

Stability of Weyl semimetals with quasiperiodic disorder

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Weyl semimetals are phases of matter with excitations effectively described by massless Dirac fermions. Their critical nature makes unclear the persistence of such phase in presence of disorder. We present a theorem ensuring the stability of the semimetallic phase in presence of weak quasiperiodic disorder. The proof relies on the subtle interplay of the relativistic Quantum Field Theory description combined with number theoretical properties used in KAM theory.

I. INTRODUCTION

Conduction electrons in metals are well described by the Schroedinger equation but in certain cases the interaction with the lattice produces an effective relativistic description in terms of massless Dirac particles; this happens, in particular, in Weyl semimetals [1], which have been recently experimentally discovered [2]-[5]. This offers the possibility of observing the counterpart of high energy phenomena at a much lower energy scale, and to have materials with unusual physical properties. The critical nature of excitations has the effect that in several cases predictions are ambiguous and sensitive to approximations. Indeed, while there is agreement that at weak coupling many body interactions do not destroy the semimetallic phase [7]-[6], it is still subject of debate the effect of disorder. Field theoretical approaches find that a weak random disorder does not destroy the semimetallic phase [10], [11] while other studies [12] based on the inclusion of rare region effects lead to the opposite conclusion, namely that even an arbitrary weak random potential destabilizes the system. Numerical investigations have been done for random [13]-[19] or quasiperiodic disorder [20],[21], but conclusions are subjected to finite size effects [22].

Rigorous results in this context are useful as can act as benchmark to check approximations or conjectures. In this paper we rigorously analyze Weyl semimetals on a lattice in presence of a weak quasiperiodic disorder. Such disorder is the one realized in cold atoms experiments [23], [24]; in addition quasiperiodic potential can effectively describe coupled Dirac systems like Moire' superlattices [25]. The effect of quasi-periodic potentials for quantum particles has been deeply studied in one dimension; in the non interacting case a very detailed mathematical knowledge has been reached [26], [27], and recently great progress in understanding the effect of the interaction has been obtained [28]-[37]. In contrast, very little is known for higher dimensional Dirac systems, with the exception of [20],[21] where numerical evidence of stability of the Weyl semimetallic phase was found. The main difficulty of quasiperiodic disorder is the presence

of infinitely many processes involving a large exchange of momentum which, due to Umklapp and incommensurability of frequencies, connect fermions with momenta close to the Weyl points. Such processes are dimensionally relevant in the Renormalization Group (RG) sense and the effect of disorder in principle increases at each RG iteration and could destroy the Weyl semimetallic phase. This phenomenon manifests in the presence in the series expansion of small divisors which could break convergence.

A similar situation is encountered in classical mechanics and in particular in Kolmogorov-Arnold-Moser (KAM) theory, where quasiperiodic solutions are written as Lindstedt series see *e.g.*[38]. Such series are plagued by small divisors but their convergence is ensured by subtle cancellations due to number theoretical properties of irrational numbers, see *e.g.*[39]. In this paper we show that a similar phenomenon allows to prove the stability of the semimetallic phase in Weyl semimetals; number theoretical properties allow to prove that the relevant terms almost connecting Weyl points are indeed ineffective. Physical quantities are written as convergent series so that non-perturbative effects due to small divisors are excluded.

The paper is organized in the following way. In §II the model is presented, in §III we describe the effect of Umklapp terms, in §IV we recall number theoretical properties of irrationals and in §V the main result is presented. Finally in §VI the Renormalization Group analysis is presented and §VII is devoted to conclusions.

II. WEYL SEMIMETALS WITH QUASIPERIODIC DISORDER

A basic model for Weyl semimetals, see [1], is obtained assuming a pair of orbitals on each site of a lattice, preserving inversion but with broken time reversal symmetry; if $x = (x_1, x_2, x_3)$ are points in a cubic three-dimensional lattice Λ , $a_{x,1}^\pm, a_{x,2}^\pm$ fermionic creation or annihilation operators, the hopping Hamiltonian is $H_0 =$

$$\sum_{x \in \Lambda} \left\{ \sum_{j=1}^2 (-1)^{j-1} [(\zeta - 1) a_{x,j}^\dagger a_{x,j} + \frac{1}{2} a_{x,j}^\dagger (-\Delta a)_{x,j}] + (1) \right. \\ \left. + \frac{it_1}{2} [a_{x,1}^\dagger (a_{x+e_1,2} - a_{x-e_1,2}) + a_{x,2}^\dagger (a_{x+e_1,1} - a_{x-e_1,1})] + \right.$$

