DISPERSIVE ESTIMATE FOR QUASI-PERIODIC SCHRÖDINGER OPERATORS ON 1-d LATTICES

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Abstract. Consider the one-dimensional discrete Schrödinger operator H_θ:

$$(H_\theta q)_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega)q_n \ , \quad n \in \mathbb{Z} \ ,$$

with $\omega \in \mathbb{R}^d$ Diophantine, and V a real-analytic function on $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. For V sufficiently small, we prove the dispersive estimate: for every $\phi \in \ell^1(\mathbb{Z})$,

$$(1) \qquad \|e^{-\mathrm{i}tH_\theta}\phi\|_{\ell^\infty} \le K_0 \frac{|\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}} \|\phi\|_{\ell^1} \ , \quad \langle t\rangle := \sqrt{1+t^2} \ ,$$

with a and K_0 two absolute constants and ε_0 an analytic norm of V. The estimate holds for every $\theta \in \mathbb{T}^d$.

1. Introduction and main results

Consider the quasi-periodic Schrödinger operator $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined as

$$(2) \qquad (H_\theta q)_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega)q_n \ , \quad n \in \mathbb{Z} \ ,$$

with $V : \mathbb{T}^d \to \mathbb{R}$ an analytic potential, $d \ge 1$, and $\omega \in \mathbb{R}^d$ a Diophantine frequency vector, it is well known that its spectrum, that we shall denote by Σ, is independent of θ. It is also well known that when the potential function V is sufficiently small, the operator H_θ has purely absolutely continuous spectrum (see e.g. [Avi08, Eli92], see also [AD08]) and that for generic potential it is a Cantor set. Furthermore, the time evolution $e^{-\mathrm{i}tH_\theta}$ presents ballistic transport (see [Zha16]).

In the present paper we prove that $e^{-\mathrm{i}tH_\theta}$ also fulfills the ℓ^1-ℓ^∞ dispersive estimate (1). As usual, from this estimate one can deduce Strichartz estimates [KT98] as well as decay and scattering for the small amplitude solutions of the nonlinear Schrödinger equation

$$(3) \qquad \mathrm{i}\dot{q}_n = (H_\theta q)_n \pm |q_n|^{p-1} q_n \ , \quad n \in \mathbb{Z} \ ,$$

provided p is large enough (see e.g. [SK05, KPS09]). Here we concentrate just on initial data in ℓ^1 and dispersive decay in ℓ^∞ and give the result for $p > 5$.

We recall that for the free Schrödinger operator,

$$(4) \qquad (-\Delta q)_n := -(q_{n+1} + q_{n-1}) \ , \quad n \in \mathbb{Z} \ ,$$

the ℓ^1-ℓ^∞ estimate

$$(5) \qquad \|e^{\mathrm{i}t\Delta}\phi\|_{\ell^\infty} \le \frac{C}{\langle t\rangle^{\frac{1}{3}}} \|\phi\|_{\ell^1} \ , \quad \forall\ \phi \in \ell^1(\mathbb{Z}) \ , \quad \langle t\rangle := \sqrt{1+t^2} \ ,$$

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is well known (see [SK05], see also [MP10]). For the operator $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(Hq)_n = -(q_{n+1} + q_{n-1}) + V_n q_n \ , \quad n \in \mathbb{Z} \ ,$$

Pelinovsky-Stefanov [PS08] have shown that

$$\|e^{-\mathrm{i}tH} P_{ac}\phi\|_{\ell^\infty} \le \frac{C}{\langle t\rangle^{\frac{1}{3}}} \|\phi\|_{\ell^1} \ , \quad \forall \ \phi \in \ell^1(\mathbb{Z}) \ , \tag{6}$$

for "generic"[1] potentials V_n decaying sufficiently fast at infinity. Here P_{ac} denotes the projection on the absolutely continuous part of the spectrum. For other related works, one can refer to [KKK06, KPS09, CT09, Bam13, EKT15].

In all these examples the continuous spectrum is the union of disjoint intervals. We emphasize that our result is the first one in which the continuous spectrum is a Cantor set.

In order to state precisely our main theorem we need a few preliminaries.

Definition 1.1. *A vector $\omega \in \mathbb{R}^d$ will be said to be Diophantine if $\exists \ \gamma > 0$ and $\tau > d-1$, s.t,*

$$\inf_{j\in\mathbb{Z}} |\langle k, \omega\rangle - j\pi| > \frac{\gamma}{|k|^\tau} \ , \quad \forall \ k \in \mathbb{Z}^d \setminus \{0\} \ , \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{R}^d$.

We will assume that there exists a positive r s.t. the potential extends to a bounded complex analytic function on $|\Im\theta| < r$. We will denote

$$\varepsilon_0 := |V|_r := \sup_{|\Im\theta|<r} |V(\theta)| \ . \tag{8}$$

We will also denote this class of functions by $\mathcal{C}^\omega_r(\mathbb{T}^d)$.

Our main result is the following theorem.

Theorem 1.2. *There exists $\varepsilon_* = \varepsilon_*(r,\gamma,\tau,d) > 0$ and two absolute constants a, $K_0 > 0$ such that if $\varepsilon_0 < \varepsilon_*$, then for any $\theta \in \mathbb{T}^d$, any $t \in \mathbb{R}$, the following estimate holds:*

$$\|e^{-\mathrm{i}tH_\theta}\phi\|_{\ell^\infty} \le K_0 \frac{|\ln\varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}} \|\phi\|_{\ell^1} \ , \quad \forall \ \phi \in \ell^1(\mathbb{Z}) \ . \tag{9}$$

It is immediate to get the following

Corollary 1. *Assume $\varepsilon_0 < \varepsilon_*$, with ε_* as in Theorem 1.2, then given any $0 < \zeta < \frac{1}{3}$, there exists $K_1 = K_1(\varepsilon_0, \zeta)$ s.t. for any $\theta \in \mathbb{T}^d$, any $t \in \mathbb{R}$,*

$$\|e^{-\mathrm{i}tH_\theta}\phi\|_{\ell^\infty} \le \frac{K_1}{\langle t\rangle^\zeta} \|\phi\|_{\ell^1} \ , \quad \forall \ \phi \in \ell^1(\mathbb{Z}) \ . \tag{10}$$

One also has the following standard corollary on the ℓ^∞ decay of the solution of (3) with $p > 5$ and small ℓ^1 initial datum.

Corollary 2. *Consider Eq. (3) with $p > 5$, assume $\varepsilon_0 < \varepsilon_*$ and fix ζ fulfilling*

$$\frac{1}{p-2} < \zeta < \frac{1}{3} \ .$$

[1] See Definition 1 of [PS08].

Then there exists $\delta_* > 0$*, with* $\delta_* = \delta_*(r, \gamma, \tau, d, \varepsilon_0, \zeta)$ *such that if the initial datum* $\phi = q(0)$ *fulfills*

$$\delta_0 := \|\phi\|_{\ell^1(\mathbb{Z})} < \delta_* \ ,$$

then the solution $q(t)$ *of* (3) *fulfills*

$$\|q(t)\|_{\ell^\infty} \leq \frac{4K_1}{\langle t\rangle^\zeta} \|\phi\|_{\ell^1(\mathbb{Z})} \ , \tag{11}$$

where K_1 *is the constant in Corollary 1.*

For the sake of completeness, we will give the proof of Corollary 2 in Section 5.

From (9) one can also deduce, as in [SK05, KPS09], Strichartz estimates as well as decay and scattering for all the solutions of the linear Schrödinger equation. From this one can also deduce scattering for all solutions of (3) with small initial data in the energy space ℓ^2, *provided* $p > 7$.

Scheme of the proof of Theorem 1.2. For the free Schrödinger operator (4), the dispersive estimate is proved by using the Fourier transform which allows to write $e^{\mathrm{i}t\Delta}\phi$ as an oscillatory integral, which is estimated through the Van der Corput lemma which gives the $t^{-\frac{1}{3}}$ decay. The variable of integration in the integral to be estimated is the wave number.

In the presence of a quasi-periodic potential, generically, the spectrum is a Cantor set and the object generalizing the wave number is the fibered rotation number of the corresponding Schrödinger cocycle (see Appendix A for a precise definition).

Now, in the quasi-periodic case the fibered rotation number (rotation number for short) can be approximated through a perturbative construction. After J steps of such a construction, the approximate rotation number ρ_J is a piecewise monotonic function defined on the union of a very large number of intervals. More precisely, there is a total number of intervals proportional to $|\ln \varepsilon_0|^{2J^2 d}$. The approximate rotation number ρ_J is of class $\mathcal{C}^k$ (in our case $k = 3$ is enough) in the interior of each interval but it behaves as $E^{\frac{1}{2}}$ at the boundaries of each interval (E being the spectral parameter), so that its derivatives diverges at such points, which in the limit are dense in the spectrum.

Following [Zha16], the idea of the proof is to stop the construction at some step, say the J-th one, and to apply Van Der Corput lemma on each one of the small intervals. Still one has to make a regularization at the boundaries of the intervals, and this will be explained in a while. First, one has to know the improper eigenfunctions of H_θ. Now, it is known how to construct such improper eigenfunctions perturbatively: they are the quasi-periodic solutions of the quasi-periodic cocycle associated to H_θ. However, it is not known how to construct the spectral measure and how to normalize the improper eigenfunctions. The idea is to choose an approximate normalization, which in some sense is the most natural one, and to modify it slightly in order to regularize the integrals to be estimated. It turns out that this is possible, and that, if one uses such "normalized" eigenfunctions to define a spectral transform, then such a transformation is not unitary, but it is bounded with a bounded inverse and thus suffices to get the result. This was done and proved in [Zha16]. Here we just recall the needed results.

In the present paper we use such a spectral transform in order to write down an approximate representation formula for the solution of the Schrödinger equation in terms of oscillatory integrals that we estimate by approximating them through integrals over intervals which in turn are estimated through the Van Der Corput Lemma. In order to get the result, the last difficulty is to estimate the errors related to the use of approximations. This is purely technical and consists in writing down all the estimates taking into account the dependence on the approximation step and on the other parameters and then to choose all the free parameters in a suitable way. The main technical lemma of the paper gives this estimate and is Lemma 3.1.

