing we can replace in the definitions of $U_j^{\mathbb{Q}^j}(a^j)$, $S^\mathbb{Q}(A)$ and $\Pi^\mathbb{Q}(A)$ the inequality in the budget constraint with an equality.
When a vector $\mathbb{Q} \in \mathscr{Q}$ is assigned, we can consider two problems. First, for each j, $U_j^{\mathbb{Q}^j}(a^j)$ is the optimal value of the classical one dimensional expected utility maximization problem with random endowment X^j under the budget constraint $\mathbb{E}_{\mathbb{Q}^j}[Y] \leq a^j$, determined by the real number a^j and the valuation operator $\mathbb{E}_{\mathbb{Q}^j}[\cdot]$

associated to $\mathbb{Q}^j$. Second, if we interpret the quantity $\sum_{j=1}^N u_j(\cdot)$ as the aggregated utility of the system, then $\Pi^{\mathbb{Q}}(A)$ is the maximal expected utility of the whole system X, among all $Y \in \mathcal{L}$ satisfying the overall budget constraint $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] \leq A$. Notice that in these problems the vector Y is not required to belong to $\mathcal{C}_{\mathbb{R}}$ (see (I.1)), but only to the vector space $\mathcal{L}$. We will show in Lemma 1.3.11 the quite obvious equality $S^{\mathbb{Q}}(A) = \Pi^{\mathbb{Q}}(A)$.

1.2 On several notions of Equilibrium

1.2.1 Pareto Allocation

Definition 1.2.1. *Given a set of feasible allocations $\mathscr{V} \subseteq \mathcal{L}$ and a vector $X \in \mathcal{L}$, $\widehat{Y} \in \mathscr{V}$ is a* Pareto allocation for $\mathscr{V}$ *if*

$$Y \in \mathscr{V} \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right] \geq \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + \widehat{Y}^j)\right] \text{ for all } n \tag{1.4}$$

imply $\mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right] = \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + \widehat{Y}^j)\right]$ *for all* j.

In general Pareto allocations are not unique and, not surprisingly, the following version of the First Welfare Theorem holds true. Define the optimization problem

$$\Pi(\mathscr{V}) := \sup_{Y \in \mathscr{V}} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right]. \tag{1.5}$$

Proposition 1.2.2. *Whenever $\widehat{Y} \in \mathscr{V}$ is the unique optimal solution of* $\Pi(\mathscr{V})$*, then it is a Pareto allocation for $\mathscr{V}$.*

Proof. Let $\widehat{Y}$ be optimal for $\Pi(\mathscr{V})$, so that $\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j + \widehat{Y}^j)\right] = \Pi(\mathscr{V})$. Suppose that there exists Y such that (1.4) holds true. As $Y \in \mathscr{V}$ we have:

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j + \widehat{Y}^j)\right] = \Pi(\mathscr{V}) \geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j + Y^j)\right] \geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j + \widehat{Y}^j)\right],$$

by (1.4). Hence also Y is an optimal solution to $\Pi(\mathscr{V})$. Uniqueness of the optimal solution implies $Y = \widehat{Y}$. □

1.2.2 Systemic utility maximization

The next definition is the utility maximization problem, in the case of a system of N agents.

Definition 1.2.3. *Fix $\mathbb{Q} \in \mathscr{Q}$. The pair $(Y_X, a_X) \in \mathcal{L} \times \mathbb{R}^N$ is a $\mathbb{Q}-$**Optimal Allocation** with budget $A \in \mathbb{R}$ if*
1) for each j, Y_X^j is optimal for $U_j^{\mathbb{Q}^j}(a_X^j)$,
2) a_X is optimal for $S^{\mathbb{Q}}(A)$,
3) $Y_X \in \mathcal{L}$.

Note that in the above definition the vector $\mathbb{Q} \in \mathscr{Q}$ is exogenously assigned. Given a total budget $A \in \mathbb{R}$, the vector $a_X \in \mathbb{R}^N$ maximizes the systemic utility

$$\sum_{j=1}^{N} U_j^{\mathbb{Q}^j}(a^j)$$

among all feasible $a \in \mathbb{R}^N$ ($\sum_{j=1}^N a^j \leq A$) and Y_X^j maximizes the single agent expected utility $\mathbb{E}_\mathbb{P}[u_j(X^j + Y)]$ among all feasible allocations $Y \in \mathcal{L}^j$ s.t. $\mathbb{E}_{\mathbb{Q}^j}[Y] \leq a_X^j$. Since $\mathbb{Q} \in \mathscr{Q}$ is given, the budget constraint $\mathbb{E}_{\mathbb{Q}^j}[Y] \leq a_X^j$ is well defined for all $Y \in \mathcal{L}$ and we do not need additional conditions of the form $Y \in \mathcal{C}_\mathbb{R}$. A generalization of the classical single agent utility maximization yields the following existence result.

Proposition 1.2.4. *Under Assumption 1.2.12 (a) select $\mathscr{Q} = \{\mathbb{Q}\}$ for some $\mathbb{Q} \in \mathcal{Q}_v$ (see* (1.11)*) with $\mathbb{Q} \sim \mathbb{P}$. Set $\mathcal{L} = L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)$ and let $X \in M^\Phi$ (see* (1.67)*). Then a $\mathbb{Q}-$Optimal Allocation exists.*

Proof. The proof can be obtained with the same arguments employed in Section 4.2 of [20]. □

Let $(Y_X, a_X) \in \mathcal{L} \times \mathbb{R}^N$ be a $\mathbb{Q}-$Optimal Allocation. Due to Lemma 1.3.11, $\Pi^\mathbb{Q}(A) = S^\mathbb{Q}(A)$ and

$$\begin{aligned}
\Pi^\mathbb{Q}(A) = S^\mathbb{Q}(A) &= \sup_{a \in \mathbb{R}^N,\ \sum_{j=1}^N a^j = A} \sum_{j=1}^{N} \sup_{Y^j \in \mathcal{L}^j} \left\{ \mathbb{E}_\mathbb{P}\left[u_j(X^j + Y^j)\right] \mid \mathbb{E}_{\mathbb{Q}^j}[Y^j] = a^j \right\} \\
&= \sum_{j=1}^{N} \sup_{Y^j \in \mathcal{L}^j} \left\{ \mathbb{E}_\mathbb{P}\left[u_j(X^j + Y^j)\right] \mid \mathbb{E}_{\mathbb{Q}^j}[Y^j] = a_X^j \right\},
\end{aligned}$$

where we replaced the inequalities with equalities in the budget constraints, as u_j are monotone. Hence the systemic utility maximization problem $\Pi^\mathbb{Q}(A)$ with overall budget constraint A reduces to the sum of j single agent maximization problems, where, however, the budget constraint of each agents is assigned by $a_X^j = \mathbb{E}_{\mathbb{Q}^j}[Y_X^j]$ and the vector a_X maximizes the overall performance of the system. We will also recover this feature in the notion of a SORTE, where the probability vector $\mathbb{Q}$ will be endogenously determined, instead of being a priori assigned, as in this case.

1.2.3 Risk Exchange Equilibrium

We here formalize Bühlmann's risk exchange equilibrium in a pure exchange economy, [32] and [33], already mentioned in conditions (a') and (b'), Item 1 of Section I.1. Let $\mathcal{Q}^1$ be the set of vectors of probability measures having all components equal:

$$\mathcal{Q}^1 := \left\{ \mathbb{Q} \in \mathcal{P}^N \mid \mathbb{Q}^1 = ... = \mathbb{Q}^N \right\}.$$

To be consistent with Definition 1.2.3 we keep the same numbering for the corresponding conditions.

Definition 1.2.5. *Fix $A \in \mathbb{R}$, $a \in \mathbb{R}^N$ such that $\sum_{j=1}^N a^j = A$. The pair $(Y_X, \mathbb{Q}_X) \in \mathcal{L} \times \mathcal{Q}^1$ is a* ***risk exchange equilibrium (with budget** A **and allocation** $a \in \mathbb{R}^N$**)*** *if:*
1) for each j, Y_X^j is optimal for $U_j^{\mathbb{Q}_X^j}(a^j)$,
3) $Y_X \in \mathcal{C}_{\mathbb{R}}$, $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}$-a.s.

Theorem 1.2.6 (Bühlmann, [33])**.** *For twice differentiable, concave, strictly increasing utilities $u_1, \dots, u_j : \mathbb{R} \to \mathbb{R}$ such that their risk aversions are positive Lipschitz and for $\mathcal{L} = (L^\infty(\mathbb{P}))^N$, $\mathscr{Q} = \mathcal{Q}^1$ and $X \in \mathcal{L}$, there exists a unique risk exchange equilibrium that is Pareto optimal.*

Proof. See [33]. □

In a risk exchange equilibrium with budget A, the vector $a \in \mathbb{R}^N$ such that $\sum_{j=1}^N a^j = A$ is exogenously assigned, while both the optimal exchange variable Y_X and the equilibrium price measure $\mathbb{Q}_X$ are endogenously determined. On the contrary, in a $\mathbb{Q}-$Optimal Allocation the pricing measure is assigned *a priori*, while the optimal allocation Y_X and optimal budget a_X are endogenously determined. We shall now introduce a notion which requires to endogenously recover the triple $(Y_X, \mathbb{Q}_X, a_X)$ from the systemic budget A.

1.2.4 Systemic Optimal Risk Transfer Equilibrium (SORTE)

The novel equilibrium concept presented in equations (I.8) (I.9) and (I.10) can now be formalized as follows. To this end, recall from (I.1) the definition of $\mathcal{C}_{\mathbb{R}}$ and fix a convex cone

$$\mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$$

of admissible allocations such that $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$.

Definition 1.2.7 (SORTE)**.** *The triple $(Y_X, \mathbb{Q}_X, a_X) \in \mathcal{L} \times \mathscr{Q} \times \mathbb{R}^N$ is a* ***Systemic Optimal Risk Transfer Equilibrium*** *with budget $A \in \mathbb{R}$ if:*
1) for each j, Y_X^j is optimal for $U_j^{\mathbb{Q}_X^j}(a_X^j)$,
2) a_X is optimal for $S^{\mathbb{Q}_X}(A)$,
3) $Y_X \in \mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$ and $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}$-a.s.

Remark 1.2.8. It follows from the monotonicity of each u_j that $\sum_{j=1}^N a_X^j = A$ and $\mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] = a_X^j$. Hence

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] = \sum_{j=1}^N a_X^j = A,$$

and

$$\sum_{j=1}^N Y_X^j = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] \ \mathbb{P}\text{-}a.s. \tag{1.6}$$

The main aim of Chapter 1 is to provide sufficient general assumptions that guarantee existence and uniqueness as well as good properties of a SORTE.

Remark 1.2.9. We emphasize that the existence of multiple equilibrium pricing measures $\mathbb{Q}_X = [\mathbb{Q}_X^1, ..., \mathbb{Q}_X^N]$ is a natural consequence of the presence of the - non trivial - constraints set $\mathcal{B}$. Indeed, even in the Bühlmann setting, if we add constraints, of a very simple nature, a single equilibrium pricing measure might not exists any more. Consider the following extension of a Bühlmann risk exchange equilibrium.
Let $\mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$ be fixed. We say that a pair $(\widetilde{Y}_X, \mathbb{Q}_X)$, with $\mathbb{Q}_X$ a single probability measure, is a **constrained** risk exchange equilibrium if:
(a2) for each j, $\widetilde{Y}_X^j$ maximizes: $\mathbb{E}_{\mathbb{P}}\left[u_j(x^j + X^j + \widetilde{Y}^j - \mathbb{E}_{\mathbb{Q}_X}[\widetilde{Y}^j])\right]$ among all variables $\widetilde{Y}^j$,
(b2) $\widetilde{Y}_X \in \mathcal{B}$ and $\sum_{j=1}^N \widetilde{Y}_X^j = 0$ $\mathbb{P}-$a.s.
We show with the next example that such an equilibrium (with one single probability $\mathbb{Q}_X$) **does not exist in general**. The example we present is rather simple, yet instructive, since it shows that the absence of the equilibrium arises not from technical assumptions, like integrability conditions, but is rather a structural problem caused by the presence of additional constraints. Here we provide the intuition for it. Suppose that two isolated systems of agents have, under suitable assumptions, their own (unconstrained) equilibria, and that such two equilibria do not coincide. As shown in the next example, we might then consider the two systems as one single larger system consisting of two isolated clusters, expressing this latter property with the addition of constraints. Then it is evident that an equilibrium (with a unique pricing measure) cannot exist for such a unified system.

Example 1.2.10. In order to ignore all integrability issues, in this example we assume that Ω is a finite set, endowed with the sigma algebra of all its subsets and the uniform probability measure. Consider $N = 4$, $u_j(x) := (1 - e^{-\alpha_j x})$, $\alpha_j > 0$, $j = 1, \dots, 4$, and some vectors $x \in \mathbb{R}^4$, and $X \in (L^\infty)^4$. Moreover take

$$\mathcal{B} = \left\{ Y \in \mathcal{C}_{\mathbb{R}} \mid Y^1 + Y^2 = 0,\, Y^3 + Y^4 = 0 \right\}.$$

Thus X and $\mathcal{B}$ model a single system of 4 agents which can exchange the risk only in a restricted way (agent 1 with agent 2, and agent 3 with agent 4), so that in effect the system consists of two isolated clusters of agents. Then a constrained risk exchange equilibrium in general does not exists. By contradiction, suppose that $(\widetilde{Y}_X, \mathbb{Q}_X)$ is a constrained risk exchange equilibrium. It is easy to verify that $([\widetilde{Y}_X^1, \widetilde{Y}_X^2], \mathbb{Q}_X)$ is a (unconstrained) risk exchange equilibrium with respect to $[X^1, X^2]$ and $[x^1, x^2]$ (i.e. it satisfies Items (a) and (b) of Section I.1 for $N = 2$). Similarly, $([\widetilde{Y}_X^3, \widetilde{Y}_X^4], \mathbb{Q}_X)$ is a (unconstrained) risk exchange equilibrium with respect to $[X^3, X^4]$ and $[x^3, x^4]$. This implies using equation (2) in Bühlmann [33] that

$$\frac{\exp\left(\eta(X^1 + X^2)\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\eta(X^1 + X^2)\right)\right]} = \frac{\mathrm{d}\mathbb{Q}_X}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(\theta(X^3 + X^4)\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\theta(X^3 + X^4)\right)\right]}, \quad \eta = \frac{1}{\alpha_1} + \frac{1}{\alpha_2},\ \theta = \frac{1}{\alpha_3} + \frac{1}{\alpha_4},$$

which clearly gives a contradiction, since X is arbitrary.
Observe, however, that in this example a constrained equilibrium exists **if we allow for possibly different pricing measures**, namely if we may replace the measure $\mathbb{Q}_X$ with a vector $[\mathbb{Q}_X^1, \dots, \mathbb{Q}_X^N]$. This would amount to replacing (a2) with (a3) below, namely to require that:

(a3) for each j, $\widetilde{Y}^j_X$ maximizes: $\mathbb{E}_{\mathbb{P}}\left[u_j(x^j+X^j+\widetilde{Y}^j-\mathbb{E}_{\mathbb{Q}^j_X}[\widetilde{Y}^j])\right]$ among all variables $\widetilde{Y}^j$,
(b2) $\widetilde{Y}_X\in\mathcal{B}$ and $\sum_{j=1}^N \widetilde{Y}^j_X=0$ $\mathbb{P}-$a.s.
Then such an equilibrium exists. Indeed, by the results in Bühlmann [33], we can guarantee the existence of the risk exchange equilibrium $([\widetilde{Y}^1_X,\widetilde{Y}^2_X],\mathbb{Q}^{12}_X)$ with respect to $[X^1,X^2]$ and $[x^1,x^2]$, and the risk exchange equilibrium $([\widetilde{Y}^3_X,\widetilde{Y}^4_X],\mathbb{Q}^{34}_X)$ with respect to $[X^3,X^4]$ and $[x^3,x^4]$. Then $([\widetilde{Y}^1_X,\widetilde{Y}^2_X,\widetilde{Y}^3_X,\widetilde{Y}^4_X],[\mathbb{Q}^{12}_X,\mathbb{Q}^{12}_X,\mathbb{Q}^{34}_X,\mathbb{Q}^{34}_X])$ satisfies (a3) and (b2). **The conclusion is that, even in the Bühlmann case, the presence of constraints implies multiple equilibrium pricing measures**.
From the mathematical point of view, this fact is very easy to understand in our setup, described in Assumption 1.2.12. More constraints implies a smaller set $\mathcal{B}_0$ of feasible vectors $\widetilde{Y}\in\mathcal{B}$ such that $\sum_{j=1}^N \widetilde{Y}^j_X=0$ and this in turn implies a larger polar set of $\mathcal{B}_0$ (which we will denote with $\mathcal{Q}$, see the definition in Section 1.3 Item 4. The equilibrium exists only if we are allowed to pick the pricing vector $\mathbb{Q}_X$ in this larger set $\mathcal{Q}$, but the elements in $\mathcal{Q}$ don't need to have all equal components. Economically, multiple pricing measures may arise because the risk exchange mechanism may be restricted to clusters of agents, as in this example, and agents from different clusters may well adopt a different equilibrium pricing measure. For further details on clustering, see the Examples 1.2.19 and 1.3.20.

Remark 1.2.11. We will show the existence of a triple $(Y_X,\mathbb{Q}_X,a_X)\in\mathcal{L}\times\mathscr{Q}\times\mathbb{R}^N$ verifying the three conditions in Definition 1.2.7. Hence, we also obtain the existence of the SORTE in the formulations given in (I.5), (I.6), (I.10) or in (I.8), (I.9), (I.10), for generic functional p^j verifying the conditions (i), (ii) and (iii) stated in the Introduction on page 11 (see also Remark 1.3.3).

In Chapter 1 we will work under the following Assumption 1.2.12.

Assumption 1.2.12.

(a) ***Utilities***: *$u_1,\dots,u_N:\mathbb{R}\to\mathbb{R}$ are strictly concave, strictly increasing differentiable functions with*

$$\lim_{x\to-\infty}\frac{u_j(x)}{x}=+\infty\qquad\lim_{x\to+\infty}\frac{u_j(x)}{x}=0,\ \textit{for any } j\in\{1,\dots,N\}.$$

Moreover we assume that the following property holds: for any $j\in\{1,\dots,N\}$ and $\mathbb{Q}^j\ll\mathbb{P}$

$$\mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]<+\infty \textit{ for some } \lambda>0\ \Leftrightarrow\ \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]<+\infty\,\forall\lambda>0,\tag{1.7}$$

where $v_j(y):=\sup_{x\in\mathbb{R}}\{u_j(x)-xy\}$ denotes the convex conjugate of u_j.

(b) ***Constraints***: *$\mathcal{B}\subseteq\mathcal{C}_{\mathbb{R}}$ is a convex cone, closed in probability, such that $\mathbb{R}^N+\mathcal{B}=\mathcal{B}$.*

Remark 1.2.13. In particular, Assumptions 1.2.12 (b) implies that all constant vectors belong to $\mathcal{B}$. The condition (1.7) is related to the Reasonable Asymptotic Elasticity condition on utility functions, which was introduced in [122]. This assumption, even though quite weak (see [21] Section 2.2), is fundamental to guarantee the existence of the optimal solution to classical utility maximization problems (see [21] and [122]).

Theorem 1.2.14. *A* ***Systemic Optimal Risk Transfer Equilibrium*** $(Y_X, \mathbb{Q}_X, a_X)$ *exists, with* $\mathbb{Q}^1_X, \dots, \mathbb{Q}^N_X$ *equivalent to* $\mathbb{P}$.

Theorem 1.2.15. *Under the additional Assumption that* $\mathcal{B}$ *is closed under truncation (Definition 1.3.13) the* ***Systemic Optimal Risk Transfer Equilibrium*** *is unique and is a Pareto optimal allocation.*

The formal statements and proofs are postponed to Section 1.3, Theorem 1.3.12 and Theorem 1.3.17.

Remark 1.2.16. A priori there are no reasons why a $\mathbb{Q}$-optimal allocation Y_X in Definition 1.2.3 would also satisfy the constraint $\sum_{j=1}^N Y^j_X \in \mathbb{R}$. The existence of a SORTE is indeed the consequence of the existence of a probability measure $\mathbb{Q}_X$ such that the $\mathbb{Q}_X$-optimal allocation Y_X in Definition 1.2.3 satisfies also the additional risk transfer constraint $\sum_{j=1}^N Y^j_X = A$ $\mathbb{P}-$a.s.

Remark 1.2.17. Without the additional feature expressed by 2) in the Definition 1.2.7, for all choices of a_X satisfying $\sum_{j=1}^N a^j_X = A$ there exists an equilibrium $(Y_X, \mathbb{Q}_X)$ in the sense of Definition 1.2.5 (see Theorem 1.2.6). The uniqueness of a SORTE is then a consequence of the uniqueness of the optimal solution in condition 2).

Remark 1.2.18. Depending on which one of the three objects $(Y, \mathbb{Q}, a) \in \mathcal{L} \times \mathscr{Q} \times \mathbb{R}^N$ we keep a priori fixed, we get a different notion of equilibrium (see the various definitions above). The characteristic features of the risk exchange equilibria and of a SORTE, compared with the more classical utility optimization problem in the systemic framework of Section 1.2.2, are the condition $\sum_{j=1}^N Y^j_X = A$ $\mathbb{P}-$a.s. and the existence of the equilibrium pricing vector $\mathbb{Q}_X$.

For both concepts of equilibrium (Definitions 1.2.5 and SORTE), each agent is behaving rationally by maximizing his expected utility given a budget constraint. The two approaches differ by the budget constraints. In Bühlmann's definition the vector $a \in \mathbb{R}^N$ that assigns the budget constraint ($\mathbb{E}_{\mathbb{Q}^j_X}[Y^j] \leq a_j$) is prescribed a priori. On the contrary, in the SORTE approach, the vector $a \in \mathbb{R}^N$, with $\sum_{j=1}^N a_j = A$, that assigns the budget constraint $\mathbb{E}_{\mathbb{Q}^j_X}[Y^j] \leq a_j$ is determined by optimizing the problem in condition 2), hence by taking into account the optimal systemic utility $S^{\mathbb{Q}_X}(A)$, which is (by definition) larger than the systemic utility $\sum_{j=1}^N U_j^{\mathbb{Q}^j_X}(a^j)$ in Bühlmann's equilibrium. The SORTE gives priority to the systemic aspects of the problem in order to optimize the overall systemic performance. A toy example showing the difference between a Bühlmann's equilibrium and a SORTE is provided in Section 1.4.2.

Example 1.2.19. We now consider the example of a cluster of agents, already introduced in [20]. For $h \in \{1, \cdots, N\}$, let $I := (I_m)_{m=1,\dots,h}$ be some partition of $\{1, \cdots, N\}$. We introduce the following family

$$\mathcal{B}^{(I)} = \left\{ Y \in L^0(\mathbb{R}^N) \mid \exists\, d = [d_1, \cdots, d_h] \in \mathbb{R}^h \,:\, \sum_{i \in I_m} Y^i = d_m \forall\, m = 1, \cdots, h \right\} \subseteq \mathcal{C}_{\mathbb{R}}. \tag{1.8}$$

For a given I, the values $(d_1, \cdots, d_h)$ may change, but the elements in each of the h groups I_m is fixed by the partition I. It is then easily seen that $\mathcal{B}^{(I)}$ is a linear space containing $\mathbb{R}^N$ and closed with respect to convergence in probability. We point out that the family $\mathcal{B}^{(I)}$ admits two extreme cases:

(i) the strongest restriction occurs when $h = N$, i.e., we consider exactly N groups, and in this case $\mathcal{B}^{(I)} = \mathbb{R}^N$ corresponds to no risk sharing;

(ii) on the opposite side, we have only one group $h = 1$ and $\mathcal{B}^{(I)} = \mathcal{C}_{\mathbb{R}}$ is the largest possible class, corresponding to risk sharing among all agents in the system. This is the only case considered in Bühlmann's definition of equilibrium.

Remark 1.2.20. As already mentioned in the Introduction, one additional feature of a SORTE, compared with the Bühlmann's notion, is the possibility to require, in addition to $\sum_{j=1}^N Y^j = A$ that the optimal solution belongs to a pre-assigned set $\mathcal{B}$ of admissible allocations, satisfying Assumption 1.2.12 (b). In particular, we allow for the selection of the sets $\mathcal{B} = \mathbb{R}^N$ or $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$. The characteristics of the optimal probability $\mathbb{Q}_X$ depend on the admissible set $\mathcal{B}$. For $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$, all the components of $\mathbb{Q}_X$ turn out to be equal. We also know (see Lemma 1.3.21) that for $\mathcal{B} = \mathcal{B}^{(I)}$ all the components $\mathbb{Q}_X^i$ of $\mathbb{Q}_X$ are equal for all $i \in I_m$, for each group I_m. Additional examples of sets $\mathcal{B}$ are provided in Section 1.3.5.

1.2.5 Explicit Formulas in the Exponential Case

We believe it is now instructive to anticipate the explicit solution to the SORTE problem in the exponential case for $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$. This is a particular case of a more general situation treated in detail in Section 1.4.

Theorem 1.2.21. *Take exponential utilities*

$$u_j(x) := 1 - \exp(-\alpha_j x),\ j = 1, \dots, N \quad \textit{for} \quad \alpha_1, \dots, \alpha_N > 0.$$

Then the SORTE for $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$ is given by

$$\begin{cases} \widehat{Y}^k = -X^k + \frac{1}{\alpha_k}\left(\frac{\overline{X}}{\beta}\right) + \frac{1}{\alpha_k}\left[\frac{A}{\beta} + \ln(\alpha_k) - \mathbb{E}_R[\ln(\alpha)]\right] & k = 1, \dots, N \\ \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\right]} =: \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} & k = 1, \dots, N \\ \widehat{a}^k = \mathbb{E}_{\widehat{\mathbb{Q}}^k}[\widehat{Y}^k] & k = 1, \dots, N \end{cases} \tag{1.9}$$

where $\beta := \sum_{j=1}^N \frac{1}{\alpha_j}$, $\overline{X} := \sum_{j=1}^N X^j$, $R(n) := \frac{\frac{1}{\alpha_j}}{\sum_{k=1}^N \frac{1}{\alpha_k}}$ *for* $j = 1, ...N$, $\alpha := (\alpha_1, ..., \alpha_N)$, $\mathbb{E}_R[\ln(\alpha)] = \sum_{j=1}^N R(n)\ln(\alpha_j)$.

1.3 Proof of Theorem 1.2.14 and Theorem 1.2.15

We need to introduce the following concepts and notations:

1. The utility functions in Assumption 1.2.12 induce an Orlicz space structure: see Section 1.5.1 for the details and the definitions of the spaces L^Φ and M^Φ. Here we just recall the following inclusions among the Banach spaces: $(L^\infty(\mathbb{P}))^N \subseteq M^\Phi \subseteq L^\Phi \subseteq (L^1(\mathbb{P}))^N$ and that for a vector of probability measures $\mathbb{Q} \ll \mathbb{P}$ the condition $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^{\Phi^*}$ implies $L^\Phi \subseteq L^1(\mathbb{Q})$. From now on in Chapter 1 we assume that $X \in M^\Phi$.

2. For any $A \in \mathbb{R}$ we set
$$\mathcal{B}_A := \mathcal{B} \cap \left\{ Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j \leq A \ \mathbb{P}\text{-}a.s. \right\}.$$
Observe that $\mathcal{B}_0 \cap M^\Phi$ is a convex cone.

3. We introduce the following problem for $X \in M^\Phi$ and for a vector of probability measures $\mathbb{Q} \ll \mathbb{P}$, with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^{\Phi^*}$,
$$\pi^{\mathbb{Q}}(A) := \sup \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[u_j \left(X^j + Y^j \right) \right] \mid Y \in M^\Phi, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \leq A \right\}. \quad (1.10)$$
Notice that in (1.10) the vector Y is not required to belong to $\mathcal{C}_{\mathbb{R}}$, but only to the vector space M^Φ. In order to show the existence of the optimal solution to the problem $\pi^{\mathbb{Q}}(A)$, it is necessary to enlarge the domain in (1.10).

4. $\mathcal{Q}$ is the set of vectors of probability measures defined by
$$\mathcal{Q} := \left\{ \mathbb{Q} \ll \mathbb{P} \mid \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^{\Phi^*}, \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right] \leq 0, \forall Y \in \mathcal{B}_0 \cap M^\Phi \right\}$$
where $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \left[\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right]$. Identifying Radon-Nikodym derivatives and measures in the natural way, $\mathcal{Q}$ turns out to be the set of normalized (i.e. with componentwise expectations equal to 1), non negative vectors in *the polar of* $\mathcal{B}_0 \cap M^\Phi$ *in the dual system* (M^Φ, L^{Φ^*}). In our N-dimensional systemic one-period setting, the set $\mathcal{Q}$ plays the same crucial role as the set of martingale measures in multiperiod stochastic securities markets.

5. We introduce the following convex subset of $\mathcal{Q}$:
$$\mathcal{Q}_v := \mathcal{Q} \ \cap \ \left\{ \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^{\Phi^*} \mid \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \geq 0 \, \forall j, \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[v_j \left(\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \right] < +\infty \right\}. \quad (1.11)$$

6. Set
$$\mathcal{L} := \bigcap_{\mathbb{Q} \in \mathcal{Q}_v} L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N), \qquad \mathscr{Q} := \mathcal{Q}_v. \quad (1.12)$$

Note that $M^\Phi \subseteq \mathcal{L}$ and that $\mathcal{L}$ has the product structure $\mathcal{L} = \mathcal{L}^1 \times \cdots \times \mathcal{L}^N$: let Proj_j denote the projection on the $n-$th component, defined on $\mathcal{Q}_v$, and take the corresponding image $\mathcal{Q}_j := \mathrm{Proj}_j(\mathcal{Q}_v)$ (consisting of a family of probability measures, all absolutely continuous with respect to $\mathbb{P}$). Set $\mathcal{L}^j := \bigcap_{\mathbb{Q}\in\mathcal{Q}_j} L^1(\mathbb{Q})$. Then $\mathcal{L} = \mathcal{L}^1 \times \cdots \times \mathcal{L}^N$.
We will consider the optimization problems (1.1), (1.2) and (1.3) with the particular choice of $(\mathcal{L}, \mathscr{Q})$ in (1.12) and will show that, with such choice, $\pi^{\mathbb{Q}}(A) = \Pi^{\mathbb{Q}}(A)$. Observe that if all utility functions are bounded from above, the requirement

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$$

is redundant, but it becomes important if we allow for utility functions to be unbounded.
We also require some additional definitions and notations:

a) $\overline{\mathcal{B}_0}$ is the polar of the cone $co(\mathcal{Q}_v)$ in the dual pair

$$\left(L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}, \bigcap_{\mathbb{Q}\in\mathcal{Q}_v} L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)\right),$$

that is

$$\overline{\mathcal{B}_0} := \left\{Y \in \bigcap_{\mathbb{Q}\in\mathcal{Q}_v} L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N) \mid \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \le 0, \forall \mathbb{Q} \in \mathcal{Q}_v\right\}.$$

It is easy to verify that

$$\mathcal{B}_0 \cap M^\Phi \subseteq \overline{\mathcal{B}_0}.$$

b) For any $A \in \mathbb{R}$ we define $\overline{\mathcal{B}_A}$ as the set

$$\overline{\mathcal{B}_A} := \left\{Y \in \bigcap_{\mathbb{Q}\in\mathcal{Q}_v} L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N) \mid \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \le A, \forall \mathbb{Q} \in \mathbb{Q}_v\right\}.$$

We will prove that $\overline{\mathcal{B}_A}$ is the correct enlargement of the domain $\mathcal{B}_A \cap M^\Phi$ in order to obtain the existence of the optimal solution of the primal problem.

c) $\{e_i\}_{i=1,\dots,N}$ is the canonical base of $\mathbb{R}^N$.

Lemma 1.3.1. *In the dual pair (M^Φ, L^{Φ^*}), consider the polar $(\mathcal{B}_0 \cap M^\Phi)^0$ of $\mathcal{B}_0 \cap M^\Phi$. Then $(\mathcal{B}_0 \cap M^\Phi)^0 \cap (L^0_+)^N$ is the cone generated by $\mathcal{Q}$.*

Proof. From the definition of $\mathcal{B}_0$ and the fact that $\mathcal{B}$ contains all constant vectors, we may conclude that all vectors in $\mathbb{R}^N$ of the form $e_i - e_j$ belong to $\mathcal{B}_0 \cap M^\Phi$. Then for all $Z \in (\mathcal{B}_0 \cap M^\Phi)^0$ and for all $i, j \in \{1, \dots, N\}$ we must have: $\mathbb{E}_{\mathbb{P}}[Z^i] - \mathbb{E}_{\mathbb{P}}[Z^j] \le 0$. As a consequence, $Z \in (\mathcal{B}_0 \cap M^\Phi)^0$ implies $\mathbb{E}_{\mathbb{P}}[Z^1] = \cdots = \mathbb{E}_{\mathbb{P}}\left[Z^N\right]$ and so

$$(\mathcal{B}_0 \cap M^\Phi)^0 \cap (L^0_+)^N = \mathbb{R}_+ \cdot \mathcal{Q}, \tag{1.13}$$

where $\mathbb{R}_+ := \{b \in \mathbb{R}, b \ge 0\}$. □

Lemma 1.3.2. $\mathcal{Q}_v^e := \{\mathbb{Q} \in \mathcal{Q}_v \ s.t.\ \mathbb{Q} \sim \mathbb{P}\} \neq \varnothing$.

Proof. The condition $\mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$ implies $\mathcal{B}_0 \cap M^\Phi \subseteq (\mathcal{C}_{\mathbb{R}} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})$, so that the polars satisfy the opposite inclusion: $(\mathcal{C}_{\mathbb{R}} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})^0 \subseteq (\mathcal{B}_0 \cap M^\Phi)^0$. Observe now that any vector $[Z, \dots, Z]$, for $Z \in L^\infty_+$, belongs to $(\mathcal{C}_{\mathbb{R}} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})^0$. In particular, $(\mathcal{B}_0 \cap M^\Phi)^0$ contains vectors in the form $\left[\frac{\varepsilon+Z}{1+\varepsilon}, \dots, \frac{\varepsilon+Z}{1+\varepsilon}\right]$ with $\varepsilon > 0$ and $Z \in L^\infty_+$, $\mathbb{E}_{\mathbb{P}}[Z] = 1$. Each component of such a vector has expectation equal to 1, belongs to L^∞_+ and satisfies $\frac{\varepsilon+Z}{1+\varepsilon} \geq \frac{\varepsilon}{1+\varepsilon}$. All these conditions imply that there exists a probability vector $\mathbb{Q} \in \mathcal{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$ $\mathbb{P}-$a.s. with $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\frac{d\mathbb{Q}^j}{d\mathbb{P}}\right)\right] < \infty$, hence $\mathcal{Q}_v^e \neq \varnothing$. □

1.3.1 Scheme of the proof

The proof of Theorem 1.2.14 is inspired by the classical duality theory in utility maximization, see for example [47] and [102] and by the minimax approach developed in [15]. More precisely, our road map will be the following:

1. First we show, in Remark 1.3.4, how we may reduce the problem to the case $A = 0$.

2. We consider

$$\pi(A) := \sup\left\{\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] \mid Y \in M^\Phi \cap \mathcal{B}, \sum_{j=1}^N Y^j \leq A \ \mathbb{P}\text{-}a.s.\right\}. \quad (1.14)$$

   In Theorem 1.3.5 we specialize the duality, obtained in Theorem 1.5.3 for a generic convex cone $\mathcal{C}$, for the maximization problem $\pi(0)$ over the convex cone $\mathcal{C} = \mathcal{B}_0 \cap M^\Phi$ and prove: (i) the existence of the optimizer $\widehat{Y}$ of $\pi(0)$, which belongs to $\mathcal{B}_0$; (ii) the existence of the optimizer $\widehat{\mathbb{Q}}$ to the dual problem of $\pi(0)$. Here we need all the assumptions on the utility functions and on the set $\mathcal{B}$ and an auxiliary result stated in Theorem 1.5.4.

3. Proposition 1.3.7 will show that also the elements in the closure of $\mathcal{B} \cap M^\Phi$ satisfy the key condition $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] \leq \sum_{j=1}^N Y^j \in \mathbb{R}$ for all $\mathbb{Q} \in \mathcal{Q}$.

4. Theorem 1.5.3 is then again applied, to a different set

$$\mathcal{C} = \left\{Y \in M^\Phi \mid \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq 0\right\}$$

   to derive Proposition 1.3.9, which establishes the duality for $\pi^{\mathbb{Q}}(0)$ and $\pi^{\mathbb{Q}}(A)$ in case a fixed probability vector $\mathbb{Q}$ is assigned.

5. The minimax duality:

$$\pi(A) = \min_{\mathbb{Q} \in \mathcal{Q}_v} \pi^{\mathbb{Q}}(A) = \pi^{\widehat{\mathbb{Q}}}(A),$$

   is then a simple consequence of the above results (see Corollary 1.3.10). This duality is the key tool to prove the existence of a SORTE (see Theorem 1.3.12).

6. Uniqueness and Pareto optimality are then proved in Theorem 1.3.17.

Remark 1.3.3. Notice that in the definition of $\pi(A)$ there is no reference to a probability vector $\mathbb{Q}$. However, the optimizer of the dual formulation of $\pi(A)$ is a probability vector $\widehat{\mathbb{Q}}$ (that will be the equilibrium pricing vector in the SORTE). Even if in the equations (I.8), (I.9), (I.10) we do not a priori require pricing functional of the form $p^j(\cdot) = \mathbb{E}_{\mathbb{Q}^j}[\cdot]$, this particular linear expression naturally appears from the dual formulation.

1.3.2 Minimax Approach

Remark 1.3.4. Only in this Remark, we need to change the notation a bit: we make the dependence of our maximization problems on the initial point explicit. To this end we will write

$$\pi_X(A) := \sup\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+Y^j\right)\right] \mid Y \in \mathcal{B}_A \cap M^\Phi\right\},$$
$$\pi_X^\mathbb{Q}(A) := \sup\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+Y^j\right)\right] \mid Y \in M^\Phi,\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \le A\right\}.$$

It is possible to reduce the maximization problem expressed by $\pi_X(A)$ (and similarly by $\pi_X^\mathbb{Q}(A)$) to the problem related to $\pi_\cdot(0)$ (respectively, $\pi_\cdot^\mathbb{Q}(0)$) by using the following simple observation: for any $a_0 \in \mathbb{R}^N$ with $\sum_{j=1}^N a_0^j = A$ consider

$$\begin{aligned}\pi_X(A) &= \sup\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+Y^j\right)\right] \mid Y \in \mathcal{B} \cap M^\Phi, \sum_{j=1}^N Y^j \le A\right\}\\ &= \sup\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+a_0^j+(Y^j-a_0^j)\right)\right] \mid Y \in \mathcal{B} \cap M^\Phi, \sum_{j=1}^N \left(Y^j-a_0^j\right) \le 0\right\}\\ &= \sup\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+a_0^j+Z^j\right)\right] \mid Z \in \mathcal{B}_0 \cap M^\Phi\right\},\end{aligned}$$

where last equality holds as we are assuming that $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$. The last line represents the original problem, but with $A = 0$ and a different initial point. This fact will be used in the conclusion of the proof of Theorem 1.3.5.

In the following Theorem we follow a minimax procedure inspired by the technique adopted in [21].

Theorem 1.3.5. *Under Assumption 1.2.12 we have*

$$\pi(A) := \sup_{Y \in \mathcal{B}_A \cap M^\Phi} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+Y^j\right)\right] = \max_{Y \in \overline{\mathcal{B}_A}} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j+Y^j\right)\right] \tag{1.15}$$
$$= \min_{\mathbb{Q}\in\mathcal{Q}} \min_{\lambda \in \mathbb{R}_{++}} \left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + A\right) + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{1.16}$$

The minimization problem in (1.16) *admits a unique optimum* $(\widehat{\lambda}, \widehat{\mathbb{Q}}) \in \mathbb{R}_{++} \times \mathcal{Q}$ *with* $\widehat{\mathbb{Q}} \sim \mathbb{P}$. *The maximization problem in (1.15) admits a unique optimum* $\widehat{Y} \in \overline{\mathcal{B}_A}$, *given by*

$$\widehat{Y}^j = -X^j - v_j'\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right),\ j = 1, ..., N, \tag{1.17}$$

which belongs to $\mathcal{B}_A$. *In addition,*

$$\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[\widehat{Y}^j\right] = A \quad \text{and} \quad \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[\widehat{Y}^j\right] \leq A \quad \forall \mathbb{Q} \in \mathcal{Q}_v. \tag{1.18}$$

Proof. We first prove the result for the case $A = 0$.
STEP 1
We first show that

$$\sup_{\mathcal{B}_0 \cap M^\Phi} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] < \sum_{j=1}^N v_j(0) = \sum_{j=1}^N u_j(+\infty)\ \ \forall X \in\ M^\Phi \tag{1.19}$$

so that we will be able to apply Theorem 1.5.3 with the choice $\mathcal{C} := \mathcal{B}_0 \cap M^\Phi$. We distinguish two possible cases: $\sum_{j=1}^N u_j(+\infty) = +\infty$ or $\sum_{j=1}^N u_j(+\infty) < +\infty$.
For $\sum_{j=1}^N u_j(+\infty) = +\infty$: observe that for any $\mathbb{Q} \in \mathcal{Q}_v$ (which is nonempty by Lemma 1.3.2) and $\lambda > 0$ we have

$$\begin{aligned}\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] &\leq \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(X^j + Y^j)\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] \\ &\leq \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right].\end{aligned}$$

We exploited above Fenchel inequality and the definition of $\mathcal{Q}_v$. Observing that the last line does not depend on Y and is finite, and using the well known relation $v_j(0) = u_j(+\infty), j = 1, \dots, N$, we conclude that

$$\sup_{\mathcal{B}_0 \cap M^\Phi} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] < +\infty = \sum_{j=1}^N v_j(0).$$

For $\sum_{j=1}^N u_j(+\infty) < +\infty$: if the inequality in (1.19) were not strict, for any maximizing sequence $(Y_m)_m$ we would have, by monotone convergence, that

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(+\infty\right)\right] - \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j_m\right)\right] = \mathbb{E}_{\mathbb{P}}\left[\left|\sum_{j=1}^N \left(u_j(+\infty) - u_j(X^j + Y^j_m)\right)\right|\right] \underset{m}{\to} 0.$$

Up to taking a subsequence we can assume the convergence is also almost sure. Since all the terms in $\sum_{j=1}^N (u_j(+\infty) - u_j(X^j + Y^j_m))$ are non negative, we also see that $u_j(X^j + Y^j_m) \to_m u_j(+\infty)$ almost surely for every $j = 1, \dots, N$. By strict monotonicity of the utilities, this would imply that, for each j, $Y^j_m \to_m +\infty$. This clearly contradicts the constraint $Y_m \in \mathcal{B}_0$.

STEP 2

We will prove equations (1.15) and (1.16), with a supremum over $\overline{\mathcal{B}_A}$ in place of a maximum, since we will show in later steps (STEP 4) that this supremum is in fact a maximum.

We observe that since $\mathcal{B}_0 \cap M^\Phi \subseteq \overline{\mathcal{B}_0}$

$$\sup_{\mathcal{B}_0 \cap M^\Phi} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right] \le \sup_{\overline{\mathcal{B}_0}} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right].$$

Moreover, by the Fenchel inequality

$$\sup_{\overline{\mathcal{B}_0}} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right] \le \inf_{\lambda \in \mathbb{R}_+, \, \mathbb{Q} \in \mathcal{Q}} \left(\lambda \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right).$$

Equations (1.15) and (1.16) follow from Theorem 1.5.3 replacing there the convex cone $\mathcal{C}$ with $\mathcal{B}_0 \cap M^\Phi$ and using equation (1.13), which shows that $(\mathcal{C}_1^0)^+ = \mathcal{Q}$.

STEP 3

We prove that if $\widehat{\lambda}$ and $\widehat{\mathbb{Q}}$ are optima in equation (1.16), then $\widehat{Y}^j := -X^j - v_j'\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)$ defines an element in $\overline{\mathcal{B}_0}$. Observe that $\widehat{\lambda}$ minimizes the function

$$\mathbb{R}_{++} \ni \gamma \mapsto \psi(\gamma) := \sum_{j=1}^N \left(\gamma \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j\right] + \mathbb{E}_\mathbb{P}\left[v_j\left(\gamma \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right)$$

which is real valued and convex. Also we have by Monotone Convergence Theorem and Lemma 1.5.2.1. that the right and left derivatives, which exist by convexity, satisfy

$$\frac{\mathrm{d}^\pm \psi}{\mathrm{d}\gamma}(\gamma) = \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j'\left(\gamma \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right],$$

hence the function is differentiable. Since $\widehat{\lambda}$ is a minimum for ψ, this implies $\psi'(\widehat{\lambda}) = 0$, which can be rephrased as

$$\sum_{j=1}^N \left(\mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \mathbb{E}_\mathbb{P}\left[v_j'\left(\widehat{\lambda} \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]\right) = 0, \tag{1.20}$$

i.e.,

$$\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[\widehat{Y}^j\right] = 0. \tag{1.21}$$

Now consider minimizing

$$\mathbb{Q} \mapsto \sum_{j=1}^N \left(\widehat{\lambda} \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \mathbb{E}_\mathbb{P}\left[v_j\left(\widehat{\lambda} \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right)$$

for fixed $\widehat{\lambda}$ and $\mathbb{Q}$ varying in $\mathcal{Q}_v$. Let again $\widehat{\mathbb{Q}}$, with $\widehat{\eta} := \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}$, be an optimum and consider another $\mathbb{Q} \in \mathcal{Q}_v$, with $\eta := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. By Assumption 1.2.12, the expression $\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]$ is finite for all choices of λ. Observe that by convexity and differentiability of v_j we have

$$\widehat{\lambda}\eta^j v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right) \leq \widehat{\lambda}\widehat{\eta}^j v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right) + v_j\left(\widehat{\lambda}\eta^j\right) - v_j\left(\widehat{\lambda}\widehat{\eta}^j\right).$$

Hence by Lemma 1.5.2.1. and $\widehat{\mathbb{Q}}, \mathbb{Q} \in \mathcal{Q}_v$ we conclude that

$$\left(\eta^j v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\right)^+ \in L^1(\mathbb{P}). \tag{1.22}$$

To prove that also the negative part is integrable, we take a convex combination of $\widehat{\mathbb{Q}}, \mathbb{Q} \in \mathcal{Q}_v$, which still belongs to $\mathcal{Q}_v$. By optimality of $\widehat{\eta}$ the function φ defined for $x \in [0,1]$ as

$$x \mapsto \varphi(x) := \sum_{j=1}^N \left(\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\left((1-x)\widehat{\eta}^j + x\eta^j\right)\right] + \mathbb{E}_\mathbb{P}\left[v_j\left(\widehat{\lambda}\left((1-x)\widehat{\eta}^j + x\eta^j\right)\right)\right]\right)$$

has a minimum at 0, thus the right derivative of φ at 0 must be non negative, so that:

$$\begin{aligned}&\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}x}\Big|_0 \left((1-x)\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\widehat{\eta}^j\right] + x\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\eta^j\right]\right)\\ &\geq -\sum_{j=1}^N \frac{\mathrm{d}}{\mathrm{d}x}\Big|_0 \mathbb{E}_\mathbb{P}\left[v_j\left((1-x)\widehat{\lambda}\widehat{\eta}^j + x\widehat{\lambda}\eta^j\right)\right].\end{aligned} \tag{1.23}$$

Define $H_j(x) := v_j\left((1-x)\widehat{\lambda}\widehat{\eta}^j + x\widehat{\lambda}\eta^j\right)$ and observe that as $x \downarrow 0$ by convexity

$$\begin{aligned}0 &\leq \left(-\frac{1}{x}\left(H_j(x) - H_j(0)\right) + H_j(1) - H_j(0)\right)\\ &\uparrow \left(-\widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j + \widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j + H_j(1) - H_j(0)\right).\end{aligned} \tag{1.24}$$

Write now explicitly equation (1.23) in terms of incremental ratios and add and subtract the real number $\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left(H_j(1) - H_j(0)\right)\right]$ to get for $\Delta_j := H_j(1) - H_j(0)$

$$\lim_{x\downarrow 0}\sum_{j=1}^N \left(\frac{1}{x}\left[\left((1-x)\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\widehat{\eta}^j\right] + x\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\eta^j\right]\right) - \widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\widehat{\eta}^j\right]\right] + \mathbb{E}_\mathbb{P}\left[\Delta_j\right]\right) \tag{1.25}$$

$$\geq \lim_{x\downarrow 0}\sum_{j=1}^N \left(\mathbb{E}_\mathbb{P}\left[-\frac{1}{x}\left(H_j(x) - H_j(0)\right) + \Delta_j\right]\right). \tag{1.26}$$

The first limit is trivial. Observe that by (1.24) and Monotone Convergence Theorem we also may compute the second limit and then deduce:

$$\begin{aligned}&\sum_{j=1}^N \left(\widehat{\lambda}\mathbb{E}_\mathbb{P}\left[X^j\left(\eta^j - \widehat{\eta}^j\right)\right] + \mathbb{E}_\mathbb{P}\left[H_j(1) - H_j(0)\right]\right)\\ &\geq \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[-\widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j + \widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j + H_j(1) - H_j(0)\right]\end{aligned}$$

and therefore

$$+\infty > \sum_{j=1}^{N} \widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\left(\eta^j - \widehat{\eta}^j\right)\right] \geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}\left(-\widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j + \widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j\right)\right]$$

$$= \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}\left(\widehat{\lambda}\left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^- - \widehat{\lambda}\left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^+ + \widehat{\lambda}v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j\right)\right].$$

Since $\sum_{j=1}^{N} v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j \in L^1(\mathbb{P})$ by Lemma 1.5.2.1, and $\sum_{j=1}^{N}\left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^+ \in L^1(\mathbb{P})$ by equation (1.22), we deduce that $\sum_{j=1}^{N}\left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^- \in L^1(\mathbb{P})$ so that

$$0 \leq \left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^- \leq \sum_{j=1}^{N}\left(v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right)^- \in L^1(\mathbb{P}).$$

We conclude that $v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j$ defines a vector in $L^1(\mathbb{P}) \times \cdots \times L^1(\mathbb{P})$, hence

$$\widehat{Y} \in L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N) \quad \forall \mathbb{Q} \in \mathcal{Q}_v. \tag{1.27}$$

Moreover equation (1.23) can be rewritten as:

$$0 \leq \sum_{j=1}^{N} \widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\left(\eta^j - \widehat{\eta}^j\right)\right] + \sum_{j=1}^{N} \widehat{\lambda}\mathbb{E}\left[v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\left(\eta^j - \widehat{\eta}^j\right)\right]. \tag{1.28}$$

Now rearrange the terms in (1.28)

$$0 \leq -\sum_{j=1}^{N}\widehat{\lambda}\left(\mathbb{E}_{\mathbb{P}}\left[X^j\widehat{\eta}^j\right] + \mathbb{E}_{\mathbb{P}}\left[v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\widehat{\eta}^j\right]\right) + \sum_{j=1}^{N}\widehat{\lambda}\left(\mathbb{E}_{\mathbb{P}}\left[X^j\eta^j\right] + \mathbb{E}_{\mathbb{P}}\left[v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\eta^j\right]\right)$$

and use (1.20):

$$0 \leq 0 - \sum_{j=1}^{N}\widehat{\lambda}\left(\mathbb{E}_{\mathbb{P}}\left[\left(-X^j - v_j'\left(\widehat{\lambda}\widehat{\eta}^j\right)\right)\eta^j\right]\right) = -\widehat{\lambda}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right].$$

This proves that

$$\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[\widehat{Y}^j\right] \leq 0 \quad \forall \mathbb{Q} \in \mathcal{Q}_v \tag{1.29}$$

and then $\widehat{Y} \in \overline{\mathcal{B}_0}$.

STEP 4 (Optimality of $\widehat{Y}$)

Under our standing Assumption 1.2.12 it is well known that $u(-v'(y)) = v(y) - yv'(y)$, $\forall y \geq 0$. As a consequence we get by direct substitution

$$u_j(X^j + \widehat{Y}^j) = u_j\left(-v_j'\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right) = -\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}v_j'\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right) + v_j\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)$$

and

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+\widehat{Y}^j\right)\right]=\widehat{\lambda}\left(-\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}v_j'\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right)+\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right].$$

Use now the expression in (1.20) to substitute in the first RHS term:

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+\widehat{Y}^j\right)\right]=\widehat{\lambda}\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[X^j]+\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right].$$

The optimality of $\widehat{Y}$ follows then by optimality of $(\widehat{\lambda},\widehat{\mathbb{Q}})$ in (1.16).
Using now our findings in STEP 2 together with optimality of $\widehat{Y}$, the proof of equation (1.15) is now complete.

STEP 5 ($\widehat{Y}\in\mathcal{B}_0$)

By Lemma 1.3.2 there exists a $\mathbb{Q}\in\mathcal{Q}_v^e:=\{\mathbb{Q}\in\mathcal{Q}_v \text{ s.t. } \mathbb{Q}\sim\mathbb{P}\}$ and from (1.27) we know that $v_j'\left(\lambda\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\in L^1(\mathbb{Q}^j)$, $\lambda>0$. Also, for every $j=1,\dots,N$, $v_j'(0+)=-\infty$, so that $\mathbb{Q}^j\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}=0\right)=0$. As $\mathbb{Q}\sim\mathbb{P}$, this in turn implies $\mathbb{P}\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}=0\right)=0$, for every $j=1,\dots,N$. Hence $\widehat{\mathbb{Q}}\sim\mathbb{P}$. Theorem 1.5.4 now can be applied to $K:=(\mathcal{B}_0\cap M^\Phi)$ and $\mathcal{Q}_v^e$ to get

$$\bigcap_{\mathbb{Q}\in\mathcal{Q}_v^e} cl_{\mathbb{Q}}\left((\mathcal{B}_0\cap M^\Phi)-L^1_+(\mathbb{Q})\right)=\left\{Z\in\bigcap_{\mathbb{Q}\in\mathcal{Q}_v^e}L^1(\mathbb{Q})\mid \sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}\left[Z^j\right]\le 0\ \forall\mathbb{Q}\in\mathcal{Q}_v^e\right\}. \tag{1.30}$$

As $\widehat{Y}\in\overline{\mathcal{B}_0}$ and $\overline{\mathcal{B}_0}$ is included in the RHS of (1.30), we deduce that $\widehat{Y}$ belongs to the LHS of (1.30). Now by equation (1.21) we see that $\widehat{Y}$ satisfies $\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]=0$, and this implies that:

$$\widehat{Y}\in cl_{\widehat{\mathbb{Q}}}\left(\mathcal{B}_0\cap M^\Phi\right), \tag{1.31}$$

the $L^1(\widehat{\mathbb{Q}}^1)\times\cdots\times L^1(\widehat{\mathbb{Q}}^1)$-(norm) closure of $\mathcal{B}_0\cap M^\Phi$. In particular $\widehat{Y}$ is a $\widehat{\mathbb{Q}}$ (hence $\mathbb{P}$)- a.s. limit of elements in $\mathcal{B}_0$ which is closed in probability $\mathbb{P}$, so that $\widehat{Y}\in\mathcal{B}_0$.

STEP 6

The conditions in (1.18) are proved in (1.21) and (1.29). We conclude with uniqueness. By the strict concavity of the utilities and the convexity of $\overline{\mathcal{B}_0}$, it is evident that the maximization problem given by $\sup_{\overline{\mathcal{B}_0}}\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+Y^j\right)\right]$ admits at most one optimum. Now clearly if $(\widehat{\lambda},\widehat{\mathbb{Q}})$ and $(\widetilde{\lambda},\widetilde{\mathbb{Q}})$ are optima for the minimax expression (1.16), they both give rise to two optima $\widehat{Y}$, $\widetilde{Y}$ as in the previous steps. Uniqueness of the solution for the primal problem implies $\widehat{Y}=\widetilde{Y}$. Under Assumption 1.2.12.(a) the functions $v_1',\dots,v_N'$ are injective and therefore we conclude that $\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}=\widetilde{\lambda}\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}$. Taking expectations we get $\widehat{\lambda}=\widetilde{\lambda}$ and then $(\widehat{\lambda},\widehat{\mathbb{Q}})=(\widetilde{\lambda},\widetilde{\mathbb{Q}})$.

Conclusion

The more general case $A\neq 0$ can be obtained using Remark 1.3.4. We just sketch one step of the proof, as the other steps follows similarly. Using a_0 as in Remark 1.3.4, in STEP 3 we see that

$$0\le-\widehat{\lambda}\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]+\widehat{\lambda}\sum_{j=1}^N a_0^j$$

which yields that $\widehat{Y} \in \overline{\mathcal{B}_A}$. $\square$

Remark 1.3.6. Notice that $Y \in \mathcal{B} \cap M^\Phi$ implies that $Z \in \mathcal{B}_0$, where Z is defined by $Z^j := Y^j - x^j \sum_{k=1}^N Y^k$ for any $x \in \mathbb{R}^N$ such that $\sum_{j=1}^N x^j = 1$. To see this, recall that we are assuming that $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$. As $\sum_{j=1}^N Y^j \in \mathbb{R}$, then $Z \in \mathcal{B}$ and, since also trivially integrability is preserved and $\sum_{j=1}^N Z^j = 0$, we conclude that $Z \in \mathcal{B}_0$.

Proposition 1.3.7. *For all $Y \in \mathcal{B} \cap M^\Phi$ and $\mathbb{Q} \in \mathcal{Q}$*

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq \sum_{j=1}^N Y^j. \tag{1.32}$$

Moreover, denoting by $cl_{\mathbb{Q}}\left(\mathcal{B} \cap M^\Phi\right)$ the $L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)$-norm closure of $\mathcal{B} \cap M^\Phi$, inequality (1.32) holds for all $Y \in cl_{\mathbb{Q}}\left(\mathcal{B} \cap M^\Phi\right)$ and $\mathbb{Q} \in \mathcal{Q}$, $\mathbb{Q} \sim \mathbb{P}$. In particular, (1.32) holds for $\widehat{\mathbb{Q}} \sim \mathbb{P}$ and $\widehat{Y} \in cl_{\widehat{\mathbb{Q}}}\left(\mathcal{B}_0 \cap M^\Phi\right)$ defined in Theorem 1.3.5.

Proof. Take $Y \in \mathcal{B} \cap M^\Phi$ and argue as in Remark 1.3.6, with the notation introduced there. By the definition of the polar, $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Z^j \varphi^j\right] \leq 0$ for all $\varphi \in (\mathcal{B} \cap M^\Phi)^0$, and in particular for all $\mathbb{Q} \in \mathcal{Q}$

$$0 \geq \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Z^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] - \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[x^j \left(\sum_{k=1}^N Y^k\right) \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]$$

where we recognize $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] - \sum_{j=1}^N Y^j$ in RHS. As to the second claim, take a sequence $(k_j)_j$ in $\mathcal{B} \cap M^\Phi$ converging both $\mathbb{Q}$-almost surely (hence $\mathbb{P}$-a.s.) and in norm to Y and apply (1.32) to k_j to get

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] = \lim_j \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[k_j^j\right] \overset{\mathbb{P}\text{-a.s.}}{\leq} \liminf_j \left(\sum_{j=1}^N k_j^j\right) \overset{\mathbb{P}\text{-a.s.}}{=} \sum_{j=1}^N Y^j. \tag{1.33}$$

$\square$

Remark 1.3.8. In particular (1.32) shows that $\forall\, \mathbb{Q} \in \mathcal{Q}$

$$\left\{Y \in \mathcal{B} \cap M^\Phi \mid \sum_{j=1}^N Y^j \leq A\right\} \subseteq \left\{Y \in M^\Phi \mid \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq A\right\}$$

and therefore $\pi(A) \leq \pi^{\mathbb{Q}}(A)$.

1.3.3 Utility Maximization with a fixed probability measure

The following represents a counterpart to Theorem 1.3.5, once a measure is fixed a priori.

Proposition 1.3.9. *Fix $\mathbb{Q} \in \mathcal{Q}_v$. If $\pi^{\mathbb{Q}}(A) < +\infty$, then*

$$\pi^{\mathbb{Q}}(A) = \Pi^{\mathbb{Q}}(A) = \sup \left\{ \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[u_j \left(X^j + Y^j \right) \right] \mid Y \in L^1(\mathbb{Q}), \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \le A \right\} \tag{1.34}$$

$$= \min_{\lambda \in \mathbb{R}_+} \left(\lambda \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[X^j \right] + A \right) + \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[v_j \left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \right] \right).$$

If additionally any of the two expressions is strictly less than $\sum_{j=1}^{N} u_j(+\infty)$, then

$$\pi^{\mathbb{Q}}(A) = \min_{\lambda \in \mathbb{R}_{++}} \left(\lambda \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[X^j \right] + A \right) + \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[v_j \left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \right] \right). \tag{1.35}$$

Proof. Again, we prove the case $A = 0$ since Remark 1.3.4 can be used to obtain the general case $A \neq 0$. From $M^{\Phi} \subseteq \mathcal{L} \subseteq L^1(\mathbb{Q})$ we obtain:

$$\pi^{\mathbb{Q}}(0) := \sup \left\{ \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[u_j \left(X^j + Y^j \right) \right] \mid Y \in M^{\Phi}, \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \le 0 \right\} \le \Pi^{\mathbb{Q}}(0)$$

$$\le \sup \left\{ \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[u_j \left(X^j + Y^j \right) \right] \mid Y \in L^1(\mathbb{Q}), \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \le 0 \right\}$$

$$\le \min_{\lambda \in \mathbb{R}_+} \left(\lambda \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[X^j \right] + \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[v_j \left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \right] \right) \tag{1.36}$$

by the Fenchel inequality. Define the convex cone

$$\mathcal{C} := \left\{ Y \in M^{\Phi} \mid \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \le 0 \right\}.$$

The hypotheses on $\mathcal{C}$ of Theorem 1.5.3 hold true and inequality (1.36) shows that $\pi^{\mathbb{Q}}(0) < +\infty$ for all $X \in M^{\Phi}$. The finite dimensional cone $\left\{ \lambda \left[\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right],\ \lambda \ge 0 \right\}$ is a closed subset of L^{Φ^*}, hence by the Bipolar Theorem

$$\mathcal{C}^0 = \left\{ \lambda \left[\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right],\ \lambda \ge 0 \right\}.$$

Hence the set $(\mathcal{C}_1^0)^+$ in the statement of Theorem 1.5.3 is exactly $\left\{ \left[\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right] \right\}$ and Theorem 1.5.3 proves that $\pi^{\mathbb{Q}}(0)$ is equal to the RHS of (1.36). We can similarly argue to prove (1.35). □

To conclude, we provide the minimax duality between the maximization problems with and without a fixed measure

Corollary 1.3.10. *The following holds:*

$$\pi(A) = \min_{\mathbb{Q}\in\mathcal{Q}_v} \pi^{\mathbb{Q}}(A) = \pi^{\widehat{\mathbb{Q}}}(A) < +\infty\,,$$

where $\widehat{\mathbb{Q}}$ is the minimax measure from Theorem 1.3.5.

Proof. It is an immediate consequence of Theorem 1.3.5 and Proposition 1.3.9. □

Lemma 1.3.11. *For all $\mathbb{Q} \in \mathscr{Q}$ we have $\Pi^{\mathbb{Q}}(A) = S^{\mathbb{Q}}(A)$ and, if $\widehat{\mathbb{Q}}$ is the minimax measure from Theorem 1.3.5, then*

$$\pi(A) = \pi^{\widehat{\mathbb{Q}}}(A) = \Pi^{\widehat{\mathbb{Q}}}(A) = S^{\widehat{\mathbb{Q}}}(A). \tag{1.37}$$

Proof. Let $Y \in \mathcal{L}$, $\mathbb{Q} \in \mathscr{Q}$, $a^j := \mathbb{E}_{\mathbb{Q}^j}[Y^j]$ and $Z^j := Y^j - a^j$. As $\mathcal{L}+\mathbb{R}^N = \mathcal{L}$, $Z^j \in \mathcal{L}^j$ and

$$\begin{aligned}
\Pi^{\mathbb{Q}}(A) &= \sup_{Y\in\mathcal{L}} \left\{ \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j+Y^j)\right] \mid \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] = A \right\} \\
&= \sup_{a\in\mathbb{R}^N,\, Z\in\mathcal{L}} \left\{ \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j+Z^j+a^j)\right] \mid \mathbb{E}_{\mathbb{Q}^j}[Z^j]=0, \sum_{j=1}^N a^j = A \right\} \\
&= \sup_{a\in\mathbb{R}^N,\, \sum_{j=1}^N a^j = A} \left\{ \sup_{Z\in\mathcal{L}\,:\,\mathbb{E}_{\mathbb{Q}^j}[Z^j]=0} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z^j+a^j)\right] \right\} \\
&= \sup_{a\in\mathbb{R}^N,\, \sum_{j=1}^N a^j = A} \sum_{j=1}^N \sup_{Z^j\in\mathcal{L}^j} \left\{ \mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z^j+a^j)\right] \mid \mathbb{E}_{\mathbb{Q}^j}[Z^j]=0 \right\} \\
&= \sup_{a\in\mathbb{R}^N,\, \sum_{j=1}^N a^j = A} \sum_{j=1}^N \sup_{Y^j\in\mathcal{L}^j} \left\{ \mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Y^j)\right] \mid \mathbb{E}_{\mathbb{Q}^j}[Y^j]=a^j \right\} \\
&= \sup_{a\in\mathbb{R}^N\ \sum_{j=1}^N a^j = A} \sum_{j=1}^N U_j^{\mathbb{Q}^j}(a^j) = S^{\mathbb{Q}}(A)\,.
\end{aligned}$$

The first equality in (1.37) follows from Corollary 1.3.10 and the second one from (1.34). □

1.3.4 Main results

Theorem 1.3.12. *Take $\mathscr{Q} = \mathcal{Q}_v$ and set $\mathcal{L} = \bigcap_{\mathbb{Q}\in\mathcal{Q}_v} L^1(\mathbb{Q})$. Under Assumption 1.2.12, for any $X \in M^\Phi$ and any $A \in \mathbb{R}$ a SORTE exists, namely $(\widehat{Y}, \widehat{\mathbb{Q}}) \in \mathcal{B}_A \times \mathcal{Q}_v$ defined in Theorem 1.3.5 and*

$$\widehat{a}^j := \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j], \quad j = 1, \dots, N, \tag{1.38}$$

satisfy:

1. *$\widehat{Y}^j$ is an optimum for $U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j)$, for each $j \in \{1, \dots, N\}$,*

2. $\widehat{a}$ is an optimum for $S^{\widehat{\mathbb{Q}}}(A)$,

3. $\widehat{Y} \in \mathcal{B}$ and $\sum_{j=1}^N \widehat{Y}^j = A$ $\mathbb{P}-a.s.$

Proof.
1): We prove that $U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j) = \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right] < u_j(+\infty)$, for all $j = 1, \dots, N$, thus showing that $\widehat{Y}^j$ is an optimum for $U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j)$. As $\widehat{Y}^j \in \mathcal{L}^j$ for all $j = 1, \dots, N$, then by definition of $U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j)$ we obtain:

$$\sup\left\{\mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Z)\right]\Big| Z \in \mathcal{L}^j,\, \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Z] \leq \widehat{a}^j\right\} =: U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j) \geq \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right].$$

If, for some index, the last inequality were strict we would obtain the contradiction

$$\pi^{\widehat{\mathbb{Q}}}(A) = \pi(A) \overset{\text{Thm. 1.3.5}}{=} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right] < \sum_{j=1}^N U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j) \leq S^{\widehat{\mathbb{Q}}}(A) = \pi^{\widehat{\mathbb{Q}}}(A)\,, \tag{1.39}$$

where we used (1.37) in the first and last equality.
In particular then $\mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right] < u_j(+\infty)$, for all $j = 1, \dots, N$. Indeed, if the latter were equal to $u_j(+\infty)$, then u_j would attain its maximum over a compact subset of $\mathbb{R}$, which is not the case.
2): From (1.18) we know that $A = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j] = \sum_{j=1}^N \widehat{a}^j$. From (1.37) we have

$$S^{\widehat{\mathbb{Q}}}(A) = \pi(A) \overset{\text{Thm. 1.3.5}}{=} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right] = \sum_{j=1}^N U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j) \leq S^{\widehat{\mathbb{Q}}}(A).$$

3): We already know that $\widehat{Y} \in \mathcal{B}_A := \mathcal{B} \cap \{Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j \leq A\}$. From Proposition 1.3.7 we deduce

$$A = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j] \leq \sum_{j=1}^N \widehat{Y}^j \leq A.$$

□

We now turn our attention to uniqueness and Pareto optimality, but we will need an additional property and an auxiliary result.

Definition 1.3.13 (Definition 4.18 in [20]). *We say that $\mathcal{B} \subseteq (L^0(\mathbb{P}))^N$ is* closed under truncation *if for each $Y \in \mathcal{B}$ there exists $m_Y \in \mathbb{N}$ and $c_Y = [c_Y^1, ..., c_Y^N] \in \mathbb{R}^N$ such that $\sum_{j=1}^N c_Y^j = \sum_{j=1}^N Y^j \in \mathbb{R}$ and for all $m \geq m_Y$*

$$Y_m := Y I_{\{\cap_{j=1}^N \{|Y^j| < m\}\}} + c_Y I_{\{\cup_{j=1}^N \{|Y^j| \geq m\}\}} \in \mathcal{B}. \tag{1.40}$$

Remark 1.3.14. We stress the fact that all the sets introduced in Example 1.2.19 satisfy closedness under truncation.

Lemma 1.3.15. *Let $\mathcal{B}$ be closed under truncation. Then for every $A \in \mathbb{R}$*

$$\mathcal{B}_A \cap \mathcal{L} \subseteq \overline{\mathcal{B}_A}.$$

Proof. Fix any $\mathbb{Q} \in \mathcal{Q}_v$ and argue as in Proposition 4.20 in [20]: let $Y \in \mathcal{B}_A \cap \mathcal{L} \subseteq L^1(\mathbb{Q})$ and consider Y_m for $m \in \mathbb{N}$ as defined in (1.40), where w.l.o.g. we assume $m_Y = 1$. Note that $\sum_{j=1}^N Y_m^j = \sum_{j=1}^N Y^j \leq A$ for all $m \in \mathbb{N}$. By boundedness of Y_m and (1.40), we have $Y_m \in \mathcal{B} \cap M^\Phi$ for all $m \in \mathbb{N}$. Further, $Y_m \to Y$ $\mathbb{P}$-a.s. for $m \to \infty$, and thus, since $|Y_m| \leq \max\{|Y|, |c_Y|\} \in L^1(\mathbb{Q})$ for all $m \in \mathbb{N}$ ($|\cdot|$, max and the inequality are meant componentwise), also $Y_m \to Y$ in $L^1(\mathbb{Q})$ for $m \to \infty$ by dominated convergence.
Now, if $\mathbb{Q} \sim \mathbb{P}$ we can directly apply Proposition 1.3.7 to get that $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] \leq \sum_{j=1}^N Y^j \leq A$. If we only have $\mathbb{Q} \ll \mathbb{P}$ we can see that (1.33) still holds, with the particular choice of $(Y_m)_m$ in place of $(k_n)_n$, because the construction of Y_m is made $\mathbb{P}$-almost surely. □

Define

$$\Pi(A) := \sup\left\{\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j + Y^j)\right] \mid Y \in \mathcal{L} \cap \mathcal{B}, \sum_{j=1}^N Y^j \leq A\right\}. \tag{1.41}$$

Lemma 1.3.16. *Let $\mathcal{B}$ be closed under truncation. If $\widehat{\mathbb{Q}}$ is the minimax measure from Theorem 1.3.5, then*

$$\pi(A) = \Pi(A) = \pi^{\widehat{\mathbb{Q}}}(A) = \Pi^{\widehat{\mathbb{Q}}}(A) = S^{\widehat{\mathbb{Q}}}(A). \tag{1.42}$$

Proof. It is clear that since $\mathcal{B}_A \cap M^\Phi \subseteq \mathcal{B}_A \cap \mathcal{L}$ we have $\pi(A) \leq \Pi(A)$ just by definitions (1.14) and (1.41). Now observe that by Lemma 1.3.15 we have $\mathcal{B}_A \cap \mathcal{L} \subseteq \overline{\mathcal{B}_A}$, so that $\Pi(A) \leq \Pi^{\widehat{\mathbb{Q}}}(A)$. The chain of equalities then follows by Lemma 1.3.11. □

Theorem 1.3.17. *Let $\mathcal{B}$ be closed under truncation. Under the same assumptions of Theorem 1.3.12, for any $X \in M^\Phi$ and $A \in \mathbb{R}$ the SORTE is unique and is a Pareto optimal allocation for both the sets*

$$\mathscr{V} = \left\{Y \in \mathcal{L} \cap \mathcal{B} \mid \sum_{j=1}^N Y^j \leq A \ \mathbb{P}\text{-a.s.}\right\} \ \text{and}\ \mathscr{V} = \left\{Y \in \mathcal{L} \mid \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\right] \leq A\right\}. \tag{1.43}$$

Proof. Use Proposition 1.3.9 and Corollary 1.3.10 to get that for any $\mathbb{Q} \in \mathcal{Q}_v$

$$\Pi^{\mathbb{Q}}(A) = \pi^{\mathbb{Q}}(A) \geq \pi(A). \tag{1.44}$$

Let $(\widetilde{Y}, \widetilde{\mathbb{Q}}, \widetilde{a})$ be a SORTE and $(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a})$ be the one from Theorem 1.3.12.
By 1) and 2) in the definition of SORTE, together with Lemma 1.3.11, we see that $\widetilde{Y}$ is an optimum for $\Pi^{\widetilde{\mathbb{Q}}}(A) = S^{\widetilde{\mathbb{Q}}}(A)$. Also, $\widetilde{Y} \in \mathcal{B}_A \cap \overline{\mathcal{B}_A}$ by Lemma 1.3.15. We can conclude by equation (1.15) that

$$\pi(A) \geq \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widetilde{Y}^j\right)\right] = \Pi^{\widetilde{\mathbb{Q}}}(A) \overset{\text{eq.}(1.34)}{=} \pi^{\widetilde{\mathbb{Q}}}(A) \overset{\text{Cor.1.3.10}}{\geq} \pi(A),$$

which tells us that $\pi(A) = \pi^{\widetilde{\mathbb{Q}}}(A) = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widetilde{Y}^j\right)\right]$.

By Theorem 1.3.5, we also have $\pi(A) = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right]$. Then $\widehat{Y}, \widetilde{Y} \in \overline{\mathcal{B}_A}$ (Lemma 1.3.15) and $\Pi(A) = \pi(A)$ (Lemma 1.3.16) imply that both $\widehat{Y}, \widetilde{Y}$ are optima for $\Pi(A)$. By strict concavity of the utilities $u_1, \dots, u_N$, $\Pi(A)$ has at most one optimum. From this, together with uniqueness of the minimax measure (see Theorem 1.3.5), we get $(\widetilde{Y}, \widetilde{\mathbb{Q}}) = (\widehat{Y}, \widehat{\mathbb{Q}})$. We infer from equation (1.38) and Remark 1.2.8 that also $\widetilde{a} = \widehat{a}$.
To prove the Pareto optimality observe that Theorem 1.3.5 proves that $\widehat{Y} \in \overline{\mathcal{B}_A} \subseteq \mathcal{L}$ is the unique optimum for $\Pi(A)$ (see Lemma 1.3.16) and so it is also the unique optimum for $\Pi^{\widehat{\mathbb{Q}}}(A)$. Pareto optimality then follows from Proposition 1.2.2, noticing that $\Pi(\mathscr{V})$ for the two sets in (1.43) are $\Pi(A)$ and $\Pi^{\widehat{\mathbb{Q}}}(A)$ respectively. □

In the proof of Theorem 1.3.17 we show that the component $\widehat{Y}$ of SORTE is an optimum for the "sup-convolution"(1.41). This implies that an optimum for such a "sup-convolution"can be realized in a two stage procedure (allocation of A at the beginning, reinsurance with $\widetilde{Y}$ at terminal time) given by SORTE, which conjugates the systemic optimality and the individual preferences.

1.3.5 Dependence of the SORTE on X and on $\mathcal{B}$

We see from the proof of Theorem 1.3.12 that the triple defining the SORTE (obviously) depends on the choice of A. We now focus on the study of how such triple depends on X. To this end, we first specialize to the case $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$.