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$$\frac{t_2}{2} [a_{x,1}^\dagger (a_{x+e_1,2} - a_{x-e_1,2}) - a_{x,2}^\dagger (a_{x+e_1,1} - a_{x-e_1,1})]$$

where in the first line Δ is the standard lattice Laplacian: $\Delta f(x) = \sum_{l=1}^3 [f(x+e_l) + f(x-e_l) - 2f(x)]$. The Hamiltonian H_0 in Fourier space can be written as $H_0 = \int \frac{dk}{(2\pi)^3} \hat{a}_k^\dagger h(k) \hat{a}_k$ with

$$h(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ \beta^*(k) & -\alpha(k) \end{pmatrix} \quad (2)$$

where $k \in (0, 2\pi]^3$, $\alpha(k) = 2 + \zeta - \cos k_1 - \cos k_2 - \cos k_3$ and $\beta(k) = t_1 \sin k_1 - it_2 \sin k_2$. We assume that $\zeta \in [0, 1)$, in which case $\hat{h}(k)$ is singular at $k = \pm p_F$, with $p_F = (0, 0, \arccos \zeta)$ called Weyl point. In the vicinity of $\pm p_F$, $k = q \pm p_F$

$$\hat{H}^0(q \pm p_F) = t_1 \sigma_1 q_1 + t_2 \sigma_2 q_2 \pm \sin p_F \sigma_3 q_3 + O(q^2) \quad (3)$$

We include now a many body interaction and quasiperiodic disorder writing

$$H = H_0 + \varepsilon \sum_x \phi_x (a_{x,1}^+ a_{x,1}^- - a_{x,2}^+ a_{x,2}^-) + \lambda \sum_{x,y} v(x-y) \rho_x \rho_y \quad (4)$$

where $v(x-y)$ is a short range potential and

$$\phi_x = \sum_n \hat{\phi}_n e^{i2\pi(\omega_1 n_1 x_1 + \omega_2 n_2 x_2 + \omega_3 n_3 x_3)} \quad (5)$$

with $n \in \mathbb{Z}^3$, $\hat{\phi}_n = \hat{\phi}_{-n}$ and $|\hat{\phi}_n| \leq C e^{-\xi(|n_1|+|n_2|+|n_3|)}$. We assume the periodicity of the potential incommensurate with the lattice periodicity, by taking ω_i *irrational*. The above potential includes the basic example of disorder like $\sum_i \cos(\omega_i x_i)$.

If $\psi_{\mathbf{x}}^\pm = e^{Hx_0} \psi_{\mathbf{x}}^\pm e^{-Hx_0}$, $\mathbf{x} = (x_0, x)$, x_0 the imaginary time, the 2-point function is given by $S(\mathbf{x}, \mathbf{y}) = \frac{\text{Tr} e^{-\beta H} T \psi_{\mathbf{x}}^- \psi_{\mathbf{y}}^+}{\text{Tr} e^{-\beta H}}$ and $\hat{S}(\mathbf{k})$ is the Fourier transform. In the non-interacting case $\lambda = \varepsilon = 0$ one has $S(\mathbf{x}, \mathbf{y})|_0 = g(\mathbf{x} - \mathbf{y})$ with

$$g(\mathbf{x}) = \frac{1}{L^3 \beta} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} (-ik_0 I + h(k))^{-1} \quad (6)$$

From (3) we see that close to the Weyl momenta the propagator $\hat{g}(\mathbf{q} \pm \mathbf{p}_F)$ is equal to the massless Dirac propagator up to corrections. By this, one can easily deduce the physical properties; for instance the real part of the zero temperature optical conductivity vanishes linearly with the frequency $\sigma(\omega) \sim \omega$.

In order to investigate the stability of the Weyl semimetallic phase in presence of incommensurate potential, it is convenient to write the interacting correlations as $S(\mathbf{x}, \mathbf{y}) = \frac{\partial^2 W}{\partial \phi_{\mathbf{x}}^- \partial \phi_{\mathbf{y}}^+}$, where $W(\phi)$ is Grassmann integral defined in the following way

$$e^{W(\phi)} = \int P(d\psi) e^V \quad (7)$$

where ϕ is an external field, $\psi_{\mathbf{x},i}^\pm$ are Grassmann variables, $P(d\psi)$ is Grassman integration with propagator $g(\mathbf{x})$ and

$$V = \lambda \int d\mathbf{p} \hat{v}(\mathbf{p}) \hat{\rho}_{\mathbf{p}} \hat{\rho}_{-\mathbf{p}} + \int d\mathbf{x} (\psi_{\mathbf{x}}^+ \phi_{\mathbf{x}}^- + \psi_{\mathbf{x}}^- \phi_{\mathbf{x}}^+) + (8)$$