The rest of paper is organized as follows. In Sect. 2 we recall some known facts on the structure of the spectrum of the Schrödinger operator and on the construction of the spectral transform. In Sect. 3 we prove the main technical lemma of the paper, namely Lemma 3.1. In Sect. 4 we conclude the proof of the main theorem. In Sect. 5 we prove Corollary 2. We also add two Appendixes. In Appendix A we recall a few facts on the rotation number, while in Appendix B we recall the version of the Van Der Corput Lemma that we use in the paper.

2. Preliminaries on Schrödinger operator and Schrödinger cocycle

In this section, we recall some basic notions and some important results for the spectrum of the quasi-periodic Schrödinger operator $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$(H_\theta q)_n = -(q_{n+1} + q_{n-1}) + V(\theta + n\omega) q_n \ , \quad n \in \mathbb{Z} \ ,$$

with V and ω given as in the statement of Theorem 1.2. We will also consider the Schrödinger cocycle $(\omega, A_0 + F_0)$:

$$\begin{pmatrix} q_{n+1} \\ q_n \end{pmatrix} = (A_0(E) + F_0(\theta + n\omega)) \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} \ , \tag{12}$$

with $A_0(E) := \begin{pmatrix} -E & -1 \\ 1 & 0 \end{pmatrix}$ and $F_0(\cdot) := \begin{pmatrix} V(\cdot) & 0 \\ 0 & 0 \end{pmatrix}$. Note that $(\omega, A_0 + F_0)$ is equivalent to the eigenvalue problem $H_\theta q = Eq$.

2.1. **Structure of the spectrum.**

We review here the KAM theory of Eliasson [Eli92] and Hadj Amor [HA09] for the reducibility of the Schrödinger cocycle $(\omega, A_0 + F_0(\cdot))$. These works relate the reducibility and the fibered rotation number (for the definition see Appendix A) globally, and improve the previous works by Dinaburg-Sinai [DS75] and Moser-Pöschel [MP84]. Here we will not prove the corresponding results (Theorem 2.1 and 2.2) referring to the work [Zha16] where a detailed proof was given. However, we will explain the strategy of proof with the aim of making the paper as self contained as possible, without adding too many details on known facts.

With $\varepsilon_0 = |V|_r$, $\sigma = \frac{1}{200}$, define, as in [HA09], the sequences:

$$\varepsilon_{j+1} = \varepsilon_j^{1+\sigma} \ , \quad N_j = 4^{j+1}\sigma |\ln \varepsilon_j| \ , \quad j \geq 0 \ .$$

All along the paper we will denote

$$\langle k \rangle_\omega := \frac{\langle k, \omega \rangle}{2} \ , \quad k \in \mathbb{Z}^d \ , \tag{13}$$

and by $|\cdot|_{\mathcal{C}^k_W(\mathcal{S})}$ the $\mathcal{C}^k$ norm of a function which is Whitney smooth on a set $\mathcal{S} \subset \mathbb{R}$, and for a function which is analytic on $\mathbb{T}^d$ (or $2\mathbb{T}^d$) and Whitney smooth on $\mathcal{S}$, we will denote by $|\cdot|_{\mathcal{C}^k_W(\mathcal{S}),\mathbb{T}^d}$ or $|\cdot|_{\mathcal{C}^k_W(\mathcal{S}),2\mathbb{T}^d}$ the supremum norm on $\mathbb{T}^d$ (or $2\mathbb{T}^d$) and $\mathcal{C}^k_W$

norm on $\mathcal{S}$. In particular, if $\mathcal{S}$ is a union of finitely many intervals, we will omit the subscript W in the above norms.

Furthermore, we denote the fibered rotation number of the Schrödinger cocycle (ω, A_0+F_0) by $\rho \equiv \rho_{(\omega,A_0+F_0)}$. It is necessary to mention that $\rho:\mathbb{R}\to[0,\pi]$ is a non-decreasing function with

$$\rho(E)\begin{cases} =0\ , & E\leq \inf\Sigma \\ \in(0,\pi)\ , & E\in(\inf\Sigma,\sup\Sigma) \\ =\pi\ , & E\geq\sup\Sigma \end{cases} ,$$

By the gap-labeling theorem [JM82], ρ is constant in a gap of Σ (i.e., an interval on $\mathbb{R}$ in the resolvent set of H_θ), and each gap is labeled with $k\in\mathbb{Z}^d$ such that $\rho=\langle k\rangle_\omega$ mod π in this gap.

Theorem 2.1. *There exists $\varepsilon_*=\varepsilon_*(\gamma,\tau,r,d)>0$ such that if $|V|_r=\varepsilon_0<\varepsilon_*$, then, for any $j\in\mathbb{N}$, there exists a Borel set $\Sigma_j\subset\Sigma$, with $\{\Sigma_j\}_j$ mutually disjoint, satisfying*

$$|\rho\left(\Sigma_{j+1}\right)|\leq 3|\ln\varepsilon_j|^{2d}\varepsilon_j^\sigma\ ,\quad j\geq 0\ ,$$
$$\left|\Sigma\setminus\widetilde{\Sigma}\right|=0\ ,\quad \widetilde{\Sigma}:=\cup_{j\geq0}\Sigma_j$$

such that the following statements hold.

(1) *The Schrödinger cocycle (ω, A_0+F_0) is reducible on $\widetilde{\Sigma}$. More precisely, there exist Z and B, with $Z:\widetilde{\Sigma}\times 2\mathbb{T}^d\to SL(2,\mathbb{R})$ analytic on $2\mathbb{T}^d$ and $B:\widetilde{\Sigma}\to SL(2,\mathbb{R})$ s.t. Z conjugates A_0+F_0 to B, namely*

$$Z(\cdot+\omega)^{-1}(A_0+F_0(\cdot))\,Z(\cdot)=B\ .$$

Furthermore B is $\mathcal{C}^1$ in the sense of Whitney on each Σ_j, and

$$|B-A_0|_{\mathcal{C}^1_W(\Sigma_0)}\leq\varepsilon_0^{\frac{1}{3}}\ ;\qquad |B|_{\mathcal{C}^1_W(\Sigma_{j+1})}\leq N_j^{10\tau}\ ,\quad j\geq0\ . \tag{14}$$

(2) *The eigenvalues of $B\big|_{\Sigma_j}$, are of the form $e^{\pm\mathrm{i}\xi}$, with $\xi\in\mathbb{R}$, and, for every $j\geq0$, there is $k_j:\widetilde{\Sigma}\to\mathbb{Z}^d$, such that*
- *$0<|k_j|\leq N_j$ on Σ_{j+1}, and $k_l=0$ on Σ_j for $l\geq j$,*
- *$\xi=\rho-\sum_{l\geq0}\langle k_l\rangle_\omega$ and $0<|\xi|_{\Sigma_{j+1}}<2\varepsilon_j^\sigma$.*

Theorem 2.1 describes the result of a KAM procedure. If one stops the procedure at a finite step one gets a picture that will be needed for our construction and which is contained in the next theorem (which of course constitutes the main step for the proof of Theorem 2.1).

Theorem 2.2. *Let $|V|_r=\varepsilon_0<\varepsilon_*$ be as in Theorem 2.1. Given any $J\in\mathbb{N}$, for $0\leq j\leq J$, there exists $\Gamma_j^{(J)}\subset[\inf\Sigma,\sup\Sigma]$, satisfying*

- *$\Sigma_j\subset\Gamma_j^{(J)}$ for $0\leq j\leq J$,*
- *$\{\Gamma_j^{(J)}\}_{j=0}^J$ are mutually disjoint and $\overline{\bigcup_{j=0}^J\Gamma_j^{(J)}}=[\inf\Sigma,\sup\Sigma]$*
- *$\bigcup_{j=0}^J\Gamma_j^{(J)}$ consists of at most $|\ln\varepsilon_0|^{2J^2d}$ open intervals,*
- *If $J\geq1$, then $\left|\rho\left(\Gamma_{j+1}^{(J)}\right)\right|\leq3|\ln\varepsilon_j|^{2d}\varepsilon_j^\sigma$ for $0\leq j\leq J-1$.*

Furthermore, the following statements hold.

(S1) *There exist* $\begin{cases} A_J : \Gamma_j^{(J)} \to SL(2,\mathbb{R}) \\ F_J : \Gamma_j^{(J)} \times \mathbb{T}^d \to gl(2,\mathbb{R}) \text{ analytic on } \mathbb{T}^d \\ Z_J : \Gamma_j^{(J)} \times 2\mathbb{T}^d \to SL(2,\mathbb{R}) \text{ analytic on } 2\mathbb{T}^d \end{cases}$ $,\ 0 \le j \le J,$ *all of which are smooth on each connected component of* $\Gamma_j^{(J)}$*, such that*

$$Z_J(\cdot+\omega)^{-1}(A_0+F_0(\cdot))\,Z_J(\cdot) = A_J + F_J(\cdot)\ ,$$

with $|F_J|_{\mathcal{C}^3(\Gamma_j^{(J)}),\mathbb{T}^d} \le \varepsilon_J$*,* $0 \le j \le J$*, and*

$$|A_J - A_0|_{\mathcal{C}^3(\Gamma_0^{(J)})} \le \varepsilon_0^{\frac{1}{2}}\ , \qquad |Z_J - Id.|_{\mathcal{C}^3(\Gamma_0^{(J)}),2\mathbb{T}^d} \le \varepsilon_0^{\frac{1}{3}}\ . \tag{15}$$

If $J \ge 1$*, then for* $0 \le j \le J-1$*,*

$$|A_J|_{\mathcal{C}^3(\Gamma_{j+1}^{(J)})} \le \varepsilon_j^{-\frac{\sigma}{6}}\ , \qquad |Z_J|_{\mathcal{C}^3(\Gamma_{j+1}^{(J)}),2\mathbb{T}^d} \le \varepsilon_j^{-\frac{\sigma}{3}}\ , \tag{16}$$

and, on $\Gamma_{j+1}^{(J)}$*,*

$$\varepsilon_j^{\frac{\sigma}{4}} \le |(\mathrm{tr}A_J)'| \le N_j^{10\tau}\ . \tag{17}$$

Moreover, for $0 \le j \le J$*,*

$$|A_J - B|_{\mathcal{C}^1_W(\Sigma_j)} \le \varepsilon_J^{\frac{1}{4}}\ , \quad |Z_J - Z|_{\mathcal{C}^1_W(\Sigma_j),2\mathbb{T}^d} \le \varepsilon_J^{\frac{1}{4}}\ . \tag{18}$$

