POLLUTION-INDUCED POVERTY TRAPS VIA HOPF BIFURCATION IN A MINIMAL INTEGRATED ECONOMIC-ENVIRONMENT MODEL

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Abstract

In this paper we present a minimal integrated environment-economic growth model. The accumulations of capital and pollution are connected by reciprocal feedbacks. Pollution is an undesirable but inevitable by-product of production, whose efficiency is hindered by pollution. The nexus between pollution and production is embodied in a damage function. The evolution of capital is enriched by the specific technology adopted here, an $\mathcal{S}$-shaped production function. The model dynamics is represented by a couple of nonlinear differential equations, whose long run behavior gives rise to multiple stationary points. A global analysis underlines the importance of a meditated choice of the relevant policy parameters. Pollution induced poverty traps emerge for low level of saving ratio and share of abated emission. Periodic behaviour of the economic and environmental variables represents an early signal of the imminent risk of being trapped in an unsatisfactory level of economic performance. Numerical examples corroborate the theoretical results of the paper.
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JEL Classification: C60, O40, Q50

1 Introduction

As the U.S. federal agency NOAA, National Oceanic and Atmospheric Administration has recently pointed out, 10 out of the last 15 years have been the hottest since reliable temperature measurements started, at the end of the 19th century. Nowadays it has been widely accepted that this global rising of temperature has an anthopogenic cause. It is the same agency that provides quarterly data about the Carbon Dioxide concentration at a global level. Putting together direct and indirect measurements, these data clearly show the exponential growth of the Carbon Dioxide concentration in the last 140 years. The bright side of this story is the unprecedented global economic growth the world has experienced in the same span of time. GDP - Gross Domestic Product - calculations go back some decades, but through indirect measurements, economists have been able to build reliable estimates of the path this proxy of economic growth has followed: similarly to the Carbon Dioxide concentration, and roughly simultaneously, the GDP has performed an exponential growth. We can wonder whether an unabated accumulation of CO_2 and other greenhouse gases - with their effects on the temperature and, consequently, on the human activities - will eventually jeopardize the future of economic growth. Early concerns about the joint evolution of economic growth and pollution[1] can be traced back to the 70s and early 80s, with the work of Mäler 1974, Brock 1973 and Luptacik

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[1]We refer to "pollution" as a general negative effect of production on the environment, not necessarily and/or utterly described by the accumulation of CO_2.

1982, among many others[2]. Relying on the idea that growth and pollution are two sides of the same coin, integrated models have been proposed, where environmental pollution has been introduced both as an input and as a by-product of production, in the framework of neoclassical growth models. In particular, the idea behind pollution as an input of production means that the more pollution is allowed, the less costly are the techniques of production (see, for examples, Bovemberg and Smulders 1995, Mohtadi 1996, Rubio 2000). Alternatively, the effect of pollution as a by-product of production has been studied through a damage function, as in Nordhaus 1992 and Nordhaus 2008 or, more recently in Brechet 2014, Capasso 2010 and La Torre 2015.

This paper is an attempt to build on the latter stream of the literature, by considering pollution as a negative production externality, whose effects can be modeled by a damage function. Let aside a stream of literature initiated with Capasso 2010, the majority of the papers in the area of integrated (economic-environment) assessment models employ an aggregate Cobb-Douglas production function approach. The consequences of this choice are twofold: there exists only one asymptotically stable equilibrium and the (multiplicative) damage function can drive the economy to the collapse only if the pollution takes infinite value. This amounts to say that however hard the damage may be, there is no pollution-induced poverty trap, no matter the initial conditions of the integrated economic-environment system. This is barely an acceptable simplification of reality in two respects: history, or initial conditions, usually matter, and the damage function has to somehow take into account the non-infinite resilience of the environment. We instead describe the economy through an $\mathcal{S}$-Shaped aggregate production function, as pioneered by Skiba 1978. Our integrated model remains minimal, but the choice of Skiba's non-neoclassical production function enriches the dynamics with a glimpse of real world complexity. The most related papers are La Torre et al 2015 and Bella et al 2019. With respect to the former work, we do not consider the spatial dimension and the related pollution diffusion. The focus of La Torre et al 2015 is indeed on pollution's spatial externalities and the main result underlines the importance of including space as a crucial dimension in integrated models. We relax the spatial hypothesis and perform a more accurate analysis of the underlying one-dimensional dynamical system. In contrast to the latter paper, where the authors dissects the parameter space to find a Bogdanov-Taken bifurcation and use this result to highlight the inherent trade-off between economic growth and environmental sustainability, we are interested in a comprehensive global analysis of the economy-environment co-evolution. In particular we focus on the inherent instabilities and delicate resilience of the interaction between the economy and the environment. We show that the choice of the policy parameters are crucial to sustainable development. In this respect, there are choices of the policy parameters where the behavior of the economic-environment system can undergo a dramatic change: guided by a slightly different set of policy choices, countries with similar initial conditions in terms of economy and environment could go very separate ways, some toward a sustainable future, and others to collapse. Moreover we argue that self sustained cycles in the system could prelude the collapse of the economy itself. To back these conclusions we perform both local and global analysis on the system of differential equations that describes our model. As for the global approach to dynamical systems in the contest of economic modelling, there is a consolidated stream of literature. We are particularly in debt to Mattana and Venturi 1999, and Neri and Venturi 2007, where the authors deals with the emergence of Hopf periodic solutions in an endogeneous economic growth model and in a IS-LM economic model respectively. In a stochastic contest, the analysis of the stability of a Hopf bifurcation has been performed by Nishimura and Shigoka 2006, among others.

This paper proceeds as follows. Section 2 introduces the model, gives details on the $\mathcal{S}$-Shaped production function, performs a local analysis of the different equilibria, and concludes with a comparative static analysis. Section 3 is devoted to the global analysis and the associated numerical examples. Section 4 presents our conclusions and opens to possible lines of future research.

[2]In this regard, a complete and precious literature review can be found in Xepapadeas 2005

2 The Model

The model we propose echoes the well-known DICE model by Nordhaus 2008, by stripping away most of its structure, nevertheless preserving the central idea: a module describing the law of motion of capital and a module depicting the accumulation of pollution. These two modules are integrated, given that pollution is a by-product of production and, simultaneously, acts as detrimental factor for productivity. The departure of our model from Nordhaus' seminal work lies in the lack of an explicit role for the temperature: in DICE model the economic activity produces greenhouse gases that contribute to the increase in temperature that, in turn, damages the production activity, in a negative production-greenhouse gases-temperature-production feedback loop. In our paper we shrink this cycle in a shorter production-pollution-production loop, and we consider a purely dynamical model, a la Solow 1956, in line with the extant literature (for a review Xepapadeas 2003 and Brock and Taylor 2005). The dynamic model is summarized by the following system of ordinary differential equations:

$$\begin{aligned} \dot{k_t} &= sf(k_t)(1-\tau)d(p_t) - \delta_k k_t \qquad (1) \\ &= \frac{sf(k_t)(1-u)^\epsilon}{1+bp(t)} - \delta_k k_t \\ \dot{p_t} &= \theta(1-u)f(k_t) - \delta_p p_t. \qquad (2) \end{aligned}$$

In equation (1) $\dot{k_t}$ represents the evolution of capital. The engine of capital accumulation lies in the production technology $f(k_t)$: pollution does directly enter the production function, but it affects the amount of output through the damage function $d(p_t)$. In the integrated models literature the word "pollution" generally means greenhouse gases. We instead interpret it in the broader sense of "undesired by-product" of economic activity . Only a portion s of the produced output goes to capital investments, so the parameter s represents the saving ratio of the economy. Moreover, a share τ of these investments is devoted to an environmental tax. The saving ratio s and the taxation rate τ naturally serve as policy variables: our model is purely dynamical, there is no layer of control upon it, but it nevertheless conveys useful information on the control space. As anticipated above, we will treat the case of a convex-concave production function, as in Skiba 1978. Given the crucial role this non-neoclassical production function takes in our paper, we will postpone a description of it to the next subsection. The damage function has been chosen as $d(p_t) = \frac{1}{1+bp(t)}$, with $b > 0$, as in La Torre 2015: this particular formulation states that the pollution externality on production is null when pollution is absent, while production falls proportionally when pollution increases. Many damage functions have been proposed in the literature of integrated assessment models, see Bretschger 2019 for an recent survey on the topic. Environmental taxation has been taken into account via the parameter τ: following La Torre 2015 and Bartz and Kelly 2008 we assume that $(1-\tau) = (1-u)^\epsilon$ with $\epsilon \geq 1$ The linear depreciation $\delta_k k$ follows the standard approach of the Solow model.
Equation (2) models the evolution of pollution. Production generates emissions which increase linearly the stock of pollution and $\theta > 0$ measures the degree of environmental inefficiency of economic activities. The abatement activities reduce a share u of emissions, thus $1-u$ represents unabated emissions.