$$\varepsilon \sum_{n,i} \hat{\phi}_n \int d\mathbf{k} (-1)^i \hat{\psi}_{i,\mathbf{k}_1}^+ \hat{\psi}_{i,\mathbf{k}_2}^- \delta_p(\mathbf{k}_1 - \mathbf{k}_2 + \bar{\omega}_n 2\pi)$$

where $\bar{\omega}_n = (0, \omega_n)$, $\omega_n = (\omega_1 n_1, \omega_2 n_2, \omega_3 n_3)$, $\rho_{\mathbf{x}} = \psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},1}^- + \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-$, $\mathbf{x} = (x_0, x)$ and $\int d\mathbf{x} = \int_{-\beta/2}^{\beta/2} dx_0 \sum_{\bar{x}}$, $\hat{\rho}_{\mathbf{p}} = \int d\mathbf{k} (\hat{\psi}_{\mathbf{k},1}^+ \hat{\psi}_{\mathbf{k}+\mathbf{p},1}^- + \hat{\psi}_{\mathbf{k},2}^+ \hat{\psi}_{\mathbf{k}+\mathbf{p},2}^-)$; moreover

$$\delta_p(\mathbf{x}) = \delta_p(x_0) \prod_{i=1}^3 \delta_p(x_i) \quad \delta_p(x_i) = L \sum_n \delta_{x_i, 2n\pi} \quad (9)$$

Note that momentum is conserved to momenta $2\pi n$ due to the presence of the lattice.

III. RELEVANT PROCESSES AND UMKLAPP TERMS

A natural way to understand the effect of the interaction and disorder is to use Renormalization Group. The physical informations are encoded in the marginal or relevant processes, that is the terms with vanishing or positive scaling dimension. The linear divergence at the Weyl points of the propagator (14) says that the scaling dimension of the interactions with n ψ fields is $D = 4 - 3n/2$, so that the only relevant terms are the bilinear ones. In absence of quasiperiodic potential $\varepsilon = 0$, there is only one relevant term corresponding to a shift in the position of the Weyl points. The irrelevance of the quartic terms has the effect that, in the weak coupling regime, the semimetallic behavior persists and the only effects of the interaction are finite renormalization of the velocities and wave function, see [6].

The presence of quasi-periodic potential produces infinitely many relevant terms quadratic in the fields, with momenta $\mathbf{k}_1, \mathbf{k}_2$ such that $k_{1,i} - k_{2,i} + 2\omega_i n_i \pi + 2l_i \pi = 0$ with l_i, n_i positive or negative integers. The factor $2\omega_i n_i \pi$ is the momentum exchanged with the quasiperiodic disorder while the factor $2l_i \pi$ is exchanged with the lattice (Umklapp). Only the terms connecting fermions with momenta close to the Weyl points are really important and, due to Umklapp, this can happen also in correspondence of a non vanishing transfer of momentum produced by the disorder. The important processes involve fermions with momenta close to the same Weyl point $\sigma = 0$ or to opposite ones $\sigma = 2$ ones; if $p_F = (0, 0, p_{F,3})$ this requires

$$n_1 \omega_1 - l_1 \sim 0 \quad n_2 \omega_2 - l_2 \sim 0 \quad n_3 \omega_3 - l_3 \pm \sigma p_{F,3} \sim 0 \quad (10)$$

Note the basic difference between periodic or quasiperiodic potentials. In the first case ω_i is rational $\omega_i =$

p/q so that the differences in (10) either are exactly vanishing or are $O(1/q)$ (if $p_F \neq n\omega_3/2$): there are no processes connecting momenta arbitrarily close to the Weyl points except the one with $n_i = 0$, a process corresponding to the shift of the chemical potential. Therefore a periodic potential is not expected to modify the physical behavior for generic values of p_F , at least for small ε .

In contrast, in the quasiperiodic case (10) can be arbitrarily close to zero, for the basic properties of irrational numbers. This means that there are infinitely many relevant processes connecting the Weyl points. Such a feature makes the case of quasiperiodic potentials very close to the random case, where the difference of momenta of relevant terms is $k_1 - k_2 = p$ with p the momentum carried by a random field $\hat{\phi}_p$ which can be arbitrarily small.