(S2) A_J *has two eigenvalues* $e^{\pm i\alpha_J}$ *with* $\alpha_J \in \mathbb{R}\cup i\mathbb{R}$*. For* $\xi_J := \Re\alpha_J$*, we have*

- $|\xi_J - \xi|_{\Sigma_j} \le \varepsilon_J^{\frac{1}{4}}$*,* $0 \le j \le J$*.*
- $|\xi_J - \rho|_{\Gamma_0^{(J)}} \le \varepsilon_J^{\frac{1}{4}}$*.*
- *If* $J \ge 1$*, then*
  - $|\xi_J|_{\Gamma_{j+1}^{(J)}} \le \frac{3}{2}\varepsilon_j^{\sigma}$*,* $0 \le j \le J-1$*.*
  - *There is* $k_j : \bigcup_{l=0}^{J}\Gamma_l^{(J)} \to \mathbb{Z}^d$*,* $0 \le j \le J-1$*, constant on each connected component of* $\bigcup_{l=0}^{J}\Gamma_l^{(J)}$*, with* $0 < |k_j| \le N_j$ *on* $\Gamma_{j+1}^{(J)}$ *and* $k_l = 0$ *on* $\Gamma_{j+1}^{(J)}$ *for* $l \ge j+1$ *such that* $\left|\xi_J + \sum_{l=0}^{J-1}\langle k_l\rangle_\omega - \rho\right|_{\Gamma_{j+1}^{(J)}} \le \varepsilon_J^{\frac{1}{4}}$*.*

(S3) $\bigcup_{j=0}^{J}\{\Gamma_j^{(J)} : |\sin\xi_J| > \frac{3}{2}\varepsilon_J^{\frac{1}{20}}\}$ *has at most* $2|\ln\varepsilon_0|^{2J^2d}$ *connected components, on which* ξ_J *is smooth with* $\xi_J' = -\frac{(\mathrm{tr}A_J)'}{2\sin\xi_J}$*. If* $J \ge 1$*, then, on* $\{\Gamma_{j+1}^{(J)} : |\sin\xi_J| > \frac{3}{2}\varepsilon_J^{\frac{1}{20}}\}$*,* $0 \le j \le J-1$*,*

$$\frac{1}{3} < \xi_J' \le \frac{N_j^{10\tau}}{|\sin\xi_J|}\ , \qquad \frac{\varepsilon_j^{\frac{3\sigma}{4}}}{4|\sin\xi_J|^3} < |\xi_J''| \le \frac{N_j^{20\tau}}{|\sin\xi_J|^3}\ . \tag{19}$$

(S4) $|\rho(\{(\inf\Sigma,\sup\Sigma) : |\sin\xi_J| \le \frac{3}{2}\varepsilon_J^{\frac{1}{20}}\})| \le \varepsilon_J^{\frac{1}{24}}$ *and for* $0 \le j \le J$*,* $|\xi_J(\Gamma_j^{(J)}\backslash\Sigma_j)| \le \varepsilon_J^{\frac{7\sigma}{8}}$*.*

From now on, we denote $\rho_J := \xi_J + \sum_{l=0}^{J-1} \langle k_l \rangle_\omega$, which gives an approximation of ρ. In particular, $\rho_0 = \xi_0$, and

$$|\rho_J - \rho|_{\Sigma_j} \le \varepsilon_j^{\frac{1}{4}} \ . \tag{20}$$

Scheme of the proof of Theorems 2.1 and 2.2. The procedure of proof is a KAM procedure in which one increases iteratively the order of the time dependent part of the cocycle. The main point is that, in order to get a quite complete description of the spectrum, one has to do the construction for a set of E's which is of full measure (not only of large measure). It is well known that the conjugacy of A_0+F_0 to a time independent cocycle can be obtained through a close to identity transformation only if some non-resonant relations are fulfilled and this is typically true only in sets of large measure. To describe the non-resonance condition, consider first the eigenvalues of A_0: they can be written in the form $e^{\pm i\rho_0}$, with $\rho_0 = \rho_0(E) := \arccos\left(-\frac{E}{2}\right)$, $|E| \le 2$. The relevant non-resonance condition in order to construct the first transformation is

$$|\rho_0(E) - \langle k \rangle_\omega| \ge \frac{\varepsilon_0^\sigma}{|k|^\tau} \ ; \quad 0 < |k| \le N_0 \ . \tag{21}$$

For the values of E s.t. (21) is fulfilled, the classical construction of the KAM step produces a close to identity transformation which conjugates $A_0 + F_0$ to $A_1 + F_1$ with $|F_1| \sim \varepsilon_1$ and $A_1 \in SL(2,\mathbb{R})$ having eigenvalues of the form $e^{\pm i\rho_1}$, with ρ_1 close to ρ_0. Such a set of E's is $\Gamma_0^{(1)}$.

Now, let k with $0 < |k| \le N_0$ be s.t. there exists a segment $\mathcal{I}_k$, on which equation (21) is violated, then it is known how to construct a time dependent matrix H_{k,A_0} (which is not close to identity) conjugating $A_0 + F_0$ to a new cocycle $\tilde{A}_0 + \tilde{F}_0$, where $\tilde{A}_0$ has eigenvalues $e^{\pm i\tilde{\rho}_0}$, with $\tilde{\rho}_0 := \rho_0 - \langle k \rangle_\omega$. Furthermore, by the fact that ω is Diophantine, there are no $\tilde{k}$ with $\tilde{k} \neq k$ s.t. (21) is violated for $E \in \mathcal{I}_k$. It follows that (21) is fulfilled by $\tilde{\rho}_0$ and therefore, on $\mathcal{I}_k$ one can conjugate $\tilde{A}_0 + \tilde{F}_0$ to a new cocycle $A_1 + F_1$ with $|F_1| \sim \varepsilon_1$, and A_1 having eigenvalues of the form $e^{\pm i\alpha_1}$, with α_1 close to $\tilde{\rho}_0$, which in turn is close to 0. It follows that for some values of E, the quantity α_1 can fail to be real. The values of E s.t. $\alpha_1(E)$ is purely imaginary are outside the approximate spectrum of H_θ, while the others belong to the approximate spectrum. We put $\xi_1(E) := \Re(\alpha_1(E))$ and $\rho_1(E) := \xi_1(E) + \langle k \rangle_\omega$.

The union of the intervals $\mathcal{I}_k$ is the set $\Gamma_1^{(1)}$.

In order to iterate we proceed as follows. For $E \in \Gamma_0^{(1)}$ one considers the non-resonance condition

$$|\rho_1(E) - \langle k \rangle_\omega| \ge \frac{\varepsilon_1^\sigma}{|k|^\tau} \ ; \quad \forall\ 0 < |k| \le N_1 \ . \tag{22}$$

The set of the $E \in \Gamma_0^{(1)}$ for which (22) is satisfied is $\Gamma_0^{(2)}$ and here one can construct a close to identity transformation conjugating $A_1 + F_1$ to $A_2 + F_2$ with $|F_2| \sim \varepsilon_2$ and $A_2 \in SL(2,\mathbb{R})$ having eigenvalues of the form $e^{\pm i\rho_2}$, with ρ_2 close to ρ_1.

Consider an element $E \in \Gamma_0^{(1)}$ s.t. (22) is violated for some k. Such E's are the first part of $\Gamma_2^{(2)}$. For such E's, one proceed as we did at the first step in $\Gamma_1^{(1)}$.

Consider now $\Gamma_1^{(1)}$. For these values of E the relevant non-resonance condition is

$$|\xi_1(E) - \langle k \rangle_\omega| \ge \frac{\varepsilon_1^\sigma}{|k|^\tau} \ ; \quad \forall\ 0 < |k| \le N_1 \ . \tag{23}$$

If it is fulfilled one proceeds as in $\Gamma_0^{(1)}$, namely, one constructs a close to identity transformation conjugating A_1+F_1 to A_2+F_2 with $|F_2|\sim\varepsilon_2$ and $A_2\in SL(2,\mathbb{R})$ having eigenvalues of the form $e^{\pm i\xi_2}$, with ξ_2 close to ξ_1. Such E's constitute $\Gamma_1^{(2)}$.

Consider now the $E\in\Gamma_1^{(1)}$ s.t. $\exists\ k$ with $0<|k|\le N_1$ s.t. (23) is violated. The union of such E's is the remaining part of $\Gamma_2^{(2)}$. Here one proceeds as we did for the first step in $\Gamma_1^{(1)}$. Iterating and adding the estimates one gets the proof of Theorem 2.2.

In order to get Theorem 2.1 one has simply to pass to the limit $J\to\infty$. We do not discuss such a limit, which is standard, but just recall that the sets Σ_j are defined as

$$\Sigma_j := \bigcap_{J:J\ge j}\Gamma_j^{(J)}\setminus\bigcup_{k\in\mathbb{Z}^d}\rho^{-1}(\langle k\rangle_\omega)\ . \tag{24}$$

2.2. **Spectral transform.**

For $E\in\Sigma$, let $\mathcal{K}(E)$ and $\mathcal{J}(E)$ be two linearly independent generalized eigenvectors of H_θ and consider the spectral transform $\mathcal{S}q$ defined as follows: for any $q\in\ell^2(\mathbb{Z})$, put

$$(\mathcal{S}q)(E) := \begin{pmatrix}\sum_n q_n\mathcal{K}_n(E)\\ \sum_n q_n\mathcal{J}_n(E)\end{pmatrix}\ . \tag{25}$$

Given any matrix of measures on $\mathbb{R}$, namely $d\varphi=\begin{pmatrix} d\varphi_{11} & d\varphi_{12}\\ d\varphi_{21} & d\varphi_{22}\end{pmatrix}$, let $\mathcal{L}^2(d\varphi)$ be the space of the vectors $G=(g_j)_{j=1,2}$, with g_j functions of $E\in\mathbb{R}$ satisfying

$$\|G\|^2_{\mathcal{L}^2(d\varphi)} := \sum_{j,k=1}^{2}\int_{\mathbb{R}} g_j\,\bar g_k\,d\varphi_{jk}<\infty\ . \tag{26}$$

Theorem 2.3 (Chapter 9 of [CL55])**.** *There exists a Hermitian matrix of measures $\mu=(\mu_{jk})_{j,k=1,2}$, with μ_{jk} non-decreasing functions, such that $\mathcal{S}:\ell^2(\mathbb{Z})\to\mathcal{L}^2(d\mu)$ is unitary.*

Remark that by this theorem the spectral transform is invertible. As anticipated in the introduction it is not known how to construct the measure $d\mu$, however in [Zha16] a procedure to construct an approximate measure was developed.