2.1 Introducing the $\mathcal{S}$-shaped production function

Before dwelling on the analysis of system 1 - 2 in its fully-fledged version, we apply the simplifying hypothesis $b = 0$. This temporary simplification serves the purpose of introducing the main ideas about the $\mathcal{S}$-shaped production function. The following results are well known from the work of Skiba (1978), but it is nevertheless helpful to report them here for clarity sake. A possible functional form of an $\mathcal{S}$-shaped production function reads as:

$$f(k_t) := \frac{\alpha_1 k_t^q}{1+\alpha_2 k_t^q},\ q > 1, \qquad (3)$$

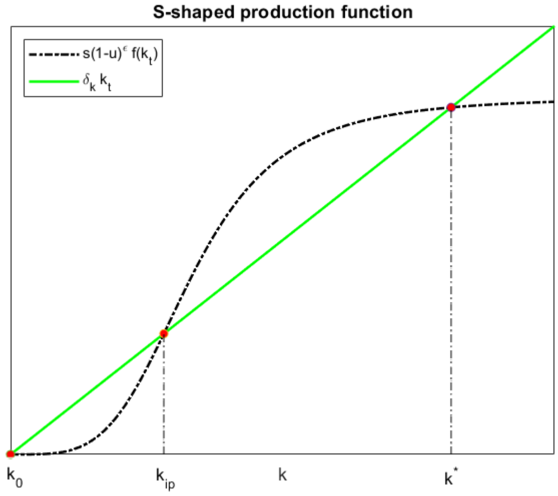

Figure 1: The $\mathcal{S}$-shaped Production Function

The interesting properties of $f(k_t)$ in 3 are guaranteed if $\alpha_1 > 0$, $\alpha_2 > 0$. The ratio $\frac{\alpha_1}{\alpha_2}$ represents the asymptotic behaviour of $f(k_t)$ when $k_t \to \infty$. Taken together, parameters α_1, α_2 and q shape the $\mathcal{S}$ function, determining the position of the inflection point and the width of the transition from low and high value of k_t. The $\mathcal{S}$-shaped production function is not a neoclassical concave production function. It starts convex and turns to concave after an inflection point at $k_{ip} = ((q-1)/(\alpha_2(q+1)))^{1/q}$. As a consequence of putting $b = 0$, equation 1 becomes decoupled from equation 2 and the evolution of capital can be studied independently from the accumulation of pollution (the opposite is not true, of course). The law of motion of capital in the presence of this aggregated technology is the following:

$$\dot{k_t} \quad = \quad s(1-u)^{\epsilon} f(k_t) - \delta_k k_t. \tag{4}$$

where s and δ_k represent the saving ratio and the depreciation rate of capital, respectively. Figure 1 shows the two curves $s(1-u)^{\epsilon} f(k_t)$ and $\delta_k\, k_t$: their intersections are the steady states of 1, namely k_0, k_{ip} and k^*. Observing Figure 1 it is straightforward to conclude that the difference between these two curves is negative before k_{th} and after k^*, while is positive in the interval $k_{ip} < k < k^*$. This means that k_{ip} is an unstable fixed point, while k^* is stable. The presence of the unstable fixed point in $k_{ip} > 0$ is precisely what makes this production function different from the neoclassical ones: the economies with starting level of capital to the left of k_{ip} cannot escape the poverty trap: the dynamics brings them back to the origin inexorably. This is not the case for a neoclassical production function, where a small but positive initial level of capital is enough to guarantee a positive economic growth.

In the remainder of the this paper, when deepening our study of the model 1 - 2 and highlighting the interaction between the economic and the environmental dynamics, the just introduced concept of "poverty trap" will be enriched by the coupled dynamics of pollution. More precisely, the trap will be characterized by a low level of economic performance, idealized by a long run value of k asymptotically approaching zero -in accordance to Skiba's idea- but the trap's occurrence will actually be heavily influenced by the dynamics of pollution, hence the definition of "pollution-induced poverty trap". In this regard the definition of pollution-induced poverty traps captures the presence of the environmental module in the model, but it does not refer to a poverty-environment trap in which the quality of both the environment and the economic performance are unsatisfactory.

2.2 Local analysis. Existence of multiple equilibria

Relaxing the hypothesis $b = 0$, and putting the $\mathcal{S}$-shaped production in, the system 1 - 2 becomes :

$$\dot{k_t} = \frac{s(1-u)\alpha k^2}{(1+bp)(1+\alpha k^2)} - \delta_k k. \tag{5}$$

$$\dot{p_t} = \frac{\theta(1-u)\alpha k^2}{1+\alpha k^2} - \delta_p p. \tag{6}$$

where we have assumed, without loosing generality, that $\epsilon = 1$, $q = 2$ and $\alpha_1 = \alpha_2 = \alpha$. A steady state (k^*, p^*) of our system is any solution that satisfies:

$$0 = \frac{s(1-u)\alpha k^2}{(1+bp)(1+\alpha k^2)} - \delta_k k. \tag{7}$$

$$0 = \frac{\theta(1-u)\alpha k^2}{1+\alpha k^2} - \delta_p p. \tag{8}$$

The vector of parameters $\omega = (\alpha, b, \theta, \delta_k, \delta_p, s, u)$ lives inside the parameter space $\Omega = \mathbb{R}^3_+ \times (0,1)^4$. First we notice that the origin is an equilibrium point $\forall \omega \in \Omega$. Origin excluded, system 7 - 8 amounts to a second order algebraic equation in k: divide both equations by k and plug $p(k)$ from 8 in 7. In order to fully characterize Ω, we define the discriminant of this algebraic equation:

$$\Delta = \alpha^2 {\delta_p}^2 s^2 (1-u)^2 - 4\,\alpha\, b\, {\delta_k}^2 \delta_p\, \theta\, (1-u) - 4\,\alpha\, {\delta_k}^2 {\delta_p}^2. \tag{9}$$

The next Proposition identifies two different regions of the parameter space Ω, depending on Δ being greater than or equal to zero, respectively. We are not interested in the $\Delta < 0$ case, because complex conjugate equilibria have not economic meaning.

Proposition 1. *The origin is a steady states $\forall \omega \in \Omega$. Let $\Omega_1 \equiv \{\omega \in \Omega : \Delta > 0\}$. Then if $\omega \in \Omega_1$ there are two additional real steady states in $\mathbb{R}^2_{++}$. Let $\Omega_2 \equiv \{\omega \in \Omega : \Delta = 0\}$. Then if $\omega \in \Omega_2$ there are two additional coincident steady states (coalescence) in $\mathbb{R}^2_{++}$.*

Proof. See Appendix A. □

Going into the details, the steady states in Ω_1 are therefore:

$$k_{eq} = 0 \tag{10}$$

$$p_{eq} = 0 \tag{11}$$

$$k_{th} = \frac{1}{2} \frac{\alpha\, \delta_p\, s(1-u) - \sqrt{\Delta}}{\alpha\, \delta_k\, \left(b\, \theta\, (1-u) + \delta_p\right)}. \tag{12}$$

$$p_{th} = \frac{\theta\,(1-u)\,\alpha\, {k_{th}}^2}{\left(\alpha\, {k_{th}}^2 + 1\right) \delta_p}. \tag{13}$$

$$k^{eq} = \frac{1}{2} \frac{\alpha\, \delta_p\, s(1-u) + \sqrt{\Delta}}{\alpha\, \delta_k\, \left(b\, \theta\, (1-u) + \delta_p\right)}. \tag{14}$$

$$p^{eq} = \frac{\theta\,(1-u)\,\alpha\, k^{eq2}}{\left(\alpha\, k^{eq2} + 1\right) \delta_p}. \tag{15}$$

2.2.1 Stability Analysis

The local stability properties of a nonlinear system are described in terms of the sign of the invariants of the associated jacobian matrix, $\boldsymbol{J}$, namely the trace, $\boldsymbol{Tr(J)}$, and the determinant, $\boldsymbol{Det(J)}$, evaluated at a hyperbolic equilibrium points $\boldsymbol{J^*} = \boldsymbol{J}(k^*, p^*)$.

Proposition 2. *Let $\omega \in \Omega_1$. The origin is a locally stable steady state. The addtional steady states are a saddle, (k_{th}, p_{th}), and non-saddle, (k^{eq}, p^{eq}).*

Proof. See Appendix A. □

The interpretation of the origin as an attractive fixed point of the system 5 - 6 is that it represents the engine of the pollution-induced poverty trap. All the trajectories whose initial conditions are in the origin's range of influence will be eventually drawn to a level of zero capital, or, in other words, to an economic pollution-induced collapse. A part from the peculiar and very unlikely case of initial condition exactly on the stable manifold of the saddle (k_{th}, p_{th}), no economy-environment system can asymptotically reach the this fixed point. We are then interested in the initial conditions of (k, p) that guarantee a dynamics to a "sustainable steady state", a situation in which a satisfactory level of economic development coexist with a positive amount of pollution. These initial conditions converge to the point (k^{eq}, p^{eq}) as long as this non-saddle maintains its stability, or to a stable limit cycle (as we will show in section 3). So we define the equilibrium (k^{eq}, p^{eq}), reported in equations 14 and 15, as the sustainable steady state or sustainable outcome.