IV. KAM THEOREM AND DIOPHANTINE CONDITIONS

In the case of random potential the issue of stability is related to the probability that certain dangerous configurations happens. In the quasiperiodic case, the problem is deterministic and related to the irrationality properties of the frequencies. Therefore quantitative estimates saying how much an irrational is close to a rational one are necessary. For instance the golden number $\omega = \frac{\sqrt{5}-1}{2}$ verifies $|q\omega - p| \geq \frac{1}{(3+\sqrt{5})2\pi q}$. If such ω is the frequency of the quasiperiodic potential, this says that, looking at (10), *only* the processes involving a *large* transfer of momentum can involve fermions *close* the Weyl points. Such a property is indeed generic. There is a class of irrationals called *Diophantine*, such that, for $q \neq 0$, $p, q \in \mathbb{Z}^2/(0,0)$

$$|q\omega - p| \geq \frac{C_0}{2\pi q^\tau} \quad (11)$$

The irrationals not verifying (11) in the unit segment have measure $O(C_0)$; as C_0 can be taken arbitrarily small, the set of Diophantine numbers is full, see e.g. [38]. Indeed the set of ω in the unit cube verifying $|q\omega - p| < \frac{C_0}{q^\tau}$ for a certain q, p is smaller than $2C_0/q^{\tau+1}$ hence summing over p (a sum bounded by $C|q|$) and q we get a set with measure bounded by $C_0 \sum_q \frac{1}{q^\tau}$ which is $O(C_0)$ for $\tau > 1$.

It is therefore not restrictive to assume the following conditions on the frequencies

$$|\omega n|_T \geq \frac{C_0}{|n|^\tau} \quad |\omega n \pm 2p_{F,3}|_T \geq \frac{C_0}{|n|^\tau} \quad n \in \mathbb{Z}/0 \quad (12)$$

where by $|\cdot|_T$ we mean the average on the torus, that is $|\omega n|_T = \inf_p |\omega n - p|$; the first condition is (11) and the second is a requirement of irrationality for $p_{F,3}$. As we will see, Diophantine conditions are crucial to prove the stability of the Weyl semimetallic phase.

Another point to stress is that in order to impose periodic boundary conditions we have to choose a sequence

of ω rational converging to an irrational in the infinite volume limit. In order to do that we start from the continued fraction representation of a number ω

$$\omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \quad (13)$$

We approximate ω by a sequence of rational numbers (*convergents*) $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$, $\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}$ and so on. Properties of the convergents imply that if ω verifies the Diophantine condition then $|\pi(n\frac{p_i}{q_i} - k)| \geq \frac{C}{2|n|^\tau}$ if $q_1 \leq n \leq \frac{q_i}{2}$ and any k . Therefore we can impose periodic boundary conditions by considering a sequence of frequencies $\omega_i = \frac{p_i}{q_i}$ and $L_i = q_i$.

Finally it is worth to recall that number theoretical conditions are unusual in condensed matter but rather common in other branches of physics. For instance planets around sun neglecting the mutual attraction have an integrable Hamiltonian dynamics which is quasiperiodic, and according to KAM theory only quasiperiodic motions with Diophantine frequencies survive in presence of perturbation breaking integrability [38]. Indeed quasiperiodic solutions are written as series in the perturbation, called Lindstedt series, whose convergence follows by subtle cancellations due to Diophantine conditions, see e.g. [39].

V. MAIN RESULT

As the interaction in general moves the location of the Weyl momentum, we write $\xi = \cos p_F + \nu$ in (4) and we choose ν so that p_F is the just the interacting Weyl momentum.

Theorem. *For λ, ε small enough and assuming that the frequencies ω_i in (5) verify (12), there exists ν such the 2-point function $\hat{S}(\mathbf{k})$ behaves as, if $\mathbf{p}_F = (0, p_F)$,*

$$S(\mathbf{q} \pm \mathbf{p}_F) = \frac{1}{Z} \begin{pmatrix} -iq_0 \pm v_3 q_3 & v_1 q_1 - iv_2 q_2 \\ v_1 q_1 + iv_2 q_2 & -iq_0 \mp v_3 q_3 \end{pmatrix}^{-1} (1 + O(\mathbf{q}))$$

with $Z = 1 + O(\lambda, \varepsilon)$, $v_1 = t_1 + O(\lambda, \varepsilon)$, $v_2 = t_2 + O(\lambda, \varepsilon)$, $v_3 = \sin p_F + O(\lambda, \varepsilon)$

This result proves the stability of the Weyl semimetallic phase, as quasiperiodic disorder does not modify qualitatively the 2-point function but produces only a finite renormalization of the parameters; no phase transition is present at small disorder. As a consequence the real part of optical conductivity vanishes as $O(\omega)$ as in the non interacting case. Even if there are infinitely many relevant terms due to quasiperiodic disorder, they do not modify the physical behavior. The result is in agreement with the numerical evidence in [20],[21].

