Fractional Paley–Wiener and Bernstein spaces

Alessandro Monguzzi[1] · Marco M. Peloso[2] · Maura Salvatori[2]

Abstract

We introduce and study a family of spaces of entire functions in one variable that generalise the classical Paley–Wiener and Bernstein spaces. Namely, we consider entire functions of exponential type a whose restriction to the real line belongs to the homogeneous Sobolev space $\dot{W}^{s,p}$ and we call these spaces fractional Paley–Wiener if $p = 2$ and fractional Bernstein spaces if $p \in (1, \infty)$, that we denote by PW_a^s and $\mathcal{B}_a^{s,p}$, respectively. For these spaces we provide a Paley–Wiener type characterization, we remark some facts about the sampling problem in the Hilbert setting and prove generalizations of the classical Bernstein and Plancherel–Pólya inequalities. We conclude by discussing a number of open questions.

Dedicated to the memory of Elias M. Stein.

✉ Marco M. Peloso
  marco.peloso@unimi.it

  Alessandro Monguzzi
  alessandro.monguzzi@unimib.it

  Maura Salvatori
  maura.salvatori@unimi.it

[1] Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Via R. Cozzi 55, 20126 Milan, Italy

[2] Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milan, Italy

---

1 Introduction and statement of the main results

A renowned theorem due to Paley and Wiener [21] characterizes the entire functions of exponential type $a > 0$ whose restriction to the real line is square-integrable in terms of the support of the Fourier transform of their restriction to the real line. An analogous characterization holds for entire functions of exponential type a whose restriction to to the real line belongs to some L^p space, $p \neq 2$ [9]. To be precise, let $\mathcal{E}_a$ be the space of entire functions of exponential type a,

$$\mathcal{E}_a = \left\{ f \in \mathrm{Hol}(\mathbb{C}) : \text{ for every } \varepsilon > 0 \text{ there exists } C_\varepsilon > 0 \text{ such that } |f(z)| \leq C_\varepsilon e^{(a+\varepsilon)|z|} \right\}. \tag{1}$$

Then, for any $p \in (1, \infty)$, the Bernstein space $\mathcal{B}_a^p$ is defined as

$$\mathcal{B}_a^p = \left\{ f \in \mathcal{E}_a : f_0 \in L^p, \ \|f\|_{\mathcal{B}_a^p} = \|f_0\|_{L^p} \right\}$$

where $f_0 := f|_{\mathbb{R}}$ denotes the restriction of f to the real line and L^p is the standard Lebesgue space. In the Hilbert setting $p = 2$, the Bernstein space $\mathcal{B}_a^2$ is more commonly known as the Paley–Wiener space and we will denote it by PW_a in place of $\mathcal{B}_a^2$.

Let $\mathcal{S}$ and $\mathcal{S}'$ denote the space of Schwartz functions and the space of tempered distributions, resp. For $f \in \mathcal{S}$ we equivalently denote by $\widehat{f}$ or $\mathcal{F}f$ the Fourier transform given by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$

The Fourier transform $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ onto itself with inverse given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} \, d\xi.$$

By Plancherel Theorem, the operator $\mathcal{F}$ extends to a surjective isometry $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

We now recall the classical Paley–Wiener characterization of the space PW_a.

Theorem [21] *Let $f \in PW_a$, then $\mathrm{supp} \widehat{f}_0 \subseteq [-a, a]$,*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f}_0(\xi) e^{iz\xi} \, d\xi$$

and $\|f\|_{PW_a} = \|\widehat{f}_0\|_{L^2([-a,a])}$. Conversely, if $g \in L^2([-a,a])$ and we define

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) e^{iz\xi} \, d\xi,$$

then $f \in PW_a$, $\widehat{f}_0 = g$ and $\|f\|_{PW_a} = \|g\|_{L^2([-a,a])}$.

In particular, the Fourier transform $\mathcal{F}$ induces a surjective isometry between the spaces $L^2([-a, a])$ and PW_a. We shall write L_a^2 instead of $L^2([-a, a])$ for short.

A similar characterization holds true for the Bernstein spaces $\mathcal{B}_a^p$, $1 < p < +\infty$. We refer the reader, for instance, to [4, Theorem 4]. We shall denote by $\mathbb{N}_0$ the set of non-negative integers.

Theorem (*Characterizations of $\mathcal{B}_a^p$*) *Let $1 < p < \infty$. Then, the following conditions on a function h defined on the real line are equivalent.*

(i) *The function h is the restriction of an entire function $f \in \mathcal{B}_a^p$ to the real line, that is, $h = f_0$;*

(ii) *$h \in L^p(\mathbb{R})$ and $\operatorname{supp}\widehat{h} \subseteq [-a, a]$;*

(iii) *$h \in C^\infty$, $h^{(n)} \in L^p$ for all $n \in \mathbb{N}_0$ and $\|h^{(n)}\|_{L^p} \leq a^n \|h\|_{L^p}$.*

The above theorem holds in the limit cases $p = 1$ and $p = +\infty$ as well, but in this paper we only focus on the range $1 < p < +\infty$.

We remark that in the Paley–Wiener characterization of the Bernstein spaces, the Fourier transform of $f_0 \in L^p(\mathbb{R})$ is to be understood in the sense of tempered distributions. Namely, $L^p(\mathbb{R}) \subseteq \mathcal{S}'$, and the Fourier transform extends to a isomorphism of $\mathcal{S}'$ onto itself, where $\widehat{f}$ is defined by the formula

$$\langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle, \qquad f \in \mathcal{S}', \varphi \in \mathcal{S}.$$

The Paley–Wiener and Bernstein spaces are classical and deeply studied for several reasons. A well-studied problem for these spaces, for instance, is the sampling problem and we refer the reader to [19, 24] and references therein. Moreover, the Paley–Wiener space PW_a is the most important example of a de Branges space, which are spaces of entire functions introduced by de Branges in [13]. They have deep connections with canonical systems and have been extensively studied in the recent years. For an overview of de Branges spaces and canonical systems we refer the reader, for instance, to [23].

In this paper we introduce a family of spaces which generalizes the classical Paley–Wiener and Bernstein spaces; we deal with spaces of entire functions of exponential type a whose restriction to the real line belongs to some homogeneous Sobolev space and we call these spaces fractional Paley–Wiener and Bernstein spaces. In the present work we start such investigation: we introduce the spaces, we study some of their structural properties, we prove a Paley–Wiener type characterization and generalizations of the classical Bernstein and Plancherel–Pólya inequalities. We also point out that classical results such as sampling theorems for the Paley–Wiener space do not necessarily extend to the fractional setting (Sect. 5). Finally we mention the papers [20, 22] in which the authors studied other generalizations of the Paley–Wiener spaces. We also point out that the investigation of these spaces has natural counterparts in the several variable setting. We refer the reader to [3] and the upcoming manuscript [17] where we present the development of this theory in the setting of several variables.

We now precisely define the function spaces we are interested in. Given a function $f \in \mathcal{S}$ and $s > 0$, we define its fractional Laplacian $\Delta^{\frac{s}{2}} f$ as

$$\Delta^{\frac{s}{2}} f := \mathcal{F}^{-1}(|\cdot|^s \mathcal{F} f)$$

and we set

$$\|f\|_{s,p} := \|\Delta^{\frac{s}{2}} f\|_{L^p}.$$

We remark that for $f \in \mathcal{S}$ the fractional Laplacian $\Delta^{\frac{s}{2}} f$ is a well-defined function and that $\|\cdot\|_{s,p}$ is a norm on the Schwartz space (see, for instance, [18]). Therefore, we define the homogeneous Sobolev space $\dot{W}^{s,p}$ as the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{s,p}$, i.e.,

$$\dot{W}^{s,p} = \overline{\mathcal{S}}^{\|\cdot\|_{s,p}}.$$

As described in [18], the space $\dot{W}^{s,p}$ turns out to be a quotient space of tempered distributions modulo polynomials of degree $m = \lfloor s - 1/p \rfloor$, where we denote by $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$ and by $\mathcal{P}_m$ the set polynomials of degree at most m, where $m \in \mathbb{N}_0$. In [18, Corollary 3.3] we prove that $f \in \dot{W}^{s,p}$ if and only if

- $f \in \mathcal{S}'/\mathcal{P}_m$;
- there exists a sequence $\{f_n\} \subseteq \mathcal{S}$ such that $f_n \to f$ in $\mathcal{S}'/\mathcal{P}_m$;
- the sequence $\{\Delta^{\frac{s}{2}} f_n\}$ is a Cauchy sequence with respect to the L^p norm.

If $f \in \dot{W}^{s,p}$ we then set

$$\Delta^{\frac{s}{2}} f = \lim_{n \to +\infty} \Delta^{\frac{s}{2}} f_n, \tag{2}$$

where the limit is to be understood as a limit in the L^p norm.

In order to avoid working in a quotient space, instead of considering the spaces $\dot{W}^{s,p}$, we consider the realization spaces $E^{s,p}$, see [18, Corollary 3.2]. Inspired by the works of G. Bourdaud [10–12], if $m \in \mathbb{N}_0 \cup \{\infty\}$ and $\dot{X}$ is a given subspace of $\mathcal{S}'/\mathcal{P}_m$ which is a Banach space, such that the natural inclusion of $\dot{X}$ into $\mathcal{S}'/\mathcal{P}_m$ is continuous, we call a subspace E of $\mathcal{S}'$ a *realization* of $\dot{X}$ if there exists a bijective linear map

$$R : \dot{X} \to E$$

such that $\lfloor R[u] \rfloor = [u]$ for every equivalence class $[u] \in \dot{X}$. We endow E of the norm given by $\|R[u]\|_E = \|[u]\|_{\dot{X}}$.

We recall the definition of the homogeneous Lipschitz space $\dot{\Lambda}^\gamma$, for $\gamma > 0$, for details see e.g. [15] and [18]. For $k \in \mathbb{N}$ and $h \in \mathbb{R} \setminus \{0\}$, let D_h^k, the difference operator of increment h and of order k, be defined as follows. If $k = 1$, $D_h f(x) = f(x+h) - f(x)$, and then inductively, for $k = 2, 3, \dots$ $D_h^k f(x) = D_h(D_h^{k-1} f)(x) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(x+jh)$. The *homogeneous Lipschitz space* is defined as

$$\dot{\Lambda}^\gamma = \left\{ [f] \in \mathcal{C}/\mathcal{P}_{\lfloor \gamma \rfloor} \; : \; \|f\|_{\dot{\Lambda}^\gamma} := \sup_{x, h \in \mathbb{R}, \, h \neq 0} \frac{|D_h^{\lfloor \gamma \rfloor + 1} f(x)|}{|h|^\gamma} < +\infty \right\}.$$

For a non-negative integer m, we denote by $\mathcal{C}^m$ the space of continuously differentiable functions of order m.

The next result describes the realization spaces $E^{s,p}$ that we will need.

Theorem [18] *For $s > 0$ and $p \in (1, +\infty)$, let $m = \lfloor s - \frac{1}{p} \rfloor$. Then, $\dot{W}^{s,p} \subseteq \mathcal{S}'/\mathcal{P}_m$. We define the spaces $E^{s,p}$ as follows.*

(i) *Let $0 < s < \frac{1}{p}$, and let $p^* \in (1, \infty)$ given by $\frac{1}{p} - \frac{1}{p^*} = s$, define*

$$E^{s,p} = \left\{ f \in L^{p^*} \: : \: \|f\|_{E^{s,p}} := \|\Delta^{s/2}f\|_{L^p} < +\infty \right\}.$$

(ii) *Let* $s - \frac{1}{p} \notin \mathbb{N}_0$, $m = \lfloor s - \frac{1}{p} \rfloor$, *and define*

$$E^{s,p} = \left\{ f \in \dot{\Lambda}^{s-\frac{1}{p}} \: : \: P_{f;m;0} = 0, \: \|f\|_{E^{s,p}} := \|\Delta^{s/2}f\|_{L^p} < +\infty \right\}.$$

Then, the space $E^{s,p}$ is a realization space for $\dot{W}^{s,p}$.

We are now ready to define the fractional Bernstein spaces.

Definition 1.1 For $a, s > 0$, $1 < p < +\infty$ and $s - \frac{1}{p} \notin \mathbb{N}_0$ the fractional Bernstein space $\mathcal{B}_a^{s,p}$ is defined as

$$\mathcal{B}_a^{s,p} = \left\{ f \in \mathcal{E}_a \: : \: f_0 \in E^{s,p} \text{ with norm } \|f\|_{\mathcal{B}_a^{s,p}} := \|f_0\|_{E^{s,p}} \right\}.$$

Remark 1.2 In this paper we restrict ourselves to the case $s - \frac{1}{p} \notin \mathbb{N}_0$, $s > 0$, $p \in (1, \infty)$. The case $s - \frac{1}{p} \in \mathbb{N}_0$ could be thought to be the *critical* case, as in the Sobolev embedding theorem. All the proofs break down for these values of s and p, although we believe that all the results in this paper extend also to case $s - \frac{1}{p} \in \mathbb{N}_0$.

Thus, the case $s - \frac{1}{p} \in \mathbb{N}_0$ remains open and is, in our opinion, of considerable interest. We will add some comments on this problem in the final Sect. 8.

Remark 1.3 We point out that from the results in the present work we can easily deduce analogous results for the *homogeneous* fractional Bernstein spaces $\dot{\mathcal{B}}_a^{s,p}$, defined as above, but without requiring that $P_{f_0;m;0} = 0$. In this way, we obtain spaces of entire functions of exponential type *modulo* polynomials of degree $m = \lfloor s - \frac{1}{p} \rfloor$.

We first consider the spaces PW_a^s, $s > 0$, and we prove some Paley–Wiener type theorems assuming that $s - \frac{1}{2} \notin \mathbb{N}_0$. For any $s > 0$ let $L_a^2(|\xi|^{2s})$ be the weighted L^2-space

$$L_a^2(|\xi|^{2s}) = \left\{ f \: : \: [-a, a] \to \mathbb{C} \text{ such that } \int_{-a}^{a} |f(\xi)|^2 |\xi|^{2s} \, d\xi < \infty \right\}.$$

We prove the following Paley–Wiener type theorems. We distinguish the case $0 < s < \frac{1}{2}$ from the case $s > \frac{1}{2}$.

