METASTABILITY PHENOMENA IN TWO-DIMENSIONAL RECTANGULAR LATTICES WITH NEAREST-NEIGHBOUR INTERACTION

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Abstract. We study analytically the dynamics of two-dimensional rectangular lattices with periodic boundary conditions. We consider anisotropic initial data supported on one low-frequency Fourier mode. We show that, in the continuous approximation, the resonant normal form of the system is given by integrable PDEs. We exploit the normal form in order to prove the existence of metastability phenomena for the lattices. More precisely, we show that the energy spectrum of the normal modes attains a distribution in which the energy is shared among a packet of low-frequencies modes; such distribution remains unchanged up to the time-scale of validity of the continuous approximation.
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Contents

1. Introduction

intro

In this paper we present an analytical study of the dynamics of two-dimensional rectangular lattices with nearest-neighbour interaction and periodic boundary conditions, for initial data with only one low-frequency Fourier mode initially excited. We give some rigorous results concerning the relaxation to a metastable state, in which energy sharing takes place among low-frequency modes only.

The study of metastability phenomena for lattices started with the numerical result by Fermi, Pasta and Ulam (FPU) [?], who investigated the dynamics of a one-dimensional chain of particles with nearest neighbour interaction. In the original simulations all the energy was initially given to a single low-frequency Fourier mode with the aim of measuring the time of relaxation of the system to the 'thermal equilibrium' by looking at the evolution of the Fourier spectrum. Classical statistical mechanics prescribes that the energy spectrum corresponding to the thermal equilibrium is a plateau (the so-called theorem of equipartition of energy). Despite the authors believed that the approach to such an equilibrium would have occurred in a short time-scale, the outcoming Fourier spectrum was far from being flat and they observed two features of the dynamics that were in contrast with their expectations: the lack of thermalization displayed by the energy spectrum and the recurrent behaviour of the dynamics.

Both from a physical and a mathematical point of view, the studies on FPU-like systems have a long and active history: a concise survey of this vast literature is discussed in the monograph [?]. For a more recent account on analytic results on the 'FPU paradox' we refer to [?].

In particular, we mention the papers [?] and [?], in which the authors used the techniques of canonical perturbation theory for PDEs in order to show that the FPU α model (respectively, β model) can be rigorously described by a system of two uncoupled KdV (resp. mKdV) equations, which are obtained as a resonant normal form of the continuous approximation of the FPU model; moreover, this result allowed to deduce a rigorous result about the energy sharing among the Fourier modes, up to the time-scales of validity of the approximation. If we denote by N the number of degrees of freedom for the lattice and by $\mu \sim \frac{1}{N} \ll 1$ the wave-number of the initially excited mode, if we assume that the specific energy $\epsilon \sim \mu^4$ (resp. $\epsilon \sim \mu^2$ for the FPU β model), then the dynamics of the KdV (resp. mKdV) equations approximates the solutions of the FPU model up to a time of order $\mathcal{O}(\mu^{-3})$. However, the relation between the specific energy and the number of degrees of freedom implies that the result does not hold in the thermodynamic limit regime, namely for large N and for fixed specific energy ϵ (such a regime is the one which is relevant for statistical mechanics).

Unlike the extensive research concerning one-dimensional systems, it seems to the authors that the behaviour of the dynamics of two-dimensional lattices is far less clear; it is expected that the interplay between the geometry of the lattice and the specific energy regime could lead to different results.

Benettin and collaborators [?] [?] [?] studied numerically a two-dimensional FPU lattice with triangular cells and different boundary conditions in order to estimate the equipartition time-scale. They found out that in the thermodynamic limit regime the equipartition is reached faster than in the one-dimensional case. The authors decided not to consider model with square cells in order to have a spectrum of linear frequencies which is different with respect to the one of the one-dimensional model; they also added (see [?], section B.(iii))

There is a good chance, however, that models with square lattice, and perhaps a different potential so as to avoid instability, behave differently from models with triangular lattice, and are instead more similar to one-dimensional models. This would correspond to an even stronger lack of universality in the two-dimensional FPU problem.

Up to the authors' knowledge, the only analytical results on the dynamics of two-dimensional lattices in this framework concern the existence of breathers [?] [?] [?] [?] [?] [?].

In this paper we study two-dimensional rectangular lattices with $(2N_1 + 1) \times (2N_2 + 1)$ sites, square cell, nearest-neighbour interaction and periodic boundary conditions, and we show the existence of metastability phenomena as in [?]. More precisely, if we denote by $\mu \ll 1$ the wave-number of the Fourier mode initially excited and by σ the ratio between the sides of the lattice, we obtain for a 2D Electrical Transmission lattice (ETL) either a system of two uncoupled KP-II equations for $\mu \ll 1$ and $\sigma = 2$, or a system of two uncoupled KdV equations for $\mu \ll 1$ and $2 < \sigma < 5$ as a resonant normal form for the continuous approximation of the lattice, while for the 2D Klein-Gordon lattice with quartic defocusing nonlinearity we obtain a one-dimensional cubic defocusing NLS equation for $\mu \ll 1$ and $1 < \sigma < 7$. Since all the above PDEs are integrable, we can exploit integrability to deduce a mathematically rigorous result on the formation of the metastable packet.

We would like to emphasize that, depending on the geometry of the lattice which is encoded in the parameter σ, the dynamics is almost 1-dimensional for highly anisotropic lattices and genuinely 2-dimensional for low values of σ. In this picture, the edge case $\sigma = 2$ has a genuinely 2-dimensional normal form equation (as it happens for $\sigma \leq 2$) which is integrable (as for $\sigma \geq 2$).

Up to the authors' knowledge, this is the first analytical result about metastable phenomena in two-dimensional Hamiltonian lattices with periodic boundary conditions; in particular, this is the first rigorous result for two-dimensional lattices in which the dynamics of the lattice in a genuinely two-dimensional regime is described by a system of two-dimensional integrable PDEs.

Some comments are in order:

i. we have that the time-scale of validity of our result is of order $\mathcal{O}(\mu^{-3})$ for the 2D ETL lattice, and of order $\mathcal{O}(\mu^{-2})$ for the 2D Klein-Gordon lattice;

ii. the ansatz about the small amplitude solutions gives a relation between the specific energy of the system ϵ and the wave-number $\mu \sim \frac{1}{N_1}$ of the Fourier mode initially excited. More precisely, we obtain $\epsilon \sim \mu^4$ for the 2D ETL lattice as in [?], and $\epsilon \sim \mu^2$ for the 2D Klein-Gordon lattice. This implies that the result does not hold in the thermodynamic limit regime;

iii. our result can be easily generalized to higher-dimensional lattices (see Remark 2.12 and Remark 2.13), such as the physical case of three-dimensional rectangular lattices with cubic cells;

iv. the upper bounds for σ in the KdV regime and in the NLS regime come from a technical assumption in the approximation results (see Proposition 6.5, Proposition 6.2 and Proposition 6.9). The approximation of solutions for the lattice with solutions of integrable PDEs in one-dimensional lattices was obtained through a detailed analysis in order to bound the error, and this is also the case for two-dimensional lattices, (see Proposition 6.5, Proposition 6.9, Appendix C and Appendix E), where one has to do very careful estimates in order to bound the different contributions to the error.

To prove our results we follow the strategy of [?]. The first step consists in the approximation of the dynamics of the lattice with the dynamics of a continuous system. As a second step we perform a normal form canonical transformation and we obtain that the effective dynamics is given by a system of integrable PDEs (KdV, KP-II, NLS depending on the lattice and the relation between N_1 and N_2). Next, we exploit the dynamics of these integrable PDEs in order to construct approximate solutions of the original discrete lattices, and we estimate the error with repect to a true solution with the corresponding initial datum. Finally, we use the known results about the dynamics of the above mentioned integrable PDEs in order to estimate the specific energies for the approximate solutions of the original lattices.

The novelties of this work are: on the one side, a mathematically rigorous proof of the approximation of the dynamics of the ETL lattice by the dynamics of certain integrable PDEs (among these integrable PDEs, there is one which is *genuinely* two-dimensional, the KP-II equation) and of the dynamics of the two-dimensional KG lattice by the dynamics of the one-dimensional nonlinear Schrödinger equation; on the other side, there are two technical differences with respect to previous works, namely the normal form theorem (which is a variant of the technique used in [?] [?] [?]) and the estimates for bounding the error between the approximate solution and the true solution of the lattice (which need a more careful study than the ones appearing in [?] [?] for the one-dimensional case).

The paper is organized as follows: in Section 2 we introduce the mathematical setting of the models and we state our main results, Theorem 2.1, Theorem 2.4 and Theorem 2.6. In Section 3 we state an abstract Averaging Theorem, which we prove in Section 3.2. In Section 4 we apply the averaging Theorem to the two-dimensional lattices, deriving the integrable approximating PDEs in the different regimes. In Section 5 we review some results about the dynamics of the normal form equation. In Section 6 we use the normal form equations in order to construct approximate solutions (see Proposition 6.5, Proposition 6.2 and Proposition 6.9), and we estimate the difference with respect to the true solutions with corresponding initial data in Proposition 6.6, Proposition 6.3 and Proposition 6.10. In Appendix A we prove the technical Lemma 3.6; in Appendix B we prove Proposition 6.5; in Appendix C we prove Proposition 6.6; in Appendix D we prove Proposition 6.9; in Appendix E we prove Proposition 6.10.

results

2. Main Results

We consider a periodic two-dimensional rectangular lattice, called ETL lattice, which in the non-periodic setting has been studied in butt2006discrete [?] (see Remark 2.9 for a physical motivation of such a model), and which can be regarded as a simpler version of a 2D rectangular FPU model. We denote

Z2
$$\mathbb{Z}^2_{N_1,N_2} := \{(j_1,j_2) : j_1, j_2 \in \mathbb{Z}, |j_1| \le N_1, |j_2| \le N_2\}; \tag{1}$$

we also write $e_1 := (1,0)$, $e_2 := (0,1)$ and $\mathbb{Z}^2_N := \mathbb{Z}^2_{N,N}$.

The Hamiltonian describing the ETL lattice is given by

HamQ
$$H(Q,P) = \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} -\frac{1}{2} P_j\,(\Delta_1 P)_j + (F(Q))_j, \tag{2}$$

Delta1
$$(\Delta_1 P)_j := (P_{j+e_1} - 2P_j + P_{j-e_1}) + (P_{j+e_2} - 2P_j + P_{j-e_2}), \tag{3}$$

FQ
$$(F(Q))_j = \frac{Q_j^2}{2} + \alpha\frac{Q_j^3}{3} + \beta\frac{Q_j^4}{4}. \tag{4}$$

We refer to (2) as $\alpha+\beta$ model (respectively, β model) if $\alpha \neq 0$ (respectively $\alpha = 0$). With the above Hamiltonian formulation the equations of motion associated to (2) are given by

$$\begin{cases} \dot{Q}_j = -(\Delta_1 P)_j \\ \dot{P}_j = -(F'(Q))_j \end{cases} ;$$

2DETLeq
$$\ddot{Q}_j = (\Delta_1 F'(Q))_j. \tag{5}$$

We also introduce the Fourier coefficients of Q via the following standard relation,

fourierQ
$$Q_j := \frac{1}{\sqrt{(2N_1+1)(2N_2+1)}} \sum_{k\in\mathbb{Z}^2_{N_1,N_2}} \hat{Q}_k e^{i\frac{j\cdot k\,2\pi}{(2N_1+1)(2N_2+1)}}, \quad j \in \mathbb{Z}^2_{N_1,N_2}, \tag{6}$$

and similarly for P_j. We denote by

EnNormMode
$$E_k := \frac{\omega_k^2|\hat{P}_k|^2 + |\hat{Q}_k|^2}{2}, \tag{7}$$

FreqNormMode
$$\omega_k^2 := 4\sin^2\left(\frac{k_1\,\pi}{2N_1+1}\right) + 4\sin^2\left(\frac{k_2\,\pi}{2N_2+1}\right), \tag{8}$$

the energy and the square of the frequency of the mode at site $k = (k_1,k_2) \in \mathbb{Z}^2_{N_1,N_2}$ (see Figure 1). For states described by real functions, one has $E_{(k_1,k_2)} = E_{(-k_1,k_2)}$ and $E_{(k_1,k_2)} = E_{(k_1,-k_2)}$ for all $k = (k_1,k_2)$, so we will consider only indexes in

$$\mathbb{Z}^2_{N_1,N_2,+} := \{(k_1,k_2) \in \mathbb{Z}^2_{N_1,N_2} : k_1, k_2 \ge 0\}.$$

As is customary in lattices with a large number of degrees of freedom, especially in relation with statistical mechanics, it is also convenient to introduce the following specific quantities,

kappa
$$\kappa := \kappa(k) = \left(\frac{k_1}{N_1+\frac{1}{2}}, \frac{k_2}{N_2+\frac{1}{2}}\right), \tag{9}$$

enkappa
$$\mathcal{E}_\kappa := \frac{E_k}{\left(N_1+\frac{1}{2}\right)\left(N_2+\frac{1}{2}\right)}, \tag{10}$$

where (10) is the specific energy of the normal mode with index κ.

We want to study the behaviour of small amplitude solutions of (5), with initial data in which only one low-frequency Fourier mode is excited.

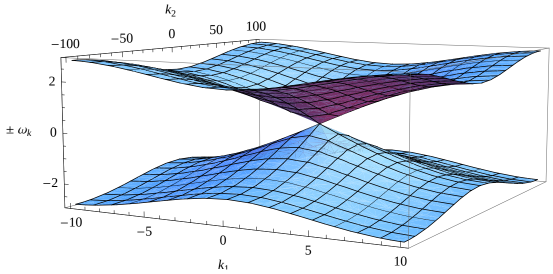

FIGURE 1. The dispersion relation for the ETL lattice (2), namely $\pm\,\omega_k$, $k = (k_1, k_2)$, vs the integer coordinates, for $N_1 = 10$ and $N_2 = 100$. fig:freq_ETL

We assume $N_1 \leq N_2$, and we introduce the quantities

mu (11)
$$\mu := \frac{2}{2N_1+1},$$

sigma (12)
$$\sigma := \log_{N_1+\frac{1}{2}}\left(N_2+\frac{1}{2}\right),$$

which play the role of parameters in our construction: we will use them in the asymptotic expansion of the dispersion relation of the continuous approximation of the lattice (see (98)-(100), and (119)-(120)) in order to derive the integrable approximating PDEs in the regimes we are considering.

We study the $\alpha+\beta$ model of (5) in the following regime:

(KP) the *weakly transverse regime*, where the effective dynamics is a described by a system of two *uncoupled* Kadomtsev-Petviashvili (KP) equation. This corresponds to taking $\mu \ll 1$ and $\sigma = 2$.

From now on, we denote by $\kappa_0 := \left(\frac{1}{N_1+\frac{1}{2}}, \frac{1}{(N_1+\frac{1}{2})^\sigma}\right) = (\mu, \mu^\sigma)$. Our main result is the following:

KPrThm **Theorem 2.1.** *Consider* (5) *with* $\alpha \neq 0$, $\sigma = 2$.

Fix $1 \leq \gamma < \frac{5}{2}$ *and two positive constants* C_0 *and* T_0, *then there exist positive constants* μ_0, C_1 *and* C_2 *(depending only on* γ, C_0 *and on* T_0*) such that the following holds. Consider an initial datum with*

KPData (13)
$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^4, \qquad \mathcal{E}_\kappa(0) = 0 \qquad \forall \kappa = (\kappa_1, \kappa_2) \neq \kappa_0,$$

and assume that $\mu < \mu_0$. *Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

EnModesKP (14)
$$\mathcal{E}_\kappa(t) \leq C_1\,\mu^4 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\,\mu^{4+\gamma}, \qquad |t| \leq \frac{T}{\mu^3}$$

for all κ.

ActAngKPrem **Remark 2.2.** *Theorem 2.1 is the first rigorous result for two-dimensional lattices in which the dynamics of the lattice in a genuinely two-dimensional regime is described by a system of two-dimensional integrable PDEs. Moreover, in Theorem 2.1 we do not mention the existence of a sequence of almost-periodic functions approximating the specific energies of the modes, and this is a difference with respect to Theorem 5.3 in* [?] (bambusi2006metastability)*. This is related to the construction of action-angle/Birkhoff coordinates for the KP equation, which is an open problem in the theory of integrable PDEs.*

InDatarem **Remark 2.3.** *For the sake of simplicity, we have proved Theorem 2.1 for initial data in which only one low-frequency Fourier mode is excited. One can also prove that a variant of Theorem 2.1 holds also in the case the higher harmonics of a low-frequency Fourier mode are excited, provided that the energy decreases exponentially with respect to* $|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|$, *and also for initial data in which the symmetrical modes of a given low-frequency Fourier mode are excited. To summarize, we are only able to prove stability of the solutions we constructed for initial data with vanishing specific energy for a time-scale* $\mathcal{O}(\mu^{-3})$.

We also point out that there are also other regimes in which the dynamics of a two-dimensional lattice can be approximated by integrable PDEs. For example, we can consider $\alpha+\beta$ model of (5) in the following regime:

(KdV) the *very weakly transverse regime*, where the effective dynamics is described by a system of two *uncoupled* Korteweg-de Vries (KdV) equations. This corresponds to taking $\mu \ll 1$ and $2 < \sigma < 5$.

The corresponding result one can prove in such a regime is the following.

KdVrThm **Theorem 2.4.** *Consider* (5) *with* $\alpha \neq 0$, $2 < \sigma < 5$.