Recall that $\sigma=\frac{1}{200}$, $|V|_r=\varepsilon_0<\varepsilon_*$ (as in Theorems 2.1 and 2.2) and the sequence $\{\varepsilon_j\}_j$ is defined by $\varepsilon_{j+1}=\varepsilon_j^{1+\sigma}$.

Proposition 2.4. *On the full measure subset $\widetilde\Sigma:=\bigcup_{j\ge0}\Sigma_j$ of the spectrum, for any fixed $\theta\in\mathbb{T}^d$, any $E\in\widetilde\Sigma$, there exist two linearly independent generalized eigenvectors $\mathcal{K}(E)$ and $\mathcal{J}(E)$ of H_θ with the following properties: define the spectral transform according to (25) and consider the matrix of measures $d\varphi$ given by*

$$d\varphi|_\Sigma := \frac{1}{\pi}\begin{pmatrix}\rho' & 0\\ 0 & \rho'\end{pmatrix}dE\ ,\qquad d\varphi|_{\mathbb{R}\setminus\Sigma}:=0\ ,$$

then we have, for any $q\in\ell^2(\mathbb{Z})$,

$$\left(1-\varepsilon_0^{\frac{\sigma^2}{10}}\right)\|q\|^2_{\ell^2(\mathbb{Z})}\le\|\mathcal{S}q\|^2_{\mathcal{L}^2(d\varphi)}\le\left(1+\varepsilon_0^{\frac{\sigma^2}{10}}\right)\|q\|^2_{\ell^2(\mathbb{Z})}\ , \tag{27}$$

and also

$$\left|\frac{1}{\pi}\int_{\Sigma}\left(g_1(E)\mathcal{K}_n(E)+g_2(E)\mathcal{J}_n(E)\right)\rho' dE - q_n\right| \leq \varepsilon_0^{\frac{\sigma^2}{10}}\|q\|_{\ell^\infty} \ . \tag{28}$$

Furthermore, the functions $\mathcal{K}(E)$ and $\mathcal{J}(E)$ have the following properties:

$$\mathcal{K}_n(E) = \sum_{n_\Delta = n, n\pm 1}\beta_{n,n_\Delta}(E)\sin n_\Delta \rho(E) \ , \quad \mathcal{J}_n(E) = \sum_{n_\Delta = n, n\pm 1}\beta_{n,n_\Delta}(E)\cos n_\Delta \rho(E) \ , \tag{29}$$

with ρ the fibered rotation number of the cocycle $(\omega, A_0 + F_0)$ and

$$|\beta_{n,n_\Delta} - \delta_{n,n_\Delta}|_{\Sigma_0} \leq \varepsilon_0^{\frac{1}{4}} \ , \qquad |\beta_{n,n_\Delta}|_{\Sigma_{j+1}} \leq \varepsilon_j^{\sigma}, \quad j \geq 0 \ .$$

Given any $J \in \mathbb{N}$, there exist β^J_{n,n_Δ}, smooth on each connected component of $\Gamma_j^{(J)}$, satisfying

$$\left|\beta^J_{n,n_\Delta} - \delta_{n,n_\Delta}\right|_{\mathcal{C}^2(\Gamma_0^{(J)})} \leq \varepsilon_0^{\frac{1}{4}} \ , \tag{30}$$

and if $J \geq 1$, then

$$|\beta^J_{n,n_\Delta}|_{\mathcal{C}^1(\Gamma_{j+1}^{(J)})} \leq \varepsilon_j^{3\sigma} \ , \quad 0 \leq j \leq J-1 \ . \tag{31}$$

Moreover,

$$\left|\beta_{n,n_\Delta} - \beta^J_{n,n_\Delta}\right|_{\Sigma_j} \leq 10\varepsilon_J^{\frac{1}{4}} \ , \quad 0 \leq j \leq J \ . \tag{32}$$

Idea of the proof. The construction and the estimates of $\mathcal{S}$ are actually given in Section 4.2 of [Zha16]. The generalized eigenvectors $\mathcal{K}$ and $\mathcal{J}$ are constructed as Bloch waves exploiting the reducibility procedure and in particular the matrices Z and B of Theorems 2.1 and 2.2. The construction naturally leads to a family of generalized eigenfunctions which do not depend in a smooth way on E (in particular Eq. (30) and (31) do not hold) so one modifies the normalization in order to get such properties. The price to pay is that $\mathcal{S}$ is no more unitary, but turns out to be just a bounded transformation with bounded inverse. For completeness we add now the details of the construction of $\mathcal{K}$ and $\mathcal{J}$, while we refer to [Zha16] for the details of the proofs of the estimates.

First we remark that one can construct Bloch-waves of Schrödinger operator H_θ on $\widetilde{\Sigma}$ using the reducibility of Schrödinger cocycle. Indeed, with an additional transform (see (3.17) of [Zha16]), one can find $\tilde{Z} : \widetilde{\Sigma} \times 2\mathbb{T}^d \to SL(2,\mathbb{C})$ and $B : \widetilde{\Sigma} \to SL(2,\mathbb{C})$, with two eigenvalues $e^{\pm \mathrm{i}\rho}$, such that

$$\tilde{Z}(\cdot+\omega)^{-1}(A_0 + F_0(\cdot))\,\tilde{Z}(\cdot) = \tilde{B} \ .$$

With the matrices $\tilde{Z} = \begin{pmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix}$, one can easily see that, defining

$$\tilde{f}_n(\theta) := \left[\tilde{Z}_{11}(\theta-\omega+n\omega)\tilde{B}_{12} - \tilde{Z}_{12}(\theta-\omega+n\omega)\tilde{B}_{11}\right]e^{-\mathrm{i}\rho} + \tilde{Z}_{12}(\theta-\omega+n\omega)$$
$$\tilde{\psi}_n = e^{\mathrm{i}n\rho}\tilde{f}_n(\theta) \ ,$$

then such a $\tilde{\psi}$ fulfills $H_\theta\tilde{\psi} = E\tilde{\psi}$ for $E \in \widetilde{\Sigma}$. In order to get smooth dependence on E we modify its normalization in Σ_j for $j \geq 1$, defining

$$\psi_n = e^{\mathrm{i}n\rho} f_n \quad \text{with} \quad f_n = \begin{cases} \tilde{f}_n \ , & E \in \Sigma_0 \\ \tilde{f}_n \sin^5 \xi, & E \in \Sigma_{j+1} \ , \quad j \geq 0 \end{cases} .$$

Then we define $\mathcal{K}_n := \Im(e^{\mathrm{i}n\rho} f_n \bar{f}_0)$ and $\mathcal{J}_n := \Re(e^{\mathrm{i}n\rho} f_n \bar{f}_0)$ on $\widetilde{\Sigma}$ and $\mathcal{K}_n|_{\mathbb{R}\setminus\widetilde{\Sigma}} = \mathcal{J}_n|_{\mathbb{R}\setminus\widetilde{\Sigma}} := 0$. By a direct calculation, we see

$$e^{\mathrm{i}n\rho} f_n \bar{f}_0 = \sum_{n_\Delta = n, n\pm 1} \beta_{n,n_\Delta} e^{\mathrm{i}n_\Delta \rho} \ ,$$

with some β_{n,n_Δ} which can be shown to fulfill the estimates claimed in the statement (for the details see [Zha16]). Thus one gets $\mathcal{K}_n = \sum_{n_\Delta} \beta_{n,n_\Delta} \sin n_\Delta \rho$, $\mathcal{J}_n = \sum_{n_\Delta} \beta_{n,n_\Delta} \cos n_\Delta \rho$.

Finally one has to show the important estimates (27) and (28). They were proved in [Zha16]. Here we just recall that the main step for its proof are the following inequalities

$$\left| \frac{1}{\pi} \int_\Sigma \left(\mathcal{K}_n^2(E) + \mathcal{J}_n^2(E) \right) \rho' dE - 1 \right| \leq \varepsilon_0^{\frac{\sigma^2}{8}} \ ,$$

$$\left| \frac{1}{\pi} \int_\Sigma \left(\mathcal{K}_m(E)\mathcal{K}_n(E) + \mathcal{J}_m(E)\mathcal{J}_n(E) \right) \rho' dE \right| \leq \frac{\varepsilon_0^{\frac{\sigma^2}{8}}}{|m-n|^{1+\frac{\sigma}{6}}} \ , \quad m \neq n \ ,$$

the second of which is obtained from an estimate of an oscillatory integral (Lemma 4.1 of [Zha16]) which is very close to the estimate given in Lemma 3.3 of the present paper. □

3. An oscillatory integral on the spectrum

In this section, by using the division of $[\inf\Sigma, \sup\Sigma]$ given in Theorem 2.2, we estimate an integral on the spectrum. This will be applied in analyzing the time evolution, and deducing dispersion in the next section.

Recall that $\sigma = \frac{1}{200}$, $|V|_r = \varepsilon_0 \leq \varepsilon_*$ and the sequence $\{\varepsilon_j\}_{j\geq 0}$ is defined by $\varepsilon_{j+1} = \varepsilon_j^{1+\sigma}$.

Lemma 3.1. *Let $h : \widetilde{\Sigma} \to \mathbb{R}$ be a function s.t. for any $J \geq 0$ there exists a function $h_J : \bigcup_{0\leq j\leq J} \Gamma_j^{(J)} \to \mathbb{R}$ which is $\mathcal{C}^1$ on each connected component of $\Gamma_j^{(J)}$, and satisfies the following assumptions*

(E0) $|h_J - h|_{\Sigma_j} \leq 10\varepsilon_J^{\frac{1}{4}}$ *for* $0 \leq j \leq J$ *and* $|h_J - h|_{\Sigma_j} \leq 2$ *for* $j \geq J+1$,

(E1) $|h_J|_{\mathcal{C}^1(\Gamma_0^{(J)})} \leq \dfrac{16}{15}$,

(E2) $|h_J|_{\mathcal{C}^1(\Gamma_{j+1}^{(J)})} \leq \varepsilon_j^{3\sigma}$, $0 \leq j \leq J-1$, *if* $J \geq 1$.