Theorem 1 *Let $0 < s < \frac{1}{2}$ and let $f \in PW_a^s$. Then, $\mathrm{supp}\widehat{f_0} \subseteq [-a, a]$, $\widehat{f_0} \in L_a^2(|\xi|^{2s})$ and*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f_0}(\xi) e^{iz\xi} \, d\xi. \tag{3}$$

Moreover, $\|f\|_{PW_a^s} = \|\widehat{f_0}\|_{L_a^2(|\xi|^{2s})}$. Conversely, let $g \in L_a^2(|\xi|^{2s})$, and define f by setting

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) e^{iz\xi} \, d\xi. \tag{4}$$

Then, $f \in PW_a^s$, $\widehat{f_0} = g$ and $\|f\|_{PW_a^s} = \|g\|_{L_a^2(|\xi|^{2s})}$.

Definition 1.4 Given $s > \frac{1}{2}$, let $m = \lfloor s - \frac{1}{2} \rfloor$, for any $g \in L^2(|\xi|^{2s})$ we define Tg by setting, for $\psi \in \mathcal{S}$,

$$\langle Tg, \psi \rangle := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \big(\psi(\xi) - P_{\psi;m;0}(\xi) \big) \, d\xi. \tag{5}$$

As we will see, Lemma 3.3, Tg is well-defined for any $\psi \in \mathcal{S}$, in particular $Tg \in \mathcal{S}'$, and $T : L^2(|\xi|^{2s}) \to \mathcal{S}'$ is a continuous operator.

We denote by $\mathcal{D}_c'$ the space of distributions with compact support, which is the dual of C^∞.

Theorem 2 *Let* $s > \frac{1}{2}$, $m = \lfloor s - \frac{1}{2} \rfloor$, *assume that* $s - \frac{1}{2} \notin \mathbb{N}$ *and set* $P_m(iz\xi) = \sum_{j=0}^{m} (iz\xi)^j / j!$. *Let* $f \in PW_a^s$, *then* $\operatorname{supp}\widehat{f_0} \subseteq [-a, a]$ *and there exists* $g \in L_a^2(|\xi|^{2s})$ *such that* $\widehat{f_0} = Tg$ *in* $\mathcal{D}_c'$, *and*

$$f(z) = \langle \widehat{f_0}, e^{iz(\cdot)} \rangle = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) \big(e^{iz\xi} - P_m(iz\xi) \big) \, d\xi. \tag{6}$$

Moreover, $\|f\|_{PW_a^s} = \|g\|_{L_a^2(|\xi|^{2s})}$. *Conversely, let* $g \in L_a^2(|\xi|^{2s})$ *and define f by setting*

$$f(z) = \langle Tg, e^{iz(\cdot)} \rangle = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) \big(e^{iz\xi} - P_m(iz\xi) \big) \, d\xi. \tag{7}$$

Then, $f \in PW_a^s$ *and* $\|f\|_{PW_a^s} = \|g\|_{L_a^2(|\xi|^{2s})}$.

Observe that in particular Theorem 1 says that, if $0 < s < \frac{1}{2}$, the Fourier transform $\mathcal{F} : PW_a^s \to L_a^2(|\xi|^{2s})$ is a surjective isomorphism, as in the case $s = 0$. On the other hand, if $s > \frac{1}{2}$, $\mathcal{F} : PW_a^s \to T(L_a^2(|\xi|^{2s}))$ is a surjective isomorphism, where $T(L_a^2(|\xi|^{2s})) \subseteq \mathcal{D}_c'$ denotes the image of $L_a^2(|\xi|^{2s})$ via the operator T, endowed with norm $\|Tg\| := \|g\|_{L_a^2(|\xi|^{2s})}$.

As a consequence of the above theorems we obtain that the spaces PW_a^s are reproducing kernel Hilbert spaces and we are able to make some interesting remarks concerning reconstruction formulas and sampling in PW_a^s for $0 < s < \frac{1}{2}$. In particular, we obtain that the spaces PW_a^s are not de Branges spaces. We refer the reader to Sect. 5 below for more details.

Then we turn our attention to the fractional Bernstein spaces $\mathcal{B}_a^{s,p}$.

Theorem 3 *Let* $s > 0$, $1 < p < \infty$ *be such that* $s - \frac{1}{p} \notin \mathbb{N}$. *Then, the fractional Bernstein spaces* $\mathcal{B}_a^{s,p}$ *are Banach spaces and the following Plancherel–Pólya estimates hold. If* $0 < s < \frac{1}{p}$, *for* $f \in \mathcal{B}_a^{s,p}$ *and* $y \in \mathbb{R}$ *we have*

$$\|f(\cdot + iy)\|_{\mathcal{B}_a^{s,p}} \leq e^{a|y|} \|f\|_{\mathcal{B}_a^{s,p}}.$$

If $s > \frac{1}{p}$ *and* $s - \frac{1}{p} \notin \mathbb{N}_0$, *for* $f \in \mathcal{B}_a^{s,p}$ *and* $y \in \mathbb{R}$ *given, define* $F(w) = f(w + iy) - P_{f(\cdot + iy);m;0}(w)$, $w \in \mathbb{C}$. *Then,* $F \in \mathcal{B}_a^{s,p}$ *and*

$$\|F\|_{\mathcal{B}_a^{s,p}} \leq e^{a|y|} \|f\|_{\mathcal{B}_a^{s,p}}.$$

Theorem 4 *Let $s > 0$ and $1 < p < \infty$ such that $s - \frac{1}{p} \notin \mathbb{N}_0$. Given a function h on the real line, the following conditions are equivalent.*

   (i) *The function h is the restriction of an entire function $f \in \mathcal{B}_a^{s,p}$ to the real line, that is, $h = f_0$;*

   (ii) $h \in E^{s,p}$ *and* $\operatorname{supp}\widehat{h} \subseteq [-a, a]$;

   (iii) $h \in C^\infty$ *and it is such that* $h^{(n)} \in E^{s,p}$ *for all* $n \in \mathbb{N}_0$ *and* $\|h^{(n)}\|_{E^{s,p}} \le a^n \|h\|_{E^{s,p}}$.

Finally, the spaces PW_a^s are closed subspaces of the Hilbert spaces $E^{s,2}$, and thus there exists a Hilbert space projection operator $\mathbf{P}_s : E^{s,2} \to PW_a^s$. It is natural to study the mapping property of the operator $\mathbf{P}_s$ with respect to the L^p norm. We prove the following result.

Theorem 5 *Let $s > 0$ and $1 < p < \infty$ such that $s - \frac{1}{2} \notin \mathbb{N}_0$, $s - \frac{1}{p} \notin \mathbb{N}_0$ and $\lfloor s - \frac{1}{2} \rfloor = \lfloor s - \frac{1}{p} \rfloor$. Then, the Hilbert space projection operator $\mathbf{P}_s : E^{s,2} \to PW_a^s$ densely defined on $E^{s,p} \cap E^{s,2}$ extends to a bounded operator $\mathbf{P}_s : E^{s,p} \to \mathcal{B}_a^{s,p}$ for all $s > 0$ and $1 < p < +\infty$.*

The paper is organized as follows. After recalling some preliminary results in Sect. 2, we prove Theorem 1 and 2 in Sect. 3. In Sect. 4 we investigate the fractional Bernstein spaces proving Theorems 3 and 4, whereas in Sect. 5 we shortly discuss the sampling problem for the fractional Paley–Wiener spaces. Finally, we prove prove Theorem 5 in Sect. 6, and conclude with further remarks and open questions in Sect. 8.

2 Preliminaries

In this section we recall some results of harmonic analysis we will need in the remaining of the paper. We omit the proofs of the results and we refer the reader, for instance, to [25]. We do not recall the results in their full generality, but only in the version we need them.

Let $0 < s < 1$ so that the function $\xi \to |\xi|^{-s}$ is locally integrable. Then, the Riesz potential operator $\mathcal{I}_s$ is defined on $\mathcal{S}$ as

$$\mathcal{I}_s f = \mathcal{F}^{-1}(|\cdot|^{-s}\widehat{f}). \tag{8}$$

Observe that if $f \in \mathcal{S}$ and $0 < s < 1$, then $f = \mathcal{I}_s \Delta^{\frac{s}{2}} f = \Delta^{\frac{s}{2}} \mathcal{I}_s f$.

For $p \in (0, \infty)$ we denote by H^p, the Hardy space on $\mathbb{R}$. Having fixed $\Phi \in \mathcal{S}$ with $\int \Phi = 1$, then

$$H^p = \left\{ f \in \mathcal{S}' : f^*(x) := \sup_{t>0} |f * \Phi_t(x)| \in L^p \right\}, \tag{9}$$

where

$$\|f\|_{H^p} = \|f^*\|_{L^p}.$$

We recall that the definition of H^p is independent of the choice of Φ and that, when $p \in (1, \infty)$, H^p coincides with L^p, with equivalence of norms.

The Riesz potential operator extends to a bounded operator on H^p, $0 < p < \infty$, according to the following theorem. Part (ii) is due to Adams, see [2].

Theorem 2.1 *Let* $0 < s < 1$, $0 < p < \infty$.

(i) *If* $s < \frac{1}{p}$ *and* $\frac{1}{p^*} = \frac{1}{p} - s$, *then,* $\mathcal{I}_s$ *extends to a bounded operator* $\mathcal{I}_s : H^p \to H^{p^*}$.

(ii) *If* $s = \frac{1}{p}$, *then,* $\mathcal{I}_s$ *extends to a bounded operator* $\mathcal{I}_s : L^p \to \text{BMO}$.

Definition 2.2 For M a nonnegative integer, define

$$\mathcal{S}_M = \left\{ f \in \mathcal{S} : \int_{\mathbb{R}} x^k f(x)\, dx = 0 \text{ for } k \in \mathbb{N}_0,\ k \leq M \right\}$$

and

$$\mathcal{S}_\infty = \left\{ f \in \mathcal{S} : \int_{\mathbb{R}} x^k f(x)\, dx = 0 \text{ for all } k \in \mathbb{N}_0 \right\}.$$

We recall that, as it is elementary to verify, $\varphi \in \mathcal{S}_M$, if and only if there exists $\Phi \in \mathcal{S}$ such that $\varphi = \Phi^{(M)}$. We will use this fact several times. We also recall that $\mathcal{S}_\infty$ is dense in H^p for all $p \in (0, \infty)$, see [25, Ch.II,5.2]. For these and other properties of the Hardy spaces see e.g. [25] or [15].

Notice that the Riesz potential operator $\mathcal{I}_s$ is also well-defined on $\mathcal{S}_\infty$ for any $s \geq 0$, since if $f \in \mathcal{S}_\infty$, then $\widehat{f}$ vanishes of infinite order at the origin. Moreover, for all $s > 0$

$$\mathcal{I}_s, \Delta^{\frac{s}{2}} : \mathcal{S}_\infty \to \mathcal{S}_\infty \tag{10}$$

are both surjective bounded isomorphisms and in fact one the inverse of the other one; see e.g. [15, Chapter 1].

3 Fractional Paley–Wiener spaces

In this section we first prove Theorems 1 and 2, we deduce that the space PW_a^s is a reproducing kernel Hilbert space for every $s > 0$, $s - \frac{1}{2} \notin \mathbb{N}_0$, and explicitly compute its reproducing kernel. We conclude this section by proving that the classical Paley–Wiener space PW_a and PW_a^s are actually isometric.

Lemma 3.1 *Let* $f \in PW_a^s$, $s > 0$. *Then,* $\text{supp}\widehat{f_0} \subseteq [-a, a]$, *so that* $\widehat{f_0} \in \mathcal{D}'_c$, *and* $\left(\widehat{f_0}\right)_{|_{\mathcal{P}_m}} = 0$, *where* $m = \lfloor s - \frac{1}{2} \rfloor$.

Proof It is clear from the description of the realization spaces $E^{s,2}$ that $f_0 \in \mathcal{S}'$, hence, once we prove that $\text{supp}\widehat{f_0} \subseteq [-a, a]$, it immediately follows that $\widehat{f_0} \in \mathcal{D}'_c$. Let $\varphi \in \mathcal{S}_M \cap C_c^\infty$, with $M \geq s$. Given $f \in PW_a^s$ we define

$$f_\varphi(z) := \int_{\mathbb{R}} f(z - t)\varphi(t)\, dt$$

and we claim that $f_\varphi \in \mathcal{E}_a$ and $(f_\varphi)_0 = f_0 * \varphi \in L^2$; where the symbol $*$ denotes the standard convolution on the real line. The function f_φ is clearly entire and, for every $\varepsilon > 0$,

$$|f_\varphi(z)| \leq \int_{\mathbb{R}} |f(z-t)||\varphi(t)|\, dt \leq C_\varepsilon e^{(a+\varepsilon)|z|} \int_{\mathbb{R}} e^{(a+\varepsilon)|t|} |\varphi(t)|\, dt \leq C e^{(a+\varepsilon)|z|}$$

where the last integral converges since φ is compactly supported. Hence, $f_\varphi \in \mathcal{E}_a$. Moreover, since $\varphi \in \mathcal{S}_M$, then $\varphi * \eta \in \mathcal{S}_M$ as well for any $\eta \in \mathcal{S}$. Therefore, if $\{\varphi_n\} \subseteq \mathcal{S}$ is such that $\varphi_n \to f_0$ in $\mathcal{S}'/\mathcal{P}_m$ and $\Delta^{\frac{s}{2}} \varphi_n \to \Delta^{\frac{s}{2}} f_0$ in L^2, for any $\eta \in \mathcal{S}$ we have[1]

$$\langle f_0 * \varphi, \eta \rangle = \langle f_0, \varphi * \eta \rangle$$
$$= \lim_{n \to +\infty} \langle \varphi_n, \varphi * \eta \rangle$$
$$= \lim_{n \to +\infty} \langle \Delta^{\frac{s}{2}} \varphi_n, \mathcal{I}_s(\varphi * \eta) \rangle.$$