2.2.2 Comparative statics

In this subsection we briefly present a comparative statics analysis with respect to s and u , in the hypothesis that $\omega \in \Omega_1$. We display the analysis of the k variable only, being the variable p proportional to k at the steady state, as evident from equations 13 and 15. We will focus on the opposite effects that marginal changes in the parameters have on the steady states (k_{th}, p_{th}) and (k^{eq}, p^{eq}).

$$\frac{\partial k_{th}}{\partial s} = \frac{1}{2}\,\frac{\delta_p\,(1-u)\left(-\alpha\,\delta_p s\,(1-u)+\sqrt{\Delta}\right)}{\sqrt{\Delta}\delta_k\,(b\theta\,(1-u)+\delta_p)}, \tag{16}$$

$$\frac{\partial k^{eq}}{\partial s} = \frac{1}{2}\,\frac{\delta_p\,(1-u)\left(\alpha\,\delta_p s\,(1-u)+\sqrt{\Delta}\right)}{\sqrt{\Delta}\delta_k\,(b\theta\,(1-u)+\delta_p)}, \tag{17}$$

$$\frac{\partial k_{th}}{\partial u} = -\frac{1}{2}\,\frac{\delta_p\left(-2\,b^2{\delta_k}^2\theta^2\,(1-u)-\alpha\,{\delta_p}^2 s^2\,(1-u)-2\,b{\delta_k}^2\delta_p\theta+\sqrt{\Delta}\delta_p s\right)}{\sqrt{\Delta}\delta_k\,(b\theta\,(1-u)+\delta_p)^2}, \tag{18}$$

$$\frac{\partial k^{eq}}{\partial u} = -\frac{1}{2}\,\frac{\delta_p\left(2\,b^2{\delta_k}^2\theta^2\,(1-u)+\alpha\,{\delta_p}^2 s^2\,(1-u)+2\,b{\delta_k}^2\delta_p\theta+\sqrt{\Delta}\delta_p s\right)}{\sqrt{\Delta}\delta_k\,(b\theta\,(1-u)+\delta_p)^2}, \tag{19}$$

s) With $\omega \in \Omega_1$ we know that both k_{th} and k^{eq} are positive. So the term $\alpha\,\delta_p\,s(1-u) \pm \sqrt{\Delta}$ is positive. This means that $\alpha\,\delta_p\,s(1-u) > \sqrt{\Delta}$. Hence, looking at the partial derivatives in 16 and 17, it is easy to note that the effect of a marginal increase in the saving ratio is negative for k_{th} and positive for k^{eq}: devoting more effort in capital investment enlarges the distance between the two equilibria. The effects on the stock of pollution are similar: in the upper equilibrium (k^{eq}, p^{eq}) the long run level of pollution stock increase as well, underlining that a trade-off between growth and environmental care is at stake in the $\mathcal{S}$-shaped technology scenario: a marginal increase in the saving ratio is always beneficial for the economy, but at the expense of environmental quality.

u) The effect of a marginal increase of the abated emission share u is described in equation 18 and 19. The key term is the paranthesis: $\sqrt{\Delta}s \pm (2\,b^2{\delta_k}^2\theta^2\,(1-u)+\alpha\,{\delta_p}^2 s^2\,(1-u)+2\,b{\delta_k}^2\delta_p\theta)$. Dividing it by s, it is immediate to note that the evaluation of the sign follows the same arguments as the previous case. Abating emissions comes at a cost: reduces the achievments of the economy and exposes to the risk of getting to a low capital cul de sac.

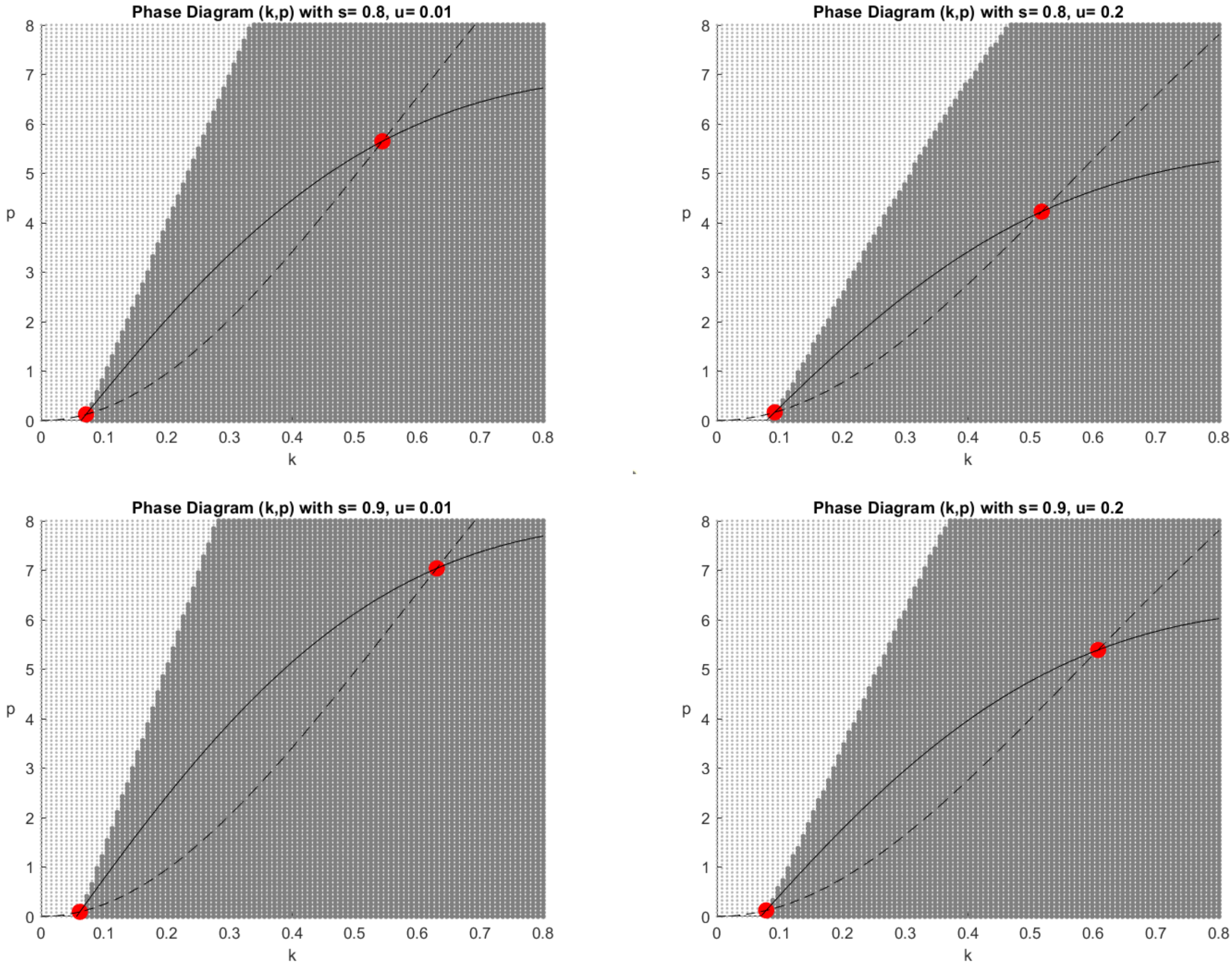

Figure 2: Comparative statics examples

Figure 2 graphically sums up the comparative statics conclusions drawn before. The light grey area represents the basin of attraction of the dire equilibrium ($k_{eq} = 0, p_{eq} = 0$), while the dark grey area represents the basin of attraction of the sustainable outcome (k^{eq}, p^{eq}). The two full circles signal the two non-zero equilibria, and they occur at the intersection of the two nullclines, $\dot{k} = 0$ and $\dot{p} = 0$ depicted with a dashed and a solid line respectively. Let us start our comparison from the upper-left panel: moving downwards means increasing the value of s, holding u constant, and this clearly makes the sustainable outcome's basin of attraction to enlarge; moving right instead means increasing u, keeping s constant, and this causes sustainable outcome's basin of attraction to shrink; moving along the diagonal means increasing both s and u, and the results depends of course on the relative dependence of the system on the two parameters. It is important to remark that the basin of attraction of (k^{eq}, p^{eq}) remains an open subset of $\mathbb{R}^2$ as long as the combination of the two parameters (s, u) does not open the door to a dramatic change in the phase plane, allowing for hopf bifurcation, as we will discuss in the next section. Apart from (s, u) that represent a couple of controls, the parameters we use in the all the simulations of Figure 2 are following:

$$\begin{cases} \delta_k = 0.05, \ \delta_p = 0.04 \\ \theta = 1, \ b = 1, \ \alpha = 1. \end{cases} \tag{20}$$

3 Global analysis.

The local stability analysis has been useful to highlight the essential characteristics of the phase space around each equilibrium point. Moreover, comparative statics gave us insights on how the long run dynamics respond to change in the relevant parameters. The model has nevertheless multiple equilibria and the interaction between them turns out to be crucial for a policy evaluation. In Figure 2 we have numerically shown that the influence the three equilibria have on the trajectories around them creates two different regions of attraction. In order to make the main point of this paper, that is stressing the risk of a pollution-induced poverty trap, we must go beyond the conventional local stability analysis and embrace a global perspective to examine the behaviour of the system 5 - 6 as a whole. This perspective is formalized in the following Lemmas and Theorems.