Then, there exists a positive constant $a < 80802$ s.t. for any $M \in \mathbb{R}$, one has

$$\left| \int_\Sigma h e^{-\mathrm{i}Et} \cos M\rho \cdot \rho' \, dE \right| \leq \frac{526 |\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t \rangle^{\frac{1}{3}}} \ . \tag{33}$$

The rest of the section is devoted to the proof of such a lemma.

From now on we assume that ε_* is such that all the smallness conditions that we will assume are satisfied.

We denote

$$\mathcal{I}_M(\mathcal{S}) := \int_{\mathcal{S}} h e^{-\mathrm{i}Et} \cos M\rho \cdot \rho' \, dE, \quad \mathcal{S} \subset \mathbb{R} \ ,$$

and

$$\mathcal{I}_M^J(\mathcal{S}) := \int_{\mathcal{S}} h_J e^{-\mathrm{i}Et} \cos M\rho \cdot \rho' \, dE, \quad \mathcal{S} \subset \mathbb{R} \ .$$

We first give three lemmas, the first of which allows to approximate $\mathcal{I}_M$ through $\mathcal{I}_M^J$. For the estimate of $\mathcal{I}_M^J$, we have to separate the cases of small M and large M: they are treated in two different lemmas. Finally we will summarize the results and deduce Lemma 3.1.

Lemma 3.2. *For any positive J and any $M \in \mathbb{R}$, under the assumptions of Lemma 3.1, one has*

$$\left|\mathcal{I}_M(\Sigma) - \mathcal{I}_M^J(\Sigma)\right| \le \varepsilon_J^{\frac{3\sigma}{4}} \ . \tag{34}$$

Proof. By the fact that $|\rho(\Sigma_{j+1})| \le 3|\ln\varepsilon_j|^{2d}\varepsilon_j^{\sigma}$, we have

$$\begin{aligned}
\sum_{j=0}^{J}\left|\int_{\Sigma_j}(h-h_J)\cos(M\rho)\cdot e^{-\mathrm{i}Et}\rho'\,dE\right| &\le 10\varepsilon_J^{\frac14}\sum_{j=0}^{J}\int_{\Sigma_j}\left|\rho'\right|dE\\
&= 10\varepsilon_J^{\frac14}\sum_{j=0}^{J}\int_{\Sigma_j}\rho' dE\\
&\le 10\varepsilon_J^{\frac14}\sum_{j=0}^{J}|\rho(\Sigma_j)|\\
&\le 10\varepsilon_J^{\frac14}\left(2\pi+\sum_{j\ge0}3\varepsilon_j^{\sigma}\left|\ln\varepsilon_j\right|^{2d}\right)\\
&\le 10\varepsilon_J^{\frac14}(2\pi+4\varepsilon_0^{\sigma})\\
&\le \frac12\varepsilon_J^{\frac16}
\end{aligned}$$

and

$$\sum_{j\ge J+1}\left|\int_{\Sigma_j}(h-h_J)\cos(M\rho)\cdot e^{-\mathrm{i}Et}\rho'\,dE\right| \le \frac12\varepsilon_J^{\frac{3\sigma}{4}} \ .$$

Hence we get that the error is bounded by

$$\frac12\varepsilon_J^{\frac16}+\frac12\varepsilon_J^{\frac{3\sigma}{4}} \le \varepsilon_J^{\frac{3\sigma}{4}} \ . \quad \square \tag{35}$$

Lemma 3.3. *Assume that for some positive $J \ge 0$ the function h_J fulfills (E2) and (E3), then for every $M \in \mathbb{R}\setminus\{0\}$ and $t \in \mathbb{R}$, we have*

$$\left|\mathcal{I}_M^J(\Sigma)\right| \le \frac{32}{15}\frac{1}{|M|}\left|\ln\varepsilon_0\right|^{2J^2d} + \frac{32}{15}\frac{1}{|M|}(\sup\Sigma-\inf\Sigma)\langle t\rangle \ . \tag{36}$$

Proof. Since $\rho' = 0$ on $[\inf\Sigma, \sup\Sigma] \setminus \Sigma$, then we have

$$\mathcal{I}_M^J(\Sigma) = \int_{\inf\Sigma}^{\sup\Sigma} h_J e^{-\mathrm{i}Et} \cos M\rho \cdot \rho' \, dE \ .$$

The above integral on the right hand side is indeed the sum of integrals over the connected component $(E_*, E_{**}) \subset \Gamma_j^{(J)}$. Since ρ is absolutely continuous, by integrating by parts on each connected component, we obtain

$$\begin{aligned} & \int_{\inf\Sigma}^{\sup\Sigma} h_J e^{-\mathrm{i}Et} \cos M\rho \cdot \rho' \, dE \\ = & \ \frac{1}{M} \sum_{j=0}^{J} \sum_{\substack{(E_*,\, E_{**}) \subset \Gamma_j^{(J)} \\ \text{connected component}}} h_J e^{-\mathrm{i}Et} \sin M\rho\big|_{(E_*, E_{**})} \\ & - \frac{1}{M} \sum_{j=0}^{J} \sum_{\substack{(E_*,\, E_{**}) \subset \Gamma_j^{(J)} \\ \text{connected component}}} \int_{E_*}^{E_{**}} (h_J e^{-\mathrm{i}Et})' \sin M\rho \, dE \ . \end{aligned}$$

Since there are at most $|\ln\varepsilon_0|^{2J^2 d}$ connected components of $\bigcup_{j=0}^J \Gamma_j^{(J)}$, we have

$$\frac{1}{|M|} \left| \sum_{j=0}^{J} \sum_{\substack{(E_*,\, E_{**}) \subset \Gamma^{(J)}{}_j \\ \text{connected component}}} h_J e^{-\mathrm{i}Et} \sin M\rho\big|_{(E_*, E_{**})} \right| \le \frac{32}{15|M|} |\ln\varepsilon_0|^{2J^2 d}$$

and

$$\frac{1}{|M|} \left| \sum_{j=0}^{J} \sum_{\substack{(E_*,\, E_{**}) \subset \Gamma_j^{(J)} \\ \text{connected component}}} \int_{E_*}^{E_{**}} (h_J e^{-\mathrm{i}Et})' \sin M\rho \, dE \right| \le \frac{32|t|}{15|M|} (\sup\Sigma - \inf\Sigma) \ . \quad \square$$

Lemma 3.4. *Assume that for some positive $J \ge 0$ the function h_J fulfills (E2) and (E3), then for every $M \in \mathbb{R}$ and $t \in \mathbb{R}$, we have*

$$\left|\mathcal{I}_M^J(\Sigma)\right| \le 512 \frac{|\ln\varepsilon_0|^{2J^2 d}}{\langle t \rangle^{\frac{1}{3}}} + \frac{1}{2} \varepsilon_J^{\frac{3\sigma}{4}} + 2\,|M|\, \varepsilon_J^{\frac{1}{4}} \ . \tag{37}$$

Proof. The proof is divided into three parts.

Step 1. Approximation

We will consider the sum of integrals

$$\sum_{j=0}^{J} \int_{\left\{\Gamma_j^{(J)} : |\sin\xi| > \varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et} \cos M\rho_J \cdot \rho_J' \, dE$$

instead of $\mathcal{I}_M^J(\Sigma)$. The error is estimated by

$$\left|\mathcal{I}_M^J(\Sigma)-\sum_{j=0}^{J}\int_{\left\{\Gamma_j^{(J)}:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\,\cos M\rho_J\cdot\rho_J'\,dE\right| \tag{38}$$

$$\leq \left|\sum_{j=0}^{J}\int_{\left\{\Sigma_j:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\left(\cos M\rho_J\cdot\rho_J'-\cos M\rho\cdot\rho'\right)\,dE\right| \tag{39}$$

$$+\left|\sum_{j=0}^{J}\int_{\left\{\Gamma_j^{(J)}\setminus\Sigma_j:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\,\cos M\rho_J\cdot\rho_J'\,dE\right| \tag{40}$$

$$+\left|\sum_{j=0}^{J}\int_{\left\{\Sigma_j:|\sin\xi|\leq\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\cos M\rho\cdot\rho'\,dE\right| \tag{41}$$

$$+\left|\sum_{j\geq J+1}\mathcal{I}_M^J(\Sigma_j)\right|\,. \tag{42}$$

- Since $|\rho(\Sigma_{j+1})|\leq 3|\ln\varepsilon_j|^{2d}\varepsilon_j^{\sigma}$, the term in (42) is bounded by
$$\frac{32}{5}|\ln\varepsilon_J|^{2d}\varepsilon_J^{\sigma}\leq\frac{1}{4}\varepsilon_J^{\frac{3\sigma}{4}}.$$
- On Σ_j, $0\leq j\leq J$, we have $|\xi_J-\xi|\leq\varepsilon_J^{\frac{1}{4}}$. So $|\sin\xi|\leq\varepsilon_J^{\frac{1}{20}}$ implies that $|\sin\xi_J|\leq\frac{3}{2}\varepsilon_J^{\frac{1}{20}}$. By the assertion **(S4)** of Theorem 2.2, the term in (41) is bounded by
$$\frac{16}{15}\varepsilon_J^{\frac{1}{24}}\leq\varepsilon_J^{5\sigma}.$$
- By the fact that $|\rho_J(\Gamma_j^{(J)}\setminus\Sigma_j)|\leq\varepsilon_J^{\frac{7\sigma}{8}}$, $0\leq j\leq J$, the term in (40) is bounded by
$$\frac{16}{15}(J+1)\cdot\varepsilon_J^{\frac{7\sigma}{8}}\leq\frac{1}{4}\varepsilon_J^{\frac{3\sigma}{4}}.$$
- On $\left\{\Sigma_j:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}$, $0\leq j\leq J$, we have $|\xi_J-\xi|\leq\varepsilon_J^{\frac{1}{4}}$, which implies $|\sin\xi_J|\geq\frac{1}{2}\varepsilon_J^{\frac{1}{20}}$. Then, by (15)–(18), we get
$$\begin{aligned}|\rho_J'-\rho'| &= \frac{1}{2}\left|\frac{(\mathrm{tr}A_J)'}{\sin\xi_J}-\frac{(\mathrm{tr}B)'}{\sin\xi}\right|\\ &= \frac{|(\mathrm{tr}A_J)'\sin\xi-(\mathrm{tr}B)'\sin\xi_J|}{2|\sin\xi||\sin\xi_{J+1}|}\\ &\leq \frac{|(\mathrm{tr}A_J)'||\sin\xi-\sin\xi_J|+|\sin\xi_J||(\mathrm{tr}B)'-(\mathrm{tr}A_J)'|}{2|\sin\xi||\sin\xi_J|}\\ &\leq 2\varepsilon_J^{-\frac{1}{10}}\cdot\left(2\varepsilon_J^{\frac{1}{4}}N_J^{10\tau}+2\varepsilon_J^{\frac{1}{4}}\right)\\ &\leq \varepsilon_J^{\frac{1}{10}}\,,\end{aligned}$$

and, using

$$|\cos M\rho_J - \cos M\rho| = 2\left|\sin\frac{M}{2}(\rho_J+\rho)\right|\left|\sin\frac{M}{2}(\rho_J-\rho)\right| \le 2|M|\cdot\varepsilon_J^{\frac{1}{4}}$$

we bound of the term (39).