Fix $1 \le \gamma \le \frac{7-\sigma}{2}$ *and two positive constants* C_0 *and* T_0, *then there exist positive constants* μ_0, C_1 *and* C_2 *(depending only on* γ, C_0 *and on* T_0*) such that the following holds. Consider an initial datum with*

KdVData
$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^4, \qquad \mathcal{E}_\kappa(0) = 0, \qquad \forall \kappa = (\kappa_1, \kappa_2) \neq \kappa_0, \tag{15}$$

and assume that $\mu < \mu_0$. *Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

EnModesKdV
$$\mathcal{E}_\kappa(t) \le C_1\,\mu^4 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\,\mu^{4+\gamma}, \qquad |t| \le \frac{T_0}{\mu^3} \tag{16}$$

for all κ. *Moreover, for any* n_2 *with* $0 \le n_2 \le N_2$ *there exists a sequence of almost-periodic functions* $(F_n)_{n=(n_1,n_2)\in\mathbb{Z}^2_{N_1,N_2,+}}$ *such that, if we denote*

AlmostPerKdV
$$\mathcal{F}_{\kappa_0} = \mu^4 F_n, \qquad \mathcal{F}_\kappa = 0 \qquad \forall \kappa \neq n\kappa_0 \tag{17}$$

then

ApprEnModesKdV
$$|\mathcal{E}_\kappa(t) - \mathcal{F}_\kappa(t)| \le C_2\,\mu^{4+\gamma}, \qquad |t| \le \frac{T_0}{\mu^3}. \tag{18}$$

Scalingrem **Remark 2.5.** *We point out that in the statement of Theorem 2.4 the assumption* $\sigma > 2$ *comes from an asymptotic expansion of the dispersion relation of the continuous approximation of the lattice (see* (98)-(100)*), while the assumption* $\sigma < 5$ *comes from a technical assumption under which we can approximate the dynamics of the lattice with the dynamics of the system of uncoupled KdV equations (see the statement of Theorem 6.6).*

We can also consider two-dimensional KG lattices, which combine the nearest-neighbour potential with an on-site one: the scalar model

Ham2KGs
$$H(Q,P) = \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} \frac{P_j^2}{2} + \frac{1}{2}\sum_{\substack{j,k\in\mathbb{Z}^2_{N_1,N_2}\\ |j-k|=1}} \frac{(Q_j - Q_k)^2}{2} + \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} U(Q_j), \tag{19}$$

potKG2
$$U(x) = m^2\frac{x^2}{2} + \beta\frac{x^{2p+2}}{2p+2}, \qquad m > 0, \qquad \beta > 0,\ p \ge 1, \tag{20}$$

can be used to describe rigid rotating molecules in the lattice plane (Q being the angle of rotation), where each molecule interacts with its neighbors and with the periodic substrate potential U; alternatively, Q can represent the transverse motion of a planar lattice rosenau2003hamiltonian [?].

Using the operator Δ_1 introduced in (3), the Hamiltonian (19) can be rewritten as

Ham2KGs2
$$H(Q,P) = \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} \frac{P_j^2}{2} + \frac{1}{2}\sum_{j\in\mathbb{Z}^2_{N_1,N_2}} Q_j\,(-\Delta_1 Q)_j + \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} U(Q_j), \tag{21}$$

the associated equations of motion are

2DKGseq
$$\ddot{Q}_j = (\Delta_1 Q)_j - m^2 Q_j - \beta Q_j^{2p+1}, \qquad j \in \mathbb{Z}^2_{N_1,N_2}. \tag{22}$$

If we take $p = 1$, we obtain a generalization of the one-dimensional ϕ^4 model.

Now introduce the Fourier coefficients of Q as in (6), and similarly for P_j, and denote by

EnNormModeKG
$$E_k := \frac{|\hat{P}_k|^2 + \omega_k^2|\hat{Q}_k|^2}{2}, \tag{23}$$

FreqNormModeKG
$$\omega_k^2 := m^2 + 4\sin^2\left(\frac{k_1\,\pi}{2N_1+1}\right) + 4\sin^2\left(\frac{k_2\,\pi}{2N_2+1}\right), \tag{24}$$

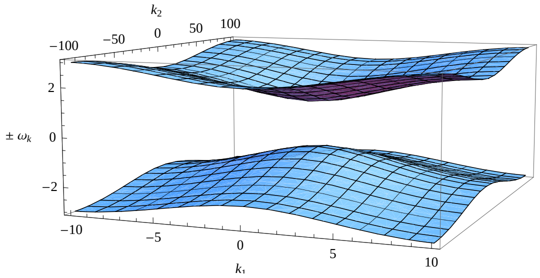

FIGURE 2. The dispersion relation for the KG lattice (21), namely $\pm\, \omega_k$, $k = (k_1, k_2)$, vs the integer coordinates, for $N_1 = 10$ and $N_2 = 100$. fig:freq_KG

the energy and the square of the frequency of the mode at site $k = (k_1, k_2) \in \mathbb{Z}^2_{N_1,N_2}$ (see Figure 2). In the rest of the paper we will assume that $m = 1$.

We consider the two-dimensional KG lattice (19) in the following regime:

(1D NLS) the *very weakly transverse regime*, where the effective dynamics is described by a cubic one-dimensional nonlinear Schrödinger (NLS) equation. This corresponds to taking $\mu \ll 1$ and $1 < \sigma < 7$.

1DNLSrThm **Theorem 2.6.** *Consider* (19) *with* $\beta > 0$, $1 < \sigma < 7$.

Fix $0 < \gamma \leq \frac{7-\sigma}{2}$ *and two positive constants* C_0 *and* T_0*, then there exist positive constants* μ_0*,* C_1 *and* C_2 *(depending only on* γ*,* C_0 *and on* T_0*) such that the following holds. Consider an initial datum with*

1DNLSData
$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^2, \qquad \mathcal{E}_\kappa(0) = 0,\ \forall \kappa = (\kappa_1, \kappa_2) \neq \kappa_0, \tag{25}$$

and assume that $\mu < \mu_0$*. Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

EnModes1DNLS
$$\mathcal{E}_\kappa(t) \leq C_1\, \mu^2 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\, \mu^{2+\gamma}, \qquad |t| \leq \frac{T_0}{\mu^2} \tag{26}$$

for all κ*. Moreover, for any* n_2 *with* $0 \leq n_2 \leq N_2$ *there exists a sequence of almost-periodic functions* $(F_n)_{n=(n_1,n_2)\in\mathbb{Z}^2_{N_1,N_2,+}}$ *such that, if we denote*

AlmostPer1DNLS
$$\mathcal{F}_{\kappa_0} = \mu^2\, F_n, \qquad \mathcal{F}_\kappa = 0 \qquad \forall \kappa \neq n\kappa_0 \tag{27}$$

then

ErrEnModes1DNLS
$$|\mathcal{E}_\kappa(t) - \mathcal{F}_\kappa(t)| \leq C_2\, \mu^{2+\gamma}, \qquad |t| \leq \frac{T_0}{\mu^2}. \tag{28}$$

InDataKGrem **Remark 2.7.** *In Theorem 2.6 we are able to prove stability of the solutions we constructed for initial data with vanishing specific energy for a time-scale* $\mathcal{O}(\mu^{-2})$*.*

ScalingKGrem **Remark 2.8.** *As for Theorem 2.4, in the statement of Theorem 2.6 the assumption* $\sigma > 1$ *comes from an asymptotic expansion of the dispersion relation of the continuous approximation of the lattice (see* (119)*-*(120)*), while the assumption* $\sigma < 5$ *comes from a technical assumption under which we can approximate the dynamics of the lattice with the dynamics of the system of uncoupled NLS equations (see the statement of Theorem 6.10).*

2.1. **Further remarks.**

ETLrem **Remark 2.9.** *The ETL lattice* (2) *describes a lossless periodic two-dimensional electrical transmission lattice (ETL), given by a rectangular configuration of repeating units, each made up of two linear inductors and a nonlinear capacitor, and lattice nodes denote the locations of capacitors. The Hamiltonian* (2) *comes from the following computations (see also* [?]*): assume that* $V_j(t)$*,* $j \in \mathbb{Z}^2_{N_1,N_2}$*, denotes the voltage across the j-th capacitor,* $Q_j(t)$ *denotes the charge stored on the j-th capacitor and* $I_j(t)$ *denotes the current through the j-th inductor along direction* e_1*. To derive the equations for the voltage* V_j *and the* *charge Q_j in the lattice one can proceed as follows. Considering a section of the lattice and applying Faraday's law and Lenz's law, the difference in shunt voltage at site j and site $j+e_1$ is given by*

Ve1
$$V_{j+e_1} - V_j = -L\frac{\mathrm{d}I_j}{\mathrm{d}t}, \tag{29}$$

where L is the inductance, which we assume to be constant. Assuming the capacitance C to be an analytic function of the voltage V we can expand it in Taylor series, obtaining for small voltages

expC
$$C_j(V) \sim C_0(1 + 2aV_j + 3bV_j^2), \tag{30}$$

where $C_0 := C_j(0)$, a and b are real constants determined by the physical realisation of the network. Using standard relations between electrical quantities we finally obtain a closed equation for the charge

eqQ2
$$\frac{\mathrm{d}^2 Q_j}{\mathrm{d}t^2} = \frac{1}{LC_0}(\Delta_1(Q + \alpha Q^2 + \beta Q^3))_j, \tag{31}$$

where Δ_1 is the operator defined in (3)*, and α, β are real parameters related to a and b. Up to a rescaling of time, we can set $LC_0 = 1$ without loss of generality; one can check that the Hamiltonian associated to* (31) *is precisely* (2)*.*

Anisorem

Remark 2.10. *The specific choice of the direction of longitudinal propagation in the regimes that we have considered is not relevant.*

FPUNLKGrem

Remark 2.11. *We point out that the time of validity of Theorem 2.6 for the KG lattice is of order $\mathcal{O}(\mu^{-2})$, which is different from the time of validity of Theorem 2.4 and Theorem 2.1 for the FPU lattice. In the one-dimensional case it has been observed that, for a fixed value of specific energy ϵ and for long-wavelength modes initially excited, the ϕ^4 model reached equipartition faster than the FPU β model (see* lichtenberg2007dynamics *[?], sec. 2.1.8).*

HigherDimRem

Remark 2.12. *Theorem 2.1 and Theorem 2.4 can be generalized to higher dimensional lattices. Indeed, let $d \le 4$, define*

Zd
$$\mathbb{Z}^d_{N_1,\dots,N_d} := \{(j_1,\dots,j_d) : j_1,\dots,j_d \in \mathbb{Z}, |j_1| \le N_1, \dots, |j_d| \le N_d\}, \tag{32}$$

and consider the d-dimensional ETL

dDHam
$$H(Q,P) = \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} -\frac{1}{2}P_j\,(\Delta_1 P)_j + (F(Q))_j, \tag{33}$$

dDF
$$(F(Q))_j = \frac{Q_j^2}{2} + \alpha\frac{Q_j^3}{3} + \beta\frac{Q_j^4}{4}, \qquad j \in \mathbb{Z}^d_{N_1,\dots,N_d}. \tag{34}$$

We assume $N_1 \le N_2, \dots, N_d$, and we introduce the quantities

dmu
$$\mu := \frac{2}{2N_1+1}, \tag{35}$$

dsigma
$$\sigma_i := \log_{N_1+\frac{1}{2}}\left(N_{i+1} + \frac{1}{2}\right), \qquad i = 1,\dots,d-1. \tag{36}$$

Then we can describe the following regimes:

(KdV-d) *the $\alpha+\beta$ model, in the very weakly transverse regime with $\mu \ll 1$ and $2 < \sigma_1, \dots, \sigma_{d-1} < 5$;*
(KP-d) *the $\alpha+\beta$ model, in the weakly transverse regime with $\mu \ll 1$ and $\sigma_1 = 2$, $2 < \sigma_2, \dots, \sigma_{d-1} < 5$.*

Moreover, in order to obtain Theorem 2.4 and Theorem 2.1 we will have to assume that

TechAssump
$$2\gamma + \sum_{i=1}^{d-1}\sigma_i < 7. \tag{37}$$

which, together with the fact that $\sigma_i > 2$ for all $i = 1,\dots,d-1$, is consistent with the assumption $d \le 4$.

Remark 2.13. *Theorem 2.6 can be generalized to higher dimensional lattices. Indeed, let $d \le 6$, define $\mathbb{Z}^d_{N_1,\dots,N_d}$ as in* (32) *and consider the d-dimensional NLKG lattice*

$$H(Q,P) = \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} \frac{P_j^2}{2} + \frac{1}{2}\sum_{\substack{j,k\in\mathbb{Z}^d_{N_1,\dots,N_d}\\ |j-k|=1}} \frac{(Q_j-Q_k)^2}{2} + \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} U(Q_j), \tag{38}$$

$$U(x) = m^2\frac{x^2}{2} + \beta\frac{x^{2p+2}}{2p+2}, \qquad m>0, \qquad \beta>0,\ p\ge 1, \tag{39}$$

We assume $N_1 \le N_2,\dots,N_{d-1}$, and we introduce the quantities μ and σ_i $(1\le i\le d-1)$ as in (35) *and* (36)*.*

Then we can describe the following regime:

(1DNLS-d) *the model* (38) *with $m = 1$ and $p = 1$ in the very weakly transverse regime, with $\mu \ll 1$, $1 < \sigma_1,\dots,\sigma_{d-1} < 7$;*

Moreover, in order to obtain Theorem 2.6 we will have to assume that

$$2\gamma + \sum_{i=1}^{d-1}\sigma_i < 7. \tag{40}$$

which, together with the fact that $\sigma_i > 1$ for all $i = 1,\dots,d-1$, is consistent with the assumption $d \le 6$.

Remark 2.14. *There are other interesting regimes for* (5) *and* (22) *especially for their relation with the modified KdV equation and two-dimensional Non-Linear Schrödinger equation respectively. These will be discussed in Remark 4.8 and Remark 4.11 respectively.*

3. Galerkin Averaging

3.1. **An Averaging Theorem.** Following [?] (see also [?] and [?]) we use a Galerkin averaging method in order to approximate the solutions of the continuous approximation of the lattice with the solutions of the system in normal form.

To this end we first have to introduce a topology in the phase space. This is conveniently done in terms of Fourier coefficients.

Definition 3.1. *Fix two constants $\rho \ge 0$ and $s \ge 0$. We will denote by $\ell^2_{\rho,s}$ the Hilbert space of complex sequences $v = (v_n)_{n\in\mathbb{Z}^2\setminus\{0\}}$ with obvious vector space structure and with scalar product*

$$\langle v, w\rangle_{\rho,s} := \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \overline{v_n} w_n e^{2\rho|n|}|n|^{2s}. \tag{41}$$

and such that

$$\|v\|^2_{\rho,s} := \langle v, v\rangle_{\rho,s} = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |v_n|^2 e^{2\rho|n|}|n|^{2s} \tag{42}$$

is finite. We will denote by ℓ^2 the space $\ell^2_{0,0}$.

We will identify a 2-periodic function v with the sequence of its Fourier coefficients $\{\hat v_n\}_n$,

$$v(y) = \frac{1}{2}\sum_{n\in\mathbb{Z}^2} \hat v_n e^{i\pi\, n\cdot y},$$

and, with a small abuse of notation, we will say that $v \in \ell^2_{\rho,s}$ if the sequence of its Fourier coefficients belong to $\ell^2_{\rho,s}$.

Now fix $\rho \ge 0$ and $s \ge 1$, and consider the scale of Hilbert spaces $\mathcal{H}^{\rho,s} := \ell^2_{\rho,s}\times\ell^2_{\rho,s} \ni \zeta = (\xi,\eta)$, endowed with one of the following symplectic forms:

$$\Omega_1 := \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}, \qquad \Omega_2 := \begin{pmatrix} -\partial_{x_1}^{-1} & 0 \\ 0 & \partial_{x_1}^{-1}\end{pmatrix}. \tag{43}$$

Observe that $\Omega_\gamma : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s+\gamma-1}$ $(\gamma = 1, 2)$ is a well-defined operator. Moreover, Ω_2 is well-defined on the space of functions with zero-average with respect to the x_1-variable, i.e. on those functions $\zeta(x_1, x_2)$ such that for every x_2 we have $\int_{-1}^{1} \zeta(x_1, x_2)\,\mathrm{d}x_1 = 0$.

If we fix $\gamma \in \{1, 2\}$, s and $U_s \subset \ell^2_{\rho,s}$ open, we define the gradient of $K \in C^\infty(U_s, \mathbb{R})$ with respect to $\xi \in \ell^2_{\rho,s}$ as the unique function s.t.

$$\langle \nabla_\xi K, h\rangle = \mathrm{d}_\xi K h, \qquad \forall h \in \ell^2_{\rho,s}.$$

Similarly, for an open set $\mathcal{U}_s \subset \mathcal{H}^{\rho,s}$ the Hamiltonian vector field of the Hamiltonian function $H \in C^\infty(\mathcal{U}_s, \mathbb{R})$ is given by

$$X_H(\zeta) = \Omega_\gamma^{-1} \nabla_\zeta H(\zeta).$$

The open ball of radius R and center 0 in $\ell^2_{\rho,s}$ will be denoted by $B_{\rho,s}(R)$; we write $\mathcal{B}_{\rho,s}(R) := B_{\rho,s}(R) \times B_{\rho,s}(R) \subset \mathcal{H}^{\rho,s}$.

Now, we introduce the Fourier projection operators $\hat{\pi}_j : \ell^2_{\rho,s} \to \ell^2_{\rho,s}$

hatpi

$$\hat{\pi}_j\big((v_n)_{n\in\mathbb{Z}^2\setminus\{0\}}\big) := \begin{cases} v_n & \text{if} \quad j-1 \le |n| < j \\ 0 & \text{otherwise} \end{cases} \quad , \qquad j \ge 1, \tag{44}$$

the operators $\pi_j : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s}$

smallpi

$$\pi_j\big((\zeta_n)_{n\in\mathbb{Z}^2\setminus\{0\}}\big) := \begin{cases} \zeta_n & \text{if} \quad j-1 \le |n| < j \\ 0 & \text{otherwise} \end{cases} \quad , \qquad j \ge 1, \tag{45}$$

and the operators $\Pi_M : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s}$

bigpi

$$\Pi_M\big((\zeta_n)_{n\in\mathbb{Z}^2\setminus\{0\}}\big) := \begin{cases} \zeta_n & \text{if} \quad |n| \le M \\ 0 & \text{otherwise} \end{cases} \quad , \qquad M \ge 0. \tag{46}$$

Lemma 3.2. *The projection operators defined in* (45) *and* (46) *satisfy the following properties for any* $\zeta \in \mathcal{H}^{\rho,s}$*:*

i. for any $j \ge 0$

$$\zeta = \sum_{j\ge 0} \pi_j \zeta;$$

ii. for any $j \ge 0$

$$\|\Pi_M \zeta\|_{\mathcal{H}^{\rho,s}} \le \|\zeta\|_{\mathcal{H}^{\rho,s}};$$

iii. the following equality holds

compnorms

$$\|\zeta\|_{\mathcal{H}^{\rho,s}} = \left\| \left[\sum_{j\in\mathbb{N}} j^{2s} |\pi_j \zeta|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \tag{47}$$

where $|\zeta|$*, for* $\zeta \in \mathcal{H}^{\rho,s}$ *is the element* $|\zeta| \in \mathcal{H}^{\rho,s}$ *whose* n*-th element is*

$$|\zeta|_n := (|\xi_n|, |\eta_n|)$$

and $(\zeta^\alpha)_n := (\xi_n^\alpha, \eta_n^\alpha)$*.*

Now we consider a Hamiltonian system of the form

Hamdecomp

$$H = h_0 + \delta F, \tag{48}$$

where we assume that

(PER) h_0 generates a linear periodic flow $\Phi^\tau_{h_0}$ with period T,

$$\Phi^{\tau+T}_{h_0} = \Phi^\tau_{h_0} \qquad \forall \tau,$$

which is analytic as a map from $\mathcal{H}^{\rho,s}$ into itself for any $s \ge 1$. Furthermore, the flow is an isometry for any $s \ge 1$.