Lemma 1. *Let $\boldsymbol{J}(k^{eq}, p^{eq})$ be the Jacobian matrix of the system 5 - 6 evaluated at the steady state (k^{eq}, p^{eq}). If $\delta_k \neq \delta_p$, then there exists a value $s = \hat{s}$ for which $\boldsymbol{J}(k^{eq}(s), p^{eq}(s))$ has a pair of pure imaginary eigenvalues.*

Proof. See Appendix B. □

Lemma 2. *Under the hypotheses of Lemma 1 the derivative of the real part of the complex conjugate eigenvalues of $\boldsymbol{J}(k^{eq}(s), p^{eq}(s))$ with respect to s, evaluated at $s = \hat{s}$, is different from zero.*

Proof. See Appendix B. □

Theorem 1. *Assume the hypotheses of Lemma 1 and Lemma 2. Then, there exists a continuous function $s(\mu)$, with $s(0) = \hat{s}$, and for all small enough $\mu \neq 0$ there exists a continuous family of non-constant positive periodic solutions $(k_t^*(\mu), p_t^*(\mu))$ for the dynamical system 5 - 6 which collapses to the stationary point (k^{eq}, p^{eq}) as $\mu \to 0$.*

Proof. The proof follows directly from the Hopf Bifurcation Theorem in Guckenheimer and Holmes. See Appendix B. □

Theorem 2. *In the hypotheses of Theorem 1 the limit cycle Hopf bifurcating from (k^{eq}, p^{eq}) is stable (supercritical Hopf bifurcation).*

Proof. See Appendix B. □

Theorem 3. *The solutions (k_t, p_t) of the system 5 - 6 cannot diverge. Cycles are not possible except around the equilibrium point (k^{eq}, p^{eq}).*

Proof. See Appendix B. □

As a matter of fact the system 5-6 has confined solutions, meaning that the trajectories cannot diverge to $+\infty$ as long as $\omega \in \Omega_1$. Moreover Index theory tells us that the system 5-6 cannot have cycles expect around (k^{eq}, p^{eq}). So if the sustainable equilibrium loses its stability, then either the near trajectories remain forever within a cycle around it or they eventually reach the origin, which is attractive $\forall \omega \in \Omega$. More into the details, adopting the saving ratio s as a bifurcation parameter (but using u, share of abated emissions, tells the same story), we uncover a stable limit cycle Hopf bifurcating around (k^{eq}, p^{eq}) at $s = \hat{s}$. This cycle's basin of attraction redefines the concept of sustainability: at the onset of the Hopf bifurcation (k^{eq}, p^{eq}) loses its stability and ceases to be the sustainable outcome, while it is exactly the region of attraction of this limit cycle that takes on the "sustainable" role. The stable limit cycle is the consequence of a Hopf bifurcation taking place for a certain value of the saving ratio, but it not the only cycle on stage: moving along decreasing value of the bifurcation parameter s, just before the appearance of the Hopf bifurcation's cycle, the basin of (k^{eq}, p^{eq}) shrinks up to a point in which it becomes a closed region, whose borders are delimited by an unstable limit cycle. This radical reshaping of the phase space means that the control's policy choice takes up a paramount importance: not only the vector (s, u) can work as a fine-tuning of the performance of the economy and the environment, as suggested by the comparative statics , but it can even reshape the dynamical landscape of the system in such a way that only initial conditions in the neighborhood of the sustainable outcome actually reach the sustainable equilibrium or a sustainable region around a limit cycle. In other words, a series of consequences emerge taking together Lemmas and Theorems. The system 5-6 has three equilibria. The origin is attractive $\forall\omega \in \Omega_1$, (k_{th}, p_{th}) is a saddle $\forall\omega \in \Omega_1$ and $det(\boldsymbol{J}(k^{eq}, p^{eq})) > 0 \ \forall\omega \in \Omega_1$. The trajectories cannot diverge, so they must reach one of the attractors: the origin, the sustainable equilibrium (k^{eq}, p^{eq}) (when $\boldsymbol{Tr}(\boldsymbol{J}(k^{eq}, p^{eq})) < 0$), and possibly an attractive cycle around (k^{eq}, p^{eq}).

3.1 Numerical examples

In this section we perform three different numerical examples, each of them with exactly the same set of parameters except for the bifurcation one, s, that takes smaller and smaller values. Next we report the parameter's choice:

$$\begin{cases} \delta_k = 0.05, \ \delta_p = 0.04 \\ \theta = 1, \ b = 1, \ \alpha = 1, \ u = 0.01 \\ s_1 = 0.59, \ s_2 = s = \hat{s} = 0.585, \ s_3 = 0.58. \end{cases} \tag{21}$$

In the first example the bifurcation parameter s takes a value slightly larger than the bifurcation one. The sustainable outcome's stationary point, (k^{eq}, p^{eq}) is stable, but its basin of attraction is now a closed region inside an unstable limit cycle, which is pictured as a black solid curve in Figure 3. There has been a major reshaping of the phase plane with respect to the example shown in the upper left panel of Figure 2: the only difference between the two simulations resides in the value of the saving ratio, decreased from $s = 0.8$ to $s = 0.59$. In particular, the left panel of Figure 3 depicts the basin of attraction of (k^{eq}, p^{eq}) in dark grey, while the right panel shows two representative trajectories, one inside and one outside the unstable limit cycle. The initial condition $IC1$ gives rise to a (red) trajectory whose final destination is the origin: starting from $IC1$ the point $(k(t), p(t))$ spirals out of the limit cycle and eventually reach the pollution-induced poverty trap's sink. The trajectory starting from $IC2$, depicted in light blue, spirals down to (k^{eq}, p^{eq}), demonstrating that initial conditions inside the limit cycle eventually reach the sustainable outcome. The second numerical example deals with the emergence of another limit cyle, this time a stable one, as a consequence of a Hopf bifurcation occurring at $s = 0.585$, as shown in Figure 4. The left panel of this picture depicts a (red) trajectory starting from a initial point just outside the unstable limit cycle: as before, initial conditions outside it give rise to trajectories that eventually reach the origin. It has to be pointed out that the region inside this unstable limit cycle is now considerably smaller than before, as it is possible to notice comparing the left panels of Figure 3 and Figure 4, which have the same scale on the axes. But it is possible to better appreciate the major differences between the two scenarios observing the right panel of Figure 4. Here it is shown a blow-up of the picture on the left, centered on the point (k^{eq}, p^{eq}). Again, two representative trajectories are depicted, but this time both of them have starting point inside the unstable limit cycle. The initial condition $IC1$ generates a trajectory (in light blue) that slowly spirals down to another limit cycle (solid back curve), the stable one created by the Hopf bifurcation. In contrast, $IC2$ is an initial condition inside the Hopf cycle, and, as predicted by Theorem 2, gives rise to a trajectory converging to to this cycle: (k^{eq}, p^{eq}) has become unstable, but still a region of sustainability persists.

The last numerical example refers to a scenario in which the saving ratio is slightly smaller than the bifurcation value. As shown in the left panel of Figure 5, now the phase plane has been completely taken