Hence, by combining the above estimates, the error given in (38) is less than

$$\frac{1}{2}\varepsilon_J^{\frac{3\sigma}{4}} + 2\,|M|\,\varepsilon_J^{\frac{1}{4}} \ .$$

Step 2. Change of variable

Recall that there are at most $2|\ln\varepsilon_0|^{2J^2d}$ connected components of

$$\bigcup_{j=0}^{J}\left\{E\in\Gamma_j^{(J)} : |\sin\xi_J| > \frac{3}{2}\varepsilon_J^{\frac{1}{20}}\right\} \ .$$

Let (E_*, E_{**}) be one of these components, on which $\rho_J(E)$ is strictly increasing. So $E = E(\rho_J)$ is well-defined with

$$\frac{dE}{d\rho_J} = \frac{1}{\rho_J'} \ , \quad \frac{d^2E}{d\rho_J^2} = -\frac{\rho_J''}{(\rho_J')^3} \ , \quad \frac{d^3E}{d\rho_J^3} = \frac{3(\rho_J'')^2}{(\rho_J')^5} - \frac{\rho_J'''}{(\rho_J')^4} \ . \tag{43}$$

Since $\rho_J' = \xi_J' > \frac{1}{3}$, we have $|\frac{dE}{d\rho_J}| < 3$. Then, for $F(\rho_J) := (h_J\circ E)(\rho_J)$, in view of the condition (E2) and (E3) for h_J, we can get

$$|F|_{\mathcal{C}^1(\rho_J(\Gamma_0^{(J)}))} \le \frac{16}{5} \ ; \qquad |F|_{\mathcal{C}^1(\rho_J(\Gamma_{j+1}^{(J)}))} \le 3\varepsilon_j^{3\sigma} \ , \quad 0\le j\le J-1 \ (\text{if } J\ge 1) \ . \tag{44}$$

By the change of variable, we get

$$\begin{aligned} &\int_{E_*}^{E_{**}} h_J e^{-\mathrm{i}Et}\,\cos M\rho_J\cdot\rho_J'\,dE \\ =\ &\frac{1}{2}\int_{\rho_J(E_*)}^{\rho_J(E_{**})} F(\rho_J)\left(e^{-\mathrm{i}t\left[E(\rho_J)+\frac{M}{t}\rho_J\right]} + e^{-\mathrm{i}t\left[E(\rho_J)-\frac{M}{t}\rho_J\right]}\right)d\rho_J \ . \end{aligned} \tag{45}$$

Step 3. Van der Corput lemma on each component

We first prove that, for any $E\in(E_*,E_{**})\subset\Gamma_0^{(J)}$, we have

$$\text{either} \quad \left|\frac{d^2E}{d\rho_J^2}\right| \quad \text{or} \quad \left|\frac{d^3E}{d\rho_J^3}\right| \ge 1-\varepsilon_0^{\frac{1}{3}} \ . \tag{46}$$

By (15) and the fact that $A_0 = \begin{pmatrix} -E & -1 \\ 1 & 0\end{pmatrix}$, we can see

$$|(\mathrm{tr}A_J)'+1| \ , \ |(\mathrm{tr}A_J)''| \ , \ |(\mathrm{tr}A_J)'''| \le 2\varepsilon_0^{\frac{1}{2}} \ \text{on } \Gamma_0 \ .$$

Since $\rho_J = \xi_J$ on $\Gamma_0^{(J)}$ and $\xi_J' = -\frac{(\mathrm{tr}A_J)'}{2\sin\xi_J}$, combining with (43), we have

$$\begin{aligned} \frac{d^2E}{d\rho_J^2} &= -\frac{4(\mathrm{tr}A_J)''\sin^2\rho_J}{(\mathrm{tr}A_J)'^3} - \frac{2\cos\rho_J}{(\mathrm{tr}A_J)'} \\ \frac{d^3E}{d\rho_J^3} &= -\frac{24(\mathrm{tr}A_J)''^2\sin^3\rho_J}{(\mathrm{tr}A_J)'^5} + \frac{8(\mathrm{tr}A_J)'''\sin^3\rho_J}{(\mathrm{tr}A_J)'^4} - \frac{12(\mathrm{tr}A_J)''\cos\rho_J\sin\rho_J}{(\mathrm{tr}A_J)'^3} + \frac{2\sin\rho_J}{(\mathrm{tr}A_J)'} \ , \end{aligned}$$

and hence

$$\left|\frac{d^2E}{d\rho_J^2}\right| + \left|\frac{d^3E}{d\rho_J^3}\right| \geq \frac{2}{|(\mathrm{tr}A_J)'|}\left(|\cos\rho_J| + |\sin\rho_J|\right) - 40\varepsilon_0^{\frac{1}{2}} \geq 2(1-\varepsilon_0^{\frac{1}{3}}) ,$$

which implies (46).

Let $\mathcal{J} \subset (E_*, E_{**})$ be the subset such that

$$\left|\frac{d^3E}{d\rho_J^3}\right| \geq 1-\varepsilon_0^{\frac{1}{3}} \text{ on } \mathcal{J} ; \quad \left|\frac{d^3E}{d\rho_J^3}\right| < 1-\varepsilon_0^{\frac{1}{3}} \text{ on } (E_*, E_{**}) \setminus \mathcal{J} .$$

Since

$$\begin{aligned} &\left|\frac{d^3E}{d\rho_J^3} - 2\sin\rho_J\right| \\ \leq\ & |\sin\rho_J|\left[2\left(\frac{1}{1-2\varepsilon_0^{\frac{1}{2}}} - 1\right) + \frac{96\varepsilon_0}{(1-2\varepsilon_0^{\frac{1}{2}})^5} + \frac{16\varepsilon_0^{\frac{1}{2}}}{(1-2\varepsilon_0^{\frac{1}{2}})^4} + \frac{24\varepsilon_0^{\frac{1}{2}}}{(1-2\varepsilon_0^{\frac{1}{2}})^3}\right] \\ \leq\ & \varepsilon_0^{\frac{1}{3}}|\sin\rho_J| , \end{aligned}$$

and $\rho_J \in \left(-\varepsilon_J^{\frac{1}{4}}, \pi + \varepsilon_J^{\frac{1}{4}}\right)$, one has that $\mathcal{J} \subset (E_*, E_{**})$ is composed at most by one sub interval (maybe empty) and $(E_*, E_{**}) \setminus \mathcal{J}$ consists of at most two sub-intervals, saying $\mathcal{S}_1$ and $\mathcal{S}_2$.

On $\mathcal{J}$, we apply Van der Corput lemma (Corollary B.2) with $k = 3$, and get (for $|t| \geq 1$)

$$\left|\int_{\rho_J(\mathcal{J})} F(\rho_J) e^{-\mathrm{i}t\left[E(\rho_J) \pm \frac{M}{t}\rho_J\right]}\, d\rho_J\right| \leq 18(1-\varepsilon_0^{\frac{1}{3}})^{-\frac{1}{3}} \cdot \frac{16}{5}(1+\pi)|t|^{-\frac{1}{3}} \leq 240|t|^{-\frac{1}{3}} .$$

On $\mathcal{S}_1$ and $\mathcal{S}_2$, we have $|\frac{d^2E}{d\rho_J^2}| \geq 1-\varepsilon_0^{\frac{1}{3}}$ in view of (47), then, by applying Corollary B.2 with $k = 2$, we get, for $l = 1, 2$, and $|t| \geq 1$

$$\left|\int_{\rho_J(\mathcal{S}_l)} F(\rho_J) e^{-\mathrm{i}t\left[E(\rho_J) \pm \frac{M}{t}\rho_J\right]}\, d\rho_J\right| \leq 8(1-\varepsilon_0^{\frac{1}{3}})^{-\frac{1}{2}} \cdot \frac{16}{5}(1+\pi)|t|^{-\frac{1}{2}} \leq 108|t|^{-\frac{1}{2}} .$$

Hence, the integral in (45) is bounded by $456|t|^{-\frac{1}{3}}$ for every connected component (E_*, E_{**}) contained in Γ_0. Recalling that there are $|\ln\varepsilon_0|^{2J^2d}$ connected components in Γ_0, we get

$$\begin{aligned} \left|\int_{\left\{\Gamma_0 : |\sin\xi| > \varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et} \cos M\rho_J \cdot \rho_J'\, dE\right| &\leq |\ln\varepsilon_0|^{2J^2d} \cdot 456\,|t|^{-\frac{1}{3}} \\ &\leq 2^{\frac{1}{6}} \cdot 456|\ln\varepsilon_0|^{2J^2d}\langle t\rangle^{-\frac{1}{3}} , \end{aligned} \tag{47}$$

since $\frac{\sqrt{1+t^2}}{|t|} \leq 2^{\frac{1}{2}}$ for $|t| \geq 1$.