(INV) for any $s \geq 1$, $\Phi^{\tau}_{h_0}$ leaves invariant the space $\Pi_j \mathcal{H}^{\rho,s}$ for any $j \geq 0$. Furthermore, for any $j \geq 0$

$$\pi_j \circ \Phi^{\tau}_{h_0} = \Phi^{\tau}_{h_0} \circ \pi_j.$$

Next, we assume that the vector field of F admits an asymptotic expansion in δ of the form

expF
$$F \sim \sum_{j\geq 1} \delta^{j-1} F_j, \tag{49}$$

expXF
$$X_F \sim \sum_{j\geq 1} \delta^{j-1} X_{F_j}, \tag{50}$$

and that the following property is satisfied

(HVF) There exists $R^* > 0$ such that for any $j \geq 1$

- X_{F_j} is analytic from $\mathcal{B}_{\rho,s+2j+\gamma}(R^*)$ to $\mathcal{H}^{\rho,s}$.

Moreover, for any $r \geq 1$ we have that

- $X_{F-\sum_{j=1}^{r}\delta^{j-1}F_j}$ is analytic from $\mathcal{B}_{\rho,s+2(r+1)+\gamma}(R^*)$ to $\mathcal{H}^{\rho,s}$.

The main result of this section is the following theorem.

gavthm **Theorem 3.3.** *Fix $R > 0$, $s_1 \gg 1$. Consider (48), and assume (PER), (INV) and (HVF). Then $\exists$ $s_0 > 0$ with the following properties: for any $s \geq s_1$ there exists $\delta_s \ll 1$ such that for any $\delta < \delta_s$ there exists $\mathcal{T}_\delta : \mathcal{B}_{\rho,s}(R/2) \to \mathcal{B}_{\rho,s}(R)$ analytic canonical transformation such that*

TransfHam
$$H_1 := H \circ \mathcal{T}_\delta = h_0 + \delta \mathcal{Z}_1 + \delta^2\, \mathcal{R}^{(1)}, \tag{51}$$

where $\mathcal{Z}_1$ is in normal form, namely

NFthm
$$\{\mathcal{Z}_1, h_0\} = 0, \tag{52}$$

and there exists a positive constant C'_s such that

$$\sup_{\mathcal{B}_{\rho,s+s_0}(R/2)} \|X_{\mathcal{Z}_1}\|_{\mathcal{H}^{\rho,s}} \leq C'_s,$$

Remthm
$$\sup_{\mathcal{B}_{\rho,s+s_0}(R/2)} \|X_{\mathcal{R}^{(1)}}\|_{\mathcal{H}^{\rho,s}} \leq C'_s, \tag{53}$$

CTthm
$$\sup_{\mathcal{B}_{\rho,s}(R/2)} \|\mathcal{T}_\delta - id\|_{\mathcal{H}^{\rho,s}} \leq C'_s\, \delta. \tag{54}$$

In particular,

average
$$\mathcal{Z}_1(\zeta) = \langle F_1 \rangle (\zeta), \tag{55}$$

where $\langle F_1 \rangle (\zeta) := \int_0^T F_1 \circ \Phi^{\tau}_{h_0}(\zeta) \frac{d\tau}{T}$.

BNFprsubsec
3.2. Proof of the Averaging Theorem.

The proof of Theorem 3.3 is actually an application of the techniques used in [?] and [?]).

First notice that by assumption (INV) the Hamiltonian vector field of h_0 generates a continuous flow Φ^{τ} which leaves $\Pi_M \mathcal{H}^{\rho,s}$ invariant.

Now we set $H = H_{1,M} + \mathcal{R}_{1,M} + \mathcal{R}_1$, where

truncsys
$$H_{1,M} := h_0 + \delta\, F_{1,M}, \tag{56}$$
$$F_{1,M} := F_1 \circ \Pi_M, \tag{57}$$

and

remsys
$$\mathcal{R}_{1,M} := h_0 + \delta F_1 - H_{1,M}, \tag{58}$$
$$\mathcal{R}_1 := \delta\,(F - F_1)\,. \tag{59}$$

The system described by the Hamiltonian (56) is the one that we will put in normal form.
In the following we will use the notation $a \lesssim b$ to mean: there exists a positive constant K independent of M and R (but eventually on s), such that $a \leq Kb$. We exploit the following intermediate results:

truncest **Lemma 3.4.** *For any $s \geq s_1$ there exists $R > 0$ such that $\forall\ \sigma > 0$, $M > 0$*

truncremt
$$\sup_{\mathcal{B}_{\rho,s+\gamma+\sigma+2}(R)} \|X_{\mathcal{R}_{1,M}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim \frac{\delta}{(M+1)^{\sigma}}, \tag{60}$$

expremest
$$\sup_{\mathcal{B}_{\rho,s+\gamma+4}(R)} \|X_{\mathcal{R}_1}(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim \delta^2. \tag{61}$$

Proof. We recall that $\mathcal{R}_{1,M} = h_0 + \delta F_j - H_{1,M}$.
We first notice that $\|id - \Pi_M\|_{\mathcal{H}^{\rho,s+\sigma} \to \mathcal{H}^{\rho,s}} = (M+1)^{-\sigma}$: indeed, using (47) we obtain

$$\begin{aligned}\left\| \sum_{j\geq M+1} \pi_j f \right\|_{\mathcal{H}^{\rho,s}} &= \left\| \left[\sum_{j\geq M+1} |j^s \pi_j f|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \leq (M+1)^{-\sigma} \left\| \left[\sum_{j\geq M+1} |j^{s+\sigma} \pi_j f|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \\ &\leq (M+1)^{-\sigma} \|f\|_{\mathcal{H}^{\rho,s+\sigma}},\end{aligned}$$

whereas the inequality $\|id - \Pi_M\|_{\mathcal{H}^{\rho,s+\sigma} \to \mathcal{H}^{\rho,\sigma}} \leq (M+1)^{-\sigma}$ is obtained with a function which has non zero components only for $|j| = M+1$, i.e. $f = \pi_{M+1} f$.

Inequality (60) follows from

$$\begin{aligned}&\sup_{(\zeta)\in\mathcal{B}_{\rho,s+\gamma+2+\sigma}(R)} \|X_{\mathcal{R}_{1,M}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \\ &\lesssim \|dX_{\delta F_1}\|_{L^\infty(\mathcal{B}_{\rho,s+2+\gamma}(R),\mathcal{H}^{\rho,s})} \|id - \Pi_M\|_{L^\infty(\mathcal{B}_{\rho,s+2+\gamma+\sigma}(R),\mathcal{B}_{\rho,s+2+\gamma}(R))} \\ &\lesssim \delta\,(M+1)^{-\sigma},\end{aligned}$$

while estimate (61) is an immediate consequence of (HVF). □

pertestlemma **Lemma 3.5.** *For any $s \geq s_1$*

$$\sup_{\mathcal{B}_{\rho,s}(R^*)} \|X_{F_{1,M}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq K^{(F)}_{1,s} M^{2+\gamma},$$

where

$$K^{(F)}_{1,s} := \sup_{\mathcal{B}_{\rho,s}(R^*)} \|X_{F_1}(\zeta)\|_{\mathcal{H}^{\rho,s-2-\gamma}} \ < \ +\infty.$$

Proof. Using (47) we have

$$\sup_{(\zeta)\in\mathcal{B}_{\rho,s}(R)} \left\| \sum_{h\leq M} \pi_h X_{F_{1,M}}(\zeta) \right\|_{\mathcal{H}^{\rho,s}} = \sup_{(\zeta)\in\mathcal{B}_{\rho,s}(R)} \left\| \left[\sum_{h\leq M} |h^s \pi_h X_{F_{1,M}}(\zeta)|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \tag{62}$$

$$\leq M^{2+\gamma} \sup_{(\zeta)\in\mathcal{B}_{\rho,s}(R)} \left\| \left[\sum_{h\leq M} |h^{s-2-\gamma} \pi_h X_{F_{1,M}}(\zeta)|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \tag{63}$$

$$\leq M^{2+\gamma} \sup_{(\zeta)\in\mathcal{B}_{\rho,s}(R)} \|X_{F_{1,M}}(\zeta)\|_{\mathcal{H}^{\rho,s-2-\gamma}} = K^{(F)}_{1,s}\, M^{2+\gamma}, \tag{64}$$

where the last quantity is finite for $R \leq R^*$ by property (HVF). □

To normalize (56) we need a slight reformulation of Theorem 4.4 in [?] (bambusi1999nekhoroshev). Here we report a statement of the result adapted to our context which is proved in Appendix A.

NFest **Lemma 3.6.** *Let $s \geq s_1 + 2 + \gamma$, $R > 0$, and consider the system (56). Assume that $\delta < \frac{1}{30}$, and that*

smallcond
$$12\, T\, K^{(F)}_{1,s} M^{2+\gamma} \delta < R \tag{65}$$

where

$$K^{(F)}_{1,s} := \sup_{(\zeta)\in\mathcal{B}_{\rho,s}(R)} \|X_{F_1}(\zeta)\|_{\mathcal{H}^{\rho,s-2-\gamma}}.$$

Then there exists an analytic canonical transformation $\mathcal{T}^{(0)}_{\delta,M} : \mathcal{B}_{\rho,s}(R/2) \to \mathcal{B}_{\rho,s}(R)$ *such that*

CTlemma
$$\sup_{\mathcal{B}_{\rho,s}(R/2)} \|\mathcal{T}^{(0)}_{\delta,M}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq 2T\, K^{(F)}_{1,s} M^{2+\gamma}\delta, \tag{66}$$

and that puts (56) *in normal form up to a small remainder,*

stepr
$$H_{1,M} \circ \mathcal{T}^{(0)}_{\delta,M} = h_0 + \delta Z^{(1)}_M + \delta^2 \mathcal{R}^{(1)}_M, \tag{67}$$

with $Z^{(1)}_M$ *in normal form, namely* $\{h_{0,M}, Z^{(1)}_M\} = 0$*, and*

itervf1
$$\sup_{\mathcal{B}_{\rho,s}(R/2)} \|X_{Z^{(1)}_M}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq K^{(F)}_{1,s} M^{2+\gamma} \tag{68}$$

vecfrem
$$\sup_{\mathcal{B}_{\rho,s}(R/2)} \|X_{\mathcal{R}^{(1)}_M}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 15 K^{(F)}_{1,s} M^{2+\gamma} \tag{69}$$

Now we conclude with the proof of the Theorem 3.3.

Proof. If we define $\delta_s := \min\{\frac{1}{30}, \frac{R}{12\,T\,K^{(F)}_{1,s} M^{2+\gamma}}\}$ and we choose

$$s_0 = \sigma + 2 + \gamma,$$
$$\sigma \geq 2,$$

then the transformation $\mathcal{T}_\delta := \mathcal{T}^{(0)}_{\delta,M}$ defined by Lemma 3.6 satisfies (51) because of (67).

Next, Eq. (52) follows from Lemma 3.6, Eq. (53) follows from (68) and (69), while (54) is precisely (66). Finally, (55) can be deduced by applying Lemma A.6 to $G = F_1$. □

KPs2Dsec

4. Applications to two-dimensional lattices

4.1. **The KP regime for the ETL lattice.** We want to study the behaviour of small amplitude solutions of (5), with initial data in which only one low-frequency Fourier mode is excited.

As a first step, we introduce an interpolating function $Q = Q(t,x)$ such that

(A1) $Q(t,j) = Q_j(t)$, for all $j \in \mathbb{Z}^2_{N_1,N_2}$;
(A2) Q is periodic with period $2N_1 + 1$ in the x_1-variable, and periodic with period $2N_2 + 1$ in the x_2-variable;
(A3) Q has zero average, $\int_{[-(N_1+\frac{1}{2}),N_1+\frac{1}{2}]\times[-(N_2+\frac{1}{2}),N_2+\frac{1}{2}]} Q(t,j)\mathrm{d}j = 0\ \forall t$;
(A4) Q fulfills

FPUeqc
$$\ddot{Q} = \Delta_1(Q + \alpha Q^2 + \beta Q^3), \tag{70}$$

Delta1c
$$\Delta_1 := 4\sinh^2\left(\frac{\partial_{x_1}}{2}\right) + 4\sinh^2\left(\frac{\partial_{x_2}}{2}\right). \tag{71}$$

It is easy to verify that (70) is Hamiltonian with Hamiltonian function

HamFPUc
$$H(Q,P) = \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^2},\frac{1}{\mu^2}\right]} \frac{-P\,\Delta_1 P + Q^2}{2} + \alpha\frac{Q^3}{3} + \beta\frac{Q^4}{4}\mathrm{d}x, \tag{72}$$

where P is a periodic function which has zero average and is canonically conjugated to Q.

We consider (70), with $\alpha \neq 0$, and we look for small amplitude solutions of the form

KPr1
$$Q(t,x) = \mu^2 q(\mu t, \mu x_1, \mu^2 x_2), \tag{73}$$

with μ as in (11). We introduce the rescaled variables $\tau = \mu t$, $y_1 = \mu x_1$, $y_2 = \mu^2 x_2$.

Plugging (73) into (70), leads to

FPUeqKPr1
$$q_{\tau\tau} = \frac{\Delta_{\mu,y_1}}{\mu^2}\left(q + \mu^2\alpha q^2\right), \tag{74}$$

Delta1mu
$$\Delta_{\mu,y_1} := 4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right) + 4\sinh^2\left(\mu^2\frac{\partial_{y_2}}{2}\right), \tag{75}$$

which is a Hamiltonian PDE corresponding to the Hamiltonian functional,

HamFPUcKP (76)
$$K_3(q,p) = \int_I \frac{-p\,\Delta_{\mu,y_1} p}{2\mu^2} + \frac{q^2}{2} + \alpha\mu^2\frac{q^3}{3} + \beta\mu^4\frac{q^4}{4}\mathrm{d}y,$$

where

I (77)
$$I := [-1,1]^2,$$

and p is the variable canonically conjugated to q.

Now, observe that the the operator Δ_{μ,y_1} admits the following asymptotic expansion up to terms of order $\mathcal{O}(\mu^4)$,

ex2Delta1 (78)
$$\frac{\Delta_{\mu,y_1}}{\mu^2} \sim \partial_{y_1}^2 + \mu^2\partial_{y_2}^2 + \frac{\mu^2}{12}\partial_{y_1}^4 + \mathcal{O}(\mu^4),$$

Therefore the Hamiltonian (76) admits the following asymptotic expansion

asexp3 (79)
$$K_3(q,p) \sim \hat{h}_0(q,p) + \mu^2\hat{F}_1(q,p) + \mu^4\hat{\mathcal{R}}(q,p),$$

h0KP (80)
$$\hat{h}_0(q,p) = \int_I \frac{-p\,(\partial_{y_1}^2 p) + q^2}{2}\mathrm{d}y,$$

F1KP (81)
$$\hat{F}_1(q,p) = \int_I -\frac{p\,\partial_{y_1}^4 p}{24} - \frac{p\,\partial_{y_2}^2 p}{2} + \alpha\frac{q^3}{3}\mathrm{d}y.$$

Following the approach of [?] (bambusi2006metastability), we can introduce the following non-canonical change of coordinates

xi (82)
$$\xi := \frac{1}{\sqrt{2}}(q + \partial_{y_1}p),$$

eta (83)
$$\eta := \frac{1}{\sqrt{2}}(q - \partial_{y_1}p).$$

which transforms the Poisson tensor into

Poisson (84)
$$J = \partial_{y_1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and Hamilton equations associated to a Hamiltonian K_1 are

$$\partial_\tau \xi = -\partial_{y_1}\frac{\delta K_1}{\delta \xi}$$
$$\partial_\tau \eta = \partial_{y_1}\frac{\delta K_1}{\delta \eta}.$$

CasimirRem **Remark 4.1.** *By the explicit expression of the Poisson tensor* (84) *we can compute straightforwardly Casimir invariants associated to* J*, which are*

CasimirKdV (85)
$$C(\xi,\eta) = A(y_2) + B(y_2)\int_{-1}^{1}\xi(\tau,y_1,y_2)\mathrm{d}y_1 + C(y_2)\int_{-1}^{1}\eta(\tau,y_1,y_2)\mathrm{d}y_1,$$

where A*,* B *and* C *are arbitrary functions of* y_2*.*

Since Casimir invariants are constants of motion, we can restrict our analysis on the subspace defined by

ZeroAvy1KdV (86)
$$\int_{-1}^{1}\xi(\tau,y_1,y_2) - \eta(\tau,y_1,y_2)\mathrm{d}y_1 = 0 \qquad \forall\tau\in\mathbb{R},\ |y_2|\le 1.$$

However, by recalling (82)*-*(83) *one sees that* (86) *implies*

ZeroAvy1KdV2 (87)
$$\int_{-1}^{1}\partial_{y_1}p(\tau,y_1,y_2)\mathrm{d}y_1 = 0 \qquad \forall\tau\in\mathbb{R},\ |y_2|\le 1,$$

which is true due to periodic boundary conditions.