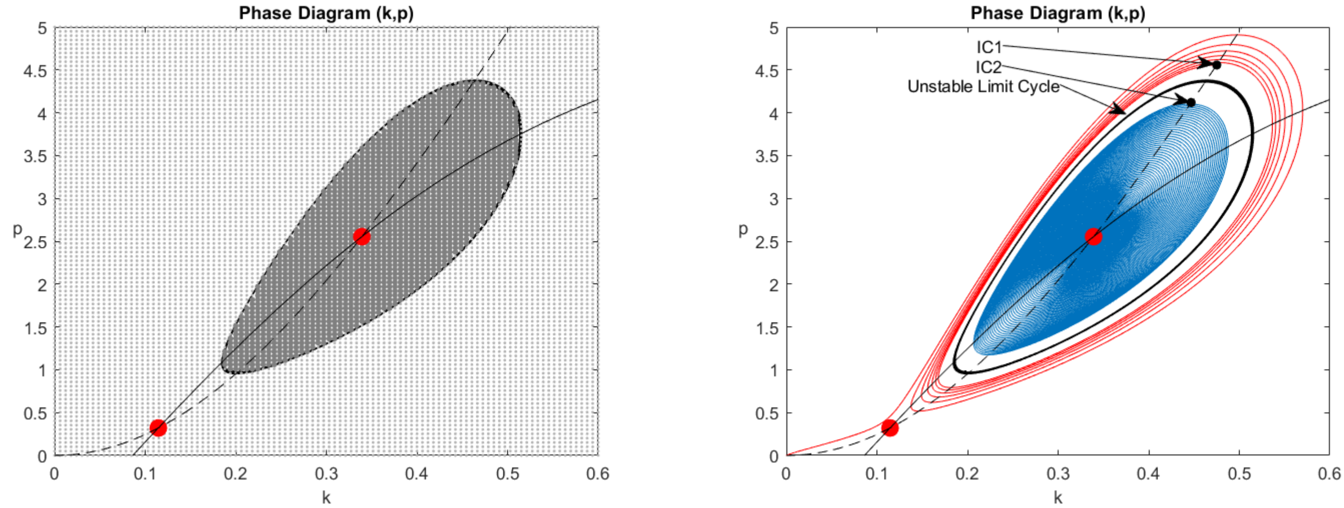

Figure 3: Unstable Limit Cycle, $s_1 = 0.59$

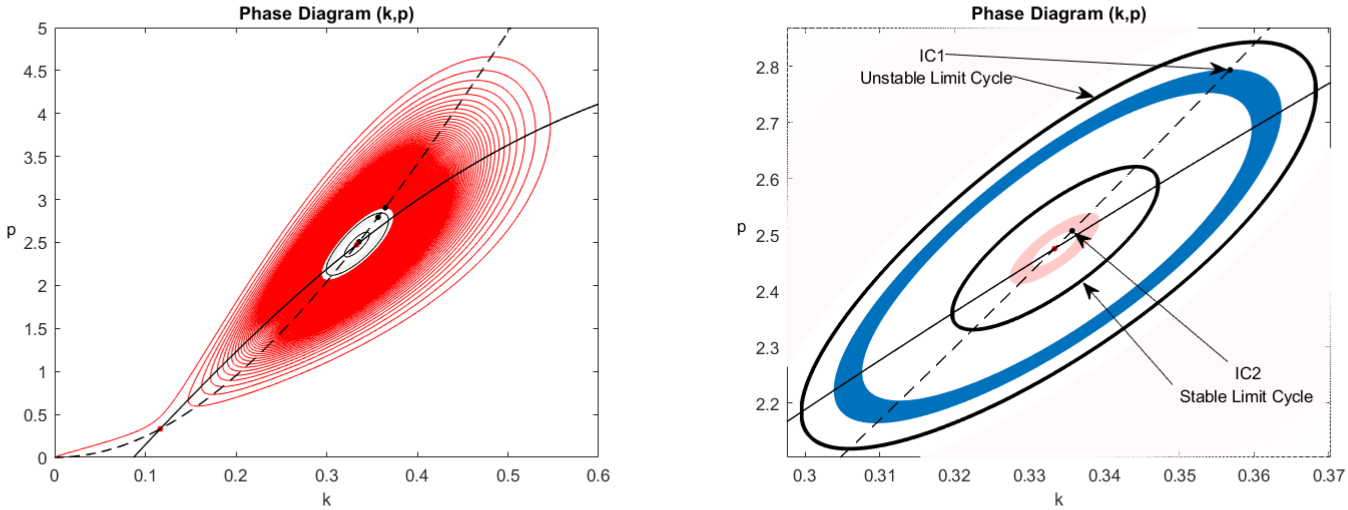

Figure 4: Hopf Bifurcation, $s_2 = \hat{s}$

over by the attraction force of the origin, the pollution-induced poverty trap's engine. On the right panel, a trajectory with the same initial condition as $IC2$ in the Hopf scenario is depicted. Summing up, the story of decreasing the saving ratio - and it can be shown that the same happens decreasing the share of abated emission - goes this way: the smaller is the saving ratio, the smaller is the basin of attraction of the sustainable equilibrium (k^{eq}, p^{eq}). The main snapshots are the following, as s decreases. First this desirable equilibrium has a open subset of the phase space as region of attraction. Then the basin becomes a closed elliptic-like region. From this point on the basin shrinks, but it does not boil down to a point with a continuous deformation: the shrinking stops when the Hopf bifurcation steps in at $\hat{s}$. After that the "safe" region disappears altogether. In this regard it is important to underline that the initial conditions that are the closest to the sustainable outcome, are exactly the ones that could suffer an abrupt change in destiny, given that the disappearing of the sustainable basin of attraction is not incremental, but happens suddenly. They still have a warning sign though: when the cyclical behavior of the economic and environmental variables set in, the regime change is around the corner.

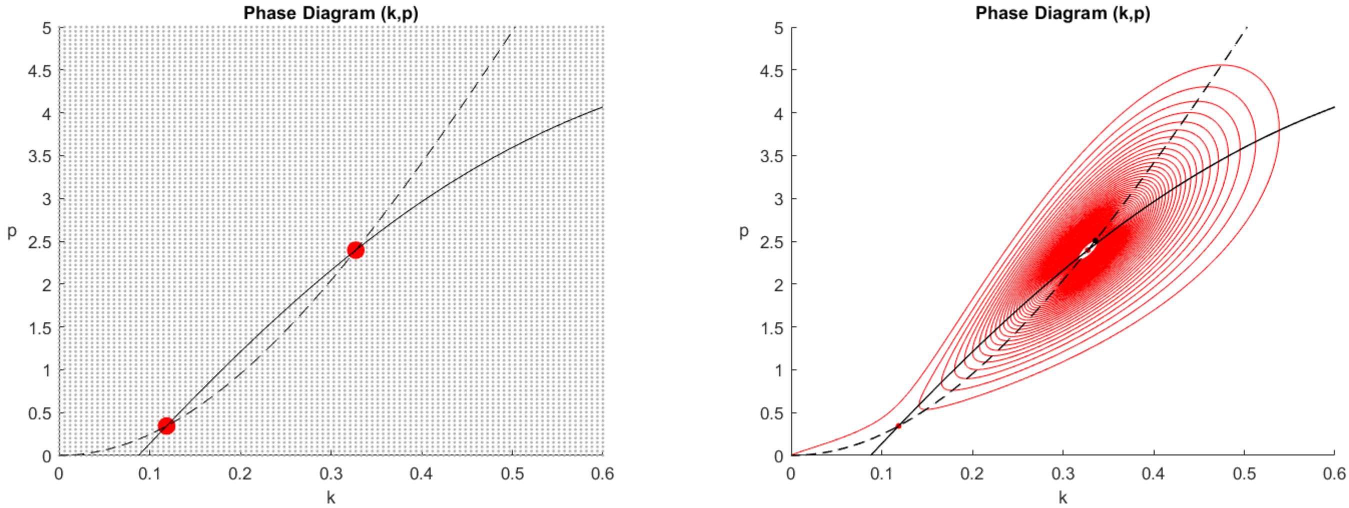

Figure 5: Collapse, $s_3 = 0.58$

4 Conclusion

In this paper we propose a minimal integrated model with an $\mathcal{S}$-shaped production function. The model is made up of two nonlinear differential equations, one describing the evolution of capital and the other describing the evolution of pollution. The two equations are coupled: the dynamics of capital accumulation is negatively affected by the presence of a damage function, whose numerical value is proportional to the level of pollution, a by-product of production. The richness of the dynamics is given by the presence of the non-neoclassical production function, that starts convex for small value of capital and then becomes concave after a threshold is trespassed. We study the existence and the stability of equilibria. We show that, on a subset of the parameters space, the system has three distinct equilibria: a stable origin, a saddle and an not-saddle equilibrium, ordered with increasing value of capital. The stable manifold of the saddle behaves as a separatrix between the basin of attraction of the origin and the basin of attraction of the upper steady state, which we call sustainable steady state. We perform a comparative statics analysis and find that the two policy choice parameters, s and u, the saving ratio and the share of abated emission respectively, are crucial to the mutual extension of the two basins of attraction. But there is more. If the saving ratio or the share of abated emissions keep on decreasing, first an unstable limit cycle encloses the region of attraction of the sustainable outcome, then a stable Hopf bifurcation steps in: a stable and an unstable limit cycle coexist that surround the upper steady state and act as a final protection from the pollution-induced poverty trap. An early sign of the imminent disappearing of the safety of the sustainable outcome lies in the kicking of cyclical behavior of the economic and environmental variables. We point out that the choice of the policy parameters, s and u, reveal itself even more crucial when a global analysis approach is performed.

As possible avenues of future research the hypothesis on the linear self-cleaning capacity of the environment δ_p could be dropped. In other words the depreciation rate of pollution, is assumed to be proportional to the stock of pollution itself, $\delta_p p_t$: this is a gross simplification, given that the coefficient δ turns out to be stock dependant actually, that is $\delta_p = \delta_p(p_t)$ (see, for example, Mäler 2003). To think about an example, we can point to the oceans absorption rate of carbon dioxide, that depends strongly on the stock of carbon dioxide itself. In this paper we have analyzed a purely dynamical model: a natural extension could be the introduction of a control layer, with a decision maker in charge of choosing the optimal values of the policy parameters.

A Proof of Proposition 1 and 2.