If $J \geq 1$, then for $(E_*, E_{**}) \subset \Gamma_{j+1}^{(J)}$, $0 \leq j \leq J-1$, (19) implies that

$$|\rho_J'| = |\xi_J'| \leq \frac{N_j^{10\tau}}{|\sin\xi_J|} , \quad |\rho_J''| = |\xi_J''| \geq \frac{\varepsilon_j^{\frac{3\sigma}{4}}}{4|\sin\xi_J|^3} ,$$

Hence the second derivative of the inverse function satisfies

$$\left|\frac{d^2E}{d\rho_J^2}\right| = \frac{|\rho_J''|}{|\rho_J'|^3} \geq \frac{\varepsilon_j^{\frac{3\sigma}{4}}}{4|\sin\xi_J|^3}\cdot\frac{|\sin\xi_J|^3}{N_j^{60\tau}} > \varepsilon_j^{\frac{7\sigma}{8}} \ . \tag{48}$$

So we apply Corollary B.2 with $k=2$, and get

$$\left|\int_{\rho(E_*)}^{\rho(E_{**})} F(\rho_J)e^{-\mathrm{i}t\left[E(\rho_J)\pm\frac{M}{t}\rho_J\right]}\,d\rho_J\right| \leq 8\varepsilon_j^{-\frac{7\sigma}{16}}\cdot 3\varepsilon_j^{3\sigma}(1+\pi)|t|^{-\frac12} \leq \varepsilon_j^{\frac{5\sigma}{2}}|t|^{-\frac12} \ .$$

Hence, we have

$$\begin{aligned}\left|\sum_{j=0}^{J-1}\int_{\left\{\Gamma_{j+1}^{(J)}:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\cos M\rho_J\cdot\rho_J'\,dE\right| \ &\leq \ |\ln\varepsilon_0|^{2J^2d}\varepsilon_0^{\frac{5\sigma}{2}}|t|^{-\frac12}\\ &\leq \ |\ln\varepsilon_0|^{2J^2d}\varepsilon_0^{2\sigma}\langle t\rangle^{-\frac13} \ .\end{aligned} \tag{49}$$

By combining (47) and (49), we get, for $|t|\geq 1$,

$$\begin{aligned}\left|\sum_{j=0}^{J}\int_{\left\{\Gamma_{j}^{(J)}:|\sin\xi|>\varepsilon_J^{\frac{1}{20}}\right\}} h_J e^{-\mathrm{i}Et}\cos M\rho_J\cdot\rho_J'\,dE\right| \ &\leq \ \left(2^{\frac16}\cdot 456+\varepsilon_0^{2\sigma}\right)|\ln\varepsilon_0|^{2J^2d}\langle t\rangle^{-\frac13}\\ &\leq \ 512|\ln\varepsilon_0|^{2J^2d}\langle t\rangle^{-\frac13} \ .\end{aligned}$$

Since the above inequality holds trivially for $|t|\leq 1$, this concludes the proof of Lemma 3.4. □

We are now ready for the

Proof of Lemma 3.1. Fix t, and choose J in such a way that the error in Lemma 3.2 satisfies

$$\varepsilon_J^{\frac{3\sigma}{4}} \leq \frac{1}{\langle t\rangle^{\frac13}} \ .$$

this gives

$$J\geq J_* := \frac{1}{\ln(1+\sigma)}\ln\left(\frac{4}{9\sigma}\frac{\ln\langle t\rangle}{|\ln\varepsilon_0|}\right) \ . \tag{50}$$

Taking $J_\sharp$ to be the smallest integer fulfilling (50), one has that, provided ε_0 is small enough, one has

$$J_\sharp \leq J_*+1 < \frac{1}{\ln(1+\sigma)}\ln\ln(2+\langle t\rangle) \leq 201\ln\ln(2+\langle t\rangle) \ . \tag{51}$$

If $|M|\geq\frac{32}{5}\langle t\rangle^{\frac43}$, then we use the estimate (36). In such a case, the second term at r.h.s of (36) is estimated by $\frac53\frac{1}{\langle t\rangle^{\frac13}}$. The first term (with $J=J_\sharp$) is estimated by $\frac{|\ln\varepsilon_0|^{2J_\sharp^2d}}{3\langle t\rangle^{\frac43}}$. Summing up we get the result for the considered values of M.

Consider now $|M|<\frac{32}{5}\langle t\rangle^{\frac43}$ and use (37). The first two terms at r.h.s. are immediately estimated. For the third one just remark that

$$2\,|M|\,\varepsilon_J^{\frac14} \leq \frac{64}{5}\langle t\rangle^{\frac43}\left(\varepsilon_J^{\frac{3\sigma}{4}}\right)^{\frac{1}{3\sigma}} \leq \frac{64}{5}\langle t\rangle^{\frac43-\frac{1}{9\sigma}} = \frac{64}{5}\frac{1}{\langle t\rangle^{\frac{200}{9}-\frac13}} \leq \frac{64}{5}\frac{1}{\langle t\rangle^{20}} \ .$$

Summing up one gets the result. □

4. Proof of dispersive estimates

Fix any $\theta \in \mathbb{T}^d$. Given $\phi \in \ell^1(\mathbb{Z})$, let $q(t) = e^{-\mathrm{i}tH_\theta}\phi$. It solves the dynamical equation $\mathrm{i}\dot{q} = H_\theta q$ with $q(0) = \phi$. Let

$$G(E,t) \equiv \begin{pmatrix} g_1(E,t) \\ g_2(E,t) \end{pmatrix} := \mathcal{S}(q(t)).$$

For a.e. $E \in \Sigma$, we have $\begin{pmatrix} g_1(E,t) \\ g_2(E,t) \end{pmatrix} = e^{-\mathrm{i}Et} \begin{pmatrix} g_1(E,0) \\ g_2(E,0) \end{pmatrix}$.

In view of eq. (28), we have

$$|q_n(t)| \leq \frac{1}{\pi} \left| \int_\Sigma \left(g_1(E,t)\mathcal{K}_n(E) + g_2(E,t)\mathcal{J}_n(E)\right) \rho' dE \right| + \varepsilon_0^{\frac{\sigma^2}{10}} \|q(t)\|_{\ell^\infty} \ , \quad \forall\, n \in \mathbb{Z}\ . \tag{52}$$

To estimate $\|q(t)\|_{\ell^\infty}$, it is sufficient to control the above integral. By a straightforward computation, we have

$$\begin{aligned} & \int_\Sigma \left(g_1(E,t)\mathcal{K}_n(E) + g_2(E,t)\mathcal{J}_n(E)\right) \rho' dE \\ = & \int_\Sigma e^{-\mathrm{i}Et} \left(g_1(E,0)\mathcal{K}_n(E) + g_2(E,0)\mathcal{J}_n(E)\right) \rho' dE \\ = & \int_\Sigma e^{-\mathrm{i}Et} \sum_{m\in\mathbb{Z}} \phi_m \left(\mathcal{K}_m(E)\mathcal{K}_n(E) + \mathcal{J}_m(E)\mathcal{J}_n(E)\right) \rho' dE \\ = & \int_\Sigma e^{-\mathrm{i}Et} \sum_{m\in\mathbb{Z}} \phi_m \sum_{m_\Delta, n_\Delta} \left(\beta_{m,m_\Delta}\beta_{n,n_\Delta} \cos(m_\Delta - n_\Delta)\rho\right) \rho' dE\ . \end{aligned} \tag{53}$$

Lemma 4.1. *Assume that $|V|_r = \varepsilon_0 \leq \varepsilon_*$ with ε_* in Theorem 2.1. For any m, m_Δ, n, n_Δ,*

$$\left| \int_\Sigma \beta_{m,m_\Delta}\beta_{n,n_\Delta} \cos(m_\Delta - n_\Delta)\rho \cdot e^{-\mathrm{i}Et} \rho' dE \right| \leq \frac{526 |\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}}\ , \quad \forall\, t \in \mathbb{R}\ .$$

Proof. We just apply Lemma 3.1 with $h = \beta_{m,m_\Delta}\beta_{n,n_\Delta}$, $M = m_\Delta - n_\Delta$ and $h_J = \beta^J_{m,m_\Delta}\beta^J_{n,n_\Delta}$. The result immediately follows. □

End of the proof of Theorem 1.2. According to (53) and Lemma 4.1, we get, for every $n \in \mathbb{Z}$,

$$\left| \int_\Sigma \left(g_1(E,t)\mathcal{K}_n(E) + g_2(E,t)\mathcal{J}_n(E)\right) \rho' dE \right| \leq \frac{9 \cdot 526 |\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}} \|q(0)\|_{\ell^1}\ .$$

Finally, by (52), we get, for every $t \in \mathbb{R}$,

$$\begin{aligned} \|q(t)\|_{\ell^\infty} & \leq \frac{9 \cdot 526}{\pi\left(1 - \varepsilon_0^{\frac{\sigma^2}{10}}\right)} \frac{|\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}} \|q(0)\|_{\ell^1} \\ & \leq 1507 \frac{|\ln \varepsilon_0|^{a(\ln\ln(2+\langle t\rangle))^2 d}}{\langle t\rangle^{\frac{1}{3}}} \|q(0)\|_{\ell^1}\ . \quad \square \end{aligned}$$

5. Proof of Corollary 2

Fix any $0 < \zeta < \frac{1}{3}$, $p > 5$ and $\theta \in \mathbb{T}^d$. Assume that $|V|_r < \varepsilon_*$ with ε_* as in Theorem 1.2. We prove Corollary 2 for $t \geq 0$, the case $t < 0$ being totally similar.

First remark that $\|H_\theta \phi\|_{\ell^\infty} \leq 3\,\|\phi\|_{\ell^\infty}$. Denote by f the map

$$f : (q_j)_j \mapsto (\mp \mathrm{i} |q_j|^{p-1} q_j)_j \ ,$$

which describes the nonlinearity in (3); one has

$$\|f(q)\|_{\ell^\infty} \leq \|q\|^p_{\ell^\infty} \ , \tag{54}$$

$$\|f(q)\|_{\ell^1} = \sum_{j\in\mathbb{Z}} |q_j|^p \leq \left(\sup_j |q_j|^{p-2}\right) \sum_j |q_j|^2 = \|q\|^{p-2}_{\ell^\infty} \|q\|^2_{\ell^2} \ . \tag{55}$$

In particular, from (54) it follows that (3) is locally well posed in ℓ^∞. Furthermore, since the solution of equation (3) fulfills

$$\|q(t)\|_{\ell^2} = \|\phi\|_{\ell^2} \ , \tag{56}$$

it is also globally well posed in ℓ^2.