VI. RENORMALIZATION GROUP

In order to prove (14) we need to evaluate the generating function $\int P(d\psi)e^{\mathcal{V}}$ with $\mathcal{V} = V + \nu \int \widehat{\psi}^+ \sigma_3 \widehat{\psi}^-$ with V given by (9) and propagator given by $g(\mathbf{x})$. We introduce two smooth cut-off functions $\chi_{\pm}(\mathbf{k})$ non vanishing in a region $|\mathbf{k} \mp \mathbf{p}_F| \leq \gamma$ and non-overlapping, γ a suitable constant: we define $\widehat{g}_{\rho}^{(\leq 0)}(\mathbf{k}) = \chi_{\rho}(\mathbf{k})\widehat{g}(\mathbf{k})$ and

$$g(\mathbf{x}) = g^{(1)}(\mathbf{x}) + \sum_{\rho=\pm} g_{\rho}^{(\leq 0)}(\mathbf{x}) \quad (14)$$

with $\widehat{g}^{(1)}(\mathbf{k}) = (1 - \sum_{\rho} \chi_{\rho})\widehat{g}(\mathbf{k})$; this induces the Grassmann variable decomposition $\psi_{\mathbf{x}} = \psi_{\mathbf{x}}^{(1)} + \sum_{\rho=\pm} \psi_{\rho}^{(\leq 0)}$ with propagators given by $g^{(1)}(\mathbf{x})$ and $g_{\rho}^{(\leq 0)}(\mathbf{x})$ respectively. Note that $\psi^{(1)}$ correspond to fermions with momenta far from the Weyl points, while $\psi_{\pm}^{(\leq 0)}$ with momenta around $\pm \mathbf{p}_F$.

We can further decompose $\widehat{g}_{\rho}(\mathbf{k}) = \sum_{h=-\infty}^0 \widehat{g}_{\rho}^{(h)}(\mathbf{k})$, $\rho = \pm$ with the cut-of function χ_{ρ} replaced by f_h with support in $\gamma^{h-1} \leq |\mathbf{k} - \rho \mathbf{p}_F| \leq \gamma^{h+1}$. After the integration of $\psi^{(1)}, \psi^{(0)}, \dots, \psi^{(h+1)}$ the generating function has the form

$$e^{W(\phi, J)} = \int P(d\psi^{(\leq h)}) e^{\mathcal{V}^{(h)}(\psi^{(\leq h)}, \phi)} \quad (15)$$

where $P(d\psi^{(\leq h)})$ has propagator

$$\widehat{g}_{\pm}^{(h)}(\mathbf{q}) = \frac{1}{Z_h} f_h(\mathbf{q}) \begin{pmatrix} -iq_0 \pm v_{3,h} q_3 & v_{1,h} q_1 - iv_{2,h} q_2 \\ v_{1,h} q_1 + iv_{2,h} q_2 & -iq_0 \mp v_{3,h} q_3 \end{pmatrix}^{-1}$$

and $\mathcal{V}^{(h)}(\psi, 0) =$

$$\sum_{m,n,\rho} \int d\mathbf{q}_1 \dots d\mathbf{q}_m W_{n,m}^{(h)}(\mathbf{q}) \psi_{\rho_1, \mathbf{q}_1}^{\varepsilon_1(\leq h)} \dots \psi_{\rho_m, \mathbf{q}_m}^{\varepsilon_m(\leq h)} \delta_{n,m}(\mathbf{q}) \quad (16)$$

where $\delta_{n,m}(\mathbf{q})$ is $L\beta$ times a periodic Kronecker delta non vanishing for $\sum_{i=1}^m \varepsilon_i q_{0,i} = 0$ and, $p_F = (0, 0, p_{F,3})$

$$\sum_{i=1}^m \varepsilon_i q_i = - \sum_{i=1}^m \varepsilon_i \rho_i p_F + 2\pi\omega_n + 2l\pi \quad (17)$$

with $l = (l_1, l_2, l_3)$ and $\omega_n = (\omega_1 n_1, \omega_2 n_2, \omega_3 n_3)$. $\mathcal{V}^{(h)}(\psi, \phi)$ has a similar expression with some ψ field replaced by an external field. The stability of the semimetallic phase relies in the fact that the sequence of effective potentials \mathcal{V}^h remains small for any RG iteration. There are however relevant terms, that is terms that could increase linearly according to dimensional analysis; they are the infinitely many terms, depending on n , with $m = 2$ in (16). One needs to show that, despite the linear divergence suggested by scaling, such terms indeed remain small due to cancellations relying on number theoretical properties.