In the new coordinates the Hamiltonian takes the form

asexp3xieta
$$K_3(\xi,\eta) \sim h_0(\xi,\eta) + \mu^2 F_1(\xi,\eta) + \mu^4 \mathcal{R}(\xi,\eta), \tag{88}$$

h0KPxieta
$$h_0(\xi,\eta) = \int_I \frac{\xi^2+\eta^2}{2}\mathrm{d}y, \tag{89}$$

F1KPxieta
$$F_1(\xi,\eta) = \int_I -\frac{[\partial_{y_1}(\xi-\eta)]^2}{48} + \frac{[\partial_{y_2}\partial_{y_1}^{-1}(\xi-\eta)]^2}{4} + \alpha\frac{(\xi+\eta)^3}{3\cdot 2^{3/2}}\mathrm{d}y, \tag{90}$$

where (90) is well defined because of (104).

Now we apply the averaging Theorem 3.3 to the Hamiltonian (88), with $\delta = \mu^2$: observe that the equations of motion of h_0 have the following simple form:

h0flow
$$\begin{cases} \xi_\tau &= -\partial_{y_1}\xi \\ \eta_\tau &= \partial_{y_1}\eta \end{cases}; \qquad \begin{cases} \xi(\tau,y) &= \xi_0(y_1-\tau,y_2) \\ \eta(\tau,y) &= \eta_0(y_1+\tau,y_2) \end{cases}. \tag{91}$$

F1avKPprop

Proposition 4.2. *The average of F_1 in* (88) *with respect to the flow of h_0 in* (88) *is given by*

F1avKP
$$\langle F_1\rangle(\xi,\eta) = \int_I -\frac{(\partial_{y_1}\xi)^2 + (\partial_{y_1}\eta)^2}{48} + \frac{(\partial_{y_2}\partial_{y_1}^{-1}\xi)^2 + (\partial_{y_2}\partial_{y_1}^{-1}\eta)^2}{4}\mathrm{d}y + \frac{\alpha}{3\cdot 2^{3/2}}([\xi^3]+[\eta^3]) \tag{92}$$

where we denote by $[f^j]$ the average $\int_I f^j(y)\frac{\mathrm{d}y}{4}$.

The proof of this proposition is a straightforward application of the following two lemmas.

lem:averaging1

Lemma 4.3. *Given two functions $u, v \in L^2([-1,1])$*

$$\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}s\, u(y\pm s)v(y\mp s) = \int_{-1}^{1}u(y)\mathrm{d}y\int_{-1}^{1}v(y)\,\mathrm{d}y.$$

Proof. Denoting with $\{\hat{u}_k\}_k$ and $\{\hat{v}_k\}_k$ the Fourier series of u and v respectively and using Plancherel theorem one obtains

$$\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}s\, u(y\pm s)v(y\mp s) = \frac{1}{2}\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}s\sum_{k,k'\in\mathbb{Z}}\hat{u}_k\hat{v}_{k'}e^{\pi i k(y\pm s)}e^{\pi i k'(y\mp s)} = \hat{u}_0\hat{v}_0$$

and thus Lemma is proved. □

lem:averaging2

Lemma 4.4. *Given a function $u \in L^1([-1,1])$ then*

$$\frac{1}{2}\int_{-1}^{1}\mathrm{d}s\int_{-1}^{1}\mathrm{d}y\; u(y\pm s) = \int_{-1}^{1}u(x)\,\mathrm{d}x$$

Proof. The thesis follows by a simple change of coordinates $x := y\pm s$. □

Proof of Proposition 4.2. For the computation of $\langle F_1\rangle(\xi,\eta)$ one can exchange the order of the integrations and apply Lemma 4.3 and 4.4. □

KPcor

Corollary 4.5. *The equations of motion associated to $h_0(\xi,\eta) + \mu^2\langle F_1\rangle(\xi,\eta)$ are given by*

KPsys
$$\begin{cases} \xi_\tau &= -\partial_{y_1}\xi - \frac{\mu^2}{24}\partial_{y_1}^3\xi - \frac{\mu^2}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\xi - \frac{\alpha\mu^2}{2\sqrt{2}}\partial_{y_1}(\xi^2) \\ \eta_\tau &= \partial_{y_1}\eta + \frac{\mu^2}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\eta + \frac{\mu^2}{24}\partial_{y_1}^3\eta + \frac{\alpha\mu^2}{2\sqrt{2}}\partial_{y_1}(\eta^2) \end{cases}. \tag{93}$$

More explicitly, we observe that (93) is a system of two uncoupled KP equations on a two-dimensional torus in translating frames.

KdVsubsec

4.2. **The KdV regime for the ETL lattice.** For this regime we consider (70), with $\alpha \neq 0$, and we look for small amplitude solutions of the form

KdVr1
$$Q(t,x) = \mu^2 q(\mu t, \mu x_1, \mu^\sigma x_2), \tag{94}$$

where $q : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ is a periodic function and μ, $\sigma > 2$ are defined in (11)-(12). We introduce the rescaled variables $\tau = \mu t$, $y_1 = \mu x_1$, $y_2 = \mu^\sigma x_2$, and we denote I is as in (77). Plugging (94) into (70), we get

FPUeqKdVr1
$$q_{\tau\tau} = \frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\left(q + \mu^2 \alpha q^2\right), \tag{95}$$

Delta1sigma
$$\Delta_{\mu,y_1,\sigma} := 4\sinh^2\left(\frac{\mu \partial_{y_1}}{2}\right) + 4\sinh^2\left(\mu^\sigma \frac{\partial_{y_2}}{2}\right), \tag{96}$$

which is a Hamiltonian PDE corresponding to the Hamiltonian functional

HamFPUcKdV
$$K_1(q,p) = \int_I \frac{-p\,\Delta_{\mu,y_1,\sigma} p}{2\mu^2} + \frac{q^2}{2} + \alpha\mu^2 \frac{q^3}{3} \mathrm{d}y, \tag{97}$$

and p is the variable canonically conjugated to q.

Now, observe that the the operator $\Delta_{\mu,y_1,\sigma}$ admits the following asymptotic expansion,

exDelta1sigma
$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2} \sim \partial_{y_1}^2 + \mu^{2(\sigma-1)}\partial_{y_2}^2 + \sum_{m\geq 1} c_m \left(\mu^{2m}\partial_{y_1}^{2(m+1)} + \mu^{2[(m+1)\sigma-1]}\partial_{y_2}^{2(m+1)}\right), \tag{98}$$

where

cm
$$c_m := \frac{2}{(2m)!}, \tag{99}$$

and considering that $\sigma > 2$, we have

ex2Delta1sigma
$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2} \sim \partial_{y_1}^2 + \frac{\mu^2}{12}\partial_{y_1}^4 + \mathcal{O}(\mu^4). \tag{100}$$

Therefore the Hamiltonian (97) admits the following asymptotic expansion

asexp1
$$K_1(q,p) \sim \hat{h}_0(q,p) + \mu^2 \hat{F}_1(q,p) + \mu^4 \hat{\mathcal{R}}(q,p), \tag{101}$$

h0
$$\hat{h}_0(q,p) = \int_I \frac{-p\,(\partial_{y_1}^2 p) + q^2}{2} \mathrm{d}y, \tag{102}$$

F1KdV
$$\hat{F}_1(q,p) = \int_I -\frac{p\,\partial_{y_1}^4 p}{24} + \alpha \frac{q^3}{3} \mathrm{d}y. \tag{103}$$

Note that the nonlinearity of degree 4 does not affect the Hamiltonian up to order $\mathcal{O}(\mu^4)$.

By exploiting again the non-canonical change of coordinates $(q,p) \mapsto (\xi,\eta)$ introduced in (82)-(83) and the Poisson tensor (84), and

ZeroAvy1
$$\int_{-1}^{1} \xi(\tau,y_1,y_2) - \eta(\tau,y_1,y_2)\mathrm{d}y_1 = 0 \qquad \forall \tau \in \mathbb{R},\ |y_2| \leq 1, \tag{104}$$

we obtain

asexp1xieta
$$K_1(\xi,\eta) \sim h_0(\xi,\eta) + \mu^2 F_1(\xi,\eta) + \mu^4 \mathcal{R}(\xi,\eta), \tag{105}$$

h0xieta
$$h_0(\xi,\eta) = \int_I \frac{\xi^2 + \eta^2}{2} \mathrm{d}y, \tag{106}$$

F1xieta
$$F_1(\xi,\eta) = \int_I -\frac{[\partial_{y_1}(\xi - \eta)]^2}{48} + \alpha \frac{(\xi+\eta)^3}{3 \cdot 2^{3/2}} \mathrm{d}y. \tag{107}$$

Now we apply the averaging Theorem 3.3 to the Hamiltonian (105), with $\delta = \mu^2$.

F1avKdVprop **Proposition 4.6.** *The average of* F_1 *in* (105) *with respect to the flow of* h_0 *in* (106) *is given by*

F1avKdV
$$\langle F_1\rangle(\xi,\eta) = -\int_I \frac{(\partial_{y_1}\xi)^2 + (\partial_{y_1}\eta)^2}{48}\mathrm{d}y + \frac{\alpha}{3\cdot 2^{3/2}}([\xi^3]+[\eta^3]), \tag{108}$$

where we denote by $[f^j]$ *the average* $\int_I f^j(y)\frac{\mathrm{d}y}{4}$.

KdVcor **Corollary 4.7.** *The equations of motion associated to* $h_0(\xi,\eta) + \mu^2\langle F_1\rangle(\xi,\eta)$ *are given by*

KdVsys
$$\begin{cases} \xi_\tau &= -\partial_{y_1}\xi - \frac{\mu^2}{24}\partial^3_{y_1}\xi - \frac{\mu^2\alpha}{2\sqrt{2}}\,\partial_{y_1}(\xi^2) \\ \eta_\tau &= \partial_{y_1}\eta + \frac{\mu^2}{24}\partial^3_{y_1}\eta + \frac{\mu^2\alpha}{2\sqrt{2}}\,\partial_{y_1}(\eta^2) \end{cases}. \tag{109}$$

The latter is a system of two uncoupled KdV equations in translating frames with respect to the y_1-direction, for each fixed value of the coordinate y_2.

mKdVrem **Remark 4.8.** *One can also study the* β *model (namely,* (70) *with* $\alpha = 0$ *and* $\beta \neq 0$*) in the following regime,*

(mKdV) *the* β *model in the very weakly transverse regime,*

mKdVr
$$Q(t,x) = \mu\, q(\mu t, \mu x_1, \mu^\sigma x_2), \tag{110}$$

where $\mu \ll 1$, $2 < \sigma$.

Let us introduce again the rescaled variables $\tau = \mu t$, $y_1 = \mu x_1$, $y_2 = \mu^\sigma x_2$, *and the domain* I *as in* (77)*; plugging* (110) *into* (70), *we get*

FPUeqmKdVr1
$$q_{\tau\tau} = \frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\left(q + \mu^2\beta q^3\right), \tag{111}$$

where $\Delta_{\mu,y_1,\sigma}$ *is the operator introduced in* (96). *Eq.* (111) *is a Hamiltonian PDE with the following corresponding Hamiltonian,*

HamFPUcmKdV
$$K_2(q,p) = \int_I \frac{-p\,\Delta_{\mu,y_1,\sigma}p}{2\mu^2} + \frac{q^2}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y, \tag{112}$$

where p *is the variable canonically conjugated to* q.

Recalling that (86) *holds true, we exploit again the non-canonical change of coordinates* (82)-(83) *and the Poisson tensor* (84), *obtaining that*

asexp2xieta
$$K_2(\xi,\eta) \sim h_0(\xi,\eta) + \mu^2 F_1(\xi,\eta) + \mu^4\mathcal{R}(\xi,\eta), \tag{113}$$

where h_0 *is the same as in* (106), *while*

F1mxieta
$$F_1(\xi,\eta) = \int_I -\frac{[\partial_{y_1}(\xi-\eta)]^2}{48} + \beta\frac{(\xi+\eta)^4}{2^4}\mathrm{d}y. \tag{114}$$

Applying Theorem 3.3 to the Hamiltonian (113) *with* $\delta = \mu^2$, *we get that the equations of motion associated to* $h_0(\xi,\eta) + \mu^2\langle F_1\rangle(\xi,\eta)$ *are given by*

mKdVsys
$$\begin{cases} \xi_\tau &= -\left(1 + \frac{3}{4}[\eta^2]\right)\partial_{y_1}\xi - \frac{\mu^2}{24}\partial^3_{y_1}\xi - \frac{\mu^2\beta}{4}\,\partial_{y_1}(\xi^3) \\ \eta_\tau &= \left(1 + \frac{3}{4}[\xi^2]\right)\partial_{y_1}\eta + \frac{\mu^2}{24}\partial^3_{y_1}\eta + \frac{\mu^2\beta}{4}\,\partial_{y_1}(\eta^3) \end{cases}. \tag{115}$$

which is a system of two uncoupled mKdV equations in translating frames with respect to the y_1*-direction. The integrability properties of the mKdV equation and the existence of Birkhoff coordinates for this model have been proved in* kappeler2008mkdv [?].

1DNLSsubsec

4.3. **The one-dimensional NLS regime for the KG Lattice.** We want to study small amplitude solutions of (22) , with initial data in which only one low-frequency Fourier mode is excited.

Analogously to the procedure of the previous sections, the first step is to introduce an interpolating function $Q = Q(t,x)$ such that

(B1) $Q(t,j) = Q_j(t)$, for all $j \in \mathbb{Z}^2_{N_1,N_2}$;
(B2) Q is periodic with period $2N_1+1$ in the x_1-variable, and periodic with period $2N_2+1$ in the x_2-variable;

(B3) Q fulfills

KGseqc
$$\ddot{Q} = \Delta_1 Q - Q - \beta Q^{2p+1}, \tag{116}$$

where Δ_1 is the operator defined in (71) (recall that we also assumed $m = 1$ in (20)).

It is easy to verify that (116) is Hamiltonian with Hamiltonian function

HamKGsc
$$H(Q,P) = \int_{[-\frac{1}{\mu},\frac{1}{\mu}]\times[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}]} \frac{P^2}{2} + \frac{Q^2}{2} - \frac{Q\,\Delta_1 Q}{2} + \beta\frac{Q^{2p+2}}{2p+2}\mathrm{d}x, \tag{117}$$

where P is a periodic function and is canonically conjugated to Q.

Starting from the Hamiltonian (19), where $p = 1$, we look for small amplitude solutions of the form

1DNLSr
$$Q(t,x) = \mu\, q(\mu^2 t, \mu x_1, \mu^\sigma x_2)\,. \tag{118}$$

where $q : \mathbb{R}\times\mathbb{T}^2 \to \mathbb{R}$ is a periodic function and σ, μ are defined respectively in (12)-(11).

We introduce the rescaled variable $y_1 = \mu x_1$ and $y_2 = \mu^\sigma x_2$, and we define I as in (77). The Hamiltonian (19) in the rescaled variable is given by

Ham2KGc1DNLS
$$K_4(q,p) = \int_I \frac{p^2}{2} + \frac{q^2}{2} - \frac{q\,\Delta_{\mu,y_1,\sigma} q}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y, \tag{119}$$

with the operator $\Delta_{\mu,y_1,\sigma}$ as in (96), and p is the variable canonically conjugated to q. The corresponding equation of motion is given by

1DKGsc2
$$q_{tt} = -q + \Delta_{\mu,y_1,\sigma} q - \beta\mu^2 q^3. \tag{120}$$

Recall that

$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2} \sim \partial_{y_1}^2 + \mu^{2(\sigma-1)}\partial_{y_2}^2 + \frac{\mu^2}{12}\partial_{y_1}^4 + \mathcal{O}(\mu^{2(2\sigma-1)}),$$

hence the Hamiltonian (119) admits the following asymptotic expansion

asexp4
$$K_4(q,p) \sim \hat{h}_0(q,p) + \mu^2\hat{F}_1(q,p) + \mu^{2(2\sigma-1)}\hat{\mathcal{R}}(q,p), \tag{121}$$

h01DNLS
$$\hat{h}_0(q,p) = \int_I \frac{p^2+q^2}{2}\mathrm{d}y, \tag{122}$$

F11DNLS
$$\hat{F}_1(q,p) = \int_I -\frac{q\,\partial_{y_1}^2 q}{2} + \beta\frac{q^4}{4}\mathrm{d}y, \tag{123}$$

and the equation of motion associated to $h_0 + F_1$ is given by the following cubic one-dimensional nonlinear Klein-Gordon (NLKG) equation,

1DNLKGeq
$$q_{tt} = -(q - \mu^2\partial_{y_1}^2 q) - \mu^2\beta q^3. \tag{124}$$

We now exploit the change of coordinates $(q,p) \mapsto (\psi,\bar{\psi})$ given by

psi
$$\psi = \frac{1}{\sqrt{2}}(q - ip), \tag{125}$$

therefore the inverse change of coordinates is given by

q
$$q = \frac{1}{\sqrt{2}}(\psi + \bar{\psi}), \tag{126}$$

p
$$p = \frac{1}{\sqrt{2}}i(\psi - \bar{\psi}), \tag{127}$$

while the symplectic form is given by $-i\mathrm{d}\psi \wedge \mathrm{d}\bar{\psi}$. With this change of variables the Hamiltonian takes the form

asexp4psi
$$K_4(\psi,\bar{\psi}) \sim h_0(\psi,\bar{\psi}) + \mu^2 F_1(\psi,\bar{\psi}) + \mu^{2(2\sigma-1)}\mathcal{R}(\psi,\bar{\psi}), \tag{128}$$

h01DNLSpsi
$$h_0(\psi,\bar{\psi}) = \int_I \psi\,\bar{\psi}\mathrm{d}y, \tag{129}$$

F11DNLSpsi
$$F_1(\psi,\bar{\psi}) = \int_I -\frac{(\psi+\bar{\psi})\,[-\partial_{y_1}^2(\psi+\bar{\psi})]}{4} + \beta\frac{(\psi+\bar{\psi})^4}{16}\mathrm{d}y. \tag{130}$$

Now we apply the averaging Theorem 3.3 to the Hamiltonian (128), with $\delta = \mu^2$. Observe that h_0 generates a periodic flow,

$$-i\partial_t \psi = \psi;$$

$$\psi(t,y) = e^{it}\psi_0(y). \tag{131}$$

Proposition 4.9. *The average of F_1 in (128) with respect to the flow of h_0 (122) is given by*

$$\langle F_1 \rangle (\psi, \bar\psi) = \int_I \frac{\bar\psi\,(-\partial_{y_1}^2 \psi)}{2} \mathrm{d}y + \frac{3\beta}{8}\int_I |\psi|^4 \mathrm{d}y. \tag{132}$$

Corollary 4.10. *The equations of motion associated to $h_0(\psi,\bar\psi) + \mu^2 \langle F_1 \rangle (\psi,\bar\psi)$ are given by a cubic one-dimensional nonlinear Schrödinger equation for each fixed value of y_2,*

$$-i\psi_t = \psi - \mu^2\, \partial_{y_1}^2 \psi + \mu^2 \frac{3\beta}{4}\,|\psi|^2\psi. \tag{133}$$

Remark 4.11. *Let us consider the Hamiltonian (19) in the following regime,*

(2-D NLS) *the scalar model (19) with $m = 1$, $p = 1$ and*

$$Q(t,x) = \mu\, q(\mu^2 t, \mu x), \tag{134}$$

where $\mu \ll 1$ and $\sigma = 1$.