Finally we recall the following well known lemma.

Lemma 5.1. *Let $0 < \zeta \leq 1$ and $\mu > 1$ be fixed, then $\exists\ C_1 > 0$ s.t.*

$$\int_0^t \frac{1}{\langle t-s\rangle^\zeta} \frac{1}{\langle s\rangle^\mu} ds < \int_0^\infty \frac{1}{\langle t-s\rangle^\zeta} \frac{1}{\langle s\rangle^\mu} ds \leq \frac{C_1}{\langle t\rangle^\zeta}, \quad \forall\ t > 0 \ . \tag{57}$$

The main step for the proof of the Corollary 2 is the next lemma.

Lemma 5.2. *Define $M := 4K_1$ and $\delta_* := (C_1 M^{p-2})^{-\frac{1}{p-1}}$. Assume that the initial datum $q(0) = \phi$ for (3) fulfills $\delta_0 = \|\phi\|_{\ell^1(\mathbb{Z})} < \delta_*$, then, if for some $T > 0$ one has*

$$\sup_{0\leq t\leq T} \langle t\rangle^\zeta \|q(t)\|_{\ell^\infty} \leq M\delta_0 \ , \tag{58}$$

the solution still fulfills the above inequality with M replaced by $\frac{M}{2}$.

Proof. By Duhamel formula the solution of (3) fulfills

$$q(t) = e^{-\mathrm{i}tH_\theta}\phi + \int_0^t e^{-\mathrm{i}(t-s)H_\theta} f(q(s)) ds \ . \tag{59}$$

Under the assumption (58), we have, for $0 < s \leq T$,

$$\|q(s)\|_{\ell^\infty} \leq \frac{\delta_0 M}{\langle s\rangle^\zeta} \ .$$

In view of (55), for $0 \le t \le T$, the integral is estimated by

$$\begin{aligned}
\left\| \int_0^t e^{-\mathrm{i}(t-s)H_\theta} f(q(s)) ds \right\|_{\ell^\infty} &\le \int_0^t \left\| e^{-\mathrm{i}(t-s)H_\theta} f(q(s)) \right\|_{\ell^\infty} ds \\
&\le \int_0^t \frac{K_1}{\langle t-s\rangle^\zeta} \|f(q(s))\|_{\ell^1} ds \\
&\le \int_0^t \frac{K_1}{\langle t-s\rangle^\zeta} \|q(s)\|_{\ell^\infty}^{p-2} \|q(s)\|_{\ell^2}^2 ds \\
&\le \int_0^t \frac{K_1}{\langle t-s\rangle^\zeta} \frac{\delta_0^{p-2} M^{p-2}}{\langle s\rangle^{\zeta(p-2)}} \|\phi\|_{\ell^2}^2 ds \\
&= \|\phi\|_{\ell^2}^2 \delta_0^{p-2} M^{p-2} K_1 \int_0^t \frac{1}{\langle t-s\rangle^\zeta} \frac{1}{\langle s\rangle^{\zeta(p-2)}} ds \\
&\le \delta_0^p M^{p-2} K_1 \frac{C_1}{\langle t\rangle^\zeta} ,
\end{aligned}$$

where we used the fact that, under the assumption of the Corollary 2, one has $\zeta(p-2) > 1$. Using again (10) in order to estimate the term $e^{-itH_\theta}\phi$ at r.h.s. of (59), one gets

$$\sup_{0\le t\le T} \|q(t)\|_{\ell^\infty} \le \frac{K_1 \delta_0}{\langle t\rangle^\zeta} \left[1 + C_1 M^{p-2} \delta_0^{p-1} \right] .$$

The choice of the constants M and δ_* made in the statement of the lemma ensures that the square bracket is smaller than 2 and therefore the proof is completed. □

End of the proof of Corollary 2. First remark that, by local well-posedness in ℓ^∞, there exists $T > 0$ s.t. (58) holds. Assume that there exists a finite T_* which is the largest time for which (58) holds, then from Lemma 5.2, there exists $T_1 > T_*$ s.t. the estimate holds (the ℓ^∞ norm takes some time to move from $\frac{\delta_0 M}{2\langle T_*\rangle^\zeta}$ to $\frac{\delta_0 M}{\langle T_*\rangle^\zeta}$) against the assumption that T_* is the largest time for which the inequality holds. Thus the solution fulfills (58) with $T = \infty$. □

Appendix A. The fibered rotation number

Related to the Schrödinger cocycle $(\omega, A_0 + F_0)$, we can define the fibered rotation number $\rho = \rho_{(\omega, A_0+F_0)}$. It was introduced originally by Herman [Her83] in this discrete case (see also Johnson-Moser [JM82]). For the precise definition, we follow the same presentation as in [HA09].

Given $A \in C(\mathbb{T}^d, SL(2,\mathbb{R}))$ with $A(\cdot) = \begin{pmatrix} a(\cdot) & b(\cdot) \\ c(\cdot) & d(\cdot) \end{pmatrix}$, we define the map

$$\begin{array}{rccc} T_{(\omega,A)}: & \mathbb{T}^d \times \frac{1}{2}\mathbb{T} & \to & \mathbb{T}^d \times \frac{1}{2}\mathbb{T} \\ & (\theta, \varphi) & \mapsto & (\theta + \omega, \phi_{(\omega,A)}(\theta,\varphi)) \end{array} ,$$

where $\frac{1}{2}\mathbb{T} := \mathbb{R}/\pi\mathbb{Z}$ and $\phi_{(\omega,A)}(\theta,\varphi) = \arctan\left(\frac{c(\theta)+d(\theta)\tan\varphi}{a(\theta)+b(\theta)\tan\varphi}\right)$. Assume that $A(\theta)$ is homotopic to identity, then the same is true for the map $T_{(\omega,A)}$ and therefore it admits a continuous lift

$$\begin{array}{rccc} \tilde{T}_{(\omega,\,A)}: & \mathbb{T}^d\times\mathbb{R} & \to & \mathbb{T}^d\times\mathbb{R} \\ & (\theta,\varphi) & \mapsto & (\theta+\omega,\,\tilde{\phi}_{(\omega,\,A)}(\theta,\varphi)) \end{array}$$

such that $\tilde{\phi}_{(\omega,\,A)}(\theta,\varphi) \bmod \pi = \phi_{(\omega,\,A)}(\theta,\varphi \bmod \pi)$. The function

$$(\theta,\varphi)\mapsto\tilde{\phi}_{(\omega,\,A)}(\theta,\varphi)-\varphi$$

is $(2\pi)^d-$periodic in θ and $\pi-$periodic in φ. We define now $\rho(\tilde{\phi}_{(\omega,\,A)})$ by

$$\rho(\tilde{\phi}_{(\omega,\,A)})=\limsup_{n\to+\infty}\frac{1}{n}(p_2\circ\tilde{T}^n_{(\omega,\,A)}(\theta,\varphi)-\varphi)\in\mathbb{R}\ ,$$

where $p_2(\theta,\varphi)=\varphi$. This limit exists for any $\theta\in\mathbb{T}^d$, $\varphi\in\mathbb{R}$, and the convergence is uniform in (θ,φ) (For the existence of this limit and its properties we can refer to [Her83]). The class of number $\rho(\tilde{\phi}_{(\omega,\,A)})$ in $\frac{1}{2}\mathbb{T}$, independent of the chosen lift, is called the **fibered rotation number** of the skew-product system

$$\begin{array}{rccc} (\omega,A): & \mathbb{T}^d\times\mathbb{R}^2 & \to & \mathbb{T}^d\times\mathbb{R}^2 \\ & (\theta,\,y) & \mapsto & (\theta+\omega,\,A(\theta)y) \end{array},$$

and we denote it by $\rho_{(\omega,\,A)}$. For further elementary properties, we refer to Appendix of [HA09].

Appendix B. Van der Corput lemma

For the convenience of readers, we give here the statement of the Van der Corput lemma and its corollary which are used in this paper, even though they can be found in many textbooks on Harmonic Analysis (see, e.g., Chapter VIII of [Ste93]).

Lemma B.1. *Suppose that ψ is real-valued and $\mathcal{C}^k$ in (a,b) for some $k\geq 2$, and*

$$|\psi^{(k)}(x)|\geq 1,\quad \forall\ x\in(a,b)\ . \tag{60}$$

For any $\lambda\in\mathbb{R}^+$, we have

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right|\leq(5\cdot 2^{k-1}-2)\lambda^{-\frac{1}{k}}\ .$$

If the hypothesis (60) in the above lemma is replaced by

$$\text{“}|\psi^{(k)}(x)|\geq c,\quad \forall\ x\in(a,b)\text{”} \tag{61}$$

for some $c>0$ independent of x, then we can derive from Lemma B.1 that

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right|\leq(5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}\lambda^{-\frac{1}{k}},\quad \forall\ \lambda\in\mathbb{R}_+\ .$$

Moreover, since (61) also holds for $-\psi$, Lemma B.1 implies that

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}dx\right|\leq(5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}|\lambda|^{-\frac{1}{k}},\quad \forall\ \lambda\in\mathbb{R}\setminus\{0\}\ .$$

Corollary B.2. *Suppose that ψ is real-valued and $\mathcal{C}^k$ in (a,b) for some $k\geq 2$, and that $|\psi^{(k)}(x)|\geq c$ for all $x\in(a,b)$. Let h be $\mathcal{C}^1$ in (a,b). Then*

$$\left|\int_a^b e^{\mathrm{i}\lambda\psi(x)}h(x)dx\right|\leq(5\cdot 2^{k-1}-2)c^{-\frac{1}{k}}\left[|h(b)|+\int_a^b|h'(x)|dx\right]|\lambda|^{-\frac{1}{k}},\quad \forall\ \lambda\in\mathbb{R}\setminus\{0\}\ .$$

This corollary is proved by writing $\int_a^b e^{\mathrm{i}\lambda\psi(x)}h(x)dx$ as $\int_a^b F'(x)\psi(x)dx$ with $F(x) := \int_a^x e^{\mathrm{i}\lambda\psi(t)}dt$, integrating by parts, and using the previous estimate

$$|F(x)| \leq (5 \cdot 2^{k-1} - 2)c^{-\frac{1}{k}}\lambda^{-\frac{1}{k}}, \quad \forall\, x \in [a, b]\ .$$

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