Note first that one can distinguish between the terms with $m = 2$ such that the l.h.s. of (17) is vanishing,

which we call resonant, from the other, which we call non-resonant. The resonant terms with $m = 2$ are possible only for $n_i = 0, l = 0$ and $\rho_1 = \rho_2$; the case $\rho_1 = -\rho_2$ would be possible if $p_{F,3} = n\omega_3/2$, a case excluded by the assumption (12). We define a localization operator

$$\mathcal{L}W_{0,2}^{(h)}(\mathbf{q}) = W_{0,2}^{(h)}(0) + \mathbf{q} \partial W_{0,2}^{(h)}(0) \quad (18)$$

and $\mathcal{L}W_{n,m}^{(h)}(\mathbf{q}) = 0$ otherwise. Note that the graphs contributing to the nondiagonal part contain an odd number of non diagonal propagators hence are vanishing; moreover $W_{0,2;11}^{(h)}(0) = -W_{0,2;22}^{(h)}(0)$. In addition the derivative with respect to 0, 3 of graphs contributing to the non diagonal part is zero, as they contain an odd number of non diagonal propagators, and the derivative with respect to 1, 2 of graphs contributing to the diagonal part is zero, as it contain an even number of non diagonal propagators.

We can write $\mathcal{V}^h = \mathcal{L}\mathcal{V}^h + \mathcal{R}\mathcal{V}^h$ with $\mathcal{R} = 1 - \mathcal{L}$ and rewrite (19) as

$$\int P(d\psi^{(\leq h)}) e^{\gamma^h \nu_h F^{(h)} + \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h)}, \phi)} \quad (19)$$

with $F^{(h)} = \int d\mathbf{x} (\psi_{\mathbf{x},1}^+ \psi_{\mathbf{x},1}^- - \psi_{\mathbf{x},2}^+ \psi_{\mathbf{x},2}^-)$. One can write $P(d\psi^{(\leq h)}) = P(d\psi^{(\leq h-1)})P(d\psi^{(h)})$ and integrate $\psi^{(h)}$ obtaining an expression similar to (16) with $h-1$ replacing h , and the procedure can be iterated.

The kernels $W_{n,m}^{(h)}$ can be written as sum of Feynman diagrams composed by vertices connected by lines, such that to each line is associated a scale label h and it corresponds to a propagator $g^{(h)}$.

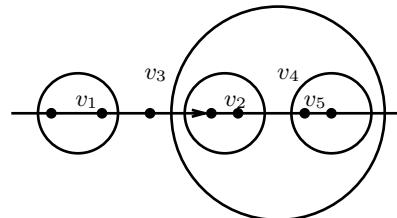


FIG. 1: An example of graph of order ε^7 with the associated clusters (the circles) contributing to $\widehat{W}_{0,2}^{(h)}(\mathbf{k})$; the value is $\widehat{g}^{h_{v_1}}(\mathbf{k}_1)\widehat{g}^{h_{v_3}}(\mathbf{k}_2)\widehat{g}^{h_{v_2}}(\mathbf{k}_3)\widehat{g}^{h_{v_4}}(\mathbf{k}_4)\widehat{g}^{h_{v_5}}(\mathbf{k}_5)$ with $k_i = k + 2\pi\omega_{n_i}$ and $h_{v_3} < h_{v_1}, h_{v_3} < h_{v_4}, h_{v_4} < h_{v_2}, h_{v_4} < h_{v_5}$.

We can therefore consider a cluster v , that is the maximally connected subset of lines corresponding to propagators with scale $h \geq h_v$ with at least a scale h_v ; n_v^e is the number of lines external to the cluster v , by definition with scale smaller than h_v (for more details see e.g. [40]). The structure of clusters induce a natural notion of subgraphs which avoid the well known problem of overlapping divergences. The clusters with $n_v^e = 2$ are such that the difference of momenta k is $2\pi(N_1\omega_1, N_2\omega_2, N_3\omega_3)$. We call v' the minimal cluster

containing v so that $h_v - h_{v'} > 0$, and S_v is the number of clusters or vertices contained in v and not in any smaller cluster.