Proposition 1.3.18. *Under the hypotheses of Theorem 1.3.12 and for $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$, the variables $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}$ and $X + \widehat{Y}$ are $\sigma(X^1 + \cdots + X^N)$ (essentially) measurable.*

Proof. By Theorem 1.3.12 and Theorem 1.3.17 we have that $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ is an optimum of the RHS of equation (1.16). Notice that in this specific case $Y := e_i 1_A - e_j 1_A \in \mathcal{B} \cap M^\Phi$ for all i, j and all measurable sets $A \in \mathcal{F}$. Let $\mathbb{Q} \in \mathcal{Q}$. Then from (1.32) $\sum_{j=1}^N (\mathbb{E}_{\mathbb{Q}^j}[Y^j] - Y^j) \leq 0$ and so $\mathbb{Q}^i(A) - 1_A - \mathbb{Q}^j(A) + 1_A \leq 0$, i.e., $\mathbb{Q}^i(A) - \mathbb{Q}^j(A) \leq 0$. Similarly taking $Y := -e_i 1_A + e_j 1_A \in \mathcal{B}$, we get $\mathbb{Q}^j(A) - \mathbb{Q}^i(A) \leq 0$. Hence all the components of vectors in $\mathcal{Q}$ are equal. Let $\mathcal{G} := \sigma(X^1 + \cdots + X^N)$. Then for any $\lambda \in \mathbb{R}_{++}$ and any $[\mathbb{Q}, \dots, \mathbb{Q}] \in \mathcal{Q}$ we have:

$$
\begin{aligned}
&\lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + A \right) + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}P}\right)\right] \\
&= \lambda \left(\mathbb{E}_{\mathbb{P}}\left[\left(\sum_{j=1}^N X^j\right) \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] + A \right) + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] \\
&= \lambda \left(\mathbb{E}_{\mathbb{P}}\left[\left(\sum_{j=1}^N X^j\right) \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\right] + A \right) + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\middle|\mathcal{G}\right]\right] \\
&\geq \lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^j \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\right] + A \right) + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\right)\right],
\end{aligned}
$$

where in the last inequality we exploited the tower property and Jensen inequality, as $v_1, \dots, v_N$ are convex. Notice now that $\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]$ defines again a probability measure (on the whole $\mathcal{F}$, the initial sigma algebra) and that this measure still belongs to $\mathcal{Q}$ since all its components are equal. As a consequence, the minimum in equation (1.16) can be equivalently taken over $\lambda \in \mathbb{R}_{++}$ (as before) and $\mathbb{Q} \in \mathcal{Q} \cap (L^0(\Omega, \mathcal{G}, \mathbb{P}))^N$. The claim for $\widehat{Y}$ follows from (1.17). □

It is interesting to notice that this dependence on the componentwise sum of X also holds in the case of Bühlmann's equilibrium (see [33] page 16 and [26]).

Remark 1.3.19. In the case a cluster of agents, see the Example 1.2.19, the above result can be clearly generalized: the i-th component of the vector $\widehat{\mathbb{Q}}$, for i belonging to the m-th group, only depends on the sum of those components of X whose corresponding indexes belong to the m-th group itself. It is also worth mentioning that if we took $\mathcal{B}^{(I)} = \mathbb{R}^N$, we would see that each component of $\widehat{\mathbb{Q}}$ and of $\widehat{Y}$ is a measurable function of the corresponding component of X. This is reasonable since, in this case, at the final time each agent would be only allowed to share and exchange risk with herself/himself and the systemic features of the model we are considering would be lost.

We provide now some additional examples, to the ones in Example 1.2.19, of possible feasible sets $\mathcal{B}$ and study the dependence of the probability measures from $\mathcal{B}$.

Example 1.3.20. Consider a measurable partition $A_1, \dots, A_K$ of Ω and a collection of partitions $I^1, \dots, I^K$ of $\{1, \dots, N\}$ as in Example 1.2.19. Take the associated clusters $\mathcal{B}^{(I^1)}, \dots, \mathcal{B}^{(I^K)}$ defined as in (1.8). Then the set

$$\mathcal{B} := \left(\sum_{i=1}^{K} \mathcal{B}^{(I^i)} 1_{A_i}\right) \cap \mathcal{C}_{\mathbb{R}} \tag{1.45}$$

satisfies Assumptions 1.2.12 and is closed under truncation, as it can be checked directly.

The set in (1.45) can be seen as a scenario-dependent clustering. A particular simple case of (1.45) is the following. For a measurable set $A_1 \in \mathcal{F}$ take $A_2 = \Omega \setminus A_1$. Then the set $\mathcal{C}_{\mathbb{R}} 1_{A_1} + \mathbb{R}^N 1_{A_2}$ is of the form (1.45) and consists of all the $Y \in (L^0)^N$ such that (i) there exists a real number $\sigma \in \mathbb{R}$ with $\sum_{j=1}^N Y^j = \sigma$ $\mathbb{P}-$a.s. on A_1, (ii) there exists a vector $b \in \mathbb{R}^N$ such that $Y = b$ $\mathbb{P}-$a.s. on A_2 and (iii) $\sigma = \sum_{j=1}^N b^j$ (recall that $Y \in \mathcal{C}_{\mathbb{R}}$ by (1.45)).
Let us motivate Example 1.3.20 with the following practical example. Suppose for each bank i a regulator establishes an excessive exposure threshold D^i. If the position of bank i falls below such threshold, we can think that it is too dangerous for the system to let that bank take part to the risk exchange. As a consequence, in the clustering example, on the event $\{X^i \leq D^i\}$ we can require the bank to be left alone. Also the symmetric situation can be considered: a bank j whose position is too good, say exceeding a value A^j, will not be willing to share risk with all others, thus entering the game only as isolated individual or as a member of the groups of "safer" banks. Both these requirements, and many others (say considering random thresholds) can be modeled with the constraints introduced in Example 1.3.20.

It is interesting to notice that, as in Example 1.2.19, assuming a constraint set of the form given in Example 1.3.20 forces a particular behavior on the probability vectors in $\mathcal{Q}_v$.

Lemma 1.3.21. *Let $\mathcal{B}$ be as in Example 1.3.20 and let $\mathbb{Q} \in \mathcal{Q}_v$. Fix any $i \in \{1, ..., K\}$ and any group I^i_m of the partition $I^i = (I^i_m)_m$. Then all the components $\mathbb{Q}^j$, $j \in I^i_m$, agree on $\mathcal{F}|_{A_i} := \{F \cap A_i,\ F \in \mathcal{F}\}$.*

Proof. We think it is more illuminating to prove the statement in a simplified case, rather than providing a fully formal proof (which would require unnecessarily complicated notation). This is "without loss of generality"in the sense that it is clear how to generalize the method. To this end, let us consider the case $K = 2$ (i.e. $A_2 = A_1^c$) and $\mathcal{B}^{(I^1)} := \mathcal{C}_\mathbb{R}$, $\mathcal{B}^{(I^2)} := \mathbb{R}^N$. For any $F \in \mathcal{F}$ and $i, j \in \{1, \dots, N\}$ we can take $Y := (1_F(e_i - e_j))\, 1_{A_1} + 0 1_{A_2}$ to obtain $Y \in \mathcal{C}_\mathbb{R} 1_{A_1} + \mathbb{R}^N 1_{A_2}$, $\sum_{j=1}^N Y^j = 0$. By definition of $\mathcal{Q}_v$ we get for any $\mathbb{Q} \in \mathcal{Q}_v$ that $\mathbb{Q}^i(A \cap F) - \mathbb{Q}^j(A \cap F) \leq 0$, and interchanging i, j yields $\mathbb{Q}^i(A \cap F) = \mathbb{Q}^j(A \cap F)$ for any $i, j = 1 \dots, N$, $F \in \mathcal{F}$. □

1.4 Exponential Case

We now specialize our analysis to the exponential setup, where

$$u_j(x) := 1 - \exp(-\alpha_j x),\ j = 1, \dots, N \quad \text{for} \quad \alpha_1, \dots, \alpha_N > 0. \tag{1.46}$$

This allows us to provide explicit formulas for a wide range of constraint sets $\mathcal{B}$ (namely, all those introduced in Example 1.2.19) and so the stability properties of SORTE, with respect to a different weighting of utilities, will be evident.

1.4.1 Explicit formulas

We consider a set of constraints of the form $\mathcal{B} = \mathcal{B}^{(I)}$ as given in Example 1.2.19. Given $X \in M^\Phi$ and $m \in \{1, \dots, h\}$, we set:

$$\begin{aligned} \beta_m &:= \sum_{j \in I_m} \frac{1}{\alpha_j} \quad \beta := \sum_{j=1}^N \frac{1}{\alpha_j} \quad \overline{X}_m := \sum_{j \in I_m} X^j, \\ R(n) &:= \frac{\frac{1}{\alpha_j}}{\sum_{k=1}^N \frac{1}{\alpha_k}},\ j = 1, ...N,\ \alpha := (\alpha_1, ..., \alpha_N),\ \mathbb{E}_R[\ln(\alpha)] = \sum_{j=1}^N R(n) \ln(\alpha_j). \end{aligned}$$

Theorem 1.4.1. *Take $u_1, \dots, u_N$ as given by (1.46) and $\mathcal{B} = \mathcal{B}^{(I)}$ as in Example 1.2.19. For $\mathcal{L}$ and $\mathcal{Q}$ defined in Theorem 1.3.12, the SORTE is given by*

$$\begin{cases} \widehat{Y}^k = -X^k + \frac{1}{\alpha_k}\left(\frac{\overline{X}_m}{\beta_m} - d_m(X)\right) + \frac{1}{\alpha_k}\left[\frac{A}{\beta} + \ln(\alpha_k) - \mathbb{E}_R[\ln(\alpha)]\right] & k \in I_m \\ \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}P} = \frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} =: \frac{\mathrm{d}\widehat{\mathbb{Q}}^m}{\mathrm{d}\mathbb{P}} & k \in I_m \\ \widehat{a}^k = \mathbb{E}_{\widehat{\mathbb{Q}}^k}[\widehat{Y}^k] & k = 1, \dots, N \end{cases} \tag{1.47}$$

where

$$d_m(X) := \left[\sum_{j=1}^{h} \frac{\beta_j}{\beta} \ln\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}_j}{\beta_j}\right)\right]\right)\right] - \ln\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right).$$

Proof. The utility functions in (1.46) satisfy Assumption 1.2.12 (a) and $\mathcal{B}$ satisfies Assumption 1.2.12 (b) and closedness under truncation, hence Theorems 1.3.12 and 1.3.17 guarantee existence and uniqueness. Recall that from this choice of $\mathcal{B}$ we have that for each $\mathbb{Q} \in \mathcal{Q}_v$, all the components of $\mathbb{Q}$ are equal in each index subset I_m.
It is easy to check that

$$v_j(\lambda y) = \frac{\lambda y}{\alpha_j} \ln \frac{\lambda}{\alpha_j} + \frac{\lambda}{\alpha_j} y \ln y - \frac{\lambda}{\alpha_j} y + 1\,. \tag{1.48}$$

Substitute now $y = \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in \mathcal{Q}_v$ in the above expressions and take expectations to get

$$\mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] = \phi_j(\lambda) + \frac{\lambda}{\alpha_j}\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \ln\left(\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right], \quad \phi_j(\lambda) = \frac{\lambda}{\alpha_j}\ln\frac{\lambda}{\alpha_j} - \frac{\lambda}{\alpha_j} + 1\,. \tag{1.49}$$

Let $K\left(\lambda, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}P}\right)$ be the functional to be optimized in (1.16). Set

$$\xi := \sum_{j=1}^{N} \frac{1}{\alpha_j} \ln\left(\frac{1}{\alpha_j}\right), \quad \phi(\lambda) = \sum_{j=1}^{N} \phi_j(\lambda) = \lambda\xi + \beta\lambda\ln\lambda - \lambda\beta + N.$$

Then from (1.49) we deduce

$$K\left(\lambda, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) = \lambda\left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + A\right) + \phi(\lambda) + \sum_{j=1}^{N} \frac{\lambda}{\alpha_j}\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \ln\left(\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]. \tag{1.50}$$

Set

$$\mu := \sum_{j=1}^{N} \frac{1}{\alpha_j}\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \ln\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\right] + A + \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[X^j]. \tag{1.51}$$

From (1.50) and (1.51)

$$K\left(\lambda, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) = \lambda\mu + \lambda\left(\xi + \beta\ln(\lambda) - \beta\right) + N\,.$$

The associated first order condition obtained differentiating in λ yields the unique solution

$$\widehat{\lambda} = \exp\left(-\frac{\mu + \xi}{\beta}\right)$$

which can be substituted in $K\left(\cdot, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$ yielding

$$K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) = -\widehat{\lambda}\beta + N\,. \tag{1.52}$$

We now **guess** that the vector of measures $\widehat{\mathbb{Q}}$ defined via (1.47) is optimal and compute the associated μ:

$$\mu = \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[X^j] + A + \sum_{j=1}^{N} \frac{1}{\alpha_j} \mathbb{E}_{\mathbb{P}} \left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \ln \left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \right) \right] = A + \sum_{j=1}^{h} \mathbb{E}_{\mathbb{P}} \left[\left(\overline{X}_j \right) \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \right] +$$

$$\sum_{j=1}^{h} \beta_j \mathbb{E}_{\mathbb{P}} \left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \ln \left(\exp \left(-\frac{\overline{X}_j}{\beta_j} \right) \right) \right] + \sum_{j=1}^{h} \beta_j \mathbb{E}_{\mathbb{P}} \left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \ln \left(\frac{1}{\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}_j}{\beta_j} \right) \right]} \right) \right].$$

Hence

$$\mu = A - \sum_{j=1}^{h} \beta_j \ln \left(\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}_j}{\beta_j} \right) \right] \right) \tag{1.53}$$

and substituting (1.52) in the explicit formula for $\widehat{\lambda}$ we get

$$K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right) = -\beta \exp \left(-\frac{1}{\beta} \left(A + \xi + \sum_{j=1}^{h} \beta_j \ln \left(\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}_j}{\beta_j} \right) \right] \right) \right) \right) + N. \tag{1.54}$$

Using equation (1.17) we **define**, for the measure given in (1.47),

$$\widehat{Y}^k = -X^k - v_j' \left(\widehat{\lambda} \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right) \quad k = 1, \dots, N.$$

By (1.48) (with $\lambda = 1$) we obtain, for $k \in I_m$, $v_k'(y) = \frac{1}{\alpha_k} \ln \left(\frac{y}{\alpha_k} \right)$ and

$$v_k' \left(\widehat{\lambda} \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right) = \frac{1}{\alpha_k} \ln \left(\frac{1}{\alpha_k} \right) + \frac{1}{\alpha_k} \ln \left(\frac{\exp \left(-\frac{\overline{X}_m}{\beta_m} - \frac{A+\mu}{\beta} \right)}{\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}_m}{\beta_m} \right) \right]} \right)$$

$$= \frac{1}{\alpha_k} \ln \left(\frac{1}{\alpha_k} \right) - \frac{1}{\alpha_k} \left(\frac{\overline{X}_m}{\beta_m} + \frac{A+\mu}{\beta} \right) - \frac{1}{\alpha_k} \ln \left(\mathbb{E}_{\mathbb{P}} \left[\exp \left(-\frac{\overline{X}_m}{\beta_m} \right) \right] \right) \overset{\text{Eq.(1.53)}}{=}$$

$$\frac{1}{\alpha_k} \ln \left(\frac{1}{\alpha_k} \right) - \frac{1}{\alpha_k} \left(\frac{\overline{X}_m}{\beta_m} + \frac{A+\xi}{\beta} \right) + \frac{1}{\alpha_k} d_m(X).$$

Hence for $k \in I_m$ we have

$$\widehat{Y}^k = -X^k + \frac{1}{\alpha_k} \left(\frac{\overline{X}_m}{\beta_m} + \frac{A+\xi}{\beta} - d_m(X) \right) - \frac{1}{\alpha_k} \ln \left(\frac{1}{\alpha_k} \right).$$

A simple computation yields $\widehat{Y} \in M^{\Phi}$, $\sum_{k \in I_m} Y^k \in \mathbb{R}$ and $\sum_{j=1}^{N} \widehat{Y}^j = A$, so that $\widehat{Y} \in \mathcal{B}_A \cap M^{\Phi}$.
Moreover

$$\exp \left(-\left(X^k + \widehat{Y}^k \right) \right) = \exp \left(-\alpha_k \left(\frac{1}{\alpha_k} \left(\frac{\overline{X}_m}{\beta_m} + \frac{A+\xi}{\beta} - d_m(X) \right) - \frac{1}{\alpha_k} \ln \left(\frac{1}{\alpha_k} \right) \right) \right)$$

$$= \frac{1}{\alpha_k} \exp\left(-\frac{\overline{X}_m}{\beta_m}\right) \exp\left(-\frac{A+\xi}{\beta}\right) \exp\left(d_m(X)\right)$$

$$= \frac{1}{\alpha_k} \frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} \exp\left(-\frac{A+\xi}{\beta}\right) \exp\left(\sum_{j=1}^{h} \frac{\beta_j}{\beta} \ln\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}_j}{\beta_j}\right)\right]\right)\right).$$

As a consequence

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[1 - \exp\left(-\alpha_j\left(X^j + \widehat{Y}^j\right)\right)\right]$$

$$= -\sum_{j=1}^{N} \frac{1}{\alpha_j} \exp\left(-\frac{1}{\beta}\left(A + \xi + \sum_{j=1}^{h} \beta_j \ln\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}_j}{\beta_j}\right)\right]\right)\right)\right) + N$$

$$\overset{\text{Eq.(1.54)}}{=} K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \tag{1.55}$$

which implies

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right] = K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right). \tag{1.56}$$

To sum up we have

$$K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \overset{\text{Eq.(1.56)}}{=} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + \widehat{Y}^j\right)\right]$$

$$\overset{\widehat{Y} \in \mathcal{B}_A \cap M^\Phi}{\leq} \sup_{Y \in \mathcal{B}_A \cap M^\Phi} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right] \overset{\text{Thm.(1.3.5)}}{=} \min_{\substack{\lambda \geq 0 \\ \mathbb{Q} \in \mathcal{Q}_v}} K\left(\lambda, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \leq K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right).$$

Consequently $\widehat{Y}$ is the (unique) optimum for the optimization problem in LHS of (1.15), and $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ is the (unique) optimum to the minimization problem in (1.16). Moreover, setting $\widehat{a}^j := \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j], \quad j = 1, \ldots, N$, the SORTE (which, as already argued, exists and is unique) is given by $\left(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a}\right)$. □

Remark 1.4.2. We observe that in the terminal part of the proof above we also got an explicit formula for the maximum systemic utility:

$$\sup_{Y \in \mathcal{B}_A \cap M^\Phi} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Y^j)\right] \overset{\text{Thm.(1.3.5)}}{=} K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \tag{1.57}$$

where $K\left(\widehat{\lambda}, \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$ is given in (1.55).

1.4.2 A toy Example

In the following two examples we compare a Bühlmann's Equilibrium with a SORTE in the simplest case where $X = 0 := (0, ..., 0)$ and $A = 0$. In the formula below we use the well known fact:

$$\sup_{Y \in L^1(\mathbb{Q})} \{\mathbb{E}_{\mathbb{P}}[u_j(Y)] \mid \mathbb{E}_{\mathbb{Q}}[Y] \leq x\} = 1 - e^{-\alpha_j x - H(\mathbb{Q},\mathbb{P})},$$

where $H(\mathbb{Q},\mathbb{P}) = E[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln(\frac{d\mathbb{Q}}{d\mathbb{P}})]$ is the relative entropy, for $\mathbb{Q} \ll \mathbb{P}$.

Example 1.4.3 (Bühlmann's equilibrium solution). As $X := 0$ then $\overline{X}_N = \sum_{k=1}^N X^k = 0$ and therefore the optimal probability measure $\mathbb{Q}_X$ defined in Bühlmann is:

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta}\overline{X}_N}}{\mathbb{E}_{\mathbb{P}}\left[e^{-\frac{1}{\beta}\overline{X}_N}\right]} = 1, \tag{1.58}$$

i.e. $\mathbb{Q}_X = \mathbb{P}$. Take $a = 0 = (0, ..., 0)$. We compute

$$U_j^{\mathbb{Q}_X}(0) = U_j^{\mathbb{P}}(0) := \sup\{\mathbb{E}_{\mathbb{P}}[u_j(0 + Y)] \mid \mathbb{E}_{\mathbb{P}}[Y] \leq 0\} = 1 - e^{-\alpha_j 0 - H(\mathbb{P},\mathbb{P})} = 1 - 1 = 0,$$

as $H(\mathbb{P},\mathbb{P}) = 0$, so that

$$\sum_{j=1}^N U_j^{\mathbb{P}}(0) = 0.$$

As a consequence, and as $u_j(0) = 0$, the optimal solution for each single n is obviously $Y_X^j = 0$.

Conclusion: The Bühlmann's equilibrium solution associated to $X := 0$ (and $A = 0$) is the couple $(Y_X, \mathbb{Q}_X) = (0, \mathbb{P})$. Here the vector a is taken a priori to be equal to $(0, ..., 0)$.

Example 1.4.4 (SORTE). From Theorem 1.4.1 with $X := 0$ and $A = 0$ we obtain for the SORTE that: the optimal probability measure $\widehat{\mathbb{Q}}$ coincides again with $\mathbb{P}$; the optimal $\widehat{Y}$ is:

$$\widehat{Y}^j = \frac{1}{\alpha_j}[\ln(\alpha_j) - \mathbb{E}_R[\ln(\alpha)]] := \widehat{a}^j. \tag{1.59}$$

Recalling that $\widehat{\mathbb{Q}}$ is in fact a minimax measure for the optimization problem $\pi_0(0)$ (see the proof of Theorem 1.3.12), we can say that

$$S^{\mathbb{P}}(0) = S^{\widehat{\mathbb{Q}}}(A) \overset{\text{Lemma.1.3.16}}{=} \pi_0(0) \overset{(1.55),(1.57)}{=} N - \beta e^{-\frac{\xi}{\beta}} \tag{1.60}$$

Notice that if the α_j are equal for all n, then $S^{\mathbb{P}}(0) = 0$, but in general

$$S^{\mathbb{P}}(0) = N - \beta e^{-\frac{\xi}{\beta}} \geq 0.$$

Indeed, by Jensen inequality:

$$e^{-\frac{\xi}{\beta}} = e^{\mathbb{E}_R[\ln(\alpha)]} \leq \mathbb{E}_R[e^{\ln(\alpha)}] = \mathbb{E}_R[\alpha] := \sum_{j=1}^N \frac{\frac{1}{\alpha_j}\alpha_j}{\sum_{k=1}^N \frac{1}{\alpha_k}} = \frac{N}{\beta}.$$

From (1.59) we deduce that the α_j are equal for all j if and only if $\widehat{a}^j = 0$ for all j, but in general $\widehat{a}^j$ may differ from 0. As $\widehat{Y}^j = \widehat{a}^j$, the same holds also for the optimal solution $\widehat{Y}$. When $\widehat{a}^j < 0$ a violation of Individual Rationality occurs.
Conclusion: The SORTE solution associated to $X := 0$ (and $A = 0$) is the triplet $(\widehat{Y}, \mathbb{P}, \widehat{a})$ where $\widehat{Y} = \widehat{a}$ is assigned in equation (1.59).

The above comparison shows that a SORTE is not a Bühlmann equilibrium, even when $X := 0$ and $A = 0$. When the α_j are all equal, then the Bühlmann and the SORTE solution coincide, as all agents are assumed to have the same risk aversion.

Remark 1.4.5. In this example, notice that we may control the risk sharing components Y^j of agent j in the SORTE by:

$$|Y^j| \leq \frac{1}{\alpha_{\min}} \left[\ln(\alpha_{\max}) - \ln(\alpha_{\min})\right].$$

Suppose that $\alpha_{\min} < \alpha_{\max}$ and consider the expression for $\widehat{Y}^j = \widehat{a}^j$ in (1.59). If $\alpha_j = \alpha_{\min}$ then the corresponding $\widehat{Y}^j < 0$ is in absolute value relatively large (divide by $\alpha_{\min}$), while if $\alpha_k = \alpha_{\max}$ the corresponding $\widehat{Y}^k > 0$ is in absolute value relatively small (divide by $\alpha_{\max}$).

1.4.3 Dependence on weights and stability

As customary in the literature on general equilibrium and risk sharing, we could have considered, in place of (I.8) and (I.9), the more general problem

$$\sup_{a\in\mathbb{R}^N} \left\{ \sum_{j=1}^N \sup_{Y^j} \left\{ \mathbb{E}_{\mathbb{P}}\left[\gamma_j u_j(X^j+Y^j)\right] \mid p^j_X(Y^j) \leq a^j \right\} \mid \sum_{j=1}^N a^j = A \right\}, \quad (1.61)$$

$$\sum_{j=1}^N Y^j_X = A \;\; \mathbb{P}-\text{a.s.}\;, \quad (1.62)$$

where the positive weights $\gamma = [\gamma_1, ..., \gamma_N] \in \mathbb{R}^N_{++}$ could have been selected exogenously, say by a social planner. In such more general problems, equilibria will generally depend on the selected weights. However, we are focused on existence, uniqueness and Pareto optimality of the equilibrium and for this analysis we may restrict, without loss of generality, our attention to the utilitarian choice $\gamma_1 = ... = \gamma_N = 1$, as we now explain. It is easy to check that given $u_1, \dots, u_N$ satisfying our assumptions (namely Assumption 1.2.12.(a)), the associated functions $x \mapsto u^\gamma_j(x) := \gamma_j u_j(x)$, $j = 1, \dots, N$ will satisfy the same Assumption 1.2.12.(a) and so (1.61) can be written as

$$\sup_{a\in\mathbb{R}^N} \left\{ \sum_{j=1}^N \sup_{Y^j} \left\{ \mathbb{E}_{\mathbb{P}}\left[u^\gamma_j(X^j+Y^j)\right] \mid p^j_X(Y^j) \leq a^j \right\} \mid \sum_{j=1}^N a^j = A \right\}, \quad (1.63)$$

Thus, technically speaking, the study of the existence, uniqueness and Pareto optimality of the equilibrium in a non-utilitarian setup ($\gamma \neq 1$) boils down to the one in (I.8) and (I.9). Of course it could be of interest to study the dependence of the optimal solution from the vector γ and to analyze the stability properties of the equilibrium

with respect to the utility functions. We address this problem for exponential utility functions, but the general case is left for future investigation.
Given $\gamma_j \in (0, +\infty)$, $j = 1, \ldots, N$ and $u_1, \ldots, u_N$ satisfying Assumption 1.2.12 (a), we recall that $u_j^\gamma(x) := \gamma_j u_j(x)$, $j = 1, \ldots, N$ and we denote by $v_j^\gamma(\cdot)$ their convex conjugates. These functions u_j^γ satisfy Assumption 1.2.12 (a).
In our exponential setup and under closedness under truncation, a different weighting only results in a translation of both allocations at initial and terminal time of a SORTE, without affecting the optimal measure:

Proposition 1.4.6. *Consider $u_1, \ldots, u_N$ as given in* (1.46) *and take the associated $u_1^\gamma, \ldots, u_N^\gamma$ as above. Suppose $\mathcal{B}$ satisfies Assumption 1.2.12 (b) and is closed under truncation. Call $\left(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a}\right)$ the unique SORTE associated to $u_1, \ldots, u_N$, and similarly define $\left(\widehat{Y}_\gamma, \widehat{\mathbb{Q}}_\gamma, \widehat{a}_\gamma\right)$ as the unique SORTE associated to $u_1^\gamma, \ldots, u_N^\gamma$. Then*

$$\begin{cases} \widehat{Y}_\gamma^k = \widehat{Y}^k + g_k(\gamma) & k = 1, \ldots, N \\ \frac{\mathrm{d}\widehat{\mathbb{Q}}_\gamma^k}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}} & k = 1, \ldots, N \\ \widehat{a}_\gamma^k = \widehat{a}^k + g_k(\gamma) & k = 1, \ldots, N \end{cases}$$

where

$$g_k(\gamma) := \frac{1}{\alpha_k} \frac{\sum_{j=1}^N \frac{1}{\alpha_j} \ln\left(\frac{1}{\gamma_j}\right)}{\sum_{j=1}^N \frac{1}{\alpha_j}} - \frac{1}{\alpha_k} \ln\left(\frac{1}{\gamma_k}\right) = \frac{1}{\alpha_k}\left(\ln(\gamma_j) - \mathbb{E}_R[\ln(\gamma)]\right) \quad k = 1, \ldots, N\,.$$

Proof. For a general set $\mathcal{B}$, we here provide only a sketch of the proof. Using the formulas for $v_1, \ldots, v_N$, after some computations one can write explicitly the minimax expression (1.16). Then use the gradient formula (1.17) to deduce (1.64). A more direct proof, that works only for sets $\mathcal{B}$ in the form described in Example 1.2.19, is based on the observation that

$$u_j^\gamma(x) := \gamma_j u_j(x) = \gamma_j - \gamma_j \exp(-\alpha_j x) = \gamma_j - \exp\left(-\alpha_j\left[x - \frac{1}{\alpha_j}\ln(\gamma_j)\right]\right).$$

Hence, $\left(\widehat{Y}_\gamma, \widehat{\mathbb{Q}}_\gamma, \widehat{a}_\gamma\right)$ can be obtained by a straightforward computation from the solution $\left(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a}\right)$, which is explicitly given in Theorem 1.4.1, using $X^j - \frac{1}{\alpha_j}\ln(\gamma_j)$, $j = 1, \ldots, N$ in place of X. □

1.5 Appendix to Chapter 1

1.5.1 Orlicz Spaces and Utility Functions

We consider the utility maximization problem defined on Orlicz spaces, see [116] for further details on Orlicz spaces. This presents several advantages. From a mathematical point of view, it is a more general setting than L^∞, but at the same time it simplifies the analysis, since the topology is order continuous and there are no singular

elements in the dual space. Furthermore, it has been shown in [22] that the Orlicz setting is the natural one to embed utility maximization problems.
Let $u : \mathbb{R} \to \mathbb{R}$ be a concave and increasing function satisfying $\lim_{x\to-\infty} \frac{u(x)}{x} = +\infty$. Consider $\phi(x) := -u(-|x|) + u(0)$. Then $\phi : \mathbb{R}_+ \to [0, +\infty)$ is a strict Young function, i.e., it is finite valued and convex on $\mathbb{R}_+$ with $\phi(0) = 0$ and $\lim_{x\to+\infty} \frac{\phi(x)}{x} = +\infty$. Notice that another popular approach to Young functions considers $\phi : \mathbb{R} \to \mathbb{R}$ requesting that ϕ is even and that $\phi|_{\mathbb{R}_+}$ satisfies our assumptions. The two approaches can be used equivalently here. The Orlicz space L^ϕ and Orlicz Heart M^ϕ are respectively defined by

$$L^\phi := \left\{X \in L^0(\mathbb{R}) \mid \mathbb{E}_\mathbb{P}[\phi(\alpha\,|X|)] < +\infty \text{ for some } \alpha > 0\right\}, \tag{1.64}$$

$$M^\phi := \left\{X \in L^0(\mathbb{R}) \mid \mathbb{E}_\mathbb{P}[\phi(\alpha\,|X|)] < +\infty \text{ for all } \alpha > 0\right\}, \tag{1.65}$$

and they are Banach spaces when endowed with the Luxemburg norm. The topological dual of M^ϕ is the Orlicz space L^{ϕ^*}, where the convex conjugate ϕ^* of ϕ, defined by

$$\phi^*(y) := \sup_{x\in\mathbb{R}_+} \{xy - \phi(x)\}, \; y \in \mathbb{R}_+,$$

is also a strict Young function. Note that

$$\mathbb{E}_\mathbb{P}[u(X)] > -\infty \text{ if } \; \mathbb{E}_\mathbb{P}[\phi(|X|)] < +\infty. \tag{1.66}$$

Remark 1.5.1. It is well known that $L^\infty(\mathbb{P};\mathbb{R}) \subseteq M^\phi \subseteq L^\phi \subseteq L^1(\mathbb{P};\mathbb{R})$. In addition, from the Fenchel inequality $xy \le \phi(x) + \phi^*(y)$ we obtain

$$(\alpha|X|)\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \le \phi(\alpha|X|) + \phi^*\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$$

for some probability measure $\mathbb{Q} \ll \mathbb{P}$ and $\lambda \ge 0$, and we immediately deduce that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^{\phi^*}$ implies $L^\phi \subseteq L^1(\mathbb{Q};\mathbb{R})$.
Given the utility functions $u_1, \cdots, u_N : \mathbb{R} \to \mathbb{R}$, satisfying the above conditions, with associated Young functions $\Phi_1, \cdots, \Phi_N$, we define

$$M^\Phi := M^{\Phi_1} \times \cdots \times M^{\Phi_N}, \quad L^\Phi := L^{\Phi_1} \times \cdots \times L^{\Phi_N}. \tag{1.67}$$

1.5.2 Auxiliary results

Lemma 1.5.2. *Let $v : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ be a convex function, and suppose that its restriction to $(0, +\infty)$ is real valued and differentiable. Let $\mathbb{Q} \ll \mathbb{P}$ be a given probability measure with $v\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \in L^1(\mathbb{P})$ for all $\lambda > 0$. Then*

1. *v' is defined on $(0, +\infty)$ and real valued there and extendable to $[0, +\infty)$ by taking $\lim_{x\to0} v'(x) \in \mathbb{R} \cup \{-\infty\}$. Also, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} v'\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \in L^1(\mathbb{P})$ for all $\lambda > 0$.*

2. *If g is such that $g + \frac{1}{g} \in L^\infty_+(\mathbb{P})$, then $v\left(g\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \in L^1(\mathbb{P})$.*

3. *If $v'(0+) = -\infty$, $v'(+\infty) = +\infty$ and v is strictly convex $F(\gamma) := \mathbb{E}_\mathbb{P}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} v'\left(\gamma\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]$ is a well defined bijection between $(0, +\infty)$ and $\mathbb{R}$.*

Proof. Lemma 2 of [21]. □

The following dual representation holds:

Theorem 1.5.3. *Let $u_1 \dots, u_j : \mathbb{R} \to \mathbb{R}$ be strictly increasing and concave functions. Let $\mathcal{C} \subseteq M^\Phi$ be a convex cone such that for every $i, j = 1, \dots, N$, $e_i - e_j \in \mathcal{C}$. Denote by $\mathcal{C}^0$ the polar of the cone $\mathcal{C}$ in the dual pair (M^Φ, L^{Φ^*})*

$$\mathcal{C}^0 := \left\{ Z \in L^{\Phi^*} \ s.t. \ \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y^j Z^j\right] \le 0 \ \forall Y \in \mathcal{C} \right\}.$$

Set

$$\mathcal{C}_1^0 := \left\{ Z \in \mathcal{C}^0 \ s.t. \ \mathbb{E}_\mathbb{P}\left[Z^1\right] = \dots = \mathbb{E}_\mathbb{P}\left[Z^N\right] = 1 \right\}$$
$$(\mathcal{C}_1^0)^+ := \left\{ Z \in \mathcal{C}_1^0 \ s.t. \ Z^j \ge 0 \ for\ all\ j \right\}$$

and suppose that

$$\pi_\mathcal{C}(X) := \sup_{Y \in \mathcal{C}} \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right] \right) < +\infty \ \ \forall X \in M^\Phi.$$

Then

$$\pi_\mathcal{C}(X) = \min_{\lambda \in \mathbb{R}_+, \, \mathbb{Q} \in (\mathcal{C}_1^0)^+} \left(\lambda \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] \right).$$

If any of the two expressions above is strictly smaller than $\sum_{j=1}^N u_j(+\infty)$, then

$$\pi_\mathcal{C}(X) = \min_{\lambda \in \mathbb{R}_{++}, \, \mathbb{Q} \in (\mathcal{C}_1^0)^+} \left(\lambda \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[v_j\left(\lambda \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] \right).$$

Proof.
Observe first that $X \mapsto \rho(X) := -\sup_{Y \in \mathcal{C}} \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right] \right)$ is a non increasing, finite valued, convex functional on the Fréchet lattice M^Φ. Only convexity is non-evident: to show it, consider $X, Z \in M^\Phi$ and $Y, W \in \mathcal{C}$. For any $0 \le \lambda \le 1$, we have by concavity

$$\begin{aligned}
&\lambda \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(X^j + Y^j\right)\right] + (1-\lambda) \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(Z^j + W^j\right)\right] \\
&\le \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(\lambda(X^j + Y^j) + (1-\lambda)(Z^j + W^j)\right)\right] \\
&= \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[u_j\left(\lambda X^j + (1-\lambda) Z^j + \left(\lambda Y^j + (1-\lambda) W^j\right)\right)\right] \le -\rho(\lambda X + (1-\lambda) Z)
\end{aligned}$$

as $\lambda Y + (1-\lambda)W \in \mathcal{C}$. Thus taking suprema over $Y, W \in \mathcal{C}$ we get

$$\lambda(-\rho(X)) + (1-\lambda)(-\rho(Z)) \leq -\rho(\lambda X + (1-\lambda)Z).$$

Now the Extended Namioka-Klee Theorem (see [23] Theorem A.3) can be applied and we obtain

$$\rho(X) = \max_{0 \leq Z \in L^{\Phi^*}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j(-Z^j)\right] - \alpha(Z) \right),$$

where

$$\begin{aligned}
\alpha(Z) &:= \sup_{X \in M^\Phi} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j(-Z^j)\right] - \rho(X) \right) \\
&= \sup_{X \in M^\Phi} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j(-Z^j)\right] + \sup_{Y \in \mathcal{C}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] \right) \right) \\
&= \sup_{Y \in \mathcal{C}} \left(\sup_{X \in M^\Phi} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j(-Z^j)\right] + \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j + Y^j\right)\right] \right) \right) \right) \\
&= \sup_{Y \in \mathcal{C}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[Y^j(Z^j)\right] + \sup_{W \in M^\Phi} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[W^j(-Z^j)\right] + \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(W^j\right)\right] \right) \right) \right).
\end{aligned} \tag{1.68}$$

Observe now that $-U(z) := \sum_{j=1}^{N} -u_j(z^j)$ for $z \in \mathbb{R}^N$ defines a continuous, convex, proper function whose Fenchel transform is

$$\begin{aligned}
(-U)^*(w) &:= \sup_{z \in \mathbb{R}^N} \left(\langle z, w \rangle - (-U(z)) \right) \\
&= \sup_{z \in \mathbb{R}^N} \left(\langle z, w \rangle + U(z) \right) = \sup_{z \in \mathbb{R}^N} \left(U(z) - \langle z, -w \rangle \right) = \sum_{j=1}^{N} v_j(-w^j).
\end{aligned}$$

Now we apply Corollary on page 534 of [118] with $L = M^\Phi$, $L^* = L^{\Phi^*}$, $F(x) = -U(x)$ to see that

$$\sup_{W \in M^\Phi} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[W^j(-Z^j)\right] + \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(W^j\right)\right] \right) = \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} v_j(Z^j) \right]$$

and replacing this in (1.68) we get:

$$\alpha(Z) = \sup_{Y \in \mathcal{C}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[Y^j Z^j\right] + \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} v_j(Z^j) \right] \right).$$

Now observe that there are two possibilities:

- either $Z \in \mathcal{C}^0$, and in this case $\alpha(Z) = \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} v_j(Z^j)\right]$ since $0 \in \mathcal{C}$
- or $\alpha(Z) = +\infty$, since $v_1, \ldots, v_N$ are bounded from below.

Hence

$$-\sup_{Y\in\mathcal{C}}\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+Y^j\right)\right]\right)=\max_{0\leq Z\in L^{\Phi^*}}\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X^j(-Z^j)\right]-\alpha(Z)\right)$$
$$=\max_{0\leq Z\in\mathcal{C}^0}\left(-\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X^jZ^j\right]+\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}v_j(Z^j)\right]\right)\right)$$
$$=-\min_{0\leq Z\in\mathcal{C}^0}\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X^jZ^j\right]+\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}v_j(Z^j)\right]\right). \tag{1.69}$$

Moreover, since for every $i,j=1,\dots,N$ $e_i-e_j\in\mathcal{C}$ we can argue as in Lemma 1.3.1 to deduce that $\mathcal{C}^0\cap(L^0_+)^N=\mathbb{R}_+\cdot(\mathcal{C}^0_1)^+$. Replacing this in the expression (1.69) we get

$$\sup_{Y\in\mathcal{C}}\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+Y^j\right)\right]\right)=\min_{\lambda\in\mathbb{R}_+,\,\mathbb{Q}\in(\mathcal{C}^0_1)^+}\left(\lambda\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]+\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]\right).$$

To prove the last claim, observe that if the optimum λ in the right hand side was 0, we would have

$$\sup_{Y\in\mathcal{C}}\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+Y^j\right)\right]\right)=\sum_{j=1}^{N}v_j\left(0\right)=\sum_{j=1}^{N}u_j(+\infty),$$

which contradicts our hypotheses. □

Theorem 1.5.4. *Let $u_1,\dots,u_N$ satisfy Assumption 1.2.12. Let $K\subseteq M^{\Phi}$ be a convex cone such that for all $i,j\in\{1,\dots,N\}$ $e_i-e_j\in K$ and suppose that $\mathcal{Q}^e_v\neq\emptyset$, where*

$$\mathcal{Q}^e_v:=\left\{\mathbb{Q}\sim\mathbb{P}\mid\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\in L^{\Phi^*_j},\mathbb{E}_{\mathbb{P}}\left[v_j\left(\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right]<+\infty,\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[k^j\right]\leq 0\ \ \forall k\in K\right\}\subseteq L^{\Phi^*}.$$

Then denoting by $cl_{\mathbb{Q}}(\dots)$ the closure in $L^1(\mathbb{Q}^1)\times\dots\times L^1(\mathbb{Q}^N)$ with respect to the norm $\|X\|_{\mathbb{Q}}:=\sum_{j=1}^{N}\|X^j\|_{L^1(\mathbb{Q}^j)}$ we have

$$\bigcap_{\mathbb{Q}\in\mathcal{Q}^e_v}cl_{\mathbb{Q}}\left(K-L^1_+(\mathbb{Q})\right)=\left\{W\in\bigcap_{\mathbb{Q}\in\mathcal{Q}^e_v}L^1(\mathbb{Q})\mid\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[W^j\right]\leq 0\ \forall\mathbb{Q}\in\mathcal{Q}^e_v\right\}.$$

Proof. We modify the procedure in [21] Theorem 4. The inclusion ($LHS\subseteq RHS$) can be checked directly. As to the opposite one ($RHS\subseteq LHS$), suppose we had a $k\in RHS$ and a $\mathbb{Q}\in\mathcal{Q}^e_v$ with $k\notin cl_{\mathbb{Q}}\left(K-L^1_+(\mathbb{Q})\right)$, that is $k\notin LHS$. We stress that by construction

$$\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[k^j\right]\leq 0\ \ \forall\mathbb{Q}\in\mathcal{Q}^e_v\,. \tag{1.70}$$

In the dual system

$$\left(L^1(\mathbb{Q}),L^\infty(\mathbb{Q})\right)$$

the set $cl_{\mathbb{Q}}\left(K - L^1_+(\mathbb{Q})\right)$ is convex and $\sigma\left(L^1(\mathbb{Q}), L^\infty(\mathbb{Q})\right)$-closed by compatibility of the latter topology with the norm topology. Thus we can use Hahn-Banach Separation Theorem to get a class $\widehat{\xi} \in L^\infty(\mathbb{Q})$ with

$$0 = \sup_{W \in (K - L^1_+(\mathbb{Q}))} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\widehat{\xi}^j W^j \frac{d\mathbb{Q}^j}{d\mathbb{P}}\right]\right) < \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\widehat{\xi}^j k^j \frac{d\mathbb{Q}^j}{d\mathbb{P}}\right]. \tag{1.71}$$

We now work componentwise. First observe that

$$[-1_{\widehat{\xi}^j<0}]_{j=1}^N \in 0 - L^\infty_+(\mathbb{Q}) \subseteq K - L^1_+(\mathbb{Q}),$$

so that $\widehat{\xi}^j \geq 0$ $\mathbb{Q}^j$-a.s. for every $j = 1, \dots, N$. Hence $\widehat{\xi}^j \frac{d\mathbb{Q}^j}{d\mathbb{P}} \geq 0$ $\mathbb{P}$-a.s. for every $j = 1, \dots, N$.
Moreover, since for all $i, j \in \{1, \dots, N\}$ $e_i - e_j \in K$, we have

$$\mathbb{E}_{\mathbb{P}}\left[\widehat{\xi}^1 \frac{d\mathbb{Q}^1}{d\mathbb{P}}\right] = \dots = \mathbb{E}_{\mathbb{P}}\left[\widehat{\xi}^N \frac{d\mathbb{Q}^N}{d\mathbb{P}}\right]. \tag{1.72}$$

It follows that for every $j = 1, \dots, N$

$$\mathbb{P}\left(\widehat{\xi}^j \frac{d\mathbb{Q}^j}{d\mathbb{P}} > 0\right) > 0$$

since if this were not the case all the terms in equation (1.72) would be null, which would yield $\widehat{\xi}^1 \frac{d\mathbb{Q}^1}{d\mathbb{P}} = \dots = \widehat{\xi}^N \frac{d\mathbb{Q}^N}{d\mathbb{P}} = 0$, a contradiction with (1.71).
Hence the vector

$$\frac{d\mathbb{Q}^j_1}{d\mathbb{P}} := \frac{1}{\mathbb{E}_{\mathbb{P}}\left[\widehat{\xi}^j \frac{d\mathbb{Q}^j}{d\mathbb{P}}\right]} \widehat{\xi}^j \frac{d\mathbb{Q}^j}{d\mathbb{P}}$$

is well defined and identifies a vector of probability measures $[\mathbb{Q}^1_1, \dots, \mathbb{Q}^N_1]$. We trivially have that

$$\mathbb{Q}^j_1 \ll \mathbb{P}, \frac{d\mathbb{Q}^j_1}{d\mathbb{P}} \in L^{\Phi^*_j},$$

and by equation (1.71), together with (1.72)

$$\sup_{W \in K} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[W^j \frac{d\mathbb{Q}^j_1}{d\mathbb{P}}\right]\right) \leq 0 < \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[k^j \frac{d\mathbb{Q}^j_1}{d\mathbb{P}}\right]. \tag{1.73}$$

We observe that if we could prove $\mathbb{Q}_1 \in \mathcal{Q}^e_v$, we would get a contradiction with (1.70). However this needs not to be true, since we cannot guarantee $\mathbb{Q}^1_1, \dots, \mathbb{Q}^N_1 \sim \mathbb{P}$.
As $\mathbb{Q} \in \mathcal{Q}^e_v$, we have $\mathbb{Q} \sim \mathbb{P}$, and for $\mathbb{Q}_1$ above we have $\mathbb{Q}_1 \ll \mathbb{Q}$, $\frac{d\mathbb{Q}^k_1}{d\mathbb{Q}^k} \in L^\infty(\mathbb{Q}^k) = L^\infty(\mathbb{P})$. Take $\lambda \in (0, 1]$ and define $\mathbb{Q}_\lambda$ via

$$\frac{d\mathbb{Q}^k_\lambda}{d\mathbb{P}} := \lambda \frac{d\mathbb{Q}^k}{d\mathbb{P}} + (1 - \lambda)\frac{d\mathbb{Q}^k_1}{d\mathbb{P}}.$$

We now prove that $\mathbb{Q}_\lambda \in \mathcal{Q}^e_v$. It is easy to check that

$$0 < \lambda \leq \frac{d\mathbb{Q}^k_\lambda}{d\mathbb{Q}^k} \leq (1 - \lambda)\frac{d\mathbb{Q}^k_1}{d\mathbb{Q}^k} + \lambda,$$

so that Lemma 1.5.2.2. with $g = g^k := \frac{d\mathbb{Q}^k_\lambda}{d\mathbb{Q}^k}$, together with $\mathbb{E}_\mathbb{P}\left[v_k\left(\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] < +\infty\,\forall\, k = 1,\dots,N$ ($\mathbb{Q} \in \mathcal{Q}^e_v$ by construction), yields for all $k \in \{1,\dots,N\}$ and $\lambda \in (0,1]$

$$\mathbb{E}_\mathbb{P}\left[v_k\left(\frac{d\mathbb{Q}^k_\lambda}{d\mathbb{P}}\right)\right] = \mathbb{E}_\mathbb{P}\left[v_k\left(\frac{d\mathbb{Q}^k_\lambda}{d\mathbb{Q}^k}\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] = \mathbb{E}_\mathbb{P}\left[v_k\left(g^k\frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] < +\infty\,.$$

Moreover $\mathbb{Q} \in \mathcal{Q}^e_v$ and $\lambda > 0$ imply $\mathbb{Q}^k_\lambda \sim \mathbb{P}$ for all $k = 1,\dots,N$. This, together with equation (1.73), yields

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[W^j\frac{d\mathbb{Q}^j_\lambda}{d\mathbb{P}}\right] \leq 0\ \ \forall W \in K, \forall\,\lambda \in (0,1]\,.$$

We can conclude that $\mathbb{Q}_\lambda \in \mathcal{Q}^e_v$, $\forall\lambda \in (0,1]$. At the same time

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[k^j\frac{d\mathbb{Q}^j_\lambda}{d\mathbb{P}}\right] = \lambda\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[k^j\frac{d\mathbb{Q}^j}{d\mathbb{P}}\right] + (1-\lambda)\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[k^j\frac{d\mathbb{Q}^j_1}{d\mathbb{P}}\right] \xrightarrow[\lambda\to 0]{} \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[k^j\frac{d\mathbb{Q}^j_1}{d\mathbb{P}}\right]$$

which, since by Equation (1.73) $\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[k^j\frac{d\mathbb{Q}^j_1}{d\mathbb{P}}\right] > 0$, gives a contradiction with Equation (1.70). We conclude that $RHS \subseteq LHS$. □

Chapter 2

Multivariate Systemic Optimal Risk Transfer Equilibrium

A Systemic Optimal Risk Transfer Equilibrium (SORTE) was introduced in Chapter 1 for the analysis of the equilibrium among financial institution or in insurance-reinsurance markets. A SORTE conjugates the classical Bühlmann's notion of an Equilibrium Risk Exchange with a capital allocation principle based on systemic expected utility optimization. In Chapter 2 we extend such a notion to the case in which the value function to be optimized has two components, one being the sum of the single agents' utility functions, the other consisting of a truly systemic component. Technically, the extension of SORTE to the new setup requires developing a theory for multivariate utility functions and selecting at the same time a suitable framework for the duality theory. Conceptually, this more general framework allows us to introduce and study a Nash Equilibrium property of the optimizer. We prove existence, uniqueness, Pareto optimality and the Nash Equilibrium property of the newly defined Multivariate Systemic Optimal Risk Transfer Equilibrium (mSORTE) with budget $A \in \mathbb{R}$ and set of admissible allocations $\mathcal{B}$. An mSORTE consists of a triple $(Y_X, \mathbb{Q}_X, a_X) \in \mathcal{L} \times \mathcal{M} \times \mathbb{R}^N$, a random vector, a vector of probability measures and a deterministic vector respectively, with $Y \in L^1(\mathbb{Q}_X)$ and such that

- (Y_X, a_X) is an optimum for

$$\sup_{\substack{a \in \mathbb{R}^N \\ \sum_{j=1}^N a_j = A}} \left(\sup \left\{ \mathbb{E}_{\mathbb{P}} \left[U(X+Y) \right] \mid Y \in \mathcal{L} \cap L^1(\mathbb{Q}_X),\, \mathbb{E}_{\mathbb{Q}_X^j} \left[Y^j \right] \leq a^j,\, \forall j \right\} \right) ,$$

- $Y_X \in \mathcal{B}$ and $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}-$a.s.

Chapter 2 is organized as follows. Multivariate utility functions we use are introduced is Section 2.1, while Section 2.2 is a short account on multivariate Orlicz spaces and on the relevant properties from functional analysis needed in the sequel of the Chapter. Section 2.3 is devoted to the specification of our notations and assumptions. The core of Chapter 2 is Section 2.4, where we formally present the key concepts and provide our main results. Most of the proofs, as well as findings of some independent interest, are deferred to Section 2.5. Section 2.6 collects some additional technical results and some of the proofs related to Section 2.2.

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2.1 Preliminary notations and Multivariate Utility

The notation regarding underlying probability space, measures on it and Lebesgue spaces is borrowed from Section 1.1. For each $j = 1, ..., N$ consider a vector subspace $\mathcal{L}^j$ with $\mathbb{R} \subseteq \mathcal{L}^j \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and set

$$\mathcal{L}:=\mathcal{L}^1 \times ... \times \mathcal{L}^N \subseteq (L^0(\mathbb{P}))^N.$$

One could take as $\mathcal{L}^j$, for example, $L^\infty(\mathbb{P})$ or some Orlicz space. With

$$\mathcal{M} \subseteq \mathcal{P}^N$$

we will denote a subset of probability vectors. Our optimization problems will be defined for the set $\mathcal{M}$ and on the vector space $\mathcal{L}$, to be specified later (see Setups A, B and C in Section 2.3).
Given a vector $y \in \mathbb{R}^N$ and $n \in \{1, \dots, N\}$ we will denote by $y^{[-n]}$ the vector in $\mathbb{R}^{N-1}$ obtained suppressing the n-th component of y for $N \geq 2$ (and $y^{[-n]} = \emptyset$ if $N = 1$) and we set

$$[y^{[-n]}; z] := \left[y^1, \dots, y^{n-1}, z, y^{n+1}, \dots, y^N\right] \in \mathbb{R}^N, \quad \text{for } z \in \mathbb{R}. \tag{2.1}$$

Finally, we will write $\langle x, y\rangle = \sum_{j=1}^N x^j y^j$ for the usual innner product of vectors $x, y \in \mathbb{R}^N$, and ∂E for the boundary of $E \subseteq \mathbb{R}^N$.

2.1.1 Multivariate Utility Functions

Definition 2.1.1. *We say that $U : \mathbb{R}^N \to \mathbb{R}$ is a* ***multivariate utility function*** *if it is strictly concave and increasing with respect to the partial componentwise order. When $N = 1$ we will use the term univariate utility function instead. For a multivariate utility function U we define the convex conjugate in the usual way by*

$$V(y) := \sup_{x\in\mathbb{R}^N} \left(U(x) - \langle x, y\rangle\right). \tag{2.2}$$

Observe that by definition $U(x) \leq \langle x, y\rangle + V(y)$ for every $x, y \in \mathbb{R}^N$, and $V(\cdot) \geq U(0)$ that is V is lower bounded. Some useful properties of V are collected in Section 2.6.3.

Definition 2.1.2 ([119] Chapter V)**.** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be concave and let $z \in \mathbb{R}^N$ be given. We define the superdifferential of f at z as*

$$\partial f(z) := \{\nu \in \mathbb{R}^N \mid f(x) - f(z) \leq \sum_{j=1}^N \nu^j (x^j - z^j) \ \forall x \in \mathbb{R}^N\}.$$

By an abuse of notation we will denote by $\nabla f(z) = \left[\frac{\partial f}{\partial x^1}(z), \dots, \frac{\partial f}{\partial x^N}(z)\right]$ a given choice of a point in $\partial f(z)$. If $N = 1$, we will write $\frac{\mathrm{d}f}{\mathrm{d}x}(z)$ for a choice of a point in $\partial f(z)$.

It is well known that $\partial f(z) \neq \emptyset$ for any $z \in \mathbb{R}^N$([119] Theorem 23.4) and that $\partial f(z)$ consists of a single point if and only if the function f is differentiable in z ([119] Theorem 25.1). More properties are collected in Section 2.6.1.

Remark 2.1.3. With the notation of Definition 2.1.2, given a concave $f : \mathbb{R}^N \to \mathbb{R}$ we can write $f(x) \leq \sum_{j=1}^N \frac{\partial f}{\partial x^j}(z)(x^j - z^j) + f(z)$ for any $x, z \in \mathbb{R}^N$. In particular, given concave nondecreasing $u_1, \dots, u_N : \mathbb{R} \to \mathbb{R}$, all null in 0, for any $x^1, \dots, x^N \geq 0$

$$\sum_{j=1}^N u_j(x^j) \leq \max_{j=1,\dots,N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}x^j}(0) \right) \sum_{j=1}^N x^j. \tag{2.3}$$

For a univariate $f : \mathbb{R} \to \mathbb{R}$ we denote the usual left and right derivatives at the point z as $\frac{\mathrm{d}^\pm f}{\mathrm{d}x}(z)$. The following assumption holds true throughout Chapter 2 without further mention.

Standing Assumption I. *We consider multivariate utility functions in the form*

$$U(x) := \sum_{j=1}^N u_j(x^j) + \Lambda(x) \tag{2.4}$$

where $u_1, \dots, u_j : \mathbb{R} \to \mathbb{R}$ are univariate utility functions and $\Lambda : \mathbb{R}^N \to \mathbb{R}$ is concave, increasing with respect to the partial componentwise order and bounded from above. Furthermore we assume that for every $\varepsilon > 0$ there exists a point $z_\varepsilon \in \mathbb{R}^N$ such that

$$\sum_{j=1}^N \left| \frac{\partial \Lambda}{\partial x^j}(z_\varepsilon) \right| < \varepsilon\,. \tag{2.5}$$

We also assume the Inada conditions

$$\lim_{x \to +\infty} \frac{u_j(x)}{x} = 0 \qquad \textit{and} \qquad \lim_{x \to -\infty} \frac{u_j(x)}{x} = +\infty \quad \forall\, j = 1, \dots, N,$$

and that, without loss of generality, $u_j(0) = 0 \;\; \forall\, j = 1, \dots, N$.

Observe that such a multivariate utility function is split in two components: the sum of single agent utility functions and a universal part Λ that could be either selected upon agreement by all the agents or could be imposed by a regulatory institution. *As Λ is not necessarily strictly convex nor strictly increasing*, we may choose $\Lambda = 0$, which corresponds to the case analyzed in Chapter 1.

Remark 2.1.4. Condition (2.5) is inspired by Asymptotic Satiability as defined in Definition 2.13 of [38]. To be more explicit and in view of Definition 2.1.2, (2.5) means: for every $\varepsilon > 0$ there exists a $z_\varepsilon \in \mathbb{R}^N$ and a selection $\nu_\varepsilon \in \partial\Lambda(z_\varepsilon)$, such that $\sum_{j=1}^N |\nu_\varepsilon^j| < \varepsilon$.

Remark 2.1.5. $U(\cdot)$ defined in (2.4) is a multivariate utility function as introduced in Definition 2.1.1 since it inherits strict concavity and strict monotonicity from $u_1, \dots, u_N$. We may assume without loss of generality that $u_j(0) = 0 \;\; \forall\, j = 1, \dots, N$, since we can always write

$$U(x) = \sum_{j=1}^N \big(u_j(x^j) - u_j(0)\big) + \left(\Lambda(x) + \sum_{j=1}^N u_j(0) \right).$$

Thus, we can always redefine the univariate utility function and the multivariate one, without affecting other assumptions, in such a way that univariate utilities are null in 0.
In the following we will make extensive use of the following properties, without explicit mention: for every $f : \mathbb{R} \to \mathbb{R}$ nondecreasing and such that $f(0) = 0$ it holds that

$$f(x) = f(x^+) + f(-x^-), \quad (f(x))^+ = f\left(x^+\right) . \tag{2.6}$$

For each $j = 1, \dots, N$ we define the convex conjugate of u_j by

$$v_j(y) := \sup_{x \in \mathbb{R}} (u_j(x) - xy) \;\; y \in \mathbb{R} . \tag{2.7}$$

Remark 2.1.6. We observe that $v_1, \dots, v_N$ are finite valued on $(0, +\infty)$ by the Inada conditions and bounded below by $u_1(0), \dots, u_N(0)$ respectively. Since V as defined in (2.2) satisfies $V(y) \le \sum_{j=1}^N v_j(y^j) + \sup_{z \in \mathbb{R}^N} \Lambda(z)$, we infer that $V(\cdot)$ is finite valued on $(0, +\infty)^N$.

2.2 Multivariate Orlicz Spaces

Given a (univariate) Young function $\phi : \mathbb{R}_+ \to \mathbb{R}$ we can associate to it its conjugate function $\phi^*(y) := \sup_{x \in \mathbb{R}_+} (x\,|y| - \phi(x))$. As in [116], we can associate to both ϕ and ϕ^* the Orlicz spaces and Hearts $L^\phi, M^\phi, L^{\phi^*}, M^{\phi^*}$.
We now introduce multivariate Orlicz functions and spaces. The following definition is a slight modification of the one in Appendix B of [7].

Definition 2.2.1. *A function $\Phi : (\mathbb{R}_+)^N \to \mathbb{R}$ is said to be a multivariate Orlicz function if it null in 0, convex, continuous, increasing in the usual partial order and satisfies: there exist $A > 0, b$ constants such that $\Phi(x) \ge A\,\|x\| - b \;\forall x \in (\mathbb{R}_+)^N$.*

For a given multivariate Orlicz function Φ we define, as in [7], the Orlicz space and the Orlicz Heart respectively:

$$L^\Phi := \left\{X \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \exists\, \lambda \in (0, +\infty), \mathbb{E}_\mathbb{P}\left[\Phi(\lambda\,|X|)\right] < +\infty\right\}$$
$$M^\Phi := \left\{X \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \forall\, \lambda \in (0, +\infty), \mathbb{E}_\mathbb{P}\left[\Phi(\lambda\,|X|)\right] < +\infty\right\} \tag{2.8}$$

where $|X| := [|X^j|]_{j=1}^N$ is the componentwise absolute value. We introduce the Luxemburg norm as the functional

$$\|X\|_\Phi := \inf\left\{\lambda > 0 \mid \mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\lambda}\,|X|\right)\right] \le 1\right\}$$

defined on $L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right)$ and taking values in $[0, +\infty]$.

Lemma 2.2.2. *Let Φ be a multivariate Orlicz function. Then*

1. *The Luxemburg norm is finite on X if and only if $X \in L^\Phi$.*

2. *The Luxemburg norm is in fact a norm on L^{Φ}, which makes it a Banach space.*

3. *M^{Φ} is a vector subspace of L^{Φ}, closed under Luxemburg norm, and is a Banach space itself if endowed with the Luxemburg norm.*

4. *L^{Φ} is continuously embedded in $(L^1(\mathbb{P}))^N$.*

5. *Convergence in Luxemburg norm implies convergence in probability.*

6. *$X \in L^{\Phi}$, $|Y^j| \le |X^j|$ $\forall j = 1, \dots, N$ implies $Y \in L^{\Phi}$, and the same holds for the Orlicz Heart. In particular $X \in L^{\Phi}$ implies $X^{\pm} \in L^{\Phi}$ and the same holds for the Orlicz Heart.*

7. *The topology of $\|\cdot\|_{\Phi}$ on M^{Φ} is order continuous and M^{Φ} is the closure of $(L^{\infty})^N$ in Luxemburg norm.*

8. *M^{Φ} and L^{Φ} are Banach lattices if endowed with the topology induced by $\|\cdot\|_{\Phi}$ and with the componentwise $\mathbb{P}$-almost sure order.*

Proof. Claims (1)-(5) follow as in [7]. (6) is trivial from the definitions. As to (7), sequential order continuity is an application of Dominated Convergence Theorem, and order continuity follows from Theorem 1.1.3 in [65]. (8) is evident. □

Now we need to work a bit on duality.

Definition 2.2.3. *The Köthe dual K_{Φ} of the space L^{Φ} is defined as*

$$K_{\Phi} := \left\{ Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \sum_{j=1}^{N} X^j Z^j \in L^1(\mathbb{P}), \forall X \in L^{\Phi} \right\}. \quad (2.9)$$

Proposition 2.2.4. *K_{Φ} can be identified with a subspace of the topological dual of L^{Φ} and is a subset of $(L^1(\mathbb{P}))^N$.*

Proof. See Section 2.6.4. □

By Proposition 2.2.4 K_{Φ} is a normed space which can be naturally endowed with the dual norm of continuous linear functionals, which we will denote by $\|Z\|_{\Phi}^{*} := \sup\left\{\mathbb{E}_{\mathbb{P}}\left[|\sum_{j=1}^{N} X^j Z^j|\right] \mid \|X\|_{\Phi} \le 1\right\}$. This norm will play here the role of the Orlicz norm, and the relation between the two norms $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\Phi}^{*}$ is well understood in the univariate case (see Theorem 2.2.9 in [65]). The following Proposition summarizes useful properties which show how the Köthe dual can play the role of the Orlicz space L^{Φ^*} for M^{Φ} in univariate theory, and are the counterparts to Corollary 2.2.10 in [65].

Proposition 2.2.5. *The following hold:*

1. $K_{\Phi} = \left\{ Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \sum_{j=1}^{N} X^j Z^j \in L^1(\mathbb{P}), \forall X \in M^{\Phi} \right\}.$

2. *The topological dual of $(M^{\Phi}, \|\cdot\|_{\Phi})$ is $(K_{\Phi}, \|\cdot\|_{\Phi}^{*})$.*

3. *Suppose $L^\Phi = L^{\Phi_1} \times \cdots \times L^{\Phi_N}$. Then we have that $K_\Phi = L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}$ where this is only meant as equality of sets.*

Proof. See Section 2.6.4. □

Definition 2.2.6. *For a multivariate utility function U specified in (2.4), we define the function Φ on $(\mathbb{R}_+)^N$ by*

$$\Phi(y) := U(0) - U(-|y|),\ y \in (\mathbb{R}_+)^N \tag{2.10}$$

and

$$\Phi_j(z) := u_j(0) - u_j(-|z|),\ z \in \mathbb{R}_+\,, \tag{2.11}$$

as the (univariate) functions associated to the univariate utilities $u_1, \dots, u_N$.

Remark 2.2.7. Notice that Φ is a multivariate Orlicz function, which generates multivariate Orlicz space and Orlicz Heart, and $\Phi_1, \dots, \Phi_N$ are univariate Orlicz functions. To prove these claims, we only need to verify the existence of $A > 0, b$ given in Definition 2.2.1. We first consider the univariate case. By Proposition 2.6.1 for $b = u_j(0)$ and $M = \frac{\mathrm{d}^+ u_j}{\mathrm{d}x}(0)$, we have $u_j(-x^j) \leq M(-x^j) + b$ for all $x^j > 0$ and $j = 1, \dots, N$. As a consequence $\Phi(x^j) \geq Mx^j + u_j(0) - b$ for all $x^j > 0$ and $j = 1, \dots, N$. We also notice that $\frac{\mathrm{d}^+ u_j}{\mathrm{d}x}(0) > 0$ by strict monotonicity of u_j. The multivariate case follows from the univariate one: we have the inequality $\Phi(x) \geq \sum_{j=1}^N \Phi_j(x^j) - \sup_{\mathbb{R}^N}(\Lambda) + \Lambda(0)$ and by assumption $u_1, \dots, u_N$ are univariate utilities.

Remark 2.2.8. The conjugate functions of $\Phi_1, \dots, \Phi_N$ will be denoted by $\Phi_1^*, \dots, \Phi_N^*$. To each of these functions $\Phi_1, \dots, \Phi_N$ and $\Phi_1^*, \dots, \Phi_N^*$ we can associate Orlicz spaces and Orlicz hearts. The relationship between the convex conjugate v_j of u_j and the conjugate Φ_j^* of Φ_j is

$$\Phi_j^*(y) = \begin{cases} 0 & y \leq \beta_j \\ v_j(y) - v_j(\beta_j) & y > \beta_j \end{cases},$$

where $\beta_j := \frac{\mathrm{d}^- u_j}{\mathrm{d}x}(0)$. When u_j is bounded from above, v_j is also bounded in a neighborhood of 0 ($v_j(0) = u(+\infty) < +\infty$), and consequently an integrability condition of the form $\mathbb{E}_\mathbb{P}[\Phi_j^*(\cdot)] < +\infty$ holds true if and only if $\mathbb{E}_\mathbb{P}[v_j(\cdot)] < +\infty$.

We now provide an example connecting the multivariate theory to the univariate classical one.

Remark 2.2.9. Even thought we will not make this assumption in the rest of the Chapter, suppose in this Remark that $\Phi(x) = \sum_{j=1}^N \Phi_j(x^j)$ for univariate Orlicz functions, that is each separately satisfying Definition 2.2.1 for $N = 1$. Then we could consider the multivariate spaces L^Φ and M^Φ as above or we could take $L^{\Phi_1} \times \cdots \times L^{\Phi_N}$ and $M^{\Phi_1} \times \cdots \times M^{\Phi_N}$.

As shown in Section 2.6.4, the following identity between sets holds:

$$M^\Phi = M^{\Phi_1} \times \cdots \times M^{\Phi_N} \quad \text{and} \quad L^\Phi = L^{\Phi_1} \times \cdots \times L^{\Phi_N}$$

and furthermore

$$\frac{1}{N} \sum_{j=1}^N \left\| X^j \right\|_{\Phi_j} \leq \|X\|_\Phi \leq N \sum_{j=1}^N \left\| X^j \right\|_{\Phi_j}. \tag{2.12}$$

Observe that in the setup of this Remark, from Proposition 2.2.5 Item 3, we have

$$K_\Phi = L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}.$$

2.3 Setup and Assumptions

Recall that $\mathcal{C}_{\mathbb{R}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{j=1}^N Y^j \in \mathbb{R} \right\}$ that is, $\mathcal{C}_{\mathbb{R}}$ is the set of random vectors such that the sum of the components is $\mathbb{P}$-a.s. a deterministic number. The following assumption holds true throughout Chapter 2 without further mention.

Standing Assumption II. *$\mathcal{B} \subseteq \mathcal{C}_{\mathbb{R}}$ is a convex cone, closed in probability, $0 \in \mathcal{B}$, $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$. The vector X belongs to the Orlicz Heart M^Φ.*

Observe that the Standing Assumption II implies that all constant vectors belong to $\mathcal{B}$, so that all (deterministic) vector in the form $e^i - e^j$ (differences of elements in the canonical base of $\mathbb{R}^N$) belong to $\mathcal{B} \cap M^\Phi$. We recall the following concept, introduced in [20] Definition 5.15 and was already used in Chapter 1.

Definition 2.3.1. *$\mathcal{B}$ is closed under truncation if for each $Y \in \mathcal{B}$ there exists $m_Y \in \mathbb{N}$ and $c_Y \in \mathbb{R}^N$ such that $\sum_{j=1}^N Y^j = \sum_{j=1}^N c_Y^j$ and for all $m \geq m_Y$*

$$Y_m := Y 1_{\{|Y^j| < m \, \forall j = 1, \dots, N\}} + c_Y 1_{\Omega \setminus \{|Y^j| < m \, \forall j = 1, \dots, N\}} \in \mathcal{B} \,.$$

Some of the following assumptions will be needed for some of our main results. However, unlike for Standing Assumptions I and II, it will be always explicitly mentioned if and when these are assumed.

Assumption 2.3.2. *$\mathcal{B}$ is closed under truncation.*

As pointed out in [20], $\mathcal{B} = \mathcal{C}_{\mathbb{R}}$ is closed under truncation. Closedness under truncation property holds true for a rather wide class of constraints. For a more detailed explanation and examples, see also Chapter 1 Example 1.2.19 and Example 1.3.20.

Assumption 2.3.3. $L^\Phi = L^{\Phi_1} \times \cdots \times L^{\Phi_N}$.

While Assumption 2.3.2 is a requirement on the set of random allocations, Assumption 2.3.3 is a request on the utility functions we allow for. It can be rephrased as: if for $X \in (L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]))^N$ there exist $\lambda_1, \dots, \lambda_N > 0$ such that $\mathbb{E}_{\mathbb{P}}[u_j(-\lambda_j |X^j|)] > -\infty$, then there exists $\alpha > 0$ such that $\mathbb{E}_{\mathbb{P}}[\Lambda(-\alpha |X|)] > -\infty$. This request is rather weak and there are many examples of choices of U and Λ that guarantee this condition is met (see Section 2.4.5). Note however that this is not a request on the topological spaces, but just an integrability requirement, and it is automatically satisfied if $\Lambda \equiv 0$.

Assumption 2.3.4. *$u_1, \dots, u_N$ satisfy $\boldsymbol{AE}_{-\infty}$, that is: $u_1, \dots, u_N$ are differentiable on $\mathbb{R}$ and*

$$\liminf_{x \to -\infty} \frac{x u_j'(x)}{u_j(x)} > 1 \;\; \forall \, j = 1, \dots, N \,.$$

Assumption 2.3.5. *The function V, defined in (2.2), satisfies the following condition: for every $\mathbb{Q} = [\mathbb{Q}_1, \dots, \mathbb{Q}_N] \ll \mathbb{P}$ with $\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$ for some $\lambda > 0$ it holds that*

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\left[\lambda_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \lambda_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right]\right)\right] < +\infty \;\forall \, \lambda_1, \dots, \lambda_N > 0 \,.$$

As explained in Section 2.4.5, in the case $N = 1$ the Assumption 2.3.5 is a condition associated to Reasonable Asymptotic Elasticity, introduced [122], and is the classical one needed for the validity of many results in the theory of univariate utility maximization, see for example [104] and [122].
In Section 2.4.5, we provide further details and sufficient conditions for these assumptions, which show that these are reasonable. We here only note that in case $\Lambda \equiv 0$ we will obtain the same results of Chapter 1 but under weaker assumptions. A more precise formulation of this fact can be found in Section 2.4.6.
We introduce the following sets:

1. For any $A \in \mathbb{R}$ consider the set of random allocations

$$\mathcal{B}_A := \mathcal{B} \cap \left\{ Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j \le A \right\} \subseteq \mathcal{C}_{\mathbb{R}} \,. \tag{2.13}$$

2. $\mathcal{Q}$ is the set of vectors of probability measures $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N]$, with $\mathbb{Q}^j \ll \mathbb{P}$ $\forall\, j = 1, \dots, N$, defined by

$$\mathcal{Q} := \left\{ \mathbb{Q} \mid \left[\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}} \dots, \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}} \right] \in K_\Phi, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \right] \le 0 \;\; \forall Y \in \mathcal{B}_0 \cap M^\Phi \right\} . \tag{2.14}$$

   Identifying Radon-Nikodym derivatives and measures in the natural way, this can be rephrased as: $\mathcal{Q}$ is the set of normalized (i.e. with componentwise expectations equal to 1), non negative vectors in the polar of $\mathcal{B}_0 \cap M^\Phi$, in the dual system (M^Φ, K_Φ).

3. $\mathcal{Q}_V$ is the following subset of $\mathcal{Q}$:

$$\mathcal{Q}_V := \left\{ \mathbb{Q} \in \mathcal{Q} \mid \mathbb{E}_{\mathbb{P}} \left[V \left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right] < +\infty \text{ for some } \lambda > 0 \right\} .$$

We are now ready to specify the framework that will be adopted in our main results. To this end, we will consider three sets of assumptions:

Setup A: Assumption 2.3.2 and Assumption 2.3.3 are fulfilled and we set $\mathcal{M} := \mathcal{Q}_V$ and $\mathcal{L} := \bigcap_{\mathbb{Q} \in \mathcal{Q}_V} L^1(\mathbb{Q})$.

Setup B Assumption 2.3.3 and Assumption 2.3.4 are fulfilled and we set $\mathcal{M} := \mathcal{Q}_V$ and $\mathcal{L} := \bigcap_{\mathbb{Q} \in \mathcal{Q}_V} L^1(\mathbb{Q})$.

Setup C Assumption 2.3.5 is fulfilled, $u_1, \dots, u_N$ are differentiable on $\mathbb{R}$, Λ is differentiable on $\mathbb{R}^N$ and we set $\mathcal{M} := \mathcal{Q}_V$ and $\mathcal{L} := (L^0(\mathbb{P}))^N$.

Recall from (2.1) that we set

$$[Y^{[-n]}; Z] := \left[Y^1, \dots, Y^{n-1}, Z, Y^{n+1}, \dots, Y^N \right] \in (L^0(\mathbb{P}))^N, \quad \text{for } Z \in L^0(\mathbb{P}).$$

Consider a multivariate utility function U. For

$$(Y, \mathbb{Q}, a, A) \in (\mathcal{L} \cap L^1(\mathbb{Q})) \times \mathcal{M} \times \mathbb{R}^N \times \mathbb{R}$$

define:

$$U_j^{Y^{[-j]}}(Z) := \mathbb{E}_{\mathbb{P}}\left[u_j(X^j + Z)\right] + \mathbb{E}_{\mathbb{P}}\left[\Lambda(X + [Y^{[-j]}, Z])\right], \quad Z \in L^0(\mathbb{P}), \ j = 1, \dots, N, \tag{2.15}$$

$$\mathbb{U}_j^{\mathbb{Q}^j, Y^{[-j]}}(a^j) := \sup\left\{U_j^{Y^{[-j]}}(Z) \mid Z \in \mathcal{L}^j \cap L^1(\mathbb{Q}^j), \ \mathbb{E}_{\mathbb{Q}^j}[Z] \leq a^j\right\}, \quad j = 1, \dots, N, \tag{2.16}$$

$$T^{\mathbb{Q}}(a) := \sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in \mathcal{L} \cap L^1(\mathbb{Q}), \ \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq a^j, \ \forall j = 1, \dots, N\right\}, \tag{2.17}$$

$$S^{\mathbb{Q}}(A) := \sup\left\{T^{\mathbb{Q}}(a) \mid a \in \mathbb{R}^N, \ \sum_{j=1}^N a^j \leq A\right\}. \tag{2.18}$$

Obviously, all such quantities depend also on X, but as X will be kept fixed throughout most of the analysis, we may avoid to explicitly specify this dependence in the notations. As $u_1, \dots, u_N, \Lambda, U$ are increasing we can replace, in the definitions (2.16), (2.17), (2.18), the inequalities in the budget constraints with equalities. Moreover, it is clear that when $\Lambda \equiv 0$ the problem $S^{\mathbb{Q}}(A)$ introduced in (2.18) coincides with $S^{\mathbb{Q}}(A)$ defined in (1.2).