The last equality follows using the Parseval identity, since $\varphi \in \mathcal{S}_M$, hence $\varphi * \eta \in \mathcal{S}_M$ as well, so that $\mathcal{I}_s(\varphi * \eta) \in L^2$. Moreover, if $\Psi \in \mathcal{S}$ is such that $\varphi * \eta = \Psi^{(M)}$, we have $\mathcal{I}_s(\varphi * \eta) = \mathcal{I}_{s-\ell}\left(R^\ell \Psi^{(M-\ell)}\right)$, where R denotes the Riesz transform, and $\ell = \lfloor s \rfloor$. Then,

$$\|\mathcal{I}_s(\varphi * \eta)\|_{L^2} = \|R^\ell \mathcal{I}_{s-\ell} \Psi^{(M-\ell)}\|_{L^2} \leq C \|\Psi^{(M-\ell)}\|_{H^p} < \infty$$

where we have used Theorem 2.1 (i) with $\frac{1}{2} = \frac{1}{p} - (s - \ell)$. Notice that p could be either greater or smaller than 1. If $p > 1$, then $\|\Psi^{(M-\ell)}\|_{H^p} \approx \|\Psi^{(M-\ell)}\|_{L^p} < \infty$ since $\Psi \in \mathcal{S}$, if $p < 1$, the fact that $\|\Psi^{(M-\ell)}\|_{H^p}$ is finite if M is sufficiently large is a well-known fact, see e.g. [25]. Therefore,

$$\langle f_0 * \varphi, \eta \rangle = \lim_{n \to +\infty} \langle \Delta^{\frac{s}{2}} \varphi_n, \mathcal{I}_s(\varphi * \eta) \rangle$$
$$= \langle \Delta^{\frac{s}{2}} f_0, \mathcal{I}_s(\varphi * \eta) \rangle$$
$$= \langle \Delta^{\frac{s}{2}} f_0 * \mathcal{I}_s \varphi, \eta \rangle$$

for any $\eta \in \mathcal{S}$. In particular $f_0 * \varphi = \Delta^{\frac{s}{2}} f_0 * \mathcal{I}_s \varphi$ in L^2 and if $\Phi \in \mathcal{S}$ is such that $\Phi^{(M)} = \varphi$,

$$\|(f_\varphi)_0\|_{L^2} = \|f_0 * \varphi\|_{L^2} = \|\Delta^{\frac{s}{2}} f_0 * R^\ell \mathcal{I}_{s-\ell} \Phi^{(M-\ell)}\|_{L^2} \leq \|f\|_{PW_a^s} \|\mathcal{I}_{s-\ell} R^\ell \Phi^{(M-\ell)}\|_{H^1}$$
$$\leq C \|f\|_{PW_a^s} \|\Phi^{(M-\ell)}\|_{H^p} < \infty,$$

(11)

for M sufficiently large, where we have used Theorem 2.1 (i) with $1 = \frac{1}{p} - (s - \ell)$. This shows that $(f_\varphi)_0 \in L^2$, and therefore, $f_\varphi \in PW_a$. Thus, $\mathrm{supp} \widehat{(f_0 * \varphi)} = \mathrm{supp}(\widehat{f_0} \cdot \widehat{\varphi}) \subseteq [-a, a]$. Since for every $\xi_0 \neq 0$, $\xi_0 \in \mathbb{R}$, there exists $\varphi \in \mathcal{S}_M$ such that $\widehat{\varphi}(\xi_0) \neq 0$, we conclude that $\mathrm{supp} \widehat{f_0} \subseteq [-a, a]$ as we wished to show.

Let now $s > \frac{1}{2}$, $s - \frac{1}{2} \notin \mathbb{N}$, and let $m = \lfloor s - \frac{1}{2} \rfloor$. Fix $\chi \in C_c^\infty$, $\chi \geq 0$, $\chi = 1$ on $[-a, a]$. For $\varepsilon > 0$, we adopt here and throughout the paper the notation

$$\chi^\varepsilon(x) = \chi(\varepsilon x), \quad \chi_\varepsilon(x) = \frac{1}{\varepsilon} \chi(x/\varepsilon).$$

---

[1] We warn the reader that, we shall denote with the same symbol $\langle \cdot, \cdot \rangle$ different *bilinear* pairings of duality, such as $\langle \mathcal{S}', \mathcal{S} \rangle$, $\langle \mathcal{D}_c', C^\infty \rangle$, $\langle L^{p'}, L^p \rangle$, etc. The actual pairing of duality should be clear from the context and there should not be any confusion.

Given $Q = \sum_{j=0}^{m} q_j x^j \in \mathcal{P}_m$, we have

$$
\begin{aligned}
\langle \hat{f}_0, Q \rangle = \langle \hat{f}_0, \chi^\varepsilon Q \rangle &= \left\langle f_0, \hat{\chi}_\varepsilon * \left(\sum_{j=0}^{m} (-1)^j q_j D^j \delta_0 \right) \right\rangle \\
&= \left\langle \sum_{j=0}^{m} q_j D^j f_0, \hat{\chi}_\varepsilon \right\rangle \\
&= \left\langle \sum_{j=0}^{m} q_j D^j f_0, \chi^\varepsilon \hat{\chi}_\varepsilon \right\rangle + \left\langle f_0, \sum_{j=0}^{m} (-1)^j q_j D^j \left((1 - \chi^\varepsilon) \hat{\chi}_\varepsilon \right) \right\rangle \\
&= I_\varepsilon + II_\varepsilon.
\end{aligned}
$$

This equality holds for all $0 < \varepsilon \leq 1$ and we observe that, since f_0 is of moderate growth, both I_ε and II_ε are given by absolutely convergent integrals. Let $M > 0$ be such that $|f_0(x)| \leq C(1 + |x|)^M$, for some $C > 0$. We have

$$
\begin{aligned}
|II_\varepsilon| &\leq C \sum_{j=0}^{m} \int_{|x| \geq \frac{a}{\varepsilon}} (1 + |x|)^M \frac{1}{\varepsilon^{j+1}} D^j(\hat{\chi})(\tfrac{x}{\varepsilon}) \, dx \\
&\leq C \sum_{j=0}^{m} \int_{|x| \geq \frac{a}{\varepsilon}} (1 + |x|)^{M+m} |D^j(\hat{\chi})(\tfrac{x}{\varepsilon})| \, dx \\
&= C\varepsilon \sum_{j=0}^{m} \int_{|t| \geq \frac{a}{\varepsilon^2}} (1 + |\varepsilon t|)^{M+m} |D^j \hat{\chi}(t)| \, dt \\
&\leq C_N \varepsilon^N,
\end{aligned}
$$

for any $N > 0$. On the other hand, using Lebesgue's dominated convergence theorem it is easy to see that, as $\varepsilon \to 0$,

$$
I_\varepsilon = \int Q(D) f_0(\varepsilon t) \chi(\varepsilon^2 t) \hat{\chi}(t) \, dt \to Q(D) f_0(0) \int \hat{\chi}(t) \, dt = 0,
$$

since $P_{f_0; m; 0} = 0$ and $Q \in \mathcal{P}_m$. Hence, $\hat{f}_0(Q) = 0$ and we are done. $\qquad\square$

We now prove our first main theorem.

Proof of Theorem 1 We start proving the second part of the statement. Let $g \in L_a^2(|\xi|^{2s})$ and define f as in (4). Then, since $0 < s < \frac{1}{2}$, for $z = x + iy$,

$$
\begin{aligned}
|f(z)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) e^{iz\xi} \, d\xi \right| \\
&\leq C \|g\|_{L_a^2(|\xi|^{2s})} \left(\int_{-a}^{a} |\xi|^{-2s} e^{-2y\xi} \, d\xi \right)^{\frac{1}{2}} \leq C e^{a|y|} \|g\|_{L_a^2(|\xi|^{2s})}.
\end{aligned}
$$

Therefore, f is well-defined, is clearly entire and belongs to $\mathcal{E}_a$. We wish to show that $f_0 \in E^{s,2}$. Observing that

$$f_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)e^{ix\xi} \, d\xi = \mathcal{I}_s \mathcal{F}^{-1}(g|\xi|^s)(x),$$

and since $\mathcal{I}_s : L^2 \to L^{2^*}$, we see that $f_0 \in L^{2^*}$. Moreover, since $\widehat{f_0} = g \in L_a^2(|\xi|^{2s})$, it follows that $\Delta^{\frac{s}{2}} f_0 \in L^2$. Hence $f_0 \in E^{s,2}$, $f \in PW_a^s$ and

$$\|f\|_{PW_a^s} = \|g\|_{L_a^2(|\xi|^{2s})}.$$

Now, let $f \in PW_a^s$. Lemma 3.1 guarantees that $\widehat{f_0} \in L_a^2(|\xi|^{2s})$ and in particular is compactly supported in $[-a, a]$. From the first part of the theorem, we know that the function

$$\widetilde{f}(z) := \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f_0}(\xi)e^{iz\xi} \, d\xi$$

is a well-defined function in PW_a^s and $\widetilde{f_0} = f_0$. Hence, f and $\widetilde{f}$ coincide everywhere as we wished to show. $\qquad \square$

Corollary 3.2 *The spaces PW_a^s, $0 < s < \frac{1}{2}$, are reproducing kernel Hilbert spaces with reproducing kernel*

$$K(w, z) = \frac{1}{2\pi} \int_{-a}^{a} e^{i(w-\bar{z})\xi} |\xi|^{-2s} \, d\xi.$$

Proof From (3) we deduce that point-evaluations are bounded on PW_a^s. In fact,

$$|f(z)| = \left| \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f_0}(\xi)e^{iz\xi} \, d\xi \right|$$

$$\leq C\|\widehat{f_0}\|_{L_a^2(|\xi|^{2s})} \left(\int_{-a}^{a} e^{-2y\xi} |\xi|^{-2s} \, d\xi \right)^{\frac{1}{2}} \leq Ce^{a|y|} \|\widehat{f_0}\|_{L_a^2(|\xi|^{2s})}.$$

This easily implies that PW_a^s is complete, hence a reproducing kernel Hilbert space. For $z \in \mathbb{C}$, the kernel function K_z satisfies[2]

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{f_0}(\xi)e^{iz\xi} \, d\xi = f(z) = \left\langle f \mid K_z \right\rangle_{PW_a^s} = \left\langle \widehat{f_0} \mid \widehat{(K_z)_0} \right\rangle_{L_a^2(|\xi|^{2s})}$$

$$= \int_{-a}^{a} \widehat{f_0}(\xi)\overline{\widehat{(K_z)_0}(\xi)}|\xi|^{2s} \, d\xi.$$

Therefore, $\widehat{(K_z)_0}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-i\bar{z}\xi} |\xi|^{-2s} \chi_{[-a,a]}(\xi)$ and the conclusion follows. $\qquad \square$

Next, we consider the case $s > \frac{1}{2}$.

Lemma 3.3 *Let $s > \frac{1}{2}$, $s - \frac{1}{2} \notin \mathbb{N}$ and let $m = \lfloor s - \frac{1}{2} \rfloor$. Given $g \in L^2(|\xi|^{2s})$ define $Tg \in \mathcal{S}'$ as*

---

[2] We denote by $\langle \cdot \mid \cdot \rangle_H$ the *Hermitian* inner product on a given Hilbert space H.

$$\langle Tg, \psi \rangle := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \big(\psi(\xi) - P_{\psi;m;0}(\xi) \big) \, d\xi \tag{12}$$

for any $\psi \in \mathcal{S}$. Then, Tg is well-defined and $T : L^2(|\xi|^{2s}) \to \mathcal{S}'$ is a continuous operator.

Proof By Hölder's inequality we have

$$|\langle Tg, \psi \rangle| \leq \|g\|_{L^2(|\xi|^{2s})} \left(\int_{\mathbb{R}} |\psi(\xi) - P_{\psi;m;0}(\xi)|^2 |\xi|^{-2s} \right)^{\frac{1}{2}}$$

$$\leq \|g\|_{L^2(|\xi|^{2s})} \left(\left(\int_{|\xi| \leq 1} + \int_{|\xi| \geq 1} \right) |\psi(\xi) - P_{\psi;m;0}(\xi)|^2 |\xi|^{-2s} \, d\xi \right)^{\frac{1}{2}}$$

Notice that, since $m = \lfloor s - \frac{1}{2} \rfloor$ and $s - \frac{1}{2} \notin \mathbb{N}$, we have $-\frac{3}{2} < m - s < -\frac{1}{2}$. Hence,

$$\int_{|\xi| \geq 1} |\psi(\xi) - P_{\psi;m;0}(\xi)|^2 |\xi|^{-2s} \, d\xi$$

$$\leq \int_{|\xi| \geq 1} |\psi(\xi)|^2 |\xi|^{2s} \, d\xi + \int_{|\xi| \geq 1} |P_{\psi;m;0}(\xi)|^2 |\xi|^{-2s} \, d\xi$$

$$\leq \int_{|\xi| \geq 1} |\psi(\xi)|^2 |\xi|^{2s} \, d\xi + \max_{0 \leq j \leq m} |\psi^{(j)}(0)| \left(\int_{|\xi| \geq 1} |\xi|^{2(m-s)} \right)^{\frac{1}{2}}$$

$$\leq C_\psi,$$

where C_ψ denotes a finite positive constant bounded by some Schwartz seminorm of ψ. Moreover,

$$\int_{|\xi| \leq 1} |\psi(\xi) - P_{\psi;m;0}(\xi)|^2 |\xi|^{-2s} \, d\xi \leq \sup_{|\xi| \leq 1} |\psi^{(m+1)}(\xi)| \int_{|\xi| \leq 1} |\xi|^{2(m+1-s)} \, d\xi < +\infty.$$

From these estimates it is clear that $T : L^2(|\xi|^{2s}) \to \mathcal{S}'$ is a continuous operator as we wished to show. □

Lemma 3.4 *Let $s > \frac{1}{2}$, $s - \frac{1}{2} \notin \mathbb{N}$ and $m = \lfloor s - \frac{1}{2} \rfloor$. Given $f \in PW_a^s$ there exists a unique $g \in L_a^2(|\xi|^{2s})$ such that $\widehat{f}_0 = Tg$ in $\mathcal{S}'$, that is,*

$$\langle \widehat{f}_0, \psi \rangle = \langle Tg, \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi) \big(\psi(\xi) - P_{\psi;m;0}(\xi) \big) \, d\xi$$

for all $\psi \in \mathcal{S}$.