If we introduce the rescaled variable $y = \mu x$ and we define I as in (77), we have that the Hamiltonian takes the following form (we denote by p the variable canonically conjugated to q)

$$K_5(q,p) = \int_I \frac{p^2}{2} + \frac{q^2}{2} - \frac{q\,\Delta_\mu q}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y, \tag{135}$$

$$\Delta_\mu := 4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right) + 4\sinh^2\left(\mu\frac{\partial_{y_2}}{2}\right). \tag{136}$$

By expanding the operator Δ_μ and by exploiting the change of variable (125), we get

$$K_5(\psi,\bar\psi) \sim h_0(\psi,\bar\psi) + \mu^2 F_1(\psi,\bar\psi) + \mu^4 \mathcal{R}(\psi,\bar\psi), \tag{137}$$

$$h_0(\psi,\bar\psi) = \int_I \psi\,\bar\psi \mathrm{d}y, \tag{138}$$

$$F_1(\psi,\bar\psi) = \int_I -\frac{(\psi+\bar\psi)\,[-\Delta(\psi+\bar\psi)]}{4} + \beta\frac{(\psi+\bar\psi)^4}{16}\mathrm{d}y. \tag{139}$$

By applying Theorem 3.3 to the Hamiltonian (128), with $\delta = \mu^2$, we obtain that the equation of motion associated to $h_0(\psi,\bar\psi) + \mu^2 \langle F_1 \rangle (\psi,\bar\psi)$ is given by the cubic nonlinear Schrödinger (NLS) equation

$$-i\psi_t = \psi - \mu^2\, \Delta\psi + \mu^2 \frac{3\beta}{4}\,|\psi|^2\psi. \tag{140}$$

The local well-posedness of the NLS equation (140) in the Sobolev space $H^s(\mathbb{T}^2)$, $s > 0$, has been discussed by Bourgain in [?]*; along with the conversation laws, this implies the global existence in the defocusing case ($\beta > 0$), and the global existence for small solutions in the focusing case ($\beta < 0$). The long time dynamics of the NLS equation has also been studied, in relation with the transfer of energy among Fourier modes and with the growth of Sobolev norms* [?] [?] [?] [?] [?].

5. Dynamics of the normal form equation

5.1. **The KP equation.** In this section we recall some known facts on the dynamics of the KP equation on the two-dimensional torus

$$\xi_\tau = -\frac{1}{24}\partial_{y_1}^3\xi - \frac{1}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\xi - \frac{\alpha}{2\sqrt{2}}\,\partial_{y_1}(\xi^2),\ \alpha = \pm 1,\ y \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2. \tag{141}$$

The KP equation has been introduced in order to describe weakly-transverse solutions of the water waves equations; it has been considered as a two-dimensional analogue of the KdV equation, since also the KP equation admits an infinite number of constants of motions [?] [?] [?]. It is customary to refer to (141) as KP-I equation when $\alpha = -1$, and as KP-II equation when $\alpha = 1$.

The global-well posedness for the KP-II equation on the two-dimensional torus has been discussed by Bourgain in bourgain1993cauchy [?]. The main point of the result by Bourgain consists in extending the local well-posedness result to a global one, even though the L^2-norm is the only constant of motion for the KP-II equation that allows an a-priori bound for the solution (see Theorem 8.10 and Theorem 8.12 in bourgain1993cauchy [?]).

BouThm

Theorem 5.1. *Consider* (141) *with* $\alpha = 1$.

Let $\rho \geq 0$ *and* $s \geq 0$, *and assume that the initial datum* $\xi(0, \cdot, \cdot) = \xi_0 \in \ell^2_{\rho,s}$. *Then* (141) *is globally well-posed in* $\ell^2_{\rho,s}$. *Moreover, the* ℓ^2*-norm of the solution is conserved,*

L2normKP

$$\|\xi(t)\|_{\ell^2} = \|\xi_0\|_{\ell^2}, \tag{142}$$

while

HsnormKP

$$\|\xi(t)\|_{\ell^2_{0,s}} \leq e^{C|t|}\|\xi_0\|_{\ell^2_{0,s}}, \tag{143}$$

where C *depends on* s.

Remark 5.2. *As pointed out by Bourgain in Sec. 10.2 of* bourgain1993cauchy [?], *a global well-posedness result for sufficiently smooth solution of the KP-I equation (namely,* (141) *with* $\alpha = -1$*) on the two-dimensional torus can be obtained by generalizing the argument in* schwarz1987periodic [?] *for small data and by using the a-priori bounds given by the constants of motion for the KP-I equation.*

For the KP equation the construction of action-angle/Birkhoff coordinates is still an open problem.

KdVdynsubsec

5.2. **The KdV equation.** In this section we recall some known facts on the dynamics of the KdV equation with periodic boundary conditions. The interested reader can find more detailed explanations and proofs in kappeler2003kdv [?].

Consider the KdV equation

KdVeq

$$\xi_\tau = -\frac{1}{24}\partial^3_{y_1}\xi - \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2), \qquad y_1 \in [0, 2]. \tag{144}$$

Through the Lax pair formulation of the evolution problem (144) one get that the periodic eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of the Sturm-Liouville operator

$$L_\xi := -\partial^2_{y_1} + 6\sqrt{2}\xi(\tau, y_1) \tag{145}$$

are conserved quantities under the evolution of the KdV equation (144). Moreover, if we define the gaps of the spectrum $\gamma_m := \lambda_{2m} - \lambda_{2m-1}$ ($m \geq 1$), it is well known that the squared spectral gaps $(\gamma^2_m)_{m\geq1}$ form a complete set of constants of motion for (144).

The following relation between the sequence of the spectral gaps and the regularity of the corresponding solution to the KdV equation holds (see Theorem 9, Theorem 10 and Theorem 11 in kappeler2008well [?]; see also poschel2011hill [?])

KapPosThm1

Theorem 5.3. *Assume that* $\xi \in L^2$, *then* $\xi \in \ell^2_{0,s}$ *if and only if its spectral gaps satisfy*

$$\sum_{m\geq1} m^{2s}|\gamma_m|^2 < +\infty.$$

Moreover if $\xi \in \ell^2_{\rho,s}$, *then*

SpecGapEst

$$\sum_{m\geq1} m^{2s}e^{2\rho m}|\gamma_m|^2 < +\infty; \tag{146}$$

conversely, if (146) *holds, then* $\xi \in \ell^2_{\rho',0}$ *for some* $\rho' > 0$.

Kappeler and Pöschel constructed the following global Birkhoff coordinates (see Theorem 1.1 in kappeler2003kdv [?])

KapPosThm2

Theorem 5.4. *There exists a diffeomorphism* $\Omega : L^2 \to \ell^2_{0,1/2} \times \ell^2_{0,1/2}$ *such that:*

- Ω *is bijective, bianalytic and canonical;*
- *for each* $s \geq 0$, *the restriction of* Ω *to* $\ell^2_{0,s}$, *namely the map*

  $$\Omega : \ell^2_{0,s} \to \ell^2_{0,s+1/2} \times \ell^2_{0,s+1/2}$$

  is bijective, bianalytic and canonical;

- *the coordinates* $(x, y) \in \ell^2_{0,3/2} \times \ell^2_{0,3/2}$ *are Birkhoff coordinates for the KdV equation, namely they form a set of canonically conjugated coordinates in which the Hamiltonian of the KdV equation* (144) *depends only on the action* $I_m := \frac{x_m^2 + y_m^2}{2}$ $(m \geq 1)$*.*

The dynamics of the KdV equation (144) in terms of the variables (x, y) is trivial: it can be immediately seen that any solution is periodic, quasiperiodic or almost periodic, depending on the number of spectral gaps (equivalently, depending on the number of actions) initially different from zero.

1DNLSdynsubsec

5.3. **The one-dimensional cubic NLS equation.** In this section we recall some known facts on the dynamics of the one-dimensional cubic defocusing NLS equation with periodic boundary conditions. The interested reader can find more detailed explanations and proofs in [?] [?].

Consider the cubic defocusing NLS equation

1DNLSeq (147)
$$i\psi_\tau = -\partial^2_{y_1}\psi + 2|\psi|^2\psi, \qquad y_1 \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}).$$

Eq. (147) is a PDE admitting a Hamiltonian structure: indeed, we can set $\mathcal{H}^{\rho,s} = \ell^2_{\rho,s} \times \ell^2_{\rho,s}$ as the phase space with elements denoted by $\phi = (\phi_1, \phi_2)$, while the associated Poisson bracket and the Hamiltonian are given by

1DNLSPoisson (148)
$$\{F, G\} := -i \int_{\mathbb{T}} (\partial_{\phi_1} F\, \partial_{\phi_2} G - \partial_{\phi_1} G\, \partial_{\phi_2} F)\, \mathrm{d}y_1,$$

1DNLSHam (149)
$$H_{NLS}(\phi_1, \phi_2) := \int_{\mathbb{T}} \partial_{y_1}\phi_1\, \partial_{y_1}\phi_2 + \phi_1^2\phi_2^2\, \mathrm{d}y_1.$$

The defocusing NLS equation (147) is obtained by restricting (149) to the invariant subspace of states of real type,

RealStates (150)
$$\mathcal{H}^{\rho,s}_r := \{\phi \in \mathcal{H}^{\rho,s} : \phi_2 = \bar{\phi}_1\}.$$

The above Hamiltonian (149) is well-defined on $\mathcal{H}^{\rho,s}$ with $s \geq 1$ and $\rho \geq 0$, while the initial value problem for the NLS equation (147) is well-posed on $\mathcal{H}^{0,0} = \ell^2 \times \ell^2$.

It is well known from the work by Zakharov and Shabat that the NLS equation (147) has a Lax pair, and that it admits infinitely many constants of motion in involution. More precisely, for any $\phi \in \mathcal{H}^{0,0}$ consider the Zakharov-Shabat operator

ZSOp (151)
$$L(\phi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_{y_1} + \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix},$$

where we call ϕ the potential of the operator $L(\phi)$. The spectrum of $L(\phi)$ on the interval $[0, 2]$ with periodic boundary conditions is pure point, and it consists of the following sequence of periodic eigenvalues

perNLSspec (152)
$$\cdots < \lambda^-_{-1} \leq \lambda^+_{-1} < \lambda^-_0 \leq \lambda^+_0 < \lambda^-_1 \leq \lambda^+_1 < \cdots,$$

where the quantities $\gamma_m := \lambda^+_m - \lambda^-_m$ $(m \in \mathbb{Z})$ are called gap lengths. It has been proved that the squared spectral lengths $(\gamma_m^2)_{m\in\mathbb{Z}}$ form a complete set of analytic constants of motion for (147).

Grébert, Kappeler and Mityagin proved the following relation between the sequence of the squared spectral gaps and the regularity of the corresponding potential (see Theorem in [?]).

GrebKapThm1

Theorem 5.5. *Let* $\rho \geq 0$ *and* $s > 0$*, then for any bounded subset* $\mathcal{B} \subset \ell^2_{\rho,s} \times \ell^2_{\rho,s}$ *there exists* $n_0 \geq 1$ *and* $M \geq 1$ *such that for any* $|k| \geq n_0$ *and any* $(\phi_1, \phi_2) \in \mathcal{B}$*, the following estimate holds*

SpecGapEstNLS (153)
$$\sum_{|k|\geq n_0} (1 + |k|)^{2s} e^{2\rho|k|} |\gamma_m|^2 \leq M.$$

Moreover, Grébert and Kappeler constructed the following global Birkhoff coordinates (see Theorem 20.1 - Theorem 20.3 in [?])

GrebKapThm2

Theorem 5.6. *There exists a diffeomorphism* $\Omega : L^2_r \to \mathcal{H}^{0,0}_r$ *such that:*

- Ω *is bianalytic and canonical;*
- *for each* $s \geq 0$*, the restriction of* Ω *to* $\mathcal{H}^{0,s}_r$*, namely the map*
$$\Omega : \mathcal{H}^{0,s}_r \to \mathcal{H}^{0,s}_r$$
is again bianalytic and canonical;

- *the coordinates* $(x, y) \in \mathcal{H}^{0,1}_r$ *are Birkhoff coordinates for the NLS equation, namely they form a set of canonically conjugated coordinates in which the Hamiltonian of the NLS equation* (147) *depends only on the action* $I_m := \frac{x_m^2 + y_m^2}{2}$ *(*$m \in \mathbb{Z}$*).*

The dynamics of the NLS equation (147) in terms of the variables (x, y) is trivial: it can be immediately seen that any solution is periodic, quasiperiodic or almost periodic, depending on the number of spectral gaps (equivalently, depending on the number of actions) initially different from zero.

ApprSec

6. Approximation results

In this section we show how to use the normal form equations in order to construct approximate solutions of (5) and (22), and we estimate the difference with respect to the true solutions with corresponding initial data.

The approach is the same for all the regimes (73), (94) and (118). First, we have to point out a relation between the energy of normal mode E_k (defined in (7) for (5), and in (7) for (22)) , $k \in \mathbb{Z}^2_{2N+1}$, and the Fourier coefficients of the solutions of the normal form equations. Then we have to prove that the approximate solutions approximate the energy of the true normal mode E_k up to the time-scale in which the continuous approximation is valid, and finally we can deduce the result about the dynamics of the lattice.

apprKPsubsec

6.1. The KP regime.

Let $I = [-1, 1]^2$ be as in (77), we define the Fourier coefficients of the function $q : I \to \mathbb{R}$ by

Fourierqcont

$$\hat{q}(j) := \frac{1}{2} \int_I q(y_1, y_2) e^{-i\pi(j_1 y_1 + j_2 y_2)} \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{154}$$

and similarly for the Fourier coefficients of the function p.

EnSpecKPLemma

Lemma 6.1. *Consider the lattice* (2) *in the regime (KP) and with interpolanting function* (73)*. Then for a state corresponding to* (q, p) *one has*

SpecEnNormModeKP

$$\mathcal{E}_\kappa = \frac{\mu^4}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2 : \mu L_1, \mu^2 L_2 \in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2 \left| \frac{\hat{p}_{K+L}}{\mu} \right|^2, \quad \forall k : \kappa(k) = (\mu K_1, \mu^2 K_2) \tag{155}$$

(where the ω_k *are defined as in* (8)*), and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. First we introduce a $(2N_1 + 1)(2N_2 + 1)$-periodic interpolating function for Q_j, namely a smooth function $Q : (t, x) \mapsto Q(t, x)$ such that

$$\begin{aligned} Q_j(t) &= Q(t, j), && \forall t, j, \\ Q(t, x_1, x_2 + 2N_2 + 1) &= Q(t, x), && \forall t, x, \\ Q(t, x_1 + 2N_1 + 1, x_2) &= Q(t, x), && \forall t, x, \end{aligned}$$

and similarly for P_j. We denote by

FourierQcont

$$\hat{Q}(j) := \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \int_{[-(N_1+\frac{1}{2}),(N_1+\frac{1}{2})]\times[-(N_2+\frac{1}{2}),(N_2+\frac{1}{2})]} Q(x) e^{-i\frac{j\cdot x\, 2\pi}{(2N_1+1)(2N_2+1)}} \mathrm{d}x, \tag{156}$$

so that by the interpolation property we obtain

$$\begin{aligned} Q_j(t) = Q(t, j) &= \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \sum_{k\in\mathbb{Z}^2} \hat{Q}(j) e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)(2N_2+1)}} \\ &= \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \\ &\quad \times \sum_{k=(k_1,k_2)\in\mathbb{Z}^2_{2N+1}} \left[\sum_{h=(h_1,h_2)\in\mathbb{Z}^2} \hat{Q}(k_1 + (2N_1+1)h_1, k_2 + (2N_2+1)h_2) \right] e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)(2N_2+1)}}, \end{aligned}$$

hence

FourierRel (157)
$$\hat{Q}_k = \sum_{h\in\mathbb{Z}^2} \hat{Q}(k_1 + (2N_1+1)h_1, k_2 + (2N_2+1)h_2).$$

The relation between $\hat{Q}(k)$ and $\hat{q}_k$ can be deduced from (94),

$$\begin{aligned}
Q(j) &= \mu^2 q(\mu j_1, \mu^2 j_2);\\
\hat{Q}_k &= \frac{1}{2}\mu^{3/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^2},\frac{1}{\mu^2}\right]} Q(x_1,x_2) e^{-i\pi(k_1x_1\mu + k_2x_2\mu^2)} \mathrm{d}x_1\,\mathrm{d}x_2\\
&= \frac{1}{2}\mu^{3/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^2},\frac{1}{\mu^2}\right]} \mu^2\, q\left(\mu x_1, \mu^2 x_2\right) e^{-i\pi(k_1x_1\mu + k_2x_2\mu^2)} \mathrm{d}x_1\,\mathrm{d}x_2\\
&\overset{(94)}{=} \frac{1}{2}\mu^{1/2} \int_I q(y) e^{-i\pi(k_1y_1+k_2y_2)} \mathrm{d}y
\end{aligned}$$

FourierRelQq (158)
$$= \mu^{1/2}\hat{q}_k,$$

and similarly

FourierRelPp (159)
$$\hat{P}_k = \mu^{-1/2}\hat{p}_k.$$

By using (7), (10) and (157)-(159) we have

$$\begin{aligned}
\mathcal{E}_\kappa &\overset{(10)}{=} \mu^3 \frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^2L_2\in 2\mathbb{Z}} |\hat{Q}_{K+L}|^2 + \omega_k^2 |\hat{P}_{K+L}|^2\\
&\overset{(158)}{=} \mu^3\, \mu \frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^2L_2\in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2 \left|\frac{\hat{p}_{K+L}}{\mu}\right|^2
\end{aligned}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$, and this leads to (155). □

KPxietaProp **Proposition 6.2.** *Fix $\rho > 0$ and $0 < \delta \ll 1$. Consider the normal form system (93), and define the Fourier coefficients of (ξ,η) through the following formula*

FourierXiKP (160)
$$\xi(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\xi}_h e^{ih\cdot y\pi},$$

FourierEtaKP (161)
$$\eta(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\eta}_h e^{ih\cdot y\pi},.$$

Consider $(\xi,\eta) \in \mathcal{H}^{\rho,0}$, and denote by $\mathcal{E}_\kappa$ the specific energy of the normal mode with index κ as defined in (9)-(10). Then for any positive μ sufficiently small

KPcoeffxieta (162)
$$\left| \mathcal{E}_\kappa - \mu^4 \frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2} \right| \le C\mu^{4+\frac{6}{5}} \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$ and $|K_1| + |K_2| \le \frac{(2+\delta)|\log\mu|}{\rho}$. Moreover,

KPSpecEnEst (163)
$$|\mathcal{E}_\kappa| \le C\,\mu^8 \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$ and $|K_1^2 + K_2^2|^{1/2} > \frac{(2+\delta)|\log\mu|}{\rho}$, and $\mathcal{E}_\kappa = 0$ otherwise.