Remark 2.3.6. From the definition of V we obtain the Fenchel inequality

$$U(X+Y) \leq \langle X+Y, \lambda Z\rangle + V(\lambda Z) \quad \mathbb{P}\text{-a.s. for all } X, Y, Z \in (L^0(\mathbb{P}))^N, \ \lambda \geq 0.$$

Recall that $M^\Phi \subseteq L^1(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{Q}$. For all $X \in M^\Phi$, for all $\mathbb{Q} \in \mathcal{Q}$ and Y such that $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq A$ we then have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] &\leq \inf_{\lambda \geq 0}\left\{\lambda \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[(X^j + Y^j)\right] + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right\} \\
&\leq \inf_{\lambda \geq 0}\left\{\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + A\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right\}
\end{aligned}$$

and the last expression is finite if $\mathbb{Q} \in \mathcal{Q}_V$. Therefore, for all $Y \in \mathcal{B}_0 \cap M^\Phi$

$$\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \leq \inf_{\mathbb{Q} \in \mathcal{Q}_V} \inf_{\lambda \geq 0}\left\{\lambda \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right\} < +\infty.$$

2.4 Multivariate Systemic Optimal Risk Transfer Equilibrium

2.4.1 Main Concepts

Here is the natural generalization of SORTE as introduced in Chapter 1:

Definition 2.4.1. *The triple* $(Y_X, \mathbb{Q}_X, a_X) \in \mathcal{L}\times\mathcal{M}\times\mathbb{R}^N$ *with* $Y_X \in L^1(\mathbb{Q}_X)$ *is a* ***Weak Multivariate Systemic Optimal Risk Transfer Equilibrium*** *(Weak mSORTE) with budget* $A \in \mathbb{R}$ *if:*
1) for each $j = 1, \dots, N$, Y_X^j *is optimal for* $\mathbb{U}_j^{\mathbb{Q}_X^j, Y_X^{[-j]}}(a_X^j)$,
2) a_X *is optimal for* $S^{\mathbb{Q}_X}(A)$,
3) $Y_X \in \mathcal{B}$ *and* $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}$- *a.s.*

Definition 2.4.2. *The triple* $(Y_X, \mathbb{Q}_X, a_X) \in \mathcal{L}\times\mathcal{M}\times\mathbb{R}^N$ *with* $Y_X \in L^1(\mathbb{Q}_X)$ *is a* ***Multivariate Systemic Optimal Risk Transfer Equilibrium*** *(mSORTE) with budget* $A \in \mathbb{R}$ *if*

1. (Y_X, a_X) *is an optimum for*

$$\sup_{\substack{a\in\mathbb{R}^N \\ \sum_{j=1}^N a_j = A}} \left(\sup \left\{ \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in \mathcal{L}\cap L^1(\mathbb{Q}_X),\, \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j\right] \le a^j,\, \forall j \right\} \right),$$

2. $Y_X \in \mathcal{B}$ *and* $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}$- *a.s.*

When $\Lambda \equiv 0$ the definition of the Weak mSORTE coincides with the one of the SORTE, as defined in Chapter 1. See Section 2.4.6 for an accurate comparison.

Remark 2.4.3. It follows from the monotonicity of the utility functions that $\sum_{j=1}^N a_X^j = A$ and $\mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] = a_X^j$. Hence

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] = \sum_{j=1}^N a_X^j = A$$

and

$$\sum_{j=1}^N Y_X^j = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j] \quad \mathbb{P} - a.s. \tag{2.19}$$

Lemma 2.4.4. *A Multivariate SORTE is a Weak Multivariate SORTE.*

Proof. Let $(Y_X, \mathbb{Q}_X, a_X) \in \mathcal{L}\times\mathcal{M}\times\mathbb{R}^N$ be an mSORTE as in Definition 2.4.2.
We prove that Item 1 in Definition 2.4.1 holds true. By Remark 2.4.3 we have $a_X^j = \mathbb{E}_{\mathbb{Q}_X^j}[Y_X^j]$, $j = 1, \dots, N$. For any $Z \in \mathcal{L}^j \cap L^1(\mathbb{Q}_X^j)$ with $\mathbb{E}_{\mathbb{Q}_X^j}[Z] \le a_X^j$ we have that $[Y_X^{[-j]}; Z]$ satisfies then the constraints of the problem

$$\sup \left\{ \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in \mathcal{L}\cap L^1(\mathbb{Q}_X),\, \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j\right] \le a_X^j,\ \forall j \right\}$$

and we have by Item 1 of Definition 2.4.2 that

$$\mathbb{E}_{\mathbb{P}}\left[U(X+Y_X)\right] \ge \mathbb{E}_{\mathbb{P}}\left[U(X+[Y_X^{[-j]}; Z]\right].$$

By simple computations, this implies $U_j^{Y_X^{[-j]}}(Y_X^j) \ge U_j^{Y_X^{[-j]}}(Z)$, yielding the required optimality.

We now move to Item 2 of Definition 2.4.1:

$$
\begin{aligned}
&\sup_{\substack{a\in\mathbb{R}^N\\ \sum_{j=1}^N a^j=A}} T^{\mathbb{Q}_X}(a)\\
&\overset{(2.17)}{=} \sup_{\substack{a\in\mathbb{R}^N\\ \sum_{j=1}^N a_j=A}} \left(\sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y\in\mathcal{L}\cap L^1(\mathbb{Q}_X),\, \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j\right]\le a^j,\ \forall j\right\}\right)\\
&\overset{\substack{\text{Def. 2.4.2}\\ \text{Item 1}}}{=} \mathbb{E}_{\mathbb{P}}\left[U(X+Y_X)\right]\\
&\overset{\text{Rem.2.4.3}}{\le} \left(\sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y\in\mathcal{L}\cap L^1(\mathbb{Q}_X),\, \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j\right]\le a_X^j,\ \forall j\right\}\right)\\
&\overset{(2.17)}{=} T^{\mathbb{Q}_X}(a_X)\le \sup_{\substack{a\in\mathbb{R}^N\\ \sum_{j=1}^N a^j=A}} T^{\mathbb{Q}_X}(a)
\end{aligned}
$$

which implies optimality of a_X.
Finally, Item 3 of Definition 2.4.1 trivially holds, since Y_X satisfies Item 2 of Definition 2.4.2. □

2.4.2 Pareto Allocation and Nash Equilibrium

For each $j=1,...,N$, let $u_j:\mathbb{R}\to\mathbb{R}$ and let $\Lambda:\mathbb{R}^N\to\mathbb{R}$. Similarly to Chapter 1 we give the following definition:

Definition 2.4.5. *Given a set of feasible allocations $\mathcal{V}\subseteq(L^0(\mathbb{P}))^N$, $Y\in\mathcal{V}$ is a* Pareto allocation for $\mathcal{V}$ *if*

$$
Z\in\mathcal{V},\quad \begin{cases}\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z^j)\right]\ge\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Y^j)\right] \text{ for all } j,\\ \mathbb{E}_{\mathbb{P}}\left[\Lambda(X+Z)\right]\ge\mathbb{E}_{\mathbb{P}}\left[\Lambda(X+Y)\right]\end{cases} \tag{2.20}
$$

imply:

$$
\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z^j)\right]=\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Y^j)\right] \text{ for all } j, \text{ and } \mathbb{E}_{\mathbb{P}}\left[\Lambda(X+Z)\right]=\mathbb{E}_{\mathbb{P}}\left[\Lambda(X+Y)\right].
$$

In general Pareto allocations are not unique and the following version of the First Welfare Theorem holds true (compare with Proposition 1.2.2).

Proposition 2.4.6. *Define the optimization problem*

$$
\Pi(\mathcal{V}):=\sup_{Z\in\mathcal{V}}\left\{\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z^j)\right]+\mathbb{E}_{\mathbb{P}}\left[\Lambda(X+Z)\right]\right\}. \tag{2.21}
$$

Whenever $Y\in\mathcal{V}$ is the unique optimal solution of $\Pi(\mathcal{V})$, then it is a Pareto allocation for $\mathcal{V}$.

Proof. Let Y be optimal for $\Pi(\mathcal{V})$, so that $\mathbb{E}\left[\sum_{j=1}^N u_j(X^j+Y^j)\right] + \mathbb{E}[\Lambda(X+Y)] = \Pi(\mathcal{V})$. Suppose that there exists Z such that (2.20) holds true. As $Z \in \mathcal{V}$ we have:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j+Y^j)\right] + \mathbb{E}_{\mathbb{P}}[\Lambda(X+Y)] \\
&= \Pi(\mathcal{V}) := \sup_{W\in\mathcal{V}}\left\{\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j(X^j+W^j)\right] + \mathbb{E}_{\mathbb{P}}[\Lambda(X+W)]\right\} \\
&\geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j+Z^j)\right] + \mathbb{E}_{\mathbb{P}}[\Lambda(X+Z)] \\
&\geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N u_j(X^j+Y^j)\right] + \mathbb{E}_{\mathbb{P}}[\Lambda(X+Y)]
\end{aligned}$$

by (2.20). Hence Z is an optimal solution to $\Pi(\mathcal{V})$. Uniqueness of the optimal solution implies $Z = Y$, and the validity of Definition 2.4.5 follows. □

We also introduce a version of a Nash Equilibrium:

Definition 2.4.7. *Given a set of feasible allocations $\mathcal{V} \subseteq (L^0(\mathbb{P}))^N$, $Y \in \mathcal{V}$ is a* Nash Equilibrium for $\mathcal{V}$ *if for every $j \in \{1,\dots,N\}$*

$$U_j^{Y^{[-j]}}(Y^j) \geq U_j^{Y^{[-j]}}(Z) \text{ for all } Z \text{ such that } [Y^{[-j]};Z] \in \mathcal{V}\,,$$

where $U_j^{Y^{[-j]}}(\cdot), j = 1,\dots,N$ are defined in (2.15).

Assuming that all agents $n \neq j$ adopt strategy $Y^{[-j]}$, in a Nash Equilibrium the strategy Y^j of agent j maximizes his own expected utility plus an additional systemic/regulatory term:

$$Y^j := \arg\max\left\{\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+\bullet)\right] + \mathbb{E}_{\mathbb{P}}\left[\Lambda(X+[Y^{[-j]};\bullet])\right]\right\}\,.$$

2.4.3 Main Results

The analysis in Chapter 1 regards existence, uniqueness and properties of a SORTE. Here we provide sufficient conditions for existence, uniqueness, Pareto optimality and the Nash Equilibrium property of a mSORTE, see Theorems 2.4.11 and 2.4.12. Such results are relatively simple consequences of the following key duality Theorem 2.4.9, whose proof in Section 2.5 will involve several steps.
We introduce the following sets of random vectors, for $A \in \mathbb{R}$:

$$\mathcal{L}_V^{(A)} := \bigcap_{\mathbb{Q}\in\mathcal{Q}_V}\left\{Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}), \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq A\right\}\,, \quad (2.22)$$

$$\mathcal{L}_V := \mathcal{L}_V^{(0)}\,.$$

Remark 2.4.8. For any $\mathbb{Q} \in \mathcal{Q}_V$

$$\mathcal{L}_V^{(A)} \subseteq \left\{ Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}), \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right] \le A \right\}$$

and then, by Fenchel inequality (using an argument similar to the one in Remark 2.3.6) we deduce that the following weak duality holds true

$$\sup_{Y \in \mathcal{L}_V^{(A)}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right] \tag{2.23}$$

$$\le \inf_{\mathbb{Q} \in \mathcal{Q}_V} \sup \left\{ \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right] \mid Y \in (L^0(\mathbb{P}))^N, \sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}),\ \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right] \le A \right\} \tag{2.24}$$

$$\le \inf_{\mathbb{Q} \in \mathcal{Q}_V} \inf_{\lambda \ge 0} \left(\lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j \right] + A \right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right] \right) \tag{2.25}$$

$$\le \inf_{\lambda \ge 0} \left(\lambda \left(\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j \right] + A \right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right) \right] \right) < +\infty, \text{ for any } \widehat{\mathbb{Q}} \in \mathcal{Q}_V. \tag{2.26}$$

Theorem 2.4.9. *In either setup A, B or C the following holds:*

$$\sup_{Y \in \mathcal{B}_A \cap M^\Phi} \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right] = \sup_{Y \in \mathcal{L}_V^{(A)}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right] \tag{2.27}$$

$$= \min_{\mathbb{Q} \in \mathcal{Q}_V} \min_{\lambda \ge 0} \left(\lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j \right] + A \right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right] \right). \tag{2.28}$$

Moreover:

1. *There exists a unique optimum* $\widehat{Y} \in \mathcal{L}_V^{(A)}$ *to the problem in RHS of* (2.27).
2. *Any optimum* $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ *of* (2.28) *satisfies* $\widehat{\lambda} > 0$ *and* $\widehat{\mathbb{Q}} \sim \mathbb{P}$.
3. *For any optimum* $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ *of* (2.28) *we have* $\widehat{Y} \in \mathcal{B}_A \cap \mathcal{L} \cap L^1(\widehat{\mathbb{Q}})$ *and*
$$\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j] = A = \sum_{j=1}^N \widehat{Y}^j, \quad \mathbb{P} - a.s..$$
4. *If* U *is differentiable, there exists a unique optimum* $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ *of* (2.28).
5. *In Setup A*
$$\sup_{Y \in \mathcal{B}_A \cap (L^\infty(\mathbb{P}))^N} \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right] = \sup_{Y \in \mathcal{B}_A \cap M^\Phi} \mathbb{E}_{\mathbb{P}}\left[U(X+Y) \right]. \tag{2.29}$$

Proof. Setup A and B: the case $A = 0$ is covered in Theorem 2.5.16. Setup C: the case $A = 0$ is covered in Theorem 2.5.17 (observe that differentiability of U is assumed in the setup C). In Section 2.5.6 we then explain how we can apply also to $A \neq 0$ the same arguments used for $A = 0$. Corollary 2.5.19 proves (2.29). □

The following result is the counterpart to Theorem 2.4.9, once a vector $\mathbb{Q} \in \mathcal{Q}_V$ is fixed.

Theorem 2.4.10. *For either $\mathcal{L} = \bigcap_{\mathbb{Q}\in\mathcal{Q}_V} L^1(\mathbb{Q})$ or $\mathcal{L} = (L^0(\mathbb{P}))^N$, for every $\mathbb{Q} \in \mathcal{Q}_V$ and $A \in \mathbb{R}$ the following holds:*

$$\sup_{\substack{Y\in\mathcal{L}\cap L^1(\mathbb{Q})\\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j]\le A}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] = \min_{\lambda\ge 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + A\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{2.30}$$

Proof. Consider first $A = 0$. By Proposition 2.5.11

$$\min_{\lambda\ge 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right) = \sup_{\substack{Y\in M^\Phi\\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j]\le 0}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right].$$

Observing that $M^\Phi \subseteq \mathcal{L}\cap L^1(\mathbb{Q}) \subseteq L^1(\mathbb{Q})$, we have

$$\begin{aligned}
\sup_{Y\in M^\Phi\, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j]\le 0} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] &\le \sup_{\substack{Y\in\mathcal{L}\cap L^1(\mathbb{Q})\\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j]\le 0}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right]\\
\le \sup_{\substack{Y\in L^1(\mathbb{Q})\\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j]\le 0}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] &\le \inf_{\lambda\ge 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right),
\end{aligned}$$

by Remark 2.3.6. The case $A = 0$ is then proved. The case $A \ne 0$, instead, follows from Section 2.5.6. $\square$

On the existence of an mSORTE and Nash Equilibrium

Theorem 2.4.11. *In either setup A, B or C a Multivariate Systemic Optimal Risk Transfer Equilibrium $(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a}) \in \mathcal{L} \times \mathcal{Q}_V \times \mathbb{R}^N$ exists. Furthermore, $\widehat{\mathbb{Q}} \sim \mathbb{P}$ and $\widehat{Y}$ is a Nash Equilibrium for both the sets*

$$\mathcal{V}_A = \mathcal{L} \cap \left\{Y \in L^1(\widehat{\mathbb{Q}}) \mid \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j] \le A\right\}$$

$$\mathcal{V}_{\widehat{a}} = \mathcal{L} \cap \left\{Y \in L^1(\widehat{\mathbb{Q}}) \mid \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j] \le \widehat{a}^j \,\forall\, j = 1, \dots, N\right\}.$$

Proof. The proof of the existence of an mSORTE consists in showing that the optimizers $(\widehat{Y}, \widehat{\mathbb{Q}})$ in Theorem 2.4.9, together with $\widehat{a}^j := \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j]$, $j = 1, \dots, N$, are an mSORTE. Let $\widehat{\mathbb{Q}}$ be an optimizer of (2.28). Then, from (2.27) and (2.28),

$$
\begin{aligned}
\sup_{Y\in\mathcal{L}_V^{(A)}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] &= \min_{\lambda\geq 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j\right]+A\right)+\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right]\right)\\
&\overset{(2.30)}{=} \sup_{\substack{Y\in\mathcal{L}\cap L^1(\widehat{\mathbb{Q}})\\ \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j]\leq A}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \qquad (2.31)\\
&= \sup_{\substack{a\in\mathbb{R}^N\\ \sum_{j=1}^N a_j=A}} \left(\sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y\in\mathcal{L}\cap L^1(\widehat{\mathbb{Q}}),\, \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\right]\leq a^j,\,\forall j\right\}\right) \qquad (2.32)\\
&= S^{\widehat{\mathbb{Q}}}(A),
\end{aligned}
$$

where (2.32) is a simple reformulation of (2.31). By Item 3 of Theorem 2.4.9, the optimizer $\widehat{Y}\in\mathcal{L}_V^{(A)}$ satisfies the constraints of the problem in (2.31), hence it is also an optimum for the problem in (2.31). We conclude that $\widehat{Y}$ and $\widehat{a}^j:=\mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j]$, $j=1,\dots,N$, provide an optimum to the problem in (2.32), so that $(\widehat{Y},\widehat{a})$ fulfills the requirements in Item 1 of Definition 2.4.2 and $\sum_{j=1}^N \widehat{a}^j=A$. Furthermore, from Item 3 Theorem 2.4.9, $\widehat{Y}$ satisfies $\widehat{Y}\in\mathcal{B}$ and $\sum_{j=1}^N \widehat{Y}^j=A$, proving Item 2 in Definition 2.4.2.

As to the Nash Equilibrium property with respect to $\mathcal{V}_A$ and $\mathcal{V}_{\widehat{a}}$: observe that given $\widehat{Y}^1,\dots,\widehat{Y}^N$ and $\widehat{a}^j=\mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j]$, $j=1,\dots,N$, we have that $\{Z\mid[\widehat{Y}^{[-k]};Z]\in\mathcal{V}_A\}=\{Z\mid[\widehat{Y}^{[-k]};Z]\in\mathcal{V}_{\widehat{a}}\}$. To check the Nash Equilibrium property, it is then enough to work on the set $\mathcal{V}_{\widehat{a}}$ only. By Lemma 2.4.4 an mSORTE is a Weak mSORTE. Item 1 in Definition 2.4.1 then yields Nash Equilibrium property for $\widehat{Y}$. □

On uniqueness of an mSORTE and Pareto Optimality

Theorem 2.4.12. *In Setup A and if U is differentiable the Multivariate SORTE $(\widehat{Y},\widehat{\mathbb{Q}},\widehat{a})$ is unique. Moreover, the vector $\widehat{Y}$ is a Pareto Allocation for $\mathcal{V}=\mathcal{B}_A\cap\mathcal{L}$.*

Proof. We claim that if $(\widehat{Y},\widehat{\mathbb{Q}},\widehat{A})$ is an mSORTE then $\widehat{Y}$ is an optimizer of RHS of (2.27) and $\widehat{\mathbb{Q}}$ is an optimizer of (2.28). Under the differentiability assumption, the uniqueness of an mSORTE is then a consequence of the uniqueness of the optimizers in (2.28) (Theorem 2.4.9 Item 4) and of the fact that, by the monotonicity of Λ, $u_1,\dots,u_N$, in an mSORTE it holds: $\widehat{a}^j=\mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j]$. To prove the claim, let $(\widehat{Y},\widehat{\mathbb{Q}},\widehat{A})$ be an mSORTE, so that $\widehat{Y}\in\mathcal{B}_A\cap\mathcal{L}$ and $\widehat{\mathbb{Q}}\in\mathcal{Q}_V$. Observe that in the Setup A the set $\mathcal{B}$ is closed under truncation. Therefore, arguing as in Lemma 1.3.15 of Chapter 1, $\mathcal{B}_A\cap\mathcal{L}\subseteq\mathcal{L}_V^{(A)}$. As a consequence, $\widehat{Y}\in\mathcal{L}_V^{(A)}$ and $\mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right]\leq$

$\sup_{Y\in\mathcal{L}_V^{(A)}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right]$. As $\widehat{\mathbb{Q}}\in\mathcal{Q}_V$, from (2.23)-(2.26) we then obtain:

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] &\leq \sup_{Y\in\mathcal{L}_V^{(A)}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] && (2.33)\\
&\leq \inf_{\lambda\geq 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j\right]+A\right)+\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right]\right) && (2.34)\\
&= \min_{\lambda\geq 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j\right]+A\right)+\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right]\right) && (2.35)\\
&\overset{\text{Thm.2.4.10}}{=} \sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y\in\mathcal{L}\cap L^1(\widehat{\mathbb{Q}}),\ \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\right]\leq A\right\}\\
&= \sup_{\substack{a\in\mathbb{R}^N\\ \sum_{j=1}^N a_j=A}}\left(\sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y\in\mathcal{L}\cap L^1(\widehat{\mathbb{Q}}),\ \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\right]\leq a^j\ \forall j\right\}\right) && (2.36)\\
&= \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] && (2.37)
\end{aligned}
$$

where the expression in (2.36) is a reformulation of the one in the previous line, and (2.37) holds true because $(\widehat{Y},\widehat{\mathbb{Q}},\widehat{a})$ is an mSORTE and therefore $\widehat{Y}$ is an optimizer of the problem in (2.36). Notice that Theorem 2.4.10 guarantees that the inf in (2.34) is a min. We then deduce that all above inequalities are equalities and $\widehat{Y}\in\mathcal{L}_V^{(A)}$ is an optimizer of RHS of (2.27) and $\widehat{\mathbb{Q}}$ is an optimizer of (2.28).
We conclude proving that $\widehat{Y}$ is a Pareto allocation: in Setup A observe that $\mathcal{L}\cap L^1(\widehat{\mathbb{Q}})=\mathcal{L}$ and $\mathcal{V}:=\mathcal{B}_A\cap\mathcal{L}\subseteq\mathcal{L}_V^{(A)}$, as already argued at the beginning of the proof. In conclusion we get

$$
\mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right]\leq\sup_{Y\in\mathcal{V}}\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right]\leq\sup_{Y\in\mathcal{L}_V^{(A)}}\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right]=\mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right],
$$

by (2.33)-(2.37). Thus $\widehat{Y}$ is the unique optimum, by the strict concavity of U, to the problem $\Pi(\mathcal{V})$ given in (2.21), and Proposition 2.4.6 can be applied. □

2.4.4 Dependence on X of mSORTE

We study here the dependence of mSORTE on the initial data X. We will work in Setup A, in such a way that both existence and uniqueness are guaranteed (see Theorem 2.4.11 and Theorem 2.4.12).

Proposition 2.4.13. *In Setup A and for $\mathcal{B}=\mathcal{C}_{\mathbb{R}}$, given an mSORTE $(\widehat{Y},\widehat{\mathbb{Q}},\widehat{a})$ the variables $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}$ and $X+\widehat{Y}$ are $\sigma(X^1+\cdots+X^N)$ (essentially) measurable.*

Proof. By Theorem 2.4.12 there exists a unique mSORTE. Recall the proof of Theorem 2.4.11, where we showed that the optimizers $(\widehat{Y},\widehat{\mathbb{Q}})$ in Theorem 2.4.9, together with $\widehat{a}^j:=\mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j]$, $j=1,\dots,N$, are the mSORTE. Notice that in this specific case $Y:=e_i 1_A-e_j 1_A\in\mathcal{B}\cap M^\Phi$ for all i,j. The same argument used in the proof of

Proposition 1.3.18 of Chapter 1 can be then applied with obvious minor modifications (i.e. using $V(\cdot)$ in place of $\sum_{j=1}^N v_j(\cdot)$ and taking any $\mathbb{Q} \in \mathcal{Q}_V$) to show that $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}$ is $\mathcal{G} := \sigma(X^1 + \cdots + X^N)$-(essentially) measurable. We stress the fact that, similarly to Proposition 1.3.18, all the components of any $\mathbb{Q} \in \mathcal{Q}_V$ are equal.
We now focus on $X+\widehat{Y}$: consider $\widehat{Z} := \mathbb{E}_\mathbb{P}[X + Y \mid \mathcal{G}] - X$ (the conditional expectation is taken componentwise). Then it is easy to check that $\sum_{j=1}^N \widehat{Z}^j = \sum_{j=1}^N \widehat{Y}^j = A$ which yields $\widehat{Z} \in \mathcal{B}_A$. We now prove that $\widehat{Z} \in \mathcal{L}_V^{(A)}$, by showing that $Z \in \mathcal{L} = \bigcap_{\mathbb{Q}\in\mathcal{Q}_V} L^1(\mathbb{Q})$ (the fact that $\widehat{Z} \in \mathcal{L}_V^{(A)}$ follows then from the fact that $\mathcal{L} \cap \mathcal{B}_A \subseteq \mathcal{L}_V^{(A)}$, as argued in the proof of Theorem 2.4.12). Since $X \in M^\Phi$, it is clearly enough to prove that $\mathbb{E}_\mathbb{P}\left[X + \widehat{Y}\middle|\mathcal{G}\right] \in \mathcal{L}$. Observe first that for any given $\mathbb{Q} \ll \mathbb{P}$, the measure $\mathbb{Q}_\mathcal{G}$ defined by $\frac{\mathrm{d}\mathbb{Q}_\mathcal{G}}{\mathrm{d}\mathbb{P}} := \mathbb{E}_\mathbb{P}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]$ satisfies

$$\mathbb{Q} \in \mathcal{Q}_V \Longrightarrow \mathbb{Q}_\mathcal{G} \in \mathcal{Q}_V\,. \tag{2.38}$$

To see this, recall that all the components of $\mathbb{Q}$ are equal, hence so are those of $\mathbb{Q}_\mathcal{G}$. Moreover

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y^j \frac{\mathrm{d}\mathbb{Q}_\mathcal{G}^j}{\mathrm{d}\mathbb{P}}\right] = \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}_\mathcal{G}^1}{\mathrm{d}\mathbb{P}}\right] = \sum_{j=1}^N Y^j \le 0 \quad \forall Y \in \mathcal{B}_0 \cap M^\Phi$$

and $\mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}_\mathcal{G}}{\mathrm{d}\mathbb{P}}\right)\right] \le \mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]$ by conditional Jensen Inequality.
Now, for any $j = 1, \dots, N$ and $\mathbb{Q} \in \mathcal{Q}_V$

$$\mathbb{E}_\mathbb{P}\left[\left|\mathbb{E}_\mathbb{P}\left[X^j + Y^j \middle|\mathcal{G}\right]\right| \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \le \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\left|X^j + Y^j\right|\middle|\mathcal{G}\right] \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\right]$$

$$= \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\left|X^j + Y^j\right| \mathbb{E}_\mathbb{P}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\middle|\mathcal{G}\right]\right] = \mathbb{E}_\mathbb{P}\left[\left|X^j + Y^j\right| \mathbb{E}_\mathbb{P}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\middle|\mathcal{G}\right]\right]\,.$$

As a consequence, since by (2.38) $\mathcal{L} \subseteq L^1(\mathbb{Q}_\mathcal{G})$ and $\widehat{Y} \in \mathcal{L}$, we get $X + \widehat{Y} \in L^1(\mathbb{Q})$, and the fact that $\widehat{Z} \in \mathcal{L}$ follows.
Finally, observe that $\mathbb{E}_\mathbb{P}\left[U\left(X + \widehat{Z}\right)\right] = \mathbb{E}_\mathbb{P}\left[U\left(\mathbb{E}_\mathbb{P}\left[X + \widehat{Y}\middle|\mathcal{G}\right]\right)\right] \ge \mathbb{E}_\mathbb{P}\left[U(X + \widehat{Y})\right]$ by conditional Jensen Inequality. Hence $\widehat{Z}$, which satisfies $\widehat{Z} \in \mathcal{L} \subseteq \mathcal{L}_V^{(A)}$, is another optimum for the optimization problem in RHS of (2.27). By Proposition 2.5.2, with $\mathcal{K} = \mathcal{L}_V^{(A)}$, we get $\widehat{Y} = \widehat{Z}$. Since $X + \widehat{Z}$ is $\mathcal{G}$-(essentially) measurable, so is clearly $X + \widehat{Y}$. □

It is interesting to notice that this dependence on the componentwise sum of X also holds in the case of SORTE (Section 1.3.5) and of Bühlmann's equilibrium (see [33] page 16, which partly inspired the proof above, and [26]).

Remark 2.4.14. In the case of clusters of agents, the above result can be clearly generalized (see Remark 1.3.19 in Chapter 1).

2.4.5 On the assumptions and examples

We introduce the following definition, inspired by Definition 2.2.1 [116].

Definition 2.4.15. *Let $u : \mathbb{R} \to \mathbb{R}$ and $\widetilde{u} : \mathbb{R} \to \mathbb{R}$. We say that $u \preceq \widetilde{u}$ if there exist $k \in \mathbb{R}$, $c \in \mathbb{R}_+$, $C \in \mathbb{R}_+$ such that $\widetilde{u}(x) \geq Cu(cx) + k$ for each $x \leq 0$.*

Note that such control is required to hold only for negative values.

Assumption 2.3.3

We now consider $\Lambda(x) := u\left(\sum_{j=1}^N \beta_j x^j\right)$ for some concave increasing (not necessarily strictly) and bounded above function $u : \mathbb{R} \to \mathbb{R}$.

Proposition 2.4.16. *Let $u_1, \dots, u_N$ be univariate utility functions and let*

$$U(x) := \sum_{j=1}^N u_j(x) + u\left(\sum_{j=1}^N \beta_j x^j\right), \text{ with } \beta_1, \dots, \beta_N \geq 0 \text{ and } \max_j \beta_j > 0,$$

satisfy Standing Assumption I. If $u_j \preceq u$, for each j, then Assumption 2.3.3 holds true.

Proof. By the concavity of u we have, for every $x \in \mathbb{R}^N$,

$$\begin{aligned} \Lambda(x) = u\left(\sum_{j=1}^N \beta_j x^j\right) &= u\left(\sum_{j=1}^N \frac{\beta_j}{\sum_{n=1}^N \beta_n}\left(\sum_{n=1}^N \beta_n\right) x^j\right) \\ &\geq \sum_{j=1}^N \frac{\beta_j}{\sum_{n=1}^N \beta_n} u\left(\left(\sum_{n=1}^N \beta_n\right) x^j\right). \end{aligned} \tag{2.39}$$

By $u_j \preceq u$, and boundedness from above of u we have for each $x \in ((-\infty, 0])^N$ and from (2.39)

$$+\infty > \sup_{z \in \mathbb{R}^N} \Lambda(z) \geq \Lambda(x) \geq \sum_{j=1}^N \frac{\beta_j}{\sum_{n=1}^N \beta_n}\left(C_j u_j\left(c_j\left(\sum_{n=1}^N \beta_n\right) x^j\right) + k_j\right). \tag{2.40}$$

If $X \in L^{\Phi_1} \times \dots \times L^{\Phi_N}$, then by definition there exists a $\lambda_0 > 0$ such that the inequality $\mathbb{E}_\mathbb{P}\left[u_j(\lambda(-|X^j|)\right] > -\infty$ holds for every $\lambda \leq \lambda_0$ and $j = 1, \dots, N$. This and (2.40) then imply the existence of some $\lambda_1 > 0$ such that $\mathbb{E}_\mathbb{P}\left[\Lambda(-\lambda|X|)\right] > -\infty$ for every $\lambda \leq \lambda_1$, that is $X \in L^\Phi$. □

The Δ_2 condition

In the theory of Orlicz spaces the well known Δ_2 condition on a Young function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ guarantees that $L^\Phi = M^\Phi$. We say that $\Phi \in \Delta_2$ if:

$$\text{There exists } y_0 \geq 0,\ K > 0 \text{ such that } \Phi(2y) \leq K\Phi(y)\ \forall\, y \text{ s.t. } |y| \geq y_0.$$

Proposition 2.4.17. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be a Young function differentiable on $\mathbb{R}_+ \setminus \{0\}$ and let $\Phi^* : \mathbb{R}_+ \to \mathbb{R}$ be its conjugate function. Then*

$$\liminf_{z\to+\infty} \frac{z\Phi'(z)}{\Phi(z)} > 1 \iff \Phi^* \in \Delta_2. \tag{2.41}$$

In particular, under Assumptions 2.3.3 and 2.3.4 we have $\Phi_1^*, \dots, \Phi_N^* \in \Delta_2$ which implies

$$K_\Phi = L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*} = M^{\Phi_1^*} \times \cdots \times M^{\Phi_N^*}. \tag{2.42}$$

Proof. The equivalence of the two conditions in (2.41) can be checked along the lines of Theorem 2.3.3 in [116], observing that the argument still works in our slightly more general setup (use Proposition 2.2 [116] in place of Theorem 2.2.(a) [116]). As to the final claim, the first equality in (2.42) comes from Assumption 2.3.3 and Proposition 2.2.5, Item (3). If $u_1, \dots, u_N$ satisfy Assumption 2.3.4 then, as can be easily checked by direct computation, Φ_j, $j = 1, \dots, N$ satisfy the condition in LHS of (2.41), so that $\Phi_j^* \in \Delta_2$, which in turns implies $L^{\Phi_j^*} = M^{\Phi_j^*}$. □

Assumption 2.3.5

First we recall the definition of Reasonable Asymptotic Elasticity that was introduced in [122].

Definition 2.4.18 ([122] Definition 1.5)**.** *Let $u : \mathbb{R} \to \mathbb{R}$ be concave, non decreasing, differentiable on $\mathbb{R}$ and satisfying the Inada conditions $u'(+\infty) = 0$, $u'(-\infty) = +\infty$. We say that u has Reasonable Asymptotic Elasticity (RAE) if the following conditions are met:*

$$\mathbf{AE}_{-\infty}: \ \liminf_{x\to-\infty} \frac{x\,u'(x)}{u(x)} > 1 \quad \textit{and} \quad \mathbf{AE}_{+\infty}: \ \limsup_{x\to+\infty} \frac{x\,u'(x)}{u(x)} < 1\,. \tag{2.43}$$

It is well known that RAE is implied by a dual formulation in terms of the conjugate of the utility function, see Corollary 4.2 [122]. We now introduce the following multivariate generalization of such dual formulation of RAE.

RAE^N: For a function $V : \mathbb{R}^N \to \mathbb{R}$ we say that $V \in \mathbf{RAE}^N$ if for all $j = 1, \dots, N$ and for any compact interval $[c_0, c_1] \subset (0, +\infty)$ there exists $\alpha^j > 0, b^j \in \mathbb{R}$ such that for all vectors $y \in \mathbb{R}^N$, with $y^i \geq 0$ for all i, we have:

$$V([y^{[-j]}, \lambda y^j]) \leq \alpha^j V(y) + b^j \quad \text{for all } \lambda \in [c_0, c_1]. \tag{2.44}$$

For $N = 1$, $\mathbf{RAE}^1$ is equivalent to such dual formulation of RAE, see [122] or [21]. We provide three sufficient conditions for Assumption 2.3.5 to hold true:

Proposition 2.4.19. *Assumption* 2.3.5 *is fulfilled under any of the following sets of conditions:*

1. *Assumption 2.3.3 and Assumption 2.3.4 hold. Additionally, $u_1, \dots, u_N$ are bounded from above.*

2. $\Lambda(x) := u\left(\sum_{j=1}^N \beta_j x^j\right)$, $u_j \preceq u$ *for each* $j = 1, \dots, N$ *(see Definition 2.4.15), and* u_j *satisfies RAE for each* j *(see Definition 2.4.18).*

3. *The convex conjugate* $V(\cdot)$ *of* $U(\cdot)$*, defined in* (2.2)*, satisfies* $V \in \mathbf{RAE}^N$.

Proof. Recall that each $v_j(\cdot)$ is bounded below. It is also easy to check that

$$V(y) \leq \sum_{j=1}^N v_j(y^j) + \sup_{\mathbb{R}^N} \Lambda\,, \tag{2.45}$$

thus to prove Item 1 and 2 it is sufficient to show that in either set of conditions, $\mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$ for some $\lambda > 0$ implies

$$\mathbb{E}_\mathbb{P}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty \;\; \forall\, \lambda > 0, \forall\, j = 1, \dots, N\,. \tag{2.46}$$

Item 1: Lemma 2.6.9 implies that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in K_\Phi$. By Proposition 2.4.17 $K_\Phi = M^{\Phi_1^*} \times \dots \times M^{\Phi_N^*}$. Then $\mathbb{E}_\mathbb{P}\left[\Phi_j^*(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}})\right] < +\infty$ for all $\lambda > 0$ and $j = 1, \dots, N$. By boundedness above of utilities and Remark 2.2.8, we then deduce (2.46).

Item 2: From the computations in (2.39) we get: for some $C_j > 0, c_j > 0$

$$\begin{aligned} V(y) &= \sup_{x\in\mathbb{R}^N}\left(\sum_{j=1}^N u_j(x^j) - x^j y^j + u\left(\sum_{j=1}^N \beta_j x^j\right)\right) \\ &\geq \sup_{x\in\mathbb{R}^N}\left(\sum_{j=1}^N u_j(x^j) - x^j y^j + \sum_{j=1}^N C_j u(c_j x^j)\right) \end{aligned}$$

which implies

$$V(y) \geq \sum_{j=1}^N \sup_{x^j\in\mathbb{R}}\left(u_j(x^j) - x^j y^j + C_j u(c_j x^j)\right)\,. \tag{2.47}$$

Observe now that since $u_j \preceq u, j = 1, \dots, N$ we can apply Lemma 2.6.8 to each term in the summation in RHS of (2.47). Calling the corresponding constants $\beta^j, B^j, K_1^j, K_2^j$, from (2.47) and $\mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$ we infer that for each $j = 1, \dots, N$,

$$\mathbb{E}_\mathbb{P}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)1_{\left\{\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \leq K_1^j\right\}}\right] < +\infty \quad \mathbb{E}_\mathbb{P}\left[v_j\left(\beta\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)1_{\left\{\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \geq K_2^j\right\}}\right] < +\infty\,.$$

Since for each $j = 1, \dots, N$ u_j satisfies RAE, so do $x \mapsto u_j(x) + 1$, $j = 1, \dots, N$. From [122] Corollary 4.2, Item (i) applied to $x \mapsto u_j(x) + 1$, $j = 1, \dots, N$, the above equations imply

$$\mathbb{E}_\mathbb{P}\left[v_j\left(\alpha\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)1_{\left\{\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \leq \frac{K_1^j}{\lambda}\right\}}\right] < +\infty \quad \mathbb{E}_\mathbb{P}\left[v_j\left(\alpha\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)1_{\left\{\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \geq \frac{K_2^j}{\lambda}\right\}}\right] < +\infty \;\forall \alpha > 0\,.$$

Since $v_1, \dots, v_N$ are continuous on $\left[\frac{K_1^j}{\lambda}, \frac{K_2^j}{\lambda}\right]$, we have for each $j = 1, \dots, N$

$$\mathbb{E}_{\mathbb{P}}\left[v_j\left(\alpha\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty \quad \forall \alpha > 0\,.$$

Item 3: Fix $\beta \in \mathbb{R}$, $\beta > 0$, and $\mathbb{Q} = [\mathbb{Q}^1, ..., \mathbb{Q}^N]$, $\mathbb{Q}^j \ll \mathbb{P}$, such that $\mathbb{E}_{\mathbb{P}}\left[V\left(\beta\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$. Take any $\lambda = [\lambda_1, ..., \lambda_N] \in \mathbb{R}^N$, with $\lambda_i > 0$ for all i, and set $c_0 := \min_i(\frac{\lambda_i}{\beta}) > 0$, $c_1 := \max_i(\frac{\lambda_i}{\beta})$. By the definition of RAE^N we then get, for any $y \in \mathbb{R}^N$ with non negative components,

$$\begin{aligned} V(\lambda_1 y^1, ..., \lambda_N y^N) &= V\left(\frac{\lambda_1}{\beta}\beta y^1, ..., \frac{\lambda_N}{\beta}\beta y^N\right) \\ &\leq \alpha^1 V\left(\beta y^1, \frac{\lambda_2}{\beta}\beta y^2, ..., \frac{\lambda_N}{\beta}\beta y^N\right) + b^1 \leq \alpha^1 \cdot ... \cdot \alpha^N V(\beta y) + \text{constant}. \end{aligned}$$

Hence

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda_1\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, ..., \lambda_N\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right)\right] \leq \alpha^1 \cdot ... \cdot \alpha^N \mathbb{E}_{\mathbb{P}}\left[V\left(\beta\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] + \text{constant} < +\infty,$$

by assumption. □

Main Examples

Suppose that $\Lambda(x) := u\left(\sum_{j=1}^N \beta_j x^j\right)$ for an increasing and concave (both not necessarily strictly) function $u : \mathbb{R} \to \mathbb{R}$, with u such that $u_j \preceq u$ for each $j = 1, \dots, N$.

- If u_j satisfies $\mathbf{AE}_{-\infty}$ for each j, then the assumptions in Setup B are fulfilled (Proposition 2.4.16) and Theorem 2.5.16 holds true.
- If u_j satisfies RAE (i.e.: $\mathbf{AE}_{+\infty}$ and $\mathbf{AE}_{-\infty}$) for each j and u is differentiable on $\mathbb{R}$, then the assumptions in Setup B and C are fulfilled (Proposition 2.4.19, Item 2) and both Theorems 2.5.16 and 2.5.17 hold true. The uniqueness of the optimal solution implies that the $\widehat{Y}$ in Theorem 2.5.17 satisfies all the conditions in Theorem 2.5.16.

It is now easy to verify that any of the multivariate utility functions described in equations (I.15) and (I.17) of the Introduction fulfill either Setups A or B or C.

2.4.6 Comparison with univariate SORTE

In this subsection we set $\Lambda \equiv 0$. It is easy to see that if an optimum exists for $\mathbb{U}_j^{Y^{[-j]},\mathbb{Q}^j}(\cdot)$ in (2.16), it no longer depends on $Y^{[-j]}$, and the optimization problem $\mathbb{U}_j^{Y^{[-j]},\mathbb{Q}^j}(\cdot)$ is in fact the same problem denoted with $U_j^{\mathbb{Q}^j}(\cdot)$ in Equation (1.1) in Chapter 1. Similarly, it can be seen that the optimization problem expressed by (2.18) is, when $\Lambda \equiv 0$, equivalent to the one in Equation (1.2), Chapter 1.

When $\Lambda \equiv 0$, Assumption 2.3.2 and Assumption 2.3.4 are left untouched, Assumption 2.3.3 is satisfied automatically, Assumption 2.3.5 can be equivalently reformulated as: for each $j = 1, \dots, N$ and any $\mathbb{Q}^j \ll \mathbb{P}$,

$$\mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty \text{ for some } \lambda > 0 \Rightarrow \mathbb{E}_{\mathbb{P}}\left[v_j\left(\lambda\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty \text{ for all } \lambda > 0\,, \tag{2.48}$$

where the convex conjugate v_j of u_j is given in (2.7). We recognize that (2.48) is Assumption 1.2.12 in Chapter 1. Thus from Theorem 2.4.11 we obtain the existence of a SORTE.

Corollary 2.4.20. *Let $\Lambda \equiv 0$ and let $u_1, \dots, u_j : \mathbb{R} \to \mathbb{R}$ be strictly increasing, strictly concave and satisfying the Inada conditions (see Standing Assumption I). Then under either Assumption 2.3.2 or 2.3.4 or 2.3.5 a SORTE exists, that is there exists a triple $(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{a}) \in \mathcal{L} \times \mathcal{M} \times \mathbb{R}^N$ such that:*

1. *$\widehat{Y}^j$ is an optimum for $U_j^{\widehat{\mathbb{Q}}^j}(\widehat{a}^j)$, for each $j \in \{1, \dots, N\}$,*

2. *$\widehat{a}$ is an optimum for $S^{\widehat{\mathbb{Q}}}(A)$,*

3. *$\widehat{Y} \in \mathcal{B}$ and $\sum_{j=1}^N \widehat{Y}^j = A$.*

In Chapter 1 the existence of a SORTE is proved assuming RAE for $u_1, \dots, u_N$ (see Definition 2.4.18). Here, such a result is generalized assuming either $\mathcal{B}$ is closed under truncation (with no differentiability requirement on $u_1, \dots, u_N$) or $\mathrm{AE}_{-\infty}$ only. Moreover, in Chapter 1 uniqueness is proved assuming additionally closedness under truncation. As Assumption 2.3.3 is satisfied automatically if $\Lambda \equiv 0$, we showed in Theorem 2.4.11 that closedness under truncation alone is in fact sufficient also for existence.

2.5 Systemic Utility Maximization and Duality

2.5.1 Preliminary Study

In this Section as well as in Sections 2.5.2 and 2.5.3 we work under the Standing Assumptions I and II only. We present here some results (well posedness and uniqueness) for generic sets $\mathcal{C}$ or $\mathcal{K}$. In subsequent sections these will be applied to specific convex cones, as $\mathcal{B}_0$ and $\mathcal{L}_V$.

Theorem 2.5.1. *Let $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ be convex, closed in probability and such that $\mathcal{C} \cap M^\Phi$ is nonempty. Assume there exists an $A \in \mathbb{R}$ such that $\sum_{j=1}^N Y^j \leq A$ for every $Y \in \mathcal{C} \cap M^\Phi$. Then for every $X \in M^\Phi$ there exists a $\widehat{Y} \in \mathcal{C} \cap L^1(\mathbb{P})$ such that*

$$-\infty < \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \leq \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] < +\infty\,. \tag{2.49}$$

Proof. First observe that $X+Y \geq -(|X|+|Y|)$ in the componentwise order, hence for $Z \in \mathcal{C} \cap M^\Phi \neq \emptyset$, as $X, Z \in M^\Phi$, we get

$$\sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \geq \mathbb{E}_\mathbb{P}\left[U(X+Z)\right] \geq \mathbb{E}_\mathbb{P}\left[U\left(-(|X|+|Z|)\right)\right] > -\infty\,.$$

Take now a maximizing sequence $(Y_n)_n$ in $\mathcal{C} \cap M^\Phi$ and observe that

$$\sup_n \left|\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j + Y_n^j\right]\right| \leq \left|\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j\right]\right| + |A| < +\infty$$

and $\mathbb{E}_\mathbb{P}\left[U(X+Y_n)\right] \geq \mathbb{E}_\mathbb{P}\left[U(X+Y_1)\right] =: B \in \mathbb{R}$. Then Lemma 2.6.4 Item 1 applies with $Z_n := X + Y_n$. Using also $|X^j| + |Y_n^j| \leq |X^j + Y_n^j| + 2\,|X^j|\,,\ j = 1, \dots, N$ we get

$$\sup_n \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[\left|X^j\right| + \left|Y_n^j\right|\right] < \infty\,.$$

Now we apply Corollary 2.6.12 with $\mathbb{P}_1 = \cdots = \mathbb{P}_N = \mathbb{P}$ and extract the subsequence $(Y_{n_h})_h$ such that for some $\widehat{Y} \in (L^1(\mathbb{P}))^N$

$$W_H := \frac{1}{H}\sum_{h=1}^H Y_{n_h} \xrightarrow[H \to +\infty]{} \widehat{Y} \quad \mathbb{P}-\text{a.s.} \quad \text{and} \quad \sup_H \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[\left|W_H^j\right|\right] < +\infty\,. \tag{2.50}$$

We observe that by convexity the random vectors W_H still belong to $\mathcal{C} \cap M^\Phi$, and $\widehat{Y} \in \mathcal{C}$ by closedness in probability. Observe now that

$$\mathbb{E}_\mathbb{P}\left[U(X+W_H)\right] \geq \frac{1}{H}\sum_{h=1}^H \mathbb{E}_\mathbb{P}\left[U(X+Y_{n_h})\right] \xrightarrow[H \to +\infty]{} \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \tag{2.51}$$

by concavity of U and the fact that $(Y_{n_h})_h$ is again a maximizing sequence. From the expression in Equation (2.51) we get that for every $\varepsilon > 0$, definitely (in H)

$$\mathbb{E}_\mathbb{P}\left[U(X+W_H)\right] \geq \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] - \varepsilon\,.$$

Apply now Lemma 2.6.4 Item 2 for $B = \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] - \varepsilon$ to the sequence $(X + W_H)_H$ for H big enough (this sequence is bounded in $(L^1(\mathbb{P}))^N$ by (2.50)) to get that for every $\varepsilon > 0$

$$\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] \geq \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] - \varepsilon\,.$$

Clearly then $\widehat{Y}$ satisfies

$$\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] \geq \sup_{Y \in \mathcal{C} \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right].$$

Now observe that by Lemma 2.6.2 for some $a > 0, b \in \mathbb{R}$

$$U(X+\widehat{Y}) \leq a\sum_{j=1}^N (X^j + \widehat{Y}^j) + a\sum_{j=1}^N (-(X^j + \widehat{Y}^j)^-) + b$$

and since RHS is in $L^1(\mathbb{P})$ we conclude that $\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] < +\infty$. □

We have also a uniqueness property.

Proposition 2.5.2. *Let $\mathcal{K} \subseteq (L^0(\mathbb{P}))^N$ be convex and $X \in M^\Phi$ be given. If*

$$\sup_{Y \in \mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] < +\infty$$

then the maximization problem $\sup_{Y\in\mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right]$ admits at most one solution. Furthermore if there exists a $\widehat{Y} \in (L^0(\mathbb{P}))^N$ such that

$$\sup_{Y \in \mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \leq \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] < +\infty$$

then we have

$$\sup_{Y \in \mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] < \sup_{z\in\mathbb{R}^N} U(z)\,.$$

Proof. The existence of one optimum at most follows from strict concavity of U (see Standing Assumption I): if two distinct optima existed, any strict convex combination of the two would belong to $\mathcal{K}$ and would produce a value for $\mathbb{E}_\mathbb{P}\left[U(X+\bullet)\right]$ strictly greater than the supremum.
The final claim is trivial if $\sup_{z\in\mathbb{R}^N} U(z) = +\infty$. Suppose that $\sup_{z\in\mathbb{R}^N} U(z) < +\infty$ and notice that

$$\sup_{Y \in \mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \leq \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] \leq \sup_{z\in\mathbb{R}^N} U(z).$$

If we had $\sup_{Y\in\mathcal{K}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \sup_{z\in\mathbb{R}^N} U(z)$, then we would also have

$$\sup_{z\in\mathbb{R}^N} U(z) = \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right]$$

so that:

$$0 = \mathbb{E}_\mathbb{P}\left[\sup_{z\in\mathbb{R}^N} U(z) - U(X+\widehat{Y})\right] = \mathbb{E}_\mathbb{P}\left[\left|\sup_{z\in\mathbb{R}^N} U(z) - U(X+\widehat{Y})\right|\right],$$

which implies $\sup_{z\in\mathbb{R}^N} U(z) = U(X+\widehat{Y})$ $\mathbb{P}$-almost surely. In particular, from the fact that $X+\widehat{Y}$ is finite almost surely, it would follow that U almost surely attains its supremum on some compact subset of $\mathbb{R}^N$, which is clearly a contradiction given that U is strictly componentwise increasing (see Standing Assumption I). □

Theorem 2.5.3. *Let $\mathcal{C} \subseteq M^\Phi$ be a convex cone with $0 \in \mathcal{C}$ and $e_i - e_j \in \mathcal{C}$ for every $i, j \in \{1, \dots, N\}$. Denote by $\mathcal{C}^0$ the polar of the cone $\mathcal{C}$ in the dual pair (M^Φ, K_Φ):*

$$\mathcal{C}^0 := \left\{Z \in K_\Phi \mid \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y^j Z^j\right] \leq 0 \;\forall\, Y \in \mathcal{C}\right\}$$

and set

$$\mathcal{C}^0_1 := \left\{Z \in \mathcal{C}^0 \mid \mathbb{E}_\mathbb{P}\left[Z^j\right] = 1 \,\forall j = 1, \dots, N\right\}$$
$$(\mathcal{C}^0_1)^+ := \left\{Z \in \mathcal{C}^0_1 \mid Z^j \geq 0 \,\forall j = 1, \dots, N\right\}\,.$$

Suppose that for every $X \in M^\Phi$

$$\sup_{Y \in \mathcal{C}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] < +\infty\,.$$

Then the following holds:

$$\sup_{Y \in \mathcal{C}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \min_{\lambda\geq 0,\,\mathbb{Q}\in(\mathcal{C}_1^0)^+} \left(\lambda\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right)\,. \tag{2.52}$$

If any of the two expressions is strictly smaller than $V(0) = \sup_{\mathbb{R}^N} U$, then the condition $\lambda \geq 0$ in (2.52) *can be replaced with the condition $\lambda > 0$.*

Proof. The proof can be obtained with minor and obvious modifications of the one in Chapter 1, Theorem 1.5.3 by replacing $\sum_{j=1}^N u_j(\cdot)$, $\sum_{j=1}^N v_j(\cdot)$, L^{Φ^*} there with $U(\cdot)$, $V(\cdot)$, K_Φ respectively. □

We also provide an analogous result when working with the pair $((L^\infty(\mathbb{P}))^N, (L^1(\mathbb{P}))^N)$ in place of (M^Φ, K_Φ), which will be used in Section 2.5.5.

Theorem 2.5.4. *Replacing M^Φ with $(L^\infty(\mathbb{P}))^N$ and K_Φ with $(L^1(\mathbb{P}))^N$ in the statement of Theorem 2.5.3, all the claims in it remain valid.*

Proof. As in Theorem 2.5.3, the proof can be obtained with minor and obvious modifications of the one in Theorem 1.5.3 of Chapter 1, using Theorem 4 of [118] in place of Corollary on page 534 of [118]. □

2.5.2 Duality

We first state some simple properties of the polar cone of $\mathcal{B}_0 \cap M^\Phi$, some of which rephrase arguments in the proofs of Lemma 1.3.1 and Lemma 1.3.2.

Remark 2.5.5. If $X \in M^\Phi$, then for any fixed $k = 1, \dots, N$ we have $[0, \dots, 0, X^k, 0, \dots, 0] \in M^\Phi$. This in turns implies that for any $Z \in K_\Phi$ and $X \in M^\Phi$, $X^jZ^j \in L^1(\mathbb{P})$ for any $j = 1, \dots, N$.

Remark 2.5.6. In the dual pair (M^Φ, K_Φ) take the polar $(\mathcal{B}_0 \cap M^\Phi)^0$ of $\mathcal{B}_0 \cap M^\Phi$. Since all (deterministic) vector in the form $e^i - e^j$ belong to $\mathcal{B}_0 \cap M^\Phi$, we have that for all $Z \in (\mathcal{B}_0 \cap M^\Phi)^0$ and for all $i, j \in \{1, \dots, N\}$ $\mathbb{E}_\mathbb{P}[Z^i] - \mathbb{E}_\mathbb{P}[Z^j] \leq 0$. It is clear that, as a consequence, $Z \in (\mathcal{B}_0 \cap M^\Phi)^0 \Rightarrow \mathbb{E}_\mathbb{P}[Z^1] = \cdots = \mathbb{E}_\mathbb{P}\left[Z^N\right]$. Recall that $\mathbb{R}_+ := \{b \in \mathbb{R}, b \geq 0\}$ and the definition of $\mathcal{Q}$ provided in (2.14). We then see:

$$(\mathcal{B}_0 \cap M^\Phi)^0 \cap (L^0_+)^N = \mathbb{R}_+ \cdot \mathcal{Q} \tag{2.53}$$

That is, $(\mathcal{B}_0 \cap M^\Phi)^0$ is the cone generated by $\mathcal{Q}$.

Remark 2.5.7. The condition $\mathcal{B} \subseteq \mathcal{C}_\mathbb{R}$ implies $\mathcal{B}_0 \cap M^\Phi \subseteq (\mathcal{C}_\mathbb{R} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})$, so that the polars satisfy the opposite inclusion: $(\mathcal{C}_\mathbb{R} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})^0 \subseteq (\mathcal{B}_0 \cap M^\Phi)^0$. Observe now that any vector $[Z, \dots, Z]$, for $Z \in L^\infty_+$, belongs to $(\mathcal{C}_\mathbb{R} \cap M^\Phi \cap \{\sum_{j=1}^N Y^j \leq 0\})^0$. In particular, as a consequence, $(\mathcal{B}_0 \cap M^\Phi)^0$ contains a vector in the form $[\frac{\varepsilon+Z}{1+\varepsilon},\dots,\frac{\varepsilon+Z}{1+\varepsilon}]$ with $\varepsilon>0$ and $Z\in L^\infty_+$, $\mathbb{E}_\mathbb{P}[Z]=1$. Each component of such a vector has expectation equal to 1, is in L^∞_+ and satisfies $\frac{\varepsilon+Z}{1+\varepsilon}\geq\frac{\varepsilon}{1+\varepsilon}$. All of these conditions together imply that in $\mathcal{Q}$ there exists a strictly positive vector $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ with $\mathbb{E}_\mathbb{P}\left[V\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]<\infty$, hence belonging to $\mathcal{Q}_V$. In particular

$$\mathcal{Q}_V\neq\emptyset$$

and there exists a $\mathbb{Q}=[\mathbb{Q}^1,\dots,\mathbb{Q}^N]\in\mathcal{Q}_V$ with $\mathbb{Q}^j\sim\mathbb{P}$, $j=1,\dots,N$ and $\varepsilon\leq\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\leq M$, $j=1,\dots,N$ for some $0<\varepsilon<M<+\infty$ real numbers.

Proposition 2.5.8 (Fairness)**.** *For all* $\mathbb{Q}\in\mathcal{Q}$

$$\sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right]\leq\sum_{j=1}^N Y^j\quad\forall Y\in\mathcal{B}\cap M^\Phi\,.$$

Proof. Let $Y\in\mathcal{B}\cap M^\Phi$. Notice that the hypothesis $\mathbb{R}^N+\mathcal{B}=\mathcal{B}$ implies that the vector Y_0, defined by $Y_0^j:=Y^j-\frac{1}{N}\sum_{k=1}^N Y^k$, belongs to $\mathcal{B}_0$. Indeed, $\sum_{k=1}^N Y^k\in\mathbb{R}$ and so $Y_0\in\mathcal{B}$ and $\sum_{j=1}^N Y_0^j=0$. By definition of polar, $\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[Y_0^jZ^j\right]\leq 0$ for all $Z\in(\mathcal{B}\cap M^\Phi)^0$, and in particular for all $\mathbb{Q}\in\mathcal{Q}$

$$0\geq\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[Y_0^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]=\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]-\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[\frac{1}{N}\left(\sum_{k=1}^N Y^k\right)\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right],$$

and we recognize $\sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}[Y^j]-\sum_{j=1}^N Y^j$ in RHS. □

Recall the Definition of $\mathcal{L}_V^{(A)}$ in (2.22) and that $\mathcal{L}_V:=\mathcal{L}_V^{(0)}$. It follows from these that

$$\mathcal{L}_V:=\bigcap_{\mathbb{Q}\in\mathcal{Q}_V}\left\{Y\in(L^0(\mathbb{P}))^N\mid\sum_{j=1}^N Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\in L^1(\mathbb{P}),\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]\leq 0\right\}.\qquad(2.54)$$

Observe that we are not requiring that each term $Y^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}$ is integrable, for $Y\in\mathcal{L}_V$.

Theorem 2.5.9.

1. *For every* $X\in M^\Phi$ *the following holds*

$$+\infty>\ \pi_0(X):=\sup_{Y\in\mathcal{B}_0\cap M^\Phi}\mathbb{E}_\mathbb{P}\left[U(X+Y)\right]=\sup_{Y\in\mathcal{L}_V}\mathbb{E}_\mathbb{P}\left[U(X+Y)\right]\qquad(2.55)$$

$$=\min_{\mathbb{Q}\in\mathcal{Q}}\min_{\lambda\geq 0}\left(\lambda\sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]+\mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right).\qquad(2.56)$$

2. *If any of the three expressions is strictly smaller than* $V(0)=\sup_{x\in\mathbb{R}^N}U(x)$, *then the condition* $\lambda\geq 0$ *in* (2.56) *can be replaced with condition* $\lambda>0$.

3. *The vector* $\widehat{Y}$ *from Theorem 2.5.1 belongs to* $\mathcal{B}_0$ *and satisfies*

$$\sum_{j=1}^N\widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\in L^1(\mathbb{P})\ \forall\mathbb{Q}\in\mathcal{Q}_V\,.$$

Proof.
Item (1): take $\mathcal{C} = \mathcal{B}_0 \cap M^\Phi$. By Theorem 2.5.1

$$\sup_{\mathcal{B}_0 \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] < +\infty \;\; \forall X \in M^\Phi\,.$$

From (2.14) we deduce $\mathcal{B}_0 \cap M^\Phi \subseteq \mathcal{L}_V$,

$$\sup_{\mathcal{B}_0 \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \leq \sup_{Y \in \mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right]$$

and by Fenchel inequality (see Remark (2.4.8))

$$\sup_{Y \in \mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \leq \inf_{\lambda \geq 0,\, \mathbb{Q} \in \mathcal{Q}} \left(\lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] \right) + \mathbb{E}_\mathbb{P}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right] \right).$$

The chain of equalities in Equations (2.55)-(2.56) then follows by Theorem 2.5.3.

Item (2): Direct substitution of $\lambda = 0$ in the expression would give a contradiction, no matter what the optimal probability measure is.
Item (3): From Theorem 2.5.1 we know that (2.49) holds. By definition of $V(\cdot)$, we have

$$U(X + \widehat{Y}) \leq V(\lambda Z) + \left\langle X + \widehat{Y}, \lambda Z \right\rangle \;\; \mathbb{P} - \text{a.s.} \;\forall\, \lambda \geq 0,\; Z \in K_\Phi\,. \tag{2.57}$$

This implies

$$(U(X + \widehat{Y}))^- \geq \left(V(\lambda Z) + \left\langle X + \widehat{Y}, \lambda Z \right\rangle \right)^-$$

so that $\left(V(\lambda Z) + \left\langle X + \widehat{Y}, \lambda Z \right\rangle \right)^- \in L^1(\mathbb{P})$.
We prove integrability also for the positive part, assuming now $Z = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$, $\mathbb{Q} \in \mathcal{Q}_V$ and taking $\lambda > 0$ such that $\mathbb{E}_\mathbb{P}\left[V(\lambda Z)\right] < +\infty$. By (2.50) $W_H \to_H \widehat{Y}\, \mathbb{P}-$a.s. so that

$$\begin{aligned}
&\mathbb{E}_\mathbb{P}\left[\left(V(\lambda Z) + \left\langle X + \widehat{Y}, \lambda Z \right\rangle \right)^+ \right] = \mathbb{E}_\mathbb{P}\left[\liminf_H \left(V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right)^+ \right] \\
&\leq \liminf_H \mathbb{E}_\mathbb{P}\left[\left(V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right)^+ \right] \\
&\leq \sup_H \left(\mathbb{E}_\mathbb{P}\left[V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right] \right) + \sup_H \left(\mathbb{E}_\mathbb{P}\left[\left(V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right)^- \right] \right).
\end{aligned} \tag{2.58}$$

Now since $\mathbb{E}_\mathbb{P}\left[\langle W_H, \lambda Z \rangle\right] \leq 0$

$$\sup_H \left(\mathbb{E}_\mathbb{P}\left[V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right] \right) \leq \mathbb{E}_\mathbb{P}\left[V(\lambda Z) + \langle X, \lambda Z \rangle \right] < +\infty\,. \tag{2.59}$$

Also by (2.57)

$$\begin{aligned}
&\sup_H \left(\mathbb{E}_\mathbb{P}\left[\left(V(\lambda Z) + \langle X + W_H, \lambda Z \rangle \right)^- \right] \right) \leq \sup_H \left(\mathbb{E}_\mathbb{P}\left[\left(U(X + W_H) \right)^- \right] \right) \\
&\leq \sup_H \left(\mathbb{E}_\mathbb{P}\left[\left(U(X + W_H) \right)^+ - U(X + W_H) \right] \right) \\
&\leq \sup_H \left(\mathbb{E}_\mathbb{P}\left[\left(U(X + W_H) \right)^+ \right] \right) - \inf_H \left(\mathbb{E}_\mathbb{P}\left[U(X + W_H) \right] \right).
\end{aligned} \tag{2.60}$$

Now use subadditivity of the function $x \mapsto x^+$ to check that

$$\sup_H \left(\mathbb{E}_{\mathbb{P}}\left[(U(X+W_H))^+\right]\right) \le \sup_H \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left(u_j(X^j+W_H^j)\right)^+\right]\right) + \sup_{z\in\mathbb{R}^N}(\Lambda(z))^+$$

$$\le \sup_H \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left((X^j+W_H^j)^+\right)\right]\right) + \sup_{z\in\mathbb{R}^N}(\Lambda(z))$$

where in the last step we used Equation (2.6). We also have, Y_1 being the first element in the maximizing sequence of Theorem 2.5.1, $\inf_H\left(\mathbb{E}_{\mathbb{P}}\left[U(X+W_H)\right]\right) \ge \mathbb{E}_{\mathbb{P}}\left[U(X+Y_1)\right]$ by construction. Thus, continuing from (2.60), we get using (2.3)

$$\sup_H \left(\mathbb{E}_{\mathbb{P}}\left[(V(\lambda Z)+\langle X+W_H,\lambda Z\rangle)^-\right]\right) \tag{2.61}$$

$$\le \sup_H \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left((X^j+W_H^j)^+\right)\right]\right) + \sup_{\mathbb{R}^N}\Lambda - \mathbb{E}_{\mathbb{P}}\left[U(X+Y_1)\right]$$

$$\le \sup_{\mathbb{R}^N}\Lambda + \max_{j=1,\dots,N}\left(\frac{\mathrm{d}u_j}{\mathrm{d}x^j}(0)\right)\sup_H\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^+\right]\right) - \mathbb{E}_{\mathbb{P}}\left[U(X+Y_1)\right] < +\infty \tag{2.62}$$

since the sequence W_H is bounded in $(L^1(\mathbb{P}))^N$ (see (2.50)) and $\mathbb{E}_{\mathbb{P}}\left[U(X+Y_1)\right] > -\infty$. From (2.58), (2.59), (2.62) we conclude that

$$\mathbb{E}_{\mathbb{P}}\left[\left(V(\lambda Z)+\left\langle X+\widehat{Y},\lambda Z\right\rangle\right)^+\right] < +\infty\,.$$

To sum up, for $Z \in \mathcal{Q}_V$ and λ s.t. $\mathbb{E}_{\mathbb{P}}\left[V(\lambda Z)\right] < +\infty$

$$\langle X,\lambda Z\rangle\,,\ V(\lambda Z),\ \left(V(\lambda Z)+\left\langle X+\widehat{Y},\lambda Z\right\rangle\right)^+,\ \left(V(\lambda Z)+\left\langle X+\widehat{Y},\lambda Z\right\rangle\right)^- \in L^1$$

which gives $\left\langle \widehat{Y},Z\right\rangle \in L^1(\mathbb{P}), \forall Z \in \mathcal{Q}_V$.

□

Remark 2.5.10. Theorem 2.5.9 shows that $\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})\ \forall \mathbb{Q} \in \mathcal{Q}_V$. However, we do not know yet if

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \le 0\ \ \forall \mathbb{Q} \in \mathcal{Q}_V. \tag{2.63}$$

This will hold under some additional conditions, as shown below in Proposition 2.5.14.

2.5.3 Optimization with fixed $\mathbb{Q} \in \mathcal{Q}_V$

The following is a counterpart to Theorem 2.5.9 when a probability measure $\mathbb{Q} \in \mathcal{Q}_V$ is fixed.

Proposition 2.5.11. *Let $X \in M^\Phi$ and $\mathbb{Q} \in \mathcal{Q}_V$ be fixed. Then*

$$+\infty > \pi_0^{\mathbb{Q}}(X) := \sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in M^\Phi, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \le 0\right\} \tag{2.64}$$

$$= \min_{\lambda \ge 0}\left(\lambda \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{2.65}$$

Furthermore if (2.64) *is strictly smaller than* $V(0)$ *then the minimum in* (2.65) *can be taken over* $(0, +\infty)$ *in place of* $[0, +\infty)$.

Proof. $\pi_0^{\mathbb{Q}}(X) < +\infty$ follows from Remark 2.3.6. The equality between (2.64) and (2.65) follows from Theorem 2.5.3, and the fact that

$$\mathcal{C} := \left\{Y \in M^\Phi, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \le 0\right\} \Longrightarrow (\mathcal{C}_1^0)^+ = \left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right\} \subseteq K_\Phi \text{ as } \mathbb{Q} \in \mathcal{Q}_V.$$

□

Corollary 2.5.12. *Let $X \in M^\Phi$ be fixed and $\pi_0(\cdot)$, $\pi_0^{\mathbb{Q}}(\cdot)$ be as in* (2.55), (2.64) *respectively. Then*

$$\pi_0(X) = \min_{\mathbb{Q} \in \mathcal{Q}_V}\left(\pi_0^{\mathbb{Q}}(X)\right). \tag{2.66}$$

Moreover, whenever $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ is an optimum for (2.56)*, then $\widehat{\mathbb{Q}}$ is an optimum for* (2.66).

Proof. We observe that in Theorem 2.5.9 the minima over $\mathcal{Q}$ can be substituted by minima over $\mathcal{Q}_{\mathcal{V}}$, since $\sup_{Y \in \mathcal{B}_0 \cap M^\Phi} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] < +\infty$ by Theorem 2.5.1. The claims then follow applying Theorem 2.5.9 Item 1 together with Proposition 2.5.11.

□

Proposition 2.5.13. *Let $\widehat{\mathbb{Q}} \in \mathcal{Q}_V$. Then the following expression is strictly smaller than $+\infty$:*

$$\sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in (L^0(\mathbb{P}))^N, \sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}), \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \le 0\right\}. \tag{2.67}$$

Suppose the optimization problem (2.67) *admits an optimum $\widehat{Y}$. Then $\widehat{\mathbb{Q}} \sim \mathbb{P}$ and*

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] = 0. \tag{2.68}$$

Proof. Define

$$\mathcal{K} := \left\{Y \in (L^0(\mathbb{P}))^N \mid \sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})\ \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \le 0\right\}.$$

By Remark 2.4.8, $\sup_{Y \in \mathcal{K}} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] < +\infty$.

We now prove that $\widehat{\mathbb{Q}} \sim \mathbb{P}$, using an argument inspired by [77] Remark 3.32: if this were not the case then $\mathbb{P}(A_k) > 0$, where $A_k := \{\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}} = 0\}$, for some component $k \in \{1, \dots, N\}$. Then the vector $\widetilde{Y}$ defined by $\widetilde{Y}^k := \widehat{Y}^k + 1_{A_k}$, $\widetilde{Y}^j := \widehat{Y}^j, j \neq k$ would still satisfy

$$\sum_{j=1}^N \widetilde{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}),\ \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widetilde{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \leq 0$$

and by monotonicity $\mathbb{E}_{\mathbb{P}}\left[U(X+\widetilde{Y})\right] \geq \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right]$. Thus $\widetilde{Y}$ would be another optimum, different from $\widehat{Y}$, contradicting uniqueness from Proposition 2.5.2 (which applies by finiteness of the supremum in (2.67)).
We now show (2.68): if this were not the case we would have $\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] < 0$ so that adding $0 < \varepsilon$ sufficiently small to each component of $\widehat{Y}$ would give a vector still satisfying the constraints but having a corresponding expected utility strictly grater that the supremum, which is a contradiction. □

2.5.4 Refined results: Existence of the optimizers

The two main Theorems in this Section show that Theorem 2.4.9 holds true, when $A = 0$, and consequently all the results in Section 2.4.3 hold true as well (note that equation (2.94) and Section 2.5.6 complete the proof of Theorem 2.4.9).
On the one hand we will provide sufficient conditions to guarantee that not only $\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})$, but also $\widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})$, for every $j = 1, \dots, N$ and every $\mathbb{Q} \in \mathcal{Q}_V$ or, at least, for $\mathbb{Q} = \widehat{\mathbb{Q}}$ (the optimum in the minimax expression (2.56)). We will rely on Theorem 2.5.1, ideally continuing the proof of it. On the other hand, in setup C, we will weaken the requirements on $\mathcal{B}$, especially the one regarding closedness under truncation.
First we show that Assumption 2.3.2 guarantees that condition (2.63) holds true for the $\widehat{Y}$ from Theorem 2.5.1.

Proposition 2.5.14. *Under Assumption 2.3.2, if $Y \in \mathcal{B}_0$ then*

$$\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})\ \forall \mathbb{Q} \in \mathcal{Q}_V \implies \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq 0\ \forall \mathbb{Q} \in \mathcal{Q}_V\,.$$

Proof. Observe that Y_m in Definition 2.3.1 satisfies

$$\left|\sum_{j=1}^N Y_m^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right| \leq \max\left(\left|\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right|, \sum_{j=1}^N \left|c_Y^j\right| \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right) \in L^1(\mathbb{P})$$

and

$$\sum_{j=1}^N Y_m^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \to_m \sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\ \mathbb{P}-a.s.$$

hence by Dominated Convergence Theorem

$$0 \geq \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y_m^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \to_m \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]$$

where the inequality for LHS comes from the fact that $Y_m \in \mathcal{B}_0 \cap (L^\infty)^N \subseteq \mathcal{B}_0 \cap M^\Phi$ and $\mathbb{Q} \in \mathcal{Q}_V$, so that by definition of $\mathcal{Q}_V \subseteq \mathcal{Q}$ (see (2.14)) $\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N Y_m^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y_m^j\right] \leq 0$. □

We prove now that, by virtue of Proposition 2.5.14 and Theorem 2.5.9, an (extended-sense) optimum exists in $\mathcal{L}_V$ under Assumption 2.3.2 alone. The Proposition 2.5.15 will be applied also in setup B, where Assumption 2.3.2 does not hold, and so it is formulated directly with condition (2.63), instead of assuming closedness under truncation.

Proposition 2.5.15. *Suppose that condition (2.63) holds true. In the notation of Theorem 2.5.9 we have*

$$\sup_{Y \in \mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right]. \tag{2.69}$$

Moreover $\widehat{Y} \in \mathcal{B}_0 \cap \mathcal{L}_V$, it is the unique optimum for the extended maximization problem expressed by (2.69) *and can be taken in such a way that $\sum_{j=1}^N \widehat{Y}^j = 0$ $\mathbb{P}-a.s.$*

Proof. Theorem 2.5.9 Item 3, together with (2.63), show that $\widehat{Y} \in \mathcal{B}_0 \cap \mathcal{L}_V \subseteq \mathcal{L}_V$. Taking $\mathcal{C} = \mathcal{B}_0$ in Theorem 2.5.1, we have

$$\begin{aligned}\sup_{Y \in \mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] &\overset{(2.55)}{=} \sup_{Y \in \mathcal{B}_0 \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \\ &\overset{(2.49)}{\leq} \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right] \overset{\widehat{Y} \in \mathcal{L}_V}{\leq} \sup_{Y \in \mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right].\end{aligned}$$

Thus we get (2.69) and optimality of $\widehat{Y}$. Uniqueness of optima to the problem in (2.69) follows from Proposition 2.5.2 for $\mathcal{K} = \mathcal{L}_V$.
As to the last claim, observe that the sequence (W_H) in Theorem 2.5.1 comes from a maximizing sequence $(Y_{n_h})_h$ (see (2.50)) for $\mathbb{E}_\mathbb{P}\left[U(X+\cdot)\right]$ over $\mathcal{B}_0 \cap M^\Phi$. We show that the sequence can be taken in such a way that $\sum_{j=1}^N W_H^j = 0$ for all H, which then implies the claim since we have (2.50). It is enough to check that the maximizing sequence (Y_{n_h}) can be taken with componentwise sum equal to 0, which can be reduced to proving

$$\sup_{Y \in \mathcal{B}_0 \cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \sup_{\substack{Y \in \mathcal{B} \cap M^\Phi \\ \sum_{j=1}^N Y^j = 0}} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right].$$

Inequaliy ($\geq$) follows from a trivial set inclusion, while ($\leq$) can be seen as follows: given any $Y \in \mathcal{B}_0 \cap M^\Phi$ with $\sum_{j=1}^N Y^j < 0$, we can add to each component an $\varepsilon > 0$ in such a way that the componentwise sum becomes equal to 0, and the corresponding expected systemic utility is strictly increased. □

Setup A and Setup B

Theorem 2.5.16.
In either Setup A or B the following hold:

1.
$$+\infty > \sup_{Y\in\mathcal{B}_0\cap M^\Phi} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \sup_{Y\in\mathcal{L}_V} \mathbb{E}_\mathbb{P}\left[U(X+Y)\right] \tag{2.70}$$
$$= \min_{\mathbb{Q}\in\mathcal{Q}_V}\min_{\lambda\geq 0}\left(\lambda\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] + \mathbb{E}_\mathbb{P}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{2.71}$$

Every optimum $(\widehat{\lambda},\widehat{\mathbb{Q}})$ *of* (2.71) *satisfies* $\widehat{\lambda}>0$ *and* $\widehat{\mathbb{Q}}\sim\mathbb{P}$*. Moreover, if* U *is differentiable,* (2.71) *admits a unique optimum* $(\widehat{\lambda},\widehat{\mathbb{Q}})$*, with* $\widehat{\mathbb{Q}}\sim\mathbb{P}$*.*

Furthermore there exists a random vector $\widehat{Y}\in(L^0(\mathbb{P}))^N$ *such that:*

2. $\widehat{Y}\in\mathcal{B}_0\cap\mathcal{L}_V$ *and it is the unique optimum to the following extended maximization problem:*
$$\sup_{Y\in\mathcal{L}_V}\mathbb{E}_\mathbb{P}\left[U(X+Y)\right] = \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y})\right]. \tag{2.72}$$

3. $\widehat{Y}$ *satisfies: for any optimizer* $(\widehat{\lambda},\widehat{\mathbb{Q}})$ *of* (2.71)
$$\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\in L^1(\mathbb{P})\ \forall\, j=1,\dots,N, \tag{2.73}$$
$$\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[\widehat{Y}^j\right] = 0 = \sum_{j=1}^N \widehat{Y}^j, \tag{2.74}$$

$$\widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\in L^1(\mathbb{P})\ \ \forall\mathbb{Q}\in\mathcal{Q}_V,\forall\, j=1,\dots,N\ \ \textit{and}\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[\widehat{Y}^j\right]\leq 0\ \ \forall\mathbb{Q}\in\mathcal{Q}_V. \tag{2.75}$$

Proof. We split the proof for the two sets of assumptions.