Proof By the results in [18], since $f_0 \in E^{s,2}$, there exists a sequence $\{\varphi_n\} \subseteq \mathcal{S}$ such that $\{\Delta^{\frac{s}{2}} \varphi_n\}$ is a Cauchy sequence in L^2, and $\varphi_n \to f_0$ in $\mathcal{S}'/\mathcal{P}_m$, where $m = \lfloor s - \frac{1}{2} \rfloor$, that is, $\langle \varphi_n, \psi \rangle \to \langle f_0, \psi \rangle = \langle \widehat{f}_0, \widehat{\psi} \rangle$, as $n \to \infty$, for all $\psi \in \mathcal{S}_m$. Therefore,

$$\langle \widehat{\varphi}_n, \eta \rangle \to \langle \widehat{f}_0, \eta \rangle$$

as $n \to \infty$, for all $\eta \in \widehat{\mathcal{S}_m} = \mathcal{S} \cap \{\eta \in \mathcal{S} : P_{\eta;m;0} = 0\}$. Moreover, there exists a unique $g \in L^2(|\xi|^{2s})$ such that $\widehat{\varphi}_n \to g$ in $L^2(|\xi|^{2s})$. Since $T : L^2(|\xi|^{2s}) \to \mathcal{S}'$ is continuous, we also

have that $T\widehat{\varphi}_n \to Tg$ in $\mathcal{S}'$. We now prove that it holds also $T\widehat{\varphi}_n \to \widehat{f}_0$ in $(\widehat{\mathcal{S}_m})'$ In fact, given $\psi \in \widehat{\mathcal{S}_m}$, we have

$$\langle T\widehat{\varphi}_n, \psi \rangle = \int_{\mathbb{R}} \widehat{\varphi}_n(\xi)\big(\psi(\xi) - P_{\psi;m;0}(\xi)\big)\,d\xi = \int_{\mathbb{R}} \widehat{\varphi}_n(\xi)\psi(\xi)\,d\xi = \int_{\mathbb{R}} \varphi_n(\xi)\widehat{\psi}(\xi)\,d\xi$$

$$\to \int_{\mathbb{R}} f_0(\xi)\widehat{\psi}(\xi)\,d\xi = \langle \widehat{f}_0, \psi \rangle.$$

Therefore, $\widehat{f}_0 = Tg$ in $(\widehat{\mathcal{S}_m})'$, that is, if $Q(D)(\delta) = \sum_{j=0}^{m} c_j \delta^{(j)}$,

$$\widehat{f}_0 = Tg + Q(D)(\delta)$$

in $\mathcal{S}'$. In particular, this implies that $\mathrm{supp}\,Tg \subset [-a, a]$, hence, $Tg \in \mathcal{D}'_c$ and $\mathrm{supp}\,g \subseteq [-a, a]$.

We now prove that $Q(D)(\delta) = 0$. Let $P \in \mathcal{P}_m$ and let $\eta \in C_c^\infty$ such that $\eta \equiv 1$ on $[-a, a]$ so that $\eta P \in \mathcal{S}$. Since $\widehat{f}_0$ is supported in $[-a, a]$ from Lemma 3.1 we get

$$\langle \widehat{f}_0, \eta P \rangle = \langle \widehat{f}_0, P \rangle = 0$$

and, since Tg is supported in $[-a, a]$ as well,

$$\langle Tg, \eta P \rangle = \int_{-a}^{a} g(\xi)\big((\eta P)(\xi) - P_{\eta P, m;0}(\xi)\big)\,d\xi = 0$$

since $P_{\eta P, m;0} = \eta P$ on $[-a, a]$. Therefore, we obtain that $\langle Q(D)(\delta), \eta P \rangle = 0$ as well and, by the arbitrariness of ηP, we conclude that $Q(D)(\delta) = 0$ as we wished to show. Thus, $\widehat{f}_0 = Tg$ in $\mathcal{S}'$. $\qquad\square$

Before proving the next lemma, we need the following definition. Given $s > 0$ and $\psi \in \widehat{\mathcal{S}_\infty}$, notice that $|\xi|^s \psi \in \widehat{\mathcal{S}_\infty}$. Then, given U in $\mathcal{S}'$, for any we define $|\xi|^s U$ by setting

$$\langle |\xi|^s U, \psi \rangle = \langle U, |\xi|^s \psi \rangle.$$

We now prove the following simple, but not obvious, lemma.

Lemma 3.5 *Let* $s > \frac{1}{2}, s - \frac{1}{2} \notin \mathbb{N}$ *and let* $f \in PW_a^s$. *Then,* $\mathcal{F}(\Delta^{\frac{s}{2}} f_0) = |\xi|^s \widehat{f}_0$, *with equality in* L_a^2.

Proof Since $f \in PW_a^s$, we already know that $\mathcal{F}(\Delta^{\frac{s}{2}} f_0) \in L^2$. We now consider $|\xi|^s \widehat{f}_0 \in (\widehat{\mathcal{S}_\infty})'$ and we show that it actually belongs to L_a^2. Then, we show it coincides with $\mathcal{F}(\Delta^{\frac{s}{2}} f_0)$. Let $\psi \in \widehat{\mathcal{S}_\infty}$. Then, from Lemma 3.4 there exists $g \in L_a^2(|\xi|^{2s})$ such that

$$\langle |\xi|^s \widehat{f}_0, \psi \rangle = \langle \widehat{f}_0, |\xi|^s \psi \rangle = \langle Tg, |\xi|^s \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)\big(|\xi|^s \psi(\xi) - P_{|\xi|^s \psi; m;0}(\xi)\big)\,d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)|\xi|^s \psi(\xi)\,d\xi,$$

since $P_{|\xi|^s \psi; m;0} = 0$. Hence,

$$|\langle |\xi|^s \widehat{f}_0, \psi \rangle| \leq C \|g\|_{L_a^2(|\xi|^{2s})} \|\psi\|_{L^2}.$$

By the density of $\widehat{\mathcal{S}_\infty}$ in L^2, we conclude that $|\xi|^s \widehat{f_0} \in (L^2)'$, that is, $|\xi|^s \widehat{f_0} \in L_a^2$ as we wished to show. Now, since $f_0 \in E^{s,2}$, there exists $\{\varphi_n\} \subseteq \mathcal{S}$ such that $\varphi_n \to f_0$ in $\mathcal{S}'/\mathcal{P}_m$ and $\{\Delta^{\frac{s}{2}} \varphi_n\}$ is a Cauchy sequence in L^2. Then, for $\psi \in \mathcal{S}_\infty$, which is dense in L^2, we have

$$\langle \Delta^{\frac{s}{2}} f, \psi \rangle = \lim_{n \to +\infty} \langle \Delta^{\frac{s}{2}} \varphi_n, \psi \rangle = \lim_{n \to +\infty} \langle |\xi|^s \widehat{\varphi_n}, \widehat{\psi} \rangle = \lim_{n \to +\infty} \langle \widehat{\varphi_n}, |\xi|^s \widehat{\psi} \rangle$$

$$= \lim_{n \to +\infty} \langle \varphi_n, \mathcal{F}^{-1}(|\xi|^s \widehat{\psi}) \rangle = \langle f_0, \mathcal{F}^{-1}(|\xi|^s \widehat{\psi}) \rangle = \langle \widehat{f_0}, |\xi|^s \widehat{\psi} \rangle$$

$$= \langle |\xi|^s \widehat{f_0}, \psi \rangle.$$

The conclusion follows from the density of $\mathcal{S}_\infty \subseteq L^2$. $\qquad\square$

Proof of Theorem 2 We first prove the second part of the theorem. Recall that $m = \lfloor s - \frac{1}{2} \rfloor$ is the integer part of $s - \frac{1}{2}$. Given f defined as in (7), we see that for every $\varepsilon > 0$

$$\left| \frac{1}{\sqrt{2\pi}} \int_{-a}^a g(\xi)(e^{iz\xi} - P_m(iz\xi)) \, d\xi \right|$$

$$\leq \left(\frac{1}{\sqrt{2\pi}} \int_{-a}^a |\xi|^{2s} |g(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left(\int_{-a}^a |\xi|^{-2s} |e^{iz\xi} - P_m(iz\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

$$\leq C e^{a|z|} |z|^{m+1} \|g\|_{L_a^2(|\xi|^{2s})} \left(\int_{-a}^a |\xi|^{2(m-s+1)} \, d\xi \right)^{\frac{1}{2}}$$

$$\leq C_\varepsilon e^{(a+\varepsilon)|z|}$$

since $2(m - s + 1) > -1$ and where we have used the inequality $\sum_{j=0}^{+\infty} r^j/(j + m + 1)! \leq e^r$, for $r > 0$. Hence, f is well-defined, clearly entire and it belongs to $\mathcal{E}_a$. Since it is clear that $P_{f;m;0} = 0$, it remains to show that $f_0 \in E^{s,2}$. We have

$$f_0^{(m)}(x + h) - f_0^{(m)}(x) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a g(\xi)(i\xi)^m (e^{i(x+h)\xi} - e^{ix\xi}) \, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a g(\xi)(i\xi)^m e^{ix\xi} (e^{ih\xi} - 1) \, d\xi,$$

so that,

$$|f_0^{(m)}(x + h) - f_0^{(m)}(x)| \leq \frac{1}{\sqrt{2\pi}} \int_{-a}^a |\xi|^s |g(\xi)| |e^{ih\xi} - 1| |\xi|^{m-s} \, d\xi$$

$$\leq C \|g\|_{L_a^2(|\xi|^{2s})}^2 \left(\int_{-a}^a |e^{ih\xi} - 1|^2 |\xi|^{2(m-s)} \, d\xi \right)^{\frac{1}{2}}$$

$$\leq C|h|^{s-\frac{1}{2}-m} \|g\|_{L_a^2(|\xi|^{2s})}^2 \left(\int_{\mathbb{R}} |e^{it} - 1|^2 |t|^{2(m-s)} \, dt \right)^{\frac{1}{2}}$$

$$\leq C|h|^{s-\frac{1}{2}-m} \|g\|_{L_a^2(|\xi|^{2s})}^2.$$

Hence,

$$\sup_{h\in\mathbb{R},h\neq 0}\frac{|f_0^{(m)}(x+h)-f_0^{(m)}(x)|}{|h|^{s-\frac{1}{2}-m}}\leq C\|g\|_{L_a^2(|\xi|^{2s})}^2$$

and we conclude that $f_0\in\dot\Lambda^{s-\frac{1}{2}}$ as we wished to show (see [15, Proposition 1.4.5]). Next, we need to show that $\Delta^{\frac{s}{2}}f_0\in L^2$ and $\|\Delta^{\frac{s}{2}}f_0\|_{L^2}=\|g\|_{L_a^2(|\xi|^{2s})}$.

To this end, let $\{\psi_n\}\subseteq\mathcal{S}$ be such that $\psi_n\to g$ in $L_a^2(|\xi|^{2s})$ and define φ_n as in (7), that is,

$$\varphi_n(x)=\frac{1}{\sqrt{2\pi}}\int_{-a}^a\psi_n(\xi)\big(e^{ix\xi}-P_m(ix\xi)\big)\,d\xi,$$

where, we recall, $m=\lfloor s-\frac{1}{2}\rfloor$. Observe that, by (12), $\varphi_n(x)=\langle T\psi_n,e^{ix(\cdot)}\rangle$. Given $\eta\in\mathcal{S}$, using Lemma 3.3, we have that all integrals in the equalities that follow converge absolutely and we have that

$$\lim_{n\to\infty}\langle\varphi_n,\eta\rangle=\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-a}^a\psi_n(\xi)\big(\hat\eta(\xi)-P_{m;\hat\eta;0}(\xi)\big)\,d\xi dx$$

$$=\lim_{n\to\infty}\langle T\psi_n,\hat\eta\rangle=\langle Tg,\hat\eta\rangle=\langle f_0,\eta\rangle.$$

Therefore, $\varphi_n\to f_0$ in $\mathcal{S}'$. Moreover, we have that $D^{m+1}\varphi_n=\mathcal{F}^{-1}\big((i\xi)^{m+1}\psi_n\big)$ and setting $s':=s-(m+1)\in(-\frac{1}{2},\frac{1}{2})$ we have that, on Schwartz functions, $\Delta^{\frac{s}{2}}=R^{m+1}\Delta^{\frac{s'}{2}}D^{m+1}$. Therefore,

$$\|\Delta^{\frac{s}{2}}\varphi_n\|_{L^2}=\|\Delta^{\frac{s'}{2}}D^{m+1}\varphi_n\|_{L^2}=\|\mathcal{F}\big(D^{m+1}\varphi_n\big)\|_{L^2(|\xi|^{2s'})}=\|\psi_n\|_{L^2(|\xi|^{2s})}.$$

It follows that $\{\Delta^{\frac{s}{2}}\varphi_n\}$ is a Cauchy sequence in L^2 and that

$$\|\Delta^{\frac{s}{2}}f_0\|_{L^2}=\lim_{n\to\infty}\|\psi_n\|_{L^2(|\xi|^{2s})}=\|g\|_{L_a^2(|\xi|^{2s})}.$$