Existence

We start rewriting the system 7 - 8 in explicit form:

$$0 = \frac{s[1-u]\alpha k(t)^2}{(1+bp(t))(1+\alpha k(t)^2)} - \delta_k k(t). \tag{22}$$

$$0 = \frac{\theta[1-u]\alpha k(t)^2}{1+\alpha k(t)^2} - \delta_p p(t). \tag{23}$$

Substituting p from the second equation into the first and going through some algebra we get:

$$0 = \alpha\,[b\,\theta(1-u)+\delta_p]\,k^2 - s\,(1-u)\,\alpha\,k + \delta_k. \tag{24}$$

$$p = \frac{\theta[1-u]\alpha k(t)^2}{(1+\alpha k(t)^2)\delta_p}. \tag{25}$$

Equation 24 has two real and positive solution as long as the discriminant Δ is greater than zero:

$$k_{th} = \frac{1}{2}\,\frac{\alpha\,\delta_p\,s(1-u) - \sqrt{\alpha^2{\delta_p}^2 s^2(1-u)^2 - 4\,\alpha\,b\,{\delta_k}^2\delta_p\,\theta\,(1-u) - 4\,\alpha\,{\delta_k}^2{\delta_p}^2}}{\alpha\,\delta_k\,\left(b\,\theta\,(1-u)+\delta_p\right)}. \tag{26}$$

$$p_{th} = \frac{\theta\,(1-u)\,\alpha\,{k_{th}}^2}{\left(\alpha\,{k_{th}}^2+1\right)\delta_p}. \tag{27}$$

$$k^{eq} = \frac{1}{2}\,\frac{\alpha\,\delta_p\,s(1-u) + \sqrt{\alpha^2{\delta_p}^2 s^2(1-u)^2 - 4\,\alpha\,b\,{\delta_k}^2\delta_p\,\theta\,(1-u) - 4\,\alpha\,{\delta_k}^2{\delta_p}^2}}{\alpha\,\delta_k\,\left(b\,\theta\,(1-u)+\delta_p\right)}. \tag{28}$$

$$p^{eq} = \frac{\theta\,(1-u)\,\alpha\,k^{eq2}}{\left(\alpha\,k^{eq2}+1\right)\delta_p}. \tag{29}$$

Summing up, if the discriminant is greater than zero, that is if

$$\Delta = \alpha^2{\delta_p}^2 s^2(1-u)^2 - 4\,\alpha\,b\,{\delta_k}^2\delta_p\,\theta\,(1-u) - 4\,\alpha\,{\delta_k}^2{\delta_p}^2 > 0. \tag{30}$$

then the solutions are both real and positive.

Stability

The Jacobian matrix $\boldsymbol{J}$ of the system 5 - 6, before the evaluation of the critical points is:

$$\boldsymbol{J} = \begin{bmatrix} \frac{2\alpha\,ks(1-u)}{(k^2\alpha+1)^2(bp+1)} - \delta_k & -\frac{\alpha\,k^2 s(1-u)b}{(\alpha\,k^2+1)(bp+1)^2} \\ \frac{2\theta\,(1-u)\alpha\,k}{(k^2\alpha+1)^2} & -\delta_p \end{bmatrix} \tag{31}$$

The origin is at the border of the domain, but given that the functions involved are sufficiently regular around it, we can still plug $(k=0, p=0)$ in the previous matrix to get that the origin is a stable equilibrium. Indeed the Jacobian matrix evaluated in (k_{eq}, p_{eq})=(0,0) gives:

$$\begin{bmatrix} -\delta_k & 0 \\ 0 & -\delta_p \end{bmatrix}$$

Now we show that

1) The determinant Jacobian matrix evaluated at (k_{th}, p_{th}) is negative, $\boldsymbol{Det(J(k_{th}, p_{th}))} < 0$: the equilibrium (k_{th}, p_{th}) is a saddle.

2) The determinant Jacobian matrix evaluated at (k^{eq}, p^{eq}) is positive, $\boldsymbol{Det(J(k^{eq}, p^{eq}))} > 0$: the equilibrium (k^{eq}, p^{eq}) is not a saddle. We numericallly show that $\boldsymbol{Tr(J(k^{eq}, p^{eq}))}$ can be both positive and negative in Ω_1, allowing for Hopf-bifurcation.

We start considering (k_{th}, p_{th}). Plugging these coordinates in $\boldsymbol{J}$, we can study the equilibrium properties of this point. The determinant of the Jacobian reads as: is:

$$\begin{aligned}\boldsymbol{Det(J(k_{th}, p_{th}))} &= C_{th}\Big[\alpha\, s\,(1-u)\left(\alpha\, s^2(1-u)^2 - 4\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right) \\ &\quad -\sqrt{\Delta}\left(\alpha\, s^2(1-u)^2 - 2\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right)\Big] \end{aligned} \tag{32}$$

where the positive constant C_{th} is:

$$C_{th} = \frac{2\delta_k\delta_p}{\alpha s(1-u)[\sqrt{\Delta} - \delta_p s(1-u)]^2} \tag{33}$$

We should show that in equation 32 the following happens:

$$\sqrt{\Delta}\left(\alpha\, s^2(1-u)^2 - 2\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right) > \alpha\, s\,(1-u)\left(\alpha\, s^2(1-u)^2 - 4\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right).$$

Observing that

$$\sqrt{\Delta} = \alpha\left(\alpha\, s^2(1-u)^2 - 4\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right).$$

we proceed dividing both member of the equation by $\sqrt{\Delta}$. We get the following semplification:

$$\alpha\, s^2(1-u)^2 - 2\,\delta_k\,[b\theta\ (1-u)+\delta_p] > s\ (1-u)\ \sqrt{\Delta}.$$

Now we divide both members by $\alpha\, s^2(1-u)^2$. After some simple algebra we get:

$$1 - \frac{2\,\delta_k\,[b\theta\ (1-u)+\delta_p]}{\alpha\, s^2(1-u)^2} > \sqrt{1 - \frac{4\,\delta_k\,[b\theta\ (1-u)+\delta_p]}{\alpha\, s^2(1-u)^2}}.$$

Now we do following scaling:

$$t = \frac{\delta_k\,[b\theta\ (1-u)+\delta_p]}{\alpha\, s^2(1-u)^2}.$$

The previous inequality becomes:

$$1 - 2t > \sqrt{1-4t}.$$

Taking the square of both members it is easy to show that the last inequality is true whenever $0 < t < 1/4$. Reversing the substitution we get:

$$0 < \frac{\delta_k\,[b\theta\ (1-u)+\delta_p]}{\alpha\, s^2(1-u)^2} < \frac{1}{4}. \tag{34}$$

If we now compare the expression of Δ to the last inequality, we find that the condition on the existence of two solutions, namely $\Delta > 0$, implies 34. So if the solutions exist, then the determinant of the Jacobian matrix evaluated at the equilibrium (k_{th}, p_{th}) is negative: the point is a saddle.

Now it is the turn of (k^{eq}, p^{eq}). The Jacobian matrix $\boldsymbol{J}$ evaluated at (k^{eq}, p^{eq}), has the following determinant:

$$\begin{aligned}\boldsymbol{Det(J(k^{eq}, p^{eq}))} &= C^{eq}\Big[\alpha\, s\,(1-u)\left(\alpha\, s^2(1-u)^2 - 4\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right) \\ &\quad +\sqrt{\Delta}\left(\alpha\, s^2(1-u)^2 - 2\,\delta_k\,[b\theta\ (1-u)+\delta_p]\right)\Big] \end{aligned} \tag{35}$$

where the positive constant C^{eq} is:

$$C^{eq} = \frac{2\delta_k \delta_p}{\alpha s(1-u)[\sqrt{\Delta + \delta_p s(1-u)}]^2} \tag{36}$$

It easy to see that the determinant is positive. Indeed, the term

$$\alpha\, s\, (1-u)\left(\alpha\, s^2(1-u)^2 - 4\,\delta_k\, [b\theta\ (1-u) + \delta_p]\right)$$

on the right-hand side of equation 35, is positive because coincides with $\frac{s(1-u)}{\delta_p}\Delta$. While the term

$$\left(\alpha\, s^2(1-u)^2 - 2\,\delta_k\, [b\theta\ (1-u) + \delta_p]\right)$$

that multiplies $\sqrt{\Delta}$ in 35, is positive because coincides with $\Delta + 2\,\delta_k\, [b\theta\ (1-u) + \delta_p]$. In conclusion the determinant of the Jacobian matrix $\boldsymbol{J}$ evaluated at (k^{eq}, p^{eq}) is positive: the point is not a saddle. QED

B Proof of Lemmas 1 and 2 and Theorems 1, 2 and 3.

We now go through the Lemmas and Theorems of Section 3. We will show that 1) solutions cannot diverge and 2) closed orbits are not possible except around (k^{eq}, p^{eq}), 3. Immediately after, we deal with Lemma 1 and Lemma 2 showing that 3) a super-critical Hopf-bifurcation occurs and a stable limit cycle emerges as a consequence, Theorems 1 and 2.