Setup A (Assumptions 2.3.2 and 2.3.3)
Item 1: Equations (2.70) and (2.71) follow from Theorem 2.5.9 Item 1, observing that minima over $\mathcal{Q}$ can be substituted with minima over $\mathcal{Q}_V$ since the expression in LHS of (2.70) is finite by Theorem 2.5.1. Again by Theorem 2.5.1, we have that the hypotheses of Proposition 2.5.2 are met with $\mathcal{K}=\mathcal{B}_0\cap M^\Phi$, hence by Theorem 2.5.9 Item 2 any optimum $(\widehat{\lambda},\widehat{\mathbb{Q}})$ of (2.71) satisfies $\widehat{\lambda}>0$. The proof of $\widehat{\mathbb{Q}}\sim\mathbb{P}$ is postponed after Item 2. Now we consider the uniqueness of the optimum for (2.71) under the additional differentiability assumption. In the notation of Theorem 2.5.3 take $\mathcal{C}:=\mathcal{B}_0\cap M^\Phi$, and observe that (2.71) can be rewritten, by (2.53), as
$$\min\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^jZ^j\right]+\mathbb{E}_\mathbb{P}\left[V(Z)\right]\mid Z\neq 0\in(\mathcal{C}_0^1)_+,\mathbb{E}_\mathbb{P}\left[V(Z)\right]<+\infty\right\}$$

which from strict convexity of $V(\cdot)$ (Lemma 2.6.5 Item 2) admits a unique optimum $0 \leq \widehat{Z} \neq 0$. We then get that, since $\widehat{\lambda} = \mathbb{E}_{\mathbb{P}}\left[\widehat{Z}\right]$, $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \frac{\widehat{Z}}{\mathbb{E}_{\mathbb{P}}[\widehat{Z}]}$ (again by (2.53)), uniqueness for optima in (2.71) follows.

Item 2: We take the vector $\widehat{Y}$ from Theorem 2.5.1 and Theorem 2.5.9. By Item 3 Theorem 2.5.9, we may apply Proposition 2.5.14 to $\widehat{Y}$ and deduce that $\mathbb{E}\left[\sum_{j=1}^N \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\widehat{Y}^j\right] \leq 0$ for all $\mathbb{Q} \in \mathcal{Q}_V$, which is condition (2.63). Now Proposition 2.5.15 yields that $\widehat{Y} \in \mathcal{B}_0 \cap \mathcal{L}_V$ is the unique optimum for (2.69), that is for (2.72):

$$\pi_0(X) := \sup_{Y \in \mathcal{B}_0 \cap M^\Phi} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] = \sup_{Y \in \mathcal{L}_V} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] = \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] \tag{2.76}$$

We claim that when $\widehat{\mathbb{Q}}$ is the optimizer of (2.71), then $\widehat{Y}$ is also an optimizer of (2.67) so that $\widehat{\mathbb{Q}} \sim \mathbb{P}$, by Proposition 2.5.13. First notice that $\widehat{Y}$ satisfy the constraint in (2.67), as $\widehat{Y} \in \mathcal{L}_V$. Moreover, as $\widehat{Y}$ is an optimizer of (2.70),

$$\mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] = \sup_{Y \in \mathcal{L}_V} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \tag{2.77}$$

$$\leq \sup\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] \mid Y \in (L^0(\mathbb{P}))^N,\, \sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}),\, \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \leq 0\right\} \tag{2.78}$$

$$\leq \inf_{\lambda \geq 0}\left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right]\right) \tag{2.79}$$

$$= \pi_0^{\widehat{\mathbb{Q}}}(X) = \pi_0(X) = \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right], \tag{2.80}$$

where the inequalities follow from (2.23), (2.24), (2.26) and the equalities in (2.80) come respectively from (2.65), (2.66) and (2.76). We conclude that $\widehat{Y}$ is an optimizer of (2.67).

Item 3: We claim that

$$\widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}) \quad \forall\, j = 1, \dots, N,\, \mathbb{Q} \in \mathcal{Q}_V\,. \tag{2.81}$$

Once this is shown, then: (2.73) holds true; the first equality in (2.74) is implied by (2.68) and the fact, just proved, that $\widehat{Y}$ is the optimizer of (2.67); the second equality in (2.74) follows from Proposition (2.5.15); the inequality in (2.75) holds true by Proposition 2.5.14.

We show (2.81): by Proposition 2.2.5 Item 3 $\mathcal{Q}_V \subseteq L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}$. Recall that in the proof of Theorem 2.5.1 we had extracted from a maximizing sequence a sequence of Césaro means $(W_H)_H$ converging to a $\widehat{Y}$ almost surely (equation (2.50)), and that the sequence $(W_H)_H$ satisfies: $(X+W_H)_H$ is bounded in $(L^1(\mathbb{P}))^N$ and $\inf_H \mathbb{E}_{\mathbb{P}}\left[U(X+W_H)\right] > -\infty$. We prove that this implies

$$\gamma := \sup_H \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\Phi_j((X^j+W_H^j)^-)\right] < +\infty\,. \tag{2.82}$$

To see this, observe that

$$U(X+W_H)=\sum_{j=1}^N u_j(X^j+W_H^j)+\Lambda(X+W_H)$$

$$=\sum_{j=1}^N u_j((X^j+W_H^j)^+)+\sum_{j=1}^N u_j(-(X^j+W_H^j)^-)+\Lambda(X+W_H)\,.$$

This implies

$$-\sum_{j=1}^N u_j(-(X^j+W_H^j)^-)\leq \max_{j=1,\dots,N}\left(\frac{\mathrm{d}u_j}{\mathrm{d}x^j}(0)\right)\sum_{j=1}^N (X^j+W_H^j)^+ +\sup_{z\in\mathbb{R}^N}\Lambda(z)-U(X+W_H)$$

where in the last line we used (2.3). By taking expectations on both sides we obtain (2.82). Indeed, $\sup_H\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^+\right]<+\infty$ by boundedness of $(W_H)_H$ in $(L^1(\mathbb{P}))^N$, and $\sup_H(-\mathbb{E}_{\mathbb{P}}\left[U(X+W_H)\right])=-\inf_H\mathbb{E}_{\mathbb{P}}\left[U(X+W_H)\right]<+\infty$.
By Fatou Lemma

$$\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[\Phi_j((X^j+\widehat{Y})^-)\right]\leq\sup_H\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[\Phi_j((X^j+W_H^j)^-)\right]\overset{\text{Eq.(2.82)}}{<}+\infty\,.$$

and hence $(X+\widehat{Y})^-$ belongs to $L^{\Phi_1}\times\dots\times L^{\Phi_N}$. Take now $0\leq Z\in L^{\Phi_1^*}\times\dots\times L^{\Phi_N^*}$, $Z\in\mathcal{Q}_V$. We will show that $(X^j+\widehat{Y}^j)^\pm Z^j=\left((X^j+\widehat{Y}^j)Z^j\right)^\pm\in L^1(\mathbb{P})$ for all $j=1,\dots,N$, which implies that condition (2.81) holds. Since again $(X+\widehat{Y})^-$ belongs to $L^{\Phi_1}\times\dots\times L^{\Phi_N}$, we have $0\leq(X^j+\widehat{Y}^j)^-Z^j\leq\sum_{j=1}^N(X^j+\widehat{Y}^j)^-Z^j\in L^1(\mathbb{P})$ for each $j=1,\dots,N$. We need now to work on the positive parts $(X^j+\widehat{Y}^j)^+Z^j$, $j=1,\dots,N$. Applying Fatou Lemma together with the trivial relation $x^+=x+x^-$ we have

$$\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+\widehat{Y}^j)^+Z^j\right]\leq\liminf_H\left(\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^+Z^j\right]\right)$$

$$\leq\sup_H\left(\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)Z^j\right]\right)+\sup_H\left(\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^-Z^j\right]\right)$$

Thus, to prove $(X^j+\widehat{Y}^j)^+Z^j\in L^1(\mathbb{P})$, $j=1,\dots,N$ it is enough to show that both suprema in the previous line are finite.
To see that

$$\sup_H\left(\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^-Z^j\right]\right)<+\infty\,,\tag{2.83}$$

observe that $(X^j+W_H^j)^-\in L^{\Phi_j}, j=1,\dots,N$ (Assumption 2.3.3), hence we have by the generalized Hölder Inequality ([65] Proposition 2.2.7)

$$\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[(X^j+W_H^j)^-Z^j\right]\leq 2\sum_{j=1}^N\left\|(X^j+W_H^j)^-\right\|_{\Phi_j}\left\|Z^j\right\|_{\Phi_j^*}$$

$$\leq 2\left(\sup_{j=1,\dots,N}\left\|Z^j\right\|_{\Phi_j^*}\right)\sup_H\left(\sum_{j=1}^N\left\|(X^j+W_H^j)^-\right\|_{\Phi_j}\right).$$

Clearly, if we show that for γ defined by (2.82)

$$\sup_H\left(\sum_{j=1}^N\left\|(X^j+W_H^j)^-\right\|_{\Phi_j}\right)\leq N\max(1,\gamma)<+\infty\,, \tag{2.84}$$

then (2.83) follows.
Splitting between the cases $\gamma\leq 1$ and $\gamma>1$, and using convexity of univariate utility functions in the latter, from equation (2.82) we infer that

$$\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[\Phi_j\left(\frac{1}{\max(1,\gamma)}(Z_n^j)^-\right)\right]\leq 1\,.$$

Then, just by definition of the Luxemburg norm (in the univariate case),

$$\left\|(X^j+W_H^j)^-\right\|_{\Phi_j}\leq\max(1,\gamma)\,,j=1,\dots,N$$

which yields (2.84), that is: the sequence $(X+W_H)^-$ is bounded in the norm $\sum_{j=1}^N\left\|\cdot\right\|_{\Phi_j}$ on $L^{\Phi_1}\times\dots\times L^{\Phi_N}$.
Going back to the optimizing sequence $(W_H)_H$ in Theorem 2.5.1, it satisfies $(W_H)_H\subseteq\mathcal{B}_0\cap M^\Phi$, so that for every $Z\in\mathcal{Q}_V$

$$\sup_H\left(\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[(X^j+W_H^j)Z^j\right]\right)\leq\sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[X^jZ^j\right]<+\infty \tag{2.85}$$

by Proposition 2.5.8.
To sum up, we proved that $(X^j+\widehat{Y}^j)^\pm Z^j\in L^1(\mathbb{P})$ for all $j=1,\dots,N$, which implies that the condition (2.81) holds.

Setup B (Assumptions 2.3.3 and 2.3.4)
Observe that Proposition 2.5.13 and Proposition 2.5.15 still apply, but Proposition 2.5.14 does not help anymore, since Assumption 2.3.2 does not hold. We will prove that (2.81) holds, and also that

$$\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N\widehat{Y}^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]=\sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}\left[\widehat{Y}^j\right]\leq 0\quad\forall\mathbb{Q}\in\mathcal{Q}_V\,. \tag{2.86}$$

As a consequence of (2.81) and (2.86), the proofs of Items 1, 2 and 3 turns then out to be identical to the one for Setup A.
In the current setup, similarly to what was done in the previous part of the proof, we can see that the sequence $(X+W_H)_H$ of Theorem 2.5.1 is bounded in $(L^1(\mathbb{P}))^N$ and (2.84) holds. Now we apply Propositions 2.4.17 and 2.6.10. Given the sequences $(X^j+W_H^j)^-$, $j=1,\dots,N$, a diagonalization argument yields a common subsequence such that $((X^j+W_H^j)^-)_H$ converges in $\sigma\left(L^{\Phi_j},M^{\Phi_j^*}\right)$ on L^{ϕ_j} for every j. Call such limit Z_j. Almost sure convergence

$$(X^j+W_H^j)^-\to(X+\widehat{Y})^-\ \mathbb{P}-a.s.$$

implies $Z = (X+Y)^-$. Indeed, if this were not the case assume without loss of generality $\mathbb{P}(Z^j > (X^j+Y^j)^-) > 0$ for some j. On a measurable subset D of the event $\{Z^j > (X^j+Y^j)^-\}$, $\mathbb{P}(D) > 0$, the convergence is uniform (by Egoroff's Theorem, Theorem 10.38 in [5]). Consequently, by Dominated Convergence Theorem plus $\sigma\left(L^\Phi(\mathcal{F}), M^{\Phi^*}(\mathcal{F})\right)$ convergence and the fact that $L^\infty \subseteq M^{\phi_j^*}$, $j=1,\dots,N$ we get $\mathbb{E}_\mathbb{P}\left[Z^j 1_D\right] = \mathbb{E}_\mathbb{P}\left[(X^j+Y^j)^- 1_D\right]$, which is a contradiction. Since by Proposition 2.4.17

$$\mathcal{Q}_V \subseteq K_\Phi = M^{\Phi_1^*} \times \cdots \times M^{\Phi_N^*}$$

we get for any $\mathbb{Q} \in \mathcal{Q}_V$:

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + W_H^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \to_H \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + \widehat{Y}^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]. \tag{2.87}$$

By Fatou Lemma and $x^+ = x + x^-$

$$\begin{aligned}
&\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + \widehat{Y}^j)^+ \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq \liminf_H \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + W_H^j)^+ \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \\
&\leq \liminf_H \left(\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N W_H^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + W_H^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]\right) \\
&\overset{\text{Prop.2.5.8}}{\leq} \liminf_H \left(\sum_{j=1}^N W_H^j + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + W_H^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]\right)\right) \\
&= \lim_H \left(\sum_{j=1}^N W_H^j\right) + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \lim_H \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + W_H^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]\right).
\end{aligned}$$

where we used Equation (2.87) and the fact that $\sum_{j=1}^N W_H^j$ is a numeric sequence converging (a.s.) to $\sum_{j=1}^N \widehat{Y}^j$ to move from lim inf to the sum of limits. As a consequence

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + \widehat{Y}^j)^+ \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq \sum_{j=1}^N \widehat{Y}^j + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + \widehat{Y}^j)^- \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]. \tag{2.88}$$

We get $Y \in L^1(\mathbb{Q})$ and rearranging terms in (2.88)

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(X^j + \widehat{Y^j}) \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq \sum_{j=1}^N \widehat{Y}^j + \sum_{n=1}^N \mathbb{E}_\mathbb{P}\left[X^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right].$$

In particular, since $\widehat{Y} \in \mathcal{B}_0$, (2.86) follows. □

Setup C

Theorem 2.5.17. *In the Setup C we have:*

1. *Equations* (2.70) *and* (2.71) *hold. Moreover* (2.71) *admits a unique optimum* $(\widehat{\lambda}, \widehat{\mathbb{Q}})$, *with* $\widehat{\lambda} > 0$ *and* $\widehat{\mathbb{Q}} \sim \mathbb{P}$.

Set $\widehat{Y} := -X - \nabla V\left(\widehat{\lambda}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right)$. *Then* $\widehat{\lambda}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = \nabla U(X+\widehat{Y})$ *and*

2. *Item 2 of Theorem 2.5.16 holds true.*

3. *Properties* (2.73) *and* (2.74) *hold true.*

Proof. We will proceed as follows: first we will establish for any optimum $(\widehat{\lambda}, \widehat{\mathbb{Q}})$ in Equation (2.71) that $\widehat{\lambda} > 0$ and the equivalence $\widehat{\mathbb{Q}} \sim \mathbb{P}$. We will then establish all the stated properties of $\widehat{Y}$ and deduce *a posteriori* the uniqueness of the optimum in Equation (2.71).

STEP 1: $\widehat{\lambda} > 0$ and $\widehat{\mathbb{Q}} \sim \mathbb{P}$.

By Proposition 2.5.2, we can apply Item 2 of Theorem 2.5.9 to guarantee that for any optimum $(\widehat{\lambda}, \widehat{\mathbb{Q}})$, $\widehat{\lambda} \neq 0$.
We now partially follow the proof of [38] Proposition 3.9. Observe that

$$\mathbb{P}\left(\widehat{\lambda}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \in \{V = +\infty\}\right) = 0$$

otherwise we would get a contradiction with (2.49): $\sup_{Y\in\mathcal{B}_0\cap M^\Phi} \mathbb{E}_{\mathbb{P}}[U(X+Y)] < +\infty$. Recall from Theorem 2.5.3 that in fact the minimizations in the dual problem of $\sup_{Y\in\mathcal{B}_0\cap M^\Phi} \mathbb{E}_{\mathbb{P}}[U(X+Y)]$ are over $(\mathcal{B}_0 \cap M^\Phi)^0$ and that by Remark 2.5.7 there exists a strictly positive vector $Z := \frac{d\mathbb{Q}}{d\mathbb{P}} \in K_\Phi$ with $\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty$. Call $\widehat{Z} = \widehat{\lambda}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}$.
By Assumption 2.3.5 then for all $\alpha \geq 0$

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\widehat{Z} + \alpha Z\right)\right] \leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[V(2\widehat{Z})\right] + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[V(2\alpha Z)\right] < +\infty$$

and clearly for all $\alpha \geq 0$ again $\widehat{Z} + \alpha Z \in (\mathcal{B}_0 \cap M^\Phi)^0$. Define for $\alpha \geq 0$ $\nu_\alpha := V\left(\widehat{Z} + \alpha Z\right)$. Observe that $\frac{\nu_\alpha - \nu_0}{\alpha}$ is monotonically decreasing as $\alpha \downarrow 0$ (by convexity of V and since $Z, \widehat{Z} \geq 0$). Applying Monotone Convergence Theorem we get for any measurable set A:

$$\mathbb{E}_{\mathbb{P}}\left[1_A \frac{\nu_\alpha - \nu_0}{\alpha}\right] \downarrow \mathbb{E}_{\mathbb{P}}\left[1_A \lim_{\alpha\downarrow 0} \frac{1}{\alpha}\left(V\left(\widehat{Z} + \alpha Z\right) - V\left(\widehat{Z}\right)\right)\right].$$

Choose now the set

$$A := \left\{\widehat{\lambda}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \in \left(\{V < +\infty\} \cap \partial((0,+\infty)^N)\right)\right\}$$

and assume by contradiction that $\mathbb{P}(A) > 0$. Since

$$1_A \frac{\nu_\alpha - \nu_0}{\alpha} = 1_A \left(\sum_{j=1}^N \frac{\partial V}{\partial x^j}(\widehat{Z} + \widetilde{\alpha} Z) Z^j\right) \text{ for some } 0 \leq \widetilde{\alpha} \leq \alpha$$

we have by Lemma 2.6.6 that $1_A \frac{\nu_\alpha - \nu_0}{\alpha} \downarrow_\alpha -\infty 1_A$. As a consequence $\mathbb{E}_{\mathbb{P}}\left[1_A \frac{\nu_\alpha - \nu_0}{\alpha}\right] \downarrow_\alpha -\infty$ which in turns yields $\mathbb{E}_{\mathbb{P}}\left[\frac{\nu_\alpha - \nu_0}{\alpha}\right] \downarrow -\infty$.

At the same time we can rewrite $\mathbb{E}_{\mathbb{P}}\left[\nu_\alpha - \nu_0\right]$ as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\left\langle X, \widehat{Z} - (\widehat{Z} + \alpha Z)\right\rangle\right] + \\
&+ \left\{\left(\mathbb{E}_{\mathbb{P}}\left[\left\langle X, \widehat{Z} + \alpha Z\right\rangle\right] + \mathbb{E}_{\mathbb{P}}\left[V(\widehat{Z} + \alpha Z)\right]\right) - \left(\mathbb{E}_{\mathbb{P}}\left[\left\langle X, \widehat{Z}\right\rangle\right] + \mathbb{E}_{\mathbb{P}}\left[V(\widehat{Z})\right]\right)\right\} \\
&\geq \mathbb{E}_{\mathbb{P}}\left[\left\langle X, \widehat{Z} - (\widehat{Z} + \alpha Z\right\rangle\right] = -\alpha \mathbb{E}_{\mathbb{P}}\left[\langle X, Z\rangle\right]
\end{aligned}$$

where the inequality comes from the fact that $\widehat{Z}, \widehat{Z}+\alpha Z \in (\mathcal{B}_0 \cap M^\Phi)^0$ and $\widehat{Z}$ minimizes

$$Z \mapsto \mathbb{E}_{\mathbb{P}}\left[\langle X, Z\rangle\right] + \mathbb{E}_{\mathbb{P}}\left[V(Z)\right], \quad Z \in (\mathcal{B}_0 \cap M^\Phi)^0,$$

so that the term $\{\dots\}$ is nonnegative. Clearly then we also get $\mathbb{E}_{\mathbb{P}}\left[\frac{\nu_\alpha - \nu_0}{\alpha}\right] \geq -\mathbb{E}_{\mathbb{P}}\left[\langle X, Z\rangle\right]$ which is a contradiction. We conclude that $\mathbb{P}(A) = 0$, and from the observations at the beginning of the proof that

$$\mathbb{P}\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \in \partial((0, +\infty)^N)\right) = 0\,.$$

This can be restated as $\widehat{\mathbb{Q}}^1, \dots, \widehat{\mathbb{Q}}^N \sim \mathbb{P}$.

STEP 2: $\widehat{Y} \in \mathcal{L}_V$.

By Lemma 2.6.5 Item 2 V is differentiable through $(0, +\infty)^N$. By STEP 1 $\widehat{\lambda} > 0$ and $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \in (0, +\infty)^N$ a.s. so that $\widehat{Y}$ is well defined. Now $\widehat{\lambda}$ minimizes, for $\mathbb{Q} = \widehat{\mathbb{Q}}$, the function

$$(0, +\infty) \ni \gamma \mapsto \psi(\gamma) := \sum_{j=1}^N \left(\gamma \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\gamma \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]$$

which is real valued and convex. Also we have by Monotone Convergence Theorem and Lemma 2.6.7 Item 1 that the right and left derivatives, which exist by convexity, satisfy

$$\frac{\mathrm{d}^\pm \psi}{\mathrm{d}\gamma}(\gamma) = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\frac{\partial V}{\partial x^j}\left(\gamma \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]$$

hence the function is differentiable. Since $\widehat{\lambda}$ is a minimum for ψ, this implies $\psi'(\widehat{\lambda}) = 0$, which can be rephrased as

$$\sum_{j=1}^N \left(\mathbb{E}_{\mathbb{P}}\left[X^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \mathbb{E}_{\mathbb{P}}\left[\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]\right) = 0\,. \tag{2.89}$$

At this point minimize over

$$\mathbb{Q} \mapsto \sum_{j=1}^N \left(\widehat{\lambda}\mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\widehat{\lambda}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]$$

where $\widehat{\lambda}$ is given above and $\mathbb{Q}$ varies in $\mathcal{Q}_V$. Let again $\widehat{\mathbb{Q}}$ be optimum and take another $\mathbb{Q} \in \mathcal{Q}_V$ (which implies by our standing assumption 2.3.5 that the expression

$\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]$ is finite for all choices of λ). Define $\widehat{\eta}$ and η to be their Radon-Nikodym derivatives with respect to $\mathbb{P}$. Take a convex combination of the two: for $0 \leq x \leq 1$ set $\xi_x := (1-x)\widehat{\eta} + x\eta$. By optimality of $\widehat{\eta}$ the function

$$x \mapsto \varphi(x) := \sum_{j=1}^{N}\left(\widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\xi_x^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\widehat{\lambda}\xi_x\right)\right]$$

has a minimum at 0, thus the right derivative of φ at 0 must be non negative:

$$0 \leq \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}x}\Big|_0\left((1-x)\widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\widehat{\eta}^j\right] + x\widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\eta^j\right]\right) + \frac{\mathrm{d}}{\mathrm{d}x}\Big|_0\mathbb{E}_{\mathbb{P}}\left[V\left((1-x)\widehat{\lambda}\widehat{\eta} + x\lambda\eta\right)\right]. \tag{2.90}$$

Differentiation in the first summation is trivial. As to the second term observe that by convexity and differentiability of V we have

$$\widehat{\lambda}\sum_{j=1}^{N}\eta^j\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right) \leq \widehat{\lambda}\sum_{j=1}^{N}\widehat{\eta}^j\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right) + V\left(\widehat{\lambda}\eta\right) - V\left(\widehat{\lambda}\widehat{\eta}\right)$$

so that by Lemma 2.6.7, Assumption 2.3.5 and $\widehat{\mathbb{Q}}, \mathbb{Q} \in \mathcal{Q}_V$ we conclude

$$\left(\sum_{j=1}^{N}\eta^j\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\right)^+ \in L^1(\mathbb{P}). \tag{2.91}$$

Define $H(x) := V\left((1-x)\widehat{\lambda}\widehat{\eta} + x\widehat{\lambda}\eta\right)$ and observe that as $x \downarrow 0$

$$\begin{aligned}0 &\leq \left(H(1) - H(0) - \frac{1}{x}\left(H(x) - H(0)\right)\right)\\ &\uparrow \left(H(1) - H(0) - \widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\eta^j + \widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\widehat{\eta}^j\right).\end{aligned}$$

Thus we have by Equation (2.90) and Monotone Convergence Theorem

$$+\infty > \mathbb{E}_{\mathbb{P}}\left[H(1) - H(0)\right] + \sum_{j=1}^{N}\widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[X^j\left(\eta^j - \widehat{\eta}^j\right)\right] \geq$$

$$\geq \mathbb{E}_{\mathbb{P}}\left[\left(H(1) - H(0) - \widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\eta^j + \widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\widehat{\eta}^j\right)\right] =$$

$$\mathbb{E}_{\mathbb{P}}\left[\left(\sum_{j=1}^{N}\widehat{\lambda}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\eta^j\right)^- - \widehat{\lambda}\left(\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\eta^j\right)^+ + R(\widehat{\eta}, \widehat{\lambda})\right]$$

for

$$R(\widehat{\eta}, \widehat{\lambda}) := H(1) - H(0) + \widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\widehat{\eta}\right)\widehat{\eta}^j.$$

This implies that

$$0 \leq \left(\sum_{j=1}^{N} \widehat{\lambda} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \eta^j \right)^{-} \in L^1(\mathbb{P})$$

since we recall that, together with (2.91), we have the following:

$$(H(1) - H(0)) \in L^1(\mathbb{P}) \text{ by Assumption 2.3.5 and } \widehat{\mathbb{Q}}, \mathbb{Q} \in \mathcal{Q}_V \,,$$

$$\sum_{j=1}^{N} \widehat{\lambda} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \widehat{\eta}^j \in L^1(\mathbb{P}) \text{ by Lemma 2.6.7 Item 1} \,.$$

We conclude that $\sum_{j=1}^{N} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right) \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})$, hence also $\sum_{j=1}^{N} \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})$ holds for all $\mathbb{Q} \in \mathcal{Q}_V$.
Moreover, in view of the integrability property we just proved, Equation (2.90) can be rewritten as:

$$0 \leq \sum_{j=1}^{N} \widehat{\lambda} \mathbb{E}_{\mathbb{P}} \left[X^j \left(\eta^j - \widehat{\eta}^j \right) \right] + \widehat{\lambda} \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \left(\eta^j - \widehat{\eta}^j \right) \right] . \tag{2.92}$$

Now rearrange the terms in (2.92) as follows

$$0 \leq -\widehat{\lambda} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j \widehat{\eta}^j \right] + \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \widehat{\eta}^j \right] \right) +$$
$$+ \sum_{j=1}^{N} \widehat{\lambda} \left(\mathbb{E}_{\mathbb{P}} \left[X^j \eta^j \right] + \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \eta^j \right] \right)$$

and use (2.89):

$$0 \leq 0 - \widehat{\lambda} \left(\mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} \left(-X^j - \frac{\partial V}{\partial x^j} \left(\widehat{\lambda} \widehat{\eta} \right) \right) \eta^j \right] \right) = -\widehat{\lambda} \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right] .$$

This proves that $\widehat{Y} \in \mathcal{L}_V$.

STEP 3: Integrability under optimal measure.

$\widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}) \, \forall \, j = 1 \, \ldots, N$ follows from $X \in M^{\Phi}$, Remark 2.5.5, Lemma 2.6.7 Item 1 and the fact that $\widehat{\lambda} > 0$.

STEP 4 Optimality of $\widehat{Y}$.

Observe that V and $-U(-\bullet)$ are convex functions conjugate to each other in the sense of Fenchel-Moreau Theorem, hence in the Legendre sense (see [119] Chapter V) on the interior of their respective domains, by [119], Theorem 26.5. Also $(\nabla(-U(-\bullet)))^{-1} = \nabla V$ ([119], Theorem 26.5 again) on $(\mathbb{R}_{++})^N$, which is the interior of $dom(V)$ by the fact that V is finite on $(\mathbb{R}_{+})^N$ by Remark 2.1.6 and equal to $+\infty$ on $\mathbb{R}^N \setminus (\mathbb{R}_{+})^N$ by Lemma 2.6.5 Item 1. Consequently, we get for $y \in (\mathbb{R}_{++})^N$ and by definition of Legendre conjugate

$$V(y) = \left\langle (\nabla(-U(-\bullet)))^{-1} (y), y \right\rangle - \left(-U \left(- (\nabla(-U(-\bullet)))^{-1} (y) \right) \right)$$

$$= \langle \nabla V(y), y \rangle + U(-\nabla V(y)) \, .$$

Equivalently $U(-\nabla V(y)) = -\langle \nabla V(y), y \rangle + V(y)$ for all $y \in (\mathbb{R}_{++})^N$.
Observe now that as a consequence

$$U(X+\widehat{Y}) = U\left(-\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right) = -\widehat{\lambda}\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} + V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) .$$

Taking expectations on both sides (both are integrable by previous arguments) we get

$$\mathbb{E}_{\mathbb{P}}\left[U\left(X+\widehat{Y}\right)\right] = \widehat{\lambda}\mathbb{E}_{\mathbb{P}}\left[-\sum_{j=1}^{N}\frac{\partial V}{\partial x^j}\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \mathbb{E}_{\mathbb{P}}\left[V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right] .$$

Use now the expression in (2.89) to substitute in the first term in RHS:

$$\mathbb{E}_{\mathbb{P}}\left[U\left(X+\widehat{Y}\right)\right] = \widehat{\lambda}\left(\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}X^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right] .$$

Recognizing in RHS the optimum value in the minimax expressions of Equation (2.56) in Theorem 2.5.9, we conclude that

$$\sup_{Y\in\mathcal{L}_V}\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] = \mathbb{E}_{\mathbb{P}}\left[U(X+\widehat{Y})\right] = \sup_{Y\in\mathcal{B}_0\cap M^\Phi}\mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] .$$

Notice that from $\nabla V = (\nabla(-U(-\bullet)))^{-1} = -(\nabla U)^{-1}$ and $X+\widehat{Y} = -\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$ we obtain:

$$\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \nabla U(X+\widehat{Y}).$$

STEP 5: $\widehat{Y}\in\mathcal{B}_0$.

The following properties hold for $K := \mathcal{B}_0\cap M^\Phi$. $K\subseteq M^\Phi$ is a convex cone such that for all $i,j\in\{1,\dots,N\}$ $e_i - e_j\in K$. If $\mathcal{S}_V^e\subseteq K_\Phi$ is defined as

$$\mathcal{S}_V^e := \left\{\mathbb{Q}\mid\mathbb{Q}\sim\mathbb{P},\,\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\in K_\Phi,\,\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]<+\infty,\,\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[k^j\right]\le 0\,\forall k\in K\right\}$$

then (use Assumption 2.3.5) $\mathcal{S}_V^e = \mathcal{Q}_V\cap\{[\mathbb{Q}^1,\dots,\mathbb{Q}^N]\mid\mathbb{Q}^j\sim\mathbb{P}\ \forall j=1,\dots,N\}$. Also for $(\widehat{\lambda},\widehat{\mathbb{Q}})$ and $\widehat{Y}$ as above:

1. $\widehat{\mathbb{Q}}\in\mathcal{S}_V^e$ and

$$\left[\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]_{j=1}^{N}\in(L^1(\mathbb{P}))^N,\quad\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]=0$$

(STEP 4 and (2.89)).

2. for all $\mathbb{Q} \in \mathcal{S}_V^e$

$$\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}), \quad \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq 0$$

(since more in general $\widehat{Y} \in \mathcal{L}_V$ by STEP 2).

As a consequence, by Theorem 2.6.13, $\widehat{Y}$ is in the closure under convergence in probability $\mathbb{P}$ of K, hence in $\mathcal{B}_0$ (which is closed in probability by Standing Assumption II).

STEP 6: uniqueness of $(\widehat{\lambda}, \widehat{\mathbb{Q}})$.

If $(\widehat{\lambda}, \widehat{\mathbb{Q}})$, $(\lambda, \mathbb{Q})$ are two optima for Equation (2.56) with $\lambda, \widehat{\lambda} > 0$, then $\widehat{Y}$ and Y defined correspondingly as above will coincide by Proposition 2.5.2. At the same time ∇V is invertible (see [119] Theorem 26.5), hence $\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. Taking expectations we get $\widehat{\lambda} = \lambda$ and $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ follows trivially.

STEP 7: $\sum_{j=1}^N \widehat{Y}^j = 0 = \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]$.

We proved in STEP 5 that $\widehat{Y} \in cl_{\widehat{\mathbb{Q}}}(\mathcal{B}_0 \cap M^\Phi)$. As a consequence, there exists a sequence $(k_n)_n \subseteq \mathcal{B}_0 \cap M^\Phi$ such that $k_n \to_n \widehat{Y}$ in $L^1(\widehat{\mathbb{Q}})$ and $\mathbb{P}$-a.s. (since $\widehat{\mathbb{Q}} \sim \mathbb{P}$). Thus, we have

$$0 \overset{(2.89)}{=} \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[\widehat{Y}^j\right] = \lim_n \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[k_n^j\right] \overset{\text{Prop. 2.5.8}}{\leq} \lim_n \sum_{j=1}^N k_n^j = \sum_{j=1}^N \widehat{Y}^j \overset{Y \in \mathcal{B}_0}{\leq} 0$$

where we used STEP 5 for last inequality. □

2.5.5 Working on $(L^\infty(\mathbb{P}))^N$

The following result is a counterpart to Theorems 2.5.16 and Theorem 2.5.17 when working in $((L^\infty(\mathbb{P}))^N, (L^1(\mathbb{P}))^N)$ in place of (M^Φ, K_Φ).

Theorem 2.5.18. *If $\mathcal{B}$ is closed under truncation the following holds:*

$$\sup_{Y \in \mathcal{B}_0 \cap (L^\infty(\mathbb{P}))^N} \mathbb{E}_{\mathbb{P}}\left[U(X+Y)\right] = \min_{\mathbb{Q} \in \mathcal{Q}_V} \min_{\lambda \geq 0} \left(\lambda\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]\right) + \mathbb{E}_{\mathbb{P}}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{2.93}$$

Proof. To check (2.93) we can apply the same argument used in proving (2.55) and (2.56), by replacing Theorem 2.5.3 with Theorem 2.5.4.
What is left to prove then is that for $\mathcal{C} = \mathcal{B}_0 \cap (L^\infty(\mathbb{P}))^N$, the set

$$\mathcal{N} := (\mathcal{C}_1^0)^+ \cap \left\{Z \in (L^1(\mathbb{P})_+)^N \mid \mathbb{E}_{\mathbb{P}}\left[V(\lambda Z)\right] < +\infty \text{ for some } \lambda > 0\right\}$$

is in fact $\mathcal{Q}_V$. To see this, observe that as consequence of Lemma 2.6.9 we have $\mathcal{N} \subseteq K_\Phi$. From this, by closedness under truncation we have for any $Y \in \mathcal{B}_0 \cap M^\Phi$

a sequence $(Y_n)_n \subseteq \mathcal{B}_0 \cap (L^\infty(\mathbb{P}))^N$ such that for each $\mathbb{Q} \in \mathcal{N}$,for each $j = 1, \dots, N$ $Y_n^j \to_n Y^j$ $\mathbb{Q}^j$-a.s. and the convergence is dominated. Thus for any $Y \in \mathcal{B}_0 \cap M^\Phi$ we have by Dominated Convergence Theorem that $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j] \leq 0$. This completes the proof that $\mathcal{N} = \mathcal{Q}_V$. □

Corollary 2.5.19. *In Setup A we have*

$$\sup_{Y \in \mathcal{B}_0 \cap (L^\infty(\mathbb{P}))^N} \mathbb{E}_\mathbb{P}[U(X+Y)] = \sup_{Y \in \mathcal{B}_0 \cap M^\Phi} \mathbb{E}_\mathbb{P}[U(X+Y)] . \tag{2.94}$$

Proof. By Theorem 2.5.16 Item 1 and Theorem 2.5.18, both LHS and RHS of (2.94) are equal to the minimax expression

$$\min_{\lambda \geq 0,\, \mathbb{Q} \in \mathcal{Q}_V} \left(\lambda \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j\right] \right) + \mathbb{E}_\mathbb{P}\left[V\left(\lambda \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right] \right) .$$

□

2.5.6 General case: total wealth $A \in \mathbb{R}$

In this section we extend previous results to cover the case in which the total wealth A might not be equal to 0.
For $A \in \mathbb{R}$ and $\mathbb{Q} \in \mathcal{Q}_V$ we define

$$\pi_A(X) := \sup \left\{ \mathbb{E}_\mathbb{P}[U(X+Y)] \mid Y \in \mathcal{B} \cap M^\Phi, \sum_{j=1}^N Y^j \leq A \right\} ,$$

$$\pi_A^\mathbb{Q}(X) := \sup \left\{ \mathbb{E}_\mathbb{P}[U(X+Y)] \mid Y \in M^\Phi, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\right] \leq A \right\} .$$

It is possible to reduce the maximization problem expressed by $\pi_A(X)$ (and similarly $\pi_A^\mathbb{Q}(X)$) to the problem related to $\pi_0(\cdot)$ (respectively, $\pi_0^\mathbb{Q}(\cdot)$).
Take any $a = [a^1, \dots, a^N] \in \mathbb{R}^N$ with $\sum_{j=1}^N a^j = A$. Then

$$\pi_A(X) = \sup \left\{ \mathbb{E}_\mathbb{P}[U(X+Y+a-a)] \mid (Y-a) \in \mathcal{B} \cap M^\Phi, \sum_{j=1}^N \left(Y^j - a^j\right) \leq 0 \right\} =$$

$$= \sup \left\{ \mathbb{E}_\mathbb{P}[U(X+Z+a)] \mid Z \in \mathcal{B}_0 \cap M^\Phi \right\} = \pi_0(X+a)$$

where last line holds since $\mathbb{R}^N + \mathcal{B} = \mathcal{B}$ under Standing Assumption II. We recognize then that $\pi_A(X)$ is just $\pi_0(\cdot)$, with different initial point $(X+a)$ in place of X.
The same technique adopted above can be exploited to show that for any $a \in \mathbb{R}^N$ with $\sum_{j=1}^N a^j = A$

$$\sup \left\{ \mathbb{E}_\mathbb{P}[U(X+Y)] \mid Y \in \mathcal{L}_V^{(A)} \right\} = \sup \left\{ \mathbb{E}_\mathbb{P}[U(X+a+Z)] \mid Z \in \mathcal{L}_V \right\} .$$

The argument above shows how to generalize Proposition 2.5.11, Theorem 2.5.16, Theorem 2.5.17, Theorem 2.5.18, Corollary 2.5.19 to cover the case $A \neq 0$, exploiting the same results with $X + a$ in place of X.

Thus the statements of Proposition 2.5.11, Theorem 2.5.16, Theorem 2.5.17, Theorem 2.5.18, Corollary 2.5.19 remain true replacing 0, $\mathcal{B}_0$, $\mathcal{L}_V$ with A, $\mathcal{B}_A$, $\mathcal{L}_V^{(A)}$ respectively, and Equation (2.65) (similarly for (2.52), (2.56), (2.71), (2.93)) with

$$\min_{\lambda\geq 0}\left(\lambda\left(\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[X^j\right]+A\right)+\mathbb{E}_{\mathbb{P}}\left[V\left(\lambda\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]\right). \tag{2.65A}$$

We will not go through all the proofs again, but only provide a hint about the methodology to be followed to obtain the results. To show that $-X-\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$ is in $\mathcal{L}_V^{(A)}$, for example, we can use the fact that $-(X+a)-\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$ is in $\mathcal{L}_V$ by Theorem 2.5.17 and then move the term $A=\sum_{j=1}^N a^j$ to LHS in

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}\left(-(X+a)-\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right)^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]\\ &=\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}\left(-X-\nabla V\left(\widehat{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right)^j\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right]-\sum_{j=1}^{N}a^j\leq 0\,.\end{aligned}$$

2.6 Appendix to Chapter 2

Throughout all this Appendix, we work under Standing Assumptions I and II without further mention.

2.6.1 Superdifferentials

Proposition 2.6.1. *Let $u:\mathbb{R}^N\to\mathbb{R}$ be concave, nondecreasing and null in 0. Let $z\in\mathbb{R}^N$. Then:*

1. *Any element in $\partial u(z)$ is nonnegative.*
2. *For $N=1$ $\frac{\mathrm{d}^\pm u}{\mathrm{d}x}(z)\in\partial u(z)$ where $\frac{\mathrm{d}^\pm u}{\mathrm{d}x}(z)$ are the left and right derivatives of u at x_0.*
3. *For $N=1$, if*

$$\lim_{x\to-\infty}\frac{u(x)}{x}=+\infty \text{ and } \lim_{x\to+\infty}\frac{u(x)}{x}=0$$

we have

$$\lim_{z\to-\infty}\frac{\mathrm{d}^-u}{\mathrm{d}x}(z)=+\infty \text{ and } \lim_{z\to+\infty}\frac{\mathrm{d}^+u}{\mathrm{d}x}(z)=0\,.$$

Proof.
Item 1: It follows from the fact that by definition $u(x)-u(z)\leq\sum_{j=1}^N\nu^j(x^j-z^j)$ for all $x\in\mathbb{R}^N$ for any $\nu\in\partial u(z)$. If for some index k $\nu^k<0$, we would get a contradiction considering $x=z+ne_k\geq z$ and taking the limit as n grows to $+\infty$.
Item 2: It follows from Theorem 23.2 in [119].

Item 3: We observe that by concavity for any $\varepsilon > 0$

$$\frac{u(z)}{z} \geq \frac{u(z+\varepsilon) - u(z)}{\varepsilon} \quad \text{for } z > 0 \text{ and } \frac{u(z)}{z} \leq \frac{u(z) - u(z-\varepsilon)}{\varepsilon} \quad \text{for } z < 0$$

taking the limit as $\varepsilon \downarrow 0$ yields

$$\frac{\mathrm{d}^+ u}{\mathrm{d}x}(z) \leq \frac{u(z)}{z} \quad \text{for } z > 0 \text{ and } \frac{\mathrm{d}^- u}{\mathrm{d}x}(z) \geq \frac{u(z)}{z} \quad \text{for } z < 0\,.$$

□

2.6.2 Additional properties of Multivariate Utilitity Functions

Lemma 2.6.2. *There exist $a > 0$, $b \in \mathbb{R}$ such that*

$$U(x) \leq a \sum_{j=1}^{N} x^j + a \sum_{j=1}^{N} (-(x^j)^-) + b \quad \forall\, x \in \mathbb{R}^N\,.$$

Proof. We start recalling that by Remark 2.1.3 for any concave $f : \mathbb{R}^N \to \mathbb{R}$ and for any $z \in \mathbb{R}^N$

$$f(x) \leq \langle \nabla f(z), (x - z) \rangle + f(z) \ \ \forall x \in \mathbb{R}^m\,.$$

We can thus write, for every $k \in \mathbb{R}$

$$\begin{aligned}
U(x) &= \sum_{j=1}^{N} u_j\left((x^j)^+\right) + \sum_{j=1}^{N} u_j\left(-(x^j)^-\right) + \Lambda(x) \\
&\leq \sum_{j=1}^{N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(0)((x^j)^+ - 0) + u_j(0) \right) + \sum_{j=1}^{N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(k)(-(x^j)^- - k) + u_j(k) \right) + \\
&+ \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(0)(x^j - 0) + \Lambda(0) \\
&= f(k) + \sum_{j=1}^{N} \frac{\mathrm{d}u_j}{\mathrm{d}z}(0)(x^j)^+ + \sum_{j=1}^{N} \frac{\mathrm{d}u_j}{\mathrm{d}z}(k)(-(x^j)^-) + \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(0)((x^j)^+ - (x^j)^-) \\
&= f(k) + \sum_{j=1}^{N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(0) + \frac{\partial \Lambda}{\partial x^j}(0) \right)(x^j)^+ + \sum_{j=1}^{N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(k) + \frac{\partial \Lambda}{\partial x^j}(0) \right)(-(x^j)^-)\,.
\end{aligned}$$

Set now

$$a := \max_j \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(0) + \frac{\partial \Lambda}{\partial x^j}(0) \right) \overset{\text{Prop.2.6.1.1}}{\geq} 0$$

and observe that since Inada conditions hold, by Proposition 2.6.1 Item 3 we can choose elements in the supergradients in such a way that

$$\min_j \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(k) \right) \xrightarrow[k \to -\infty]{} +\infty\,.$$

Hence for some $\widehat{k} < 0$ we have

$$\min_j \left(\frac{\mathrm{d}u_j}{\mathrm{d}z}(\widehat{k}) \right) + \min_j \left(\frac{\partial \Lambda}{\partial x^j}(0) \right) \geq 2a\,.$$

As a consequence

$$U(x) \leq f(\widehat{k}) + a\sum_{j=1}^{N}(x^j)^+ + 2a\sum_{j=1}^{N}(-(x^j)^-) = a\sum_{j=1}^{N} x^j + a\sum_{j=1}^{N}(-(x^j)^-) + b$$

once we set $f(\widehat{k}) = b$. □

Lemma 2.6.3. *For every $\varepsilon > 0$ there exist a constant b_ε such that*

$$U(x) \leq 2\varepsilon \sum_{j=1}^{N}(x^j)^+ + b_\varepsilon \quad \forall x \in \mathbb{R}^N. \tag{2.95}$$

Proof. Fix $\varepsilon > 0$. From the fact that the Inada conditions hold, again by Proposition 2.6.1 Item 3 we can choose elements in the supergradients in such a way that

$$\max_j \left(\frac{\mathrm{d}u_j}{\mathrm{d}x}(k) \right) \xrightarrow[k\to+\infty]{} 0\,.$$

As a consequence, given $\varepsilon > 0$, we have for some function $\psi : \mathbb{R} \to \mathbb{R}$ and some $k_\varepsilon > 0$ that $\forall x \in \mathbb{R}^N$

$$\sum_{j=1}^{N} u_j\left(x^j\right) \leq \sum_{j=1}^{N} u_j((x^j)^+) \leq \max_j \left(\frac{\mathrm{d}u_j}{\mathrm{d}x}(k) \right) \sum_{j=1}^{N}(x^j)^+ \psi(k_\varepsilon) \leq \varepsilon \sum_{j=1}^{N}(x^j)^+ + \psi(k_\varepsilon). \tag{2.96}$$

The concavity of Λ implies that for any fixed $z \in \mathbb{R}^N$ and any $x \in \mathbb{R}^N$, by Remark 2.1.3,

$$\Lambda(x) \leq \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z)(x^j - z^j) + \Lambda(z) \leq$$

$$\leq \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z)(x^j)^+ + \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z)(-(x^j)^-) + \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z)(-z^j) + \Lambda(z)\,.$$

Since Λ is nondecreasing each element in its supergradient is componentwise nonnegative (Proposition 2.6.1 Item 1) and so $\sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z)(-(x^j)^-) \leq 0$. Also, for any $\varepsilon > 0$ we can now take z_ε as in Standing Assumption I and reformulate what we found as

$$\Lambda(x) \leq \sum_{j=1}^{N} \frac{\partial \Lambda}{\partial x^j}(z_\varepsilon)(x^j)^+ + \xi(z_\varepsilon) \leq \varepsilon \left(\sum_{j=1}^{N}(x^j)^+ \right) + \xi(z_\varepsilon) \quad \forall x \in \mathbb{R}^N \tag{2.97}$$

for some function $\xi : \mathbb{R}^N \to \mathbb{R}$. We conclude from (2.96) and (2.97) that

$$U(x) = \sum_{j=1}^{N} u_j\left(x^j\right) + \Lambda(x) \leq 2\varepsilon \sum_{j=1}^{N}(x^j)^+ + \xi(z_\varepsilon) + \psi(k_\varepsilon) \quad \forall x \in \mathbb{R}^N\,.$$

When $\varepsilon > 0$ is fixed $\xi(z_\varepsilon) + \psi(k_\varepsilon) =: b_\varepsilon$ is a constant and we find (2.95). □

Lemma 2.6.4. *Let $(Z_n)_n$ be a sequence of random variables taking values in $\mathbb{R}^N$ such that $\mathbb{E}_{\mathbb{P}}[U(Z_n)] \geq B$ for all n, for some $B \in \mathbb{R}$.*

1. *If* $\sup_n \left|\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Z_n^j]\right| < +\infty$ *then* $\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[|Z_n^j|] < \infty$.

2. *If* $Z_n \to Z$ *a.s. and* $\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^+] < +\infty$ *then* $\mathbb{E}_{\mathbb{P}}[U(Z)] \geq B$.

Proof.
Item 1. Suppose that

$$\sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left|Z_n^j\right|\right]\right) = \sup_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^+\right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^-\right]\right) = +\infty\,.$$

From the boundedness of

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Z_n^j\right] = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^+\right] - \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^-\right]$$

we conclude that $\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[(Z_n^j)^-] = +\infty$. Select a, b as in Lemma 2.6.2 . Then we have

$$B \leq \mathbb{E}_{\mathbb{P}}[U(Z_n)] \leq a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Z_n^j\right] - a \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^-\right] + b$$

which is clearly a contradiction.
Item 2. For $\varepsilon > 0$ define the function Γ_ε as

$$\Gamma_\varepsilon(x) := 2\varepsilon\left(\sum_{j=1}^N (x^j)^+\right) + b_\varepsilon - U(x)$$

where the coefficient b_ε is the one in Lemma 2.6.3. Then $\Gamma_\varepsilon \geq 0$ and by Fatou Lemma we have

$$\begin{aligned}
&2\varepsilon\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z^j)^+\right]\right) + b_\varepsilon - \mathbb{E}_{\mathbb{P}}[U(Z)] = \mathbb{E}_{\mathbb{P}}[\Gamma_\varepsilon(Z)] \leq \liminf_n \mathbb{E}_{\mathbb{P}}[\Gamma_\varepsilon(Z_n)] \\
&= \liminf_n \left(2\varepsilon\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^+\right]\right) + b_\varepsilon - \mathbb{E}_{\mathbb{P}}[U(Z_n)]\right) \\
&\leq -B + b_\varepsilon + 2\varepsilon \liminf_n \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^+\right]\right)\,.
\end{aligned}$$

As a consequence

$$\mathbb{E}_{\mathbb{P}}[U(Z)] \geq B + 2\varepsilon\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z^j)^+\right] - \sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Z_n^j)^+\right]\right)\,.$$

Since the term multiplying ε is finite by hypothesis and the inequality holds for all $\varepsilon > 0$ we conclude that $\mathbb{E}_{\mathbb{P}}[U(Z)] \geq B$. □

2.6.3 Additional properties of Conjugates of Multivariate Utility Functions

Lemma 2.6.5.

1. *The conjugate V given in Definition 2.1.1 is convex and componentwise convex, where by the latter we mean that for every given $k \in \{1, \dots, N\}$ and $y \in \mathbb{R}^N$ the map over $\mathbb{R}$ defined by $z \mapsto V([y^{[-k]}; z])$ is convex. Moreover $V = +\infty$ on $\mathbb{R}^N \setminus [0, +\infty)^N$.*

2. *If U is differentiable, V is strictly convex and differentiable on the interior of its domain int(dom(V)) $= (0, +\infty)^N$. On $(0, +\infty)^N$, $\nabla V = -(\nabla U)^{-1}$ and for every sequence $(y_n)_n \subseteq$ int(dom(V)) converging to some element y in the boundary of int(dom(V))*

$$\lim_{n \to +\infty} \sum_{j=1}^{N} \left| \frac{\partial V}{\partial x^j}(y^n) \right| = +\infty \,.$$

Proof. Item 1: convexity and componentwise convexity are trivial. As to $V = +\infty$ on $\mathbb{R}^N \setminus [0, +\infty)^N$, take $y \in \mathbb{R}^N \setminus [0, +\infty)^N$ and let $y^k < 0$ (this must happen for at least one component). Then $V(y) \geq U(ne_k) - ny^k \uparrow_n +\infty$. The fact that the interior of *dom*(V) is $(0, +\infty)^N$ follows from what we just proved and from Remark 2.1.6. As to Item 2, differentiability, strict convexity and gradient property hold by [119] Theorem 26.5 applied to U, which is differentiable and strictly convex by assumption. □

Lemma 2.6.6. *If U is differentiable, the function V satisfies: for every $b \in (0, +\infty)^N$ and $a \in \partial((0, +\infty)^N)$*

$$\sum_{j=1}^{N} \frac{\partial V}{\partial x^j}(a + \lambda b)\, b^j \downarrow -\infty \quad \text{as } \lambda \downarrow 0 \,.$$

Proof. Follows from Lemma 26.2 in [119] setting "x"$=a$, "a"$=a+b$ and observing that V is differentiable. The fact that $\sum_{j=1}^{N} \frac{\partial V}{\partial x^j}(a + \lambda b)\, b^j$ decreases to $-\infty$ monotonically follows from convexity of $\lambda \mapsto V(a + \lambda b)$. □

Lemma 2.6.7. *Assume that U is differentiable and that for $\mathbb{Q} \ll \mathbb{P}$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in K_\Phi$ we have*

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\left[\lambda_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \lambda_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right] \right) \right] < +\infty$$

for all $\lambda_1, \dots, \lambda_N > 0$. Then the following hold:

1. *$\frac{\partial V}{\partial x^j}\left(\left[\lambda_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \lambda_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right] \right) \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P})$ for all $\lambda_1, \dots, \lambda_N > 0$.*

2. *If $g \in (L^0_+(\mathbb{P}))^N$ is such that $g^j + \frac{1}{g^j} \in L^\infty_+(\mathbb{P})$, $\forall\, j = 1, \dots, N$ and $\mathbb{Q} \sim \mathbb{P}$ then*

$$V\left(\left[g_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, g_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}} \right] \right) \in L^1(\mathbb{P}) \,.$$

Proof. Observe first that V is differentiable by lemma 2.6.5 Item 2.
Item 1: for every fixed $y \in (\mathbb{R}_{++})^N$, by componentwise convexity (see Lemma 2.6.5 Item 1)

$$\frac{\partial V}{\partial x^j}(y) \leq \frac{V([y^{[-j]}; \alpha y^j]) - V(y)}{(\alpha - 1)y^j} \ \ \forall \alpha \in (1, +\infty).$$

$$\frac{\partial V}{\partial x^j}(y) \geq \frac{V([y^{[-j]}; \alpha y^j]) - V(y)}{(\alpha - 1)y^j} \ \ \forall \alpha \in (0, 1].$$

The result then follows multiplying each term by y^j and replacing

$$y \rightsquigarrow \left[\lambda_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \lambda_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right].$$

Item 2: to begin with, observe that for any $z \in (0, +\infty)^N$ and $0 < \varepsilon < M$ the function $\alpha \mapsto \varphi(\alpha) := V([\alpha^1 z^1, \dots, \alpha^N z^N])$ on $[\varepsilon, M]^N$ is convex and continuous. By Bauer Maximum Principle (see [5] Theorem 7.69) φ has a maximum on an extreme point of $[\varepsilon, M]^N$, which is a point belonging to the set $\{\varepsilon, M\}^N$. We conclude that

$$\sup_{\alpha \in [\varepsilon, M]^N} \varphi(\alpha) \leq \sum_{\alpha \in \{\varepsilon, M\}^N} \varphi(\alpha).$$

Now observe that by hypothesis there exist $\varepsilon, M > 0$ such that for every $j = 1, \dots, N$ $\varepsilon \leq g^j \leq M$ $\mathbb{P}-$ almost surely. Hence

$$V\left(\left[g_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, g_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right]\right) \leq \sum_{\alpha \in \{\varepsilon, M\}^N} V\left(\left[\alpha^1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, \alpha^N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right]\right).$$

The term in RHS has finite expectation by hypotheses and V is bounded from below since $U(0) < +\infty$, so that we conclude $V\left(\left[g_1 \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}, \dots, g_N \frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}\right]\right) \in L^1(\mathbb{P})$. □

Lemma 2.6.8. *Let $u, \widetilde{u} : \mathbb{R} \to \mathbb{R}$ be convex, nondecreasing, differentiable on $\mathbb{R}$ and such that $u(0) = 0 = \tilde{u}(0)$. Assume $u \preceq \widetilde{u}$ (see Section 2.4.5 for the definition). Let $d, D > 0$ be given. Then there exist constants $K_1, K_2, \beta, B > 0, b \in \mathbb{R}$ such that, for v the convex conjugate of u,*

$$\sup_{x \in \mathbb{R}} (u(x) - xy + D\widetilde{u}(dx)) \geq \begin{cases} v\,(y) & 0 \leq y \leq K_1 \\ 0 & K_1 \leq y \leq K_2 \\ B\,v\,(\beta\,y) + b & y \geq K_2 \end{cases}. \tag{2.98}$$

Proof. We first observe that

$$\begin{aligned} &\sup_{x \in \mathbb{R}} (u(x) - xy + D\widetilde{u}(dx)) \\ &= \max\left(\sup_{x \leq 0} (u(x) - xy + D\widetilde{u}(dx)), \sup_{x \geq 0} (u(x) - xy + D\widetilde{u}(dx))\right). \end{aligned} \tag{2.99}$$

We work on the supremum over $(-\infty, 0]$: since $u \preceq \widetilde{u}$, we have that for constants $h, H \geq 0, b \in \mathbb{R}$

$$\sup_{x \leq 0} (u(x) - xy + D\widetilde{u}(dx)) \geq \sup_{x \leq 0} (u(x) - xy + DHu(dhx)) + b.$$

Setting $\beta_1 := \max(dh, 1)$ and $B_1 := \max(DH, 1)$ with simple computations $u(x) + DHu(dhx) \geq 2B_1 u(\beta_1 x) \forall x \leq 0$. Hence

$$\sup_{x\leq 0}\left(u(x) - xy + D\widetilde{u}(dx)\right) \geq \sup_{x\leq 0}\left(2B_1 u(\beta_1 x) - xy\right) + b = \sup_{x\leq 0}\left(2B_1 u(x) - x\frac{y}{\beta_1}\right) + b.$$

From concavity of u it is easy to see that $2B_1 u(x) - x\frac{y}{\beta} \leq (2B_1 u'(0) - \frac{y}{\beta_1})x$ for every $x \in \mathbb{R}$, where $u'(0)$ stands for the right derivative of u at 0 (which exists by concavity). This in turns implies that for $y \geq 2\beta_1 B_1 u'(0) =: K_2$ we have

$$\sup_{x\leq 0}\left(2B_1 u(x) - x\frac{y}{\beta_1}\right) = \sup_{x\in\mathbb{R}}\left(2B_1 u(x) - x\frac{y}{\beta_1}\right) = Bv\left(\beta y\right)$$

where we set $B := 2B_1$ and $\beta = \frac{1}{\beta_1 B}$.
We now move to the supremum over $[0, +\infty)$: by monotonicity and $\widetilde{u}(0) = 0$ we have

$$\sup_{x\geq 0}\left(u(x) - xy + D\widetilde{u}(dx)\right) \geq \sup_{x\geq 0}\left(u(x) - xy\right) .$$

It is then clear that, similarly to what we did before, for $y \leq u'(0) =: K_1$

$$\sup_{x\geq 0}\left(u(x) - xy\right) = \sup_{x\in\mathbb{R}}\left(u(x) - xy\right) = v(y)\,.$$

To sum up, from (2.99) we have then

$$\sup_{x\in\mathbb{R}}\left(u(x) - xy + D\widetilde{u}(dx)\right) \geq \begin{cases} v\,(y) & 0 \leq y \leq K_1 \\ B\,v\,(\beta\, y) + b & y \geq K_2 \end{cases} .$$

To conclude the proof, we just observe that $\sup_{x\in\mathbb{R}}\left(u(x) - xy + D\widetilde{u}(dx)\right) \geq 0$ for any $y \in [K_1, K_2]$. □

2.6.4 Results on Multivariate Orlicz Spaces

Proof of Proposition 2.2.4. We show that K_Φ is a subspace of the topological dual of L^Φ and is a subset of $(L^1(\mathbb{P}))^N$.
For $Z \in K_\Phi$ consider the well defined linear map $\phi : L^\Phi \to L^1(\mathbb{P})$, $X \mapsto \sum_{j=1}^N X^j Z^j$. Suppose $X_n \to X$ in L^Φ and $\phi(X_n) \to W$, then we can extract a subsequence (X_{n_k}) converging almost surely to X, since convergence in Luxemburg norm implies convergence in probability (Lemma 2.2.2 Item 5). It is then clear that $\phi(X_{n_k}) = \sum_{j=1}^N X^j_{n_k} Z^j \to_k \sum_{j=1}^N X^j Z^j = W\,\mathbb{P}-$a.s., thus the graph of ϕ is closed in $L^\Phi \times L^1(\mathbb{P})$ (endowed with product topology). By Closed Graph Theorem ([5] Theorem 5.20) the map is then continuous, thus any vector in K_Φ identifies a continuous linear functional on L^Φ. Finally since $[sign(Z^j)]_{j=1}^N \in (L^\infty(\mathbb{P}))^N \subseteq M^\Phi \subseteq L^\Phi$, $\sum_{j=1}^N |Z^j| \in L^1(\mathbb{P})$ yielding $K_\Phi \subseteq L^1(\mathbb{P})$. □

Proof of Proposition 2.2.5 Item 1. We show that for any extended real valued vector $Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right)$

$$\sup_{X\in L^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] = \sup_{X\in M^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] . \tag{2.100}$$

and that, moreover

$$K_\Phi = \left\{ Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \sum_{j=1}^N X^j Z^j \in L^1(\mathbb{P}),\, \forall\, X \in M^\Phi \right\}.$$

Argue as in Proposition 2.2.8 of [65]: take any $X \in L^\Phi$ and $Z \in (L^0(\mathbb{P}))^N$ and assume w.l.o.g. both are componentwise nonnegative (multiplying by signum functions will not affect Luxemburg norms by definition). Take sequences of simple functions $(Y^j_n)_n$, $j = 1, \dots, n$ each converging to X^j monotonically from below. Clearly $\|Y_n\|_\Phi \leq \|X\|_\Phi$ for each n and by Monotone Convergence Theorem

$$\mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] = \lim_n \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|Y^j_n Z^j\right|\right].$$

This implies that

$$\sup_{X \in L^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right]$$

$$\leq \sup_{X \in L^\infty, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] \leq \sup_{X \in M^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right]$$

since $(L^\infty(\mathbb{P}))^N \subseteq M^\Phi$. The converse inequality is evident, so that (2.100) follows. Now suppose

$$Z \in \left\{ Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \sum_{j=1}^N X^j Z^j \in L^1(\mathbb{P}),\, \forall\, X \in M^\Phi \right\}.$$

Observe (by using $|X^j|\, sgn(Z^j)$ in place of X^j in RHS below) that

$$\sup_{X \in M^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] = \sup_{X \in M^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N X^j Z^j\right] < +\infty.$$

where we used a Closed Graph Theorem argument similar to the one in the proof of Proposition 2.2.4, with M^Φ in place of L^Φ, to show finiteness of RHS: since $X \mapsto \sum_{j=1}^N X^j Z^j$ is well defined and continuous on M^Φ it must have finite operator norm, i.e. RHS. Now it follows that

$$\sup_{X \in L^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] \overset{(2.100)}{=} \sup_{X \in M^\Phi, \|X\|_\Phi \leq 1} \mathbb{E}_\mathbb{P}\left[\sum_{j=1}^N \left|X^j Z^j\right|\right] < +\infty$$

which in turns provides $Z \in K_\Phi$. □

Proof of Proposition 2.2.5 Item 2. We prove that the topological dual of $(M^\Phi, \|\cdot\|_\Phi)$ is $(K_\Phi, \|\cdot\|^*_\Phi)$. By order continuity, for a given linear functional ϕ in the topological dual of M^Φ we have that $A \mapsto \phi([0, \dots, 0, 1_A, 0, \dots, 0])$ defines a (finite) absolutely

continuous measure with respect to $\mathbb{P}$. This gives by Radon-Nikodym Theorem a vector $Z \in (L^1(\mathbb{P}))^N$ satisfying: for every vector of simple functions $s \in (L^\infty(\mathbb{P}))^N$ $\phi(s) = \sum_{j=1}^N \mathbb{E}_\mathbb{P}[s^j Z^j]$ We now prove that Z belongs to K_Φ: take $X \geq 0$ and a sequence $(Y_n)_n$ of non negative simple functions (vectors of simple functions more precisely) converging to X from below.
By order continuity of the topology on M^Φ we have

$$\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[sgn(Z^j)Y_n^j Z^j\right] = \phi\left(\left[sign(Z^j)Y_n^j\right]_{j=1}^N\right) \xrightarrow[n]{\|\cdot\|_\Phi} \phi\left(\left[sign(Z^j)X^j\right]_{j=1}^N\right) < +\infty\,.$$

Thus by Monotone Convergence Theorem

$$+\infty > \lim_n \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[sgn(Z^j)Y_n^j Z^j\right] = \lim_n \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y_n^j \left|Z^j\right|\right] = \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[X^j \left|Z^j\right|\right]\,.$$

This proves that $Z \in K_\phi$, since the argument above can be applied to any $0 \leq X \in M^\Phi$ and subsequently to any $X \in M^\Phi$. Finally, the norm we use on K_Φ is exactly the usual one for continuous linear functionals, so $(K_\Phi, \|\cdot\|_\Phi^*)$ is isometric to the topological dual of $(M^\Phi, \|\cdot\|_\Phi)$. □

Proof of Proposition 2.2.5 Item 3. We show that if we suppose

$$L^\Phi = L^{\Phi_1} \times \cdots \times L^{\Phi_N}\,, \tag{2.101}$$

then we have that $K_\Phi = L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}$. To see this, observe that

$$K_\Phi := \left\{Z \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \sum_{j=1}^N X^j Z^j \in L^1(\mathbb{P}),\, \forall\, X \in L^\Phi\right\}$$

$$\overset{(2.101)}{=} \left\{Z \in L^0\left(\mathbb{P}; [-\infty, +\infty]\right)^N \mid \sum_{j=1}^N X^j Z^j \in L^1(\mathbb{P}),\, \forall\, X \in L^{\Phi_1} \times \cdots \times L^{\Phi_N}\right\}$$

$$= \left\{Z \in L^0\left(\mathbb{P}; [-\infty, +\infty]\right)^N \mid X^j Z^j \in L^1(\mathbb{P}),\, \forall\, X^j \in L^{\Phi_j},\, \forall j = 1 \ldots, N\right\}\,.$$

Now apply Corollary 2.2.10 in [65] componentwise. □

Proof of Remark 2.2.9. To prove the claims, observe that $M^\Phi \subseteq M^{\Phi_1} \times \cdots \times M^{\Phi_N}$ follows from the fact that $\mathbb{E}_\mathbb{P}[\Phi_j(\lambda |X^j|)] \leq \mathbb{E}_\mathbb{P}[\Phi(\lambda |X|)]$, while the converse ($\supseteq$) is trivial.
We now prove inequalities (2.12). First observe that for $X \in M^\Phi$ and for every $j = 1, \ldots, N$ the functions $\gamma \mapsto \mathbb{E}_\mathbb{P}\left[\Phi(\frac{1}{\gamma}|X|)\right]$ and $\gamma \mapsto \mathbb{E}_\mathbb{P}\left[\Phi_j(\frac{1}{\gamma}|X^j|)\right]$ are continuous by Dominated Convergence Theorem, hence for $\|X\|_\Phi \neq 0$ and every $j = 1, \ldots, N$

$$\mathbb{E}_\mathbb{P}\left[\Phi_j\left(\frac{1}{\|X^j\|_{\Phi_j}}\left|X^j\right|\right)\right] \leq \mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\|X\|_\Phi}|X|\right)\right] = 1\,.$$

Since also for $\|X\|_\Phi = 0$ we have $X = 0$ and as a consequence $\|X^j\|_{\Phi_j} = 0$, $j = 1, \dots, N$, we have

$$\left\|X^j\right\|_{\Phi_j} \leq \|X\|_\Phi \; j = 1, \dots, N\,. \tag{2.102}$$

Moreover for $X \neq 0$ set $\lambda := \max_j \left(\|X^j\|_{\Phi_j}\right)$. Then

$$\mathbb{E}_\mathbb{P}\left[\Phi_j\left(\frac{1}{N\lambda}\left|X^j\right|\right)\right] \leq \frac{1}{N}\mathbb{E}_\mathbb{P}\left[\Phi_j\left(\frac{1}{\lambda}\left|X^j\right|\right)\right] \leq \frac{1}{N}\,.$$

Hence for $X \neq 0$

$$\|X\|_\Phi \leq N \max_j \left(\left\|X^j\right\|_{\Phi_j}\right)$$

and the same trivially holds for $X = 0$. In general then

$$\|X\|_\Phi \leq N \max_j \left(\left\|X^j\right\|_{\Phi_j}\right) \leq N \sum_{j=1}^N \left\|X^j\right\|_{\Phi_j}\,. \tag{2.103}$$

Now inequalities (2.12) follow from inequalities (2.102) and (2.103) and the claims are proved. □

Lemma 2.6.9. *Let $Z \in (L^1(\mathbb{P}))^N$ be such that for some $\lambda > 0$ $\mathbb{E}_\mathbb{P}[V(\lambda Z)] < +\infty$. Then $Z \in K_\Phi$.*

Proof. By definition of V we have for any $x, z \in \mathbb{R}^N$ $-\langle x, z\rangle \leq V(z) - U(X)$. Take Z with $\mathbb{E}_\mathbb{P}[V(\lambda Z)] < +\infty$ for some $\lambda > 0$. For any $X \in M^\Phi$ consider $\widehat{X}$ defined as

$$\widehat{X}^j := -sgn(X^j)sgn(Z^j)X^j,\; j = 1, \dots, N$$

and observe that $\widehat{X} \in M^\Phi$. Moreover we have $\lambda\langle|X|, |Z|\rangle = -\left\langle\widehat{X}, \lambda Z\right\rangle \leq V(\lambda Z) - U(\widehat{X})$. If $\widehat{X} \in M^\Phi$ then, by (2.10), $\mathbb{E}_\mathbb{P}\left[U(\widehat{X})\right] > -\infty$. Since $V(\lambda Z) \in L^1(\mathbb{P})$ by hypothesis, we conclude that $\langle X, Z\rangle \in L^1(\mathbb{P})$ for every $X \in M^\Phi$, which in turns yields $Z \in K_\Phi$ by Proposition 2.2.5 Item 1. □

Sequential w*-compactness in Orlicz Spaces

The following is partly inspired by [55], page 26, Chap. II, proof of Theorem 24. A similar result is stated in [111], proof of Theorem 1, with a more technical (even though shorter) proof. Throughout Section 2.6.4 we will put more emphasis on sigma algebras rather than on probability measures (we work under $\mathbb{P}$). Thus, when considering Orlicz spaces and Orlicz Hearts as well as Lebesgue spaces, we will explicitly mention the underlying sigma algebra ($\mathcal{F}$ up to now, and we will introduce $\mathcal{G} \subseteq \mathcal{F}$ soon).

Proposition 2.6.10. *On a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that Φ, Φ^* are (univariate) conjugate Young functions, both everywhere finite valued. Then the balls in $L^\Phi(\mathcal{F})$, endowed with Orlicz norm, are $\sigma\left(L^\Phi(\mathcal{F}), M^{\Phi^*}(\mathcal{F})\right)$ sequentially compact.*

Proof. First recall that under these assumptions $L^\infty(\mathcal{F}) \subseteq M^{\Phi^*}(\mathcal{F}) \neq \{0\}$, $M^{\Phi^*}(\mathcal{F})$ is order continuous and the norm dual of $M^{\Phi^*}(\mathcal{F})$ is isometric to $L^\Phi(\mathcal{F})$, endowed with the Orlicz norm $\|\cdot\|_{L^\Phi(\mathcal{F})}$. Consider a ball

$$B_r(\mathcal{F}) := \left\{ X \in L^\Phi(\mathcal{F}) \mid \| X \|_{L^\Phi(\mathcal{F})} \leq r \right\}$$

and a sequence $(X_n)_n \subseteq B_r(\mathcal{F})$. Observe that, by Banach Alaoglu Theorem ([5] Theorem 6.21) $B_r(\mathcal{F})$ is w^*-compact (i.e. $\sigma\left(L^\Phi(\mathcal{F}), M^{\Phi^*}(\mathcal{F})\right)$-compact), hence $B_r(\mathcal{F})$ is also w^*-closed, as $\left(L^\Phi(\mathcal{F}), \sigma\left(L^\Phi(\mathcal{F}), M^{\Phi^*}(\mathcal{F})\right)\right)$ is a Hausdorff topological space. We now prove that there exists a subsequence of $(X_n)_n$ converging in the w^*-topology to an element $X \in L^\Phi(\mathcal{F})$, which then implies the thesis, as $B_r(\mathcal{F})$ is w^*-closed.
Set $\mathcal{G} := \sigma((X_n)_n)$ and observe that $\mathcal{G}$ is countably generated ([55] page 10, Chap. I, for definitions and page 26, Chap. 2, in the proof of Theorem 24). Then a standard argument yields that $M^\Phi(\mathcal{G})$ and $M^{\Phi^*}(\mathcal{G})$ are separable. Therefore, the w^*-topology $\sigma(L^\Phi(\mathcal{G}), M^{\Phi^*}(\mathcal{G}))$ on balls $B_r(\mathcal{G}) \subseteq L^\Phi(\mathcal{G})$, is metrizable (Theorem 6.30 [5]). Applying again the Banach Alaoglu Theorem, we deduce that the balls $B_r(\mathcal{G})$ are also $\sigma(L^\Phi(\mathcal{G}), M^{\Phi^*}(\mathcal{G}))$ compact, hence sequentially $\sigma(L^\Phi(\mathcal{G}), M^{\Phi^*}(\mathcal{G}))$-compact, by metrizability of the w^*-topology on $B_r(\mathcal{G})$ ([5] Theorem 6.30). As Φ is convex and increasing on $\mathbb{R}_+$, by Jensen inequality we obtain

$$\mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\lambda}\left|\mathbb{E}_\mathbb{P}\left[X|\mathcal{G}\right]\right|\right)\right] \leq \mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\lambda}|X|\right)\right]$$

and it follows that

$$\|\mathbb{E}_\mathbb{P}\left[X|\mathcal{G}\right]\|_{\Phi(\mathcal{G})} \leq \|X\|_{\Phi(\mathcal{F})},$$

where

$$\|X\|_{\Phi(\cdot)} := \inf\left(\lambda > 0 \mid \mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\lambda}|X|\right)\right] \leq 1\right)$$

is the Luxemburg norm in $L^\Phi(\cdot)$. Consider the conditional operator T

$$T : \left(L^\Phi(\mathcal{F}), \|\cdot\|_{L^\Phi(\mathcal{F})}\right) \to \left(L^\Phi(\mathcal{G}), \|\cdot\|_{L^\Phi(\mathcal{G})}\right)$$
$$X \mapsto T(X) := \mathbb{E}_\mathbb{P}\left[X|\mathcal{G}\right].$$

By the equivalence of the Orlicz norm with the Luxemburg norm, T is then well defined and norm-continuous:

$$\| T(X) \|_{L^\Phi(\mathcal{G})} \leq K \| X \|_{L^\Phi(\mathcal{F})} \tag{2.104}$$

for some positive constant K. As $X_n \in L^\Phi(\mathcal{F})$ and X_n is $\mathcal{G}$-measurable, $X_n \in L^\Phi(\mathcal{G})$. As $X_n \in B_r(\mathcal{F})$, then, $X_n = \mathbb{E}_\mathbb{P}[X_n|\mathcal{G}] = T(X_n) \in B_{Kr}(\mathcal{G})$, by (2.104). By the sequential compactness of $B_{Kr}(\mathcal{G})$ proven above, we can extract a subsequence $(X_{n_k})_k$ that is $\sigma(L^\Phi(\mathcal{G}), M^{\Phi^*}(\mathcal{G}))$-converging to some $X \in L^\Phi(\mathcal{G})$.
Now for every $W \in M^{\Phi^*}(\mathcal{F})$ we have that $\mathbb{E}_\mathbb{P}[W|\mathcal{G}] \in M^{\Phi^*}(\mathcal{G})$ (because of the inequality $\mathbb{E}_\mathbb{P}\left[\Phi^*\left(\lambda\left|\mathbb{E}_\mathbb{P}[W|\mathcal{G}]\right|\right)\right] \leq \mathbb{E}_\mathbb{P}\left[\Phi^*\left(\lambda|W|\right)\right]$), and from $(X_{n_k})_k \to X$ with respect to $\sigma(L^\Phi(\mathcal{G}), M^{\Phi^*}(\mathcal{G}))$ we obtain:

$$\mathbb{E}_\mathbb{P}[X_{n_k}W] = \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}[X_{n_k}W|\mathcal{G}]\right] =$$
$$\mathbb{E}_\mathbb{P}\left[X_{n_k}\mathbb{E}_\mathbb{P}[W|\mathcal{G}]\right] \to_n \mathbb{E}_\mathbb{P}\left[X\mathbb{E}_\mathbb{P}[W|\mathcal{G}]\right] = \mathbb{E}_\mathbb{P}[XW],$$

so that

$$(X_{n_k})_k \to X \text{ in } \sigma(L^\Phi(\mathcal{F}), M^{\Phi^*}(\mathcal{F})).$$

□

2.6.5 On Komlós Theorem

We now recall the original Komlós Theorem:

Theorem 2.6.11 (Komlós). *Let $(f_n)_n \subseteq L^1(\mathbb{P})$ be a sequence with bounded $L^1(\mathbb{P})$-norms. Then there exists a subsequence $(f_{n_k})_k$ and a g again in $L^1(\mathbb{P})$ such that for any further subsequence the Césaro means satisfy:*

$$\frac{1}{N}\sum_{i\leq N} f_{n_{k_i}} \to g \ \ \mathbb{P}- \ \ a.s. \ \ as \ N\to+\infty\,.$$

Proof. See [103] Theorem 1a. □

Corollary 2.6.12. *Let a sequence $(Y_n)_n$ be given in $L^1(\mathbb{P}_1)\times\dots\times L^1(\mathbb{P}_N)$ such that for probabilities $\mathbb{P}_1,\dots,\mathbb{P}_N \ll \mathbb{P}$*

$$\sup_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left|Y_n^j\right|\frac{\mathrm{d}\mathbb{P}_j}{\mathrm{d}\mathbb{P}}\right] < \infty\,.$$

Then there exists a subsequence $(Y_{n_h})_h$ and an $\widehat{Y}\in L^1(\mathbb{P}_1)\times\dots\times L^1(\mathbb{P}_N)$ such that every further subsequence $(Y_{n_{h_k}})_k$ satisfies

$$\frac{1}{K}\sum_{k=1}^K Y^j_{n_{h_k}} \to \widehat{Y}^j \ \ \mathbb{P}_j - a.s. \ \ \forall\, j=1,\dots N \ \ as \ K\to+\infty\,.$$

Proof. We suppose $N=2$, the argument can be iterated. The result follows from a diagonal argument: take the first component, we have a subsequence and an $\widehat{Y}^1$ s.t. each further subsequence has $\mathbb{P}_1-$a.s. converging Césaro means as in Theorem 2.6.11. Now take this sequence in place of the one we began with, and do the same for the second component. Notice that in the end we get a subsequence for the second component too, and the corresponding indices yield a subsequence of the one we extracted for the first component. The claim follows. □

2.6.6 Integrability Issues

The following is a variant of Theorem 1.5.4 in Chapter 1.