Let us consider now $f\in PW_a^s$. From Lemmas 3.1 and 3.4 we know that $\operatorname{supp}\hat f_0\subseteq[-a,a]$, and that there exists a unique $g\in L_a^2(|\xi|^{2s})$ such that $\hat f_0=Tg$ in $\mathcal{S}'$. Hence, the function

$$\tilde f(z):=\frac{1}{\sqrt{2\pi}}\int_{-a}^a g(\xi)\big(e^{iz\xi}-P_m(iz\xi)\big)\,d\xi$$

is a well-defined function in PW_a^s by the first part of the proof. Moreover,

$$D_z^{m+1}\tilde f_0(x)=\frac{1}{\sqrt{2\pi}}\int_{-a}^a(i\xi)^{m+1}g(\xi)e^{ix\xi}\,d\xi,$$

so that $\mathcal{F}(D^{m+1}\tilde f_0)=(i\xi)^{m+1}g$. On the other hand, we also have that

$$\mathcal{F}(f_0^{(m+1)})=(i\xi)^{m+1}\hat f_0=(i\xi)^{m+1}Tg.$$

Now, for $\psi\in\mathcal{S}$,

$$\langle (i\xi)^{m+1} Tg, \psi \rangle = \langle Tg, (i\xi)^{m+1}\psi \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)\big((i\xi)^{m+1}\psi(\xi) - P_{(i\xi)^{m+1}\psi;m;0}(\xi)\big)\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)(i\xi)^{m+1}\psi(\xi)\, d\xi$$

$$= \langle \mathcal{F}(D^{m+1}\tilde{f}_0), \psi \rangle,$$

hence $\mathcal{F}(f_0^{(m+1)}) = \mathcal{F}(D^{m+1}\tilde{f}_0)$, i.e., $f_0^{(m+1)} = \tilde{f}_0^{(m+1)}$. Therefore, f and $\tilde{f}$ coincide up to a polynomial of degree at most m. Since $P_{f;m;0} = P_{\tilde{f};m;0} \equiv 0$, we get $\tilde{f} = f$; in particular,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g(\xi)\big(e^{iz\xi} - P_m(iz\xi)\big)\, d\xi$$

and $\|\Delta^{\frac{s}{2}} f_0\|_{L^2} = \|g\|_{L_a^2(|\xi|^{2s})}$, as we wished to show. $\qquad \square$

As in the case $s < \frac{1}{2}$, we have the following

Corollary 3.6 *For $s > \frac{1}{2}$, $s - \frac{1}{2} \notin \mathbb{N}$, the spaces PW_a^s, are reproducing kernel Hilbert spaces with reproducing kernel*

$$K(w, z) = \frac{1}{2\pi} \int_{-a}^{a} \big(e^{iw\xi} - P_m(iw\xi)\big)\big(e^{-i\bar{z}\xi} - P_m(-i\bar{z}\xi)\big)|\xi|^{-2s}\, d\xi.$$

Notice that, since $K_z = K(\cdot, z) \in PW_a^s$, $P_{K_z;m;0} = 0$, that is, K_z vanishes of order m at the origin, where $m = \lfloor s - \frac{1}{2} \rfloor$.

Proof From the previous theorem we know that the Fourier transform is a surjective isometry from PW_a^s onto $T(L_a^2(|\xi|^{2s}))$, the closed subspace of $\mathcal{S}'$ endowed with norm $\|Tg\| := \|g\|_{L_a^2(|\xi|^{2s})}$. Therefore, PW_a^s are Hilbert spaces.

Similarly to the proof of Corollary 3.2 we deduce from the representation formula (6) that the spaces PW_a^s are reproducing kernel Hilbert spaces. Then,

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \hat{f}_0(\xi)\big(e^{iz\xi} - P_m(iz\xi)\big)\, d\xi = f(z) = \big\langle f \mid K_z \big\rangle_{PW_a^s} = \int_{-a}^{a} |\xi|^{2s}\hat{f}_0(\xi)\overline{\widehat{(K_z)_0}(\xi)}\, d\xi$$

and therefore,

$$\widehat{(K_z)_0}(\xi) = \frac{1}{\sqrt{2\pi}}\big(e^{-i\bar{z}\xi} - P_m(-i\bar{z}\xi)\big)|\xi|^{-2s}.$$

From this identity and (6), the conclusion follows. $\qquad \square$

The following lemma is obvious and we leave the details to the reader (or see the proof of Lemma 4.1).

Lemma 3.7 *The space $\{f \in PW_a : f_0 \in \mathcal{S}_\infty\}$ is dense in PW_a.*

We now show that the fractional Laplacian $\Delta^{\frac{s}{2}}$ induces a surjective isometry from PW_a^s onto PW_a.

Theorem 3.8 *Let $s > 0$ and assume $s - \frac{1}{2} \notin \mathbb{N}_0$. Then, the operator $\Delta^{\frac{s}{2}} : PW_a^s \to PW_a$ is a surjective isometry, whose inverse is $\mathcal{I}_s$ if $0 < s < \frac{1}{2}$, whereas if $s > \frac{1}{2}$ the inverse is given by*

$$\left(\Delta^{\frac{s}{2}}\right)^{-1} h(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{h}_0(\xi) |\xi|^{-s} \left(e^{iz\xi} - P_m(iz\xi)\right) d\xi, \tag{13}$$

with $h \in PW_a$.

Proof We only need to prove the theorem in the case $s > \frac{1}{2}$, $s - \frac{1}{2} \notin \mathbb{N}$. We recall that from Lemma 3.5, if $f \in PW_a^s$, $\mathcal{F}(\Delta^{\frac{s}{2}} f_0) = |\xi|^s \widehat{f}_0 \in L_a^2$. Hence, the map $f \mapsto \Delta^{\frac{s}{2}} f_0$ is clearly an isometry, and $\text{supp}\left(\mathcal{F}(\Delta^{\frac{s}{2}} f_0)\right) \subseteq [-a, a]$. By the classical Paley–Wiener theorem, $\Delta^{\frac{s}{2}} f_0$ extends to a function in PW_a, that we denote by $\Delta^{\frac{s}{2}} f$.

Let us focus on the surjectivity. Let $h \in PW_a$, then, by the previous lemma, there exists a sequence $\{\varphi_n\} \subseteq \{h \in PW_a : h_0 \in \mathcal{S}_\infty\}$ such that $\|h - \varphi_n\|_{PW_a} \to 0$ as $n \to \infty$. Set

$$\Phi_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{\varphi_n}(\xi) |\xi|^{-s} \left(e^{iz\xi} - P_m(iz\xi)\right) d\xi. \tag{14}$$

We observe that, since $\varphi_n \in \mathcal{S}_\infty$, we can write

$$\Phi_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{\varphi_n}(\xi) |\xi|^{-s} e^{iz\xi} d\xi - \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{\varphi_n}(\xi) |\xi|^{-s} P_m(iz\xi) d\xi$$

since both the integrals converge absolutely. We are going to show that $\Phi_n \in PW_a^s$, $\{\Phi_n\}$ is a Cauchy sequence in PW_a^s, and that $\Delta^{\frac{s}{2}} \Phi_n \to g$ in L^2. From these facts the surjectivity follows at once. As in the proofs of Theorems 1 and 2, we see that $\Phi_n \in PW_a^s$. Moreover, using [18, Corollary 3.4] and the fact that $\varphi_n \in \mathcal{S}_\infty$, we see that

$$\Delta^{\frac{s}{2}} \Phi_n = \Delta^{\frac{s}{2}} \left(\mathcal{F}^{-1}(\widehat{\varphi}_n |\xi|^{-s})\right) = \varphi_n.$$

Hence, $\Delta^{\frac{s}{2}} \Phi_n \to g$ in L^2, and the surjectivity follows.

In order to show that the inverse of $\Delta^{\frac{s}{2}}$ has the expression (13), we observe that $\widehat{h}_0(\xi) |\xi|^{-s} \in L_a^2(|\xi|^{2s})$, so that arguing as in the proof of Theorem 2, we see that $F = \left(\Delta^{\frac{s}{2}}\right)^{-1} h \in PW_a^s$. Now, if $\{g_n\} \subseteq C_c^\infty\left(\{\delta_n \leq |\xi| \leq a - \delta_n\}\right)$ are such that $\delta_n \to 0$ and $g_n \to \widehat{h}_0(\xi) |\xi|^{-s}$ in $L_a^2(|\xi|^{2s})$, using [18, Corollary 3.4] again we have

$$\mathcal{F}\left(\Delta^{\frac{s}{2}} F_0\right)(t) = \lim_{n \to \infty} \mathcal{F}\left(\Delta^{\frac{s}{2}} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g_n(\xi) \left(e^{ix\xi} - P_m(ix\xi)\right) d\xi\right)(t)$$

$$= \lim_{n \to \infty} \mathcal{F}\left(\Delta^{\frac{s}{2}} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} g_n(\xi) e^{ix\xi} d\xi\right)(t)$$

$$= \lim_{n \to \infty} |t|^s g_n(t) = \widehat{h}_0(t).$$

Thus, $\Delta^{\frac{s}{2}} F = h$ and the surjectivity follows. □

4 Fractional Bernstein spaces

In this section we study the fractional Bernstein spaces and we first show that the spaces $\mathcal{B}_a^{s,p}$ are isometric to the classical Bernstein spaces $\mathcal{B}_a^p$. The proof is similar to the Hilbert case, but we have to overcome the fact that Plancherel and Parseval's formulas are no longer available.

We need the following density lemma.

Lemma 4.1 *Let $1 < p < \infty$. Then, the space $\mathcal{T} = \{f \in \mathcal{E}_a : f_0 \in \mathcal{S}_\infty\}$ is dense in $\mathcal{B}_a^p$.*

The proof of such lemma is somewhat elementary but not immediate and it is postponed to Sect. 7.

Theorem 4.2 *Let $s > 0$ such that $s - \frac{1}{p} \notin \mathbb{N}_0$. Then, the operator $\Delta^{\frac{s}{2}}$ is a surjective isometry*

$$\Delta^{\frac{s}{2}} : \mathcal{B}_a^{s,p} \to \mathcal{B}_a^p$$

and the inverse is as in (13) *(with $h \in \mathcal{B}_a^p$).*

Proof We first notice that $\Delta^{\frac{s}{2}}$ is injective on $\mathcal{B}_a^{s,p}$ since these spaces are defined using the realizations $E^{s,p}$ of the homogeneous Sobolev spaces $\dot{W}^{s,p}$.

We now prove that $\Delta^{\frac{s}{2}}f \in \mathcal{B}_a^p$ whenever $f \in \mathcal{B}_a^{s,p}$. Due to the characterization of the Bernstein spaces, $\Delta^{\frac{s}{2}}f$ is in $\mathcal{B}_a^p$ if and only if $\Delta^{\frac{s}{2}}f_0 \in L^p$ and $\operatorname{supp}\Delta^{\frac{s}{2}}f_0 \subseteq [-a,a]$. Let $\varphi \in C_c^\infty \cap \mathcal{S}_M$, with M to be chosen later. Given $f \in \mathcal{B}_a^{s,p}$ we set

$$f_\varphi(z) := \int_{\mathbb{R}} f(z-t)\varphi(t)\,dt$$

and we claim that $f_\varphi \in \mathcal{E}_a$ and $(f_\varphi)_0 \in L^p(\mathbb{R})$. In fact, f_φ is clearly entire and, for every $\varepsilon > 0$,

$$|f_\varphi(z)| \le \int_{\mathbb{R}} |f(z-t)\varphi(t)|\,dt \le C_\varepsilon e^{(a+\varepsilon)|z|} \int_{\mathbb{R}} e^{(a+\varepsilon)|t|}|\varphi(t)|\,dt < +\infty,$$

where the last integral converges since φ is continuous and compactly supported. Hence, f_φ is of exponential type a. Now, let $(f_\varphi)_0 = f_0 * \varphi$ be the restriction of f_φ to the real line. Since $f_0 \in E^{s,p}$ there exists a sequence $\{\varphi_n\} \subseteq \mathcal{S}$ such that $\varphi_n \to f_0$ in $\mathcal{S}'/\mathcal{P}_m$, where $m = \lfloor s - 1/p \rfloor$ and $\{\Delta^{\frac{s}{2}}\varphi_n\}$ is a Cauchy sequence in L^p. We now argue as in (11). Since $\varphi \in \mathcal{S}_M$, let $\Phi \in \mathcal{S}$ be such that $\varphi = \Phi^{(M)}$, so that $\mathcal{I}_s\varphi = R^\ell \mathcal{I}_{s-\ell}\Phi^{(M-\ell)}$, where R denotes the Riesz transform. Then we have

$$\int_{\mathbb{R}} |(\varphi_n * \varphi)(x)|^p\,dx = \int_{\mathbb{R}} |\Delta^{\frac{s}{2}}\varphi_n * \mathcal{I}_s\varphi(x)|^p\,dx$$

$$\le \|\Delta^{\frac{s}{2}}\varphi_n\|_{L^p} \|(R^\ell \mathcal{I}_{s-\ell}\Phi^{(M-\ell)})\|_{H^1}$$

$$\le C\|\Delta^{\frac{s}{2}}\varphi_n\|_{L^p} \|\Phi^{(M-\ell)}\|_{H^q}$$

$$\le C_\varphi \|\Delta^{\frac{s}{2}}\varphi_n\|_{L^p},$$

where $1 = \frac{1}{q} - (s - \ell)$, choosing $\ell = \lfloor s \rfloor$ in such a way that $q \leq 1$. In particular, we get that $\{\varphi_n * \varphi\}$ is a Cauchy sequence in L^p. Since $\varphi_n * \varphi \to f_0 * \varphi$ in $\mathcal{S}'/\mathcal{P}_m$ and $\mathcal{S}_m$ is dense in $L^{p'}$, we see that $\varphi_n * \varphi \to f_0 * \varphi = (f_\varphi)_0$ in L^p.

Therefore, f_φ is an exponential function of type a whose restriction to the real line is L^p-integrable. Hence, $f_\varphi \in \mathcal{B}^p_a$ and $\mathrm{supp}(\widehat{f_\varphi})_0 = \mathrm{supp}(\widehat{f_0}\widehat{\varphi}) \subseteq [-a,a]$. From the arbitrariness of φ we conclude that $\mathrm{supp}\widehat{f_0} \subseteq [-a,a]$.