The proof of the above Proposition is deferred to Appendix B.

Now, consider the following systems of uncoupled KP equations

KPsys1 (164)
$$\xi_\tau = -\frac{1}{24}\partial_{y_1}^3\xi - \frac{1}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\xi - \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2),$$

KPsys2 (165)
$$\eta_\tau = \frac{1}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\eta + \frac{1}{24}\partial_{y_1}^3\eta + \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\eta^2).$$

and consider a solution $(\tau, y) \mapsto (\widetilde{\xi_a}(\tau,y), \widetilde{\eta_a}(\tau,y))$ such that it belongs to $\mathcal{H}^{\rho,n}$, for some $n \ge 1$.

We consider the approximate solutions (Q_a, P_a) of the FPU model (70)

Qappr2
$$Q_a(\tau, y) := \frac{\mu^2}{\sqrt{2}} \left[\widetilde{\xi}_a(\mu^2\tau, y_1 - \tau, y_2) + \widetilde{\eta}_a(\mu^2\tau, y_1 + \tau, y_2)\right] \tag{166}$$

Pappr2
$$\partial_{y_1} P_a(\tau, y) := \frac{\mu}{\sqrt{2}} \left[\widetilde{\xi}_a(\mu^2\tau, y_1 - \tau, y_2) - \widetilde{\eta}_a(\mu^2\tau, y_1 + \tau, y_2)\right], \tag{167}$$

We need to compare the difference between the approximate solution (166)-(167) and the true solution of (5). Let consider an initial datum (Q_0, P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k}, \hat{P}_{0,k})$ given by (6), where

InDatumHyp21
$$Q_{0,k} \neq 0 \text{ only if } \kappa(k) = (\mu K_1, \mu^2 K_2). \tag{168}$$

We also assume that there exist C, $\rho > 0$ such that

InDatumHyp22
$$\frac{|\hat{Q}_{0,k}|^2 + \omega_k^2 |\hat{P}_{0,k}|^2}{N} \leq C e^{-2\rho|(\kappa_1(k)/\mu, \kappa_2(k)/\mu^2)|}. \tag{169}$$

Moreover, we define an interpolating function for the initial datum (Q_0, P_0) by

$$Q_0(y) = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{K:\left(\mu^2|K_1|^2 + \mu^4|K_2|^2\right)^{1/2} = |\kappa(k)| \leq 1} \hat{Q}_{0,k} e^{i\pi(\mu K_1 y_1 + \mu^2 K_2 y_2)},$$

and similarly for $y \mapsto P_0(y)$.

ApprPropKP
Proposition 6.3. *Consider (5) with $\sigma = 2$, and fix $1 \leq \gamma \leq \frac{5}{2}$. Let us assume that the initial datum for (5) satisfying (168)-(169), and denote by $(Q(t), P(t))$ the corresponding solution. Consider the approximate solution $(\widetilde{\xi}_a, \widetilde{\eta}_a)$ with the corresponding initial datum. Assume that $(\widetilde{\xi}_a, \widetilde{\eta}_a) \in \mathcal{H}^{\rho,n}$ for some $\rho > 0$ and for some $n \geq 1$ for all times, and fix $T_0 > 0$ and $0 < \delta \ll 1$.*

Then there exists $\mu_0 = \mu_0(T_0, \|(\widetilde{\xi}_a(0), \widetilde{\eta}_a(0))\|_{\mathcal{H}^{\rho,n}})$ such that, if $\mu < \mu_0$, we have that there exists $C > 0$ such that

KPapprDiscrCont
$$\sup_j |Q_j(t) - Q_a(t,j)| + |P_j(t) - P_a(t,j)| \leq C\mu^\gamma, \quad |t| \leq \frac{T_0}{\mu^3}, \tag{170}$$

where (Q_a, P_a) are given by (182)-(183). Moreover,

LowModesApprKP
$$\left|\mathcal{E}_\kappa - \mu^4 \frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2}\right| \leq C\mu^{4+\gamma} \tag{171}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$ and $|K_1| + |K_2| \leq \frac{(2+\delta)|\log \mu|}{\rho}$. Moreover,

HighModesApprKP
$$|\mathcal{E}_\kappa| \leq \mu^{4+\gamma} \tag{172}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log \mu|}{\rho}$, and $\mathcal{E}_\kappa = 0$ otherwise.

The proof of the above Proposition is deferred to Appendix C.

Proof of Theorem (2.1). First we prove (14).

We consider an initial datum as in (13); when passing to the continuous approximation (70), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0,n}$ for some $\rho_0 > 0$ and $n \geq 1$. By Theorem 5.1 the corresponding solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.1). Applying Proposition 6.3, we can deduce the corresponding result for the discrete model (5) and the specific quantities (10). □

apprKdVsubsec

6.2. **The KdV regime.** Similarly to Lemma 6.1, Proposition 6.2 we can prove the following results

EnSpecKdVLemma

Lemma 6.4. *Consider the lattice* (2) *in the regime (KdV) and with interpolanting function* (94). *Then for a state corresponding to* (q,p) *one has*

cEnNormModeKdV

$$\mathcal{E}_\kappa = \frac{\mu^4}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2 \left|\frac{\hat{p}_{K+L}}{\mu}\right|^2, \qquad \forall k : \kappa(k) = (\mu K_1, \mu^\sigma K_2) \tag{173}$$

(where the ω_k *are defined as in* (8) *and the* $\mathcal{E}_\kappa$ *in* (10)*), and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. As in Lemma 6.4 we introduce a $(2N_1+1, 2N_2+1)$-periodic interpolating function for Q_j and P_j. We denote $\hat{Q}(j)$ and $\hat{Q}_k$ as in (156) and (157).

The relation between $\hat{Q}(k)$ and $\hat{q}_k$ can be deduced from (94),

ourierRelQqKdV

$$\begin{aligned}
Q(j) &= \mu^2 q(\mu j_1, \mu^\sigma j_2);\\
\hat{Q}_k &= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} Q(x_1,x_2) e^{-i\pi(k_1 x_1 \mu + k_2 x_2 \mu^\sigma)} \mathrm{d}x_1\,\mathrm{d}x_2\\
&= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} \mu^2\, q\left(\mu x_1, \mu^\sigma x_2\right) e^{-i\pi(k_1 x_1 \mu + k_2 x_2 \mu^\sigma)} \mathrm{d}x_1\,\mathrm{d}x_2\\
&\overset{(94)}{=} \frac{1}{2}\mu^{(3-\sigma)/2} \int_I q(y) e^{-i\pi(k_1 y_1 + k_2 y_2)} \mathrm{d}y\\
&= \mu^{(3-\sigma)/2} \hat{q}_k, \qquad (174)
\end{aligned}$$

and similarly

ourierRelPpKdV

$$\hat{P}_k = \mu^{(1-\sigma)/2} \hat{p}_k. \tag{175}$$

By using (7), (10) and (174)-(175) we have

$$\begin{aligned}
\mathcal{E}_\kappa &\overset{(10)}{=} \mu^{\sigma+1} \frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{Q}_{K+L}|^2 + \omega_k^2 |\hat{P}_{K+L}|^2\\
&\overset{(174),(174)}{=} \mu^{\sigma+1}\, \mu^{3-\sigma} \frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2 \left|\frac{\hat{p}_{K+L}}{\mu}\right|^2
\end{aligned}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$, and this leads to (173). □

KdVxietaProp

Proposition 6.5. *Fix* $\rho > 0$ *and* $0 < \delta \ll 1$. *Consider the normal form system* (109), *and define the Fourier coefficients of* (ξ,η) *through the following formula*

FourierXiKdV

$$\xi(y) = \frac{1}{2} \sum_{h\in\mathbb{Z}^2} \hat{\xi}_h e^{ih\cdot y\pi}, \tag{176}$$

FourierEtaKdV

$$\eta(y) = \frac{1}{2} \sum_{h\in\mathbb{Z}^2} \hat{\eta}_h e^{ih\cdot y\pi}, . \tag{177}$$

Consider $(\xi,\eta) \in \mathcal{H}^{\rho,0}$, *and denote by* $\mathcal{E}_\kappa$ *the specific energy of the normal mode with index* κ *as defined in* (9)-(10). *Then for any positive* μ *sufficiently small*

KdVcoeffxieta

$$\left| \mathcal{E}_\kappa - \mu^4 \frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2} \right| \le C\mu^{4+\frac{6}{5}} \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{178}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| \le \frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

KdVSpecEnEst

$$|\mathcal{E}_\kappa| \le C\,\mu^8 \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{179}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, *and* $\mathcal{E}_\kappa = 0$ *otherwise.*

We defer the proof of the above Proposition to Appendix B.

Now, consider the following system of uncoupled KdV equations

KdVsys1
$$\xi_\tau = -\frac{1}{24}\partial^3_{y_1}\xi - \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2), \tag{180}$$

KdVsys2
$$\eta_\tau = \frac{1}{24}\partial^3_{y_1}\eta + \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\eta^2), \tag{181}$$

and consider a solution $(\tau, y) \mapsto (\widetilde{\xi_a}(\tau, y), \widetilde{\eta_a}(\tau, y))$ such that it belongs to $\mathcal{H}^{\rho,n}$, for some $n \geq 1$.
We consider the approximate solutions (Q_a, P_a) of the FPU model (70)

Qappr
$$Q_a(\tau, y) := \frac{\mu^2}{\sqrt{2}}\left[\widetilde{\xi_a}(\mu^2\tau, y_1 - \tau, y_2) + \widetilde{\eta_a}(\mu^2\tau, y_1 + \tau, y_2)\right] \tag{182}$$

Pappr
$$\partial_{y_1}P_a(\tau, y) := \frac{\mu}{\sqrt{2}}\left[\widetilde{\xi_a}(\mu^2\tau, y_1 - \tau, y_2) - \widetilde{\eta_a}(\mu^2\tau, y_1 + \tau, y_2)\right], \tag{183}$$

We need to compare the difference between the approximate solution (182)-(183) and the true solution of (5). Let consider an initial datum (Q_0, P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k}, \hat{P}_{0,k})$ given by (6), where

InDatumHyp1
$$Q_{0,k} \neq 0 \ \text{ only if } \ \kappa(k) = (\mu K_1, \mu^\sigma K_2). \tag{184}$$

We also assume that there exist C, $\rho > 0$ such that

InDatumHyp2
$$\frac{|\hat{Q}_{0,k}|^2 + \omega_k^2|\hat{P}_{0,k}|^2}{(2N_1+1)(2N_2+1)} \leq Ce^{-2\rho|(\kappa_1(k)/\mu, \kappa_2(k)/\mu^\sigma)|}. \tag{185}$$

Moreover, we define an interpolating function for the initial datum (Q_0, P_0) by

$$Q_0(y) = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{K:\left(\mu^2|K_1|^2 + \mu^{2\sigma}|K_2|^2\right)^{1/2} = |\kappa(k)| \leq 1} \hat{Q}_{0,k} e^{i\pi(\mu K_1 y_1 + \mu^\sigma K_2 y_2)},$$

and similarly for $y \mapsto P_0(y)$.

ApprPropKdV

Proposition 6.6. *Consider* (5) *with* $\sigma > 2$ *and* $\gamma \geq 1$ *such that* $\sigma + 2\gamma < 7$. *Let us assume that the initial datum satisfies* (184)-(185), *and denote by* $(Q(t), P(t))$ *the corresponding solution. Consider the approximate solution* $(\widetilde{\xi_a}(t, x), \widetilde{\eta_a}(t, x))$ *with the corresponding initial datum. Assume that* $(\widetilde{\xi_a}, \widetilde{\eta_a}) \in \mathcal{H}^{\rho,n}$ *for some* $\rho > 0$ *and for some* $n \geq 1$ *for all times, and fix* $T_0 > 0$ *and* $0 < \delta \ll 1$.

Then there exists $\mu_0 = \mu_0(T_0, \|(\widetilde{\xi_a}(0), \widetilde{\eta_a}(0))\|_{\mathcal{H}^{\rho,n}})$ *such that, if* $\mu < \mu_0$, *we have that there exists* $C > 0$ *such that*

apprDiscrCont
$$\sup_j |Q_j(t) - Q_a(t, j)| + |P_j(t) - P_a(t, j)| \leq C\mu^\gamma, \ \ |t| \leq \frac{T_0}{\mu^3}, \tag{186}$$

where (Q_a, P_a) *are given by* (182)-(183). *Moreover,*

LowModesAppr
$$\left|\mathcal{E}_\kappa - \mu^4\frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2}\right| \leq C\,\mu^{4+\gamma} \tag{187}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| \leq \frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

HighModesAppr
$$|\mathcal{E}_\kappa| \leq \mu^{4+\gamma} \tag{188}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, *and* $\mathcal{E}_\kappa = 0$ *otherwise.*

We defer the proof to Appendix C.

assKdVrem

Remark 6.7. *The conditions* $\sigma + 2\gamma < 7$, *which, together with* $\gamma > 1$, *implies the upper bound* $\sigma < 5$ *found in the statement of Theorem* (2.4), *is the consequence of a technical condition which allows to estimate the error in the proof of Proposition 6.6 (see Claim 2, together with* (268)-(269)*).*

Proof of Theorem (2.4). First we prove (16).

We consider an initial datum as in (15); when passing to the continuous approximation (70), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0,n}$ for some $\rho_0 > 0$ and $n \geq 1$. By Theorem 5.3 the corresponding sequence of gaps belongs to $\mathcal{H}^{\rho_0,n}$, and that the solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.4). Applying Proposition 6.6, we can deduce the corresponding result for the discrete model (5) and the specific quantities (10).