Theorem 2.6.13. *Under Standing Assumption I and Assumption 2.3.5, let $K\subseteq M^\Phi$ be a convex cone such that for all $i,j\in\{1,\dots,N\}$ $e_i-e_j\in K$ and consider the subset of K_Φ defined by*

$$\mathcal{S}^e_V := \left\{\mathbb{Q}\mid \mathbb{Q}\sim\mathbb{P},\, \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\in K_\Phi,\, \mathbb{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right]<+\infty,\, \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[k^j\right]\leq 0\,\forall k\in K\right\}.$$

Suppose $\widehat{Y}\in (L^0(\mathbb{P}))^N$ satisfies:

1. *for some $\widehat{\mathbb{Q}}\in\mathcal{S}^e_V$*

$$\left[\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]_{j=1}^N \in (L^1(\mathbb{P}))^N,\ \ \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]=0\,.$$

2. for all $\mathbb{Q} \in \mathcal{S}_V^e$

$$\sum_{j=1}^{N} \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^1(\mathbb{P}), \quad \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}}\right] \leq 0\,.$$

Then $\widehat{Y}$ *is in the* $L^1(\widehat{\mathbb{Q}}^1) \times \cdots \times L^1(\widehat{\mathbb{Q}}^N)$*-norm closure of* K*. In particular,* $\widehat{Y}$ *is in the closure of* K *under convergence in probability* $\mathbb{P}$ *.*

Proof. We first prove by contradiction that $\widehat{Y}$ belongs to the $L^1(\widehat{\mathbb{Q}})$-norm closure of $K - L^\infty_+(\widehat{\mathbb{Q}}) = K - (L^\infty_+(\mathbb{P}))^N$ (equality holds by equivalence of the probabilities). Suppose this were not the case. Then $\widehat{Y} \notin cl_{\widehat{\mathbb{Q}}}\left(K - (L^\infty_+(\mathbb{P}))^N\right)$, which is norm closed and convex (being closure of a convex sets). By convexity, $cl_{\widehat{\mathbb{Q}}}\left(K - (L^\infty_+(\mathbb{P}))^N\right)$ is also closed in the topology induced on $L^1(\widehat{\mathbb{Q}}^1) \times \cdots \times L^1(\widehat{\mathbb{Q}}^N)$ by the pairing

$$\left(L^1(\widehat{\mathbb{Q}}^1) \times \cdots \times L^1(\widehat{\mathbb{Q}}^N), L^\infty(\widehat{\mathbb{Q}}^1) \times \cdots \times L^\infty(\widehat{\mathbb{Q}}^N)\right)\,.$$

As a consequence, we can apply Hahn-Banach Separation Theorem to get a $\xi \in L^\infty(\widehat{\mathbb{Q}}) = (L^\infty(\mathbb{P}))^N$ with

$$0 = \sup_{k \in K - (L^\infty_+(\mathbb{P}))^N} \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} \xi^j k^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] < \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} \xi^j \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]\,. \tag{2.105}$$

We now work componentwise. First observe that

$$[-1_{\xi^j<0}]_{j=1}^N \in 0 - (L^\infty_+(\mathbb{P}))^N \subseteq K - (L^\infty_+(\mathbb{P}))^N$$

so that $\xi^j \geq 0$ $\widehat{\mathbb{Q}}^j$ (hence $\mathbb{P}$)-a.s. for every $j = 1, \dots, N$. Hence $\xi^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} \geq 0$ $\mathbb{P}$-a.s. for every $j = 1, \dots, N$.
Moreover since $e_i - e_j \in K$ for all $i, j \in \{1, \dots, N\}$ we have

$$\mathbb{E}_{\mathbb{P}}\left[\xi^1 \frac{\mathrm{d}\widehat{\mathbb{Q}}^1}{\mathrm{d}\mathbb{P}}\right] = \cdots = \mathbb{E}_{\mathbb{P}}\left[\xi^N \frac{\mathrm{d}\widehat{\mathbb{Q}}^N}{\mathrm{d}\mathbb{P}}\right]\,. \tag{2.106}$$

It follows that for every $j = 1, \dots, N$

$$\mathbb{P}\left(\xi^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} > 0\right) > 0$$

since, if this were not the case, one of the values in the chain in Equation (2.106) would be 0, hence all of them would be 0. Non negativity would then imply that $\xi^1 \frac{\mathrm{d}\widehat{\mathbb{Q}}^1}{\mathrm{d}\mathbb{P}} = \cdots = \xi^N \frac{\mathrm{d}\widehat{\mathbb{Q}}^N}{\mathrm{d}\mathbb{P}} = 0$, which yields a contradiction with strict inequality in Equation (2.105). We conclude that the vector

$$\frac{\mathrm{d}\mathbb{Q}_1^j}{\mathrm{d}\mathbb{P}} := \frac{1}{\mathbb{E}_{\mathbb{P}}\left[\xi^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right]} \xi^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}$$

is well defined and identifies a vector of probability measures $\mathbb{Q}_1 = [\mathbb{Q}_1^1, \dots, \mathbb{Q}_1^N] \ll \mathbb{P}$. Observe that $\xi \in (L^\infty(\mathbb{P}))^N$ and $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \in K_\Phi$ implies

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} \in K_\Phi \,.$$

Equations (2.105) and (2.106) yield

$$\sup_{k\in K}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[k^j \frac{\mathrm{d}\mathbb{Q}_1^j}{\mathrm{d}\mathbb{P}}\right]\right) \le 0 < \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}_1^j}{\mathrm{d}\mathbb{P}}\right]. \tag{2.107}$$

We observe that if we could prove $\mathbb{Q}_1 \in \mathcal{S}_V^e$, we would get a contradiction with Item 2 in the hypothesis. However this needs not to be true, and some more work is necessary, as shown in the subsequent arguments.
Let us now fix $k \in \{1, \dots, N\}$, and observe that since $\widehat{\mathbb{Q}} \in \mathcal{S}_V^e$ we have $\widehat{\mathbb{Q}} \sim \mathbb{P}$, and for $\mathbb{Q}_1$ above we have $\mathbb{Q}_1^k \ll \widehat{\mathbb{Q}}^k$, $\frac{\mathrm{d}\mathbb{Q}_1^k}{\mathrm{d}\widehat{\mathbb{Q}}^k} = \xi^k \in L^\infty(\mathbb{Q}) = (L^\infty(\mathbb{P}))^N$. Take $\lambda \in [0,1)$ and define $\mathbb{Q}_\lambda \sim \mathbb{P}$ via

$$\frac{\mathrm{d}\mathbb{Q}_\lambda^k}{\mathrm{d}\mathbb{P}} := (1-\lambda)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}} + \lambda \frac{\mathrm{d}\mathbb{Q}_1^k}{\mathrm{d}\mathbb{P}} \,.$$

It is easy to check that

$$0 < 1-\lambda \le \frac{\mathrm{d}\mathbb{Q}_\lambda^k}{\mathrm{d}\widehat{\mathbb{Q}}^k} \le \lambda \frac{\mathrm{d}\mathbb{Q}_1^k}{\mathrm{d}\widehat{\mathbb{Q}}^k} + (1-\lambda) \,.$$

Apply Lemma 2.6.7 Item 2 with $g^k = \frac{\mathrm{d}\mathbb{Q}_\lambda^k}{\mathrm{d}\widehat{\mathbb{Q}}^k}$ and $g^k \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}_\lambda^k}{\mathrm{d}\mathbb{P}}$, $k = 1, \dots, N$, to deduce from $\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty$ that

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mathbb{Q}_\lambda}{\mathrm{d}\mathbb{P}}\right)\right] < +\infty, \;\; \forall\, \lambda \in [0,1) \,.$$

Moreover by $\widehat{\mathbb{Q}} \in \mathcal{S}_V^e$ and Equation (2.107)

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[k^j \frac{\mathrm{d}\mathbb{Q}_\lambda^j}{\mathrm{d}\mathbb{P}}\right] \le 0 \;\; \forall k \in K, \forall\, \lambda \in [0,1)$$

which yields $\mathbb{Q}_\lambda \in \mathcal{S}_V^e$, $\forall \lambda \in [0,1)$. At the same time

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}_\lambda^j}{\mathrm{d}\mathbb{P}}\right] &= (1-\lambda)\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] + \lambda \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}_1^j}{\mathrm{d}\mathbb{P}}\right] \\ &\xrightarrow[\lambda\to 1]{} \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\mathbb{Q}_1^j}{\mathrm{d}\mathbb{P}}\right] \overset{\text{Eq.(2.105)}}{>} 0 \,, \end{aligned}$$

which is a contradiction (with Item 2 in the hypotheses).

Now we prove that in fact $\widehat{Y} \in cl_{\widehat{\mathbb{Q}}}(K)$: observe that there exists sequences $(k_n)_n \subseteq K$ and $(f_n)_n \in (L^\infty_+(\mathbb{P}))^N$ such that $(k_n - f_n) \to \widehat{Y}$ in $L^1(\widehat{\mathbb{Q}})$-norm and $\mathbb{P}$-almost surely. Now we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N k_n^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] - \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N f_n^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \left(k_n^j - f_n^j\right) \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \to_n \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \widehat{Y}^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] = 0 \end{aligned} \tag{2.108}$$

by Item 1 in the hypothesis. As $\widehat{\mathbb{Q}} \in \mathcal{S}_V^e$ the first sum in LHS of (2.108) is non positive, while the second summation is non negative ($f_n \in (L^\infty_+(\mathbb{P}))^N$). We then get that both these summations tend to zero. In particular

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N f_n^j \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] = \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \left|f_n^j\right| \frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right] \to 0 \text{ as } n \to +\infty\,.$$

Thus $f_n \to_n 0$ in $L^1(\widehat{\mathbb{Q}})$-norm, which gives us that $\widehat{Y}$ is the $L^1(\widehat{\mathbb{Q}})$-norm limit of a sequence in K. Finally, $\widehat{Y}$ is in the closure under convergence in probability $\mathbb{P}$ of K: just extract a subsequence $(k_{n_h})_h$ with $k^j_{n_h} \to \widehat{Y}^j\ \widehat{\mathbb{Q}}^j -$a.s. for every $j = 1, \dots, N$ and notice that $\widehat{\mathbb{Q}}^j \sim \mathbb{P}$ for each $j = 1, \dots, N$. □

Chapter 3

Dynamic Systemic Risk Measures

In Chapter 3 we investigate to which extent the relevant features of (static) Systemic Risk Measures can be extended to a conditional setting. After providing a general dual representation result of the form

$$\rho(X) = \max_{\substack{\mathbb{Q}\in\mathcal{M}_1 \\ \mathbb{Q}=[\mathbb{Q}^1,\dots,\mathbb{Q}^N]}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_j}\left[-X^j \middle| \mathcal{G}\right] - \rho^*(-\mathbb{Q}) \right) \quad X \in (L^\infty(\Omega,\mathcal{F},\mathbb{P}))^N$$

where

$$\rho^*(-\mathbb{Q}) = \sup_{X\in(L^\infty(\Omega,\mathcal{F},\mathbb{P}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}_j}\left[-X^j \middle| \mathcal{G}\right] - \rho(X) \right)$$

we analyze in greater detail Conditional Shortfall Systemic Risk Measures. In the particular case of exponential preferences, we provide explicit formulas that also allow us to show a time consistency property. Finally, we provide an interpretation of the allocations associated to Conditional Shortfall Systemic Risk Measures as suitably defined equilibria. Conceptually, the generalization from static to Conditional Systemic Risk Measures can be achieved in a natural way, even though the proofs become more technical than in the unconditional framework.
As to the structure of Chapter 3, in Section 3.1 we recap some results for the static setup and present the general conditional setup in Section 3.2, where we also show a general dual representation result (Theorem 3.2.9). In Section 3.3 we recall some previously obtained results on multivariate utility functions and Orlicz spaces, while Section 3.4 collects definitions and results on Conditional Shortfall Systemic Risk Measures (Definition 3.4.3 and Theorem 3.4.4). The case of exponential utility functions is treated in Section 3.5, where we provide explicit formulas (Theorem 3.5.6) and we study time consistency properties (Theorem 3.5.8). Finally, in Section 3.6 we generalize mSORTE to the conditional setup and study its relations with Conditional Shortfall Systemic Risk Measures and its time consistency properties in the exponential case. Section 3.7 is the Appendix to Chapter 3.

3.1 Static Setup

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We take two vector subspaces of $(L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$, call them $L_{\mathcal{F}}$, L^*.

Definition 3.1.1. *We say that a proper convex functional $F : L_{\mathcal{F}} \to \mathbb{R}$ is* ***nicely representable*** *if Fenchel-Moreau Theorem holds for F with respect to the $\sigma(L_{\mathcal{F}}, L^*)$ topology and in the representation of F the supremum is a maximum, that is*

$$F(X) = \max_{Y \in L^*} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j Y^j \right] - F^*(Y) \right) \quad \forall X \in L_{\mathcal{F}}$$

where

$$F^*(Y) = \sup_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j Y^j \right] - F(X) \right) \quad Y \in L^* .$$

Remark 3.1.2. Some sufficient conditions for nice representability are:

1. $L_{\mathcal{F}} = (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ $L^* = (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$, $F(\cdot)$ is monotone, convex, monetary and satisfies the Lebesgue property: see [77], Corollary 4.35 for $N = 1$ and Theorem 3.1.4 for $N > 1$.

2. $L_{\mathcal{F}} = M^\Phi \neq \emptyset$, $L^* = L^{\Phi^*}$ (see Section 2.2 for the definitions), $F(\cdot)$ is monotone, convex, monetary: in this case F is real valued and Extended Namioka-Klee Theorem of [23] applies.

We will now extend classical results to our systemic setup. Only slight modifications are needed in the proofs, but we add them for sake of completeness.

Definition 3.1.3. *A functional $\rho : (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N \to \mathbb{R}$ will be called* ***(static) Systemic Risk Measure*** *if it satisfies:* ***Monotonicity****, that is $X \leq Y$ componentwise $\Rightarrow$ $\rho(X) \geq \rho(Y)$,* ***Convexity****, that is $0 \leq \lambda \leq 1$ $\Rightarrow \rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ and* ***Monetary property*** *(or Cash Additivity), that is $X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N, c \in \mathbb{R}^N$ $\Rightarrow$ $\rho(X + c) = \rho X - \sum_{j=1}^{N} c^j$.*

We will denote by ba_1 the set of N-dimensional vectors of finitely additive functionals on $\mathcal{F}$ taking values in $[0, 1]$ and taking value 1 on Ω, and by $\mathcal{M}_1$ the set of vectors of probability measures $\{\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \mid \mathbb{Q} \ll \mathbb{P}\}$.We can now state the following Theorem, which generalizes well known results in the one dimensional case (see [77]).

Theorem 3.1.4. *Let $\rho : (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N \to \mathbb{R}$ be a (static) Systemic Risk Measure. Suppose additionally that ρ is continuous from below, that is $X_n \uparrow_n X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ for a sequence $(X_n)_n \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N$ implies $\rho(X) = \lim_n \rho(X_n)$. Then ρ has the following dual representation:*

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{M}_1} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}_j} \left[-X^j \right] - \rho^*(-\mathbb{Q}) \right) \quad X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N \tag{3.1}$$

where

$$\rho^*(-\mathbb{Q}) = \sup_{X \in (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^N} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}_j} \left[-X^j \right] - \rho(X) \right) .$$

Proof. See Section 3.7.2. □

Corollary 3.1.5. *For $Y \in (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$ we set*

$$\rho^*(Y) := \sup_{X \in (L^\infty(\Omega,\mathcal{F},\mathbb{P}))^N} \left(\mathbb{E}_{\mathbb{P}}[XY] - \rho(X)\right) .$$

In the hypotheses and notation of Theorem 3.1.4, we have

$$\rho(X) = \max_{Y \in (L^1(\Omega,\mathcal{F},\mathbb{P}))^N} \left(\mathbb{E}_{\mathbb{P}}[XY] - \rho^*(Y)\right) .$$

Proof. See Section 3.7.2. □

3.2 Conditional Systemic Risk Measures

3.2.1 Setup and Notation

We let $\mathcal{G} \subseteq \mathcal{F}$ be a sub sigma algebra. Throughout all Chapter 3 we will often need to change underlying sigma algebras. In order to avoid unnecessarily heavy notation, we will explicitly specify the one or the other only when some confusion might arise. For example, $L^\infty(\mathcal{F})$, $L^\infty(\mathcal{G})$ stand for $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ respectively.

Remark 3.2.1. In the following (MON) and (DOM) are references to Monotone and Dominated Convergence Theorem respectively. (cMON) and (cDOM) refer to their conditional counterparts. We will use without explicit mention the properties of essential suprema (and essential infima) collected in Section 3.7.1.

Definition 3.2.2. *$L_\mathcal{F}$ is $\mathcal{G}$-decomposable if $(L^\infty(\mathcal{F}))^N \subseteq L_\mathcal{F}$ and for any vectors $Y \in (L^\infty(\mathcal{G}))^N$ and $X \in L_\mathcal{F}$, the vector Z defined as $Z^j = X^j Y^j$, $j = 1, \dots, N$ belongs to $L_\mathcal{F}$.*

Remark 3.2.3. Observe that by decomposability whenever $A \in \mathcal{G}$ and $X, Y \in L_\mathcal{F}$ we also have $X1_A + Y1_{A^c} \in L_\mathcal{F}$. We stress the fact that $\mathcal{G}$-decomposability is a very mild requirement, which is clearly satisfied for example if $L_\mathcal{F} = (L^p)^N$ for some $p \in [1, +\infty]$ or $L_\mathcal{F}$ is an Orlicz space (see Section 2.2).

Definition 3.2.4. *A subset $\mathcal{C} \subseteq L_\mathcal{F}$ is:*

- *$\mathcal{G}-$**conditionally convex** if for any $\lambda \in L^0(\mathcal{G})$, $0 \leq \lambda \leq 1$ and any $X, Y \in \mathcal{C}$ $\lambda X + (1-\lambda)Y \in \mathcal{C}$.*
- ***a** $\mathcal{G}-$**conditional cone** if for any $0 \leq \lambda \in L^\infty(\mathcal{G})$ and any $X \in \mathcal{C}$, $\lambda X \in \mathcal{C}$.*
- ***closed under** $\mathcal{G}-$ **truncation** if for any $Y \in \mathcal{C}$ there exists $k_Y \in \mathbb{N}$ and a $Z_Y \in L^\infty(\mathcal{F})$ such that $\sum_{j=1}^N Z_Y^j = \sum_{j=1}^N Y^j$ and for any $k \geq k_Y$, $k \in \mathbb{N}$*

$$Y_{(k)} := Y 1_{\bigcap_j \{|Y^j| \leq k\}} + Z_Y 1_{\bigcup_j \{|Y^j| > k\}} \in \mathcal{C}. \tag{3.2}$$

We will explicitly specify the sigma algebra ($\mathcal{G}$ in the notation above) with respect to which the properties are required to hold only when some confusion might arise.

Definition 3.2.5. *A map $\rho_\mathcal{G}(\cdot) : L_\mathcal{F} \to L^1(\mathcal{G})$ is a **Conditional Systemic Risk Measure** (CSRM) if it satisfies*

1. ***Monotonicity***, *that is*

$$X \leq Y \text{ componentwise } \mathbb{P} - a.s. \;\Rightarrow\; \rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y) \;\; \mathbb{P} - a.s. \tag{3.3}$$

2. ***Conditional Convexity***, *that is for every* $X, Y \in L_{\mathcal{F}}$

$$0 \leq \lambda \leq 1, \lambda \in L^{\infty}(\mathcal{G}) \;\Rightarrow \rho_{\mathcal{G}}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{\mathcal{G}}(X) + (1-\lambda)\rho_{\mathcal{G}}(Y) \;\; \mathbb{P}-a.s. \tag{3.4}$$

3. ***Conditional $\mathcal{G}$-Additivity*** *(or conditional Monetary property), that is*

$$X \in (L^{\infty}(\mathcal{F}))^{N}, Y \in (L^{\infty}(\mathcal{G}))^{N} \;\Rightarrow\; \rho_{\mathcal{G}}(X+Y) = \rho_{\mathcal{G}}(X) - \sum_{j=1}^{N} Y^{j} \;\mathbb{P} - a.s. \tag{3.5}$$

Definition 3.2.6. *For the particular choice* $L_{\mathcal{F}} = (L^{\infty}(\mathcal{F}))^{N}$ *we say that a CSRM* $\rho_{\mathcal{G}}(\cdot) : (L^{\infty}(\mathcal{F}))^{N} \to L^{0}(\mathcal{G})$ *is*

- ***continuous from above*** *if for any sequence* $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^{N}$ *and* $X \in (L^{\infty}(\mathcal{F}))^{N}$ *such that for each* $j = 1, \ldots, N$ $X_n^j \downarrow_n X^j$ *we have* $\rho_{\mathcal{G}}(X_n) \uparrow_n \rho_{\mathcal{G}}(X)$ $\mathbb{P}-a.s.$

- ***continuous from below*** *if for any sequence* $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^{N}$ *and* $X \in (L^{\infty}(\mathcal{F}))^{N}$ *such that for each* $j = 1, \ldots, N$ $X_n^j \uparrow_n X^j$ *we have* $\rho_{\mathcal{G}}(X_n) \downarrow_n \rho_{\mathcal{G}}(X)$ $\mathbb{P}-a.s.$

- ***Lebesgue continuous*** *(or that* $\rho_{\mathcal{G}}(\cdot)$ *has the Lebesgue property) if for any sequence* $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^{N}$ *and* $X \in (L^{\infty}(\mathcal{F}))^{N}$ *such that for each* $j = 1, \ldots, N$ $\sup_n \|X_n^j\|_{\infty} < +\infty$ *and* $X_n^j \to_n X^j$ $\mathbb{P}-a.s.$ *we have* $\rho_{\mathcal{G}}(X_n) \to_n \rho_{\mathcal{G}}(X)$ $\mathbb{P}-a.s.$

Remark 3.2.7. Observe that continuity from above and continuity from below of a CSRM $\rho_{\mathcal{G}}(\cdot)$ yield the Lebesgue property. This can be seen, analogously to the classical case $N = 1$, as follows: assuming continuity from above and from below (c.a. and c.b. respectively in short), take a sequence $(X_n)_n \subseteq (L^{\infty}(\mathcal{F}))^{N}$ and $X \in (L^{\infty}(\mathcal{F}))^{N}$ such that for each $j = 1, \ldots, N$ $\sup_n \|X_n^j\|_{\infty} < +\infty$ and $X_n^j \to_n X^j$ $\mathbb{P}-$a.s. Then we have

$$\liminf_n \rho_{\mathcal{G}}(X_n) \geq \lim_n \rho_{\mathcal{G}}\left(\sup_{N \geq n} X_N\right) \stackrel{\text{c.a.}}{=} \rho_{\mathcal{G}}\left(\inf_n \sup_{N \geq n} X_N\right) = \rho_{\mathcal{G}}(X)$$

where the suprema and infima for vectors are taken componentwise. Similarly using c.b. we get $\rho_{\mathcal{G}}(X) \geq \limsup_n \rho_{\mathcal{G}}(X_n)$. These inequalities together yield

$$\rho_{\mathcal{G}}(X) \geq \limsup_n \rho_{\mathcal{G}}(X_n) \geq \liminf_n \rho_{\mathcal{G}}(X_n) \geq \rho_{\mathcal{G}}(X)$$

and the convergence follows. Also, the norm boundedness of the sequence together with (3.5) implies that the convergence $\rho_{\mathcal{G}}(X_n) \to_n \rho_{\mathcal{G}}(X)$ is dominated in $L^{\infty}(\mathcal{G})$.

3.2.2 Dual Representation of Conditional Systemic Risk Measures

This section follows the lines of the scalarization procedure in [57] and [112].

Assumption 3.2.8. *We assume that $L_{\mathcal{F}}$ is $\mathcal{G}$-decomposable and that $\sum_{j=1}^{N} X^j Z^j \in L^1(\mathcal{F})$ for any $X \in L_{\mathcal{F}}$, $Z \in L^*$.*

Theorem 3.2.9. *Suppose Assumption 3.2.8 holds and $\rho_{\mathcal{G}}(\cdot) : L_{\mathcal{F}} \to L^1(\mathcal{G})$ satisfies Monotonicity, Conditional Convexity and Conditional Additivity (that is, $\rho_{\mathcal{G}}(\cdot)$ is a CSRM). Suppose $\rho_0(\cdot) := \mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(\cdot)] : L_{\mathcal{F}} \to \mathbb{R}$ is nicely representable.*
Define

$$\rho^*_{\mathcal{G}}(Y) := \operatorname*{ess\,sup}_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X) \right). \tag{3.6}$$

Then $\rho_{\mathcal{G}}(\cdot)$ admits the following dual representation:

$$\rho_{\mathcal{G}}(X) = \operatorname*{ess\,sup}_{Y \in L^*} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j | \mathcal{G}\right] - \rho^*_{\mathcal{G}}(Y) \right). \tag{3.7}$$

Furthermore, there exists $\widehat{Y} \in L^*$ such that

$$\widehat{Y}^j \leq 0, \quad \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j \middle| \mathcal{G}\right] = -1 \;\; \forall\, j = 1, \ldots, N,$$

$$\rho_{\mathcal{G}}(X) = \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j \middle| \mathcal{G}\right] - \rho^*_{\mathcal{G}}(\widehat{Y}). \tag{3.8}$$

Proof.
STEP 1: $\rho_{\mathcal{G}}(\cdot)$ has the local property, i.e. for any $A \in \mathcal{G}$ and $X \in L_{\mathcal{F}}$ $\rho_{\mathcal{G}}(X)1_A = \rho_{\mathcal{G}}(1_A X)1_A$.
Observe that

$$\rho_{\mathcal{G}}(1_A X) \overset{(3.4)}{\leq} \rho_{\mathcal{G}}(X)1_A + \rho_{\mathcal{G}}(0)1_{A^c}$$

$$\overset{(3.4)}{\leq} 1_A\left(\rho_{\mathcal{G}}(1_A X)1_A + \rho_{\mathcal{G}}(X 1_{A^c})1_{A^c}\right) + \rho_{\mathcal{G}}(0)1_{A^c} = \rho_{\mathcal{G}}(1_A X)1_A + \rho_{\mathcal{G}}(0)1_{A^c},$$

then multiply by 1_A.

STEP 2: for every $Y \in L^*$ the set $\{\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}[X^j Y^j | \mathcal{G}] - \rho_{\mathcal{G}}(X), X \in L_{\mathcal{F}}\}$ is upward directed.
This can be checked directly using STEP 1: for $X, Z \in L_{\mathcal{F}}$ we set

$$\xi_X := \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j | \mathcal{G}\right] - \rho_{\mathcal{G}}(X), \qquad \xi_Z := \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[Z^j Y^j | \mathcal{G}\right] - \rho_{\mathcal{G}}(Z)$$

$$A := \{\xi_X \geq \xi_Z\}, \qquad W := X 1_A + Z 1_{A^c}.$$

By Remark 3.2.3 $W \in L_{\mathcal{F}}$ and by STEP 1

$$\rho_{\mathcal{G}}(W)1_A = \rho_{\mathcal{G}}(W 1_A)1_A = \rho_{\mathcal{G}}(X 1_A)1_A = \rho_{\mathcal{G}}(X)1_A$$

and an analogous argument can be applied with Z and A^c. It is then immediate to see that

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[W^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(W) = \xi_X 1_A + \xi_Z 1_{A^c} = \max(\xi_X, \xi_Y)$$

proving the claim.
Observe that as a consequence there exists a sequence $(X_n)_n$ in $L_{\mathcal{F}}$ such that

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X_n^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X_n) \uparrow_n \operatorname{ess\,sup}\left\{\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X), X \in L_{\mathcal{F}}\right\}$$

and in particular

$$\mathbb{E}_{\mathbb{P}}\left[\operatorname{ess\,sup}\left\{\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X), X \in L_{\mathcal{F}}\right\}\right]$$

$$\overset{\text{(MON)}}{=} \lim_n \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X_n^j Y^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X_n)\right] \leq \sup_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j\right] - \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X)\right]\right).$$

This allows us to state that

$$\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}^*(Y)\right] \leq \rho_0^*(Y) \ \ \forall Y \in L^*. \tag{3.9}$$

STEP 3: dual representation for $\rho_0(\cdot) := \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(\cdot)\right]$.

Consider the conjugate of ρ_0:

$$\rho_0^*(Y) := \sup_{X \in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j\right] - \rho_0(X)\right).$$

By nicely representability assumption

$$\rho_0(X) = \max_{Y \in L^*} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j\right] - \rho_0^*(Y)\right) = \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j\right] - \rho_0^*(\widehat{Y}).$$

STEP 4: $\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}^*\right] = \rho_0^*$ on L^*.

By (3.9) we have $\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}^*\right] \leq \rho_0^*$. At the same time $\rho_{\mathcal{G}}^*(Y) \geq \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j | \mathcal{G}\right] - \rho_{\mathcal{G}}(X)$ for all $X \in L_{\mathcal{F}}$. Taking expectations and a supremum over $X \in L_{\mathcal{F}}$ in RHS of the expression above, we get the inequality $\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}^*\right] \geq \rho_0^*$.

STEP 5: we prove (3.7) and (3.8).

Observe that trivially $\rho_{\mathcal{G}}(X) \geq \operatorname{ess\,sup}_{Y \in L^*}\left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j Y^j | \mathcal{G}\right] - \rho_{\mathcal{G}}^*(Y)\right)$ by definition of $\rho_{\mathcal{G}}^*$. In particular then

$$\rho_{\mathcal{G}}(X) \geq \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}^*(\widehat{Y}). \tag{3.10}$$

Additionally we have for $\widehat{Y}$ obtained in STEP 3 (which obviously depends on X)

$$\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X)\right] \overset{\text{STEP 3}}{=} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j\right] - \rho_0^*(\widehat{Y}) \overset{\text{STEP 4}}{=} \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}^*(\widehat{Y})\right].$$

This, together with (3.10), proves the claim.

STEP 6: $\widehat{Y}^j \leq 0$ and $\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j \middle| \mathcal{G}\right] = -1$ for every $j = 1, \dots, N$.

Recall that e_k is the $k-$th element of the canonical basis of $\mathbb{R}^N$. By definition of $\rho_{\mathcal{G}}^*$ and (3.3) we have

$$\rho_{\mathcal{G}}^*(\widehat{Y}) \geq \lambda \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[1_{\{\widehat{Y}^j \geq 0\}} \widehat{Y}^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(0), \ \ \forall\, 0 \leq \lambda \in L^{\infty}(\mathcal{G})$$

and by (3.5)

$$\rho_{\mathcal{G}}^*(\widehat{Y}) \geq \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[(\gamma e_k^j) \widehat{Y}^j | \mathcal{G}\right] - \rho_{\mathcal{G}}(\gamma e_k) = \gamma\left(\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k \middle| \mathcal{G}\right] + 1\right) - \rho_{\mathcal{G}}(0), \ \ \forall\, \gamma \in L^{\infty}(\mathcal{G}).$$

By (3.8) $\rho_{\mathcal{G}}^*(\widehat{Y}) = \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X) \in L^1$, hence from the inequalities above we get a contradiction unless for every $j = 1, \dots, N$ we have $\widehat{Y}^j \leq 0$ and $\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^j \middle| \mathcal{G}\right] = -1$. □

Corollary 3.2.10. *Under the hypotheses of Theorem 3.2.9, assume additionally that $\rho_{\mathcal{G}}(X) \in L_{\mathcal{F}}$ for any $X \in L_{\mathcal{F}}$. Define the set*

$$\mathscr{Q}_{\mathcal{G}} := \left\{\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \ll \mathbb{P} \mid \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \in L^*, \ \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \middle| \mathcal{G}\right] = 1 \ \forall\, j = 1, \dots, N\right\} \tag{3.11}$$

and set

$$\alpha(\mathbb{Q}) := \operatorname*{ess\,sup}_{X \in L_{\mathcal{F}}, \rho_{\mathcal{G}}(X) \leq 0} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j}\left[-X | \mathcal{G}\right], \quad X \in L_{\mathcal{F}}. \tag{3.12}$$

Then $\rho_{\mathcal{G}}(\cdot)$ admits the following dual representation:

$$\rho_{\mathcal{G}}(X) = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \alpha(\mathbb{Q})\right). \tag{3.13}$$

Furthermore, there exists $\widehat{\mathbb{Q}} \in \mathscr{Q}_{\mathcal{G}}$ such that $\rho_{\mathcal{G}}(X) = \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[-X^j | \mathcal{G}\right] - \alpha(\widehat{\mathbb{Q}})$.

Proof. For a given $X \in L_{\mathcal{F}}$ we see that setting $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} := -\widehat{Y}$ for $\widehat{Y}$ provided in Theorem 3.2.9 we have $\widehat{\mathbb{Q}} \in \mathscr{Q}_{\mathcal{G}}$ and

$$\sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X | \mathcal{G}] - \rho_{\mathcal{G}}^*\left(-\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right) \overset{\text{STEP 6}}{=} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[X^j \widehat{Y}^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}^*(\widehat{Y}) \overset{\text{STEP 5}}{=} \rho_{\mathcal{G}}(X)$$

$$\overset{(3.7)}{\geq} \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathscr{Q}_{\mathcal{G}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] - \rho_{\mathcal{G}}^* \left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right) \geq \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X^j|\mathcal{G}] - \rho_{\mathcal{G}}^* \left(-\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \right).$$

Also, for every $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}$

$$\rho_{\mathcal{G}}^* \left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) := \operatorname*{ess\,sup}_{X\in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] - \rho_{\mathcal{G}}(X) \right)$$

$$\overset{\text{Def.}\mathscr{Q}_{\mathcal{G}}}{=} \operatorname*{ess\,sup}_{X\in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[X^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] + \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[\frac{1}{N} \rho_{\mathcal{G}}(X) \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] \right)$$

$$= \operatorname*{ess\,sup}_{X\in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[\left(X^j + \frac{1}{N} \rho_{\mathcal{G}}(X) \right) \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] \right)$$

$$\leq \operatorname*{ess\,sup}_{Z\in L_{\mathcal{F}},\rho_{\mathcal{G}}(Z)\leq 0} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[Z^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] \right)$$

$$\leq \operatorname*{ess\,sup}_{Z\in L_{\mathcal{F}},\rho_{\mathcal{G}}(Z)\leq 0} \sum_{j=1}^{N} \left(\mathbb{E}_{\mathbb{P}} \left[Z^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] - \frac{1}{N} \rho_{\mathcal{G}}(Z) \right)$$

$$\leq \operatorname*{ess\,sup}_{Z\in L_{\mathcal{F}}} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[Z^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] - \rho_{\mathcal{G}}(Z) \right) = \rho_{\mathcal{G}}^* \left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right).$$

This implies that

$$\begin{aligned} \alpha(\mathbb{Q}) &:= \operatorname*{ess\,sup}_{X\in L_{\mathcal{F}},\rho_{\mathcal{G}}(X)\leq 0} \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j} \left[-X^j \middle| \mathcal{G} \right] \\ &= \operatorname*{ess\,sup}_{Z\in L_{\mathcal{F}},\rho_{\mathcal{G}}(Z)\leq 0} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[Z^j \left(-\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \right) \middle| \mathcal{G} \right] \right) = \rho_{\mathcal{G}}^* \left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \end{aligned} \tag{3.14}$$

for all $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}$, and the proof of the corollary is complete. □

The assumption $\rho_{\mathcal{G}}(\cdot) \in L_{\mathcal{F}}$ in Corollary 3.2.10 is satisfied by the Conditional Shortfall Systemic Risk Measure (see Definition 3.4.3 and Theorem 3.4.4).

3.3 Multivariate Utility Functions and Induced Orlicz Spaces

3.3.1 Multivariate Utility Functions

For the properties and technical results on multivariate utility functions and multivariate Orlicz spaces we will greatly rely on the work in Sections 2.1, 2.2 and 2.6 of Chapter 2.

The following assumption, which is the same of Standing Assumption I in Chapter 2, holds true throughout Chapter 3 without further mention.

Standing Assumption I. *We will consider multivariate utility functions in the form*

$$U(x) := \sum_{j=1}^{N} u_j(x^j) + \Lambda(x) \tag{3.15}$$

where $u_1, \dots, u_j : \mathbb{R} \to \mathbb{R}$ *are univariate utility function and* $\Lambda : \mathbb{R}^N \to \mathbb{R}$ *is concave, increasing with respect to the partial componentwise order and bounded from above. Inspired by Asymptotic Satiability as defined in Definition 2.13 [38] we will furthermore assume that for every* $\varepsilon > 0$ *there exist a point* $z_\varepsilon \in \mathbb{R}^N$ *and a selection* $\nu_\varepsilon \in \partial\Lambda(z_\varepsilon)$*, such that* $\sum_{j=1}^{N} |\nu_\varepsilon| < \varepsilon$ *.*
We also assume the Inada conditions

$$\lim_{x\to+\infty} \frac{u_j(x)}{x} = 0 \qquad \textit{and} \qquad \lim_{x\to-\infty} \frac{u_j(x)}{x} = +\infty \quad \forall\, j = 1, \dots, N$$

and that, without loss of generality, $u_j(0) = 0 \;\; \forall\, j = 1, \dots, N$*.*

Observe again that such multivariate utility function is split in two components: the sum of single agent utility functions and a universal part Λ. *As* Λ *is not necessarily strictly convex nor strictly increasing*, we may choose $\Lambda \equiv 0$, which corresponds to the case analyzed in [20] for the non conditional case.

3.3.2 Multivariate Orlicz Spaces

We recall that (see Definition 2.2.1) a function $\Phi : (\mathbb{R}_+)^N \to \mathbb{R}$ is said to be a multivariate Orlicz function if it null in 0, convex, continuous, increasing in the usual partial order and satisfies: there exist $A > 0, b$ constants such that $\Phi(x) \geq A\,\|x\| - b \; \forall x \in (\mathbb{R}_+)^N$.
For a given multivariate Orlicz function Φ we define, as in [7], the Orlicz space and the Orlicz Heart respectively (recall Section 2.2).

$$L^\Phi := \left\{ X \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \exists\, \lambda \in (0, +\infty), \mathbb{E}_\mathbb{P}\left[\Phi(\lambda\,|X|)\right] < +\infty \right\}$$

$$M^\Phi := \left\{ X \in L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right) \mid \forall\, \lambda \in (0, +\infty), \mathbb{E}_\mathbb{P}\left[\Phi(\lambda\,|X|)\right] < +\infty \right\}$$

where $|X| := [|X^j|]_{j=1}^N$ is the componentwise absolute value. We recall the Luxemburg norm as the functional

$$\|X\|_\Phi := \inf\left\{ \lambda > 0 \mid \mathbb{E}_\mathbb{P}\left[\Phi\left(\frac{1}{\lambda}\,|X|\right)\right] \leq 1 \right\}$$

defined on $L^0\left((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]^N\right)$ and taking values in $[0, +\infty]$.

3.4 Conditional Shortfall Systemic Risk Measures on $(L^\infty(\mathcal{F}))^N$

Given a sub sigma algebra $\mathcal{G} \subseteq \mathcal{F}$ we introduce the set

$$\mathscr{D}_{\mathcal{G}} := \left\{ Y \in (L^0(\Omega, \mathcal{F}, \mathbb{P}))^N \mid \sum_{j=1}^{N} Y^j \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \right\} . \tag{3.16}$$

We would like to consider as the set of admissible allocations a subset

$$\mathscr{B}_{\mathcal{G}} \subseteq \mathscr{D}_{\mathcal{G}}$$

satisfying appropriate conditions (see the Standing Assumption II).
At the same time, we observe that the constraints in (3.16) can be interpreted saying that the risk can be shared by all the agents in the single group $\mathbf{I} := \{1, \cdots, N\}$. This can be generalized by introducing the set of constraints corresponding to a cluster of agents conditional on the information in $\mathcal{G}$, inspired by an example in [20] for the static case.

Definition 3.4.1. *For $h \in \{1, \cdots, N\}$, let $\mathbf{I} := (I_m)_{m=1,\dots,h}$ be some partition of $\{1, \cdots, N\}$. Then we set*

$$\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} := \left\{ Y \in (L^0(\mathcal{F}))^N \mid \ \exists\ d = (d_1, \cdots, d_h) \in (L^0(\mathcal{G}))^h \mid \sum_{i \in I_m} Y^i = d_m, m = 1, \cdots, h \right\}, \tag{3.17}$$

$$\mathcal{B}_{\mathcal{G}}^{(\mathbf{I}),\infty} := \left\{ Y \in (L^0(\mathcal{F}))^N \mid \ \exists\ d = (d_1, \cdots, d_h) \in (L^\infty(\mathcal{G}))^h \mid \sum_{i \in I_m} Y^i = d_m, m = 1, \cdots, h \right\}. \tag{3.18}$$

We stress that the family $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$ admits two extreme cases:

- (i) when we have only one group $h = 1$ then $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} = \mathscr{D}_{\mathcal{G}}$ is the largest possible class, corresponding to risk sharing among all agents in the system;
- (ii) on the opposite side, the strongest restriction occurs when $h = N$, i.e., we consider exactly N groups, and in this case $\mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} = (L^0(\mathcal{G}))^N$ corresponds to no risk sharing.

Suppose now a partition $\mathbf{I}$ has been fixed. We will consider a subset

$$\mathscr{B}_{\mathcal{G}} \subseteq \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$$

and note that each component of $Y \in \mathscr{B}_{\mathcal{G}}$ is required to be $\mathcal{F}$-measurable, while the sums $\sum_{i \in I_m}^{N} Y^i$ are $\mathcal{G}$-measurable, and so is consequently $\sum_{j=1}^{N} Y^j$. Thus $\mathscr{B}_{\mathcal{G}} \subseteq \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})} \subseteq \mathscr{D}_{\mathcal{G}}$.
We define

$$\mathscr{C}_{\mathcal{G}} := \mathscr{B}_{\mathcal{G}} \cap \mathcal{B}_{\mathcal{G}}^{(\mathbf{I},\infty)} \cap (L^1(\Omega, \mathcal{F}, \mathbb{P}))^N . \tag{3.19}$$

We add this second Standing Assumption, which again holds true throughout Chapter 3 without further mention.

Standing Assumption II. *$\mathscr{B}_{\mathcal{G}}$ is closed in probability, it is conditionally convex and it is a conditional cone. Moreover $\mathscr{B}_{\mathcal{G}}+(L^0(\mathcal{G}))^N = \mathscr{B}_{\mathcal{G}}$ and the set $\mathscr{C}_{\mathcal{G}}$ is closed under $\mathcal{G}$-truncation. We finally consider a $B \in L^\infty(\mathcal{G})$ with $\|B\|_\infty < \sup_{z\in\mathbb{R}^N} U(z) \leq +\infty$.*

Example 3.4.2. It is easily seen that taking $\mathscr{B}_{\mathcal{G}} = \mathcal{B}_{\mathcal{G}}^{(\mathbf{I})}$ and consequently $\mathscr{C}_{\mathcal{G}} = \mathcal{B}_{\mathcal{G}}^{(\mathbf{I},\infty)} \cap (L^1(\Omega,\mathcal{F},\mathbb{P}))^N$ Standing Assumption II is satisfied. Closedness under truncation in particular is verified as follows: for $Y \in \mathscr{C}_{\mathcal{G}}$, for $j \in I_m$ we can take $Z_Y^j = \frac{1}{|I_m|}\sum_{i\in I_m} Y^i$ where $|I_m|$ is the cardinality of I_m. Then it is easily verified that $Y_{(k)}$ defined as in (3.2) satisfies for every $m = 1,\dots,h$

$$\sum_{i\in I_m} Y^i_{(k)} = \left(\sum_{i\in I_m} Y^i\right) 1_{\bigcap_j\{|Y^j|\leq k\}} + \left(\sum_{i\in I_m}\left(\frac{1}{|I_m|}\sum_{i\in I_m} Y^i\right)\right) 1_{\bigcup_j\{|Y^j|>k\}} = \sum_{i\in I_m} Y^i \in L^\infty_{\mathcal{G}}$$

which proves that $Y_{(k)} \in \mathcal{B}_{\mathcal{G}}^{(\mathbf{I}),\infty} = \mathscr{C}_{\mathcal{G}}$ and that also

$$\sum_{j=1}^N Y^j_{(k)} = \sum_{m=1}^h \sum_{i\in I_m} Y^i_{(k)} = \sum_{m=1}^h \sum_{i\in I_m} Y^i = \sum_{j=1}^N Y^j \,.$$

Finally, we point out that we can cover the setup of [20] in our framework (clearly, here we work with bounded positions and not in an Orlicz setup). Indeed, we may take the trivial partition $\mathbf{I} = \{\{1,\dots,N\}\}$ and, to cover the static case, we may choose $\mathcal{G} = \{\emptyset,\Omega\}$. Then we select the set $\mathscr{B}_{\mathcal{G}}$ equal to the set $\mathcal{C}_0$, defined in [20], which is assumed to be closed under truncation in the sense of [20] Definition 4.18. Then our assumptions here are satisfied as well.

Definition 3.4.3. *We define the functional*

$$\rho_{\mathcal{G}}^\infty(\cdot) : (L^\infty(\Omega,\mathcal{F},\mathbb{P}))^N \to L^0\left((\Omega,\mathcal{G},\mathbb{P});\,[-\infty,+\infty]\right)$$

as

$$\rho_{\mathcal{G}}^\infty(X) := \operatorname{ess\,inf}\left\{\sum_{j=1}^N Y^j \mid Y \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N, \mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B\right\} \tag{3.20}$$

which we call ***Conditional Shortfall Systemic Risk Measure*** *associated to the multivariate utility function U and the set of allocations $\mathscr{C}_{\mathcal{G}}$.*

We state now our main result concerning Conditional Shortfall Systemic Risk Measures. The proof, which is quite lengthy, is split in separate results in the following Section 3.4.1.

Theorem 3.4.4. *The functional $\rho_{\mathcal{G}}^\infty(\cdot)$ satisfies:*

1. *$\rho_{\mathcal{G}}^\infty(X) \in L^\infty(\mathcal{G})$ for all $X \in (L^\infty(\mathcal{F}))^N$. $\rho_{\mathcal{G}}^\infty(\cdot)$ is monotone* (3.3)*, conditionally convex* (3.4) *and conditionally monetary* (3.5)*. It is also continuous from above and from below in the sense of Definition 3.2.6.*

2. *For every $X \in (L^\infty(\mathcal{F}))^N$ we have*

$$\rho^\infty_{\mathcal{G}}(X) = \operatorname{ess\,inf}\left\{\sum_{j=1}^N Y^j \mid Y \in \mathscr{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}\left[U(X+Y)|\mathcal{G}\right] \geq B\right\} =: \rho_{\mathcal{G}}(X) \quad (3.21)$$

and the essential infimum in the central expression is attained.

3. *Define for every $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N] \ll \mathbb{P}$*

$$\alpha^1(\mathbb{Q}) := \operatorname*{ess\,sup}_{\substack{Z \in (L^\infty(\mathcal{F}))^N, \\ \mathbb{E}_{\mathbb{P}}[U(Z)|\mathcal{G}] \geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-Z^j \middle| \mathcal{G}\right]\right). \quad (3.22)$$

Then $\rho_{\mathcal{G}}(\cdot)$ admits the following dual representation:

$$\rho_{\mathcal{G}}(X) = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \alpha^1(\mathbb{Q})\right), \quad \forall X \in (L^\infty(\mathcal{F}))^N, \quad (3.23)$$

where

$$\mathscr{Q}^1_{\mathcal{G}} := \left\{ \mathbb{Q} \in \mathscr{Q}_{\mathcal{G}} \;\middle|\; \begin{array}{l} \alpha^1(\mathbb{Q}) \in L^1(\mathcal{G}) \quad \textit{and} \\ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j \middle| \mathcal{G}\right] \leq \sum_{j=1}^N Y^j, \ \forall Y \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{G}))^N \end{array} \right\}. \quad (3.24)$$

Furthermore, for every $X \in (L^\infty(\mathcal{F}))^N$ there exists $\widehat{\mathbb{Q}} \in \mathscr{Q}^1_{\mathcal{G}}$ such that

$$\rho_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[-X^j \middle| \mathcal{G}\right] - \alpha^1(\widehat{\mathbb{Q}}).$$

3.4.1 Proof of Theorem 3.4.4

In the notation (3.21), the expression $\mathbb{E}_{\mathbb{P}}\left[U(X+Y)|\mathcal{G}\right] \geq B$ stands for a shortened version of the following set of conditions: $U(X+Y) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and the conditional expectation $\mathbb{E}_{\mathbb{P}}\left[U(X+Y)|\mathcal{G}\right]$, which is well defined, is not smaller than B $\mathbb{P}-$a.s. Recall also that for any random variable W, taking values in $[0, +\infty]$, $\mathbb{E}_{\mathbb{P}}\left[W|\mathcal{G}\right]$ is always well defined via the Radon-Nikodym Theorem (see [11], Theorems 17.10-11), and in this case the notation $\mathbb{E}_{\mathbb{P}}\left[W|\mathcal{G}\right]$ will be used with this meaning.
For technical reasons we first study the functional $\rho_{\mathcal{G}}(\cdot)$ defined in (3.21). We will first prove all the properties in Theorem 3.4.4 Item 1, made exception for continuity from below, and existence of an allocation for $\rho_{\mathcal{G}}(\cdot)$ (Claim 3.4.5). We will then show that $\rho_{\mathcal{G}}(\cdot) \equiv \rho^\infty_{\mathcal{G}}(\cdot)$ on $(L^\infty(\mathcal{F}))^N$ (Claim 3.4.6), which yields Theorem 3.4.4 Item 2, and move on proving continuity from below (Claim 3.4.7). Finally, in Claim 3.4.8 we prove Theorem 3.4.4 Item 3.

Claim 3.4.5. *The functional $\rho_{\mathcal{G}}(\cdot)$ on $(L^\infty(\mathcal{F}))^N$ takes values in $L^\infty(\mathcal{G})$, the infimum is attained by a $\widehat{Y} \in \mathscr{C}_{\mathcal{G}}$, it is monotone* (3.3) *conditionally convex* (3.4) *and conditionally monetary* (3.5). *Furthermore, it is continuous from above in the sense of Definition 3.2.6.*

Proof.
STEP 1: the functional takes values in $L^\infty(\mathcal{G})$.
First we see that the set over which we take the essential infimum defining $\rho_{\mathcal{G}}(\cdot)$ is nonempty. We have by monotonicity (for m an N-dimensional deterministic vector) $\mathbb{E}_{\mathbb{P}}[U(X+m)|\mathcal{G}] \geq U(-\|X\|_\infty + m)$ where $\|X\|_\infty$ stands for the vector $[\|X^1\|_\infty, \dots, \|X^N\|_\infty] \in \mathbb{R}^N$. Since by assumption

$$\sup_{m\in\mathbb{R}^N} U(-\|X\|_\infty + m) = \sup_{z\in\mathbb{R}^N} U(z) > \|B\|_\infty ,$$

for some $m \in \mathbb{R}^N$ we have consequently $\mathbb{E}_{\mathbb{P}}[U(X+m)|\mathcal{G}] \geq B$.
We claim that the set over which we take the essential infimum is downward directed. To show this, suppose that $Z, Y \in (L^1(\mathcal{F}))^N$ are such that $\sum_{j=1}^N Y^j, \sum_{j=1}^N Z^j \in L^\infty(\mathcal{G})$ and

$$\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B, \mathbb{E}_{\mathbb{P}}[U(X+Z)|\mathcal{G}] \geq B .$$

Define the set $A := \{\sum_{j=1}^N Y^j \leq \sum_{j=1}^N Z^j\} \in \mathcal{G}$ and the random variable $W := 1_A Y + 1_{A^c} Z \in (L^1(\mathcal{F}))^N \cap \mathscr{B}_{\mathcal{G}}$ (observe that it belongs to $\mathscr{B}_{\mathcal{G}}$ since $\mathscr{B}_{\mathcal{G}}$ is conditionally convex, thus local). It is easy to see that $\sum_{j=1}^N W^j = 1_A \sum_{j=1}^N Y^j + 1_{A^c} \sum_{j=1}^N Z^j = \min\left(\sum_{j=1}^N Y^j, \sum_{j=1}^N Z^j\right) \in L^\infty(\mathcal{G})$, so that the set is downward directed. Furthermore

$$\mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}]1_A + \mathbb{E}_{\mathbb{P}}[U(X+W)|\mathcal{G}]1_{A^c} =$$

$$= \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}]1_A + \mathbb{E}_{\mathbb{P}}[U(X+Z)|\mathcal{G}]1_{A^c} \geq B1_A + B1_{A^c} = B$$

which concludes the proof of our claim.
Since the set is downward directed, there exists a minimizing sequence $(Y_n)_n \subseteq \mathscr{C}_{\mathcal{G}}$ such that $\sum_{j=1}^N Y_n^j \downarrow_n \rho_{\mathcal{G}}(X)$ and, having $\rho_{\mathcal{G}}(X) \leq \sum_{j=1}^N Y_1^j \in L^\infty$, we conclude that $\|(\rho_{\mathcal{G}}(X))^+\|_\infty < +\infty$. Suppose now by contradiction that for a sequence $k_n \uparrow +\infty$ we had $\mathbb{P}(\rho_{\mathcal{G}}(X) \leq -k_n) > 0$ for all n. Then, since $\{\rho_{\mathcal{G}}(X) \leq -k_n\} \in \mathcal{G}$, we would have for all $M \in \mathbb{N}$

$$-\|B\|_\infty \mathbb{P}(\rho_{\mathcal{G}}(X) \leq -k_n) \leq \mathbb{E}_{\mathbb{P}}\left[U(X+Y_M)1_{\{\rho_{\mathcal{G}}(X)\leq -k_n\}}\right]$$

$$\overset{\text{Cor.3.7.4}}{\leq} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left(a\left(X^j + Y_M^j\right) + b\right)1_{\{\rho_{\mathcal{G}}(X)\leq -k_n\}}\right]$$

$$\leq \left(a\sum_{j=1}^N \left\|X^j\right\|_\infty + b\right)\mathbb{P}(\rho_{\mathcal{G}}(X) \leq -k_n) + a\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N Y_M^j 1_{\{\rho_{\mathcal{G}}(X)\leq -k_n\}}\right] .$$

Consequently

$$-\|B\|_\infty \mathbb{P}(\rho_{\mathcal{G}}(X) \leq -k_n)$$

$$\le \left(a \sum_{j=1}^{N} \left\| X^j \right\|_\infty + b \right) \mathbb{P} \left(\rho_{\mathcal{G}} (X) \le -k_n \right) + a \lim_M \mathbb{E}_{\mathbb{P}} \left[\sum_{j=1}^{N} Y_M^j 1_{\{\rho_{\mathcal{G}}(X) \le -k_n\}} \right]$$

$$\overset{\text{(MON)}}{=} \left(a \sum_{j=1}^{N} \left\| X^j \right\|_\infty + b \right) \mathbb{P} \left(\rho_{\mathcal{G}} (X) \le -k_n \right) + a \mathbb{E}_{\mathbb{P}} \left[\rho_{\mathcal{G}} (X) 1_{\{\rho_{\mathcal{G}}(X) \le -k_n\}} \right]$$

$$\le \left(a \sum_{j=1}^{N} \left\| X^j \right\|_\infty + b \right) \mathbb{P} \left(\rho_{\mathcal{G}} (X) \le -k_n \right) - k_n a \mathbb{P} \left(\rho_{\mathcal{G}} (X) \le -k_n \right) .$$

Dividing by $\mathbb{P}(\rho_{\mathcal{G}}(X) \le -k_n)$ and sending n to infinity we would get a contradiction. This proves that $\|(\rho_{\mathcal{G}}(X))^-\|_\infty < +\infty$. Recalling that we already proved $\|(\rho_{\mathcal{G}}(X))^+\|_\infty < +\infty$, we obtain $\rho_{\mathcal{G}}(X) \in L^\infty(\mathcal{G})$.

STEP 2: the infimum is attained.
For the minimizing sequence $(Y_n)_n$, from the budget constraint $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \ge B$ and the fact that $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[X^j + Y_n^j]$ is bounded in n because of what we just proved ($L^\infty(\mathcal{G}) \ni \rho_{\mathcal{G}}(X) \le \sum_{j=1}^N Y_n^j \le \sum_{j=1}^N Y_1^j \in L^\infty(\mathcal{G})$) , we obtain that the sequence $(Y_n)_n$ is bounded in $(L^1(\mathcal{F}))^N$ using Lemma 2.6.4 Item 1.
Applying Corollary 2.6.12 we can find a subsequence and a $\widehat{Y} \in (L^1(\mathcal{F}))^N$ such that

$$W_H := \frac{1}{H} \sum_{h=1}^{H} Y_{n_h} \xrightarrow[H\to\infty]{\mathbb{P}-\text{a.s.}} \widehat{Y} .$$

Furthermore $\sum_{j=1}^N Y^j \in L^1(\mathcal{G})$, $W_H \in \mathscr{B}_{\mathcal{G}}$ by convexity of the set and $\widehat{Y} \in \mathscr{B}_{\mathcal{G}}$ since this set is closed in probability. Additionally we have that

$$L^\infty(\mathcal{G}) \ni \rho_{\mathcal{G}}(X) \le \sum_{j=1}^{N} \widehat{Y}^j = \lim_H \frac{1}{H} \sum_{h=1}^{H} \sum_{j=1}^{N} Y_{n_h}^j \le \frac{1}{H} \sum_{h=1}^{H} \sum_{j=1}^{N} Y_{n_1}^j = \sum_{j=1}^{N} Y_{n_1}^j \in L^\infty(\mathcal{G})$$

which yields that also $\sum_{j=1}^N \widehat{Y}^j \in L^\infty(\mathcal{G})$. To prove that $\widehat{Y} \in \mathscr{C}_{\mathcal{G}}$ we need to show that $\sum_{i \in I_m} \widehat{Y}^i \in L^\infty(\mathcal{G})$ for every $m = 1, \dots, h$. This will be a consequence of Proposition 3.7.6, once we show that $\mathbb{E}_{\mathbb{P}}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \ge B$. Hence we now focus on the latter inequality. We observe now that setting $Z_H := X + \frac{1}{H}\sum_{h=1}^H Y_{n_h}$ and $Z = X + \widehat{Y}$ Items 2 and 3 in Lemma 3.7.5 are satisfied. Moreover if we take

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[Z_H^j | \mathcal{G} \right] = \sum_{j=1}^{N} X^j + \frac{1}{H} \sum_{h=1}^{H} \sum_{j=1}^{N} Y_{n_h}^j$$

we see that the first term in the sum in RHS does not depend on H, while the Césaro means almost surely converge. Hence also Item 1 in Lemma 3.7.5 is satisfied, and we get that $\mathbb{E}_{\mathbb{P}}[U(Z)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \ge B$. As mentioned above, we now get also $\widehat{Y} \in \mathscr{C}_{\mathcal{G}}$. We finally observe that

$$\sum_{j=1}^{N} \widehat{Y}^j = \lim_H \frac{1}{H} \sum_{h=1}^{H} \sum_{j=1}^{N} Y_{n_h}^j \overset{\text{Rem.3.7.3}}{=} \lim_h \sum_{j=1}^{N} Y_{n_h}^j = \rho_{\mathcal{G}}(X)$$

so that the infimum is in fact attained at $\widehat{Y}$, which satisfies the constraints for $\rho_{\mathcal{G}}(X)$.

STEP 3: equations (3.3), (3.4), (3.5).

These have to be checked directly using definition of $\rho_{\mathcal{G}}(\cdot)$. We start with (3.3): if $X \leq Z$ componentwise a.s. then, for all $Y \in (L^1(\mathbb{P}))^N$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$, we have automatically (by monotonicity of U) that

$$\mathbb{E}_{\mathbb{P}}[U(Z+Y)|\mathcal{G}] \geq \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$$

so that

$$\begin{aligned}&\left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B\right\}\\ &\subseteq \left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathcal{G}}, \mathbb{E}_{\mathbb{P}}[U(Z+Y)|\mathcal{G}] \geq B\right\}\end{aligned}$$

and taking essential infima equation (3.3) follows.

As to (3.4), fix $0 \leq \lambda \leq 1$, $\lambda \in L^\infty(\mathcal{G})$ and $X, Z \in (L^\infty(\mathcal{F}))^N$. For $Y, W \in \mathscr{C}_{\mathcal{G}}$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$, $\mathbb{E}_{\mathbb{P}}[U(Z+W)|\mathcal{G}] \geq B$ we then have by concavity of utilities and $\mathcal{G}$-measurability of λ

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}[U(\lambda X + (1-\lambda)Z + \lambda Y + (1-\lambda)W)|\mathcal{G}]\\ &= \mathbb{E}_{\mathbb{P}}[U(\lambda(X+Y) + (1-\lambda)(Z+W))|\mathcal{G}]\\ &\geq \lambda\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] + (1-\lambda)\mathbb{E}_{\mathbb{P}}[U(Z+W)|\mathcal{G}]\\ &\geq \lambda B + (1-\lambda)B = B\,.\end{aligned}$$

Moreover obviously $\lambda Y + (1-\lambda)W \in \mathscr{C}_{\mathcal{G}}$, so that by definition

$$\rho_{\mathcal{G}}(\lambda X + (1-\lambda)Z) \leq \lambda \sum_{j=1}^{N} Y^j + (1-\lambda)\sum_{j=1}^{N} W^j\,.$$

Taking essential infima in RHS over Y and W yields equation (3.4).

Finally we come to (3.5). For $Y \in (L^\infty(\mathcal{G}))^N$ the assumption (see Standing Assumption II) $\mathscr{B}_{\mathcal{G}} + L^0(\mathcal{G}) = \mathscr{B}_{\mathcal{G}}$ implies that $W := Z + Y \in \mathscr{C}_{\mathcal{G}}$ for all $Z \in \mathscr{C}_{\mathcal{G}}$. Hence

$$\begin{aligned}\rho_{\mathcal{G}}(X+Y) &= \operatorname{ess\,inf}\left\{\sum_{j=1}^{N} Z^j \mid Z \in \mathscr{C}_{\mathcal{G}},\ \mathbb{E}_{\mathbb{P}}[U(U(X+Y+Z))|\mathcal{G}] \geq B\right\}\\ &= \operatorname{ess\,inf}\left\{\sum_{j=1}^{N}(W^j - Y^j) \mid W \in \mathscr{C}_{\mathcal{G}},\ \mathbb{E}_{\mathbb{P}}[U(U(X+W))|\mathcal{G}] \geq B\right\}\\ &= \rho_{\mathcal{G}}(X) - \sum_{j=1}^{N} Y^j\,.\end{aligned}$$

STEP 4: continuity from above.

Take $X_n \downarrow X$ and first fix n. We can take a minimizing sequence such that $\sum_{j=1}^N Y_k^j \downarrow_k \rho_{\mathcal{G}}(X_n)$. By (MON) we have that $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y_k^j\right] \downarrow_k \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X_n)\right]$, hence we can assume without loss of generality that $\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y_1^j\right] - \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X_n)\right] < 1$. Furthermore, by Egorov Theorem (see Theorem 3.7.2) we can select a set $A_n \in \mathcal{G}$, $\mathbb{P}(A_n) \leq \frac{1}{2^n}$ and an element in the sequence, which we will call Y_n (even if it is not necessarily the $n-$th element) such that

$$\left\| \left(\sum_{j=1}^N Y_n^j - \rho_{\mathcal{G}}(X_n) \right) 1_{(A_n)^c} \right\|_\infty < \frac{1}{n}.$$

Observe that

$$\rho_{\mathcal{G}}(X_1) \leq \rho_{\mathcal{G}}(X_n) \to_n \lim_n \rho_{\mathcal{G}}(X_n) \leq \rho_{\mathcal{G}}(X) \tag{3.25}$$

(which proves that $\lim_n \rho_{\mathcal{G}}(X_n) \in L^\infty$) and by Borel Cantelli Lemma almost all $\omega \in \Omega$ lie definitely in $(A_n)^c$: if we set $B_K := \bigcap_{n \geq K} (A_n)^c$ ($B_K \in \mathcal{G}$) we have

$$\mathbb{P}\left(\bigcup_K B_K \right) = 1 \tag{3.26}$$

which implies (by Borel Cantelli Lemma we know that for almost every $\omega \in \Omega$ $\left| \sum_{j=1}^N Y_n^j(\omega) - \rho_{\mathcal{G}}(X_n)(\omega) \right| \leq \frac{1}{n}$ definitely in n) that

$$\sum_{j=1}^N Y_n^j \xrightarrow[n \to +\infty]{\mathbb{P}\text{-a.s.}} \lim_n \rho_{\mathcal{G}}(X_n). \tag{3.27}$$

Observe that by construction we have

$$\left| \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y_n^j\right] - \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}(X_n)\right] \right| < 1 \text{ for each } n. \tag{3.28}$$

Take B_1 and observe that by definition of it and (3.25) we have that the sequence

$$\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left(X_n^j + Y_n^j\right) 1_{B_1}\right] \right)_n$$

is bounded and $\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[U(X_n + Y_n)|\mathcal{G}\right] 1_{B_1}\right] \geq -\|B\|_\infty \mathbb{P}(B_1)$. In fact (only the first claim is non trivial),

$$\left| \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left(X_n^j + Y_n^j\right) 1_{B_1}\right] \right| = \left| \mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^N \left(X_n^j + Y_n^j\right) 1_{B_1} \right] \right|$$

$$\leq \mathbb{E}_{\mathbb{P}}\left[\left(\left| \sum_{j=1}^N X_n^j \right| + \left| \sum_{j=1}^N Y_n^j - \rho_{\mathcal{G}}(X_n) \right| + \left| \rho_{\mathcal{G}}(X_n) \right| \right) 1_{B_1} \right]$$

$$\leq \sup_n \left(\sum_{j=1}^N \left\| X_n^j \right\|_\infty + \mathbb{P}(B_1) + \sup_n \left\| \rho_{\mathcal{G}}(X_n) \right\|_\infty \right) < +\infty.$$

Thus we can apply Lemma 2.6.4 and Corollary 2.6.12 to find a subsequence such that Césaro means of any further subsequence converge a.s. on B_1 to a (common) random variable Y_1. Now take B_2 ($B_1 \subseteq B_2$) and replicate the argument, taking the first subsequence we extracted in place of the one we began with. Observe that in this way the Césaro means will converge to a new random Y_2 on B_2, and that $Y_2 1_{B_1} = Y_1 1_{B_1}$. Iterating and applying a diagonal argument yields a $Y \in L^0(\mathcal{F})$ with $\sum_{j=1}^N Y^j \in L^0(\mathcal{G})$ and a subsequence such that

$$\frac{1}{H}\sum_{h=1}^{H} Y_{n_h} 1_{B_k} \to_H Y_K 1_{B_K} = Y 1_{B_K}\ \forall\, K,\ \ \mathbb{P}-a.s.$$

From $\mathbb{P}(\bigcup_K B_K) = 1$ we obtain

$$\frac{1}{H}\sum_{h=1}^{H} Y_{n_h} \to_H Y\ \ \mathbb{P}-a.s.$$

Equations (3.25) and (3.27) together with Remark 3.7.3, yield

$$\sum_{j=1}^{N} Y^j = \lim_n \rho_{\mathcal{G}}\left(X_n\right) \in L^\infty(\mathcal{G})\,.$$

Observe now that

$$\begin{aligned}\sup_n\left|\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X_n^j + Y_n^j\right]\right| &\leq \sup_n \sum_{j=1}^{N}\left\|X_n^j\right\|_\infty + \\ +\sup_n\left|\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[Y_n^j\right] - \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}\left(X_n\right)\right]\right| &+ \sup_n\left\|\rho_{\mathcal{G}}\left(X_n\right)\right\|_\infty < +\infty\end{aligned} \tag{3.29}$$

using (3.28) and the fact that $(X_n)_n$ is norm bounded (by definition), and so is $(\rho_{\mathcal{G}}(X_n))_n$ as a consequence (by monotonicity).
We stress the fact that the "pasting over subsets" procedure we used to obtain Y does not guarantee integrability of Y, which we prove now. We will show that setting

$$Z_H := \frac{1}{H}\sum_{h=1}^{H}(X_{n_h}^j + Y_{n_h}^j)$$

we have $\sup_H \sum_{j=1}^N \left\|Z_H^j\right\|_1 < +\infty$, which in turns yields $\sum_{j=1}^N \|X^j + Y^j\|_1 < +\infty$ (Fatou lemma) and finally (from $X \in (L^\infty(\mathcal{F}))^N$) $\sum_{j=1}^N \|Y^j\|_1 < +\infty$. To see $\sup_H \sum_{j=1}^N \left\|Z_H^j\right\|_1 < +\infty$, observe that

$$\sup_H\left|\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[Z_H^j\right]\right| \leq \sup_H \frac{1}{H}\sum_{h=1}^{H}\left|\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X_{n_h}^j + Y_{n_h}^j\right]\right| \overset{\text{Eq.(3.29)}}{<} +\infty$$

and that

$$\mathbb{E}_{\mathbb{P}}\left[U(Z_H)\right] \geq \frac{1}{H}\sum_{h=1}^{H}\mathbb{E}_{\mathbb{P}}\left[U(X_{n_h}+Y_{n_h})\right] \geq B \ \ \forall\, H$$

thus we can apply Lemma 2.6.4 Item 1 and the required norm boundedness follows. We conclude that $Y \in (L^1(\mathcal{F}))^N$, and we also know that $Y \in \mathscr{B}_{\mathcal{G}}$ since Y is an a.s. limit of convex combinations of elements of $\mathscr{C}_{\mathcal{G}}$, which is convex and closed in probability. We now prove that $\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B$ applying Lemma 3.7.5 to $Z_H \to_H Z := X + \widehat{Y}$. Lemma 3.7.5 Item 2 and 3 are readily verified. As to Lemma 3.7.5 Item 1 we see that

$$\begin{aligned}
\sup_H \left|\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[Z_H^j|\mathcal{G}\right]\right| &\leq \sup_H \left|\frac{1}{H}\sum_{h=1}^{H}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[X_{n_h}^j|\mathcal{G}\right] + \frac{1}{H}\sum_{h=1}^{H}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[Y_{n_h}^j|\mathcal{G}\right]\right| \\
&\leq \sup_H \frac{1}{H}\sum_{h=1}^{H}\sup_n\left(\sum_{j=1}^{N}\left\|X_n^j\right\|_\infty\right) + \sup_H\left|\frac{1}{H}\sum_{h=1}^{H}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[Y_{n_h}^j|\mathcal{G}\right]\right| \\
&\leq \sup_n\left(\sum_{j=1}^{N}\left\|X_n^j\right\|_\infty\right) + \sup_H\left|\frac{1}{H}\sum_{h=1}^{H}\mathbb{E}_{\mathbb{P}}\left[\sum_{j=1}^{N}Y_{n_h}^j\middle|\mathcal{G}\right]\right| \\
&= \sup_n\left(\sum_{j=1}^{N}\left\|X_n^j\right\|_\infty\right) + \sup_H\left|\frac{1}{H}\sum_{h=1}^{H}\sum_{j=1}^{N}Y_{n_h}^j\right| < +\infty \ \mathbb{P}-a.s.
\end{aligned}$$

using in the last steps the fact that $\sum_{j=1}^{N}Y_{n_h}^j \in L^\infty(\mathcal{G})$ and that $\sum_{j=1}^{N}Y_{n_h}^j \to_h \rho_{\mathcal{G}}\left(X\right)$ a.s. Thus all the assumptions of Lemma 3.7.5 are satisfied and we conclude that $\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] = \mathbb{E}_{\mathbb{P}}\left[U\left(Z\right)|\mathcal{G}\right] \geq B$. Consequently, since Y satisfies the constraints for $\rho_{\mathcal{G}}\left(X\right)$, we conclude that

$$\rho_{\mathcal{G}}\left(X\right) \leq \sum_{j=1}^{N}Y^j = \lim_n \rho_{\mathcal{G}}\left(X_n\right) \leq \rho_{\mathcal{G}}\left(X\right)$$

which proves continuity from above.

□

Claim 3.4.6. *We have that $\rho_{\mathcal{G}}^{\infty}\left(X\right) = \rho_{\mathcal{G}}\left(X\right)$ for every $X \in (L^\infty(\mathcal{F}))^N$.*

Proof. It is clear that

$$\rho_{\mathcal{G}}\left(X\right) \leq \operatorname{ess\,inf}\left\{\sum_{j=1}^{N}Y^j \mid Y \in \mathscr{C}_{\mathcal{G}} \cap \left(L^\infty(\mathcal{F})\right)^N, \mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B\right\}$$

since the infimum on RHS is taken over a smaller set.
We prove now the reverse inequality: by Claim 3.4.5 an allocation exists, call it Y. Use closedness under truncation to see that for $k \geq k_Y$ $Y_{(k)} \in \mathscr{C}_{\mathcal{G}}$ where $Y_{(k)}$, defined as in (3.2), satisfies $Y_{(k)} \to_k Y$ a.s.. We want to show that the convergence $U\left(X + Y_{(k)} + \varepsilon\mathbf{1}\right) \to_k U\left(X + Y + \varepsilon\mathbf{1}\right)$ is dominated, where $\mathbf{1}$ is the $N-$components

vector with all components equal to 1. To see this observe that $|U(X+Y+\varepsilon\mathbf{1})|$ and $|U(X+Z_Y+\varepsilon\mathbf{1})|$ are integrable:

$$L^1(\mathcal{F}) \ni a\left(\sum_{j=1}^{N}(X^j+Y^j)\right)+aN\varepsilon+b \overset{\text{Cor.3.7.4}}{\geq} U(X+Y+\varepsilon\mathbf{1}) \geq U(X+Y) \in L^1(\mathcal{F})$$

while integrability of $|U(X+Z_Y+\varepsilon\mathbf{1})|$ is trivial by boundedness of the vectors X, Z_Y and continuity of U. Moreover

$$\left|U\left(X+Y_{(k)}+\varepsilon\mathbf{1}\right)\right| = \left|U(X+Y+\varepsilon\mathbf{1})\,1_{\cap_j\{|Y^j|\leq k\}}+U(X+Z_Y+\varepsilon\mathbf{1})\,1_{\cup_j\{|Y^j|>k\}}\right| \leq$$

$$\max\left(|U(X+Y+\varepsilon\mathbf{1})|\,,\,|U(X+Z_Y+\varepsilon\mathbf{1})|\right) \leq |U(X+Y+\varepsilon\mathbf{1})|+|U(X+Z_Y+\varepsilon\mathbf{1})|\,.$$

Applying(cDOM) we then get that for all $\varepsilon>0$

$$\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y_{(k)}+\varepsilon\mathbf{1}\right)\middle|\mathcal{G}\right] \to_k \mathbb{E}_{\mathbb{P}}\left[U(X+Y+\varepsilon\mathbf{1})|\mathcal{G}\right] > B\,.$$

From the last expression we infer that

$$\mathbb{P}\left(\Gamma_K := \bigcap_{k\geq K}\left\{\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y_{(k)}+\varepsilon\mathbf{1}\right)\middle|\mathcal{G}\right]\geq B\right\}\right)\uparrow_K 1\,. \tag{3.30}$$

Fix K and take $\alpha_K\in\mathbb{R}^N$ with

$$U\left(-\|X\|_\infty-\left\|Y_{(K)}\right\|_\infty+\varepsilon\mathbf{1}+\alpha_K\right)\geq\|B\|_\infty$$

where again $\|X\|_\infty$ denotes the vector $[\|X^1\|_\infty,\dots,\left\|X^N\right\|_\infty]$ and similar notation is used for $\left\|Y_{(K)}\right\|_\infty$. Notice that such an α_K exists since $\sup_{z\in\mathbb{R}^N}U(z)>\|B\|_\infty$. Define Z_K by $Z_K^j := Y_{(K)}^j+\varepsilon+\alpha_K 1_{\Gamma_K^c}$, $j=1,\dots,N$ and observe that since $\Gamma_K\in\mathcal{G}$, $Z_K\in\mathscr{C}_{\mathcal{G}}\cap(L^\infty(\mathcal{F}))^N$. Furthermore

$$\mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}]\,1_{\Gamma_K}+\mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}]\,1_{\Gamma_K^c}$$

and

$$\mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}]\,1_{\Gamma_K} = \mathbb{E}_{\mathbb{P}}\left[U\left(X+Y_{(K)}+\varepsilon\mathbf{1}\right)\middle|\mathcal{G}\right]1_{\Gamma_K}\geq B1_{\Gamma_K}$$

by definition of Γ_K and the fact that 1_{Γ_K} can be moved inside conditional expectation. Moreover by definition of α_K

$$\mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}]\,1_{\Gamma_K^c} = \mathbb{E}_{\mathbb{P}}\left[U\left(X+Y_{(K)}+\varepsilon\mathbf{1}+\alpha_K\right)\middle|\mathcal{G}\right]1_{\Gamma_K^c}\geq B1_{\Gamma_K^c}\,.$$

Hence we have that $Z_K\in\mathscr{C}_{\mathcal{G}}\cap(L^\infty(\mathcal{F}))^N$, $\mathbb{E}_{\mathbb{P}}[U(X+Z_K)|\mathcal{G}]\geq B$, and we conclude that

$$\rho_{\mathcal{G}}^\infty(X)\leq\sum_{j=1}^N Z_K^j\,. \tag{3.31}$$

Now, by (3.30), for almost all $\omega\in\Omega$ there exists a $K(\omega)\in\mathbb{N}$ such that $\omega\in\Gamma_K$ for all $K\geq K(\omega)$, which implies for all $j=1,\dots,N$ $Z_K^j(\omega)=Y_{(K)}^j(\omega)+\varepsilon\ \ \forall\,K\geq K(\omega)$.