We now argue as in the proof of Lemma 3.5, to show that also $\mathrm{supp}\widehat{\Delta^{\frac{s}{2}}f_0} \subseteq [-a,a]$. Let $\Phi_n \in C_c^\infty(\{\delta_n \leq |\xi| \leq a\})$, $\Phi_n \to \widehat{f_0}$ in $\mathcal{S}'$, and setting $\eta_n = \mathcal{F}^{-1}(\Phi_n)$ we have $\eta_n \in \mathcal{S}_\infty$. Now, for $\psi \in \mathcal{S}$, as $n \to \infty$ we have

$$\langle \eta_n, \psi \rangle = \langle \Phi_n, \widehat{\psi} \rangle \to \langle \widehat{f_0}, \widehat{\psi} \rangle = \langle f_0, \psi \rangle,$$

where the pairings are in $\mathcal{S}'$. Thus, $\eta_n \to f_0$ in $\mathcal{S}'$, which implies that $\Delta^{\frac{s}{2}}\eta_n \to \Delta^{\frac{s}{2}}f_0$ in $\mathcal{S}'/\mathcal{P}$, so that

$$\mathrm{supp}\widehat{\Delta^{\frac{s}{2}}f_0} \subseteq \bigcup_n \mathrm{supp}\widehat{\Delta^{\frac{s}{2}}\eta_n} \cup \{0\} \subseteq [-a,a],$$

as we wished to show.

Since $\Delta^{\frac{s}{2}}f_0 \in L^p$ by hypothesis, we conclude that $\Delta^{\frac{s}{2}}f_0$ is the restriction to the real line of a function in $\mathcal{B}^p_a$, function that we denote by $\Delta^{\frac{s}{2}}f$. Moreover, we trivially have the equality $\|f\|_{\mathcal{B}^{s,p}_a} = \|\Delta^{\frac{s}{2}}f\|_{\mathcal{B}^p_a}$.

It remains to prove that $\Delta^{\frac{s}{2}}$ is surjective. Let $h \in \mathcal{B}^p_a$. Then, by Lemma 4.1, there exists a sequence $\{h_n\} \subseteq \mathcal{T} = \{h \in \mathcal{B}^p_a : h_0 \in \mathcal{S}_\infty\}$ such that $h_n \to h$ in $\mathcal{B}^p_a$.

Let $0 < s < \frac{1}{p}$ and set

$$F_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{(h_n)_0}(\xi)|\xi|^{-s}e^{iz\xi}\,d\xi.$$

Then, $F_n \in \mathcal{E}_a$, $\Delta^{\frac{s}{2}}(F_n)_0 = (h_n)_0$ and $\{F_n\}$ is a Cauchy sequence in $\mathcal{B}^{s,p}_a$ since $\|F_n\|_{\mathcal{B}^{s,p}_a} = \|h_n\|_{\mathcal{B}^p_a}$. In particular, this means that $\{(F_n)_0\}$ is a Cauchy sequence in $E^{s,p}$. Hence, there exists a limit function $\widetilde{F} \in E^{s,p}$. We need to prove that $\widetilde{F}$ is the restriction to the real line of a function in $\mathcal{B}^{s,p}_a$.

Since $s < \frac{1}{p}$, by Parseval's identity, we have

$$F_n(z) = \int_{\mathbb{R}} h_n(x)\mathcal{F}^{-1}\left(|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right)(x)\,dx,$$

so that

$$|F_n(z)| \leq \|(h_n)_0\|_{L^p}\|\mathcal{F}^{-1}\left(|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right)\|_{L^{p'}},$$

where p, p' are conjugate indices. Observing that

$$\mathcal{F}^{-1}\left(|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right) = \mathcal{I}_s\left(\mathcal{F}^{-1}(e^{iz\xi}\chi_{[-a,a]})\right),$$

from Theorem 2.1 we obtain

$$\|\mathcal{F}^{-1}\left(|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right)\|_{L^{p'}} \leq C\|\mathcal{F}^{-1}\left(e^{iz\xi}\chi_{[-a,a]}\right)\|_{L^{\frac{p'}{1+sp'}}},$$

since $\frac{p'}{1+sp'} > 1$. However, for z fixed,

$$\mathcal{F}^{-1}\left(e^{iz\xi}\chi_{[-a,a]}\right)(t) = \frac{a}{2\pi}\text{sinc}(a(z+t))$$

belongs to $\mathcal{B}_a^q$ for any $q \in (1, \infty)$. Therefore, by the classical Plancherel–Pólya Inequality, we obtain

$$\|\mathcal{F}^{-1}\left(|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right)\|_{L^{p'}} \le Ce^{a|y|}\|\text{sinc}(a\xi)\|_{L^{\frac{p'}{1+sp'}}},$$

where $y = \text{Im}z$. In conclusion,

$$|F_n(z)| \le Ce^{a|y|}\|(h_n)_0\|_{L^p}.$$

Since $\|(h_n)_0\|_{L^p} = \|h_n\|_{\mathcal{B}_a^p}$, we just proved that the $\mathcal{B}_a^p$-convergence of $\{h_n\}$ implies the uniform convergence on compact subsets of $\mathbb{C}$ of $\{F_n\}$ to a function F of exponential type a. Necessarily, $F|_{\mathbb{R}} = \widetilde{F}$ as we wished to show. Notice that we also have that

$$\Delta^{\frac{s}{2}}F_0 = \lim_{n\to+\infty}\Delta^{\frac{s}{2}}(F_n)_0 = \lim_{n\to+\infty}(h_n)_0 = h_0,$$

that is, the inverse is given by equation (13).

Suppose now that $s > \frac{1}{p}, s - \frac{1}{p} \notin \mathbb{N}_0$. Again, let $h \in \mathcal{B}_a^p, \{h_n\} \subseteq \mathcal{T}, h_n \to h$ in $\mathcal{B}_a^p$ and set

$$F_n(z) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\widehat{(h_n)_0}(\xi)|\xi|^{-s}\left(e^{iz\xi} - P_m(iz\xi)\right)d\xi,$$

where $m = \lfloor s - 1/p \rfloor$. Then, $F_n \in \mathcal{E}_a$ and $\Delta^{\frac{s}{2}}(F_n)_0 = (h_n)_0$ by [18, Corollary 3.4]. Thus, $\{F_n\}$ is a Cauchy sequence in $\mathcal{B}_a^{s,p}$, that is, $\{(F_n)_0\}$ is a Cauchy sequence in $E^{s,p}$, hence there exists a limit function $\widetilde{F} \in E^{s,p}$. We need to prove that $\widetilde{F}$ is the restriction of some entire function of exponential type a.

Differentiating $m+1$ times, since $s' := m+1-s \in (-1/p, 1/p')$ the integrals below converge absolutely so that

$$\begin{aligned}
F_n^{(m+1)}(z) &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\widehat{(h_n)_0}(\xi)|\xi|^{-s}(i\xi)^{m+1}e^{iz\xi}\,d\xi \\
&= \int_{\mathbb{R}}h_n(t)\mathcal{F}^{-1}\left((i\xi)^{m+1}|\xi|^{-s}e^{iz\xi}\chi_{[-a,a]}\right)(t)\,dt.
\end{aligned} \tag{15}$$

Then, if $-\frac{1}{p} < s' < 0$, the term on the right hand side in (15) equals

$$\int_{\mathbb{R}}h_n(t)R^{m+1}\mathcal{I}_{s'}\text{sinc}(z-t)\,dt,$$

and by Theorem 2.1 we obtain

$$\begin{aligned}
|F_n^{(m+1)}(z)| &\le C\|(h_n)_0\|_{L^p}\|\text{sinc}(z-\cdot)\|_{L^q} \\
&\le Ce^{a|y|}\|(h_n)_0\|_{L^p},
\end{aligned} \tag{16}$$

where $\frac{1}{q} = \frac{1}{p'} - s'$, by the classical Plancherel–Pólya inequality. If $0 \le s' < \frac{1}{p'}$, we repeat the same argument with $s'-1$ in place of s', observing that the the term on the right hand side in (15) equals

$$\int_{\mathbb{R}} h_n(t) R^m \mathcal{I}_{s'-1}\big(D\mathrm{sinc}(z - \cdot)\big)(t)\, dt\,,$$

and using the classical Bernstein inequality as well.

Therefore, the convergence of $\{h_n\}$ in $\mathcal{B}_a^p$ implies the uniform convergence on compact subsets of $\mathbb{C}$ of $\{F_n^{(m+1)}\}$ and, in particular, the limit function G_{m+1} is of exponential type a. Then, F is the anti-derivative of G_{m+1} such that $P_{F;m,0} = 0$ and $F|_{\mathbb{R}} = \widetilde{F}$, as we wished to show. This shows that the inverse is as in (13) and concludes the proof of the theorem.

$\square$

Corollary 4.3 *Let* $s > 0, 1 < p < \infty$ *and* $s - \frac{1}{p} \notin \mathbb{N}_0$. *Then, norm convergence in* $\mathcal{B}_a^{s,p}$ *implies uniform convergence on compact subsets of* $\mathbb{C}$.

Proof Let $f \in \mathcal{B}_a^{s,p}$. From the identity, (13) with $h \in \mathcal{B}_a^p$, arguing as in (16), we obtain that

$$|f^{(m+1)}(z)| \le e^{a|y|} \|\Delta^{\frac{s}{2}} f\|_{\mathcal{B}_a^p} = e^{a|y|} \|f\|_{\mathcal{B}_a^{s,p}}.$$

Since $P_{f;m;0} = 0$, it follows that for any compact $K \subseteq \mathbb{C}$,

$$\sup_{z \in K} |f(z)| \le C_K \|f\|_{\mathcal{B}_a^{s,p}}.$$

$\square$

We are now ready to prove Theorems 3 and 4.

Proof of Theorem 3 We observe that the completeness follows from the above corollary, or from the surjective isometry between $\mathcal{B}_a^{s,p}$ and $\mathcal{B}_a^p$. For the second part of the theorem we argue as follows. Let $h \in \mathcal{T}$, and as in the proof of Theorem 4.2 define

$$f(w) = (\Delta^{\frac{s}{2}})^{-1} h(w) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{h}_0(\xi) |\xi|^{-s} \big(e^{iw\xi} - P_m(iw\xi)\big)\, d\xi\,,$$

and therefore

$$P_{f(\cdot+iy);m;0}(w) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{h}_0(\xi) |\xi|^{-s} \big(e^{-y\xi} P_m(iw\xi) - P_m(i(w + iy)\xi)\big)\, d\xi.$$

Hence,

$$\begin{aligned}
F(w) &= f(w + iy) - P_{f(\cdot+iy);m;0}(w) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{h}_0(\xi) e^{-y\xi} |\xi|^{-s} \big(e^{iw\xi} - P_m(i(w + iy)\xi)\big)\, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \widehat{h}_0(\cdot + iy)(\xi) |\xi|^{-s} \big(e^{iw\xi} - P_m(iw\xi)\big)\, d\xi = (\Delta^{\frac{s}{2}})^{-1} \big(h(\cdot + iy)\big)(w).
\end{aligned}$$

Hence, from (13), we conclude that $F \in \mathcal{B}_a^{s,p}$ since, by the classical Plancherel–Pólya inequality ([26]), $h(\cdot + iy) \in \mathcal{B}_a^p$ and by Theorem 4.2 we obtain

$$\|F\|_{\mathcal{B}_a^{s,p}} = \|h(\cdot + iy)\|_{\mathcal{B}_a^p} \leq e^{a|y|}\|h\|_{\mathcal{B}_a^p} = e^{a|y|}\|f\|_{\mathcal{B}_a^{s,p}}$$

as we wished to show. The conclusion now follows from Lemma 4.1. □

Proof of Theorem 4 If $f \in \mathcal{B}_a^{s,p}$, then $f_0 \in E^{s,p}$ and $\mathrm{supp}\widehat{f_0} \subseteq [-a, a]$, hence (i) implies (ii).

If (ii) holds, then $\Delta^{\frac{s}{2}}h \in L^p$ and $\mathrm{supp}\widehat{\Delta^{\frac{s}{2}}h} \subseteq \mathrm{supp}\widehat{h} \subseteq [-a, a]$. Hence, it follows that $\Delta^{\frac{s}{2}}h = f_0$, for some $f \in \mathcal{B}_a^p$. Setting $F = (\Delta^{\frac{s}{2}})^{-1}f$ we have $F \in \mathcal{B}_a^{s,p}$. Hence, $\Delta^{\frac{s}{2}}F_0 = f_0 = \Delta^{\frac{s}{2}}h$. Since $F_0, h \in E^{s,p}$ and $\Delta^{\frac{s}{2}}$ is injective on $E^{s,p}$, it follows that $F_0 = h$, that is, (ii) implies (i).

By applying the classical characterization of Bernstein spaces to $\Delta^{\frac{s}{2}}h$ and Theorem 4.2 we easily see that (ii) and (iii) are equivalent. □

5 Reconstruction formulas and sampling in PW_a^s

In this small section we make some comments and observations on reconstruction formulas and sampling for the fractional Paley–Wiener spaces PW_a^s. In particular we conclude that the fractional Paley–Wiener spaces are not de Branges spaces.

Proposition 5.1 *Let* $0 < s < \frac{1}{2}$. *Then, the set* $\{\psi(\cdot - n\pi/a)\}_{n\in\mathbb{Z}}$,

$$\psi(z - n\pi/a) = \frac{1}{2\sqrt{a\pi}} \int_{-a}^a e^{in\frac{\pi}{a}\xi}e^{iz\xi}|\xi|^{-s}\, d\xi$$

is an orthonormal basis for PW_a^s.

If $s > \frac{1}{2}, s - \frac{1}{2} \notin \mathbb{N}$, *the set* $\{\psi(\cdot - n\pi/a)\}_{n\in\mathbb{Z}}$,

$$\psi(z - n\pi/a) = \frac{1}{2\sqrt{a\pi}} \int_{-a}^a e^{in\frac{\pi}{a}\xi}\left(e^{iz\xi} - P_m(iz\xi)\right)|\xi|^{-s}\, d\xi$$

is an orthonormal basis for PW_a^s.

Proof It is a well known fact that the family of functions $\{\varphi_n\}_{n\in\mathbb{Z}}$,

$$\varphi_n(z) = \frac{1}{2\sqrt{a\pi}} \int_{-a}^a e^{it(n\frac{\pi}{a}-z)}\, dt = \sqrt{a/\pi}\,\mathrm{sinc}\left(a(z - n\pi/a)\right)$$

is an orthonormal basis for PW_a. The conclusion follows from Theorem 3.8.