Next, we prove (18). In order to do so, we exploit the Birkhoff coordinates (x, y) introduced in Theorem 5.4; indeed, by rewriting the normal form system (109) in Birkhoff coordinates we get that every solution is almost-periodic in time. Now, let us introduce the quantities

$$E_K^{(1)} := \left|\hat{\xi}_K\right|^2, \qquad E_K^{(2)} := |\hat{\eta}_K|^2,$$

then $\tau \mapsto E_K^{(1)}(x(\tau), y(\tau))$ and $\tau \mapsto E_K^{(2)}(x(\tau), y(\tau))$ are almost-periodic. If we set $E_K := \frac{1}{2}\left(E_K^{(1)} + E_K^{(2)}\right)$, we can exploit (187) of Proposition 6.6 to translate the results in terms of the specific quantities $\mathcal{E}_\kappa$, and we get the thesis. □

pr1DNLSsubsec

6.3. **The one-dimensional NLS regime.** Let $\beta > 0$ and let I be as in (77), we define the Fourier coefficients of the function $q : I \to \mathbb{R}$ by

ourierqcontKG (189)
$$\hat{q}(j) := \frac{1}{2}\int_I q(y_1, y_2) e^{-i\pi(j_1 y_1 + j_2 y_2)} \mathrm{d}y_1\, \mathrm{d}y_2,$$

and similarly for the Fourier coefficients of the function p.

pec1DNLSLemma **Lemma 6.8.** *Consider the lattice* (19) *in the regime (1D NLS) and with interpolanting function* (118). *Then for a state corresponding to* (q, p) *one has*

NormMode1DNLS (190)
$$\mathcal{E}_\kappa = \frac{\mu^2}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2 : \mu L_1, \mu^\sigma L_2 \in 2\mathbb{Z}} |\hat{p}_{K+L}|^2 + \omega_k^2 |\hat{q}_{K+L}|^2, \quad \forall k : \kappa(k) = (\mu K_1, \mu^\sigma K_2)$$

(where the ω_k *are defined as in* (24)*), and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. We introduce a $(2N_1+1, 2N_2+1)$-periodic interpolating function for Q_j and P_j. We denote $\hat{Q}(j)$ and $\hat{Q}_k$ as in (156) and (157). By the interpolation property we obtain

FourierRel2 (191)
$$\hat{Q}_k = \sum_{h\in\mathbb{Z}^2} \hat{Q}(k_1 + (2N_1+1)h_1, k_2 + (2N_2+1)h_2).$$

The relation between $\hat{Q}(k)$ and $\hat{q}_k$ can be deduced from (118),

$$\begin{aligned}
Q(j) &= \mu q(\mu j_1, \mu^\sigma j_2);\\
\hat{Q}_k &= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu}, \frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma}, \frac{1}{\mu^\sigma}\right]} Q(x_1, x_2) e^{-i\pi(k_1 x_1 \mu + k_2 x_2 \mu^\sigma)} \mathrm{d}x_1\, \mathrm{d}x_2\\
&\overset{(118)}{=} \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu}, \frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma}, \frac{1}{\mu^\sigma}\right]} \mu\, q\left(\mu x_1, \mu^\sigma x_2\right) e^{-i\pi(k_1 x_1 \mu + k_2 x_2 \mu^\sigma)} \mathrm{d}x_1\, \mathrm{d}x_2\\
&= \frac{1}{2}\mu^{(1-\sigma)/2} \int_I q(y) e^{-i\pi(k_1 y_1 + k_2 y_2)} \mathrm{d}y
\end{aligned}$$

FourierRelQq2 (192)
$$= \mu^{(1-\sigma)/2} \hat{q}_k,$$

and similarly

FourierRelPp2 (193)
$$\hat{P}_k = \mu^{(1-\sigma)/2} \hat{p}_k.$$

By using (23), (10) and (191)-(193) we have

$$\mathcal{E}_\kappa \overset{(10)}{=} \mu^{\sigma+1}\frac{1}{2}\sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{P}_{K+L}|^2+\omega_k^2|\hat{Q}_{K+L}|^2$$
$$\overset{(192)}{=} \mu^{\sigma+1}\mu^{1-\sigma}\frac{1}{2}\sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{p}_{K+L}|^2+\omega_k^2|\hat{q}_{K+L}|^2$$

for all k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$, and this leads to (190). □

1DNLSpsiProp **Proposition 6.9.** *Fix $\rho>0$ and $0<\delta\ll 1$. Consider the normal form equation (133), and define the Fourier coefficients of $(\psi,\bar{\psi})$ through the following formula*

FourierPsi1D (194)
$$\psi(y)=\frac{1}{2}\sum_{h\in\mathbb{Z}^2}\hat{\psi}_h e^{ih\cdot y\pi},$$

Consider $(\psi,\bar{\psi})\in\mathcal{H}^{\rho,0}$, and denote by $\mathcal{E}_\kappa$ the specific energy of the normal mode with index κ as defined in (9)-(10). Then for any positive μ sufficiently small

1DNLScoeffpsi (195)
$$\left|\mathcal{E}_\kappa-\mu^2\frac{|\hat{\psi}_K|^2}{2}\right|\le C\mu^{2+\frac{6}{5}}\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}$$

for all k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|\le\frac{(2+\delta)|\log\mu|}{\rho}$. Moreover,

1DNLSSpecEnEst (196)
$$|\mathcal{E}_\kappa|\le C\,\mu^8\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,1}}$$

for all k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}$, and $\mathcal{E}_\kappa=0$ otherwise.

We defer the proof of the above Proposition to Appendix D.

Now, consider the normal form equation, namely the following cubic defocusing one-dimensional NLS

1DimNLSeq1 (197)
$$-i\psi_t=-\,\partial^2_{y_1}\psi+\frac{3\beta}{4}\,|\psi|^2\psi.$$

and consider a solution $(\widetilde{\psi_a},\widetilde{\bar{\psi}_a})$ such that it belongs to $\mathcal{H}^{\rho,n}$, for some $n>0$.

We consider the approximate solutions (Q_a,P_a) of the KG lattice (19) (in the following $\tau=\mu^2t$)

Qappr3 (198)
$$Q_a(\tau,y):=\frac{\mu}{\sqrt{2}}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2)+e^{-i\tau}\widetilde{\bar{\psi}_a}(\tau,y_1,y_2)\right]$$

Pappr3 (199)
$$P_a(\tau,y):=\frac{\mu}{\sqrt{2}i}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2)+e^{-i\tau}\widetilde{\bar{\psi}_a}(\tau,y_1,y_2)\right]$$

(200)

We need to compare the difference between the approximate solution (182)-(183) and the true solution of (19). Let consider an initial datum (Q_0,P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k},\hat{P}_{0,k})$ given by (6), where

InDatumHyp31 (201)
$$Q_{0,k}\neq 0\ \text{ only if }\ \kappa(k)=(\mu K_1,\mu^\sigma K_2).$$

We also assume that there exist C, $\rho>0$ such that

InDatumHyp32 (202)
$$\frac{|\hat{P}_{0,k}|^2+\omega_k^2|\hat{Q}_{0,k}|^2}{N}\le Ce^{-2\rho|(\kappa_1(k)/\mu,\kappa_2(k)/\mu^\sigma)|}.$$

Moreover, we define an interpolating function for the initial datum (Q_0,P_0) by

$$Q_0(y)=\frac{1}{(2N_1+1)(2N_2+1)}\sum_{K:\left(\mu^2|K_1|^2+\mu^{2\sigma}|K_2|^2\right)^{1/2}=|\kappa(k)|\le 1}\hat{Q}_{0,k}e^{i\pi(\mu K_1y_1+\mu^\sigma K_2y_2)},$$

and similarly for $y\mapsto P_0(y)$.

ApprProp1DNLS **Proposition 6.10.** *Consider* (19) *with* $\sigma > 1$ *and* $\gamma > 0$ *such that* $\sigma + 2\gamma < 7$. *Let us assume that the initial datum satisfies* (201)-(202), *and denote by* $(Q(t), P(t))$ *the corresponding solution. Consider the approximate solution* $(\widetilde{\psi_a}(t,x), \widetilde{\bar\psi_a}(t,x))$ *with the corresponding initial datum. Assume that* $(\widetilde{\psi_a}, \widetilde{\bar\psi_a}) \in \mathcal{H}^{\rho,n}$ *for some* $\rho > 0$ *and for some* $n \geq 0$ *for all times, and fix* $T_0 > 0$ *and* $0 < \delta \ll 1$.

Then there exists $\mu_0 = \mu_0(T_0, \|(\widetilde{\psi_a}(0), \widetilde{\bar\psi_a}(0))\|_{\mathcal{H}^{\rho,n}})$ *such that, if* $\mu < \mu_0$, *we have that there exists* $C > 0$ *such that*

pprDiscrCont3

$$\sup_j |Q_j(t) - Q_a(t,j)| + |P_j(t) - P_a(t,j)| \leq C\mu^\gamma, \quad |t| \leq \frac{T_0}{\mu^2}, \tag{203}$$

where (Q_a, P_a) *are given by* (198)-(199). *Moreover,*

wModesApprNLS

$$\left| \mathcal{E}_\kappa - \mu^2 \frac{|\hat\psi_K|^2}{2} \right| \leq C\,\mu^{2+\gamma} \tag{204}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| \leq \frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

hModesApprNLS

$$|\mathcal{E}_\kappa| \leq \mu^{2+\gamma} \tag{205}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, *and* $\mathcal{E}_\kappa = 0$ *otherwise.*

We defer the proof to Appendix E.

assNLSrem **Remark 6.11.** *The conditions* $\sigma + 2\gamma < 7$, *which, together with* $\gamma > 0$, *implies the upper bound* $\sigma < 7$ *found in the statement of Theorem* (2.6), *is the consequence of a technical condition which allows to estimate the error in the proof of Proposition 6.10 (see Claim 2, together with* (315)-(317)*).*

Proof of Theorem (2.6). First we prove (26).

We consider an initial datum as in (25); when passing to the continuous approximation (117), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0,n}$. By Theorem 5.5 the corresponding sequence of gaps belongs to $\mathcal{H}^{\rho_0,n}$, and that the solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.6). Applying Proposition 6.10, we can deduce the corresponding result for the discrete model (22) and the specific quantities (10).

Next, we prove (28). In order to do so, we exploit the Birkhoff coordinates (x, y) introduced in Theorem 5.6; indeed, by rewriting the normal form system (133) in Birkhoff coordinates we get that every solution is almost-periodic in time. Now, let us introduce the quantity

$$E_K := \frac{1}{2}\left|\hat\psi_K\right|^2,$$

then $\tau \mapsto E_K(x(\tau), y(\tau))$ is almost-periodic. Hence we can exploit (204) of Proposition 6.10 to translate the results in terms of the specific quantities $\mathcal{E}_\kappa$, and we get the thesis. ☐

Appendix A. Proof of Lemma 3.6

BNFest

This appendix is devoted to the proof of the Lemma 3.6, which is a key step to normalize the system (56). Its proof is an adaptation of Theorem 4.4 in [?] (bambusi1999nekhoroshev) and it is based on the method of Lie transform, briefly recalled in the following. Throughout this Section, we consider $s \geq s_1$ and $\rho \geq 0$ to be fixed quantities.

Given an auxiliary function χ analytic on $\mathcal{H}^{\rho,s}$, we consider the auxiliary differential equation

auxDE

$$\dot\zeta = X_\chi(\zeta) \tag{206}$$

and denote by Φ^t_χ its flow at time t.

cauchylemma **Lemma A.1.** *Let* χ *and its vector field be analytic in* $\mathcal{B}_{\rho,s}(R)$. *Fix* $\delta < R$, *and assume that*

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \delta.$$

Then, if we consider the time-t flow Φ^t_χ *of* X_χ *we have that for* $|t| \leq 1$

$$\sup_{\mathcal{B}_{\rho,s}(R-\delta)} \|\Phi^t_\chi(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}.$$

Definition A.2. *The map* $\Phi_\chi := \Phi^1_\chi$ *is called the* Lie transform *generated by* χ*.*

Given G analytic on $\mathcal{H}^{\rho,s}$, let us consider the differential equation

orDE

$$\dot{\zeta} = X_G(\zeta), \tag{207}$$

where by X_G we denote the vector field of G. Now define

$$\Phi^*_\chi G(\tilde{\zeta}) := G \circ \Phi_\chi(\tilde{\zeta}).$$

By exploiting the fact that Φ_χ is a canonical transformation, we have that in the new variable $\tilde{\zeta}$ defined by $\zeta = \Phi_\chi(\tilde{\zeta})$ equation (207) is equivalent to

pullbDE

$$\dot{\tilde{\zeta}} = X_{\Phi^*_\chi G}(\tilde{\zeta}). \tag{208}$$

Using the relation

eq:CompositionRelation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_\chi G = \Phi^*_\chi\{\chi, G\}, \tag{209}$$

and the Poisson bracket formalism $\{G_1, G_2\}(\zeta) := \mathrm{d}G_1(\zeta)[X_{G_2}(\zeta)]$ we formally get

lieseries

$$\begin{aligned} \Phi^*_\chi G &= \sum_{\ell=0}^{\infty} G_\ell, \\ G_0 &:= G, \\ G_\ell &:= \frac{1}{\ell}\{\chi, G_{\ell-1}\}, \quad \ell \geq 1. \end{aligned} \tag{210}$$

In order to estimate the vector field of the terms appearing in (210), we exploit the following results

lem:liebrest

Lemma A.3. *Let* $R > 0$*, and assume that* χ*,* G *are analytic on* $\mathcal{B}_{\rho,s}(R)$ *as well as their vector fields. Then, for any* $d \in (0, R)$ *we have that* $\{\chi, G\}$ *is analytic on* $\mathcal{B}_{\rho,s}(R-d)$*, and*

liebrest

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\{\chi,G\}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \frac{2}{d}\left(\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)\left(\sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}\right). \tag{211}$$

Proof. Observe that

$$\begin{aligned} \|X_{\{\chi,G\}}(\zeta)\|_{\mathcal{H}^{\rho,s}} &= \|\mathrm{d}X_\chi(\zeta)\ X_G(\zeta) - \mathrm{d}X_G(\zeta)\ X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \\ &\leq \|\mathrm{d}X_\chi(\zeta)\ X_G(\zeta)\|_{\mathcal{H}^{\rho,s}} + \|\mathrm{d}X_G(\zeta)\ X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}, \end{aligned}$$

and since for any $d \in (0, R)$ Cauchy inequality gives

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|\mathrm{d}X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} \leq \frac{1}{d}\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}},$$

we finally get

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|\mathrm{d}X_\chi(\zeta)\ X_G(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \frac{1}{d}\left(\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)\left(\sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}\right).$$

With a similar estimate for the other term we obtain the thesis. □

lem:Liesrest

Lemma A.4. *Let* $R > 0$*, and assume that* χ*,* G *are analytic on* $\mathcal{B}_{\rho,s}(R)$ *as well as their vector fields. Let* $\ell \geq 1$*, and consider* G_ℓ *as defined in* (210)*; for any* $d \in (0, R)$*,* G_ℓ *is analytic on* $\mathcal{B}_{\rho,s}(R-d)$ *as well as it vector field, and*

lieserest

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{G_\ell}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \left(\frac{2e}{d}\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)^\ell \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}. \tag{212}$$

Proof. Fix ℓ, and denote $\delta := d/\ell$. We look for a sequence $C_m^{(\ell)}$ such that

$$\sup_{\mathcal{B}_{\rho,s}(R-m\delta)} \|X_{G_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \le C_m^{(\ell)}, \;\; \forall m \le \ell.$$

Lemma A.3 ensures that the following sequence satisfies this property.

$$\begin{aligned} C_0^{(\ell)} &:= \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}, \\ C_m^{(\ell)} &= \frac{2}{\delta m} C_{m-1}^{(\ell)} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \\ &= \frac{2\ell}{dm} C_{m-1}^{(\ell)} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}. \end{aligned}$$

One has

$$C_\ell^{(\ell)} = \frac{1}{\ell!}\left(\frac{2\ell}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)^\ell \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}},$$

and by using the inequality $\ell^\ell < \ell! e^\ell$ one obtains the estimate (212). □

Before stating the next Lemma, we point out that the Poisson tensor Ω_2^{-1}, obtained by inversion from the associated symplectic form Ω_2 in (43), is not a bounded operator on $\mathcal{H}^{\rho,s}$. We thus have to weaken the hypothesis of Theorem 4.4 in bambusi1999nekhoroshev [?]; indeed, we just assume that

$$\|\Omega^{-1} f\|_{\mathcal{H}^{\rho,s}} \le \|f\|_{\mathcal{H}^{\rho,s+1}}.$$

This property is satisfied by both Ω_1^{-1} and Ω_2^{-1}.

Transformation **Lemma A.5.** *Let χ and F be analytic on $\mathcal{B}_{\rho,s}(R)$ as well as their vector fields. Fix $d \in (0, R)$, and assume also that*

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \le d/3.$$

Then for $|t| \le 1$

vfest (213)
$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{(\Phi_\chi^t)^*F-F}(\zeta)\|_{\mathcal{H}^{\rho,s}} \le \frac{9}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}.$$

Proof. Since the bound on the norm of X_χ implies that $\Phi_\chi^t(\zeta) \in \mathcal{B}_{\rho,s}(R)$ when $\zeta \in \mathcal{B}_{\rho,s}(R - d/3)$, using Cauchy inequality and Lemma A.1

$$\begin{aligned} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|d\Phi_\chi^{-t}(\Phi_\chi^t(\zeta)) - id\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} &\le \sup_{\mathcal{B}_{\rho,s}(R-2d/3)} \|d\Phi_\chi^{-t}(\zeta) - id\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} \\ &\le \frac{3}{d} \sup_{\mathcal{B}_{\rho,s}(R-d/3)} \|\Phi_\chi^{-t}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \\ &\le \frac{3}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \end{aligned}$$

Since Φ_χ^t is a canonical transformation, a direct computation shows

$$\Omega^{-1} d(F \circ \Phi_\chi^t)(\zeta) = (d\Phi_\chi^{-t}(\Phi_\chi^t(\zeta)) - id)\Omega^{-1} dF(\Phi_\chi^t) + \Omega^{-1} dF(\Phi_\chi^t(\zeta))$$

whence

$$\begin{aligned}
\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{(\Phi^t_\chi)^*F-F}(\zeta)\|_{\mathcal{H}^{\rho,s}} &= \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\Omega^{-1}d(F(\Phi^t_\chi(\zeta)) - F(\zeta))\|_{\mathcal{H}^{\rho,s}}\\
&\le \sup_{\mathcal{B}_{\rho,s}(R-d)} \|(d\Phi^{-t}_\chi(\Phi^t_\chi(\zeta)) - id)\Omega^{-1}dF(\Phi^t_\chi) + \Omega^{-1}d(F(\Phi^t_\chi(\zeta)) - F(\zeta))\|_{\mathcal{H}^{\rho,s}}\\
&\le \sup_{\mathcal{B}_{\rho,s}(R-d)} \|d\Phi^{-t}_\chi(\Phi^t_\chi(\zeta)) - id\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_F(\Phi^t_\chi(\zeta))\|_{\mathcal{H}^{\rho,s}}\\
&+ \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_F(\Phi^t_\chi(\zeta)) - X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}\\
&\le \frac{3}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}\\
&+ \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^t [X_\chi, X_F](\Phi^s_\chi(\zeta))\mathrm{d}s\|_{\mathcal{H}^{\rho,s}}
\end{aligned}$$

To estimate the last term we use Cauchy inequality

$$\begin{aligned}
\sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^t [X_\chi, X_F](\Phi^s_\chi(\zeta))\mathrm{d}s\|_{\mathcal{H}^{\rho,s}} &\le 2 \sup_{\mathcal{B}_{\rho,s}(R-2d/3)} \|[X_\chi, X_F](\zeta)\|_{\mathcal{H}^{\rho,s}}\\
&\le \frac{6}{2d} 2 \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}\\
&\le \frac{6}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}\,.
\end{aligned}$$