By definition $Y_{(K)} \to_K Y$ a.s., so that by (3.31) we can write for almost all $\omega \in \Omega$:

$$\rho_{\mathcal{G}}^{\infty}(X) \leq \liminf_{K\to+\infty} \sum_{j=1}^{N} Z_K^j = \liminf_{K\to+\infty} \left(\sum_{j=1}^{N} \left(Y_{(K)}^j + \varepsilon \right) \right)$$
$$= \lim_{K\to+\infty} \sum_{j=1}^{N} Y_{(K)}^j + N\varepsilon = \rho_{\mathcal{G}}(X) + N\varepsilon \,.$$

It is then straightforward to see that $\rho_{\mathcal{G}}^{\infty}(X) \leq \rho_{\mathcal{G}}(X)$ a.s., which implies $\rho_{\mathcal{G}}^{\infty}(X) = \rho_{\mathcal{G}}(X)$ a.s. □

Claim 3.4.7. *The functional $\rho_{\mathcal{G}}(\cdot)$ on $(L^\infty)^N$ is continuous from below.*

Proof. Consider a sequence $X_n \uparrow_n X$ and take any $Y \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty)^N$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$. Then for any $\varepsilon > 0$

$$B < \mathbb{E}_{\mathbb{P}}[U(X+Y+\varepsilon\mathbf{1})|\mathcal{G}] \overset{\text{(cMON)}}{=} \lim_n \mathbb{E}_{\mathbb{P}}[U(X_n+Y+\varepsilon\mathbf{1})|\mathcal{G}] \,.$$

Hence the sequence $(A_K)_K$, where

$$A_K := \{\mathbb{E}_{\mathbb{P}}[U(X_n+Y+\varepsilon\mathbf{1})|\mathcal{G}] \geq B, \ \forall n \geq K\}$$

satisfies $\mathbb{P}(A_K) \uparrow_K 1$. Take $\alpha_K \in \mathbb{R}^N$ such that

$$U(-\|X_n\|_\infty - \|Y\|_\infty + \varepsilon\mathbf{1} + \alpha_K) \geq \|B\|_\infty \ \forall n \geq K$$

where the notation for $\|X_n\|_\infty$ and $\|Y\|_\infty$ is the same as in the proof of Claim 3.4.6. Define $Z_K \in (L^\infty(\mathcal{F}))^N$ by $Z_K^j := Y^j + \varepsilon\mathbf{1} + \alpha_K 1_{A_K^c}$ for $j = 1, \dots, N$. Since $A_K \in \mathcal{G}$ we have $Z \in \mathscr{C}_{\mathcal{G}}$. Furthermore for all $n \geq K$

$$\mathbb{E}_{\mathbb{P}}[U(X_n+Z_K)|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[U(X_n+Z_K)|\mathcal{G}]\,1_{A_K} + \mathbb{E}_{\mathbb{P}}[U(X_n+Z_K)|\mathcal{G}]\,1_{A_K^c}$$
$$\geq B 1_{A_K} + \|B\|_\infty 1_{A_K^c} \geq B \,.$$

Hence by definition of $\rho_{\mathcal{G}}(X_n)$

$$\rho_{\mathcal{G}}(X_n) \leq \sum_{j=1}^{N} Z_K^j = \sum_{j=1}^{N} Y^j + N\varepsilon + \sum_{j=1}^{N} \alpha_K^j 1_{A_K^c} \,,$$

$$\lim_n \rho_{\mathcal{G}}(X_n) \leq \liminf_K \left(\sum_{j=1}^{N} Y^j + N\varepsilon + \sum_{j=1}^{N} \alpha_K^j 1_{A_K^c} \right) .$$

Recall now that $\mathbb{P}(A_K) \to_K 1$ and $A_K \subseteq A_{K+1}$. Hence almost all $\omega \in \Omega$ are such that $1_{A_K^c}(\omega) = 0$ definitely in K. As a consequence

$$\liminf_K \left(\sum_{j=1}^{N} Y^j + N\varepsilon + \sum_{j=1}^{N} \alpha_K^j 1_{A_K^c} \right) = \liminf_K \left(\sum_{j=1}^{N} Y^j + N\varepsilon \right) = \sum_{j=1}^{N} Y^j + N\varepsilon.$$

It follows that

$$\lim_n \rho_{\mathcal{G}}(X_n) \leq \sum_{j=1}^{N} Y^j \ \mathbb{P}-a.s.$$

and this holds for all $Y \in \mathscr{C}_{\mathcal{G}}$ such that $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$. Taking essential infimum on RHS for $Y \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$, $\mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B$ by Claim 3.4.6 we obtain

$$\lim_n \rho_{\mathcal{G}}(X_n) \leq \rho_{\mathcal{G}}(X) \overset{(3.3)}{\leq} \lim_n \rho_{\mathcal{G}}(X_n) \ .$$

□

Theorem 3.2.9 yields a dual representation result for $\rho_{\mathcal{G}}(\cdot)$ using $L_{\mathcal{F}} := (L^\infty(\mathcal{F}))^N$ and $L^* := (L^1(\mathcal{G}))^N$: since we have continuity from above (Claim 3.4.5) and from below (Claim 3.4.7), we can apply Corollary 3.1.5 to prove nice representability of $\mathbb{E}_{\mathbb{P}}[\rho_{\mathcal{G}}(\cdot)] : (L^\infty(\mathcal{F}))^N \to \mathbb{R}$. However, in view of Claim 3.4.6, we can apply an argument inspired by [84] Proposition 3.6 to get a more specific dual representation. Observe that in this setup Corollary 3.2.10 applies and the set $\mathscr{Q}_{\mathcal{G}}$ defined there takes the form:

$$\mathscr{Q}_{\mathcal{G}} := \left\{ \mathbb{Q} \ll \mathbb{P} : \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in (L^1(\mathcal{F}))^N, \ \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \middle| \mathcal{G} \right] = 1 \ \forall\, j = 1, \dots, N \right\} . \tag{3.32}$$

Claim 3.4.8. *Let $\rho_{\mathcal{G}}(\cdot) : (L^\infty(\mathcal{F}))^N \to L^\infty(\mathcal{G})$ be defined by (3.21) and take $\alpha^1(\cdot)$ as in (3.22). Then the following are equivalent for fixed $p \in \{0, 1\}$ and $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}$:*

1. $\rho^*_{\mathcal{G}}\left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \in L^p(\mathcal{F})$,
2. $\alpha(\mathbb{Q}) \in L^p(\mathcal{F})$, *where $\alpha(\cdot)$ is defined in (3.12) for $L_{\mathcal{F}} = (L^\infty(\mathcal{F}))^N$ and $L^* = (L^1(\mathcal{F}))^N$,*
3. $\alpha^1(\mathbb{Q}) \in L^p(\mathcal{F})$ *and* $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] \leq \sum_{j=1}^N Y^j$ *for all* $Y \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty(\mathcal{F}))^N$.

Moreover $\rho_{\mathcal{G}}(\cdot)$ admits the dual representation in (3.23) for $\mathcal{Q}^1_{\mathcal{G}}$ defined in (3.24) and for every $X \in (L^\infty(\mathcal{F}))^N$ there exists $\widehat{\mathbb{Q}} \in \mathscr{Q}^1_{\mathcal{G}}$ such that $\rho_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X^j|\mathcal{G}] - \alpha^1(\widehat{\mathbb{Q}})$.

Proof. Recall that in this specific setup we have by Claim 3.4.5 that $\rho_{\mathcal{G}}(\cdot) \in L^\infty(\mathcal{G})$ and that $\rho_{\mathcal{G}}(\cdot) = \rho^\infty_{\mathcal{G}}(\cdot)$ on $(L^\infty(\mathcal{F}))^N$ by Claim 3.4.6. Hence using Corollary 3.2.10 and (3.14) we see that

$$\alpha(\mathbb{Q}) = \rho^*_{\mathcal{G}}\left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \tag{3.33}$$

for all $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}$. Moreover we have:

$$\rho^*_{\mathcal{G}}\left(-\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) = \operatorname*{ess\,sup}_{X \in (L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \rho_{\mathcal{G}}(X) \right)$$

$$\overset{(3.33)}{=} \operatorname*{ess\,sup}_{X\in(L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \operatorname*{ess\,inf}_{\substack{Y\in\mathscr{C}_\mathcal{G}\cap(L^\infty(\mathcal{F}))^N\\ \mathbb{E}_\mathbb{P}[U(X+Y)|\mathcal{G}]\geq B}} \left(\sum_{j=1}^N Y^j\right)\right)$$

$$= \operatorname*{ess\,sup}_{\substack{X,Y\in(L^\infty(\mathcal{F}))^N\\ Y\in\mathscr{C}_\mathcal{G},\mathbb{E}_\mathbb{P}[U(X+Y)|\mathcal{G}]\geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \sum_{j=1}^N Y^j\right)$$

$$= \operatorname*{ess\,sup}_{\substack{Z,Y\in(L^\infty(\mathcal{F}))^N,\\ Y\in\mathscr{C}_\mathcal{G},\mathbb{E}_\mathbb{P}[U(Z)|\mathcal{G}]\geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-(Z^j-Y^j) \middle| \mathcal{G}\right] - \sum_{j=1}^N Y^j\right).$$

We conclude that

$$\alpha(\mathbb{Q}) = \operatorname*{ess\,sup}_{\substack{Z\in(L^\infty(\mathcal{F}))^N,\\ \mathbb{E}_\mathbb{P}[U(Z)|\mathcal{G}]\geq B}} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-Z^j \middle| \mathcal{G}\right]\right) + \operatorname*{ess\,sup}_{Y\in\mathscr{C}_\mathcal{G}\cap(L^\infty(\mathcal{F}))^N} \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j \middle| \mathcal{G}\right] - \sum_{j=1}^N Y^j\right). \tag{3.34}$$

The equivalence between Item 1-2-3 is now clear, once we observe that for every $\mathbb{Q} \in \mathscr{Q}_\mathcal{G}$ such that $\alpha(\mathbb{Q}) \in L^0(\mathcal{F})$ we must have $\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}[Y^j|\mathcal{G}] - \sum_{j=1}^N Y^j \leq 0$ $\mathbb{P}-$a.s. since $\mathscr{C}_\mathcal{G} \cap (L^\infty(\mathcal{F}))^N$ is a conditional cone.
All the claims then follow from Corollary 3.2.10, observing that for the optimum $\widehat{\mathbb{Q}}$ provided there we must have $\alpha(\mathbb{Q}) \in L^1(\mathcal{G})$ (since $\rho_\mathcal{G}(X) \in L^\infty(\mathcal{G})$). □

Remark 3.4.9. We stress the fact that by Claim 3.4.8 we have for every $Y \in \mathscr{C}_\mathcal{G} \cap (L^\infty(\mathcal{F}))^N$

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j \middle| \mathcal{G}\right] \leq \sum_{j=1}^N Y^j \quad \mathbb{P}-\text{a.s.} \quad \text{for all } \mathbb{Q}\in\mathscr{Q}_\mathcal{G} \text{ such that } \alpha(\mathbb{Q})\in L^0(\mathcal{G}). \tag{3.35}$$

3.4.2 Uniqueness and Integrability of optima of $\rho_\mathcal{G}(\cdot)$

Uniqueness

Assumption 3.4.10. *The function $U : \mathbb{R}^N \to \mathbb{R}$ satisfies:*

$$\begin{cases} X \in (L^1(\Omega,\mathcal{F},\mathbb{P}))^N \\ (U(X))^- \in L^1(\Omega,\mathcal{F},\mathbb{P}) \end{cases} \Rightarrow \exists\,\delta>0 \text{ s.t. } (U(X-\varepsilon\mathbf{1}))^- \in L^1(\Omega,\mathcal{F},\mathbb{P})\,\forall\, 0<\varepsilon<\delta. \tag{3.36}$$

Observe that for example taking $\alpha_1,\dots,\alpha_N,\beta_1,\dots,\beta_N > 0$ the function

$$U(x) := \sum_{j=1}^N \left(1-\exp\left(-\alpha_j x_j\right)\right) + \left(1-\exp\left(-\sum_{j=1}^N \beta_j x_j\right)\right)$$

satisfies Assumption 3.4.10.

Proposition 3.4.11. *Under Assumption 3.4.10 $\rho_{\mathcal{G}}(X)$ admits a unique optimum in $\mathscr{C}_{\mathcal{G}}$ in the extended sense of Theorem 3.4.4, for every $X \in (L^\infty(\mathcal{F}))^N$.*

Proof. Suppose $\widehat{Y}_1 \neq \widehat{Y}_2$ were two optima. Then clearly so is $\widehat{Y}_\lambda = \lambda \widehat{Y}_1 + (1-\lambda)\widehat{Y}_2$ for $\lambda \in \mathbb{R}, 0 < \lambda < 1$ by concavity of U. At the same time, we have that $\Gamma := \left\{\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_\lambda)|\mathcal{G}\right] > \lambda\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_1)|\mathcal{G}\right] + (1-\lambda)\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_2)|\mathcal{G}\right]\right\} \in \mathcal{G}$ satisfies $\mathbb{P}(\Gamma) = 1$ by strict concavity: if this were not the case, from concavity and

$$\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_\lambda)1_{\Gamma^c}\right] = \lambda\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_1)1_{\Gamma^c}\right] + (1-\lambda)\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_2)1_{\Gamma^c}\right]$$

we would get that on Γ^c, which is not empty, $U(X+\widehat{Y}_\lambda) = \lambda U(X+\widehat{Y}_1) + (1-\lambda)U(X+\widehat{Y}_2)$ which contradicts strict concavity of U.
Now for some $\varepsilon > 0$, recalling that $\lambda\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_1)\middle|\mathcal{G}\right] + (1-\lambda)\mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_2)\middle|\mathcal{G}\right] \geq B$, we would have by monotonicity of U, Assumption 3.4.10, and Egorov Theorem that definitely in $H \in \mathbb{N}$ by (cMON), on a set Ξ of positive measure,

$$\mathbb{E}_\mathbb{P}\left[U\left(X + \widehat{Y}_\lambda - \frac{1}{H}1_\Xi\mathbf{1}\right)|\mathcal{G}\right] \geq B$$

($\Xi \in \mathcal{G}$ can be taken as a set where $\mathbb{E}_\mathbb{P}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B + \varepsilon$ and the convergence

$$\mathbb{E}_\mathbb{P}\left[U\left(X + \widehat{Y}_\lambda - \frac{1}{H}\mathbf{1}\right)\middle|\mathcal{G}\right] \uparrow_H \mathbb{E}_\mathbb{P}\left[U(X+\widehat{Y}_\lambda)|\mathcal{G}\right]$$

is uniform, which exists by Egorov Theorem). But also $\widehat{Y}_\lambda - \frac{1}{H}1_\Xi\mathbf{1} \in \mathscr{C}_{\mathcal{G}}$ and by definition $\rho_{\mathcal{G}}(X)1_\Xi \leq \left(\sum_{j=1}^N \widehat{Y}^j_\lambda - N\frac{1}{H}\right)1_\Xi < \left(\sum_{j=1}^N \widehat{Y}^j_\lambda\right)1_\Xi$ which contradicts the optimality of $\widehat{Y}$ (recall again that $\mathbb{P}(\Xi) > 0$). □

Integrability

Proposition 3.4.12. *There exists an extension, ρ_0^Φ, of $\rho_0(\cdot) := \mathbb{E}_\mathbb{P}\left[\rho_{\mathcal{G}}(\cdot)\right]$ to M^Φ which is convex, nondecreasing and $\|\cdot\|_\Phi$-continuous.*

Proof. Observe that because of the downward directedness proved in STEP 2 of Claim 3.4.5, together with (MON), we have

$$\rho_0(X) = \inf\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y^j\right] \mid Y \in \mathscr{C}_{\mathcal{G}}, \mathbb{E}_\mathbb{P}\left[U(X+Y)|\mathcal{G}\right] \geq B\right\} \quad X \in (L^\infty(\mathcal{F}))^N.$$

Define now

$$\rho_0^\Phi(X) := \inf\left\{\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[Y^j\right] \mid Y \in \mathscr{C}_{\mathcal{G}}, \mathbb{E}_\mathbb{P}\left[U(X+Y)|\mathcal{G}\right] \geq B\right\} \quad X \in M^\Phi. \tag{3.37}$$

We easily see that $\rho_0^\Phi(X) < +\infty$ for every $X \in M^\Phi$ (since the set over which we take infima in (3.37) is nonempty). Moreover $\rho_0^\Phi(X) > -\infty$ since if this were the case, for a minimizing sequence $(Y_n)_n$ we would have $\inf_n \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}[Y_n^j] = -\infty$. Now, by Lemma 2.6.2

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[B] \leq \mathbb{E}_{\mathbb{P}}[U(X+Y_n)] \\
&\leq a\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^j\right] + a\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y_n^j\right] - a\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(Y_n^j)-\right] + b \\
&\leq const + a\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[Y_n^j\right]
\end{aligned}$$

which gives a contradiction. Clearly, mimicking what we did in the proof of Claim 3.4.5 Step 3, we can check that $\rho_0^{\Phi}(\cdot)$ is also convex and nondecreasing. Now by Extended Namioka-Klee Theorem in [23]) it is also norm continuous. □

Lemma 3.4.13. *For any $Z \in (L^1(\mathcal{F}))^N$ we have that $\rho_{\mathcal{G}}^*(Z) \in L^1(\mathcal{G})$ if and only if $\rho_0^*(Z) < +\infty$ and, if any of the two conditions is met, we have $Z \in K_\Phi$.*

Proof. Observe that for any $Z \in (L^1(\mathcal{F}))^N$ we have $\rho_0^*(Z) = \mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}^*(Z)\right]$. The first claim then follows. Suppose now $\rho_{\mathcal{G}}^*(Z) \in L^1(\mathcal{G})$. By [5] Theorem 5.43 Item 3 $\rho_0^{\Phi}(\cdot)$ is bounded on a ball B_ε (defined using the norm $\|\cdot\|_\Phi$) centered at 0. We have as a consequence

$$+\infty > \sup_{X\in B_\varepsilon}\left(\rho_0^{\Phi}(X) + \rho_0^*(Z)\right) .$$

Now we use the fact that $\rho_0^{\Phi}(\cdot)$, when restricted to $(L^\infty(\mathcal{F}))^N$, coincides with $\rho_0(\cdot)$, and continuity of ρ_0^{Φ} (by Proposition 3.4.12) to see that

$$+\infty > \sup_{X\in B_\varepsilon}\left(\rho_0^{\Phi}(X) + \rho_0^*(Z)\right) \geq \sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\rho_0^{\Phi}(X) + \rho_0^*(Z)\right) =$$

$$= \sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\rho_0(X) + \rho_0^*(Z)\right) \geq \sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^jZ^j\right]\right)$$

where we used Fenchel inequality to obtain the last inequality. Furthermore, using the fact that given Z, for any $X \in B_\varepsilon \cap (L^\infty(\mathcal{F}))^N$ the vector $\widehat{X}$ defined by $\widehat{X}^j = sgn(Z^j)\left|X^j\right|$ still belongs to $B_\varepsilon \cap (L^\infty(\mathcal{F}))^N$, we have

$$\sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[X^jZ^j\right]\right) = \sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left|X^jZ^j\right|\right]\right) .$$

To conclude, we observe that an approximation with simple functions yields:

$$\sup_{X\in B_\varepsilon\cap(L^\infty(\mathcal{F}))^N}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left|X^jZ^j\right|\right]\right) = \sup_{X\in B_\varepsilon}\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[\left|X^jZ^j\right|\right]\right) .$$

This completes the proof using Proposition 2.2.5 Item 1. □

Assumption 3.4.14. $L^{\Phi} = L^{\Phi_1} \times \cdots \times L^{\Phi_N}$.

Assumption 3.4.14, which is the same of Assumption 2.3.3 in Chapter 2, is a request on the utility functions we allow for. It can be rephrased as: if for $X \in (L^0((\Omega, \mathcal{F}, \mathbb{P}); [-\infty, +\infty]))^N$ there exist $\lambda_1, \dots, \lambda_N > 0$ such that $\mathbb{E}_{\mathbb{P}}[u_j(-\lambda_j |X^j|)] > -\infty$, then there exists $\alpha > 0$ such that $\mathbb{E}_{\mathbb{P}}[\Lambda(-\alpha |X|)] > -\infty$. This request is rather weak and there are many examples of choices of U and Λ that guarantee this condition is met, see Section 2.4.5. Note again that however this is not a request on the topological spaces, but just an integrability requirement, and it is automatically satisfied if $\Lambda \equiv 0$.

Lemma 3.4.15. *Suppose Assumption 3.4.14 holds. Let $Z \in (L^1(\mathcal{F}))^N$ be given and suppose that $\mathbb{E}_{\mathbb{P}}[U(Z) \mid \mathcal{G}] \geq B$. Then for any $W \in K_{\Phi}$ we have $\sum_{j=1}^N (Z^j)^- W^j \in L^1(\mathcal{F})$.*

Proof. Observe that $\mathbb{E}_{\mathbb{P}}[U(Z)] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[U(Z) \mid \mathcal{G}]] \geq \mathbb{E}_{\mathbb{P}}[B]$. Furthermore

$$U(Z) = \sum_{j=1}^N u_j(Z^j) + \Lambda(Z) = \sum_{j=1}^N u_j((Z^j)^+) + \sum_{j=1}^N u_j(-(Z^j)^-) + \Lambda(Z).$$

This implies

$$-\sum_{j=1}^N u_j(-(Z^j)^-) \leq \max_{j=1,\dots,N} \left(\frac{\mathrm{d}u_j}{\mathrm{d}x^j}(0) \right) \sum_{j=1}^N (Z^j)^+ + \sup_{z \in \mathbb{R}^N} \Lambda(z) - U(Z)$$

where in the last line we used (2.3). It then follows that $\sum_{j=1}^N (-u_j(-(Z^j)^-)) \in L^1(\mathcal{F})$, which in turns yields $(Z)^- \in L^{\Phi_1} \times \cdots \times L^{\Phi_N}$. Since by Proposition 2.2.5 we have $W \in L^{\Phi_1^*} \times \cdots \times L^{\Phi_N^*}$, we get by [65] Proposition 2.2.7 that $(Z^j)^- W^j \in L^1(\mathcal{F})$ for every $j = 1, \dots, N$, and the last claim is proved.

□

Proposition 3.4.16. *Suppose Assumption 3.4.14 is fulfilled. Then for any $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}^1$ (defined in (3.24)) the optimum $\widehat{Y}$ from Theorem 3.4.4 satisfies $\widehat{Y} \in L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)$ and*

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[\widehat{Y}^j \middle| \mathcal{G}\right] \leq \sum_{j=1}^N \widehat{Y}^j \quad \mathbb{P}-a.s.$$

Proof. By Lemma 3.4.13 we get that for any $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}^1$ we have $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in K_{\Phi}$, so that by Lemma 3.4.15 we have for any $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}^1$ that $[(\widehat{Y}^1)^-, \dots, (\widehat{Y}^N)^-] \in L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)$. Moreover, given $\widehat{Y}_{(k)}$ using (3.2) for $k \geq k_{\widehat{Y}}$, by Fatou Lemma we have for any $\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}^1$.

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[(\widehat{Y}^j)^+ \middle| \mathcal{G}\right] \overset{\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}}{=} \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(\widehat{Y}^j)^+ \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \middle| \mathcal{G}\right]$$

$$\leq \liminf_k \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[(\widehat{Y}_{(k)}^j)^+ \frac{\mathrm{d}\mathbb{Q}^j}{\mathrm{d}\mathbb{P}} \middle| \mathcal{G}\right] \overset{\mathbb{Q} \in \mathscr{Q}_{\mathcal{G}}}{=} \liminf_k \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[(\widehat{Y}_{(k)}^j)^+ \middle| \mathcal{G}\right]$$

$$\leq \liminf_k \left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[\widehat{Y}^j_{(k)} \middle| \mathcal{G} \right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j_{(k)})^- \middle| \mathcal{G} \right] \right)$$

$$\overset{\mathbb{Q}\in\mathscr{Q}_{\mathcal{G}}}{\leq} \liminf_k \left(\sum_{j=1}^N \widehat{Y}^j_{(k)} + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j_{(k)})^- \middle| \mathcal{G} \right] \right) .$$

We conclude that

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j)^+ \middle| \mathcal{G} \right] \leq \sum_{j=1}^N \widehat{Y}^j + \lim_k \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j_{(k)})^- \middle| \mathcal{G} \right] \overset{\text{(cDOM)}}{=} \sum_{j=1}^N \widehat{Y}^j + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[(\widehat{Y}^j)^- \middle| \mathcal{G} \right]$$

where we used in the last step that $Y_{(k)} \to \widehat{Y}\, \mathbb{P}-$ a.s. and that $(\widehat{Y})^- \in L^1(\mathbb{Q}^1) \times \cdots \times L^1(\mathbb{Q}^N)$ to apply (cDOM): $(\widehat{Y}^j_{(k)})^- \leq \max\left((\widehat{Y}^j)^-, (Z^j_Y)^- \right) \in L^1(\mathbb{Q}^j)$, $j = 1, \dots, N$. This yields both integrability and the fact that, rearranging terms,

$$\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[\widehat{Y}^j \middle| \mathcal{G} \right] \leq \sum_{j=1}^N \widehat{Y}^j .$$

□

3.4.3 Optimization with a fixed measure $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}$

We will now need to change underlying probability measures. Thus from now on, given a vector of probability measures $\mathbb{Q} = [\mathbb{Q}^1, \dots, \mathbb{Q}^N]$ and a number $p \in \{0, 1, \infty\}$, we set

$$L^p(\mathcal{F}, \mathbb{Q}) := L^p(\Omega, \mathcal{F}, \mathbb{Q}^1) \times \cdots \times L^p(\Omega, \mathcal{F}, \mathbb{Q}^N) .$$

Similarly, when some confusion might arise, we will write explicitly also the measure $\mathbb{P}$, that is we will use $L^p(\mathcal{F}, \mathbb{P}) := L^p(\Omega, \mathcal{F}, \mathbb{P})$ in place of the shortened $L^p(\mathcal{F})$. Define the following optimization problem for a given $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}$:

$$\rho^{\mathbb{Q}}_{\mathcal{G}}(X) := \operatorname*{ess\,inf}_{\substack{Y \in (L^\infty(\mathcal{F}, \mathbb{P}))^N \\ \mathbb{E}_{\mathbb{P}}[U(X+Y)|\mathcal{G}] \geq B}} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[Y^j \middle| \mathcal{G} \right] . \tag{3.38}$$

Proposition 3.4.17. *The following holds for any $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}$:*

$$\rho^{\mathbb{Q}}_{\mathcal{G}}(X) = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j} \left[-X^j \middle| \mathcal{G} \right] - \alpha^1(\mathbb{Q}) . \tag{3.39}$$

Moreover for any $X \in (L^\infty(\mathcal{F}, \mathbb{P}))^N$ we have

$$\rho_{\mathcal{G}}(X) = \max_{\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}} \rho^{\mathbb{Q}}_{\mathcal{G}}(X) = \rho^{\widehat{\mathbb{Q}}}_{\mathcal{G}}(X) \text{ for any optimum } \widehat{\mathbb{Q}} = \widehat{\mathbb{Q}}_X \text{ of (3.23).} \tag{3.40}$$

Proof. We first prove (3.39): observe that by (3.22) we have for any $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}$

$$\alpha^1(\mathbb{Q}) = \operatorname*{ess\,sup}_{\substack{W \in (L^\infty(\mathcal{F},\mathbb{P}))^N \\ \mathbb{E}_\mathbb{P}[U(W)|\mathcal{G}] \geq B}} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-W^j \middle| \mathcal{G}\right]$$

$$= - \operatorname*{ess\,inf}_{\substack{W-X \in (L^\infty(\mathcal{F},\mathbb{P}))^N \\ \mathbb{E}_\mathbb{P}[U(X+(W-X))|\mathcal{G}] \geq B}} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[(W^j - X^j) \middle| \mathcal{G}\right] + \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[X^j \middle| \mathcal{G}\right] \right\}$$

$$= \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \middle| \mathcal{G}\right] - \operatorname{ess\,inf} \left\{ \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Z^j \middle| \mathcal{G}\right] \mid Z \in (L^\infty(\mathcal{F},\mathbb{P}))^N,\ \mathbb{E}_\mathbb{P}\left[U\left(X+Z\right)|\mathcal{G}\right] \geq B \right\}.$$

As a consequence by definition of $\alpha^1(\cdot)$ in (3.22) and of $\rho^{\mathbb{Q}}_{\mathcal{G}}(X)$ in (3.38)

$$\alpha^1(\mathbb{Q}) = \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \mid \mathcal{G}\right] - \rho^{\mathbb{Q}}_{\mathcal{G}}(X).$$

Observe now that by Theorem 3.4.4 Item 3 $\rho_{\mathcal{G}}(X) \geq \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[-X^j \mid \mathcal{G}\right] - \alpha^1(\mathbb{Q})$ for every $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}(\mathbb{Q})$, and equality holds for any optimum $\widehat{\mathbb{Q}}$ of (3.39). Direct substitution yields then (3.40). □

Proposition 3.4.18. *Suppose Assumption 3.4.14 is fulfilled. Then for any $\mathbb{Q} \in \mathscr{Q}^1_{\mathcal{G}}$ we have*

$$\rho^{\mathbb{Q}}_{\mathcal{G}}(X) = \operatorname*{ess\,inf}_{\substack{Y \in L^1(\mathcal{F},\mathbb{Q}) \cap (L^1(\mathcal{F},\mathbb{P}))^N \\ \mathbb{E}_\mathbb{P}[U(X+Y)|\mathcal{G}] \geq B}} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j \middle| \mathcal{G}\right]. \tag{3.41}$$

Proof. Clearly the inequality ($\geq$) is trivial, since we are enlarging the set over which we take the essential infimum. As to the converse ($\leq$), observe that whenever $Y \in L^1(\mathcal{F},\mathbb{Q}) \cap (L^1(\mathcal{F},\mathbb{P}))^N$ is given with $\mathbb{E}_\mathbb{P}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B$, we have

$$\mathbb{E}_\mathbb{P}\left[U\left(X+Y+\varepsilon\mathbf{1}\right)|\mathcal{G}\right] > B \ \mathbb{P}-a.s.$$

by monotonicity of U. Hence, given $Y_{(k)}$ as in (3.2), $k \geq k_Y$, defining

$$\Gamma_K := \bigcap_{k \geq K} \left\{ \mathbb{E}_\mathbb{P}\left[U\left(X + Y_{(k)} + \varepsilon\mathbf{1}\right) \middle| \mathcal{G}\right] \geq B \right\} \in \mathcal{G}$$

we have that $\Gamma_K \subseteq \Gamma_{k+1}$ and $\mathbb{P}(\cup_K \Gamma_K) = 1$: the argument is similar to the one in the proof of Claim 3.4.6. As a consequence, outside of a set E_0 of zero $\mathbb{P}$-measure, we have

$$1_{\Gamma^c_K} = 0 \text{ definitely in } K \ \mathbb{P}-a.s. \tag{3.42}$$

Take for each K a vector $\alpha_K \in \mathbb{R}^N$ such that

$$U(-\left\|X\right\|_\infty - \left\|Y_{(K)}\right\|_\infty + \varepsilon\mathbf{1} + \alpha_K) \geq \left\|B\right\|_\infty$$

where the notation for the vectors $\|X\|_\infty, \|Y_{(K)}\|_\infty$ is the same used in the proof of Claim 3.4.6 and define

$$Z_K := Y_{(K)} + \varepsilon\mathbf{1} + \alpha_K 1_{\Gamma_K^c} \in (L^\infty(\mathcal{F},\mathbb{P}))^N .$$

Then clearly

$$\mathbb{E}_\mathbb{P}\left[U\left(X+Y_{(K)}\right)|\mathcal{G}\right]1_{\Gamma_K} + \mathbb{E}_\mathbb{P}\left[U\left(Y_{(K)}+\varepsilon\mathbf{1}+\alpha_K 1_{\Gamma_K^c}\right)|\mathcal{G}\right]1_{\Gamma_K^c} \geq B$$

which implies $\mathbb{E}_\mathbb{P}\left[U\left(X+Z_K\right)|\mathcal{G}\right] \geq B$. Hence

$$\begin{aligned}
\rho_\mathcal{G}^\mathbb{Q}(X) &\leq \liminf_K \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Z_K^j\middle|\mathcal{G}\right] \\
&= \liminf_K \left(\left(\sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y_{(K)}^j\middle|\mathcal{G}\right]\right)1_{\Gamma_K} + \left(\sum_{j=1}^N \alpha_K^j\right)1_{\Gamma_K^c}\right) + N\varepsilon \\
&= \lim_K \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y_{(K)}^j\middle|\mathcal{G}\right] + \lim_K\left(\sum_{j=1}^N \alpha_K^j 1_{\Gamma_K^c}\right) + N\varepsilon \\
&\overset{\substack{(3.42)\\(\text{cDOM})}}{=} \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}^j}\left[Y^j\middle|\mathcal{G}\right] + N\varepsilon .
\end{aligned}$$

Taking essential infimum in RHS over $Y \in L^1(\mathcal{F},\mathbb{Q}) \cap (L^1(\mathcal{F},\mathbb{P})^N$ such that the budget constraint $\mathbb{E}_\mathbb{P}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B$ holds, then over $\varepsilon > 0$ we get the desired inequality. □

Theorem 3.4.19. *Suppose Assumption 3.4.14 is fulfilled. Then for any optimum* $\widehat{\mathbb{Q}}$ *of* (3.23)*, the optimum* $\widehat{Y} \in \mathscr{C}_\mathcal{G}$ *of Theorem 3.4.4 is an optimum for* $\rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X)$ *in the following extended sense:* $\widehat{Y} \in L^1(\mathcal{F},\mathbb{Q}) \cap (L^1(\mathcal{F},\mathbb{P}))^N, \mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \geq B$ *and*

$$\rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X) = \operatorname*{ess\,inf}_{\substack{Y\in L^1(\mathcal{F},\mathbb{Q})\cap(L^1(\mathbb{P},\mathcal{F}))^N\\ \mathbb{E}_\mathbb{P}[U(X+Y)|\mathcal{G}]\geq B}} \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j|\mathcal{G}\right] = \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[\widehat{Y}^j\middle|\mathcal{G}\right] .$$

Moreover,

$$\rho_\mathcal{G}(X) = \rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X).$$

Proof. By Proposition 3.4.16 we have that $\widehat{Y} \in L^1(\mathcal{F},\widehat{\mathbb{Q}}) \cap (L^1(\mathcal{F},\mathbb{P}))^N$. We also know that $\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \geq B$. Hence we have $\rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X) \leq \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j|\mathcal{G}] \overset{\text{Prop.3.4.16}}{\leq} \sum_{j=1}^N \widehat{Y}^j$. We also know, by optimality of $\widehat{Y}$ for $\rho_\mathcal{G}(X)$ and (3.40), that $\sum_{j=1}^N \widehat{Y}^j = \rho_\mathcal{G}(X) = \rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X)$. We then conclude jointly optimality in the extended sense of $\widehat{Y}$ for $\rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X)$ and the remaining claim. □

Corollary 3.4.20. *Under the same assumptions of Theorem 3.4.19, we have also that* $\widehat{Y} \in (L^1(\mathcal{F},\mathbb{P}))^N \cap \bigcap_{\mathbb{Q}\in\mathscr{Q}_\mathcal{G}^1} L^1(\mathcal{F},\mathbb{Q})$, $\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \geq B$ *and hence* $\rho_\mathcal{G}^{\widehat{\mathbb{Q}}}(X)$

coincides with

$$\operatorname{ess\,inf}\left\{\sum_{j=1}^{N}\mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\middle|\mathcal{G}\right] \mid Y\in(L^1(\mathcal{F},\mathbb{P}))^N\cap\bigcap_{\mathbb{Q}\in\mathscr{Q}^1_{\mathcal{G}}}L^1(\mathcal{F},\mathbb{Q}),\,\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right]\geq B\right\} \tag{3.43}$$

and $\widehat{Y}$ *attains the essential infimum in* (3.43)*.*

Proof. It is enough to observe that

$$\left\{\sum_{j=1}^{N}\mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\middle|\mathcal{G}\right] \mid Y\in(L^1(\mathcal{F},\mathbb{P}))^N\cap\bigcap_{\mathbb{Q}\in\mathscr{Q}^1_{\mathcal{G}}}L^1(\mathcal{F},\mathbb{Q}),\,\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right]\geq B\right\}$$

is a subset of

$$\left\{\sum_{j=1}^{N}\mathbb{E}_{\widehat{\mathbb{Q}}^j}\left[Y^j\middle|\mathcal{G}\right] \mid Y\in(L^1(\mathcal{F},\mathbb{P}))^N\cap L^1(\mathcal{F},\mathbb{Q}),\,\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right]\geq B\right\}$$

for every $\mathbb{Q}\in\mathscr{Q}^1_{\mathcal{G}}$ and apply at this point Theorem 3.4.19. □

3.5 The exponential case

Throughout the whole Section 3.5 we take $u_j(x)=-e^{-\alpha_j x}, j=1,\dots,N$ for real numbers $\alpha_1,\dots,\alpha_N>0$ and $\Lambda\equiv 0$. M^{Φ} in this setup takes the form $M^{\Phi}=M^{\Phi_1}\times\dots\times M^{\Phi_N}$ where for every $j=1,\dots,N$

$$M^{\Phi_j}=M^{\exp}=\left\{X\in L^0(\Omega,\mathcal{F},\mathbb{P})\mid \mathbb{E}_{\mathbb{P}}\left[\exp(\lambda\left|X\right|)\right]<+\infty\,\forall\lambda>0\right\}.$$

We also introduce the following constants:

$$\beta:=\sum_{j=1}^{N}\frac{1}{\alpha_j};\;\; A_k:=\frac{1}{\alpha_k}\log\left(\frac{1}{\alpha_k}\right);\;\; A:=\sum_{j=1}^{N}A_j\,. \tag{3.44}$$

Finally, we consider the case $\mathscr{B}_{\mathcal{G}}=\mathscr{D}_{\mathcal{G}}$ (recall (3.16) and (3.19)).

3.5.1 Explicit formulas

Finitely generated $\mathcal{G}$

Remark 3.5.1. Consider $\mathcal{G}=\sigma(E_1,\dots,E_P)$ for some finite partition of Ω. Define on E_k the conditioned probability as usual and denote by $\mathbb{E}_{\mathbb{P}^k}\left[\dots\right]$ the expectation with respect to it. If Z is a random variable on Ω, $\mathbb{E}_{\mathbb{P}^k}\left[Z\right]$ will denote the expectation of its restriction to E_k with the induced sigma algebra and conditioned probability. Observe that $\mathbb{E}_{\mathbb{P}}\left[Z|\mathcal{G}\right]=\sum_{k=1}^{P}\mathbb{E}_{\mathbb{P}^k}\left[Z\right]1_{E_k}$. We will use below the non conditional explicit formulas obtained in [20] Theorem 6.2, observing that to move from utilities of the form $u_j(x)=-\frac{1}{\alpha_j}\exp(-\alpha_j x)$ to the ones we are to use here, it is enough to substitute X in the formulas of [20] Theorem 6.2 with $X^j-\frac{1}{\alpha_j}\log\left(\frac{1}{\alpha_j}\right), j=1,\dots,N$. The formulas we use here, obtained with the substitution above, are the ones that can be found in Biagini et al, "On fairness of Systemic Risk Measures", arXiv:1803.09898v3, 2018, Section 6.

Theorem 3.5.2. *Take $\mathcal{G}=\sigma(E_1,\dots,E_P)$, $B=\sum_{k=1}^P B_k 1_{E_k}\in L^\infty(\mathcal{G})$ with $B_k\in\mathbb{R}$ and $\|B\|_\infty<0$. Define the functional*

$$\rho_{\mathcal{G}}(X):=\operatorname{ess\,inf}\left\{\sum_{j=1}^N Y^j \mid \sum_{j=1}^N Y^j\in L^\infty(\mathcal{G}), Y\in(L^\infty(\mathcal{F}))^N, \sum_{j=1}^N \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+Y^j\right)\middle|\mathcal{G}\right]\geq B\right\}$$

on M^Φ. Then $\rho_{\mathcal{G}}$ takes values in $L^\infty(\mathcal{G})$ and satisfies $\rho_{\mathcal{G}}(X)=\sum_{k=1}^P d^k_{\mathcal{G}}(X)1_{E_k}$ for

$$d^k_{\mathcal{G}}(X):=\inf\left\{\sum_{j=1}^N Y^j \mid \sum_{j=1}^N Y^j\in\mathbb{R}, Y\in(L^\infty(E_k))^N, \sum_{j=1}^N \mathbb{E}_{\mathbb{P}^k}\left[u_j(X^j+Y^j)\right]\geq B_k\right\}$$

$$=\beta\log\left(-\frac{\beta}{B_k}\mathbb{E}_{\mathbb{P}^k}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\right]\right)-A\quad \text{where}\quad \overline{X}:=\sum_{j=1}^N X^j\,.$$

Moreover the following is an optimum for $\rho_{\mathcal{G}}(X)$:

$$Y^j=-X^j+\frac{1}{\beta\alpha_k}\overline{X}+\frac{1}{\beta\alpha_k}\rho_{\mathcal{G}}(X)+\left(\frac{1}{\beta\alpha_k}A-A_k\right).$$

As to the conjugate of $\rho_{\mathcal{G}}(\cdot)$ we have

$$\rho_{\mathcal{G}}(X)=\operatorname*{ess\,sup}_{Y\in(L^1(\mathbb{P},\mathcal{F}))^N}\left(\sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[X^jY^j\middle|\mathcal{G}\right]-\rho^*(Y)\right)=\sum_{j=1}^N\mathbb{E}_{\mathbb{Q}^j}\left[-X^j\middle|\mathcal{G}\right]-\rho^*_{\mathcal{G}}\left(-\frac{d\mathbb{Q}}{d\mathbb{P}}\right)$$

for

$$\frac{d\mathbb{Q}^j}{d\mathbb{P}}=\sum_{k=1}^P\frac{e^{-\frac{\overline{X}}{\beta}}}{\mathbb{E}_{\mathbb{P}^k}\left[e^{-\frac{\overline{X}}{\beta}}\right]}1_{E_k}\ \forall\, j=1,\dots,N \tag{3.45}$$

and

$$\rho^*_{\mathcal{G}}(Y)=\sum_{k=1}^P d^k_{\mathcal{G}}(Y|_{E_k})^*1_{E_k}$$

denoting by $d^k_{\mathcal{G}}(\cdot)^*$ the conjugate of $d^k_{\mathcal{G}}(\cdot)$ as a functional on $(L^\infty(E_k,\mathcal{F}|_{E_k},\mathbb{P}_k))^N$ under the duality $\left((L^\infty(E_k,\mathcal{F}|_{E_k},\mathbb{P}_k))^N,(L^1(E_k,\mathcal{F}|_{E_k},\mathbb{P}_k))^N\right)$.

Proof. Observe that for $\mathcal{G}=\sigma(E_1,\dots,E_P)$ we have

$$\mathscr{D}_{\mathcal{G}}=\left\{Y\in(L^\infty(\mathcal{F}))^N \mid \sum_{j=1}^N Y^j\in L^\infty(\mathcal{G})\right\}$$

and

$$\operatorname{ess\,inf}\left\{\sum_{j=1}^N Y^j \mid Y\in\mathcal{C}_{\mathcal{G}}, \sum_{j=1}^N\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Y^j)\mid\mathcal{G}\right]\geq B\right\}=$$

$$\sum_{k=1}^{P} \inf\left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathbb{R}}(E_k), Y \in (L^\infty(E_k))^N \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}^k}\left[u_j(X|_{E_k} + Y^j)\right] \geq B_k\right\} 1_{E_k} \tag{3.46}$$

where

$$\mathcal{C}_{\mathbb{R}}(E_k) = \left\{Y \in (L^\infty(E_k, \mathbb{P}_k))^N \mid \sum_{j=1}^{N} Y^j \in \mathbb{R}\, \mathbb{P}_k - \text{a.s.}\right\}.$$

Based on the fact that $\mathcal{C}_{\mathbb{R}}(E_k)$ is closed under truncation in the deterministic sense (see [20] Definition 4.18), we see that for any $X \in M^\Phi$

$$\rho_k(X|_{E_k}) := \inf\left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathbb{R}}(E_k) \cap (L^\infty(E_k))^N, \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}^k}\left[u_j(X|_{E_k} + Y^j)\right] \geq B_k\right\}$$

$$= \inf\left\{\sum_{j=1}^{N} Y^j \mid Y \in \mathscr{C}_{\mathbb{R}}(E_k), Y \in M^\Phi(E_k), \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}^k}\left[u_j(X|_{E_k} + Y^j)\right] \geq B_k\right\}. \tag{3.47}$$

Furthermore, by norm density of $(L^\infty(E_k))^N$ in $M^\Phi(E_k)$, we have

$$\rho_k^*(Z|_{E_k}) = \sup_{X \in (L^\infty(E_k))^N} \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}^k}\left[X^j Z|_{E_k}\right] - \rho_k(X^j)\right).$$

Each ρ_k can be treated individually as a Systemic Risk Measure as defined in [20] on the probability space $(E_k, \mathcal{F}|_{E_k}, \mathbb{P}^k)$. All the results in [20] can be applied, in particular Theorem 6.2. This proves the formulas for $d_{\mathcal{G}}^k(\cdot)$ which in turns yield the formula for $\rho_{\mathcal{G}}(X)$, and the optimality of Y as defined in the statement. Observe furthermore that an analogous argument can be applied for the conjugate $\rho_{\mathcal{G}}^*$, which can be expressed as

$$\rho_{\mathcal{G}}^*(Z) = \sum_{k=1}^{P} \rho_k^*(Z|_{E_k}) 1_{E_k} \quad Z \in K_\Phi$$

(recall Definition 2.9). This implies that in fact any optimum for the dual representation of ρ_k in the dual pair $(M^\Phi(E_k), K_\Phi(E_k))$ is an optimum for the dual representation of the restriction $\rho_k|_{(L^\infty(\mathcal{F}))^N}$ which coincides to the $k-$th coefficient of the sum in (3.46). We then conclude the optimality of $\mathbb{Q}$ in (3.45).
This concludes the proof. □

Remark 3.5.3. We can also see from [20] Example 4.14 or by direct computation that

$$\rho_{\mathcal{G}}^*\left(-\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \sum_{j=1}^{N} \frac{1}{\alpha_j} \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}^j}{d\mathbb{P}}\right)\middle|\mathcal{G}\right] + \sum_{j=1}^{N} \frac{1}{\alpha_j} \log\left(-\frac{B}{\beta\alpha_j}\right)$$

$$= \beta \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}^j}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}^j}{d\mathbb{P}}\right)\middle|\mathcal{G}\right] + A + \beta \log\left(-\frac{B}{\beta}\right) = \sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}^j}[-X^j|\mathcal{G}] - \rho_{\mathcal{G}}(X). \tag{3.48}$$

Countably generated $\mathcal{G}$: explicit formulas

For $X \in (L^\infty(\mathcal{F}))^N$ and a filtration of sigma algebras $(\mathcal{F}_t)_{t\in\mathbb{N}}$ which are finitely generated set $\mathcal{G} := \bigvee_t \mathcal{F}_t$. Take $B \in L^\infty(\mathcal{G}), \|B\|_\infty < 0$ and define $\rho_{\mathcal{F}_t}(\cdot)$ using $\mathbb{E}_\mathbb{P}[B|\mathcal{F}_t]$ in place of B.

Lemma 3.5.4. *The following limits hold, if notation for parameters is used as in* (3.44)

$$\begin{aligned}\lim_{t\uparrow+\infty}\rho_{\mathcal{F}_t}(X) &= \lim_{t\uparrow+\infty}\left[\beta\log\left(-\frac{\beta}{\mathbb{E}_\mathbb{P}[B|\mathcal{F}_t]}\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{F}_t\right]\right)-A\right]\\ &= \beta\log\left(-\frac{\beta}{B}\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]\right)-A\,,\\ \lim_{t\uparrow+\infty}&\left[-X^k+\frac{1}{\beta\alpha_k}\left(\overline{X}+\rho_{\mathcal{F}_t}(X)+A\right)-A^k\right]\\ &= -X^k+\frac{1}{\beta\alpha_k}\left(\overline{X}+\rho_{\mathcal{G}}(X)+A\right)-A^k =: Y^k_\infty\,.\end{aligned}$$

Also,

$$\begin{gathered}Y_\infty \in (L^\infty(\mathcal{F}))^N\,,\\ \sum_{j=1}^N Y^j_\infty = \beta\log\left(-\frac{\beta}{B}\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]\right)-A\in L^\infty(\mathcal{G})\,,\\ \sum_{j=1}^N\mathbb{E}_\mathbb{P}\left[\exp\left(-\alpha_j\left(X^j+Y^j_\infty\right)\right)\big|\mathcal{G}\right]=B\,.\end{gathered}$$

Proof. Existence of the limits reduces to a convergence argument for uniformly integrable martingales. The formulas can be checked by hand. □

Proposition 3.5.5. *For a countably generated $\mathcal{G}$, we have that*

$$\rho_{\mathcal{G}}(X) = \beta\log\left(-\frac{\beta}{B}\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]\right)-A$$

and Y_∞ is an optimal allocation for $\rho_{\mathcal{G}}(X)$. Moreover

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} := \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]}\quad j=1,\dots,N$$

defines an optimum for the dual representation of $\rho_{\mathcal{G}}(X)$.

Proof. From Lemma 3.5.4 it is clear, since Y_∞ satisfies the constraints in the definition of $\rho_{\mathcal{G}}(\cdot)$, that

$$\rho_{\mathcal{G}}(X) \le \beta\log\left(-\frac{\beta}{B}\mathbb{E}_\mathbb{P}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]\right)-A\,.$$

We need to prove the opposite inequality. We will proceed showing first that for every $\widehat{Y} \in (L^{\infty}(\mathcal{F}))^N$ such that

$$\sum_{j=1}^{N} \widehat{Y}^j \in L^{\infty}(\mathcal{G})$$

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+\widehat{Y}^j\right)\Big|\mathcal{G}\right] > B$$

it holds that

$$\lim_t \rho_{\mathcal{F}_t}(X) \leq \sum_{j=1}^{N} \widehat{Y}^j .$$

The claim in the proposition will then follow by minor refinements.

STEP 1: to prove what we just mentioned, let us observe that we can write $\widehat{Y}^N = Z - \sum_{j=1}^{N-1} \widehat{Y}^j$ for some $Z \in L^{\infty}(\mathcal{G})$, and that we can define the vector Y_t by $Y_t^j = \widehat{Y}^j$, $j = 1, \dots, N-1$, $Y_t^N = \mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t] - \sum_{j=1}^{N-1} \widehat{Y}^j$, in such a way that $\sum_{j=1}^{N} Y_t^j \in L^{\infty}(\mathcal{F}_t)$. Also, it is clear that $Y_t \to_t \widehat{Y}$ a.s. and that $\sup_t \sum_{j=1}^{N} \left\|Y_t^j\right\|_{\infty} < \infty$.
Using the properties of the exponential we observe that we can rewrite

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+Y_t^j\right)\right)\big|\mathcal{F}_t\right]$$

as

$$\begin{aligned}
&\sum_{j=1}^{N-1} \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+\widehat{Y}^j\right)\right)\Big|\mathcal{F}_t\right] + \\
&+ \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_N\left(X^N+\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t] - \sum_{j=1}^{N-1}\widehat{Y}^j\right)\right)\Bigg|\mathcal{F}_t\right] \\
&= \sum_{j=1}^{N-1} \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+\widehat{Y}^j\right)\right)\Big|\mathcal{F}_t\right] + \\
&+ \exp\left(-\alpha_N\left(\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t]\right)\right)\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_N\left(X^N - \sum_{j=1}^{N-1}\widehat{Y}^j\right)\right)\Bigg|\mathcal{F}_t\right].
\end{aligned}$$

Take now the limit as $t \to \infty$. We get:

$$\begin{aligned}
&\sum_{j=1}^{N-1} \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+\widehat{Y}^j\right)\right)\Big|\mathcal{G}\right] + \\
&\exp\left(-\alpha_N(Z)\right)\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_N\left(X^N - \sum_{j=1}^{N-1}\widehat{Y}^j\right)\right)\Bigg|\mathcal{G}\right] \\
&= \sum_{j=1}^{N-1} \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+\widehat{Y}^j\right)\right)\Big|\mathcal{G}\right] + \\
&+ \mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_N\left(X^N + Z - \sum_{j=1}^{N-1}\widehat{Y}^j\right)\right)\Bigg|\mathcal{G}\right].
\end{aligned}$$

Thus we have:

$$\lim_{t\uparrow+\infty}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+Y_t^j\right)\right)\middle|\mathcal{F}_t\right]=\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+\widehat{Y}^j\right)\right)\middle|\mathcal{G}\right]>B.$$

From the latter inequality we conclude that setting

$$\Theta_t:=\left\{\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+Y_t^j\right)\right)\middle|\mathcal{F}_t\right]\geq\mathbb{E}_{\mathbb{P}}\left[B|\mathcal{F}_t\right]\right\}$$

we have that, almost surely, $1_{\Theta_t^c}$ is null definitely in t (i.e. for almost all ω there exist a $T(\omega)$ such that this indicator is null for $t\geq T(\omega)$). This holds since $\mathbb{E}_{\mathbb{P}}\left[B|\mathcal{F}_t\right]\to_t \mathbb{E}_{\mathbb{P}}\left[B|\mathcal{G}\right]=B\in L^\infty(\mathcal{G})$. Observe also that $\Theta_{t_n}^c\in\mathcal{F}_{t_n}$ and there exists a constant K such that

$$\sum_{j=1}^{N}\exp\left(-\alpha_j\left(-\left\|X^j\right\|_\infty-\sup_t\left\|Y_t^j\right\|_\infty+K\right)\right)\geq\mathbb{E}_{\mathbb{P}}\left[B|\mathcal{F}_t\right].$$

We conclude that we can set $W_t^j=Y_t^j+K1_{\Theta_t}, j=1,\dots,N$ obtaining an element of $(L^\infty(\mathcal{F}))^N$ satisfying:

$$\sum_{j=1}^{N}W_t^j\in L^\infty(\mathcal{F}_t)\,,$$

$$\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+W_t^j\right)\right)\middle|\mathcal{F}_t\right]\geq\mathbb{E}_{\mathbb{P}}\left[B|\mathcal{F}_t\right].$$

This yields

$$\rho_{\mathcal{F}_t}(X)\leq\sum_{j=1}^{N}W_t^j$$

and also

$$\lim_t\rho_{\mathcal{F}_t}(X)=\limsup_t\rho_{\mathcal{F}_t}(X)\leq\liminf_t\sum_{j=1}^{N}W_t^j=\liminf_t\sum_{j=1}^{N}Y_t^j=\sum_{j=1}^{N}\widehat{Y}^j\,.$$

STEP 2: refinement in proving inequality.
For any $Y\in(L^\infty(\mathcal{F}))^N$ such that $\sum_{j=1}^{N}Y^j\in L^\infty(\mathcal{G})$ and

$$\sum_{j=1}^{N}\mathbb{E}_{\mathbb{P}}\left[-\exp\left(-\alpha_j\left(X^j+Y^j\right)\right)\middle|\mathcal{G}\right]\geq B \tag{3.49}$$

it clearly holds that $\widehat{Y}^j=Y^j+\varepsilon,\ j=1,\dots,N$ satisfies

$$\sum_{j=1}^{N}\widehat{Y}^j\in L^\infty(\mathcal{G})\,,$$

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}}\left[u_j\left(X^j+\widehat{Y}^j\right)\Big|\mathcal{G}\right] > B$$

for all $\varepsilon > 0$. An easy limiting argument then gives us: $\lim_t \rho_{\mathcal{F}_t}(X) \le \sum_{j=1}^N Y^j$ for $Y \in (L^\infty(\mathcal{F}))^N$ such that $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ and (3.49) holds so that by definition of $\rho_{\mathcal{G}}(X)$ and of essential infimum we get $\lim_t \rho_{\mathcal{F}_t}(X) \le \rho_{\mathcal{G}}(X)$.

STEP 3: optimality of Y_∞. From the first part of this proof we have an explicit formula for $\rho_{\mathcal{G}}(X)$, now optimality can be checked just by summing the explicit formulas for Y_∞ we got in Lemma 3.5.4.

STEP 4: optimality of $\widehat{\mathbb{Q}}$. By Theorem 3.5.2 we have an explicit expression (3.45). A limiting argument can then be repeated as above using martingale convergence Theorem: observe that using (3.48), together with the fact that $X \in (L^\infty(\mathcal{F}))^N$, we get

$$\beta\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}_t^j}{\mathrm{d}\mathbb{P}}\log\left(\frac{\mathrm{d}\mathbb{Q}_t^j}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}_t\right]+A+\beta\log\left(-\frac{\mathbb{E}_{\mathbb{P}}[B|\mathcal{F}_t]}{\beta}\right)\to_t$$

$$\to_t \beta\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]+A+\beta\log\left(-\frac{B}{\beta}\right)$$

where $\mathbb{Q}_t$ is defined as in (3.45) for $\mathcal{F}_t$ is place of $\mathcal{G}$. The same computations of Remark 3.5.3, using the explicit formula for $\rho_{\mathcal{G}}(X)$ we obtained above, show that

$$\beta\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]+A+\beta\log\left(\frac{B}{\beta}\right)=\sum_{j=1}^{N}\mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X^j|\mathcal{G}]-\rho_{\mathcal{G}}(X)\,. \tag{3.50}$$

As can directly be checked, $\widehat{\mathbb{Q}} \in \mathcal{Q}^1_{\mathcal{G}}$, and by the proof of Claim 3.4.8 we get as a consequence that $\alpha^1(\widehat{\mathbb{Q}}) = \rho^*_{\mathcal{G}}\left(-\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)$. Consequently, by definition of $\rho^*_{\mathcal{G}}$ and using (3.50),

$$\alpha^1(\widehat{\mathbb{Q}})\ge\beta\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}^j}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]+A+\beta\log\left(\frac{B}{\beta}\right)\,. \tag{3.51}$$

We now prove the converse inequality in (3.51). To do so, observe that by Fenchel inequality for any $\lambda \in L^\infty(\mathcal{G})$ with $a \le \lambda$ a.s. for some $a \in (0,+\infty)$ we have

$$\alpha^1(\widehat{\mathbb{Q}}) = \operatorname*{ess\,sup}_{\substack{W\in(L^\infty(\mathcal{F}))^N\\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}]\ge B}}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^j}\left[-W^j \mid \mathcal{G}\right] \overset{\widehat{\mathbb{Q}}\in\mathcal{Q}^1_{\mathcal{G}}}{=} \operatorname*{ess\,sup}_{\substack{W\in(L^\infty(\mathcal{F}))^N\\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}]\ge B}}\sum_{j=1}^{N}\lambda\mathbb{E}_{\mathbb{P}}\left[-W^j\left(\frac{1}{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]$$

$$\le \operatorname*{ess\,sup}_{\substack{W\in(L^\infty(\mathcal{F}))^N\\ \mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}]\ge B}}\left(\lambda\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{1}{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]-\lambda\mathbb{E}_{\mathbb{P}}[U(W)|\mathcal{G}]\right)\le\lambda\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{1}{\lambda}\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]-\lambda B$$

$$=\sum_{j=1}^{N}\frac{1}{\alpha_j}\log\left(\frac{1}{\alpha_j}\right)-\sum_{j=1}^{N}\frac{1}{\alpha_j}+\sum_{j=1}^{N}\frac{1}{\alpha_j}\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\Bigg|\mathcal{G}\right]+\sum_{j=1}^{N}\frac{1}{\alpha_j}\log\left(\frac{1}{\lambda}\right)-\lambda B$$

$$= A - \beta + \beta \log\left(\frac{1}{\lambda}\right) - \lambda B + \beta \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\middle|\mathcal{G}\right].$$

We can now choose $\lambda = -\frac{\beta}{B}$ and obtain that

$$\alpha^1(\widehat{\mathbb{Q}}) \leq A - \beta + \beta \log\left(-\frac{B}{\beta}\right) - \left(-\frac{\beta}{B}\right) B + \beta \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\middle|\mathcal{G}\right]$$

$$= \beta \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \log\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\middle|\mathcal{G}\right] + A + \beta \log\left(-\frac{B}{\beta}\right)$$

which proves the converse inequality to (3.51). Combining now (3.50) and (3.51) we conclude

$$\alpha^1(\widehat{\mathbb{Q}}) = \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[-X^j|\mathcal{G}] - \rho_{\mathcal{G}}(X)$$

which, since as we already mentioned $\widehat{\mathbb{Q}} \in \mathcal{Q}^1_{\mathcal{G}}$, proves the optimality of $\widehat{\mathbb{Q}}$ in the dual representation of $\rho_{\mathcal{G}}(X)$. □

General $\mathcal{G}$

Theorem 3.5.6. *Take a general sub sigma algebra $\mathcal{G} \subseteq \mathcal{F}$, $X \in (L^\infty(\mathcal{F}))^N$, $B \in L^\infty(\mathcal{G}), \|B\|_\infty < 0$. Then $\rho_{\mathcal{G}}(X) = \beta \log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]\right) - A$ and Y_∞ is an optimal allocation for $\rho_{\mathcal{G}}(X)$. Moreover $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} := \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]}$ defines an optimum for the dual representation of $\rho_{\mathcal{G}}(X)$.*

Proof. Fix $\mathcal{G}$ and argue as in [129] Section 14.13 Step II to see that there exists a sequence of countably generated sigma algebras $(\mathcal{F}_t)_t$ such that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{F}_t\right] \xrightarrow[t\to+\infty]{L^1(\mathcal{F})} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right].$$

For each of such $\mathcal{F}_t$ Proposition 3.5.5 applies and the argument of existence of limits in Lemma 3.5.4 can be replicated. The proof of Proposition 3.5.5 can be then replicated in a step-by-step way. □

Remark 3.5.7. We covered for the explicit formulas the case $\mathscr{B}_{\mathcal{G}} = \mathscr{D}_{\mathcal{G}}$. We believe that similar formulas can be obtained for the cluster cases in Example 3.4.2, using the corresponding deterministic formulas in [20] and substituting expectations with conditional expectations. We believe the proof can be obtained in a step by step way similarly to the case we treated, with a more complicated notation.

3.5.2 Time consistency

We consider now two sub sigma algebras $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. In this subsection we will need to exploit explicitly the dependence of optimal allocations and minimax measures given by Theroem 3.5.6 on initial datum and sub sigma algebras. To fix notation, given $X \in (L^\infty(\mathcal{F}))^N$ and $\mathcal{G} \subseteq \mathcal{F}$ we define:

$$\widehat{Y}^k(\mathcal{G}, X) := -X^k + \frac{1}{\beta\alpha^k}\left(\sum_{j=1}^N X^j + \rho_{\mathcal{G}}(X) + A\right) - A^k, \tag{3.52}$$

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^k(\mathcal{G}, X)}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]}, \tag{3.53}$$

$$\widehat{a}^k(\mathcal{G}, X) = \mathbb{E}_{\widehat{\mathbb{Q}}^k(\mathcal{G}, X)}[\widehat{Y}^k(\mathcal{G}, X) \mid \mathcal{G}], \tag{3.54}$$

$$\rho_{\mathcal{G}}(X) = \beta\log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\Big|\mathcal{G}\right]\right) - A = \sum_{j=1}^N \widehat{Y}^j(\mathcal{G}, X) = \sum_{j=1}^N \widehat{a}^j(\mathcal{G}, X). \tag{3.55}$$

Theorem 3.5.8. *The following time consistency property holds whenever $B \in L^\infty(\mathcal{H})$ is given: for every $k = 1, \dots, N$*

$$\widehat{Y}^k\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right) = \widehat{Y}^k(\mathcal{H}, X) + \widehat{Y}^k(\mathcal{H}, 0), \tag{3.56}$$

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{G}, X)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right) = \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{G}, X)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, X), \tag{3.57}$$

$$\widehat{a}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = \widehat{a}^k(\mathcal{H}, X) + \widehat{a}^k(\mathcal{H}, 0). \tag{3.58}$$

Proof.
Equation (3.56): we start observing that a straightforward computation yields

$$\widehat{Y}^k(\mathcal{G}, X) = \widehat{Y}^k(\mathcal{H}, X) + \frac{1}{\beta\alpha_k}(\rho_{\mathcal{G}}(X) - \rho_{\mathcal{H}}(X)) \ \ \forall\, k = 1, \dots, N. \tag{3.59}$$

We also have, recalling $\sum_{j=1}^N \widehat{Y}^j(\mathcal{G}, X) = \rho_{\mathcal{G}}(X)$ and fixing k, that

$$\begin{aligned}
&\widehat{Y}^k\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right)\\
&= \widehat{Y}^k(\mathcal{G}, X) + \frac{1}{\beta\alpha_k}\left(-\rho_{\mathcal{G}}(X) + \rho_{\mathcal{H}}\left(-\widehat{Y}(\mathcal{G}, X)\right)\right) + \frac{1}{\beta\alpha_k}A - A^k\\
&\overset{\text{Eq.(3.59)}}{=} \widehat{Y}^k(\mathcal{H}, X) + \frac{1}{\beta\alpha_k}(-\rho_{\mathcal{H}}(X)) +\\
&\frac{1}{\beta\alpha_k}\left(\rho_{\mathcal{H}}\left(-\widehat{Y}(\mathcal{G}, X)\right) - \rho_{\mathcal{H}}(0)\right) + \frac{1}{\beta\alpha_k}(\rho_{\mathcal{H}}(0) + A) - A^k.
\end{aligned}$$

It is then enough to show that $\rho_{\mathcal{H}}\left(-\widehat{Y}(\mathcal{G}, X)\right) = \rho_{\mathcal{H}}(X) + \rho_{\mathcal{H}}(0)$, since $\widehat{Y}^k(\mathcal{H}, 0) = \frac{1}{\beta\alpha_k}(\rho_{\mathcal{H}}(0) + A) - A^k$. A direct computation yields

$$\rho_{\mathcal{H}}\left(-\widehat{Y}(\mathcal{G}, X)\right) = \beta\log\left(-\frac{\beta}{B}\right) - A + \beta\log\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\beta}\left(-\sum_{j=1}^N \widehat{Y}^j(\mathcal{G}, X)\right)\right)\Big|\mathcal{H}\right]\right)$$

$$\overset{\text{Eq.(3.55)}}{=} \beta \log\left(-\frac{\beta}{B}\right) - A + \beta \log\left(-\frac{A}{\beta}\right) +$$
$$+ \beta \log\left(\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\beta}{\beta}\log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\beta}\overline{X}\right)\middle|\mathcal{G}\right]\right)\right)\middle|\mathcal{H}\right]\right)$$
$$= \rho_{\mathcal{H}}(0) - A + \beta \log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\beta}\overline{X}\right)\middle|\mathcal{G}\right]\middle|\mathcal{H}\right]\right) .$$

Hence we have

$$\rho_{\mathcal{H}}\left(-\widehat{Y}(\mathcal{G}, X)\right) = \rho_{\mathcal{H}}(0) + \rho_{\mathcal{H}}(X) . \tag{3.60}$$

Equation (3.57): we have by (3.53) and using (3.55) that

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{G}, X)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right) = \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]}\frac{\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)\middle|\mathcal{H}\right]} . \tag{3.61}$$

We now see, just using (3.55), that

$$\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right) = \left(-\frac{\beta}{B}\right)\exp\left(-\frac{A}{\beta}\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right] ,$$

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)\middle|\mathcal{H}\right] = \left(-\frac{\beta}{B}\right)\exp\left(-\frac{A}{\beta}\right)\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{H}\right] .$$

Direct substitution in (3.61) yields

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{G}, X)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right) = \frac{\exp\left(-\frac{\overline{X}}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{H}\right]} \overset{\text{Eq.(3.53)}}{=} \frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, X) .$$

Equation (3.57): by definition (3.54) and using the fact that

$$\mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H}, -\widehat{Y}(\mathcal{G}, X)\right)\middle|\mathcal{H}\right] = 1 \;\; \forall\, k = 1, \ldots, N$$

we have

$$\widehat{a}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X)) = \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right] = E+F+G+H$$

where

$$E := \mathbb{E}_{\mathbb{P}}\left[-\left(-\widehat{a}(\mathcal{G}, X)\,k\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right] ,$$

$$F := \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\beta\alpha_k}\sum_{j=1}^{N}\left(-\widehat{a}^j(\mathcal{G}, X)\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}(\mathcal{H}, -\widehat{a}(\mathcal{G}, X))\middle|\mathcal{H}\right] ,$$

$$G := \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\beta\alpha_k}\rho_{\mathcal{H}}\left(-\widehat{a}\left(\mathcal{G},\, X\right)\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)\middle|\mathcal{H}\right],$$

$$H := \mathbb{E}_{\mathbb{P}}\left[\left(\frac{1}{\beta\alpha_k}A - A^k\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)\middle|\mathcal{H}\right] = \frac{1}{\beta\alpha_k}A - A^k .$$

We now work separately on each of the above random variables:

- considering E, by (3.54) and observing that $\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)$ is in $L^\infty(\mathcal{G})$ we have

$$\begin{aligned}
E &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k\left(\mathcal{G},\, X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{G},\, X\right)\middle|\mathcal{G}\right]\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)\middle|\mathcal{H}\right]\\
&= \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k\left(\mathcal{G},\, X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{G},\, X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)\middle|\mathcal{H}\right]\\
&\overset{\text{Eq.(3.57)}}{=} \mathbb{E}_{\mathbb{P}}\left[\widehat{Y}^k\left(\mathcal{G},\, X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, X\right)\middle|\mathcal{H}\right]\\
&\overset{\text{Eq.(3.59)}}{=} \mathbb{E}_{\mathbb{P}}\left[\left[\widehat{Y}^k\left(\mathcal{H},\, X\right) + \frac{1}{\beta\alpha_k}\left(\rho_{\mathcal{G}}\left(X\right) - \rho_{\mathcal{H}}\left(X\right)\right)\right]\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, X\right)\middle|\mathcal{H}\right].
\end{aligned}$$

  Using now the fact that $\rho_{\mathcal{H}}\left(X\right) \in L^\infty(\mathcal{H})$ and (3.54) we get

$$E = \widehat{a}^k\left(\mathcal{H},\, X\right) + \frac{1}{\beta\alpha_k}\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}\left(X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, X\right)\middle|\mathcal{H}\right] - \frac{1}{\beta\alpha_k}\rho_{\mathcal{H}}\left(X\right). \tag{3.62}$$

- We now move to F. First let us compute $\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right)$:

$$\begin{aligned}
\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\, -\widehat{a}\left(\mathcal{G},\, X\right)\right) &= \frac{\exp\left(-\frac{1}{\beta}\left(\sum_{j=1}^N -\widehat{a}^j\left(\mathcal{G},\, X\right)\right)\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{1}{\beta}\left(\sum_{j=1}^N -\widehat{a}^j\left(\mathcal{G},\, X\right)\right)\right)\middle|\mathcal{H}\right]}\\
&\overset{\text{Eq.(3.55)}}{=} \frac{\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\rho_{\mathcal{G}}(X)}{\beta}\right)\middle|\mathcal{H}\right]}\\
&= \text{computations similar to the ones for (3.57)}\\
&= \frac{\exp\left(-\frac{A}{\beta}\right)}{\exp\left(-\frac{A}{\beta}\right)}\frac{\exp\left(\frac{\beta}{\beta}\log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]\right)\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\beta}{\beta}\log\left(-\frac{\beta}{B}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]\right)\right)\middle|\mathcal{H}\right]}\\
&= \frac{-\frac{\beta}{B}}{-\frac{\beta}{B}}\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\overline{X}}{\beta}\right)\middle|\mathcal{G}\right]\middle|\mathcal{H}\right]}.
\end{aligned}$$

In conclusion

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\,-\widehat{a}\left(\mathcal{G},\,X\right)\right)=\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\,. \tag{3.63}$$

Now we have as a consequence

$$\begin{aligned}
F&=\mathbb{E}_{\mathbb{P}}\left[\frac{1}{\beta\alpha_k}\sum_{j=1}^{N}\left(-\widehat{a}^j\left(\mathcal{G},\,X\right)\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\,-\widehat{a}\left(\mathcal{G},\,X\right)\right)\Bigg|\mathcal{H}\right]\\
&=\frac{1}{\beta\alpha_k}\mathbb{E}_{\mathbb{P}}\left[-\rho_{\mathcal{G}}\left(X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\,-\widehat{a}\left(\mathcal{G},\,X\right)\right)\Bigg|\mathcal{H}\right]\\
&=\frac{1}{\beta\alpha_k}\mathbb{E}_{\mathbb{P}}\left[-\rho_{\mathcal{G}}\left(X\right)\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\Bigg|\mathcal{H}\right]\\
&=\frac{1}{\beta\alpha_k}\frac{\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[-\rho_{\mathcal{G}}\left(X\right)\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{G}\right]\Big|\mathcal{H}\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\\
&=\frac{1}{\beta\alpha_k}\frac{\mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[-\rho_{\mathcal{G}}\left(X\right)\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{G}\right]\Big|\mathcal{H}\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\\
&=\frac{1}{\beta\alpha_k}\mathbb{E}_{\mathbb{P}}\left[-\rho_{\mathcal{G}}\left(X\right)\frac{\exp\left(-\frac{X}{\beta}\right)}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\Bigg|\mathcal{H}\right]\,.
\end{aligned}$$

Recognizing the explicit expression of $\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\,X\right)$ (see (3.53)) we get

$$F=-\frac{1}{\beta\alpha_k}\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{G}}\left(X\right)\frac{\mathrm{d}\widehat{\mathbb{Q}}^k}{\mathrm{d}\mathbb{P}}\left(\mathcal{H},\,X\right)\Bigg|\mathcal{H}\right]\,. \tag{3.64}$$

- To compute G we first see that by an easy check $\rho_{\mathcal{H}}\left(-\widehat{a}\left(\mathcal{G},\,X\right)\right)=\rho_{\mathcal{H}}\left(-\widehat{Y}\left(\mathcal{G},\,X\right)\right)$. We can thus exploit (3.60) to see that $\rho_{\mathcal{H}}\left(-\widehat{a}\left(\mathcal{G},\,X\right)\right)=\rho_{\mathcal{G}}\left(0\right)+\rho_{\mathcal{H}}\left(X\right)$. Using also (3.63) we get

$$\begin{aligned}
G&=\frac{1}{\beta\alpha_k}\left(\rho_{\mathcal{H}}\left(0\right)+\mathbb{E}_{\mathbb{P}}\left[\rho_{\mathcal{H}}\left(X\right)\frac{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{X}{\beta}\right)\Big|\mathcal{H}\right]}\Bigg|\mathcal{H}\right]\right)\\
&\overset{\rho_{\mathcal{H}}(X)\in L^\infty(\mathcal{H})}{=}\frac{1}{\beta\alpha_k}\left(\rho_{\mathcal{H}}\left(0\right)+\rho_{\mathcal{H}}\left(X\right)\right)\,.
\end{aligned} \tag{3.65}$$

- Recalling (3.54) we have $\widehat{a}^k\left(\mathcal{H},\,0\right)=H+\frac{1}{\beta\alpha_k}\rho_{\mathcal{H}}\left(0\right)$ hence

$$H=\widehat{a}^k\left(\mathcal{H},\,0\right)-\frac{1}{\beta\alpha_k}\rho_{\mathcal{H}}\left(0\right)\,. \tag{3.66}$$

Summing (3.62), (3.64), (3.65), (3.66) most terms simplify and we get

$$\widehat{a}^k\left(\mathcal{H},\,-\widehat{a}\left(\mathcal{G},\,X\right)\right)=E+F+G+H=\widehat{a}^k\left(\mathcal{H},\,X\right)+\widehat{a}^k\left(\mathcal{H},\,0\right)\quad k=1,\dots,N\,.$$

□

3.6 Conditional Shortfall Systemic Risk Measures and equilibrium: Dynamic mSORTE

In Chapter 1 and Chapter 2 the equilibrium concepts of Systemic Optimal Risk Transfer Equilibrium (SORTE) and of its multivariate extension Multivariate Systemic Optimal Risk Transfer Equilibrium (mSORTE) were introduced and analyzed in a static setup. Here we show that a generalization to the conditional setting is possible and prove the existence of a time consistent family of dynamic mSORTE in the exponential setup. Consider a multivariate utility function U. For each $j=1,...,N$ consider a vector subspace $\mathcal{L}^j$ with $L^\infty(\mathcal{G})\subseteq\mathcal{L}^j\subseteq L^0(\Omega,\mathcal{F},\mathbb{P})$ and set

$$\mathcal{L}:=\mathcal{L}^1\times...\times\mathcal{L}^N\subseteq(L^0(\Omega,\mathcal{F},\mathbb{P}))^N.$$

With

$$\mathcal{M}\subseteq\mathscr{Q}_{\mathcal{G}}$$

we will denote a subset of probability vectors.