- The solutions $(k(t), p(t))$ of the system 5 - 6 cannot diverge. Below are equations 5 and 6:

$$\dot{k_t} = \frac{s[1-u]\alpha k(t)^2}{(1+bp(t))(1+\alpha k(t)^2)} - \delta_k k(t). \tag{37}$$

$$\dot{p_t} = \frac{\theta[1-u]\alpha k(t)^2}{1+\alpha k(t)^2} - \delta_p p(t). \tag{38}$$

We want to compare 37 and 38 to the following system of two linear non homogeneous equations:

$$\dot{\tilde{k_t}} = s[1-u] - \delta_k k(t). \tag{39}$$

$$\dot{\tilde{p_t}} = \theta[1-u] - \delta_p p(t). \tag{40}$$

The above linear system has a globally stable equilibrium in $\left(\frac{s[1-u]}{\delta_k}, \frac{\theta[1-u]}{\delta_p}\right)$. It's immediate to see that, $\forall t \geq 0$ holds the following:

$$\dot{\tilde{k_t}} \geq \dot{k_t}. \tag{41}$$

$$\dot{\tilde{p_t}} \geq \dot{p_t}. \tag{42}$$

With well-known comparison theorem applied to 41 and 42, starting from the same initial conditions $(k(0), p(0))$ we can conclude that $\forall t \geq 0$:

$$k_t \leq \tilde{k_t}. \tag{43}$$

$$p_t \leq \tilde{p_t}. \tag{44}$$

From 43 and 44 follows that the solutions of the system 5 - 6 cannot diverge.

- As for the existence of cycles, we know from index theory that any closed orbit in the phase plane must enclose fixed points whose indeces sum up to +1. For the uniqueness of the solutions, we cannot enclose the origin. Ruling out cycles that do not include any fixed points, three possibilities emerge: a cycle around the saddle (k_{th}, p_{th}), a cycle around the non-saddle (k^{eq}, p^{eq}), or a cycle that include both the saddle and the non-saddle. A cycle around the saddle contradicts the theory, because the index number of a saddle is -1, and a cycle embracing the saddle and the non-saddle would have index number $I = 0$. The only possibility left is a cycle around (k^{eq}, p^{eq}).

- The Hopf bifurcation occurs when two complex conjugates eigenvalues smoothly cross the imaginary axis for some value of the bifurcation parameter. We choose the saving rate s as bifurcation parameter, and we call $s = \hat{s}$ the bifurcation value (the same could be done for the other policy parameter, the abated emissions share u). Clearly the Jacobian matrix evaluated at the equilibrium point (k^{eq}, p^{eq}) depends on the bifurcation parameter:

$$\boldsymbol{J(k^{eq}, p^{eq})} = \boldsymbol{J(k^{eq}(s), p^{eq}(s))}. \tag{45}$$

In order to discuss the existence of the Hopf bifurcation at $s = \hat{s}$, we follow closely the Hopf bifurcation theorem as proposed by Lorenz 1993. As shown in the Stability subsection above, $\boldsymbol{Det(J(k^{eq}, p^{eq}))} > 0$ in Ω_1, so the implicit function theorem ensures the existence of the two smooth functions $(k^{eq}(s), p^{eq}(s))$ in the neighborhood of $\hat{s}$. When applied to two dimensions, the theorem states that if

- the Jacobian matrix evaluated at $s = \hat{s}$, i.e. $\boldsymbol{J(k^{eq}(\hat{s}), p^{eq}(\hat{s}))}$, has a pair of purely imaginary eigenvalues, that is $Re[\lambda(\hat{s})] = 0$ and
- the derivative of real part of the eigenvalues is different from zero when the bifurcation parameter hits the bifurcation point, that is $\frac{dRe[\lambda(s)]}{ds} \neq 0$ at $s = \hat{s}$

then there exists some periodic solutions bifurcating from $(k^{eq}(s), p^{eq}(s))$ at $s = \hat{s}$. In other words, the first conditions of the theorem guarantees that at $s = \hat{s}$ the two eigenvalues are on the imaginary axis and the second that they are not moving on the imaginary axis, but across it. From the characteristic equation the eigenvalues can be generally expressed in terms of $\boldsymbol{Det(J)}$ and $\boldsymbol{Tr(J)}$ as follows:

$$\lambda_{1,2} = \frac{\boldsymbol{Tr(J)} + \sqrt{\boldsymbol{Tr(J)}^2 - 4\boldsymbol{Det(J)}}}{2}. \tag{46}$$

Thanks to equation 46 we can express the theorem's conditions on the eigenvalues in an alternative way: at the bifurcation value we must have that

$$\boldsymbol{Tr(J}(k^{eq}(s = \hat{s}), p^{eq}(s = \hat{s}))) = 0. \tag{47}$$

$$\frac{1}{2}\frac{d\boldsymbol{Tr(J(k^{eq}(s = \hat{s}), p^{eq}(s = \hat{s})))}}{ds} \neq 0. \tag{48}$$

Equations 47 and 48 tell us that in the neighborhood of $s = \hat{s}$ the sign of real part of the eigenvalues is determined by the sign of the trace $\boldsymbol{Tr(J(k^{eq}, p^{eq}))}$: if the trace is negative the eigenvalues have negative real part (stable focus); while if the trace is positive the real part is positive (unstable focus). The bifurcation occurs when the trace becomes zero, at $s = \hat{s}$. Pluggin (k^{eq}, p^{eq}) from 28 - 29 into 31, equations 47 and 48 give us the bifurcation value and the associated derivative:

$$\hat{s} = \frac{(b\,\theta\;(1-u)\,(\delta_k - \delta_p) + 2\,\delta_k\,\delta_p)\,\delta_k}{\sqrt{\alpha\;({\delta_k}^2 - {\delta_p}^2)\,(1-u)\,\delta_p}}. \tag{49}$$

$$\frac{dRe[\lambda(s = \hat{s})]}{ds} = \frac{1}{2}\frac{\delta_p\,(1-u)\,\sqrt{\alpha}\,({\delta_k}^2 - {\delta_p}^2)^{3/2}}{(b\,\theta\;(1-u)\,(\delta_p - \delta_k) + 2\,{\delta_p}^2)\,{\delta_k}^2}. \tag{50}$$

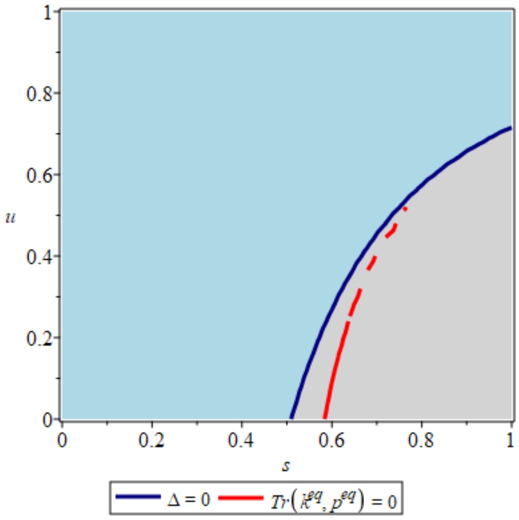

Figure 6:

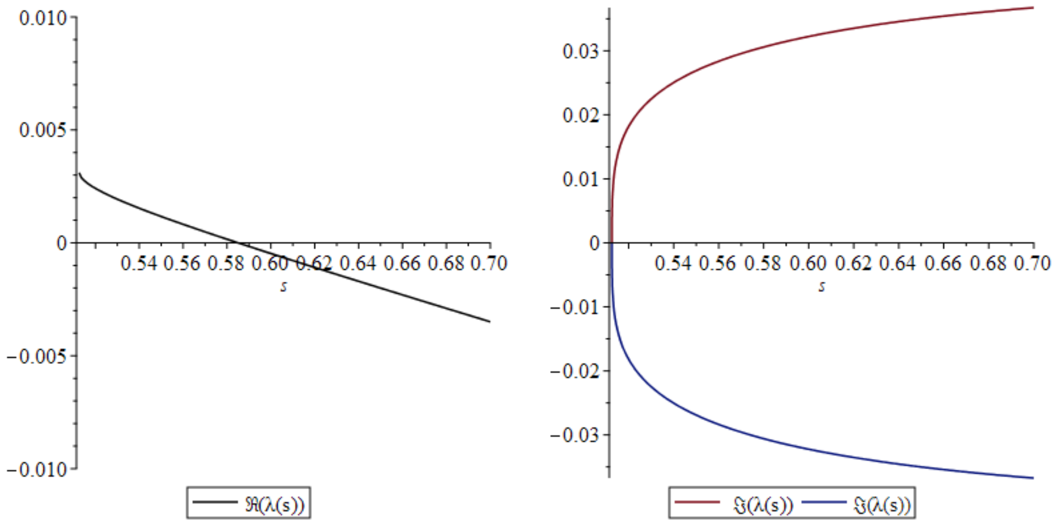

Figure 7:

Studying the results reported in equations 49 and 50 we can conclude that if $\hat{s} \in \Omega_1$ and $\delta_k \neq \delta_p$ than the Hopf bifurcation occurs at $\hat{s}$ around (k^{eq}, p^{eq}). In the remainder we numerically show that tuning the bifurcation parameter s it is possible move the complex conjugate eigenvalues across the imaginary axis at $s = \hat{s}$, where a subcritical Hopf-bifurcation occurs. As anticipated before, complex conjugates eigenvalues that cross the imaginary axis are the "signature" of Hopf-bifurcation. In Figure 6 we show the plane (s, u), with $\alpha = 1$, $b = 1$, $\theta = 1$, $\delta_k = 0.05$ and $\delta_p = 0.04$. The gray region that lies below the blue curve "$\Delta = 0$" represents Ω_1: two distinct equilibria exist, as discussed in section 2.2. Above the red curve "$\boldsymbol{Tr(k^{eq}, p^{eq})} = 0$" the trace is negative, while below the trace is positive. Setting, for example, $u = 0.01$, the eigenvalues at (k^{eq}, p^{eq}) for $s = 0.6$ and $s = 0.58$ are reported in the table below. The bifurcation occurs at $s = \hat{s} = 0.5850168348$, as it is possible to observe in the pictures of Figure 7 where the real part (left) and the imaginary part (right) of the eigenvalues are depicted.