Using that the propagator $g^{(h)}(\mathbf{x})$ is bounded by γ^{3h} and the integral of the propagator over coordinates by γ^{-h} , a graph is bounded by $C^s(\max(\varepsilon, \lambda, \nu_h))^s$ times

$$\prod_v \gamma^{-4h_v(S_v-1)} \prod_v \gamma^{3h_v n_v} \prod_v \gamma^{z_v(h_{v'}-h_v)} \quad (20)$$

where n_v is the number of propagators in the cluster v but not in any smaller one, and $z_v = 2$ if v is a resonant cluster with $n_v^e = 2$ and zero otherwise; the last term in the above expression is produced by the renormalization \mathcal{R} .

By using the relations

$$\begin{aligned} \sum_v (h_v - h)(S_v - 1) &= \sum_v (h_v - h_{v'})(m_v^4 + m_v^2 - 1) \\ \sum_v (h_v - h)n_v &= \sum_v (h_v - h_{v'})(2m_v^4 + m_v^2 - n_v^e/2) \end{aligned}$$

one gets

$$\gamma^{Dh} \prod_v \gamma^{(h_v-h_{v'})(D_v-z_v)} \prod_v \gamma^{2h_v \bar{m}_v^4} \prod_v \gamma^{-h_v \bar{m}_v^2} \left[\prod_i e^{-\xi|\bar{n}_i|} \right] \quad (21)$$

where \bar{m}_v^4 is the number of vertices λ contained in v and not in any smaller cluster, \bar{m}_v^2 is the number of vertices ε contained in v and not in any smaller cluster, s is the order, $D_v = 4 - 3n_v^e/2$; the last term is due to the decay factors $\hat{\phi}_n$. Note that the same bound is valid for the sum over all Feynman graphs from the cancellations due to Pauli principle; no combinatorial problems arise for the number of graphs, see [40]

One needs to sum over all the choices of scales $\{h\}$. By looking to (30) we see indeed that if for all v one has $D_v - z_v < 0$ then one can sum over all the choices of scales, that is $\sum_{\{h\}} \prod_v \gamma^{(h_v-h_{v'})(D_v-z_v)}$ is bounded by C^s (remember that $h_v - h_{v'} > 0$). There are however clusters with $D_v - z_v = 0$, actually the non resonant clusters with $n_v^e = 2$, and this produces a divergent bound $|h|^s$. Such divergence may suggest that the Weyl semimetallic behavior is unstable.

We need however to take into account the number theoretical properties of the frequencies. Let us consider a subgraph with two external lines (see e.g. Fig. 2), associated to propagators with momenta $\mathbf{k}_1, \mathbf{k}_2$. If q_1, q_2 are the momenta measured from Weyl points, $k = q \pm p_F$, external to a cluster v , one has $|q| \leq \gamma^{h_{v'}}$ for the compact support properties of the propagator. Moreover $k_1 - k_2 = 2\pi(N_1\omega_1, N_2\omega_2, N_3\omega_3)$ so that, if $|q|_T = \sqrt{|q_1|_T^2 + |q_2|_T^2 + |q_3|_T^2}$

$$2\gamma^{h_{v'}} \geq |q_1|_T + |q_2|_T \geq |q_1 - q_2|_T \quad (22)$$

where we have used the triangular inequality on the torus. Now we use the Diophantine property (12), $\varepsilon =$

$0, \pm$

$$2\gamma^{h_{v'}} \geq \sqrt{|\omega_1 N_1|_T^2 + |\omega_2 N_2|_T^2 + |\omega_3 N_3 + \varepsilon 2p_{F,3}|_T^2} \geq \frac{3C_0}{\bar{N}^\tau} \quad (23)$$

so that, if $\bar{N} = \max(N_1, N_2, N_3)$ then

$$\bar{N} \geq C\gamma^{-h_{v'}/\tau} \quad (24)$$

This inequality says that if the momenta external to a non resonant cluster are very small, than the momentum transferred is very large. On the other hand by conservation of momentum $N_v = \sum_i n_i$ where n_i is the momentum associated with each ε vertex in the cluster, and

$$\prod_i e^{-\xi|n_i|} \leq e^{-\xi\bar{N}} \leq e^{-C\gamma^{-h_{v'}/\tau}} \quad (25)$$

as $\sum_i |n_i| \geq |\sum_i n_i| \geq \bar{N}$. This relation implies a dramatic improvement with respect to the dimensional bound.

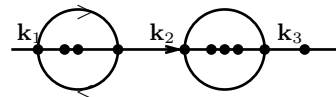


FIG. 2: A contribution order $\lambda^4 \varepsilon^6$ to $\widehat{W}_{0,2}^{(h)}(\mathbf{k})$; if $k_2 - k_3 = 2\pi\omega_n$ and $|q_2| \sim \gamma^{-h_{v'}}$, $|q_3| \sim \gamma^{-h_{v'}}$ then $\max_i |n_i| \geq \gamma^{-h/\tau}$.