Remark 3.6.1. We impose the condition $\mathcal{M}\subseteq\mathscr{Q}_{\mathcal{G}}$ in order to guarantee that

$$(L^1(\mathcal{G},\mathbb{P}))^N=L^1(\mathcal{G},\mathbb{Q})\,.$$

For $(Y,\mathbb{Q},\,\alpha,\,A)\in(\mathcal{L}\cap L^1(\mathcal{F},\mathbb{Q}))\times\mathcal{M}\times(L^1(\mathcal{G},\mathbb{P}))^N\times L^\infty(\mathcal{G})$ define for $j=1,\dots,N$

$$Y^{[-j]}:=[Y^1,\dots,Y^{j-1},Y^{j+1},\dots,Y^N]\in L^0(\mathcal{F},\mathbb{P})^{N-1},$$
$$[Y^{[-j]};Z]:=[Y^1,\dots,Y^{j-1},Z,Y^{j+1},\dots,Y^N],\quad Z\in L^0(\mathcal{F},\mathbb{P}),$$

$$U_j^{Y^{[-j]}}(Z):=\mathbb{E}_{\mathbb{P}}\left[u_j(X^j+Z)\middle|\mathcal{G}\right]+\mathbb{E}_{\mathbb{P}}\left[\Lambda(X+[Y^{[-j]};Z])\middle|\mathcal{G}\right],\quad Z\in L^0(\mathcal{F},\mathbb{P})\,,\quad(3.67)$$

$$\mathbb{U}_j^{\mathbb{Q}^j,Y^{[-j]}}(\alpha^j):=\operatorname{ess\,sup}\left\{U_j^{Y^{[-j]}}(Z)\mid Z\in\mathcal{L}^j\cap L^1(\Omega,\mathcal{F},\mathbb{Q}^j),\,\mathbb{E}_{\mathbb{Q}^j}[Z|\mathcal{G}]\le\alpha^j\right\},\quad(3.68)$$

and

$$T^{\mathbb{Q}}(\alpha):=\operatorname{ess\,sup}\left\{\mathbb{E}_{\mathbb{P}}\left[U(X+Y)|\mathcal{G}\right]\mid Y\in\mathcal{L}\cap L^1(\mathcal{F},\mathbb{Q}),\,\mathbb{E}_{\mathbb{Q}^j}\left[Y^j|\mathcal{G}\right]\le\alpha^j,\,\forall j\right\},\quad(3.69)$$

$$S^{\mathbb{Q}}(A):=\operatorname{ess\,sup}\left\{T^{\mathbb{Q}}(\alpha)\mid\alpha\in(L^1(\mathcal{G},\mathbb{P}))^N,\,\sum_{j=1}^N\alpha^j\le A\right\}.\quad(3.70)$$

Obviously, all such quantities depend also on X, but as X will be kept fixed throughout most of the analysis, we may avoid to explicitly specify this dependence in the notations. As $u_1,\dots,u_N,\Lambda,U$ are increasing we can replace, in the definitions (3.68), (3.69), (3.70), the inequalities in the budget constraints with equalities.

Definition 3.6.2. *The triple* $(Y_X, \mathbb{Q}_X, \alpha_X) \in \mathcal{L}\times\mathcal{M}\times(L^1(\mathcal{G},\mathbb{P}))^N$ *with* $Y \in L^1(\mathcal{F}, \mathbb{Q}_X)$ *is a* ***Dynamic Multivariate Systemic Optimal Risk Transfer Equilibrium*** *(Dynamic mSORTE) with budget* $A \in L^\infty(\mathcal{G})$ *if*

1. (Y_X, α_X) *is an optimum for*

$$\operatorname*{ess\,sup}_{\substack{\alpha\in(L^1(\mathbb{P},\mathcal{G}))^N \\ \sum_{j=1}^N \alpha_j = A}} \left\{ \operatorname*{ess\,sup}_{\substack{Y\in\mathcal{L}\cap L^1(\mathcal{F},\mathbb{Q}_X) \\ \mathbb{E}_{\mathbb{Q}_X^j}\left[Y^j|\mathcal{G}\right]\le\alpha^j,\ \forall j}} \mathbb{E}_\mathbb{P}\left[U(X+Y)|\mathcal{G}\right] \right\} \tag{3.71}$$

2. $Y_X \in \mathscr{C}_\mathcal{G}$ *and* $\sum_{j=1}^N Y_X^j = A$ $\mathbb{P}$*- a.s.*

Theorem 3.6.3. *Suppose Assumption 3.4.10 and Assumption 3.4.14 hold. Let* $\widehat{Y}$ *be the optimum from Theorem 3.4.4 and let* $\widehat{\mathbb{Q}}$ *be an optimum of* (3.23)*. Define* $\widehat{\alpha}^j := \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widehat{Y}^j|\mathcal{G}]$*. Then* $(\widehat{Y}, \widehat{\mathbb{Q}}, \widehat{\alpha})$ *is a Dynamic mSORTE for* $\mathcal{Q} := \mathscr{Q}^1_\mathcal{G}$*,* $\mathcal{L} := (L^1(\mathcal{F},\mathbb{P}))^N \cap \bigcap_{\mathbb{Q}\in\mathscr{Q}^1_\mathcal{G}} L^1(\mathcal{F},\mathbb{Q})$*,* $A := \rho_\mathcal{G}(X)$*.*

Proof.
STEP 1: Item 2 of Definition 3.6.2 is satisfied. We start observing that by Theorem 3.4.4, $\widehat{Y} \in \mathscr{C}_\mathcal{G}$ and trivially being an optimum it satisfies $\sum_{j=1}^N \widehat{Y}^j = \rho_\mathcal{G}(X) =: A$.
STEP 2: we prove that for any optimum $\widehat{\mathbb{Q}} \in \mathscr{Q}^1_\mathcal{G}$ of (3.23), the optimization problem

$$\pi_A^{\mathcal{G},\mathbb{Q}}(X) := \operatorname*{ess\,sup}\left\{ \mathbb{E}_\mathbb{P}\left[U(X+Y)|\mathcal{G}\right] \;\middle|\; \begin{array}{l} Y \in (L^1(\mathcal{F},\mathbb{P}))^N \cap \bigcap_{\mathbb{Q}\in\mathcal{Q}^1_\mathcal{G}} L^1(\mathcal{F},\mathbb{Q}) \ \text{ and} \\ \sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j|\mathcal{G}] \le A \end{array} \right\}. \tag{3.72}$$

satisfies $\pi_A^{\mathcal{G},\widehat{\mathbb{Q}}}(X) = B$.
We start showing that the optimal allocation $\widehat{Y}$ for $\rho_\mathcal{G}(X)$ provided by Theorem 3.4.4 satisfies $\sum_{j=1}^N \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j|\mathcal{G}] = A$ (directly from Theorem 3.4.19) and

$$\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] = B\,.$$

To see the latter equality, observe that we already know that $\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \ge B$. If on a set Ξ of positive measure we had that the inequality is strict, we would have for some $N > 0$ that $\Xi_N := \{\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] > B + \frac{1}{N}\} \in \mathcal{G}$ has positive probability. By Assumption 3.4.10 we have using (cDOM)

$$\mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}-\frac{1}{H}\right)\middle|\mathcal{G}\right] \uparrow_H \mathbb{E}_\mathbb{P}\left[U\left(X+\widehat{Y}\right)\middle|\mathcal{G}\right] \text{ on } \Xi_N\,.$$

On a $\mathcal{G}$-measurable subset $\Theta_N \in \mathcal{G}$ of Ξ_N, $\mathbb{P}(\Theta_N) > 0$, the convergence is uniform (in H) by Theorem 3.7.2. Hence definitely in H $\mathbb{E}_{\mathbb{P}}\left[U\left(X + \widehat{Y} - \frac{1}{H}1_{\Theta_N}\right)\Big|\mathcal{G}\right] \geq B$. Clearly then $\rho_{\mathcal{G}}(X) \leq \sum_{j=1}^{N} \widehat{Y}^j - \frac{N}{H}1_{\Theta_N}$, since

$$\widehat{Y} - \frac{1}{H}1_{\Theta_N}\mathbf{1} \in \mathscr{C}_{\mathcal{G}},\, \mathbb{E}_{\mathbb{P}}\left[U\left(X + \widehat{Y} - \varepsilon 1_{\Theta_N}\mathbf{1}\right)\Big|\mathcal{G}\right] \geq B\,.$$

We then get a contradiction, since $\widehat{Y}$ is supposed to be an optimum for $\rho_{\mathcal{G}}(X)$.
Now we prove that $\pi_A^{\mathcal{G},\mathbb{Q}}(X) = B$: for $\widehat{Y}$ as before, which we stress satisfies $\widehat{Y} \in (L^1(\mathcal{F},\mathbb{P}))^N$ by Theorem 3.4.4 and $\widehat{Y} \in L^1(\mathcal{F},\mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ by Proposition 3.4.16, we have as we showed above that $\sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j|\mathcal{G}] = A$ and $\mathbb{E}_{\mathbb{P}}\left[U\left(X + \widehat{Y}\right)\Big|\mathcal{G}\right] = B$. Hence by (3.72) we have $\pi_A^{\mathcal{G},\mathbb{Q}}(X) \geq B$. Let now $(Y_n)_n$ be a maximizing sequence for $\pi_A^{\mathcal{G},\mathbb{Q}}(X)$ such that $\sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y_n^j|\mathcal{G}] = A$ (w.l.o.g.), which exists since the set over which we take essential supremum to define $\pi_A^{\mathcal{G},\mathbb{Q}}(X)$ is upward directed. Suppose for $\Delta := \{\pi_A^{\mathcal{G}}(X) > B\}$ we had $\mathbb{P}(\Delta) > 0$. Then setting $\Delta_N := \{\pi_A^{\mathcal{G}}(X) > B + \frac{1}{N}\} \in \mathcal{G}$ we have $\mathbb{P}(\Delta_N) > 0$ for some N big enough . By Theorem 3.7.2, we have that on a subset $\widetilde{\Delta}_N$ of Δ_N, having positive probability, the pointwise convergence of $\mathbb{E}_{\mathbb{P}}\left[U\left(X + Y_n\right)|\mathcal{G}\right]$ to the essential supremum is uniform. Hence given $\varepsilon > 0$ small enough, for n big enough and for $\widetilde{Y}_n = Y_n - \varepsilon 1_{\widetilde{\Delta}_N}\mathbf{1} \in \mathscr{C}_{\mathcal{G}} \cap (L^\infty(\mathbb{P},\mathcal{F}))^N$ we have $\mathbb{E}_{\mathbb{P}}\left[U\left(X + \widetilde{Y}_n\right)\Big|\mathcal{G}\right] \geq B$. Clearly $\sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widetilde{Y}_n^j|\mathcal{G}] < A$ with positive probability, by definition of $\widetilde{Y}_n$. Now we use (3.41) for $\mathbb{Q} = \widehat{\mathbb{Q}}$ and obtain that with positive probability (i.e. on $\widetilde{\Delta}_N$)

$$\rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X) \leq \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[\widetilde{Y}_n^j|\mathcal{G}] < A\,.$$

We then get a contradiction to $A := \rho_{\mathcal{G}}(X)$, since by (3.40) we would have $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}^{\widehat{\mathbb{Q}}}(X)$.
STEP 3: the optimal allocation $\widehat{Y}$ for $\rho_{\mathcal{G}}(X)$ given in Theorem 3.4.4 (which is an optimum by Theorem 3.4.19) is an optimum for RHS of (3.72), for the given $\mathbb{Q} = \widehat{\mathbb{Q}}$. This follows trivially from the arguments in the previous steps: $\widehat{Y} \in (L^1(\mathcal{F},\mathbb{P}))^N$ by Theorem 3.4.4 and $\widehat{Y} \in L^1(\mathcal{F},\mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{G}}^1$ by Proposition 3.4.16, $\sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y^j|\mathcal{G}] = A$, thus $\widehat{Y}$ satisfies the constraints of RHS of (3.72) Moreover we proved in STEP 1 that $\mathbb{E}_{\mathbb{P}}\left[U\left(X + \widehat{Y}\right)\Big|\mathcal{G}\right] = B = \pi_A^{\mathcal{G},\mathbb{Q}}(X)$.
STEP 4: conclusion. We easily see $\widehat{Y}$ is an optimum for

$$\operatorname{ess\,sup}\left\{\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] \mid Y \in \mathcal{L}, \sum_{j=1}^{N} \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y|\mathcal{G}] \leq A\right\} =$$

$$\operatorname*{ess\,sup}_{\substack{\alpha \in (L^1(\mathbb{P},\mathcal{G}))^N \\ \sum_{j=1}^{N}\alpha_j = A}} \left(\operatorname{ess\,sup}\left\{\mathbb{E}_{\mathbb{P}}\left[U\left(X+Y\right)|\mathcal{G}\right] \mid Y \in \mathcal{L}, \mathbb{E}_{\widehat{\mathbb{Q}}^j}[Y|\mathcal{G}] \leq \alpha^j\, \forall\, j = 1, \dots, N\right\}\right)\,. \tag{3.73}$$

Hence $(\widehat{Y}, \widehat{\alpha})$ are optimum for (3.71), and also Item 1 of Definition 3.6.2 is satisfied. This completes the proof. □

Corollary 3.6.4. *For the exponential utilities there exists a time consistent family of Dynamic mSORTEs.*

Proof. Follows from Theorem 3.5.8 and Theorem 3.6.3. □

3.7 Appendix to Chapter 3

3.7.1 Essential suprema and infima

In this Section 3.7.1 we write $L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$ for the set of (equivalence classes of) $[-\infty, +\infty]$-valued random variables. $L^0(\Omega, \mathcal{F}, \mathbb{P}; [0, +\infty))$ is defined analogously.

Proposition 3.7.1. *Let $\mathcal{A}, \mathcal{B} \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$ be nonempty, $0 \le \lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$, $f : \mathcal{A} \times \mathcal{B} \to L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$, $g : \mathcal{A} \to L^0(\Omega, \mathcal{F}, \mathbb{P}; [-\infty, +\infty])$, a sequence $(\alpha_n)_n \subseteq \mathcal{A}$ be given. Then:*

1. $$\operatorname*{ess\,sup}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} (\alpha + \beta) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \alpha + \operatorname*{ess\,sup}_{\beta\in\mathcal{B}} \beta = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \left(\alpha + \operatorname*{ess\,sup}_{\beta\in\mathcal{B}} \beta \right) ,$$

2. $$\operatorname*{ess\,sup}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}} f(\alpha, \beta) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \operatorname*{ess\,sup}_{\beta\in\mathcal{B}} f(\alpha, \beta) = \operatorname*{ess\,sup}_{\beta\in\mathcal{B}} \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} f(\alpha, \beta) ,$$

3. $$\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \lambda g(\alpha) = \lambda \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} g(\alpha) ,$$

4. $$\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \alpha \ge \limsup_n \alpha_n ,$$

5. $$\operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} -g(\alpha) = -\operatorname*{ess\,inf}_{\alpha\in\mathcal{A}} g(\alpha) .$$

3.7.2 Proofs: Static Systemic Risk Measures

Proof of Theorem 3.1.4. Apply Namioka-Klee Theorem in [23] together with a standard argument regarding Monetary property, to see that

$$\rho(X) = \max_{\mu\in\mathrm{ba}_1} \left(\sum_{j=1}^N \mu^j(-X^j) - \rho^*(-\mu) \right) = \sum_{j=1}^N \widehat{\mu}^j(-X^j) - \rho^*(-\widehat{\mu}) \tag{3.74}$$

for a $\widehat{\mu} \in \mathrm{ba}_1$. We follow the lines of [77], Theorem 4.22 and Lemma 4.23.
Fix any $c > -\rho(0)$ and define the set

$$\Lambda_c := \{\mu \in \mathrm{ba}_1 \text{ s.t. } \rho^*(-\mu) \le c\} .$$

Take any sequence $(X_n)_n$ in $\mathcal{L}^\infty$ such that

1. $0 \leq X_n \leq 1 \, \forall \, j \in \{1, \dots, N\}, \forall \, n$,
2. $\rho(\lambda X_n e^k) \to_n \rho(\lambda e^k)$ for each $\lambda \geq 1$ and $k \in \{1, \dots, N\}$.

Then $\inf_{\mu \in \Lambda_c} \mu^k(X_n) \to_n 1$ for all $k \in \{1, \dots, N\}$ (using the notation $\mu = [\mu^1, \dots, \mu^N]$). To see this, observe that by definition of ρ^* for any $Y \in \mathcal{L}^\infty$ and fixed $k \in \{1, \dots, N\}$

$$c \geq \rho^*(-\mu) \geq \mathbb{E}_{\mathbb{Q}}\left[(-\lambda Y)e^k\right] - \rho(\lambda Y e^k)$$

which implies

$$\inf_{\mu \in \Lambda_c} \mu(Y e^k) = \inf_{\mu \in \Lambda_c} \mu^k(Y) \geq \frac{1}{\lambda}\left(-c - \rho(\lambda Y e^k)\right) .$$

Now consider a sequence as above and observe that, by what we just saw,

$$\liminf_n \left(\inf_{\mu \in \Lambda_c} \mu^k(X_n)\right) \geq \lim_n \frac{1}{\lambda}\left(-c - \rho(\lambda X_n e^k\right) = \frac{1}{\lambda}\left(-c - \rho(\lambda e^k)\right) \overset{\text{Monet. prop.}}{=} 1 - \frac{c}{\lambda} .$$

Letting $\lambda \to +\infty$ we see that

$$\inf_{\mu \in \Lambda_c} \mu^k(X_n) \to_n 1, \quad k = 1, \dots, N .$$

Now take any optimum $\widehat{\mu}$ in the dual representation (3.74) and $c > -\rho(0)$ big enough, so that $\widehat{\mu}$ belongs to Λ_c. For any sequence of sets $(A_n)_n$ in $\mathcal{F}$ increasing to Ω we have by continuity from below that for the sequence $(X_n := 1_{A_n})_n$ Items 1 and 2 are satisfied, and we conclude that $\widehat{\mu}^k(A_n) \to_n 1$ for every $k = 1, \dots, N$. Since this happens for any sequence $(A_n)_n$ in $\mathcal{F}$ increasing to Ω, we conclude σ-additivity holds for the functional $A \mapsto \widehat{\mu}^k(1_A)$ for $k = 1, \dots, N$.
We proved that any optimum of (3.74) belongs to $\mathcal{M}_1$, which as a consequence can replace ba_1 in (3.74).

□

Proof of Corollary 3.1.5. By monotonicity and Monetary property we have that condition $\rho^*(Y) < +\infty$ implies that $Y^j \leq 0, \mathbb{E}_{\mathbb{P}}[Y^j] = -1$ for every $j = 1, \dots, N$. The claim then follows from Theorem 3.1.4 since by Radon-Nikodym Theorem we have a one to one correspondence between $\{Y \mid \rho^*(Y) < +\infty\}$ and $\mathcal{M}_1$. □

3.7.3 Miscellaneous Results

Theorem 3.7.2 (Egorov)**.** *Let $(X_n)_n$ be a sequence in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ almost surely converging to an $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$. For every $\varepsilon > 0$ there exists A_ε measurable, $\mathbb{P}(A_\varepsilon) < \varepsilon$ such that*

$$\left\|(X_n - X)\, 1_{(A_\varepsilon)^c}\right\|_\infty \to_n 0 .$$

Proof. See [5] Theorem 10.38. □

Remark 3.7.3. Observe that for any sequence of real numbers $(a_n)_n$ converging to an $a \in \mathbb{R}$ and for any sequence $(N_h)_h \uparrow +\infty$ we have $\frac{1}{N_h}\sum_{j \leq N_h} a_j \to_h a$. This can be seen as follows: for $\varepsilon > 0$ fixed take K s.t. $|a_j - a| \leq \varepsilon$ for all $j \geq K$. Take h big enough to have $N_h >> K$. Then

$$\left|\frac{1}{N_h}\sum_{j \leq N_h} a_j - a\right| \leq \frac{1}{N_h}\sum_{j \leq N_h} |a_j - a| \leq \frac{K}{N_h}\sup_{j \leq K}|a_j - a| + \frac{N_h - K}{N_h}\varepsilon$$

and we can send h to infinity.

3.7.4 Additional properties of multivariate utility functions

In this Section 3.7.4 we work under Standing Assumption I.

Corollary 3.7.4. *There exist* $a > 0$, $b \in \mathbb{R}$ *such that*

$$U(x) \leq a \sum_{j=1}^{N} x^j + b \quad \forall\, x \in \mathbb{R}^N .$$

Proof. Use Lemma 2.6.2 and observe that

$$a \sum_{j=1}^{N} x^j + a \sum_{j=1}^{N} (-(x^j)^-) + b \leq a \sum_{j=1}^{N} x^j + b .$$

□

Lemma 3.7.5. *Suppose* $(Z_n)_n$ *is a sequence in* $(L^1(\Omega, \mathcal{F}, \mathbb{P}))^N$. *Suppose furthermore that the following conditions are met for some* $B \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$*:*

1. $\sup_n \left| \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} [Z_n | \mathcal{G}] \right| < +\infty$ $\mathbb{P}-$*a.s.*
2. $\inf_n \mathbb{E}_{\mathbb{P}} [U(Z_n^j) | \mathcal{G}] \geq B$ $\mathbb{P}-$*a.s.*
3. $Z_n \to_n Z$ $\mathbb{P}-$*a.s.*

Then $\mathbb{E}_{\mathbb{P}} [U(Z) | \mathcal{G}] \geq B$ $\mathbb{P}-$*a.s.*

Proof.

STEP 1: $\sup_n \left(\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} [(Z_n^j)^+ | \mathcal{G}] \right) < +\infty$ $\mathbb{P}-$a.s.
Define the sets

$$A^+ := \left\{ \sup_n \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(Z_n)^+ \middle| \mathcal{G} \right] = +\infty \right\} \quad A^- := \left\{ \sup_n \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(Z_n)^- \middle| \mathcal{G} \right] = +\infty \right\} .$$

We prove that $\mathbb{P}(A^-) = 0$: suppose by contradiction that $\mathbb{P}(A^-) > 0$. Apply Item 2 together with the fact that A^- is $\mathcal{G}$ measurable to see that for some $a > 0, b \in \mathbb{R}$

$$B 1_{A^-} \leq \mathbb{E}_{\mathbb{P}} \left[U\left(Z_n^j\right) \middle| \mathcal{G} \right] 1_{A^-} \overset{\text{Lemma 2.6.2}}{\leq} \left(a \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[Z_n^j \mid \mathcal{G} \right] - a \sum_{j=1}^{N} (Z_n^j)^- + b \right) 1_{A^-}$$

which is a contradiction, by definition of A^- and boundedness of B. Hence $\mathbb{P}(A^-) = 0$. By Item 1, together with

$$\sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} [Z_n | \mathcal{G}] = \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(Z_n)^+ \middle| \mathcal{G} \right] - \sum_{j=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(Z_n)^- \middle| \mathcal{G} \right]$$

we have that the symmetric difference $A^+ \Delta A^-$ is $\mathbb{P}$-null (equivalently $1_{A^+} = 1_{A^-}$), so that from $\mathbb{P}(A^-) = 0$ we get $\mathbb{P}(A^+) = 0$, and the claim follows.

STEP 2: Fatou Lemma and conclusion.
By Lemma 2.6.3 for every $\varepsilon > 0$ there exist $b_\varepsilon > 0$ such that $\Gamma_\varepsilon(x) := -U(x) + 2\varepsilon \sum_{j=1}^N (x^j)^+ + b_\varepsilon \geq 0$ for all $x \in \mathbb{R}^N$. By Fatou Lemma (Γ_ε is continuous) we have that

$$-\mathbb{E}_\mathbb{P}\left[U\left(Z\right)|\mathcal{G}\right] + \varepsilon \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z^j)^+\middle|\mathcal{G}\right] + b_\varepsilon = \mathbb{E}_\mathbb{P}\left[\Gamma_\varepsilon(Z)|\mathcal{G}\right] = \mathbb{E}_\mathbb{P}\left[\liminf_n \Gamma_\varepsilon(Z_n)\middle|\mathcal{G}\right]$$

$$\leq \liminf_n \mathbb{E}_\mathbb{P}\left[\Gamma_\varepsilon(Z_n)|\mathcal{G}\right] = \liminf_n \left(-\mathbb{E}_\mathbb{P}\left[U\left(Z_n\right)|\mathcal{G}\right] + \varepsilon \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z_n^j)^+\middle|\mathcal{G}\right] + b_\varepsilon\right).$$

This chain of inequalities yields, since b_ε disappears on both sides:

$$-\mathbb{E}_\mathbb{P}\left[U\left(Z\right)|\mathcal{G}\right] + \varepsilon \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z^j)^+\middle|\mathcal{G}\right] \leq \liminf_n \left(-\mathbb{E}_\mathbb{P}\left[U\left(Z_n^j\right)|\mathcal{G}\right] + \varepsilon \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z_n^j)^+\middle|\mathcal{G}\right]\right).$$

We can thus exploit Item 2 in RHS to get

$$-\mathbb{E}_\mathbb{P}\left[U\left(Z^j\right)\middle|\mathcal{G}\right] + \varepsilon \sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z^j)^+\middle|\mathcal{G}\right] \leq \left(-B + \varepsilon \sup_n \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z_n^j)^+\middle|\mathcal{G}\right]\right)\right).$$

The latter inequality might be rewritten as

$$-\mathbb{E}_\mathbb{P}\left[U\left(Z^j\right)\middle|\mathcal{G}\right] \leq -B + \varepsilon \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z^j)^+\middle|\mathcal{G}\right] + \sup_n \left(\sum_{j=1}^N \mathbb{E}_\mathbb{P}\left[(Z_n^j)^+\middle|\mathcal{G}\right]\right)\right)$$

which holds $\mathbb{P}-$a.s. for all $\varepsilon > 0$. We observe now that the term multiplying ε in the last expression is a.s. finite by Item 3 and STEP 1. hence a standard limiting procedure over a decreasing sequence of values for ε yields the desired inequality $-\mathbb{E}_\mathbb{P}\left[U\left(Z^j\right)|\mathcal{G}\right] \leq -B$. □

Proposition 3.7.6. *Suppose the vectors $X \in (L^\infty(\mathcal{F}))^N$ and $Y \in (L^1(\mathcal{F}))^N$ satisfy $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ and*

$$\mathbb{E}_\mathbb{P}\left[U(X+Y) \mid \mathcal{G}\right] \geq B\,.$$

Suppose $\sum_{i\in I} Y^i \in L^0(\mathcal{G})$ for some family of indexes $I \subseteq \{1, \dots, N\}$. Then $\sum_{i\in I} Y^i \in L^\infty(\mathcal{G})$.

Proof. Set $A_H := \{\sum_{i\in I} Y^i < -H\} \in \mathcal{G}$ for $H > 0$ and suppose $\mathbb{P}(A_H) > 0$ for all $H > 0$. Then we have by Lemma 2.6.2 and $\mathbb{E}_\mathbb{P}\left[U\left(X+Y\right)|\mathcal{G}\right] \geq B$ that

$$B1_{A_H} \leq a\left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j + b\right)1_{A_H} - a\left(\sum_{i\in I}(X^i+Y^i)^-\right)1_{A_H}\,. \tag{3.75}$$

At the same time from $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ we must have for some index $k \in I$ (depending on H) that $A_H^k := \{Y^k < -\frac{1}{N+1}H\} \cap A_H \subseteq A_H$ satisfies $\mathbb{P}(A_H^k) > 0$ (otherwise we would get that $\sum_{i\in I} Y^i \geq -\frac{N}{N+1}H$ on A_H, which is a contradiction). From (3.75) and H big enough we also have

$$B1_{A_H^k} \leq a\left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j + b\right)1_{A_H^k} + a(-(X^k+Y^k)^-)1_{A_H^k}$$
$$\leq a\left(\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j + b\right)1_{A_H^k} + a(-(\left\|X^k\right\|_\infty + Y^k)^-)1_{A_H^k}\,.$$

Hence

$$B1_{A_H^k} \leq a\left(\left\|\sum_{j=1}^N X^j + \sum_{j=1}^N Y^j\right\|_\infty + b\right)1_{A_H^k} + a\left(\left\|X^k\right\|_\infty - \frac{H}{N+1}\right)1_{A_H^k}\,. \tag{3.76}$$

Possibly taking an even bigger H, in (3.76) we then get a contradiction.
Now set $B_H := \{\sum_{i\in I} Y^i > H\}$. Assume that $\mathbb{P}(B_H) > 0$ for all $H > 0$. Then from $\sum_{j=1}^N Y^j \in L^\infty(\mathcal{G})$ we get that

$$\mathbb{P}\left(\left\{\sum_{i\notin I} Y^i < \sum_{j=1}^N Y^j - H\right\}\right) > 0\,.$$

The argument in the first part of the proof can then be replicated, since $\sum_{i\notin I} Y^i \in L^0(\mathcal{G})$, yielding a contradiction. □

Chapter 4

Entropy Martingale Optimal Transport

In Chapter 4 we develop the duality

$$A := \inf_{\mathbb{Q}\in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c\right] + \mathcal{D}_U(\mathbb{Q})\right) = \sup_{\Delta\in\mathcal{H}} \sup_{\varphi\in\mathbf{\Phi}_\Delta(c)} S^U\left(\varphi\right) := B$$

between the Entropy Martingale Optimal Transport problem (A) and an associated optimization problem (B). Problem (A) is inspired by Entropy Optimal Transport (EOT) of of Liero et al. [108]. As customary in Martingale Optimal Transport (MOT) theory, we take the infimum of the cost functional over martingale probability measures and we consider fairly general penalty terms $\mathcal{D}_U$, which may not have a divergence formulation, unlike in [108]. In (B) the objective functional is related to the term $\mathcal{D}_U$ via Fenchel conjugacy and is not linear in general, in contrast to the classical theory of Optimal Transport (OT) or MOT. We provide several examples of such functionals and associated penalty terms, from those induced by utility functions to those determined by penalization with market prices. Our results allow us to establish a novel nonlinear and robust pricing-hedging duality, which covers many known results as special cases. The setup we consider for hedging is rather general, and allows in principle for considering semistatic trading strategies with path dependent options, beyond the classical vanilla ones.
We summarize the introductory discussion in Section I.4 in the following Table 4.1 and we point out that in Chapter 4 we develop the duality theory sketched in the last row of the Table. Differently from rows 1, 2, 5, 6, in rows 3, 4, 7, 8, the financial market is present and martingale measures are involved in the dual formulation. In rows 1, 2, 3, 4 we illustrate the classical setting, where the conditions in the functional form hold $\mathbb{P}$-a.s., while in the last four rows Optimal Transport is applied to treat the robust versions, where the inequalities holds for all elements of Ω.

Table 4.1: Prob(Ω) is the set of all probabilities on Ω; $\mathcal{P}(\mathbb{P}) = \{\mathbb{Q} \in \text{Prob}(\Omega) \mid \mathbb{Q} \ll \mathbb{P}\}$; Mart($\Omega$) is the set of all martingale probabilities on Ω; $\mathcal{M}(\mathbb{P}) = \text{Mart}(\Omega) \cap \mathcal{P}(\mathbb{P})$; $\Pi(\mathbb{Q}_1, \mathbb{Q}_2) = \{\mathbb{Q} \in \text{Prob}(\Omega) \text{ with given marginals}\}$; $\text{Mart}(\mathbb{Q}_1, \mathbb{Q}_2) = \{\mathbb{Q} \in \text{Mart}(\Omega) \text{ with given marginals}\}$; Meas($\Omega$) is the set of all positive finite measures on Ω; Sub(c) is the set of static parts of semistatic subhedging strategies for c; U is a concave proper utility functional and S^U is the associated Generalized Optimized Certainty Equivalent.

		FUNCTIONAL FORM	SUBLINEAR	CONVEX
1	- Coherent R.M.	$-\inf\{m \mid c + m \in \mathcal{A}\}$, $\mathcal{A}$ cone	$\inf_{\mathbb{Q} \in \mathcal{Q} \subseteq \mathcal{P}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[c]$	
2	- Convex R.M.	$-\inf\{m \mid c + m \in \mathcal{A}\}$, $\mathcal{A}$ convex		$\inf_{\mathbb{Q} \in \mathcal{P}(\mathbb{P})} (\mathbb{E}_{\mathbb{Q}}[c] + \alpha_{\mathcal{A}}(\mathbb{Q}))$
3	Subreplic. price	$\sup\{m \mid \exists \Delta : m + I^{\Delta}(X) \leq c\}$	$\inf_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} \mathbb{E}_{\mathbb{Q}}[c]$	
4	Indiff. price	$\sup\{m \mid U(c - m) \geq U(0)\}$		$\inf_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} (\mathbb{E}_{\mathbb{Q}}[c] + \alpha_U(\mathbb{Q}))$
5	O.T.	$\sup_{\varphi + \psi \leq c} (\mathbb{E}_{\mathbb{Q}_1}[\varphi] + \mathbb{E}_{\mathbb{Q}_2}[\psi])$	$\inf_{\mathbb{Q} \in \Pi(\mathbb{Q}_1, \mathbb{Q}_2)} \mathbb{E}_{\mathbb{Q}}[c]$	
6	E.O.T.	$\sup_{\varphi + \psi \leq c} U(\varphi, \psi)$		$\inf_{\mathbb{Q} \in \text{Meas}(\Omega)} (\mathbb{E}_{\mathbb{Q}}[c] + \mathcal{D}_U(\mathbb{Q}))$
7	M.O.T.	$\sup_{[\varphi, \psi] \in \text{Sub}(c)} (\mathbb{E}_{\mathbb{Q}_1}[\varphi] + \mathbb{E}_{\mathbb{Q}_2}[\psi])$	$\inf_{\mathbb{Q} \in \text{Mart}(\mathbb{Q}_1, \mathbb{Q}_2)} \mathbb{E}_{\mathbb{Q}}[c]$	
8	E.M.O.T.	$\sup_{[\varphi, \psi] \in \text{Sub}(c)} S^U(\varphi, \psi)$		$\inf_{\mathbb{Q} \in \text{Mart}(\Omega)} (\mathbb{E}_{\mathbb{Q}}[c] + \mathcal{D}_U(\mathbb{Q}))$

Chapter 4 is structured as follows: in Section 4.1, after introducing the necessary notation and setup, we state and prove our main results, namely Theorem 4.1.3 and Theorem 4.1.4. In Section 4.2 we analyze in detail the case of additive penalizations and valuation functionals. Section 4.3.1 collects various applications of our main results to the problems of nonlinear subhedging and superhedging. In particular we present the cases of valuation and divergences induced by utility functions (Corollaries 4.3.3 and 4.3.5), and valuation induced by penalization with market prices (Proposition 4.3.9). In Section 4.3.2 we present a dual representation for Generalized OCE associated to the indirect utility function (Theorem 4.3.13). Section 4.4 collects some auxiliary results.

4.1 A Generalized Optimal Transport Duality

For unexplained concepts on Measure Theory we refer to the Section 4.4.1. We let Ω be a Polish Space and define the following sets:

$$\begin{aligned}
\text{ca}(\Omega) &:= \{\gamma : \mathcal{B}(\Omega) \to (-\infty, +\infty) \mid \gamma \text{ is finite signed Borel measures on } \Omega\}, \\
\text{Meas}(\Omega) &:= \{\mu : \mathcal{B}(\Omega) \to [0, +\infty) \mid \mu \text{ is a non negative finite Borel measures on } \Omega\}, \\
\text{Prob}(\Omega) &:= \{\mathbb{Q} : \mathcal{B}(\Omega) \to [0, 1] \mid \mathbb{Q} \text{ is a probability Borel measures on } \Omega\}, \\
\mathcal{C}_b(\Omega, \mathbb{R}^M) &:= (\mathcal{C}_b(\Omega))^M = \{\varphi : \Omega \to \mathbb{R}^M \mid \varphi \text{ is bounded and continuous on } \Omega\}.
\end{aligned}$$

We let $\mathcal{E} \subseteq \mathcal{C}_b(\Omega; \mathbb{R}^{M+1})$ be a vector subspace, $U : \mathcal{E} \to [-\infty, +\infty)$ be a proper concave functional and set

$$V(\varphi) := -U(-\varphi).$$

We define $\mathcal{D} : \mathrm{ca}(\Omega) \to (-\infty, +\infty]$ by

$$\mathcal{D}(\gamma) := \sup_{\varphi \in \mathcal{E}} \left(U(\varphi) - \sum_{m=0}^{M} \int_\Omega \varphi_m \mathrm{d}\gamma \right) = \sup_{\varphi \in \mathcal{E}} \left(\sum_{m=0}^{M} \int_\Omega \varphi_m \mathrm{d}\gamma - V(\varphi) \right), \quad \gamma \in \mathrm{ca}(\Omega). \tag{4.1}$$

$\mathcal{D}$ is a convex functional and is $\sigma(\mathrm{ca}(\Omega), \mathcal{E})$- lower semicontinuous, even if we do not require that U is $\sigma(\mathcal{E}, \mathrm{ca}(\Omega))$-upper semicontinuous.
The following Assumption will hold throughout all Chapter 4 without further mention.

Standing Assumption 4.1.1. *$\mathcal{D}$ is proper, i.e.*

$$\mathrm{dom}(\mathcal{D}) = \{\gamma \in \mathrm{ca}(\Omega) \mid \mathcal{D}(\gamma) < +\infty\} \neq \emptyset.$$

Remark 4.1.2. Another way to introduce our setting, that will be used in Subsection 4.3.1, is to start initially with a proper convex functional $\mathcal{D} : \mathrm{ca}(\Omega) \to (-\infty, +\infty]$ which is $\sigma(\mathrm{ca}(\Omega), \mathcal{E})$-lower semicontinuous for some vector subspace $\mathcal{E} \subseteq \mathcal{C}_b(\Omega, \mathbb{R}^{M+1})$. By Fenchel-Moreau Theorem we then have the representation

$$\mathcal{D}(\gamma) = \sup_{\varphi \in \mathcal{E}} \left(\sum_{m=0}^{M} \int_\Omega \varphi_m \, \mathrm{d}\gamma - V(\varphi) \right),$$

where now V is the Fenchel-Moreau (convex) conjugate of $\mathcal{D}$, namely

$$V(\varphi) := \sup_{\gamma \in \mathrm{ca}(\Omega)} \left(\sum_{m=0}^{M} \int_\Omega \varphi_m \, \mathrm{d}\,\gamma - \mathcal{D}(\gamma) \right). \tag{4.2}$$

Setting

$$U(\varphi) := -V(-\varphi), \quad \varphi \in \mathcal{E}, \tag{4.3}$$

we get back that $\mathcal{D}$ satisfies (4.1) and additionally that U is $\sigma(\mathcal{E}, \mathrm{ca}(\Omega))$-upper semicontinuous. In conclusion, a pair $(U, \mathcal{D})$ satisfying (4.1) might be defined either providing a proper concave $U : \mathcal{E} \to [-\infty, +\infty)$, as described at the beginning of this section, or assigning a proper convex and $\sigma(\mathcal{E}, \mathrm{ca}(\Omega))$-lower semicontinuous functional $\mathcal{D} : \mathrm{ca}(\Omega) \to (-\infty, +\infty]$ as explained in this Remark.
We set

$$\mathrm{dom}(U) := \{\varphi \in \mathcal{E} \mid U(\varphi) > -\infty\}. \tag{4.4}$$

Theorem 4.1.3. *Let $c : \Omega \to (-\infty, +\infty]$ be proper lower semicontinuous with compact sublevel sets and assume the following holds:*

$$\text{There exists a sequence } (k^n)_n \subseteq \mathbb{R}^{M+1} \text{ with } \limsup_n \sum_{m=0}^{M} k_m^n = +\infty \text{ such that } U(-k^n) > -\infty \ \forall\, n. \tag{4.5}$$

Then

$$\inf_{\mu \in \mathrm{Meas}(\Omega)} \left(\int_\Omega c \, \mathrm{d}\mu + \mathcal{D}(\mu) \right) = \sup_{\varphi \in \mathbf{\Phi}(c)} U(\varphi),$$

where

$$\mathbf{\Phi}(c) := \left\{ \varphi \in \mathrm{dom}(U) \mid \sum_{m=0}^{M} \varphi_m(x) \leq c(x) \ \forall x \in \Omega \right\}. \tag{4.6}$$

Proof. We start applying (4.1) to get that

$$\int_\Omega c\,\mathrm{d}\mu + \mathcal{D}(\mu) = \int_\Omega c\,\mathrm{d}\mu + \sup_{\varphi\in\mathcal{E}} \left(U(\varphi) - \sum_{m=0}^{M} \int_\Omega \varphi_m \,\mathrm{d}\mu \right).$$

We then consider $\mathcal{L} : \mathrm{Meas}(\Omega) \times \mathrm{dom}(U) \to (-\infty, +\infty]$ defined by

$$\mathcal{L}(\mu,\varphi) := \int_\Omega \left(c - \sum_{m=0}^{M} \varphi_m \right) \mathrm{d}\,\mu + U(\varphi)$$

and we set $M := \{\mu \in \mathrm{Meas}(\Omega) \mid \int_\Omega c\,\mathrm{d}\mu < +\infty\}$. We observe that $\mathcal{L}$ is real valued on $M \times \mathrm{dom}(U)$ and for any $\mu \in \mathrm{Meas}(\Omega) \setminus M$ we have $\mathcal{L}(\mu,\varphi) = +\infty$ for all $\varphi \in \mathrm{dom}(U)$ (since c is bounded from below). We also see that setting $\mathcal{C} := \mathrm{dom}(U)$

$$\inf_{\mu\in\mathrm{Meas}(\Omega)} \left(\int_\Omega c\mathrm{d}\mu + \mathcal{D}(\mu) \right) = \inf_{\mu\in\mathrm{Meas}(\Omega)} \sup_{\varphi\in\mathcal{C}} \mathcal{L}(\mu,\varphi) = \inf_{\mu\in M} \sup_{\varphi\in\mathcal{C}} \mathcal{L}(\mu,\varphi). \tag{4.7}$$

The aim is now to interchange sup and inf in RHS of (4.7), using Theroem 4.4.8. To this end, without loss of generality we can assume $\alpha := \sup_{\varphi\in\mathcal{C}} \inf_{\mu\in\mathrm{Meas}(\Omega)} \mathcal{L}(\mu,\varphi) < +\infty$ and we have to find $\varphi \in \mathcal{C}$ and $C > \alpha$ such that the sublevel set $\mu_C := \{\mu \in \mathrm{Meas}(\Omega) \mid \mathcal{L}(\mu,\varphi) \le C\}$ is weakly compact. The functional c is proper, lower semicontinuous and has compact sublevel sets, hence it attains a minimum on Ω. Therefore, for any $\varepsilon > 0$ we can choose, by Assumption (4.5), a deterministic vector $\varphi \in \mathcal{C}$ having all components φ_m equal to some constant $-k_m^n < 0$, such that $\varphi \in \mathrm{dom}(U)$ and

$$\inf_{x\in\Omega} \left(c(x) - \sum_{m=0}^{M} \varphi_m(x) \right) > \varepsilon > 0.$$

For such choice of φ and for a sufficiently big constant $C > \alpha$ there exists another constant $D := C - U(\varphi) \ge 0$, independent of μ, such that

$$\begin{aligned} \mu_C &= \left\{ \mu \in \mathrm{Meas}(\Omega) \mid \int_\Omega \left(c - \sum_{m=0}^{M} \varphi_m \right) \mathrm{d}\mu \le D \right\} \\ &= \left\{ \mu \in \mathrm{Meas}(\Omega) \mid \int_\Omega \left(c - \sum_{m=0}^{M} \varphi_m - \varepsilon \right) \mathrm{d}\mu + \varepsilon\mu(\Omega) \le D \right\}. \end{aligned} \tag{4.8}$$

Consequently, the set μ_C is:

1. Nonempty, as the measure $\mu \equiv 0$ belongs to μ_C.

2. Narrowly closed. Indeed, for each $\varphi \in \mathcal{C}$ the function $c - \varphi$ is lower semicontinuous on Ω, and so it is the pointwise supremum of bounded continuous functions $(c_n)_n \subseteq \mathcal{C}_b(\Omega)$. For each n, $\mu \mapsto \int_\Omega c_n \,\mathrm{d}\mu$ is narrowly lower semicontinuous on $\mathrm{Meas}(\Omega)$, by definition. Hence by Monotone Convergence Theorem the map $\mu \mapsto \int_\Omega \left(c - \sum_{m=0}^{M} \varphi_m \right) \mathrm{d}\mu$ is the pointwise supremum of narrowly lower semicontinuous functions, and is lower semicontinuous with respect to the narrow topology itself. We conclude that for each $\varphi \in \mathcal{C}$ the functional $\mathcal{L}(\,\cdot\,,\varphi)$ is narrowly lower semicontinuous, and has closed sublevel sets. This implies that in particular μ_C is narrowly closed, using the central expression in (4.8).

3. Bounded: having a sequence of measures in μ_C with unbounded total mass would result in a contradiction with the constraint in the last item of (4.8), taking into account that $c - \sum_{m=0}^{M} \varphi_m - \varepsilon \geq 0$ and $\varepsilon > 0$.

4. Tight: let $0 \leq f := c - \sum_{m=0}^{M} \varphi_m - \varepsilon$. Since $\varepsilon\mu(\Omega) \geq 0$ for all $\mu \in \mathrm{Meas}(\Omega)$, by the central expression of (4.8) the inclusion $\mu_C \subseteq \{\mu \in \mathrm{Meas}(\Omega) \mid \int_\Omega f \mathrm{d}\mu \leq D\}$ holds. Now it is easy to check that for all $\mu \in \mu_C$ and $\alpha > 0$

$$D \geq \int_\Omega f \mathrm{d}\mu \geq \int_{f>\alpha} f \mathrm{d}\mu \geq \alpha\mu(\{f > \alpha\}) .$$

   Observing that the sublevels of f are compact, by lower semicontinuity of c and compactness of its sublevel sets, we see that $\{f > \alpha\}$ are complementaries of compact subsets of Ω and can be taken with arbitrarily small measure, just by increasing α, uniformly in $\mu \in \mu_C$. Thus tightness follows.

5. A subset of M.

These properties in turns yield narrow compactness of μ_C in $\mathrm{Meas}(\Omega)$, by Theorem 4.4.6, and therefore $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$ compactness (recalling that weak and narrow topology coincide in our setup). As a consequence, by Item 5 , μ_C is compact in the relative topology $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_M$. We now may apply Theorem 4.4.8. Indeed, $\mathcal{L}$ is real valued on $M \times \mathcal{C}$. Items 1 and 2 of Theorem 4.4.8 are fulfilled for: $A = M$ endowed with the topology $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_M$; $B = \mathcal{C}$; and C taken as above. We only justify explicitly lower semicontinuity $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_M$ for Item 1, which can be obtained arguing as in Item 4 above and observing that narrow topology and weak topology coincide in our setup (see Proposition 4.4.4). Hence we may interchange sup and inf in RHS of (4.7), obtaining

$$\inf_{\mu\in M} \sup_{\varphi\in\mathcal{C}} \mathcal{L}(\mu, \varphi) = \sup_{\varphi\in\mathcal{C}} \inf_{\mu\in M} \mathcal{L}(\mu, \varphi) = \sup_{\varphi\in\mathcal{C}} \inf_{\mu\in\mathrm{Meas}(\Omega)} \mathcal{L}(\mu, \varphi) \tag{4.9}$$

where the last equality follows from the fact that $\mathcal{L}(\mu, \varphi) = +\infty$ on the complementary of M in $\mathrm{Meas}(\Omega)$ for every $\varphi \in \mathcal{C}$. It is now easy to check that for every $\varphi \in \mathcal{C}$

$$\inf_{\mu\in\mathrm{Meas}(\Omega)} \mathcal{L}(\mu, \varphi) = \begin{cases} U(\varphi) & \text{if } \sum_{m=0}^{M} \varphi_m(x) \leq c(x) \, \forall \, x \in \Omega \\ -\infty & \text{otherwise} \end{cases} ,$$

thus

$$\sup_{\varphi\in\mathcal{C}} \inf_{\mu\in\mathrm{Meas}(\Omega)} \mathcal{L}(\mu, \varphi) = \sup_{\varphi\in\mathbf{\Phi}(c)} U(\varphi)$$

which concludes the proof, given (4.7) and (4.9). □

4.1.1 The Entropy Martingale Optimal Transport Duality

In order to describe a suitable theory to develop the Entropy Optimal Transport duality in a dynamic setting, in this Section we will adopt a particular product structure of the set Ω.

To this end, in addition to the notations already introduced in Section 4.1, we consider $T \in \mathbb{N}$, $T \geq 1$, and

$$\Omega := K_0 \times \cdots \times K_T \tag{4.10}$$

for $K_0, \dots, K_T \subseteq \mathbb{R}$. We denote with $X_0, \dots, X_T$ the canonical projections $X_t : \Omega \to K_t$ and we set $X = [X_0, \dots, X_T] : \Omega \to \mathbb{R}^{T+1}$, to be considered as discrete-time stochastic process. We denote with:

$$\mathrm{Mart}(\Omega) := \{\text{Martingale measures for the canonical process of } \Omega\}.$$

When $\mu \in \mathrm{Meas}(K_0 \times \cdots \times K_T)$, its marginals will be denoted with: $\mu_0, \dots, \mu_T$.
We recall, respectively from (I.37) and (I.38), that $\mathcal{H}$ is the set of admissible trading strategies and $\mathcal{I}$ is the set of elementary stochastic integral. We take $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ where $\mathcal{E}_t \subseteq \mathcal{C}_b(K_0 \times \cdots \times K_t)$ is a vector subspace, for every $t = 0, \dots, T$. Then $\mathcal{E}$ is clearly a vector subspace of $\mathcal{C}_b(\Omega; \mathbb{R}^{T+1})$, and in the stochastic processes interpretation its elements are processes adapted to the natural filtration of the process $(X_t)_t$.
We suppose that $U : \mathcal{E} \to [-\infty, +\infty)$ is proper and concave, $\mathcal{D} : \mathrm{Meas}(\Omega) \to (-\infty, +\infty]$ is defined in (4.1) and, as in (I.50),the **Generalized Optimized Certainty Equivalent** (Generalized OCE) associated to U is

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \quad \varphi \in \mathcal{E}.$$

Theorem 4.1.4. *Assume that* $\Omega := K_0 \times \cdots \times K_T$ *for compact sets* $K_0, \dots, K_T \subseteq \mathbb{R}$, *that* $c : \Omega \to (-\infty, +\infty]$ *is lower semicontinuous, that* $\mathcal{D} : \mathrm{Meas}(\Omega) \to (-\infty, +\infty]$ *is lower bounded on* $\mathrm{Meas}(\Omega)$ *and proper. Suppose also* U *satisfies* (4.5), *and that*

$$\mathcal{N} := \left\{ \mu \in \mathrm{Meas}(\Omega) \cap \mathrm{dom}(\mathcal{D}) \mid \int_\Omega c \, \mathrm{d}\mu < +\infty \right\} \neq \emptyset, \ \mathrm{dom}(U) + \mathbb{R}^{T+1} \subseteq \mathrm{dom}(U)\,. \tag{4.11}$$

Then the following holds:

$$\inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c(X)] + \mathcal{D}(\mathbb{Q}) \right) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} S^U(\varphi) \tag{4.12}$$

where for each $\Delta \in \mathcal{H}$

$$\mathbf{\Phi}_\Delta(c) := \left\{ \varphi \in \mathrm{dom}(U) \mid \sum_{t=0}^{T} \varphi_t(x_t) + \sum_{t=0}^{T-1} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) \leq c(x) \ \ \forall\, x \in \Omega \right\}. \tag{4.13}$$

Proof. The first part of the proof in inspired by [14] Equations (3.4)-(3.3)-(3.2)-(3.1).

$$\inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c(X)] + \mathcal{D}(\mathbb{Q}) \right) \tag{4.14}$$

$$= \inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \sup_{\Delta \in \mathcal{H}} \left(\mathbb{E}_{\mathbb{Q}} \left[c(X) - \sum_{t=0}^{T-1} \Delta_t(X_0, \dots, X_t)(X_{t+1} - X_t) \right] + \mathcal{D}(\mathbb{Q}) \right) \tag{4.15}$$

$$= \inf_{\mathbb{Q} \in \mathrm{Prob}(\Omega)} \sup_{\Delta \in \mathcal{H}} \left(\mathbb{E}_{\mathbb{Q}} \left[c(X) - \sum_{t=0}^{T-1} \Delta_t(X_0, \dots, X_t)(X_{t+1} - X_t) \right] + \mathcal{D}(\mathbb{Q}) \right) \tag{4.16}$$

$$= \inf_{\mathbb{Q}\in\text{Prob}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}}} \left(\mathbb{E}_{\mathbb{Q}} \left[c(X) - \sum_{t=0}^{T-1} \Delta_t(X_0,\dots,X_t)(X_{t+1}-X_t) + \lambda \right] - \lambda + \mathcal{D}(\mathbb{Q}) \right) \tag{4.17}$$

$$= \inf_{\mu\in\text{Prob}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}}} \left(\int_\Omega \left[c(x) - I^\Delta(x) + \lambda \right] \mathrm{d}\mu(x) - \lambda + \mathcal{D}(\mu) \right) \tag{4.18}$$

$$= \inf_{\mu\in\text{Meas}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}}} \left(\int_\Omega \left[c - I^\Delta + \lambda \right] \mathrm{d}\mu - \lambda + \mathcal{D}(\mu) \right) \tag{4.19}$$

$$= \inf_{\mu\in\text{Meas}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}^{T+1}}} \left(\int_\Omega \left[c - I^\Delta + \sum_{t=0}^{T} \lambda_t \right] \mathrm{d}\mu - \sum_{t=0}^{T} \lambda_t + \mathcal{D}(\mu) \right). \tag{4.20}$$

The equality chain above is justified as follows: (4.14)=(4.15) is trivial; (4.15)=(4.16) follows using the same argument as in [14] Lemma 2.3, which yields that the inner supremum explodes to $+\infty$ unless $\mathbb{Q}$ is a martingale measure on Ω; (4.16)=(4.17) and (4.17)=(4.18) are trivial; (4.18)=(4.19) follows observing that the inner supremum over $\lambda \in \mathbb{R}$ explodes to $+\infty$ unless $\mu(\Omega) = 1$; (4.19)=(4.20) is trivial.
We define now $\mathcal{K} : \text{Meas}(\Omega) \times (\mathcal{H} \times \mathbb{R}^M) \to (-\infty, +\infty]$ as

$$\mathcal{K}(\mu, \Delta, \lambda) := \int_\Omega \left[c - I^\Delta + \sum_{t=0}^{T} \lambda_t \right] \mathrm{d}\mu - \sum_{t=0}^{T} \lambda_t + \mathcal{D}(\mu).$$

From (4.11), we observe that $\mathcal{K}$ is real valued on $\mathcal{N} \times (\mathcal{H} \times \mathbb{R}^{T+1})$ and that $K(\mu, \Delta, \lambda) = +\infty$ if $\mu \in \text{Meas}(\Omega) \setminus \mathcal{N}$, for all $(\Delta, \lambda) \in \mathcal{H} \times \mathbb{R}^{T+1}$. This, together with our previous computations, provides

$$\inf_{\mathbb{Q}\in\text{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \mathcal{D}(\mathbb{Q}) \right) = \inf_{\mu\in\text{Meas}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}^{T+1}}} \mathcal{K}(\mu, \Delta, \lambda) = \inf_{\mu\in\mathcal{N}} \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}^{T+1}}} \mathcal{K}(\mu, \Delta, \lambda). \tag{4.21}$$

As in the proof of Theorem 4.1.3, we wish to apply the Minimax Theorem 4.4.8 in order to interchange inf and sup in RHS of (4.21) and without loss of generality we can assume that $\alpha := \sup_{\substack{\Delta\in\mathcal{H}\\ \lambda\in\mathbb{R}^{T+1}}} \inf_{\mu\in\mathcal{N}} \mathcal{K}(\mu, \Delta, \lambda) < +\infty$. The functional $\mathcal{K}$ is real valued on $\mathcal{N} \times (\mathcal{H} \times \mathbb{R}^{T+1})$ and convexity in Item 1, concavity in Item 2 of Theorem 4.4.8 are clearly satisfied. We have to find $\Delta \in \mathcal{H}$, $\lambda \in \mathbb{R}^{T+1}$ and $C > \alpha$ such that the sublevel set $M_C := \{\mu \in \text{Meas}(\Omega) \mid \mathcal{K}(\mu, \Delta, \lambda) \leq C\}$ is weakly compact. Fix a $\varepsilon > 0$. As the functional c is lower semicontinuous on the compact Ω, it is lower bounded on Ω and we can take $\Delta = 0$ and λ sufficiently big in such a way that $\inf_{x\in\Omega}(c(x) + \sum_{t=0}^{T} \lambda_t) > \varepsilon$. For such a choice of (Δ, λ) we have that M_C is a subset of

$$\left\{ \mu \in \text{Meas}(\Omega) \mid \int_\Omega \left[c + \sum_{t=0}^{T} \lambda_t - \varepsilon \right] \mathrm{d}\mu(x) + \varepsilon\mu(\Omega) \leq C + \sum_{t=0}^{T} \lambda_t - \inf_{\mu\in\text{Meas}(\Omega)} \mathcal{D}(\mu) =: D \right\}$$

where $D \in \mathbb{R}$ since $\mathcal{D}(\cdot)$ is lower bounded by hypothesis. By (4.11) and for large enough C, the set M_C is nonempty, and the same arguments in Items 2, 3 and 4 of the proof of Theorem 4.1.3 can be applied to conclude that the set M_C is narrowly closed, bounded and tight, hence narrowly and $\sigma(\text{Meas}(\Omega), \mathcal{C}_b(\Omega))$-compact. Moreover we see that $M_C \subseteq \mathcal{N}$, hence it is also compact in the topology $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_{\mathcal{N}}$. We finally verify $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_{\mathcal{N}}$-lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ on $\mathcal{N}$ for every $(\Delta, \lambda) \in (\mathcal{H} \times \mathbb{R}^{T+1})$. To see this, observe that arguing as in Item 2 of the proof of Theorem 4.1.3 we get that $\mu \mapsto \int_\Omega \left[c - I^\Delta + \sum_{t=0}^T \lambda_t\right] \mathrm{d}\mu - \sum_{t=0}^T \lambda_t$ is $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))|_{\mathcal{N}}$-lower semicontinuous, while $\mathcal{D}$ is by definition $\sigma(\mathrm{ca}(\Omega), \mathcal{E})|_{\mathcal{N}}$ lower semicontinuous (being supremum of linear functionals each continuous in such a topology). Since sum of lower semicontinuous functions is lower semicontinuous, the desired lower semicontinuity of $\mathcal{K}(\cdot, \Delta, \lambda)$ follows. All the hypotheses of Theorem 4.4.8 are now verified, and we may then interchange sup and inf in RHS of (4.21) and obtain

$$\inf_{\mu \in \mathcal{N}} \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \mathcal{K}(\mu, \Delta, \lambda) = \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \inf_{\mu \in \mathcal{N}} \mathcal{K}(\mu, \Delta, \lambda) \overset{(\star)}{=} \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \inf_{\mu \in \mathrm{Meas}(\Omega)} \mathcal{K}(\mu, \Delta, \lambda)$$

$$= \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \inf_{\mu \in \mathrm{Meas}(\Omega)} \left(\int_\Omega \left[c - I^\Delta + \sum_{t=0}^T \lambda_t \right] \mathrm{d}\mu + \mathcal{D}(\mu) \right) - \sum_{t=0}^T \lambda_t, \tag{4.22}$$

where in $(\star)$ we used the fact that $\mathcal{K}(\mu, \Delta, \lambda) = +\infty$ on the complementary of $\mathcal{N}$ in $\mathrm{Meas}(\Omega)$, for every $(\Delta, \lambda) \in \mathcal{H} \times \mathbb{R}^{T+1}$.
We apply now Theorem 4.1.3 to the inner infimum with the cost functional $c - I^\Delta + \sum_{t=0}^T \lambda_t$, observing that, since we are assuming $\mathrm{dom}(U) + \mathbb{R}^{T+1} = \mathrm{dom}(U)$ (see (4.11)), the condition (4.5) is satisfied. We get that

$$(4.22) = \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \sup_{\varphi \in \mathbf{\Phi}_{\Delta, \lambda}(c)} \left(U(\varphi) - \sum_{t=0}^T \lambda_t \right)$$

where $\mathbf{\Phi}_{\Delta, \lambda}(c)$, which depends on $\Delta, \lambda \in \mathcal{H} \times \mathbb{R}^{T+1}$, is defined according to (4.6) by

$$\mathbf{\Phi}_{\Delta, \lambda}(c) = \left\{ \varphi \in \mathrm{dom}(U), \ \sum_{t=0}^T \varphi_t(x) \leq c(x) - I^\Delta(x) + \sum_{t=0}^T \lambda_t \ \ \forall \, x \in \Omega \right\} .$$

From (4.11), $(\varphi_t - \lambda_t)_t \in \mathrm{dom}(U)$ and we can absorb λ in φ obtaining

$$\mathbf{\Phi}_{\Delta, \lambda}(c) = \mathbf{\Phi}_\Delta(c) + \lambda, \ \forall \, \lambda \in \mathbb{R}^{T+1}, \Delta \in \mathcal{H} \,,$$

with $\mathbf{\Phi}_\Delta(c)$ given in (4.13), so that

$$(4.22) = \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} \left(U(\varphi + \lambda) - \sum_{t=0}^T \lambda_t \right) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} \sup_{\lambda \in \mathbb{R}^{T+1}} \left(U(\varphi + \lambda) - \sum_{t=0}^T \lambda_t \right).$$

We now recognize the expression in (I.50) and we conclude that

$$\inf_{\mu \in \mathrm{Meas}(\Omega)} \sup_{\substack{\Delta \in \mathcal{H} \\ \lambda \in \mathbb{R}^{T+1}}} \mathcal{K}(\mu, \Delta, \lambda) = (4.22) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} S^U(\varphi) \,,$$

and consequently, recalling our minimax argument,

$$\inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_\mathbb{Q}\left[c(X)\right] + \mathcal{D}(\mathbb{Q}) \right) \overset{\text{Eq.(4.21)}}{=} (4.22) = \sup_{\Delta \in \mathcal{H}} \sup_{\varphi \in \mathbf{\Phi}_\Delta(c)} S^U(\varphi) \,.$$

□

Remark 4.1.5. (i) The assumptions of Theorem 4.1.4 are reasonably weak and are satisfied, for example, if: $\mathrm{dom}(U) = \mathcal{E}$, there exists a $\widehat{\mu} \in \mathrm{Meas}(\Omega) \cap \partial U(0)$ such that $c \in L^1(\widehat{\mu})$, and c is lower semicontinuous. Indeed, for all $\mu \in \mathrm{Meas}(\Omega)$, $\mathcal{D}(\mu) \geq U(0) - 0 > -\infty$. Clearly $\mathrm{dom}(U) + \mathbb{R}^{T+1} = \mathrm{dom}(U)$. Finally, $\widehat{\mu} \in \mathcal{N}$, because $c \in L^1(\widehat{\mu})$ and $-\infty < U(0) \leq \mathcal{D}(\widehat{\mu}) \leq 0$, by definition of $\mathcal{D}$.
(ii) The step (4.15)=(4.16) is the crucial point where compactness of the sets $K_0, \dots, K_T \subseteq \mathbb{R}$ is necessary for a smooth argument, since integrability of the underlying stock process is in this case automatically satisfied for all $\mathbb{Q} \in \mathrm{Prob}(\Omega)$, not only for $\mathbb{Q} \in \mathrm{Mart}(\Omega)$. Also, compactness is key in guaranteeing that the cost functional $c - I^\Delta + \sum_t \lambda_t$ is bounded from below, in order to apply Theorem 4.1.3.

Proposition 4.1.6. *Suppose that LHS of* (4.12) *is finite and that* $\mathcal{D}|_{\mathrm{Meas}(\Omega)}$ *is lower semicontinuous with respect to the topology* $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$. *Then, under the same the assumptions of Theorem 4.1.4, the problem in LHS of* (4.12) *admits an optimum.*

Proof. Similarly to what we argued in Item 2 of the proof of Theorem 4.1.3, the map $\mu \mapsto \int_\Omega c \,\mathrm{d}\mu$ is $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$-lower semicontinuous, and we deduce the lower semicontinuity of

$$\mathbb{Q} \mapsto \mathcal{J}(\mathbb{Q}) := \mathbb{E}_\mathbb{Q}[c] + \sum_{t=0}^{T} \mathcal{D}(\mathbb{Q}), \quad \mathbb{Q} \in \mathrm{Mart}(\Omega)\,.$$

Moreover for C big enough the sublevel $\{\mathbb{Q} \in \mathrm{Mart}(\Omega) \mid \mathcal{J}(\mathbb{Q}) \leq C\}$ is nonempty (since we are assuming LHS of (4.12) is finite), hence $\mathcal{J}$ is proper on $\mathrm{Mart}(\Omega)$. Since $K_0, ..., K_T$ are compact, $\mathrm{Prob}(\Omega)$ is $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$-compact (see [5] Theorem 15.11), and $\mathrm{Mart}(\Omega)$ is $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$-closed because, arguing as in [14] Lemma 2.3,

$$\mathrm{Mart}(\Omega) = \bigcap_{\Delta \in \mathcal{H}} \left\{ \mathbb{Q} \in \mathrm{Prob}(\Omega) \mid \int_\Omega \left(\sum_{t=0}^{T-1} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) \right) \mathrm{d}\mathbb{Q}(x) \leq 0 \right\}.$$

We conclude that $\mathrm{Mart}(\Omega)$ is $\sigma(\mathrm{Meas}(\Omega), \mathcal{C}_b(\Omega))$-compact, and $\mathcal{J}$ is lower semicontinuous and proper on it, hence it attains a minimum. ☐

Remark 4.1.7. The lower semicontinuity assumption in Proposition 4.1.6 is satisfied in many cases, as it will become clear in Section 4.2.

4.1.2 A useful rephrasing of Theorem 4.1.4

We now rephrase our findings in Theorem 4.1.4, with minor additions, to get the formulation in Corollary 4.1.8 which will simplify our discussion of Section 4.3. In particular, this reformulation will come in handy when dealing with subhedging and superhedging dualities in Corollaries 4.3.3-4.3.8 and Proposition 4.3.9.
For a given proper concave $U : \mathcal{E} \to \mathbb{R}$, we recall the definition of S^U in (I.50)

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \quad \varphi \in \mathcal{E}$$

and, for $V(\cdot) = -U(-\cdot)$, we define $\mathrm{dom}(V) := \{\varphi \in \mathcal{E} \mid V(\varphi) < +\infty\} = -\mathrm{dom}(U)$ and

$$S_V(\varphi) := \inf_{\lambda \in \mathbb{R}^{T+1}} \left(V(\varphi + \lambda) - \sum_{t=0}^{T} \lambda_t \right) = -S^U(-\varphi), \quad \varphi \in \mathrm{dom}(V)\,.$$

Furthermore, given functions $c : \Omega \to (-\infty, +\infty]$, $d : \Omega \to [-\infty, +\infty)$ we introduce the sets

$$\mathcal{S}_{sub}(c) := \left\{ \varphi \in \mathrm{dom}(U) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi(x_t) + I^{\Delta}(x) \le c(x) \ \ \forall x \in \Omega \right\}, \tag{4.23}$$

$$\mathcal{S}_{sup}(d) := \left\{ \varphi \in \mathrm{dom}(V) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi(x_t) + I^{\Delta}(x) \ge d(x) \ \ \forall x \in \Omega \right\}. \tag{4.24}$$

Corollary 4.1.8. *Suppose that the assumptions in Theorem 4.1.4 are satisfied, that $d : \Omega \to [-\infty, +\infty)$ is upper semicontinuous and that $\{\mu \in \mathrm{Meas}(\Omega) \cap \mathrm{dom}(\mathcal{D}) \mid \int_\Omega d \, \mathrm{d}\mu > -\infty\} \neq \emptyset$. Then the following hold*

$$\inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \mathcal{D}(\mathbb{Q}) \right) = \sup_{\varphi \in \mathcal{S}_{sub}(c)} S^U(\varphi)\,, \tag{4.25}$$

$$\sup_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[d(X)\right] - \mathcal{D}(\mathbb{Q}) \right) = \inf_{\varphi \in \mathcal{S}_{sup}(d)} S_V(\varphi)\,. \tag{4.26}$$

Proof. Equation (4.25) is an easy rephrasing of the corresponding (4.12). As to (4.26), we observe that for $c := -d$ we get from (4.25)

$$\sup_{\varphi \in \mathcal{S}_{sub}(-d)} S^U(\varphi) = \inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[-d(X)\right] + \mathcal{D}(\mathbb{Q}) \right) = - \sup_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[d(X)\right] - \mathcal{D}(\mathbb{Q}) \right).$$

Observing that

$$\mathcal{S}_{sup}(d) = -\mathcal{S}_{sub}(-d)$$

and that $S_V(\cdot) = -S^U(-\cdot)$ on $\mathrm{dom}(V)$ we get

$$\sup_{\varphi \in \mathcal{S}_{sub}(-d)} S^U(\varphi) = - \inf_{\varphi \in \mathcal{S}_{sup}(d)} S_V(\varphi)\,.$$

This completes the proof. □

4.2 Additive structure

In Section 4.1, we did not require any particular structural form of the functionals $\mathcal{D}, U$. Here instead, we will assume in addition to (4.10) also an additive structure of U and, complementarily, an additive structure of $\mathcal{D}$. In the whole Section 4.2 we take for each $t = 0, \dots, T$ a vector subspace $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ such that $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ and set $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$. Observe that we automatically have $\mathcal{E} + \mathbb{R}^{T+1} = \mathcal{E}$. It is also clear that $\mathcal{E}$ is a subspace of $\mathcal{C}_b(\Omega, \mathbb{R}^{T+1})$, if we interpret $\mathcal{E}_0, \dots, \mathcal{E}_T$ as subspaces of $\mathcal{C}_b(\Omega)$.

4.2.1 Additive Structure of U

Setup 4.2.1. *For every $t = 0, \dots, T$ we consider a proper concave functional $U_t : \mathcal{E}_t \to [-\infty, +\infty)$. We define $\mathcal{D}_t$ on $\mathrm{ca}(K_t)$ similarly to (4.1) as*

$$\mathcal{D}_t(\gamma_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left(U_t(\varphi_t) - \int_{K_t} \varphi_t \,\mathrm{d}\gamma_t \right) \quad \gamma_t \in \mathrm{ca}(K_t)$$

and observe that $\mathcal{D}_t$ can also be thought to be defined on $\mathrm{ca}(\Omega)$ using for $\gamma \in \mathrm{ca}(\Omega)$ the marginals $\gamma_0, \dots, \gamma_T$ and setting $\mathcal{D}_t(\gamma) := \mathcal{D}_t(\gamma_t)$. We may now define, for each $\varphi \in \mathcal{E}$, $U(\varphi) := \sum_{t=0}^T U_t(\varphi_t)$ and define $\mathcal{D}$ on $\mathrm{ca}(\Omega)$ using (4.1) with $M = T$. Recall from (I.50)

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^T \xi_t \right), \ \varphi \in \mathcal{E}\,,$$
$$S^{U_t}(\varphi_t) := \sup_{\alpha \in \mathbb{R}} \left(U_t(\varphi_t + \alpha) - \alpha \right), \ \ \varphi_t \in \mathcal{E}_t\,.$$

Lemma 4.2.2. *In Setup 4.2.1 we have*

$$\mathcal{D}(\gamma) = \sum_{t=0}^T \mathcal{D}_t(\gamma) = \sum_{t=0}^T \mathcal{D}_t(\gamma_t), \quad \forall\, \gamma \in \mathrm{ca}(\Omega), \quad S^U(\varphi) = \sum_{t=0}^T S^{U_t}(\varphi_t) \text{ for all } \varphi \in \mathcal{E}, \tag{4.27}$$

and for all $\varphi \in \mathcal{E}$

$$S^U(\varphi + \beta) = S^U(\varphi) + \sum_{t=0}^T \beta_t, \text{ for } \beta \in \mathbb{R}^{T+1}, \quad S^{U_t}(\varphi_t + d) = S^{U_t}(\varphi_t) + d, \text{ for } d \in \mathbb{R}.$$

Proof. See Section 4.4.2. □

4.2.2 Duality for the general Cash Additive setup

Theorem 4.2.3. *Suppose for each $t = 0, \dots, T$ $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ is a vector subspace satisfying $\mathrm{Id}_t \in \mathcal{E}_t$ and $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$ and that $S_t : \mathcal{E}_t \to \mathbb{R}$ is a concave, cash additive functional null in 0. Consider for every $t = 0, \dots, T$ the penalizations*

$$\mathcal{D}_t(\mathbb{Q}_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left(S_t(\varphi_t) - \int_{K_t} \varphi_t \,\mathrm{d}\mathbb{Q}_t \right) \quad \text{for } \mathbb{Q}_t \in \mathrm{Prob}(K_t),$$

and set $\mathcal{D}(\mathbb{Q}) := \sum_{t=0}^T \mathcal{D}_t(\mathbb{Q}_t)$. Let $c : \Omega \to (-\infty, +\infty]$ be lower semicontinuous and let $\mathfrak{D}(c)$ and $\mathfrak{P}(c)$ be defined respectively in (I.42) *and* (I.47)*. If $\mathcal{N} := \left\{\mu \in \mathrm{Meas}(\Omega) \cap \mathrm{dom}(\mathcal{D}) \mid \int_\Omega c \,\mathrm{d}\mu < +\infty\right\} \neq \emptyset$ then $\mathfrak{P}(c) = \mathfrak{D}(c)$.*

Proof. Set $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ and $U(\varphi) := \sum_{t=0}^T S_t(\varphi_t)$, for $\varphi \in \mathcal{E}$, and let $\mathcal{D}$ defined as in (4.1) for $M = T$. For any $\mu \in \mathrm{Meas}(\Omega)$ we have $\mathcal{D}(\mu) \geq \sum_{t=0}^T S_t(0) - 0 = 0$ hence $\mathcal{D}$ is lower bounded on $\mathrm{Meas}(\Omega)$. Observe that $\mathrm{dom}(U) = \mathcal{E}$, which implies $\mathrm{dom}(U) + \mathbb{R}^{T+1} = \mathrm{dom}(U)$, and that we are in Setup 4.2.1. Lemma 4.2.2 tells us that $S^U(\varphi) = \sum_{t=0}^T S_t^{U_t}(\varphi_t) = \sum_{t=0}^T S_t(\varphi_t)$, since $S_0, \dots, S_T$ are Cash Additive, and that $\mathcal{D}$ coincides on $\mathrm{Mart}(\Omega)$ with the penalization term $\mathbb{Q} \mapsto \sum_{t=0}^T \mathcal{D}_t(\mathbb{Q}_t)$, as provided in the statement of this Theorem. Since all the assumptions of Theorem 4.1.4 are fulfilled, we can apply Corollary 4.1.8, which yields exactly $\mathfrak{D}(c) = \mathfrak{P}(c)$. □

4.2.3 Additive Structure of $\mathcal{D}$.

The results of this subsection will be applied in Subsection 4.3.1. In the spirit of Remark 4.1.2, we may now reverse the procedure taken in the previous subsection: we start from some functionals $\mathcal{D}_t$ on $\mathrm{ca}(K_t)$, for $t = 0, \ldots, T$, and build an additive functional $\mathcal{D}$ on $\mathrm{ca}(\Omega)$. Our aim is to find the counterparts of the results in Section 4.2.1.

Setup 4.2.4. *For every $t = 0, \ldots, T$ we consider a proper, convex, $\sigma(\mathrm{ca}(K_t), \mathcal{E}_t)$-lower semicontinuous functional $\mathcal{D}_t : \mathrm{ca}(K_t) \to (-\infty, +\infty]$. We can then extend the functionals $\mathcal{D}_t$ to $\mathrm{ca}(\Omega)$ by using, for any $\gamma \in \mathrm{ca}(\Omega)$, the marginals $\gamma_0, \ldots, \gamma_T$. If $\gamma \in \mathrm{ca}(\Omega)$, we set*

$$\mathcal{D}_t(\gamma) := \mathcal{D}_t(\gamma_t) \qquad \text{and} \qquad \mathcal{D}(\gamma) := \sum_{t=0}^{T} \mathcal{D}_t(\gamma) = \sum_{t=0}^{T} \mathcal{D}_t(\gamma_t)\,.$$

We define $V(\varphi)$ for $\varphi \in \mathcal{E}$ and $V_t(\varphi_t)$ for $\varphi_t \in \mathcal{E}_t$, for $t = 0, \ldots, T$ similarly to (4.2), as

$$V(\varphi) := \sup_{\gamma \in \mathrm{ca}(\Omega)} \left(\int_\Omega \left(\sum_{t=0}^{T} \varphi_t \right) \mathrm{d}\gamma - \mathcal{D}(\gamma) \right) \text{ and } V_t(\varphi_t) := \sup_{\gamma \in \mathrm{ca}(K_t)} \left(\int_{K_t} \varphi_t \,\mathrm{d}\gamma - \mathcal{D}_t(\gamma) \right).$$

We define on $\mathcal{E}$ the functional $U(\cdot) = -V(-\cdot)$, as in (4.3), and similarly $U_t(\cdot) = -V_t(-\cdot)$ on $\mathcal{E}_t$, for $t = 0, \ldots, T$. Finally, $S^U(\varphi)$, $S^{U_0}(\varphi_0), \ldots, S^{U_T}(\varphi_T)$ are defined as in Setup 4.2.1.