We have the following consequences.

Corollary 5.2 *For every* $f \in PW_a^s$ *we have the orthogonal expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} \Delta^{\frac{s}{2}}f(n\pi/a)\psi(z - n\pi/a)\,, \tag{17}$$

where the series converges in norm and uniformly on compact subsets of $\mathbb{C}$. *Moreover,*

$$\|f\|^2_{PW^s_a} = \frac{a}{\pi} \sum_{n\in\mathbb{Z}} |\Delta^{\frac{s}{2}} f(n\pi/a)|^2. \tag{18}$$

Proof By the classical theory of Hilbert spaces and Plancherel's formula, we get

$$
\begin{aligned}
f &= \sum_{n\in\mathbb{Z}} \langle f \mid \psi(\cdot - n\pi/a) \rangle_{PW^s_a} \, \psi(\cdot - n\pi/a) \\
&= \sum_{n\in\mathbb{Z}} \frac{a}{\pi} \Big(\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} |\xi|^s \widehat{f_0}(\xi) e^{in\frac{\pi}{a}\xi} \, d\xi \Big) \psi(\cdot - n\pi/a) \\
&= \sum_{n\in\mathbb{Z}} \Delta^{\frac{s}{2}} f(n\pi/a) \psi(\cdot - n\pi/a),
\end{aligned}
$$

where the series convergences in PW^s_a-norm, hence uniformly convergence on the compact subsets of $\mathbb{C}$. Finally, formula (18) follows from Theorem 3.8 and the classical Shannon–Kotelnikov formula. $\qquad\square$

A few comments are in order. Both the reconstruction formula (17) and the norm identity (18) resemble some known results for the Paley–Wiener space PW_a. In particular, equation (18) can be thought as a substitute of the Shannon–Kotelnikov sampling theorem in the setting of fractional Paley–Wiener spaces. However, these results are somehow unsatisfactory: we recover the function f and its norm from point evaluations of the fractional Laplacian $\Delta^{\frac{s}{2}} f$ and not of the function itself. Hence, it is a very natural question if we can do something better. Indeed, this is the case for the reconstruction formula (17), but we cannot really improve (18). For simplicity, we now restrict ourselves to the case $0 < s < \frac{1}{2}$.

Proposition 5.3 *Let* $f \in PW^s_a$, $0 < s < \frac{1}{2}$. *Then, for* $z \in \mathbb{C}$,

$$f(z) = \sum_{n\in\mathbb{Z}} f(n\pi/a)\mathrm{sinc}\big(a(z - n\pi/a)\big),$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Proof Let $f \in PW^s_a$ and let $\{f_k\} \subseteq PW_a$ be a sequence such that $f_k \to f$ in PW^s_a. Then, by the Shannon–Kotelnikov theorem, we have that

$$f_k(x) = \sum_{n\in\mathbb{Z}} f_k(n\pi/a)\mathrm{sinc}\big(a(x - n\pi/a)\big),$$

where the series converges absolutely and in PW_a-norm. However, norm convergence in PW_a implies uniform convergence on compact subsets of $\mathbb{C}$. Thus, we obtain

$$
\begin{aligned}
f(x) &= \lim_{k\to+\infty} f_k(x) = \lim_{k\to+\infty} \sum_{n\in\mathbb{Z}} f_k(n\pi/a)\mathrm{sinc}\big(a(x - n\pi/a)\big) \\
&= \sum_{n\in\mathbb{Z}} f(n\pi/a)\mathrm{sinc}\big(a(x - n\pi/a)\big).
\end{aligned}
$$

$\qquad\square$

We now point out that, in general, we cannot improve (17) with point evaluations of f instead of its fractional Laplacian. More generally, we would like to know if it is possible

to have a real sampling sequence in PW_a^s. We will see that, at least in the case $0 < s < \frac{1}{2}$, this is not the case.

Definition 5.4 Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$. The sequence Λ is a sampling sequence for PW_a^s if there exists two positive constants A, B such that

$$A\|f\|_{PW_a^s}^2 \leq \sum_{\lambda_n \in \Lambda} |f(\lambda_n)|^2 = \sum_{\lambda_n \in \Lambda} |\langle f \mid K_{\lambda_n} \rangle_{PW_a^s}|^2 \leq B\|f\|_{PW_a^s}^2,$$

where K_{λ_n} is the reproducing kernel of PW_a^s, as in Corollary 3.2.

Since for $0 < s < \frac{1}{2}$ the space PW_a^s can be identified with $L_a^2(|\xi|^{2s})$ via the Fourier transform, the sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is a sampling sequence for PW_a^s if and only if the family of functions $\{\widehat{K}_{\lambda_n}\}_{\lambda_n \in \Lambda}$ is a frame for $L_a^2(|\xi|^{2s})$, that is, if and only if there exist two positive constants A, B such that

$$A\|\widehat{f_0}\|_{L_a^2(|\xi|^{2s})}^2 \leq \sum_{\lambda_n \in Z} |\langle \widehat{f_0} \mid \widehat{(K_{\lambda_n})_0} \rangle_{L_a^2(|\xi|^{2s})}|^2 \leq B\|\widehat{f_0}\|_{L_a^2(|\xi|^{2s})}^2.$$

From Corollary 3.2, when $0 < s < \frac{1}{2}$ we obtain that $\widehat{(K_{\lambda_n})_0}(\xi) = \frac{1}{\sqrt{2\pi}}e^{-i\lambda_n\xi}|\xi|^{-2s}\chi_{[-a,a]}(\xi)$ and the following result is easily proved.

Proposition 5.5 *The family $\{e^{-i\bar{\lambda}_n\xi}|\xi|^{-2s}\}_{\lambda_n \in \Lambda}$ is a frame for $L_a^2(|\xi|^{2s})$ if and only if the family $\{e^{-i\bar{\lambda}_n\xi}|\xi|^{-s}\}_{\lambda_n \in \Lambda}$ is a frame for L_a^2.*

Proof Assume that $\{e^{-i\bar{\lambda}_n\xi}|\xi|^{-2s}\}_{\lambda_n \in \Lambda}$ is a frame for $L_a^2(|\xi|^{2s})$ and let f be a function in PW_a. Then,

$$\|\widehat{f_0}\|_{L_a^2}^2 = \||\xi|^{-s}\widehat{f_0}\|_{L_a^2(|\xi|^{2s})}^2 \approx \sum_{\lambda_n \in \Lambda} \Big| \int_{-a}^a |\xi|^{-s}\widehat{f_0}(\xi)e^{i\lambda_n\xi}|\xi|^{-2s}|\xi|^{2s} \, d\xi \Big|^2$$

$$= \sum_{\lambda_n \in \Lambda} \Big| \int_{-a}^a \widehat{f_0}(\xi)e^{i\lambda_n\xi}|\xi|^{-s} \Big|^2 \, d\xi,$$

hence, $\{e^{-i\bar{\lambda}_n\xi}|\xi|^{-s}\}_{\lambda_n \in \Lambda}$ is a frame for $L_a^2(|\xi|^{2s})$. The reverse implication is similarly proved. $\square$

Therefore, the sampling problem for PW_a^s, $0 < s < \frac{1}{2}$, is equivalent to study windowed frames for L_a^2. The following result, due to [16] (see also [14]) implies that we cannot have real sampling sequences for PW_a^s. Hence, we cannot obtain an analogue of (18) with point evaluations of the function instead of point evaluations of its fractional Laplacian.

Theorem [14, 16] *The family $\left\{ g(\xi)e^{i\lambda_n\xi} \right\}_{\lambda_n \in \Lambda}$ is a frame for L_a^2 for some sequence of points $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R}$ if and only if there exist positive constants m, M such that $m \leq g(\xi) \leq M$.*

We conclude the section with one last comment about fractional Paley–Wiener spaces and de Brange spaces. These latter spaces were introduced by de Branges [13] and have been extensively studied in the last years. Among others, we recall the papers [1, 5–8,

19]. The space PW_a is the model example of a de Branges space. A classical result, see [13], states that de Brange spaces always admit a real sampling sequence, or, equivalently, always admit a Fourier frame of reproducing kernels. The above discussion proves that this is not the case for the spaces PW_a^s, $0 < s < \frac{1}{2}$. Therefore, the following result holds.

Theorem 5.6 *The fractional Paley–Wiener spaces PW_a^s, $0 < s < \frac{1}{2}$, are not de Branges spaces.*

6 Boundedness of the orthogonal projection

In the previous sections we proved that the spaces PW_a^s can be equivalently described as

$$PW_a^s = \left\{ f \in E^{s,2} \,:\, \mathrm{supp}\widehat{f} \subseteq [-a,a] \right\},$$

thus, it is clear that the spaces PW_a^s are closed subspaces of the Hilbert spaces $E^{s,2}$. Therefore, we can consider the Hilbert space projection operator $\mathbf{P}_s : E^{s,2} \to PW_a^s$.

In the case $0 < s < \frac{1}{2}$ we get from Theorem 1 that the projection operator $\mathbf{P}_s$ is explicitly given by the formula

$$\mathbf{P}_s f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(\xi) \chi_{[-a,a]}(\xi) e^{ix\xi} \, d\xi. \tag{19}$$

Similarly, we explicitly deduce $\mathbf{P}_s$ in the case $s > \frac{1}{2}$ from Theorem 2,

$$\mathbf{P}_s f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \chi_{[-a,a]}(\xi) \big(e^{ix\xi} - P_m(ix\xi)\big) \, d\xi. \tag{20}$$

A very natural question is to investigate whether the operator $\mathbf{P}_s$ densely defined on $E^{s,p} \cap E^{s,2}$ extends to a bounded operator $\mathbf{P}_s : E^{s,p} \to \mathcal{B}_a^{s,p}$ assuming that $s - \frac{1}{2} \notin \mathbb{N}_0$, $s - \frac{1}{p} \notin \mathbb{N}_0$ and $\lfloor s - \frac{1}{2} \rfloor = \lfloor s - \frac{1}{p} \rfloor$. This is the content of Theorem 5 which we now prove.

Proof of Theorem 5 We first assume $0 < s < \frac{1}{2}$. Let f be a function in $E^{s,p} \cap E^{s,2}$. By definition of $E^{s,p} \cap E^{s,2}$, we can assume f to be in the Schwartz space $\mathcal{S}$. Then, the projection $\mathbf{P}_s f$ is given by (19). The function $\mathbf{P}_s f$ clearly extends to an entire function of exponential type a, which we still denote by $\mathbf{P}_s f$. Moreover, we assumed $f \in \mathcal{S}$, so that, for instance, $\mathbf{P}_s f$ is a well-defined L^2 function with a well-defined Fourier transform. Thus,

$$\Delta^{\frac{s}{2}} \mathbf{P}_s f(x) = \mathcal{F}^{-1} \big(|\cdot|^s \chi_{[-a,a]} \widehat{f} \big) = \mathcal{F}^{-1} \big(\chi_{[-a,a]} \widehat{\Delta^{\frac{s}{2}} f} \big).$$

Hence

$$\|\mathbf{P}_s f\|_{\mathcal{B}_a^{s,p}} = \|\Delta^{\frac{s}{2}} \mathbf{P}_s f\|_{L^p(\mathbb{R})} \le C \|\Delta^{\frac{s}{2}} f\|_{L^p(\mathbb{R})} = C \|f\|_{E^{s,p}},$$

where the inequality holds since $\chi_{[-a,a]}$ is an L^p-Fourier multiplier for any $1 < p < +\infty$. Therefore, $\mathbf{P}_s$ extends to a bounded operator $\mathbf{P}_s : E^{s,p} \to \mathcal{B}_a^{s,p}$ when $0 < s < \frac{1}{2}$.

Assume now $s > \frac{1}{2}$. Then, given $f \in E^{s,p} \cap E^{s,2} \cap \mathcal{S}$, the projection $\mathbf{P}_s f$ is given by (20), that is,

$$\mathbf{P}_s f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \chi_{[-a,a]}(\xi)(e^{ix\xi} - P_m(ix\xi)) \, d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \chi_{[-a,a]}(\xi)e^{ix\xi} \, d\xi - \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) \chi_{[-a,a]}(\xi) P_m(ix\xi) \, d\xi$$

$$=: (\mathbf{P}_s f)^1(x) + (\mathbf{P}_s f)^2(x).$$

As before, $(\mathbf{P}_s f)^1$ is a well-defined L^2 function with a well-defined Fourier transform, whereas $(\mathbf{P}_s f)^2$ is a polynomial of degree $m = \lfloor s - 1/2 \rfloor < s$, thus its fractional Laplacian $\Delta^{\frac{s}{2}}$ is zero. Therefore,

$$\Delta^{\frac{s}{2}} \mathbf{P}_s f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^s \widehat{f}(\xi) \chi_{[-a,a]}(\xi)e^{ix\xi} \, d\xi = \mathcal{F}^{-1}\big(\chi_{[-a,a]} \widehat{\Delta^{\frac{s}{2}}f}\big)(x).$$

Once again we have

$$\|\mathbf{P}_s f\|_{\mathcal{B}_a^{s,p}} = \|\Delta^{\frac{s}{2}} \mathbf{P}_s f\|_{L^p(\mathbb{R})} \le C \|\Delta^{\frac{s}{2}} f\|_{L^p(\mathbb{R})} = C\|f\|_{E^{s,p}},$$

since $\chi_{[-a,a]}$ is a L^p-Fourier multiplier for any $1 < p < \infty$. □

7 Proof of Lemma 4.1

Proof of Lemma 4.1 We recall that, given a function φ on $\mathbb{R}$, for $t > 0$ we set $\varphi_t = \frac{1}{t}\varphi(\cdot/t)$. We also set $\varphi^t = \varphi(t\cdot)$ and observe that $\mathcal{F}(\varphi_t) = (\mathcal{F}\varphi)^t$. Moreover, it is easy to see that for all $p \in [1, \infty)$, $\varphi_r, \varphi^r \to \varphi$ in L^p, as $r \to 1$.