Saving rate s	Eigenvalues
0.6	$-0.4729239105 \cdot 10^{-3} \pm 0.3223957334 \cdot 10^{-1}$I
0.58	$0.1608291087 \cdot 10^{-3} \pm 0.3059286732 \cdot 10^{-1}$I

Only the region with $s \in \Omega_1$ is shown: with this parameters' setting, the minimum value for s is $s = 0.5125705012$. Slightly decreasing the saving rate from $s = 0.6$ to $s = 0.58$ brings the complex conjugate eigenvalues to cross the imaginary axis: at $s = \hat{s}$ the real part changes sign, while the two imaginary parts are obviously symmetric with respect to the s axis. The value of the derivative is $\frac{dRe[\lambda(s=\hat{s})]}{ds} = -0.03191641792$.

As for the check on the stability properties of the Hopf's limit cycle, we follow the procedure described in Lorenz 1993, with some details from Guckenheimer et al. 1983 and Kuznetsov 2004. There are four main steps to take: 1) the system has to be centered in the stationary point; 2) the linear and the non-linear part of the centered system have to be separated; 3) a change of variables has to be performed in order to obtain the normal form of the Hopf bifurcation; 4) finally, a well-known formula applied to the nonlinear part of the normal form provides the answer about the cycle's stability. We start centering the system 5 - 6 around the equilibrium (k^{eq}, p^{eq}) and the bifurcation value $\hat{s}$, introducing the new variables

$$\tilde{k}_t = k_t - k^{eq} \quad (51)$$

$$\tilde{p}_t = p_t - p^{eq} \quad (52)$$

$$\tilde{s} = s - \hat{s} \quad (53)$$

The system 5 - 6 becomes:

$$\dot{\tilde{k}}_t = \frac{\alpha(s+\hat{s})(1-u)(\tilde{k}_t + k^{eq})^2}{(1+b(\tilde{p}_t + p^{eq}))(1+\alpha(\tilde{k}_t + k^{eq})^2)} - \delta_k(\tilde{k}_t + k^{eq}). \quad (54)$$

$$\dot{\tilde{p}}_t = \frac{\alpha\theta(1-u)(\tilde{k}_t + k^{eq})^2}{1+\alpha(\tilde{k}_t + k^{eq})^2} - \delta_p(\tilde{p}_t + p^{eq}). \quad (55)$$

System 54 - 55 has a stationary point in the origin, and a bifurcation for $\tilde{s} = 0$. A first order Taylor expansion on $\left(\tilde{k}, \tilde{p}, \tilde{s}\right)$ allows the system to be rephrased in matrix form as:

$$\begin{bmatrix} \dot{\tilde{k}} \\ \dot{\tilde{p}} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \tilde{k} \\ \tilde{p} \end{bmatrix} + \begin{bmatrix} F_1\left(\tilde{k}, \tilde{p}, \tilde{s}\right) \\ F_2\left(\tilde{k}, \tilde{p}, \tilde{s}\right) \end{bmatrix} \quad (56)$$

The matrix $\mathbf{J}$ contains the first order terms, while the vector $\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ groups all the nonlinear terms. In particular:

$$\mathbf{J} = \begin{bmatrix} -\frac{p^{eq}\,\delta_k\left(\alpha\,{k^{eq}}^2+1\right)^2 b+\alpha^2 {k^{eq}}^4\delta_k+2\,\alpha\,{k^{eq}}^2\delta_k+2\,\alpha\,k^{eq}\,(s+\hat{s})(-1+u)+\delta_k}{\left(\alpha\,{k^{eq}}^2+1\right)^2(bp^{eq}+1)} & \frac{b\alpha\,{k^{eq}}^2(s+\hat{s})(-1+u)}{\left(\alpha\,{k^{eq}}^2+1\right)(bp^{eq}+1)^2} \\ -2\,\frac{\alpha\,(-1+u)\theta\,k^{eq}}{\left(\alpha\,{k^{eq}}^2+1\right)^2} & -\delta_p \end{bmatrix} \quad (57)$$

$$\mathbf{F} = \begin{bmatrix} \frac{\alpha\left(\tilde{k}+k^{eq}\right)^2(s+\hat{s})(1-u)}{\left(\alpha\left(\tilde{k}+k^{eq}\right)^2+1\right)(b(\tilde{p}+p^{eq})+1)} - \delta_k\left(\tilde{k}+k^{eq}\right) + \\ \frac{\tilde{k}\left(p^{eq}\,\delta_k\left(\alpha\,{k^{eq}}^2+1\right)^2 b+\alpha^2 {k^{eq}}^4\delta_k+2\,\alpha\,{k^{eq}}^2\delta_k+2\,\alpha\,k^{eq}\,(s+\hat{s})(-1+u)+\delta_k\right)}{\left(\alpha\,{k^{eq}}^2+1\right)^2(bp^{eq}+1)} - \\ \frac{\alpha\,{k^{eq}}^2(s+\hat{s})(-1+u)b\tilde{p}}{\left(\alpha\,{k^{eq}}^2+1\right)(bp^{eq}+1)^2} \\ \frac{\theta\,(1-u)\alpha\left(\tilde{k}+k^{eq}\right)^2}{\alpha\left(\tilde{k}+k^{eq}\right)^2+1} - (\tilde{p}+p^{eq})\,\delta_p + 2\,\frac{\theta\,(-1+u)\alpha\,k^{eq}\,\tilde{k}}{\left(\alpha\,{k^{eq}}^2+1\right)^2} + \tilde{p}\delta_p \end{bmatrix} \quad (58)$$

Where k^{eq}, p^{eq} and $\hat{s}$ have been defined in equations 28, 29 and 49 respectively. As in Lorenz 1993, we proceed with a coordinate transformation induced by the following transformation matrix $\mathbf{D}$:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{D} \begin{bmatrix} \tilde{k} \\ \tilde{p} \end{bmatrix} \tag{59}$$

where

$$\mathbf{D} = \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{-(\mathbf{J}(1,1))^2 - 4\,\mathbf{J}(1,2)\mathbf{J}(2,1) + (\mathbf{J}(2,2))^2}}{2\mathbf{J}(1,2)} & -\frac{\mathbf{J}(1,1) - \mathbf{J}(2,2)}{2\mathbf{J}(1,2)} \end{bmatrix} \tag{60}$$

System 56 can be expressed in terms of the new variables (w_1, w_2) as:

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \mathbf{D^{-1}J} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \mathbf{D^{-1}} \begin{bmatrix} F_1\,(w_1, w_2) \\ F_2\,(w_1, w_2) \end{bmatrix} \tag{61}$$

Now we define

$$\begin{bmatrix} G^1(w_1, w_2) \\ G^2(w_1, w_2) \end{bmatrix} = \mathbf{D^{-1}} \begin{bmatrix} F_1\,(w_1, w_2) \\ F_2\,(w_1, w_2) \end{bmatrix} \tag{62}$$

Finally, as reported in Lorenz 1993, the last step is to define the coefficient μ:

$$\mu = \frac{1}{16}(G^1_{\omega_1\omega_1\omega_1} + G^1_{\omega_1\omega_2\omega_2} + G^2_{\omega_1\omega_1\omega_2} + G^2_{\omega_2\omega_2\omega_2}) + \frac{1}{16\beta}(G^1_{\omega_1\omega_2}(G^1_{\omega_1\omega_1} + G^1_{\omega_2\omega_2})) + \tag{63}$$

$$\frac{1}{16\beta}(G^1_{\omega_2\omega_2}G^2_{\omega_2\omega_2} - G^2_{\omega_1\omega_2}(G^2_{\omega_1\omega_1} + G^2_{\omega_2\omega_2}) - G^1_{\omega_1\omega_1}G^2_{\omega_1\omega_1}) \tag{64}$$

Where β is the imaginary part of the eigenvalues of $\mathbf{J}$. All the previous partial derivative have to be evaluated at $(\omega_1 = 0, \omega_2 = 0)$. The sign of μ decides the stabilty of limit cycle: if $\mu < 0$ the limit cycle is stable and the Hopf bifurcation is super-critical, otherwise, if $\mu > 0$, the limit cycle is unstable and the Hopf bifurcation is sub-critical. In our study case the calculations give

$$\mu = -0.06479284879 \tag{65}$$

so the limit cycles is stable and the Hopf bifurcation is super-critical.
All The analytical expression of the G^i functions, together with their partial derivatives, are omitted for clarity purposes, but remain available from the authors upon request.

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