Lemma 4.2.5. *In Setup 4.2.4 we have:*

1. *$\mathcal{D}_0, \ldots, \mathcal{D}_T$, as well as $\mathcal{D}$, are $\sigma(\mathrm{ca}(\Omega), \mathcal{E})$-lower semicontinuous.*
2. *Under the additional assumption that $\mathrm{dom}(\mathcal{D}_t) \subseteq \mathrm{Prob}(K_t)$ for every $t = 0, \ldots, T$, for any $\varphi = [\varphi_0, \ldots, \varphi_T] \in \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$*

$$U(\varphi) = \sum_{t=0}^{T} U_t(\varphi_t) = \sum_{t=0}^{T} -V_t(-\varphi_t)\,, \tag{4.28}$$

$$S^U(\varphi) = \sum_{t=0}^{T} S^{U_t}(\varphi_t)\,. \tag{4.29}$$

Proof. See Section 4.4.2. □

4.2.4 Divergences induced by utility functions

Assumption 4.2.6. *We consider concave, upper semicontinuous nondecreasing functions $u_0, \ldots, u_T : \mathbb{R} \to [-\infty, +\infty)$ with $u_0(0) = \cdots = u_T(0) = 0$, $u_t(x) \le x\ \forall\, x \in \mathbb{R}$ (that is $1 \in \partial u_0(0) \cap \cdots \cap \partial u_T(0)$). For each $t = 0, \ldots, T$ we define $v_t(x) := -u_t(-x)$, $x \in \mathbb{R}$ and*

$$v_t^*(y) := \sup_{x \in \mathbb{R}} (xy - v_t(x)) = \sup_{x \in \mathbb{R}} (u_t(x) - xy)), \quad y \in \mathbb{R}\,. \tag{4.30}$$

We observe that $v_t(y) = v_t^{**}(y) = \sup_{x\in\mathbb{R}}(xy - v_t^*(y))$ for all $y \in \mathbb{R}$ by Fenchel-Moreau Theorem and that v_t^* is convex, lower semicontinuous and lower bounded on $\mathbb{R}$.

Example 4.2.7. Assumption 4.2.6 is satisfied by a wide range of functions. Just to mention a few with various peculiar features, we might take u_t of the following forms: $u_t(x) = 1 - \exp(-x)$, whose convex conjugate is given by $v_t^*(y) = -\infty$ for $y < 0$, $v_t^*(0) = 0$, $v_t^*(y) = (y\log(y) - y + 1)$ for $y > 0$; $u_t(x) = \alpha x 1_{(-\infty,0]}(x)$ for $\alpha \geq 1$, so that $v_t^*(y) = +\infty$ for $y < 0$, $v_t^*(y) = 0$ for $y \in [0, \alpha]$, $v_t^*(y) = +\infty$ for $y > \alpha$; $u_t(x) = \log(x + 1)$ for $x > -1$, $u_t(x) = -\infty$ for $x \leq -1$, so that $v_t^*(y) = +\infty$ for $y \leq 0$, $v_t^*(y) = y - \log(y) - 1$ for $y > 0$; $u_t(x) = -\infty$ for $x \leq -1$, $u_t(x) = \frac{x}{x+1}$ for $x > -1$ so that $v_t^*(y) = -\infty$ for $y < 0$, $v_t^*(y) = y - 2\sqrt{y} + 1$ for $y \geq 0$; $u_t(x) = -\infty$ for $x < 0$, $u_t(x) = 1 - \exp(-x)$ for $x \geq 0$, so that $v_t^*(y) = +\infty$ for $y < 0$, $v_t^*(y) = y\log(y) - y + 1$ for $0 \leq y \leq 1$, $v_t^*(y) = 0$ for $y > 1$.

Fix $\widehat{\mu}_t \in \mathrm{Meas}(K_t)$. We pose for $\mu \in \mathrm{Meas}(K_t)$

$$\mathcal{D}_{v_t^*,\widehat{\mu}_t}(\mu) := \begin{cases} \int_{K_t} v_t^*\left(\frac{\mathrm{d}\mu}{\mathrm{d}\widehat{\mu}_t}\right) \mathrm{d}\widehat{\mu}_t & \text{if } \mu \ll \widehat{\mu}_t \\ +\infty & \text{otherwise} \end{cases}. \tag{4.31}$$

In the next two propositions, whose proofs are postponed to the Section 4.4.2, we provide the dual representation of the divergence terms.

Proposition 4.2.8. *Take $u_0, \dots, u_T$ satisfying Assumption 4.2.6, and suppose*

$$\mathrm{dom}(u_0) = \cdots = \mathrm{dom}(u_T) = \mathbb{R}.$$

Let $\widehat{\mu}_t \in \mathrm{Meas}(K_t)$ and $v_t(\cdot) := -u_t(-\cdot)$, $t = 0, \dots, T$. Then for every $\mu \in \mathrm{Meas}(K_t)$

$$\mathcal{D}_{v_t^*,\widehat{\mu}_t}(\mu) = \sup_{\varphi_t\in\mathcal{C}_b(K_t)} \left(\int_{K_t} \varphi_t(x_t)\,\mathrm{d}\mu(x_t) - \int_{K_t} v_t(\varphi_t(x_t))\,\mathrm{d}\widehat{\mu}_t(x_t) \right). \tag{4.32}$$

Set:

$$(v_t^*)'_\infty := \lim_{y\to+\infty} \frac{v_t^*(y)}{y}, \quad t = 0, \dots, T.$$

As $\mathrm{dom}(u) \supseteq [0, +\infty)$, $(v_t^*)'_\infty \in [0, +\infty]$. Let $\widehat{\mathbb{Q}}_t \in \mathrm{Prob}(K_t)$ and, for $\mu \in \mathrm{Meas}(K_t)$, let $\mu = \mu_a + \mu_s$ be the Lebesgue Decomposition of μ with respect to $\widehat{\mathbb{Q}}_t$, where $\mu_a \ll \widehat{\mathbb{Q}}_t$ and $\mu_s \perp \widehat{\mathbb{Q}}_t$. Then we can define for $\mu \in \mathrm{Meas}(K_t)$

$$\mathcal{F}_t(\mu \mid \widehat{\mathbb{Q}}_t) := \int_{K_t} v_t^*\left(\frac{\mathrm{d}\mu_a}{\mathrm{d}\widehat{\mathbb{Q}}_t}\right) \mathrm{d}\widehat{\mathbb{Q}}_t + (v_t^*)'_\infty \mu_s(K_t)$$

where we use the convention $\infty \times 0 = 0$, in case $(v_t^*)'_\infty = +\infty$, $\mu_s(K_t) = 0$. Observe that the restriction of $\mathcal{F}(\cdot \mid \widehat{\mathbb{Q}}_t)$ to $\mathrm{Meas}(K_t)$ coincides with the functional in (2.35) of [108] with $F = v_t^*$, and that whenever $\mathrm{dom}(u_t) = \mathbb{R}$ we have $(v_t^*)'_\infty = \lim_{y\to+\infty} \frac{v_t^*(y)}{y} = +\infty$ and $\mathcal{F}_t(\cdot \mid \widehat{\mathbb{Q}}_t)$ coincides with $\mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\cdot)$ (see (4.31)) on $\mathrm{Meas}(K_t)$.

Proposition 4.2.9. *Suppose that $u_0, \dots, u_T : \mathbb{R} \to [-\infty, +\infty)$ satisfy Assumption 4.2.6 and $v_t(\cdot) := -u_t(-\cdot)$. If $\widehat{\mathbb{Q}}_t \in \mathrm{Prob}(K_t)$, $t \in \{0, \dots, T\}$, has full support then for every $\mu \in \mathrm{Meas}(K_t)$*

$$\mathcal{F}_t(\mu \mid \widehat{\mathbb{Q}}_t) = \sup_{\varphi_t \in \mathcal{C}_b(K_t)} \left(\int_{K_t} \varphi_t(x_t) \, \mathrm{d}\mu(x_t) - \int_{K_t} v_t(\varphi_t(x_t)) \, \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) \right) . \tag{4.33}$$

Example 4.2.10. The requirement that $\widehat{\mathbb{Q}}_0, \dots, \widehat{\mathbb{Q}}_T$ have full support is crucial for the proof of Proposition 4.2.9. We provide a simple example to the fact that (4.33) does not hold in general when such an assumption is not fulfilled. To this end, take $K = \{-2, 0, 2\}$, $\widehat{\mathbb{Q}} = \frac{1}{2}\delta_{\{-2\}} + \frac{1}{2}\delta_{\{+2\}}$, $\mu = \delta_{\{0\}}$, $u(x) := \frac{x}{x+1}$ for $x \geq -1$ and $u(x) = -\infty$ for $x < -1$. It is easy to see that the associated v^* via (4.30) is defined by $v^*(y) = 1 + y - 2\sqrt{y}$ for $y \geq 0$ and $v^*(y) = -\infty$ for $y < 0$, so that $(v_t^*)'_\infty = 1$. It is also easy to see that $\mu \perp \widehat{\mathbb{Q}}$, hence in the Lebesgue decomposition with respect to $\widehat{\mathbb{Q}}$, $\mu_a = 0$ and $\mu_s = \mu$. Hence $\mathcal{F}(\mu \mid \widehat{\mathbb{Q}}) = 1 + 1\mu(K) = 2$. At the same time we see that taking $\varphi_N \in \mathcal{C}_b(K)$ defined via $\varphi_N(-2) = \varphi_N(2) = 0, \varphi_N(0) = -N$ (observe that for N sufficiently large $u(\varphi_N) \notin \mathcal{C}_b(K)$) we have

$$\begin{aligned}
&\sup_{\varphi \in \mathcal{C}_b(K)} \left(\int_K \varphi \, \mathrm{d}\mu - \int_K v(\varphi) \, \mathrm{d}\widehat{\mathbb{Q}} \right) = \sup_{\varphi \in \mathcal{C}_b(K)} \left(\int_K u(\varphi) \, \mathrm{d}\widehat{\mathbb{Q}} - \int_K \varphi \, \mathrm{d}\mu \right) \\
&\geq \sup_N \left(\int_K u(\varphi_N) \, \mathrm{d}\widehat{\mathbb{Q}} - \int_K \varphi_N \, \mathrm{d}\mu \right) \geq \sup_N \left((0)\,\frac{1}{2} + (0)\,\frac{1}{2} - (-N) \right) = +\infty.
\end{aligned}$$

4.3 Applications of the Main Theorems of Section 4.1

In this Section 4.3 we suppose the following requirements are fulfilled:

Standing Assumption 4.3.1. $\Omega := K_0 \times \dots \times K_T$ *for compact sets* $K_0, \dots, K_T \subseteq \mathbb{R}$ *and* $K_0 = \{x_0\}$*; the functional* $c : \Omega \to (-\infty, +\infty]$ *is lower semicontinuous and* $d : \Omega \to [-\infty, +\infty)$ *is upper semicontinuous;* $\mathrm{Mart}(\Omega) \neq \emptyset$*;* $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ *is a given probability measure with marginals* $\widehat{\mathbb{Q}}_0, \dots, \widehat{\mathbb{Q}}_T$*;* $c, d \in L^1(\widehat{\mathbb{Q}})$.

4.3.1 Subhedging and Superhedging

As it will become clear from the proofs, in all the results in Section 4.3.1 the functional U is real valued on the whole $\mathcal{E}$, that is $\mathrm{dom}(U) = \mathcal{E}$. Thus we will exploit Theorem 4.1.4 and Corollary 4.1.8, in particular (4.25) and (4.26), in the case $\mathrm{dom}(U) = \mathrm{dom}(V) = \mathcal{E}$.
We set for $\varphi_t \in \mathcal{C}_b(K_t)$

$$U_{\widehat{\mathbb{Q}}_t}(\varphi_t) = \sup_{\alpha, \lambda \in \mathbb{R}} \left(\int_{K_t} u_t(\varphi_t(x_t) + \alpha \mathrm{Id}_t(x_t) + \lambda) \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - (\alpha x_0 + \lambda) \right) , \tag{4.34}$$

$$V_{\widehat{\mathbb{Q}}_t}(\varphi_t) = -U_{\widehat{\mathbb{Q}}_t}(-\varphi_t) = \inf_{\alpha, \lambda \in \mathbb{R}} \left(\int_{K_t} v_t(\varphi_t(x_t) + \alpha \mathrm{Id}_t(x_t) + \lambda) \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - (\alpha x_0 + \lambda) \right) .$$

We observe that Assumption 4.2.6 does **not** impose that the functions u_t are real valued on the whole $\mathbb{R}$. Nevertheless, for the functionals $U_{\widehat{\mathbb{Q}}_t}, V_{\widehat{\mathbb{Q}}_t}$ we have:

Lemma 4.3.2. *Under Assumption 4.2.6, for each* $t = 0, \dots, T$

1. $U_{\widehat{\mathbb{Q}}_t}$ *and* $V_{\widehat{\mathbb{Q}}_t}$ *are real valued on* $\mathcal{C}_b(K_t)$ *and null in* 0.
2. $U_{\widehat{\mathbb{Q}}_t}$ *and* $V_{\widehat{\mathbb{Q}}_t}$ *are concave and convex respectively, and both nondecreasing.*
3. $U_{\widehat{\mathbb{Q}}_t}$ *and* $V_{\widehat{\mathbb{Q}}_t}$ *are stock additive on* $\mathcal{C}_b(K_t)$, *namely for every* $\alpha_t, \lambda_t \in \mathbb{R}$ *and* $\varphi_t \in \mathcal{C}_b(K_t)$.

$$U_{\widehat{\mathbb{Q}}_t}(\varphi_t+\alpha_t \mathrm{Id}_t+\lambda_t) = U_{\widehat{\mathbb{Q}}_t}(\varphi_t)+\alpha_t x_0+\lambda_t, \;\; V_{\widehat{\mathbb{Q}}_t}(\varphi_t+\alpha_t \mathrm{Id}_t+\lambda_t) = V_{\widehat{\mathbb{Q}}_t}(\varphi_t)+\alpha_t x_0+\lambda_t .$$

Proof. Since $V_{\widehat{\mathbb{Q}}_t}(\varphi_t) = -U_{\widehat{\mathbb{Q}}_t}(-\varphi_t)$, w.l.o.g. we prove the claims only for $U_{\widehat{\mathbb{Q}}_t}$. Clearly $U_{\widehat{\mathbb{Q}}_t}(\varphi_t) > -\infty$, as we may choose $\lambda_t \in \mathbb{R}$ so that $(\varphi_t+0\mathrm{Id}_t+\lambda_t) \in \mathrm{dom}(u) \supseteq [0, +\infty)$. Furthermore,

$$\begin{aligned} U_{\widehat{\mathbb{Q}}_t}(\varphi_t) &\overset{1\in\partial U_t(0)}{\leq} \sup_{\alpha,\lambda\in\mathbb{R}} \left(\int_{K_t} (\varphi_t + \alpha \mathrm{Id}_t + \lambda) \, \mathrm{d}\widehat{\mathbb{Q}}_t - (\alpha x_0 + \lambda) \right) \\ &\overset{\widehat{\mathbb{Q}}\in\mathrm{Mart}(\Omega)}{=} \sup_{\alpha,\lambda\in\mathbb{R}} \left(\int_{K_t} \varphi_t \, \mathrm{d}\widehat{\mathbb{Q}}_t + (\alpha x_0 + \lambda - \alpha x_0 - \lambda) \right) \leq \|\varphi_t\|_\infty . \end{aligned}$$

Finally, $0 = \int_{K_t} u\,(0) \, \mathrm{d}\widehat{\mathbb{Q}}_t \leq U_{\widehat{\mathbb{Q}}_t}(0) \leq \|0\|_\infty$.
Item 2: trivial from the definitions. Item 3: we see that

$$\begin{aligned} &U_{\widehat{\mathbb{Q}}_t}(\varphi_t + \alpha_t \mathrm{Id}_t + \lambda_t) \\ &= \sup_{\substack{\alpha\in\mathbb{R}\\ \lambda\in\mathbb{R}}} \left(\int_{K_t} u_t\left(\varphi_t(x_t) + (\alpha+\alpha_t)x_t + (\lambda+\lambda_t)\right) \, \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - (\alpha x_0 + \lambda) \right) + \alpha_t x_0 + \lambda_t + \\ &+ = \sup_{\substack{\alpha\in\mathbb{R}\\ \lambda\in\mathbb{R}}} \left(\int_{K_t} u_t\left(\varphi_t(x_t) + (\alpha+\alpha_t)x_t + (\lambda+\lambda_t)\right) \, \mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - ((\alpha_t+\alpha)x_0 + (\lambda_t+\lambda)) \right) \end{aligned}$$

in which we recognize the definition of $U_{\widehat{\mathbb{Q}}_t}(\varphi_t) + \alpha_t x_0 + \lambda_t$. □

As in [14], **in the next two Corollaries** we suppose that the elements in $\mathcal{E}_t$ represent portfolios obtained combining call options with maturity t, units of the underlying stock at time t (x_t) and deterministic amounts, that is $\mathcal{E}_t$ consists of all the functions in $\mathcal{C}_b(K_t)$ with the following form:

$$\varphi_t(x_t) = a + bx_t + \sum_{n=1}^{N} c_n (x_t - k_n)^+, \text{ for } a, b, c_n, k_n \in \mathbb{R}, \; x_t \in K_t$$

and take $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$. As shown in the proof, one could as well take $\mathcal{E} = \mathcal{C}_b(K_0) \times \cdots \times \mathcal{C}_b(K_T)$ preserving validity of (4.35), (4.36), (4.37) and (4.38).

Corollary 4.3.3. *Take $u_0, \dots, u_T$ satisfying Assumption 4.2.6, and suppose*

$$\mathrm{dom}(u_0) = \dots = \mathrm{dom}(u_T) = \mathbb{R}\,.$$

Then the following equalities hold:

$$\inf_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \sum_{t=0}^{T} \mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t) \right) = \sup\left\{ \sum_{t=0}^{T} U_{\widehat{\mathbb{Q}}_t}(\varphi_t) \mid \varphi\in\mathcal{S}_{sub}(c) \right\}\,, \quad (4.35)$$

$$\sup_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[d(X)\right] - \sum_{t=0}^{T} \mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t) \right) = \inf\left\{ \sum_{t=0}^{T} V_{\widehat{\mathbb{Q}}_t}(\varphi_t) \mid \varphi\in\mathcal{S}_{sup}(d) \right\}\,. \quad (4.36)$$

Proof. We prove (4.35), since (4.36) can be obtained in a similar fashion. Set $U(\varphi) = \sum_{t=0}^{T} U_{\widehat{\mathbb{Q}}_t}(\varphi_t)$ for $\varphi\in\mathcal{E}$. We observe that $\mathcal{E}_t$ consists of all piecewise linear functions on K_t, which are norm dense in $\mathcal{C}_b(K_t)$. By Lemma 4.3.2 for each $t = 0, \dots, T$ the monotone concave functional $\varphi_t \mapsto U_{\widehat{\mathbb{Q}}_t}(\varphi_t)$ is actually well defined, finite valued, concave and nondecreasing on the whole $\mathcal{C}_b(K_t)$. Hence, by the Extended Namioka-Klee Theorem (see [23]) it is norm continuous on $\mathcal{C}_b(K_t)$ and we can take $\mathcal{E} = \mathcal{C}_b(K_t)\times\dots\times\mathcal{C}_b(K_T)$ in place of $\mathcal{E}_0\times\dots\times\mathcal{E}_T$ in the RHS of (4.35) and prove equality to LHS in this more comfortable case (notice that $\mathcal{S}_{sub}(c)$ depends on $\mathcal{E}$). We also observe that in this case we are in Setup 4.2.1. Define $\mathcal{D}$ as in (4.1) with $M = T$. Using the facts that if $\varphi_t\in\mathcal{E}_t$, $\alpha,\lambda\in\mathbb{R}$ then $(\varphi_t + \alpha\mathrm{Id}_t + \lambda)\in\mathcal{E}_t$, that $\mathbb{Q}\in$Mart(Ω) and that $v_t(\cdot) := -u_t(-\cdot)$ one may easily check that

$$\begin{aligned}
\mathcal{D}(\mathbb{Q}) &:= \sup_{\varphi\in\mathcal{E}} \left(U(\varphi) - \sum_{t=0}^{T}\int_{K_t}\varphi_t\,\mathrm{d}\mathbb{Q}_t \right)\\
&= \sup_{\varphi\in\mathcal{E}} \left(\sum_{t=0}^{T}\int_{K_t} u_t(\varphi_t(x_t))\,\mathrm{d}\widehat{\mathbb{Q}}_t(x_t) - \sum_{t=0}^{T}\int_{K_t}\varphi_t\,\mathrm{d}\mathbb{Q}_t \right)\\
&= \sum_{t=0}^{T}\sup_{\psi_t\in\mathcal{E}_t}\left(\int_{K_t}\psi_t\,\mathrm{d}\mathbb{Q}_t - \int_{K_t} v_t(\psi_t(x_t))\,\mathrm{d}\widehat{\mathbb{Q}}_t(x_t) \right)\\
&= \sum_{t=0}^{T}\mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t),\ \ \forall\mathbb{Q}\in\mathrm{Mart}(\Omega)
\end{aligned}$$

where the last equality follows from Proposition 4.2.8 Equation (4.32). The Standing Assumption 4.1.1 is satisfied. Indeed, from Assumption 4.2.6 we have

$$v_0^*(1), \dots, v_T^*(1) < +\infty\,,$$

hence $\mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\widehat{\mathbb{Q}}_t) = \int_{K_t} v_t^*\left(\frac{\mathrm{d}\widehat{\mathbb{Q}}_t}{\mathrm{d}\widehat{\mathbb{Q}}_t}\right)\mathrm{d}\widehat{\mathbb{Q}}_t < +\infty$ and therefore $\widehat{\mathbb{Q}}\in\mathrm{dom}(\mathcal{D})$. Recalling that $c\in L^1(\widehat{\mathbb{Q}})$, this in turns yields

$$\widehat{\mathbb{Q}}\in\mathcal{N} = \left\{ \mu\in\mathrm{Meas}(\Omega)\cap\mathrm{dom}(\mathcal{D}) \mid \int_{\Omega} c\,\mathrm{d}\mu < +\infty \right\}\,.$$

Moreover, by Lemma 4.3.2 Item 1, $\mathrm{dom}(U) = \mathcal{E}$, and for every $\mu\in\mathrm{Meas}(\Omega)$ $\mathcal{D}(\mu) \geq U(0) - 0 = 0$, hence $\mathcal{D}$ is lower bounded on the whole Meas(Ω). We conclude that U and $\mathcal{D}$ satisfy the assumptions of Theorem 4.1.4.

Using Lemma 4.2.2 and the fact that $U_{\widehat{\mathbb{Q}}_0}, \dots, U_{\widehat{\mathbb{Q}}_T}$ are cash additive we get $S^U(\varphi) = \sum_{t=0}^T S^{U_{\widehat{\mathbb{Q}}_t}}(\varphi_t) = \sum_{t=0}^T U_{\widehat{\mathbb{Q}}_t}(\varphi_t) = U(\varphi)$, and by Corollary 4.1.8 Equation (4.25) we obtain

$$\inf_{\mathbb{Q}\in \mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \sum_{t=0}^{T} \mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t) \right) = \sup \left\{ \sum_{t=0}^{T} U_{\widehat{\mathbb{Q}}_t}(\varphi_t) \mid \varphi \in \mathcal{S}_{sub}(c) \right\}.$$

□

Remark 4.3.4. At this point we can add some more discussion on the compactness condition of the underlying sets $K_0, \dots, K_T$ in Assumption 4.3.1. In the classical non-robust setup the requirement of (essential) boundedness of the underlying stock is quite common. We observe that in our canonical setup for the underlying space, compactness is essentially tantamount to requiring that the stock $(\mathrm{Id}_t)_t$ is bounded (everywhere). But we can actually say more: when $\mathcal{D}$ is taken as in Corollary 4.3.3, we automatically have that $\mathbb{Q}_t \ll \widehat{\mathbb{Q}}_t$, whenever $\mathcal{D}(\mathbb{Q}) < +\infty$. If the marginals $\widehat{\mathbb{Q}}_t$, for every $t = 0, \dots, T$, satisfy $\mathrm{Id}_t \in L^\infty(\widehat{\mathbb{Q}}_t)$ we then get automatically that $\mathrm{Id}_t \in L^\infty(\mathbb{Q}_t)$, with $\|\mathrm{Id}_t\|_{L^\infty(\mathbb{Q}_t)} \le \|\mathrm{Id}_t\|_{L^\infty(\widehat{\mathbb{Q}}_t)}$, for any $\mathbb{Q} \in \mathrm{dom}(\mathcal{D})$. Thus, it is possible to reformulate the hypotheses in Corollary in 4.3.3 using $K_0 = \dots = K_T = \mathbb{R}$ but requesting that the marginals $\widehat{\mathbb{Q}}_0, \dots, \widehat{\mathbb{Q}}_T$ have compact support. A version of Corollary 4.3.3 should hold even without the compactness requirement in Assumption 4.3.1, but it is a delicate issue. It would require a modification of the settings, as the set of continuous functions would not work well any more, and a generalization of [108] Theorem 2.7, which is not trivial. We leave these interesting issues for future research.

We stress the fact that in Corollary 4.3.3 we assume that all the functions $u_0, \dots, u_T$ are real valued on the whole $\mathbb{R}$. A more general result can be obtained when weakening this assumption, but it requires an additional assumption on the marginals of $\widehat{\mathbb{Q}}$.

Corollary 4.3.5. *Suppose Assumption 4.2.6 is fulfilled. Assume $\widehat{\mathbb{Q}}_0, \dots, \widehat{\mathbb{Q}}_T$ have full support on $K_0, \dots, K_T$ respectively. Then Equations* (4.35), (4.36) *hold true replacing $\mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t)$ with $\mathcal{F}_t(\mathbb{Q}_t|\widehat{\mathbb{Q}}_t)$.*

Proof. The proof can be carried over almost literally as the proof of Corollary 4.3.3, with the exception of replacing the reference to Proposition 4.2.8 with the reference to Proposition 4.2.9. □

Remark 4.3.6. Observe that we are requesting the full support property on $K_0, \dots, K_T$ with respect to their induced (Euclidean) topology. In particular, this means that whenever $k_t \in K_t$ is an isolated point, $\widehat{\mathbb{Q}}_t(\{k_t\}) > 0$. This is consistent with our assumption $K_0 = \{x_0\}$, which implies $\mathrm{Prob}(K_0)$ reduces to the Dirac measure, $\mathrm{Prob}(K_0) = \{\delta_{\{x_0\}}\}$.

We now take $u_t(x) = x$ for each $t = 0, \dots, T$, and get $U_{\widehat{\mathbb{Q}}_t}(\varphi_t) = V_{\widehat{\mathbb{Q}}_t}(\varphi_t) = \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t]$. Hence with an easy computation we have

$$\mathcal{D}_{v_t^*,\widehat{\mathbb{Q}}_t}(\mathbb{Q}_t) = \begin{cases} 0 & \text{if } \mathbb{Q}_t \equiv \widehat{\mathbb{Q}}_t \\ +\infty & \text{otherwise} \end{cases} \quad \text{for all } \mathbb{Q} \in \mathrm{Mart}(\Omega).$$

If $\text{Mart}(\widehat{\mathbb{Q}}_1,\dots,\widehat{\mathbb{Q}}_T) = \{\mathbb{Q} \in \text{Mart}(\Omega) \mid \mathbb{Q}_t \equiv \widehat{\mathbb{Q}}_t \ \forall\, t = 0,\dots,T\}$, from Corollary 4.3.3 we can recover the following result of [14] (under more stringent assumptions on the underlying space).

Corollary 4.3.7 ([14] Theorem 1.1 and Corollary 1.2)**.** *The following equalities hold:*

$$\inf_{\mathbb{Q}\in\text{Mart}(\widehat{\mathbb{Q}}_1,\dots,\widehat{\mathbb{Q}}_T)} \mathbb{E}_{\mathbb{Q}}[c] = \sup\left\{\sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] \mid \varphi \in \mathcal{S}_{sub}(c)\right\}, \tag{4.37}$$

$$\sup_{\mathbb{Q}\in\text{Mart}(\widehat{\mathbb{Q}}_1,\dots,\widehat{\mathbb{Q}}_T)} \mathbb{E}_{\mathbb{Q}}[d] = \inf\left\{\sum_{t=0}^{T} \mathbb{E}_{\widehat{\mathbb{Q}}_t}[\varphi_t] \mid \varphi \in \mathcal{S}_{sup}(d)\right\}. \tag{4.38}$$

Superhedging and Subhedging without Options

Corollary 4.3.8. *The following equalities hold:*

$$\inf_{\mathbb{Q}\in\text{Mart}(\Omega)} \mathbb{E}_{\mathbb{Q}}[c] = \sup\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H} \ s.t.\ m + I^{\Delta} \le c\right\} := \Pi^{sub}(c), \tag{4.39}$$

$$\sup_{\mathbb{Q}\in\text{Mart}(\Omega)} \mathbb{E}_{\mathbb{Q}}[d] = \inf\left\{m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H} \ s.t.\ m + I^{\Delta} \ge d\right\} := \Pi^{sup}(d). \tag{4.40}$$

Proof. We take $\mathcal{E}_0 = \cdots = \mathcal{E}_T = \mathbb{R}$ and $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T = \mathbb{R}^{T+1}$. We first focus on (4.39). For each $\varphi \in \mathcal{E}$ with $\varphi = [m_1,\dots,m_T]$, $m \in \mathbb{R}^{T+1}$ we select $U(\varphi) := \sum_{t=0}^{T} m_t$ (we notice that when $\mathcal{E} = \mathbb{R}^{T+1}$, $u_t(x_t) = x_t$, $t = 0,...,T$, and $\widehat{\mathbb{Q}} \in \text{Mart}(\Omega)$, the functional $U_{\widehat{\mathbb{Q}}_t}$ defined in (4.34) is given by $U_{\widehat{\mathbb{Q}}_t}(m_t) = m_t$ and so $U(m) = \sum_{t=0}^{T} U_{\widehat{\mathbb{Q}}_t}(m_t) = \sum_{t=0}^{T} m_t$ for all $m \in \mathcal{E}$). Then applying the definition of $\mathcal{D}$ in (4.1) we get

$$\mathcal{D}(\gamma) = \begin{cases} 0 & \text{for } \gamma \in \text{ca}(\Omega) \text{ s.t. } \gamma(\Omega) = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

In particular $\mathcal{D}(\mathbb{Q}) = 0$ for every $\mathbb{Q} \in \text{Mart}(\Omega)$. Moreover we observe that $\mathcal{S}^U(\varphi) = U(\varphi)$ for every $\varphi \in \mathcal{E}$. Applying Corollary 4.1.8 (whose assumptions are clearly satisfied here), from Equation (4.25) we get that

$$\inf_{\mathbb{Q}\in\text{Mart}(\Omega)} \mathbb{E}_{\mathbb{Q}}[c] = \sup\left\{\sum_{t=0}^{T} m_t \mid m_0,\dots,m_T \in \mathbb{R} \text{ s.t. } \exists\, \Delta \in \mathcal{H} \text{ with } \sum_{t=0}^{T} m_t + I^{\Delta} \le c\right\}.$$

We recognize in the RHS above the RHS of (4.39). Equation (4.40) can be obtained in a similar way using Corollary 4.1.8 Equation (4.26). □

Penalization with market prices

In this section we change our perspective. Instead of starting from a given U, we will give a particular form of the penalization term $\mathcal{D}$ and proceed in identifying the corresponding U in the spirit of Remark 4.1.2. For each $t = 0,\dots,T$ we suppose that finite sequences $(c_{t,n})_{1\le n\le N_t} \subseteq \mathbb{R}$ and $(f_{t,n})_{1\le n\le N_t} \subseteq \mathcal{C}_b(K_t)$ are given. The functions $(f_{t,n})_{1\le n\le N_t} \subseteq \mathcal{C}_b(K_t)$ represent payoffs of options whose prices $(c_{t,n})_{1\le n\le N_t} \subseteq \mathbb{R}$ are known from the market. Furthermore, we consider penalization functions $\Psi_{n,t} : \mathbb{R} \to (-\infty,+\infty]$ which are convex, null in 0, symmetric in

0, proper and lower semicontinuous. For such functions we define the conjugates $\Psi^*_{t,n} : \mathbb{R} \to (-\infty, +\infty]$ as $\Psi^*_{t,n}(y) = \sup_{x\in\mathbb{R}}(xy - \Psi_{t,n}(x))$.
Define

$$\mathrm{Mart}_t(K_t) = \{\gamma_t \in \mathrm{Prob}(K_t) \mid \exists \mathbb{Q} \in \mathrm{Mart}(\Omega) \text{ with } \gamma_t \equiv \mathbb{Q}_t\} \subseteq \mathrm{ca}(K_t)$$

and for $\gamma_t \in \mathrm{ca}(K_t)$

$$\mathcal{D}^{\Psi}_t(\gamma_t) := \begin{cases} \sum_{n=1}^{N_t} \Psi_{t,n}\left(\left|\int_{K_t} f_{t,n}\,\mathrm{d}\gamma_t - c_{t,n}\right|\right) & \text{for } \gamma_t \in \mathrm{Mart}_t(K_t) \\ +\infty & \text{otherwise} \end{cases}$$

Proposition 4.3.9. *Suppose that the martingale measure* $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ *in Standing Assumption 4.3.1 also satisfies* $\left|\int_{K_t} f_{t,n}\,\mathrm{d}\widehat{\mathbb{Q}}_t - c_{t,n}\right| \in \mathrm{dom}(\Psi_{t,n})$ *for every* $n = 0, \dots, N_t$, $t = 0, \dots, T$. *Then setting for* $n = 1, \dots, N_t, t = 0, \dots, T$ $g_{t,n} := f_{t,n} - c_{t,n} \in \mathcal{C}_b(K_t)$ *we have*
Subhedging Duality*:*

$$\inf_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[c] + \sum_{t=0}^{T} \mathcal{D}^{\Psi}_t(\mathbb{Q}_t)\right) = \sup\left\{\sum_{t=0}^{T} U^{\Psi}_t(\varphi_t) \mid \varphi \in \mathcal{S}_{sub}(c)\right\}, \tag{4.41}$$

where

$$U^{\Psi}_t(\varphi_t) := \sup_{y_t\in\mathbb{R}^{N_t}} \left(\Pi^{sub}\left(\varphi_t + \sum_{n=1}^{N_t} y_{t,n}g_{t,n}\right) - \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|)\right)$$

is stock additive and Π^{sub} *is given in* (4.39).
Superhedging Duality*:*

$$\sup_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}[d] - \sum_{t=0}^{T} \mathcal{D}^{\Psi}_t(\mathbb{Q}_t)\right) = \inf\left\{\sum_{t=0}^{T} V^{\Psi}_t(\varphi_t) \mid \varphi \in \mathcal{S}_{sup}(d)\right\}, \tag{4.42}$$

where

$$V^{\Psi}_t(\varphi_t) = -U^{\Psi}_t(-\varphi_t) = \inf_{y_t\in\mathbb{R}^{N_t}} \left(\Pi^{sup}\left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n}g_{t,n}\right) + \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|)\right)$$

is stock additive and Π^{sup} *is given in* (4.40).

Before providing a proof, we state an auxiliary result.

Lemma 4.3.10. *Suppose* $K_0, \dots, K_t \subseteq \mathbb{R}$ *are compact. Then* $\mathrm{Mart}_t(K_t)$ *is compact in the topology* $\sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$.

Proof. We see that $\mathrm{Mart}(\Omega)$ is a $\sigma(\mathrm{ca}(\Omega), \mathcal{C}_b(\Omega))$-closed subset of the $\sigma(\mathrm{ca}(\Omega), \mathcal{C}_b(\Omega))$-compact set $\mathrm{Prob}(\Omega)$ (which is compact since Ω is a compact Polish space, see [5] Theorem 15.11), hence it is compact himself. $\mathrm{Mart}_t(K_t)$ is then the image of a compact set via the marginal map $\gamma \mapsto \gamma_t$ which is $\sigma(\mathrm{ca}(\Omega), \mathcal{C}_b(\Omega)) - \sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$ continuous, hence it is $\sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$-compact. □

Proof of Proposition 4.3.9. We focus on (4.41) first.
STEP 1: for any $t \in \{0, \dots, T\}$ we prove the following: the functional $\mathcal{D}_t^\Psi$ is $\sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$-lower semicontinuous and for every $\varphi_t \in \mathcal{C}_b(K_t)$ its Fenchel-Moreau (convex) conjugate satisfies

$$\begin{aligned} V_t^\Psi(\varphi_t) &:= \sup_{\gamma_t \in \mathrm{ca}(K_t)} \left(\int_{K_t} \varphi_t \, \mathrm{d}\gamma_t - \mathcal{D}_t^\Psi(\gamma_t) \right) \\ &= \inf_{y_t \in \mathbb{R}^{N_t}} \left(\Pi^{\mathrm{sup}} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) + \sum_{n=1}^{N_t} \Psi_{t,n}^*(|y_{t,n}|) \right) , \end{aligned}$$

and thus

$$U_t^\Psi(\varphi_t) := -V_t^\Psi(-\varphi_t) = \sup_{y_t \in \mathbb{R}^{N_t}} \left(\Pi^{\mathrm{sub}} \left(\varphi_t + \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) - \sum_{n=1}^{N_t} \Psi_{t,n}^*(|y_{t,n}|) \right) . \quad (4.43)$$

We observe that $\mathcal{D}_t^\Psi$ is $\sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$-lower semicontinuous (it is a sum of functions, each being composition of a lower semicontinuous function and a continuous function on $\mathrm{Mart}_t(K_t)$ which is $\sigma(\mathrm{ca}(K_t), \mathcal{C}_b(K_t))$-compact by Lemma 4.3.10). We now need to compute

$$V_t^\Psi(\varphi_t) = \sup_{\gamma_t \in \mathrm{ca}(K_t)} \left(\int_{K_t} \varphi_t \, \mathrm{d}\gamma_t - \mathcal{D}_t^\Psi(\gamma_t) \right) = \sup_{\mathbb{Q}_t \in \mathrm{Mart}_t(K_t)} \left(\int_{K_t} \varphi_t \, \mathrm{d}\mathbb{Q}_t - \mathcal{D}_t^\Psi(\mathbb{Q}_t) \right) .$$

Recall now that from Fenchel-Moreau Theorem and symmetry

$$\Psi_{t,n}(|x|) = \Psi_{t,n}(x) = \sup_{y \in \mathbb{R}} (xy - \Psi_{t,n}^*(y)) = \sup_{y \in \mathbb{R}} (xy - \Psi_{t,n}^*(|y|)) .$$

Hence, setting $g_{t,n} = f_{t,n} - c_{t,n}$,

$$\begin{aligned} V_t^\Psi(\varphi_t) &= \sup_{\mathbb{Q}_t \in \mathrm{Mart}_t(K_t)} \left(\int_{K_t} \varphi_t \, \mathrm{d}\mathbb{Q}_t - \sum_{n=1}^{N_t} \sup_{y_{t,n} \in \mathbb{R}} \left(y_{t,n} \int_{K_t} g_{t,n} \, \mathrm{d}\mathbb{Q}_t - \Psi_{t,n}^*(|y_{t,n}|) \right) \right) \\ &= \sup_{\mathbb{Q}_t \in \mathrm{Mart}_t(K_t)} \left(\int_{K_t} \varphi_t \, \mathrm{d}\mathbb{Q}_t - \sum_{n=1}^{N_t} \sup_{y_{t,n} \in \mathrm{dom}(\Psi_{t,n}^*)} \left(y_{t,n} \int_{K_t} g_{t,n} \, \mathrm{d}\mathbb{Q}_t - \Psi_{t,n}^*(|y_{t,n}|) \right) \right) \\ &= \sup_{\mathbb{Q}_t \in \mathrm{Mart}_t(K_t)} \inf_{y_t \in \mathrm{dom}} \left(\int_{K_t} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) \mathrm{d}\mathbb{Q}_t + \sum_{n=1}^{N_t} \Psi_{t,n}^*(|y_{t,n}|) \right) \\ &=: \sup_{\mathbb{Q}_t \in \mathrm{Mart}(K_t)} \inf_{y_t \in \mathrm{dom}} \mathcal{T}(y_t, \mathbb{Q}_t), \end{aligned}$$

where $\mathrm{dom} = \mathrm{dom}(\Psi_{t,1}^*) \times \cdots \times \mathrm{dom}(\Psi_{t,N_t}^*) \subseteq \mathbb{R}^{N_t}$. We now see that $\mathcal{T}$ is real valued on $\mathrm{dom} \times \mathrm{Mart}_t(K_t)$, is convex in the first variable and concave in the second. Moreover, $\{\mathcal{T}(y_t, \cdot) \geq C\}$ is $\sigma(\mathrm{Mart}_t(K_t), \mathcal{C}_b(K_t))$-closed in $\mathrm{Mart}_t(\Omega)$ for every $y_t \in \mathrm{dom}$, and $\mathrm{Mart}_t(K_t)$ is $\sigma(\mathrm{Mart}_t(K_t), \mathcal{C}_b(K_t))$-compact (by Lemma 4.3.10). As a consequence $\mathcal{T}(y_t, \cdot)$ is $\sigma(\mathrm{Mart}_t(K_t), \mathcal{C}_b(K_t))$-lower semicontinuous on $\mathrm{Mart}_t(K_t)$. We can apply [124] Theorem 3.1 with $A = \mathrm{dom}$ and $B = \mathrm{Mart}_t(K_t)$ endowed with the topology

$\sigma(\text{Mart}_t(K_t), \mathcal{C}_b(K_t))$, and interchange inf and sup. From our previous computations we then get

$$
\begin{aligned}
V_t^\Psi(\varphi_t) &= \sup_{\mathbb{Q}_t \in \text{Mart}_t(K_t)} \inf_{y_t \in \text{dom}} \mathcal{T}(y_t, \mathbb{Q}_t) = \inf_{y_t \in \text{dom}} \sup_{\mathbb{Q}_t \in \text{Mart}_t(K_t)} \mathcal{T}(y_t, \mathbb{Q}_t) \\
&= \inf_{y_t \in \text{dom}} \left(\sup_{\mathbb{Q}_t \in \text{Mart}_t(K_t)} \int_{K_t} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) \mathrm{d}\mathbb{Q}_t + \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|) \right) \\
&= \inf_{y_t \in \text{dom}} \left(\sup_{\mathbb{Q} \in \text{Mart}(\Omega)} \int_{\Omega} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) \mathrm{d}\mathbb{Q} + \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|) \right) \\
&\overset{(4.40)}{=} \inf_{y_t \in \text{dom}} \left(\Pi^{\text{sup}} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) + \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|) \right) \\
&= \inf_{y_t \in \mathbb{R}^{N_t}} \left(\Pi^{\text{sup}} \left(\varphi_t - \sum_{n=1}^{N_t} y_{t,n} g_{t,n} \right) + \sum_{n=1}^{N_t} \Psi^*_{t,n}(|y_{t,n}|) \right).
\end{aligned}
$$

Equation (4.43) can be obtained with minor manipulations.
STEP 2: conclusion. We are clearly in the setup of Theorem 4.1.4 with $\mathcal{D}$ given as in Setup 4.2.4 from $\mathcal{D}_0^\Psi, \dots, \mathcal{D}_T^\Psi$, and by definition $\text{dom}(\mathcal{D}_t^\Psi) \subseteq \text{Prob}(K_t)$ for each $t = 0, \dots, T$. Using Lemma 4.2.5 Item 2, together with the computations in STEP 1 and the fact that clearly $S^{U_t^\Psi} \equiv U_t^\Psi$ by Cash Additivity of U_t^Ψ, we get the desired equality from Corollary 4.1.8 Equation (4.25): observe that our assumption on the existence of the measure $\widehat{\mathbb{Q}} \in \text{Mart}(\Omega)$ guarantees, together with the fact that $\mathcal{D}$ is clearly lower bounded on $\text{Meas}(\Omega)$, that the hypotheses of Theorem 4.1.4 are satisfied (hence so is Standing Assumption 4.1.1). Equality (4.42) can now be obtained similarly to (4.41). □

Remark 4.3.11. Our assumption of existence of a particular $\widehat{\mathbb{Q}} \in \text{Mart}(\Omega)$ in Proposition 4.3.9 expresses the fact that we are assuming our market prices $(c_{t,n})_{t,n}$ are close enough to those given by expectations under some martingale measure.

Remark 4.3.12. Proposition 4.3.9 covers a wide range of penalizations. For example, we might use power-like penalizations, i.e. $\Psi_{t,n}(x) = \frac{|x|^{p_{t,n}}}{p_{t,n}}$ for $p_{t,n} \in (1, +\infty)$. In such a case $\Psi^*_{t,n}(x) = \frac{|x|^{q_{t,n}}}{q_{t,n}}$ for $\frac{1}{p_{t,n}} + \frac{1}{q_{t,n}} = 1$. Alternatively, we might impose a threshold for the fitting, that is take into account only those martingale measure $\mathbb{Q}$ such that $\left| \int_\Omega f_{t,n} \,\mathrm{d}\mathbb{Q}_t - c_{t,n} \right| \leq \varepsilon_{t,n}$ for some $\varepsilon_{t,n} > 0$. To express this, we might take for $x, y \in \mathbb{R}$

$$
\Psi_{t,n}(x) = \begin{cases} 0 & \text{if } |x| \leq \varepsilon_{t,n} \\ +\infty & \text{otherwise} \end{cases} \quad \Longrightarrow \quad \Psi^*_{t,n}(y) = \varepsilon_{t,n} |y| \,.
$$

4.3.2 Beyond uniperiodal semistatic hedging

We now explore the versatility of Corollary 4.1.8, which can be used beyond the semistatic subhedging and superhedging problems in Section 4.3.1. Note that in Section 4.3.1 we chose for static hedging portfolios the sets $\mathcal{E}_t, t = 0, \dots, T$ consisting of deterministic amounts, units of underlying stock at time t and call options with

different strike prices and same maturity t. This affected the primal problem in the fact that the penalty $\mathcal{D}$ turned out to depend solely on the (one dimensional) marginals of $\widehat{\mathbb{Q}}$. Nonetheless, Theorem 4.1.4 allows to choose for each $t = 0, \dots, T$ a subspace $\mathcal{E}_t \subseteq \mathcal{C}_b(K_0 \times \cdots \times K_t)$, potentially allowing to consider also Asian and path-dependent options in the sets $\mathcal{E}_t$. We expect that this would translate in the penalty $\mathcal{D}$ depending no more only on the one dimensional marginals of $\widehat{\mathbb{Q}}$. The study of these less restrictive, yet technically more complex cases is left for future research.
In the following we will treat a slightly different problem, which however helps understanding how also the extreme case $\mathcal{E}_t = \mathcal{C}_b(K_0 \times \cdots \times K_t), t = 0, \dots, T$ is of interest.

Dual representation for Generalized OCE associated to the indirect utility function

Theorem 4.1.4 yields the following dual robust representation of the Generalized Optimized Certainty Equivalent associated to the indirect utility function. We stress here the fact that, again, $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ is a fixed martingale measure, but we will not focus anymore on its marginals only, as will become clear in the following.

Theorem 4.3.13. *Take $u : \mathbb{R} \to \mathbb{R}$ such that $u_0 = \dots, u_T := u$ satisfy Assumption 4.2.6 and let v^* be defined in* (4.30) *with u in place of u_t. Let $U^{\mathcal{H}}_{\widehat{\mathbb{Q}}} : \mathcal{C}_b(\Omega) \to \mathbb{R}$ be the associated indirect utility*

$$U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}(\varphi) := \sup_{\Delta \in \mathcal{H}} \int_\Omega u(\varphi + I^\Delta)\, \mathrm{d}\widehat{\mathbb{Q}} .$$

and $S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}$ be the associated Generalized Optimized Certainty Equivalent defined according to (I.50)*, namely*

$$S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}(\varphi) := \sup_{\xi \in \mathbb{R}} \left(U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}(\varphi + \xi) - \xi \right) \quad \varphi \in \mathcal{C}_b(\Omega) .$$

Then for every $c \in \mathcal{C}_b(\Omega)$

$$S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}(c) = \inf_{\mathbb{Q} \in \mathrm{Mart}(\Omega)} \left(\int_\Omega c\, \mathrm{d}\mathbb{Q} + \mathcal{D}_{\widehat{\mathbb{Q}}}(\mathbb{Q}) \right)$$

where for $\mu \in \mathrm{Meas}(\Omega)$

$$\mathcal{D}_{\widehat{\mathbb{Q}}}(\mu) := \begin{cases} \int_\Omega v^* \left(\frac{\mathrm{d}\mu}{\mathrm{d}\widehat{\mathbb{Q}}} \right) \mathrm{d}\widehat{\mathbb{Q}} & \text{if } \mu \ll \widehat{\mathbb{Q}} \\ +\infty & \text{otherwise} \end{cases} .$$

Proof. Take $\mathcal{E}_t = \mathcal{C}_b(K_0 \times \cdots \times K_t)$ for $t = 0, \dots, T$. Define for $\psi \in \mathcal{E} = \mathcal{E}_0 \times ... \times \mathcal{E}_T$ $U(\psi) := U^{\mathcal{H}}_{\widehat{\mathbb{Q}}} \left(\sum_{t=0}^T \psi_t \right)$. Clearly $U(\psi) > -\infty$ for any $\psi \in \mathcal{E}$, and since $\widehat{\mathbb{Q}} \in \mathrm{Mart}(\Omega)$ and $u(x) \leq x$ for all $x \in \mathbb{R}$ we also have $U(\psi) \leq \sum_{t=0}^T \|\varphi_t\|_\infty < +\infty$. Moreover it is easy to verify that defining $\mathcal{D}$ as in (4.1) for any $\mathbb{Q} \in \mathrm{Mart}(\Omega)$ we have

$$\mathcal{D}(\mathbb{Q}) := \sup_{\psi \in \mathcal{E}} \left(U(\psi) - \int_\Omega \left(\sum_{t=0}^T \psi_t \right) \mathrm{d}\mathbb{Q} \right) = \sup_{\varphi \in \mathcal{C}_b(\Omega)} \left(\int_\Omega u(\varphi)\, \mathrm{d}\widehat{\mathbb{Q}} - \int_\Omega \varphi\, \mathrm{d}\mathbb{Q} \right)$$

and arguing as in Proposition 4.2.8 we get $\mathcal{D}(\mathbb{Q}) = \mathcal{D}_{\widehat{\mathbb{Q}}}(\mathbb{Q})$. From the fact that $u(x) \leq x$ for every $x \in \mathbb{R}$ we have $v^*(1) < +\infty$, hence from Assumption 4.3.1 $\widehat{\mathbb{Q}} \in \mathrm{dom}(\mathcal{D})$. This and $c \in L^1(\widehat{\mathbb{Q}})$ in turns yields $\widehat{\mathbb{Q}} \in \mathcal{N}$ (see (4.11)). Moreover $\mathrm{dom}(U) = \mathcal{E}$ and by definition of $\mathcal{D}$ for any $\mu \in \mathrm{Meas}(\Omega)$ we have $\mathcal{D}(\mu) \geq U(0) - 0 = 0$, hence $\mathcal{D}$ is lower bounded on the whole $\mathrm{Meas}(\Omega)$. We conclude that U and $\mathcal{D}$ satisfy the assumptions of Theorem 4.1.4. We then get

$$\inf_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \mathcal{D}_{\widehat{\mathbb{Q}}}(\mathbb{Q})\right) = \inf_{\mathbb{Q}\in\mathrm{Mart}(\Omega)} \left(\mathbb{E}_{\mathbb{Q}}\left[c(X)\right] + \mathcal{D}(\mathbb{Q})\right) = \sup_{\Delta\in\mathcal{H}} \sup_{\psi\in\mathbf{\Phi}_\Delta(c)} S^U(\psi)\,.$$

Observe now that S^U satisfies

$$S^U(\psi) := \sup_{\lambda\in\mathbb{R}^{T+1}} \left(U(\psi+\lambda) - \sum_{t=0}^{T}\lambda_t\right) = \sup_{\lambda\in\mathbb{R}^{T+1}} \left(U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}\left(\sum_{t=0}^{T}\psi_t + \sum_{t=0}^{T}\lambda_t\right) - \sum_{t=0}^{T}\lambda_t\right)$$

$$= \sup_{\xi\in\mathbb{R}} \left(U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}\left(\sum_{t=0}^{T}\psi_t + \xi\right) - \xi\right) =: S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}\left(\sum_{t=0}^{T}\psi_t\right).$$

$S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}} : \mathcal{C}_b(\Omega) \to \mathbb{R}$ is (IA) and is nondecreasing, thus

$$\sup_{\Delta\in\mathcal{H}} \sup_{\psi\in\mathbf{\Phi}_\Delta(c)} S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}\left(\sum_{t=0}^{T}\psi_t\right) = \sup_{\Delta\in\mathcal{H}} \sup_{\psi\in\mathbf{\Phi}_\Delta(c)} S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}\left(\sum_{t=0}^{T}\psi_t + I^{\Delta}\right) = S^{U^{\mathcal{H}}_{\widehat{\mathbb{Q}}}}(c)$$

by definition of $\mathbf{\Phi}_\Delta(c)$ and since $c \in \mathcal{C}_b(\Omega)$. □

4.4 Appendix to Chapter 4

4.4.1 Setting

Measures

We start fixing our setup and some notation. Let Ω be a Polish space and endow it with the Borel sigma algebra $\mathcal{B}(\Omega)$ generated by its open sets. A set function $\mu : \mathcal{B}(\Omega) \to \mathbb{R}$ is a **finite signed measure** if $\mu(\emptyset) = 0$ and μ is countably additive. A **finite measure** μ is a finite signed measure such that $\mu(B) \geq 0$ for all $B \in \mathcal{B}(\Omega)$. A finite measure μ such that $\mu(\Omega) = 1$ will be called a **probability measure**. Recall from Section 4.1 the notations for $\mathrm{ca}(\Omega)$, $\mathrm{Meas}(\Omega)$, $\mathrm{Prob}(\Omega)$. The following result is well known, see e.g. [24] Theorem 1.1 and 1.3.

Proposition 4.4.1. *Every finite measure μ on $\mathcal{B}(\Omega)$ is a Radon Measure, that is for every $B \in \mathcal{B}(\Omega)$ and every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subseteq B$ such that $\mu(B \setminus K_\varepsilon) \leq \varepsilon$.*

A measure $\mu \in \mathrm{Meas}(\Omega)$ has **full support** if $\mu(A) > 0$ for every nonempty open set $A \subseteq \Omega$. We also introduce for $M \in \mathbb{N}$, $M \geq 1$ the sets

$$\mathcal{C}_b(\Omega) := \mathcal{C}_b(\Omega,\mathbb{R}) = \{\varphi : \Omega \to \mathbb{R} \mid \varphi \text{ is bounded, continuous}\},$$
$$\mathcal{C}_b(\Omega,\mathbb{R}^M) := (\mathcal{C}_b(\Omega))^M = \{\varphi : \Omega \to \mathbb{R}^M \mid \varphi \text{ is bounded, continuous}\},$$
$$\mathrm{LSC}_b(\Omega) := \mathrm{LSC}_b(\Omega,\mathbb{R}) = \{\varphi : \Omega \to \mathbb{R} \mid \varphi \text{ is bounded, lower semicontinuous}\}.$$

Given a vector subspace $\mathcal{E} \subseteq \mathcal{C}_b(\Omega, \mathbb{R}^{M+1})$ we will consider the dual pair

$$(\text{ca}(\Omega), \mathcal{C}_b(\Omega, \mathbb{R}^{M+1}))$$

with pairing given by the bilinear functional $(\gamma, \varphi) \mapsto \int_\Omega \left(\sum_{m=0}^M \varphi_m\right) \mathrm{d}\gamma$. We will induce on $\text{ca}(\Omega)$ the topology $\sigma(\text{ca}(\Omega), \mathcal{E})$, which is the coarsest topology on $\text{ca}(\Omega)$ making the functional $\gamma \mapsto \int_\Omega \left(\sum_{m=0}^M \varphi_m\right) \mathrm{d}\gamma$ continuous for each $\varphi \in \mathcal{E}$. Similarly, we will induce on $\mathcal{E}$ the topology $\sigma(\mathcal{E}, \text{ca}(\Omega))$ which is the coarsest topology on $\mathcal{E}$ making the functional $\gamma \mapsto \int_\Omega \left(\sum_{m=0}^M \varphi_m\right) \mathrm{d}\gamma$ continuous for each $\gamma \in \text{ca}(\Omega)$.

Weak and Narrow Topology

Definition 4.4.2. *The* ***Weak Topology*** *on* $\text{Meas}(\Omega)$ *is the coarsest (Hausdorff) topology for which all maps* $\mu \mapsto \int_\Omega \varphi \,\mathrm{d}\mu$ *are continuous, for all* $\varphi \in \mathcal{C}_b(\Omega)$. *The* ***Narrow Topology*** *is the coarsest (Hausdorff) topology for which all maps* $\mu \mapsto \int_\Omega \varphi \,\mathrm{d}\mu$ *are lower semicontinuous, for all* $\varphi \in LSC_b(\Omega)$.

Remark 4.4.3. The weak topology on $\text{Meas}(\Omega)$ is the topology $\sigma(\text{Meas}(\Omega), \mathcal{C}_b(\Omega, \mathbb{R})$, which is the relative topology $\sigma(\text{ca}(\Omega), \mathcal{C}_b(\Omega, \mathbb{R})|_{\text{Meas}(\Omega)}$ induced by $\sigma(\text{ca}(\Omega), \mathcal{C}_b(\Omega, \mathbb{R})$ on $\text{Meas}(\Omega) \subseteq \text{ca}(\Omega)$ (see [5] Lemma 2.53).

Proposition 4.4.4. *When* Ω *is a Polish space, the weak and narrow topologies coincide.*

Proof. See [123] page 371. □

Remark 4.4.5. Even though the two topologies coincide in our setting, because of their different definitions we will find more convenient to exploit the one or the other topology in our proofs.

We now turn our attention to compactness issues in $\text{Meas}(\Omega)$ under the narrow topology. We recall first that a family $\Gamma \subseteq \text{Meas}(\Omega)$ is **bounded** if $\sup_{\mu\in\Gamma} \mu(\Omega) < +\infty$ and **tight** if for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subseteq \Omega$ such that $\sup_{\mu\in\Gamma} \mu(\Omega \setminus K_\varepsilon) \le \varepsilon$. The following generalization of Prokhorov's Theorem holds:

Theorem 4.4.6. *If a subset* $\Gamma \subseteq \text{Meas}(\Omega)$ *is bounded and tight, it is relatively compact in the narrow topology.*

Proof. See [123] Theorem 3 page 379. □

4.4.2 Auxiliary Results and Proofs

Lemma 4.4.7. *Take compact* $K_1, \dots, K_T \subseteq \mathbb{R}$, *and suppose that* $K_0 = \{x_0\}$ *and* $\text{card}(K_{t+1}) \ge \text{card}(K_t)$ *for every* $t = 0, \dots, T-1$. *Take* $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$ *for vector subspaces* $\mathcal{E}_t \subseteq \mathcal{C}_b(K_t)$ *such that* $\text{Id}_t \in \mathcal{E}_t$ *and* $\mathcal{E}_t + \mathbb{R} = \mathcal{E}_t$, *for* $t = 0, \dots, T$. *Suppose there exist* $\varphi, \psi \in \mathcal{E}$ *and* $\Delta \in \mathcal{H}$, *where* $\mathcal{H}$ *is defined in* (I.37), *such that* $\sum_{t=0}^T \varphi_t = \sum_{t=0}^T \psi_t + I^\Delta$. *Then there exist constants* $k_0, \dots, k_T, h_0, \dots, h_T \in \mathbb{R}$ *such* *that for each* $t = 0, \dots, T$ $\psi_t(x_t) = \varphi_t(x_t) + k_t x_t + h_t$, $\forall x_t \in K_t$. *In particular for* $S_t : \mathcal{E}_t \to \mathbb{R}$, $t = 0, \dots, T$ *stock additive functionals we have*

$$\sum_{t=0}^{T} S_t(\varphi_t) = \sum_{t=0}^{T} S_t(\psi_t)\,,$$

and for $\mathcal{V} := \sum_{t=0}^{T} \mathcal{E}_t + \mathcal{I}$ *(see* (I.38)*) the map*

$$v = \sum_{t=0}^{T} \varphi_t + I^{\Delta} \mapsto S(v) := \sum_{t=0}^{T} S_t(\varphi_t)$$

is well defined on $\mathcal{V}$*, (CA) and (IA).*

Proof.
STEP 1: we prove that if $\sum_{t=0}^{T} \varphi_t = \sum_{t=0}^{T} \psi_t + I^{\Delta}$ then $\Delta = [\Delta_0, \dots, \Delta_{T-1}] \in \mathcal{H}$ is a deterministic vector $\Delta \in \mathbb{R}^T$. If $\text{card}(K_T) = 1$ this is trivial. We can then suppose $\text{card}(K_T) \geq 2$ We see that

$$\varphi_T(x_T) - \psi_T(x_T) = \sum_{t=0}^{T-1} (\psi(x_t) - \varphi_t(x_t)) + \sum_{t=0}^{T-2} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) +$$
$$+ \Delta_{T-1}(x_0, \dots, x_{T-1})(x_T - x_{T-1}) = f(x_0, \dots, x_{T-1}) + \Delta_{T-1}(x_0, \dots, x_{T-1}) x_T$$

for some function f. If Δ_{T-1} were not constant, on two points it would assume values $a \neq b$, with corresponding values of f that we call f_a, f_b. Then $f_a + a x_T = f_b + b x_T$ has a unique solution, contradicting the fact that all the equalities need to hold on the whole $K_0, \dots, K_T$ and in particular for two different values of x_T. We proceed one step backward. If $\text{card}(K_{T-1}) = 1$, the claim trivially follows, given our previous step. If $\text{card}(K_{T-1}) \geq 2$, similarly to the previous computation

$$\varphi_{T-1}(x_{T-1}) - \psi_{T-1}(x_{T-1}) = \sum_{s \neq T-1} (\psi_s(x_s) - \varphi_s(x_s)) + \sum_{t=0}^{T-3} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) +$$
$$+ \Delta_{T-2}(x_0, \dots, x_{T-2})(x_{T-1} - x_{T-2}) + \Delta_{T-1}(x_T - x_{T-1})$$
$$= f(x_s, s \neq T-1) + (\Delta_{T-2}(x_0, \dots, x_{T-2}) - \Delta_{T-1}) x_{T-1}\,.$$

An argument similar to the one we used in the previous time step shows that

$$\Delta_{T-2}(x_0, \dots, x_{T-2}) - \Delta_{T-1}$$

is constant, hence so is Δ_{T-2}. Our argument can be clearly be iterated up to Δ_0.
STEP 2: we prove existence of the vectors $k, h \in \mathbb{R}^{T+1}$, as stated in the Lemma. From Step 1 it is clear that there exist constants $k_0, \dots, k_T$ such that $I^{\Delta}(x) = \sum_{t=0}^{T} k_t x_t$. Hence $\sum_{t=0}^{T} \varphi_t(x_t) = \sum_{t=0}^{T} (\psi_t(x_t) + k_t x_t)$ for all $x \in \Omega$, which yields for each $t = 0, \dots, T$ that $\varphi_t(x_t) - (\psi_t(x_t) + k_t x_t)$ does not depend on x_t, hence is constant, call it $-h_t$. Then $k_0, \dots, k_T, h_0, \dots, h_T \in \mathbb{R}$ satisfy our requirements. The last claim $\sum_{t=0}^{T} S_t^U(\varphi_t) = \sum_{t=0}^{T} S_t^U(\psi_t)$ is then an easy consequence of Stock Additivity.
STEP 3: well posedness and properties of S. Observe that whenever $\varphi, \psi \in \mathcal{E}$, $\Delta, H \in \mathcal{H}$ are given with $\sum_{t=0}^{T} \varphi_t + I^{\Delta} = \sum_{t=0}^{T} \psi_t + I^{H}$ we have by Steps 1-2 that $\sum_{t=0}^{T} S_t^U(\varphi_t) = \sum_{t=0}^{T} S_t^U(\psi_t)$. As a consequence, S is well defined. Cash Additivity is inherited from $S_0, \dots, S_T$ while Integral Additivity is trivial from the definition. □

Proof of Lemma 4.2.2. We will only focus on (4.27), since the remaining claims are easily checked. We have that

$$\begin{aligned}\mathcal{D}(\gamma) &= \sup_{\varphi\in\mathcal{E}}\left(\sum_{t=0}^{T} U_t(\varphi_t) - \sum_{t=0}^{T}\int_{K_t}\varphi_t\,\mathrm{d}\gamma\right) = \sum_{t=0}^{T}\sup_{\varphi_t\in\mathcal{E}_t}\left(U_t(\varphi_t) - \int_{K_t}\varphi_t\,\mathrm{d}\gamma\right)\\ &= \sum_{t=0}^{T}\mathcal{D}_t(\gamma_t) = \sum_{t=0}^{T}\mathcal{D}_t(\gamma)\,.\end{aligned}$$

As to the second claim in (4.27), we observe that

$$\sup_{\xi\in\mathbb{R}^{T+1}}\left(U(\varphi+\xi) - \sum_{t=0}^{T}\xi_t\right) = \sum_{t=0}^{T}\sup_{\xi\in\mathbb{R}}\left(U_t(\varphi_t+\xi)-\xi\right) = \sum_{t=0}^{T} S^{U_t}(\varphi_t).$$

□

Proof of Lemma 4.2.5.
Item 1. For each $t = 0,\dots,T$ $\mathcal{D}_t(\gamma) = \mathcal{D}_t\circ\pi_t(\gamma)$, where $\mathcal{D}_t$ is $\sigma(\mathrm{ca}(K_t),\mathcal{E}_t)$-lower semicontinuous and π_t, the projection to the t-th marginal, is $\sigma(\mathrm{ca}(\Omega),\mathcal{E}) - \sigma(\mathrm{ca}(K_t),\mathcal{E}_t)$ continuous. Hence, for each $t = 0,\dots,T$ $\gamma\mapsto\mathcal{D}_t(\gamma)$ is $\sigma(\mathrm{ca}(\Omega),\mathcal{E})$-lower semicontinuous. Lower semicontinuity of $\mathcal{D}$ is then a consequence of the fact that the sum of lower semicontinuous functions is lower semicontinuous.
Item 2, equation (4.28). We have that for $\psi = -\varphi$

$$\begin{aligned}&-U(\varphi) = V(\psi) = \sup_{\mu\in\mathrm{ca}(\Omega)}\left(\int_\Omega\left(\sum_{t=0}^{T}\psi_t\right)\mathrm{d}\mu - \mathcal{D}(\mu)\right)\\ &= \sup_{\mu\in\mathrm{ca}(\Omega)}\sum_{t=0}^{T}\left(\int_{K_t}\psi_t\mathrm{d}\mu - \mathcal{D}_t(\mu_t)\right)\\ &\overset{(i)}{=}\sup\left\{\sum_{t=0}^{T}\left(\int_{K_t}\psi_t\mathrm{d}\gamma_t - \mathcal{D}_t(\gamma_t)\right) \mid \gamma\in\mathrm{ca}(\Omega)\text{ with }\gamma_t\in\mathrm{Prob}(K_t)\ \forall t=0,\dots,T\right\}\\ &\overset{(ii)}{=}\sup\left\{\sum_{t=0}^{T}\left(\int_{K_t}\psi_t\mathrm{d}\mathbb{Q}_t - \mathcal{D}_t(\mathbb{Q}_t)\right) \mid [\mathbb{Q}_0,\dots,\mathbb{Q}_T]\in\mathrm{Prob}(K_0)\times\cdots\times\mathrm{Prob}(K_T)\right\}\\ &= \sum_{t=0}^{T}\sup_{\mathbb{Q}_t\in\mathrm{Prob}(K_t)}\left(\int_{K_t}\psi_t\mathrm{d}\mathbb{Q}_t - \mathcal{D}_t(\mathbb{Q}_t)\right) \overset{(iii)}{=} \sum_{t=0}^{T}\sup_{\gamma_t\in\mathrm{ca}(K_t)}\left(\int_{K_t}\psi_t\mathrm{d}\gamma_t - \mathcal{D}_t(\gamma_t)\right)\\ &= \sum_{t=0}^{T}V_t(\psi_t) = \sum_{t=0}^{T}-U_t(\varphi_t)\,.\end{aligned}$$

Note that (i) follows from $\mathrm{dom}(\mathcal{D})\subseteq\mathcal{Z} := \{\gamma\in\mathrm{ca}(\Omega)\mid\gamma_t\in\mathrm{Prob}(K_t)\ \forall t=0,\dots,T\}$. In (ii) we used the facts that any vector of probability measures $(\mathbb{Q}_0,\dots,\mathbb{Q}_T)$ with $\mathbb{Q}_t\in\mathrm{Prob}(K_t)$, $t=0,\dots,T$, identifies $\gamma := \mathbb{Q}_0\otimes\cdots\otimes\mathbb{Q}_T\in\mathcal{Z}$ with $\mathcal{D}(\gamma) = \sum_{t=0}^{T}\mathcal{D}_t(\mathbb{Q}_t)$ (note that this does not hold for a general vector of signed measures, which is why we need the additional assumption on the domains of the penalization functionals for Item 2) and that for every $\gamma \in \mathscr{Z}$, setting $\mathbb{Q}_t := \gamma_t \in \mathrm{Prob}(K_t)$, we have $\mathcal{D}(\gamma) = \sum_{t=0}^T \mathcal{D}_t(\mathbb{Q}_t)$. Equality (iii) follows from $\mathrm{dom}(\mathcal{D}_t) \subseteq \mathrm{Prob}(K_t)$ for each $t = 0, \dots, T$.

Item 2, equation (4.29). The argument is identical to the one in the proof of Lemma 4.2.2, using the additive structure of U we obtained in the previous step of the proof. □

Proof of Proposition 4.2.8. We will use [108] Theorem 2.7 and [108] Remark 2.8. To do so, let us rename $F := v_t^*$ ((see (4.30) for the definition of v^*), which implies that $F^\circ(y) := -F^*(-y)$ of [108] Equation (2.45) satisfies $F^\circ(y) := -F^*(-y) = -v_t^{**}(-y) = -v_t(-y) = u_t(y)$, by Fenchel-Moreau Theorem. All the assumptions of [108] Section 2.3 on F are satisfied, since for every $y \geq 0$ $F(y) \geq u_t(0) - 0y = 0$ and $F(1) = \sup_{x\in\mathbb{R}}(u_t(x) - x) \leq 0$ (recall $u_t(x) \leq x$, $\forall\, x \in \mathbb{R}$). Also, since $\mathrm{dom}(u_t) = \mathbb{R}$, $\lim_{y\to+\infty} \frac{F(y)}{y} = F'_\infty = +\infty$. We can then apply [108] Theorem 2.7 and [108] Remark 2.8, obtaining (4.32). We stress the fact that since u_t is finite valued on the whole $\mathbb{R}$, it is continuous there and for every $\varphi_t \in \mathcal{C}_b(K_t)$, $F^\circ(\varphi_t) = u_t(\varphi_t) \in \mathcal{C}_b(K_t)$, hence the additional constraint $F^\circ(\varphi_t) \in \mathcal{C}_b(K_t)$ (below [108] (2.49)) would be redundant in our setup. □

Proof of Proposition 4.2.9. We will exploit again [108] Theorem 2.7 and [108] Remark 2.8 (with u_t in place of F°) , as we explain now. Since u_t is nondecreasing, either its domain is in the form $[M, +\infty)$ or $(M, +\infty)$, with $M \leq 0$. Given a $\varphi_t \in \mathcal{C}_b(K_t)$ and a $\mu \in \mathrm{Meas}(K_t)$

- Either $\inf(\varphi_t(\mathbb{R})) > M$, in which case $u_t(\varphi_t) \in \mathcal{C}_b(K_t)$ since u_t is continuous on the interior of its domain.
- Or $\inf(\varphi_t(\mathbb{R})) < M$, in which case $\{\varphi_t < M\}$ is open nonempty and hence has positive $\widehat{\mathbb{Q}}_t$ measure, as $\widehat{\mathbb{Q}}_t$ has full support. Thus $\int_{K_t} u_t(\varphi_t)\,\mathrm{d}\widehat{\mathbb{Q}}_t = -\infty$.
- Or $\inf(\varphi_t(\mathbb{R})) = M$ in which case $u_t(\varphi_t) = \lim_{\varepsilon\downarrow 0} u_t(\max(\varphi_t, M+\varepsilon))$ (since u_t is nondecreasing and upper semicontinuous) $u_t(\max(\varphi_t, M+\varepsilon)) \in \mathcal{C}_b(K_t)$ (see first bullet) and by Monotone Convergence Theorem
$$\int_{K_t} u_t(\varphi_t)\,\mathrm{d}\widehat{\mathbb{Q}}_t - \int_{K_t} \varphi_t\,\mathrm{d}\mu$$
$$= \lim_{\varepsilon\downarrow 0}\left(\int_{K_t} u_t(\max(\varphi_t, M+\varepsilon))\,\mathrm{d}\widehat{\mathbb{Q}}_t - \int_{K_t} \max(\varphi_t, M+\varepsilon)\,\mathrm{d}\mu\right).$$
Then we infer that
$$\sup_{\varphi_t\in\mathcal{C}_b(K_t)}\left(\int_{K_t}\varphi_t\,\mathrm{d}\mu - \int_{K_t} v_t(\varphi_t)\,\mathrm{d}\widehat{\mathbb{Q}}_t\right) = \sup_{\varphi_t\in\mathcal{C}_b(K_t)}\left(\int_{K_t} u_t(\varphi_t)\,\mathrm{d}\widehat{\mathbb{Q}}_t - \int_{K_t}\varphi_t\,\mathrm{d}\mu\right)$$
$$= \sup\left\{\int_{K_t} u_t(\varphi_t)\,\mathrm{d}\widehat{\mathbb{Q}}_t - \int_{K_t}\varphi_t\,\mathrm{d}\mu \mid \varphi_t, u_t(\varphi_t) \in \mathcal{C}_b(K_t)\right\}, \tag{4.44}$$
and from [108] Theorem 2.7, [108] Remark 2.8 and (4.44) we conclude the thesis.

□

4.4.3 On Minimax Duality Theorem

The following theorem is stated, without the proof, in [108], Theorem 2.4. For the sake of completeness and without claiming any originality, we here provide the short proof.

Theorem 4.4.8 (Minimax Duality Theorem)**.** *Let A, B be nonempty convex subsets of some vector spaces and suppose A is endowed with a Hausdorff topology. Let $L : A \times B \to \mathbb{R}$ be a function such that*

1. *$a \mapsto L(a,b)$ is convex and lower semicontinuous in A for every $b \in B$,*
2. *$b \mapsto L(a,b)$ is concave in B for every $a \in A$.*

When $\alpha := \sup_{b\in B} \inf_{a\in A} L(a,b) < +\infty$, suppose that there exist $C > \alpha$ and $b^\star \in B$ such that $\{a \in A \mid L(a,b^\star) \le C\}$ is compact in A. Then

$$\inf_{a\in A} \sup_{b\in B} L(a,b) = \sup_{b\in B} \inf_{a\in A} L(a,b) \,. \tag{4.45}$$

Proof. We start observing that in general $\inf_{a\in A}\sup_{b\in B} L(a,b) \ge \sup_{b\in B}\inf_{a\in A} L(a,b)$, hence if $\alpha = +\infty$ then (4.45) trivially holds. We then assume $\alpha < +\infty$ and modify the proof of [124] Theorem 3.1. Let $b_1, \dots, b_N \in B$ be given and set $b_0 = b^\star$. By [124] Lemma 2.1.(a), using $f_i(\cdot) := L(\cdot, b_i)$ we get constants $\lambda_0, \dots, \lambda_N \ge 0$ with $\sum_{i=0}^N \lambda_i = 1$ such that

$$\begin{aligned}
\inf_{a\in A}\left(\max_{i=0,\dots,N} L(a,b_i)\right) &= \inf_{a\in A}\left(\sum_{i=0}^N \lambda_i L(a,b_i)\right) \\
&\le \inf_{a\in A} L\left(a, \sum_{i=0}^N \lambda_i b_i\right) \le \sup_{b\in B}\inf_{a\in A} L(a,b) = \alpha \,,
\end{aligned}$$

where we used the concavity in B to obtain the first inequality. We now observe that for all $\varepsilon > 0$ there exists an $a \in A$ such that

$$\begin{aligned}
a \in \left\{\max_{i=0,\dots,N} L(a,b_i) \le \alpha + \varepsilon\right\} &= \bigcap_{i=0}^N \{L(a,b_i) \le \alpha+\varepsilon\} \\
&= \bigcap_{i=1}^N \{L(a,b_i) \le \alpha+\varepsilon\} \cap \{L(a,b^\star) \le \alpha+\varepsilon\} \,.
\end{aligned}$$

Hence for $A^\star = \{L(a,b^\star) \le \alpha+\varepsilon\}$ the family $A_b^\varepsilon := \{a \in A^\star \mid L(a,b) \le \alpha+\varepsilon\}$ is a collection of closed subsets of $A^\star$ having the finite intersection property. Now take $\varepsilon > 0$ such that $\alpha + \varepsilon < C$. Then $A^\star$ is Hasudorff and compact, being a closed subset of the compact set $\{a \in A \mid L(a,b^\star) \le C\}$. As a consequence $\bigcap_{b\in B} A_b^\varepsilon \ne \emptyset$. This yields the existence of an $a^\star$ such that $a^\star \in A^\star$ and $L(a^\star, b) \le \alpha+\varepsilon \;\forall\, b \in B$. Hence

$$\inf_{a\in A}\sup_{b\in B} L(a,b) \le \sup_{b\in B} L(a^\star,b) \le \varepsilon + \alpha$$

and letting $\varepsilon \downarrow 0$ we get

$$\inf_{a\in A}\sup_{b\in B} L(a,b) \le \sup_{b\in B}\inf_{a\in A} L(a,b) \le \inf_{a\in A}\sup_{b\in B} L(a,b) \,.$$

□

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