We first claim that the subspace $\bigcup_{\delta > 0} \{f \in \mathcal{B}_{a-\delta}^p : f_0 \in \mathcal{S}\}$ is dense in $\mathcal{B}_a^p$, $1 < p < \infty$. Let $f \in \mathcal{B}_a^p$ be given. Then $\mathrm{supp}\widehat{f_0} \subseteq [-a, a]$ and if $0 < r < 1$, $f^r \in \mathcal{B}_{ar}^p$ so that $\mathrm{supp}\widehat{f_0^r} \subseteq [-ar, ar]$. Let $\delta = (1 - r)a/2$ and let $\varphi \in C_c^\infty[-1, 1]$, $\varphi = 1$ on $[-\frac{1}{4}, \frac{1}{4}]$, $\int \varphi = 1$. Then $\widehat{f_0^r} * \varphi_\delta \in C_c^\infty$ and $\mathrm{supp}\widehat{f_0^r} * \varphi_\delta \subseteq [-a + \delta, a - \delta]$. Therefore, $\mathcal{F}^{-1}\big(\widehat{f_0^r} * \varphi_\delta\big) \in \mathcal{S}$ extends to a function $f_{(\delta)} \in \mathcal{B}_{a-\delta}^p$ and $f_{(\delta)} \to f$ in $\mathcal{B}_a^p$ as $\delta \to 0$. This proves the claim.

Next, let $f \in \mathcal{B}_{a-\delta}^p$ be such $f_0 \in \mathcal{S}$, for $\delta > 0$. Let $\eta_{(\delta)} \in C_c^\infty[-\delta, \delta]$, $\eta = 1$ on $[-\delta/2, \delta/2]$. Then $(1 - \eta_{(\delta)})\widehat{f_0} \in C_c^\infty$ and has support in $\{\xi : \delta/2 \le |\xi| \le a\}$. Therefore, $\mathcal{F}^{-1}\big((1 - \eta_{(\delta)})\widehat{f_0}\big) \in \mathcal{S}_\infty$ and extends to a function in $\mathcal{B}_a^p$. Thus, it suffices to show that $\|\mathcal{F}^{-1}\big(\eta_{(\delta)}\widehat{f_0}\big)\|_{L^p} \to 0$ as $\delta \to 0$. This fact follows by observing that we may choose

$$\eta_{(\delta)} = (\chi * \varphi)^{1/\delta}$$

where $\varphi \in C_c^\infty[-\frac{1}{2}, \frac{1}{2}]$ with $\int \varphi = 1$. Then, it is clear that $\eta_{(\delta)} \in C_c^\infty[-\delta, \delta]$, and $\eta = 1$ on $[-\delta/2, \delta/2]$. Finally, for $q \in (1, \infty)$, it is easy to see that

$$\|\mathcal{F}^{-1}\eta_{(\delta)}\|_{L^q} = \delta^{1-1/q}\|\widehat{\chi}\widehat{\varphi}\|_{L^q} \to 0$$

as $\delta \to 0$. □

Corollary 7.1 *Let* $s > 0$, $p \in (1, \infty)$, $s - \frac{1}{p} \notin \mathbb{N}_0$, *and set* $m = \lfloor s - \frac{1}{p} \rfloor$. *For* $s > \frac{1}{p}$, *set* $\mathcal{T}_m = \{f \in \mathcal{E}_a : f_0 \in \mathcal{S}_\infty, P_{f;m;0} = 0\}$. *Then, if* $0 < s < \frac{1}{p}$ *the subspace* $\mathcal{T}$ *is dense in* $\mathcal{B}_a^{s,p}$, *whereas if* $s > \frac{1}{p}$ *the subspace* $\mathcal{T}_m$ *is dense in* $\mathcal{B}_a^{s,p}$ *if* $s > \frac{1}{p}$, $s - \frac{1}{p} \notin \mathbb{N}_0$.

Proof We only prove the case $s > \frac{1}{p}$, $s - \frac{1}{p} \notin \mathbb{N}_0$, the other case being easier. By Lemma 4.1 and Theorem 4.2 we have that $(\Delta^{\frac{s}{2}})^{-1}(\mathcal{T})$ is dense in $\mathcal{B}_a^{s,p}$. Thus, it suffices to show that this latter space is contained in $\mathcal{T}_m$. Let $h \in \mathcal{T}$ and let $f = \Delta^{-\frac{s}{2}} h$ be given by (13). It is clear that $P_{f;m;0} = 0$. Moreover, $f_0^{(m+1)} = \mathcal{F}^{-1}\left((i\xi)^{m+1}|\xi|^{-s}\hat{h}_0\right) \in \mathcal{S}_\infty$ and extends to a function in $\mathcal{B}_a^{s,p}$, by Theorem 4. This easily implies that $f_0 \in \mathcal{S}_\infty$ and the conclusion follows. $\qquad\square$

8 Final remarks and open questions

We believe that the fractional spaces we introduced are worth investigating and, as we mentioned, they arise naturally in a several variable setting [17]. We mention a number of questions that remain open.

First of all, it is certainly of interest to consider the cases $s - \frac{1}{p} \in \mathbb{N}_0$. As we pointed out already, these cases correspond to the critical cases in the Sobolev embedding theorem. From [18, Theorem 3 (iii)] we also have a description of the realization spaces $E^{s,p}$ of $\dot{W}^{s,p}$, as follows.

Let $s - \frac{1}{p} \in \mathbb{N}_0$. Fix the bounded interval $Q = [0, 2\pi]$. If $s = \frac{1}{p}$, let

$$E^{s,p} = \left\{ f \in \mathrm{BMO} : f_Q = 0, \|f\|_{E^{s,p}} := \|\Delta^{\frac{s}{2}} f\|_{L^p} < +\infty \right\},$$

while, if $m = s - \frac{1}{p}$ is a positive integer, let

$$E^{s,p} = \left\{ f \in \mathcal{S}' \cap \mathcal{C}^{m-1} : P_{f;m-1;0} = 0, f^{(m)} \in \mathrm{BMO}, f_Q^{(m)} = 0, \|f\|_{E^{s,p}} := \|\Delta^{\frac{s}{2}} f\|_{L^p} < +\infty \right\}.$$

By [18, Theorem 3 (iii)] the space $E^{s,p}$ is a realization space for $\dot{W}^{s,p}$. Then, for $s - \frac{1}{p} \in \mathbb{N}_0$ we may still define the spaces $\mathcal{B}_a^{s,p}$ as in (1.1), that is,

$$\mathcal{B}_a^{s,p} = \left\{ f \in \mathcal{E}_a : f_0 \in E^{s,p} \text{ with norm } \|f\|_{\mathcal{B}_a^{s,p}} := \|f_0\|_{E^{s,p}} \right\}.$$

However, all proofs break down in the cases $s - \frac{1}{p} \in \mathbb{N}_0$, due to the presence of the BMO condition. Since for $f \in \mathcal{E}_a$, f_0 is a smooth function, a more natural definition of $\mathcal{B}_a^{s,p}$ seems to be

$$\mathcal{B}_a^{s,p} = \left\{ f \in \mathcal{E}_a : [f_0]_m \in \dot{W}^{s,p} \text{ and if } m \geq \frac{1}{p}, P_{f_0;m;0} = 0 \right\},$$

where $[f_0]_m$ denotes the equivalence class of f_0 in $\mathcal{S}'/\mathcal{P}_m$, and endow it with the norm $\|f\|_{\mathcal{B}_a^{s,p}} := \|f_0\|_{\dot{W}^{s,p}}$. Such definition would require to prove a different description of a realization space for $\dot{W}^{s,p}$ when $s - \frac{1}{p} \in \mathbb{N}_0$.

To further elaborate on this point, we point out that, as shown by Bourdaud, when $s - \frac{1}{p} \notin \mathbb{N}_0$, the realization spaces $E^{s,p}$ of $\dot{W}^{s,p}$ are the unique realization spaces whose norms are homogeneous with respect the natural dilations. On the other hand, when $s - \frac{1}{p} \in \mathbb{N}_0$ there exists no realization space of $\dot{W}^{s,p}$ whose norm is homogeneous. Finally, a "good" definition of $\mathcal{B}_a^{s,p}$ in these critical cases should coincide with the interpolating space between two spaces with $s - \frac{1}{p} \notin \mathbb{N}_0$. In any event, these spaces remain to be investigated.

Naturally, another question that remains open is the boundedness of the orthogonal projection $\mathbf{P} : E^{s,p} \to \mathcal{B}_a^{s,p}$ in the cases $s - \frac{1}{p} \in \mathbb{N}_0$. Such boundedness would allow one to explicitly describe the dual space of $\mathcal{B}_a^{s,p}$, for the whole scale $s > 0$ and $p \in (1, \infty)$.

The Paley–Wiener space is a very special instance of a de Branges spaces. These spaces where introduced by de Branges also in connection with the analysis of the canonical systems, see e.g. [13, 23]. It would be interesting to determine whether the fractional Paley–Wiener spaces PW_a^s also arise to the solution of a canonical system defined in terms of the fractional derivative.

In [7] it is shown that the Paley–Wiener space, and, more generally, any de Branges space, coincides as set with a Fock-type space with non-radial weight. The Paley–Wiener (or de Branges) norm given by an integral on the real line is replaced by an equivalent weighted integral on the complex plane. We wonder if an analogous result holds true for the fractional Paley–Wiener spaces.

Another important fact about the classical Paley–Wiener space is that, up to a multiplication by an inner function, it admits a representation as a *model space* of $H^2(\mathbb{C}_+)$, the Hardy space of the upper half-plane. We recall that a model subspace of $H^2(\mathbb{C}_+)$ is defined as $K_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$ where Θ is an inner function in $\mathbb{C}_+$. By means of the Weyl–Titchmarsh transform, it is possible to interpret the completeness problem for eigenfunctions of the Schrödinger equations as the completeness problem for a system of reproducing kernel $\{k_\lambda\}_\lambda$ in some model space K_Θ. Thus, there exists a close link between model spaces, in particular the Paley–Wiener spaces. It would be interesting to understand if an analogous link exists between the fractional canonical systems and the fractional Paley–Wiener spaces.

References

1. Abakumov, E., Baranov, A., Belov, Y.: Krein-type theorems and ordered structure for Cauchy–de Branges spaces. J. Funct. Anal. **277**(1), 200–226 (2019)
2. Adams, D.R.: A note on Riesz potentials. Duke Math. J. **42**(4), 765–778 (1975)
3. Arcozzi, N., Monguzzi, A., Peloso, M.M., Salvatori, M.: Paley–Wiener theorems on the Siegel upper half-space. J. Fourier Anal. Appl. **25**(4), 1958–1986 (2019)
4. Andersen, N.B.: Entire L^p-functions of exponential type. Expo. Math. **32**(3), 199–220 (2014)
5. Baranov, A., Belov, Y., Borichev, A.: Spectral synthesis in de Branges spaces. Geom. Funct. Anal. **25**(2), 417–452 (2015)
6. Baranov, A., Belov, Y., Borichev, A.: Fock type spaces with Riesz bases of reproducing kernels and de Branges spaces. Stud. Math. **236**(2), 127–142 (2017)
7. Baranov, A., Bommier-Hato, H.: De Branges spaces and Fock spaces. Complex Var. Elliptic Equ. **63**(7–8), 907–930 (2018)

8. Baranov, A., Belov, Y., Poltoratski, A.: De Branges functions of Schroedinger equations. Collect. Math. **68**(2), 251–263 (2017)
9. Bernstein, S.N.: Sur une propriete des fonctions entières. C. R. Acad. Sci. **176**, 1603–1605 (1923)
10. Bourdaud, G.: Réalisations des espaces de Besov homogènes. Ark. Math. **26**(1), 41–54 (1988)
11. Bourdaud, G.: Realizations of homogeneous Sobolev spaces. Complex Var. Elliptic Equ. **56**(10–11), 857–874 (2011)
12. Bourdaud, G.: Realizations of homogeneous Besov and Lizorkin–Triebel spaces. Math. Nachr. **286**(5–6), 476–491 (2013)
13. de Branges, L.: Hilbert Spaces of Entire Functions. Prentice-Hall Inc, Englewood Cliffs (1968)
14. Gabardo, J.-P., Lai, C.-K.: Frames of multi-windowed exponentials on subsets of $\mathbb{R}^d$. Appl. Comput. Harmon. Anal. **36**(3), 461–472 (2014)
15. Grafakos, L.: Modern Fourier analysis, Graduate Texts in Mathematics, vol. 250, 2nd edn. Springer, New York (2014)
16. Lai, C.-K.: On Fourier frame of absolutely continuous measures. J. Funct. Anal. **261**(10), 2877–2889 (2011)
17. Monguzzi, A., Peloso, M.M., Salvatori, M.: Spaces of entire functions in $\mathbb{C}^{n+1}$ (in preparation) (2020)
18. Monguzzi, A., Peloso, M.M., Salvatori, M.: Fractional Laplacian, homogeneous Sobolev spaces and their realizations. Ann. Mat. Pura Appl. (4) **199**(6), 2243–2261 (2020)
19. Ortega-Cerdà, J., Seip, K.: Fourier frames. Ann. Math. (2) **155**(3), 789–806 (2002)
20. Pesenson, I.: Sampling of band-limited vectors. J. Fourier Anal. Appl. **7**(1), 93–100 (2001)
21. Paley, R.E.A.C., Wiener, N.: Fourier transforms in the complex domain, American Mathematical Society Colloquium Publications, vol. 19, American Mathematical Society, Providence, RI, 1987[1934], Reprint of the 1934 original
22. Pesenson, I., Zayed, A.I.: Paley–Wiener subspace of vectors in a Hilbert space with applications to integral transforms. J. Math. Anal. Appl. **353**(2), 566–582 (2009)
23. Romanov, R.: Canonical systems and de Branges spaces, arXiv e-prints (2014), arXiv:1408.6022
24. Seip, K.: Interpolation and sampling in spaces of analytic functions, University Lecture Series, vol. 33. American Mathematical Society, Providence, RI (2004)
25. Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III
26. Young, R.M.: An Introduction to Nonharmonic Fourier Series, 1st edn. Academic Press Inc, San Diego, CA (2001)