There is however in general a sequence of clusters enclosed one in the other and ε vertices are generally internal to several clusters. In order to get a decay factor for each cluster we can write

$$e^{-\xi|n|/2} = \prod_{h=-\infty}^{-1} e^{-\xi 2^h |n|/2} \quad (26)$$

We can therefore associate to each relevant non-resonant cluster v a factor $e^{-\xi \bar{N}_v 2^{h_{v'}}}$, so that, see Fig. 3

$$e^{-\xi \bar{N}_v 2^{h_{v'}}} \leq e^{-\xi 2^{h_{v'}} \gamma^{-h_{v'}/\tau}} \quad (27)$$

If we choose $\gamma^{1/\tau} = 4$ then $e^{-\xi 2^{h_{v'}}} \leq (Ne/\xi)^{N} 2^{N 2^{h_{v'}}}$, by using $e^{-\alpha x} x^N \leq (\frac{N\varepsilon}{\alpha})^N$. Therefore, choosing N so that $2^N = \gamma$ ($N = 2\tau$)

$$\left[\prod_i e^{-\xi|\bar{n}_i|/2} \right] \leq C^s \prod_v \gamma^{h_v 2S_v^{NR}} \quad (28)$$

where S_v^{NR} is the number of non resonant clusters or vertices in v and not in any smaller cluster and m is the order. Using that

$$\prod_v^* \gamma^{-2(h_{v'}-h_v)} \prod_v \gamma^{-h_v 2\bar{m}_v^2} \leq \prod_v \gamma^{-h_v 2S_v^{NR}} \quad (29)$$

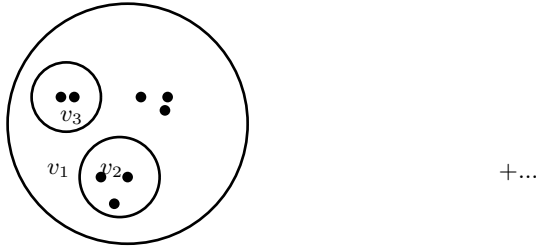


FIG. 3: Three clusters v_1, v_2, v_3 . If the points are associated to ϕ_{n_i} , assume 1, 2, 3 in v_2 , 3, 4 in v_3 and 4, 5, 6 in v_1 . Hence $\prod_{i=1}^8 e^{-|n_i|}$ is bounded by $e^{-(|n_1|+|n_2|+|n_2|)2^{h v_2}} e^{-(|n_3|+|n_4|)2^{h v_3}} e^{-(|n_1|+\dots+|n_8|)2^{h v_1}}$

where the first product is over the non resonant relevant v we get that (30) is replaced by

$$\gamma^{Dh} \prod_v \gamma^{(h_v - h_{v'}) (D_v - \bar{z}_v)} \prod_v \gamma^{h_v \bar{m}_v^4} \left[\prod_i e^{-\xi |\bar{n}_i|/2} \right] \quad (30)$$

with $\bar{z}_v = 2$ for $n_v^e = 2$ and zero otherwise.

As $D_v - \bar{z}_v > 1$ we can sum over all the scale choices getting a bound $O(C^s \max(\lambda, \varepsilon, \nu_h)^s)$ from which convergence of the series expansions follows, provided that ν is chosen so that ν_h vanishes as $h \rightarrow -\infty$.

Finally we note that the velocities verify a recursive relation $v_{h-1} = v_h + \beta_v^h$ and the wave function renormalization verifies $Z_{h-1} = Z_h + \beta_z^h$; note that the Feynman graphs contributing to β_h have at least a λ vertex so that $\beta^h = O(\lambda \gamma^h)$ and $v_{-\infty} = v_0 + O(\lambda)$, $Z_{-\infty} = 1 + O(\lambda)$. The correlations are therefore close to the non-interacting ones up to finite renormalizations.

VII. CONCLUSION

We have rigorously established the stability of the Weyl semimetallic phase in presence of weak interaction and quasiperiodic disorder. Even if the infinitely many relevant terms produced by the disorder could possibly destabilize the semimetallic phase, this is avoided by subtle cancellations due to number theoretical properties. The physical properties appear to be determined by the interplay of relativistic Quantum Field Theory with classical mechanics and KAM theory. There are no phase transition for weak quasiperiodic disorder, where rare region effects are absent. If a similar rigorous RG analysis can be performed in the case of random disorder is a very interesting open question.

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