Then the thesis follows. □

homeqlemma

Lemma A.6. *Assume that G is analytic on $\mathcal{B}_{\rho,s}(R)$ as well as its vector field, and that h_0 satisfies PER. Then there exists χ analytic on $\mathcal{B}_{\rho,s}(R)$ and Z analytic on $\mathcal{B}_{\rho,s}(R)$ with Z in normal form, namely $\{h_0, Z\} = 0$, such that*

homeq

$$\{\chi, h_0\} \;+\; G \;=\; Z. \tag{214}$$

Such Z and χ are given explicitly by

eq:ExplicitZ

$$Z(\zeta) \;=\; \frac{1}{T}\int_0^T G(\Phi^t_{h_0}(\zeta))\,\mathrm{d}t\,, \tag{215}$$

eq:ExplicitChi

$$\chi(\zeta) = \frac{1}{T}\int_0^T t\,\left[Z(\Phi^t_{h_0}(\zeta)) - G(\Phi^t_{h_0}(\zeta))\right]\mathrm{d}t\,. \tag{216}$$

Furthermore, we have that the vector fields of χ and Z are analytic on $\mathcal{B}_{\rho,s}(R)$, and satisfy

vfhomeq

$$\begin{aligned}
\sup_{\mathcal{B}_{\rho,s}(R)} \|X_Z(\zeta)\|_{\mathcal{H}^{\rho,s}} &\le \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}},\\
\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} &\le 2T \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}.
\end{aligned} \tag{217}$$

Proof. We check directly that the solution of (214) is (216). Indeed,

$$\begin{aligned}
\{\chi, h_0\}(\zeta) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\chi(\Phi^s_{h_0}(\zeta))\\
&= \frac{1}{T}\int_0^T t\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\left[Z(\Phi^{t+s}_{h_0}(\zeta)) - G(\Phi^{t+s}_{h_0}(\zeta))\right]\mathrm{d}t\\
&= \frac{1}{T}\int_0^T t\frac{\mathrm{d}}{\mathrm{d}t}\left[Z(\Phi^t_{h_0}(\zeta)) - G(\Phi^t_{h_0}(\zeta))\right]\mathrm{d}t\\
&= \frac{1}{T}\left[tZ(\Phi^t_{h_0}(\zeta)) - tG(\Phi^t_{h_0}(\zeta))\right]_{t=0}^T - \frac{1}{T}\int_0^T \left[Z(\Phi^t_{h_0}(\zeta)) - G(\Phi^t_{h_0}(\zeta))\right]\mathrm{d}t\\
&= Z(\zeta) - G(\zeta).
\end{aligned}$$

In the last step we used the explicit expression of Z provided in (215). Finally, the first estimate in (217) follows from the explicit expression of Z in (215) while for the second estimate we write explicitly the vector field X_χ:

$$X_\chi(\zeta) = \frac{1}{T}\int_0^T t\, D\Phi_{h_0}^{-t}(\Phi_{h_0}^t(\zeta)) \circ X_{Z-G}(\Phi_{h_0}^t(\zeta))\, \mathrm{d}t\,.$$

Hypothesis (PER) guarantees that $\Phi_{h_0}^t$ as well as its derivatives and the inverses are uniformly bounded as operators from $\mathcal{H}^{\rho,s}$ into itself. Moreover, for any $t \in \mathbb{R}$, the map $\zeta \mapsto \Phi_{h_0}^t(\zeta)$ is a diffeomorphism of $\mathcal{B}_{\rho,s}(R)$ into itself. Thus

$$\begin{aligned}\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} &\le T \sup_{t\in[0,T]} \sup_{\zeta\in\mathcal{H}^{\rho,s}} \left(\|(D\Phi_{h_0}^t(\zeta))^{-1}\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}}\right) \sup_{\mathcal{B}_{\rho,s}} (\|X_Z(\zeta)\|_{\mathcal{H}^{\rho,s}} + \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}) \\ &\le 2T \sup_{t\in[0,T]} \sup_{\zeta\in\mathcal{H}^{\rho,s}} \left(\|(D\Phi_{h_0}^t(\zeta))^{-1}\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}}\right) \sup_{\mathcal{B}_{\rho,s}} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}\end{aligned}$$

where in the last step we used the first inequality in (217). Since by assumption (PER) $\Phi_{h_0}^t$ is an isometry, $\sup_{t\in[0,T]}\sup_{\zeta\in\mathcal{H}^{\rho,s}}\left(\|(D\Phi_{h_0}^t(\zeta))^{-1}\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}}\right) = 1$ and the thesis follows. □

Lemma A.7. *Assume that G and its vector fields are analytic on $\mathcal{B}_{\rho,s}(R)$, and that h_0 satisfies PER. Let χ and its vector field be analytic on $\mathcal{B}_{\rho,s}(R)$, and assume that χ solves (214). For any $\ell \ge 1$ denote by $h_{0,\ell}$ the functions defined recursively as in (210) from h_0. Then for any $d \in (0,R)$ one has that $h_{0,\ell}$ and its vector field are analytic on $\mathcal{B}_{\rho,s}(R-d)$, and*

lieseriesh0 (218)
$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{h_{0,\ell}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \le 2 \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}} \left(\frac{9}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)^\ell .$$

Proof. By using (214) one gets that $h_{0,1} = Z - G$ is analytic on $\mathcal{B}_{\rho,s}(R)$. Then by exploiting (213) one gets the result. □

Poissonlemma **Lemma A.8.** *Assume that G and its vector field are analytic on $\mathcal{B}_{\rho,s}(R)$, and that h_0 satisfies PER. Let χ be the solution of (214), denote by Φ_χ^t the flow of the Hamiltonian vector field associated to χ and by Φ_χ the corresponding time-one map. Moreover, denote by*

$$\mathcal{F}(\zeta) := h_0(\Phi_\chi(\zeta)) - h_0(\zeta) - \{\chi, h_0\}(\zeta).$$

Let $d < R$, and assume that

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \le d/3\,.$$

Then we have that $\mathcal{F}$ and its vector field are analytic on $\mathcal{B}_{\rho,s}(R-d)$, and

vfPois (219)
$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\mathcal{F}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \le \frac{18}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}.$$

Proof. Since

$$\begin{aligned} h_0(\Phi_\chi(\zeta)) - h_0(\zeta) &= \int_0^1 \{\chi, h_0\} \circ \Phi_\chi^t(\zeta)\, \mathrm{d}t \\ &\overset{(214)}{=} \int_0^1 Z(\Phi_\chi^t(\zeta)) - G(\Phi_\chi^t(\zeta))\, \mathrm{d}t, \end{aligned}$$

if we define $F(\zeta) := Z(\zeta) - G(\zeta)$, we get

$$\mathcal{F}(\zeta) = \int_0^1 F(\Phi_\chi^t(\zeta)) - F(\zeta) \mathrm{d}t.$$

Now, we have

$$\begin{aligned}
&\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\mathcal{F}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&= \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\Omega^{-1} d\left(\int_0^1 F(\Phi^t_\chi(\zeta)) - F(\zeta)\mathrm{d}t\right)\|_{\mathcal{H}^{\rho,s}} \\
&\leq \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^1 (d\Phi_\chi^{-t}(\Phi^t_\chi(\zeta)) - id)\Omega^{-1} dF(\Phi^t_\chi) + \Omega^{-1} d(F(\Phi^t_\chi(\zeta)) - F(\zeta))\,\mathrm{d}t\|_{\mathcal{H}^{\rho,s}} \\
&\leq \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^1 (d\Phi_\chi^{-t}(\Phi^t_\chi(\zeta)) - id)\Omega^{-1} dF(\Phi^t_\chi)\,\mathrm{d}t\|_{\mathcal{H}^{\rho,s}} \\
&+ \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^1 X_F(\Phi^t_\chi(\zeta)) - X_F(\zeta)\,\mathrm{d}t\|_{\mathcal{H}^{\rho,s}}
\end{aligned}$$

and by dominated convergence we can bound the last quantity by

$$\begin{aligned}
&\sup_{\mathcal{B}_{\rho,s}(R-d)} \sup_{t\in[0,1]} \|d\Phi_\chi^{-t}(\Phi^t_\chi(\zeta)) - id\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_F(\Phi^t_\chi(\zeta))\|_{\mathcal{H}^{\rho,s}} \\
&+ \sup_{\mathcal{B}_{\rho,s}(R-d)} \sup_{t\in[0,1]} \|X_F(\Phi^t_\chi(\zeta)) - X_F(\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&\leq \sup_{t\in[0,1]} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|d\Phi_\chi^{-t}(\Phi^t_\chi(\zeta)) - id\|_{\mathcal{H}^{\rho,s}\to\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_F(\Phi^t_\chi(\zeta))\|_{\mathcal{H}^{\rho,s}} \\
&+ \sup_{t\in[0,1]} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_F(\Phi^t_\chi(\zeta)) - X_F(\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&\leq \frac{3}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}} + \sup_{t\in[0,1]} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^t [X_\chi, X_F](\Phi^s_\chi(\zeta))\mathrm{d}s\|_{\mathcal{H}^{\rho,s}},
\end{aligned}$$

where we can estimate the last term by Cauchy inequality

$$\begin{aligned}
\sup_{\mathcal{B}_{\rho,s}(R-d)} \|\int_0^t [X_\chi, X_F](\Phi^s_\chi(\zeta))\mathrm{d}s\|_{\mathcal{H}^{\rho,s}} &\leq 2 \sup_{\mathcal{B}_{\rho,s}(R-2d/3)} \|[X_\chi, X_F](\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&\leq \frac{6}{2d} 2 \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&\leq \frac{6}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}.
\end{aligned}$$

By the above computations and (217) we obtain

$$\begin{aligned}
\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\mathcal{F}}(\zeta)\|_{\mathcal{H}^{\rho,s}} &\leq \frac{9}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}} \\
&\overset{(217)}{\leq} \frac{18}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}.
\end{aligned}$$

□

itlemma **Lemma A.9.** *Let $s \geq s_1 \gg 1$, $R > 0$, $m \geq 0$, and consider the Hamiltonian*

Hm (220) $$H^{(m)}(\zeta) = h_0(\zeta) + \delta Z^{(m)}(\zeta) + \delta^{m+1} F^{(m)}(\zeta).$$

Assume that h_0 satisfies PER and INV, and that

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_{F^{(0)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F.$$

Fix $d < \frac{R}{m+1}$, *and set* $R_m := R - md$ $(m \geq 1)$.
Assume also that $Z^{(m)}$ *is analytic on* $\mathcal{B}_{\rho,s}(R_m)$, *and that*

$$\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{Z^{(0)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} = 0,$$

eq:estimateZm

$$\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{Z^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F \sum_{i=0}^{m-1} \delta^i K_0^i, \quad m \geq 1, \tag{221}$$

stepm

$$\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F\, K_0^m, \quad m \geq 1, \tag{222}$$

with $K_0 \geq 15$ *and* $d > 3T\delta F$.
Then, if $\delta K_0 < 1/2$ *there exists a canonical transformation* $\mathcal{T}_\delta^{(m)}$ *analytic on* $\mathcal{B}_{\rho,s}(R_{m+1})$ *such that*

CTm

$$\sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|\mathcal{T}_\delta^{(m)}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq 2T\delta^{m+1} K_0^m F, \tag{223}$$

$H^{(m+1)} := H^{(m)} \circ \mathcal{T}^{(m)}$ *has the form* (220) *and satisfies* (222) *with* m *replaced by* $m+1$.

Proof. The key point of the proof is to look for $\mathcal{T}_\delta^{(m)}$ as the time-one map of the Hamiltonian vector field of an analytic function $\delta^{m+1}\chi_m$. Hence, consider the differential equation

chim

$$\dot{\zeta} = X_{\delta^{m+1}\chi_m}(\zeta). \tag{224}$$

By standard theory we have that, if $\|X_{\delta^{m+1}\chi_m}\|_{\mathcal{B}_{\rho,s}(R_m)}$ is small enough (e.g. $\|X_{\delta^{m+1}\chi_m}\|_{\mathcal{B}_{\rho,s}(R_m)} \leq \frac{md}{m+1}$) and $\zeta_0 \in \mathcal{B}_{\rho,s}(R_{m+1})$, then the solution of (224) exists for $|t| \leq 1$.

Therefore we can define $\mathcal{T}_{m,\delta}^t : \mathcal{B}_{\rho,s}(R_{m+1}) \to \mathcal{B}_{\rho,s}(R_m)$, and in particular the corresponding time-one map $\mathcal{T}_\delta^{(m)} := \mathcal{T}_{m,\delta}^1$, which is an analytic canonical transformation, δ^{m+1}-close to the identity. We have

$$(\mathcal{T}_\delta^{(m)})^* \left(h_0 + \delta Z^{(m)} + \delta^{m+1} F^{(m)}\right) = h_0 + \delta Z^{(m)}$$

$$+ \delta^{m+1} \left[\{\chi_m, h_0\} + F^{(m)}\right] +$$

nonnorm1

$$+ \left(h_0 \circ \mathcal{T}_\delta^{(m)} - h_0 - \delta^{m+1}\{\chi_m, h_0\}\right) + \delta\left(Z^{(m)} \circ \mathcal{T}_\delta^{(m)} - Z^{(m)}\right) \tag{225}$$

nonnorm2

$$+ \delta^{m+1}\left(F^{(m)} \circ \mathcal{T}_\delta^{(m)} - F^{(m)}\right). \tag{226}$$

It is easy to see that the first two terms are already normalized, that the term in the second line is the non-normalized part of order $m+1$ that can be normalized through the choice of a suitable χ_m, and that (225)-(226) contain all the terms of order higher than $m+1$.

In order to normalize the terms in the second line we solve the homological equation

$$\{\chi_m, h_0\} + F^{(m)} \;=\; Z_{m+1},$$

with Z_{m+1} in normal form. Lemma A.6 ensures the existence of χ_m and Z_{m+1} as well as their explicit expressions:

$$Z_{m+1}(\zeta) = \frac{1}{T}\int_0^T F^{(m)}(\Phi_{h_0}^t(\zeta))\,\mathrm{d}t\,,$$

$$\chi_m(\zeta) = \frac{1}{T}\int_0^T t[F^{(m)}(\Phi_{h_0}^t(\zeta)) - Z_{m+1}(\Phi_{h_0}^t(\zeta))]\,\mathrm{d}t\,.$$

The explicit expression of X_{χ_m} can be computed following the argument of Lemma A.6. Using this explicit expression, the analyticity of the flow $\Phi_{h_0}^t$ ensured by (PER) and (217) one has

ateVectorField

$$\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,\sigma}} \leq 2T \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}\|_{\mathcal{H}^{\rho,\sigma}} \leq 2TK_0^m F\,. \tag{*}$$

Straightforwardly, from the explicit expression of $Z_{m+1}(\zeta)$ and (222) one has

$$\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{Z_{m+1}}\|_{\mathcal{H}^{\rho,s}} \leq K_0^m F$$

Now define $Z^{(m+1)} := Z^{(m)} + \delta^m Z_{m+1}$ and notice that as a consequence of the latter estimate and (221) we have

$$\sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|X_{Z^{(m+1)}}(\zeta)\| \leq \sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|X_{Z^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} + \sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|X_{\delta^m Z_{m+1}}(\zeta)\|_{\mathcal{H}^{\rho,s}}$$
$$\leq F\left(\sum_{j=0}^{m-1} \delta^j K_0^j + \delta^m K_0^m\right)$$

Defining now $\mathcal{T}_\delta^{(m)}(\zeta) := \Phi^1_{\delta^{m+1}\chi_m}(\zeta)$ we can apply Lemma A.1 and (*) to obtain

$$\sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|\mathcal{T}_\delta^{(m)}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} = \sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|\Phi^1_{\delta^{m+1}\chi_m}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}}$$
$$\leq \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\delta^{m+1}\chi_m}\|_{\mathcal{H}^{\rho,s}} \leq 2T\delta^{m+1}K_0^m F\,.$$

Let us set now $\delta^{m+2}F^{(m+1)} := (225) + (226)$. Using Lemma A.5 one can estimate separately the three pieces. We notice that $\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\delta^{m+1}\chi_m}\|_{\mathcal{H}^{\rho,s}} \leq 2T\delta^{m+1}K_0^m F$ and since $\delta K_0 < \frac{1}{2}$ we have $\sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\delta^{m+1}\chi_m}\|_{\mathcal{H}^{\rho,s}} < T\delta F < \frac{d}{3} \leq \frac{(m+1)d}{3}$. We can thus apply Lemma A.5 and Lemma A.8 to get

$$\sup_{\mathcal{B}(R_{m+1})} \|X_{Z^{(m)}\circ\mathcal{T}_\delta^{(m)} - Z^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \frac{27\,\delta^{m+1}}{(m+1)d} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{Z^{(m)}}\|_{\mathcal{H}^{\rho,s}},$$
$$\sup_{\mathcal{B}(R_{m+1})} \|X_{F^{(m)}\circ\mathcal{T}_\delta^{(m)} - F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \frac{27\,\delta^{m+1}}{(m+1)d} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}\|_{\mathcal{H}^{\rho,s}},$$
$$\sup_{\mathcal{B}(R_{m+1})} \|X_{h_0\circ\mathcal{T}_\delta^{(m)} - h_0 - \delta^{m+1}\{\chi_m, h_0\}}\|_{\mathcal{H}^{\rho,s}} \leq \frac{18\,\delta^{2m+2}}{(m+1)d} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}}.$$

By means of these inequalities, with the additional information $\|X_{\delta^{m+1}\chi_m}\|_{\mathcal{H}^{\rho,s}} \leq \frac{(m+1)d}{3}$ and the hypotheses (221) and (222), we can estimate

$$\sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|X_{\delta^{m+2}F^{(m+1)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 9\delta^{m+2} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{Z^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} + 9\,\delta^{2m+2} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}}$$
$$+ 6\,\delta^{2m+2} \sup_{\mathcal{B}_{\rho,s}(R_m)} \|X_{F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}}$$
$$\leq 9\,\delta^{m+2} F \sum_{i=0}^{m-1} \delta^i K_0^i + 9\,\delta^{2m+2} F\,K_0^m + 6\,\delta^{2m+2} F\,K_0^m$$
$$= \delta^{m+2}\left(9F \sum_{i=0}^{m-1} \delta^i K_0^i + 9\delta^m F\,K_0^m + 6\,\delta^m F\,K_0^m\right)$$

If $m = 0$ the first term is not present and then

$$\sup_{\mathcal{B}_{\rho,s}(R_1)} \|X_{\delta^2 F^{(1)}}\|_{\mathcal{H}^{\rho,s}} \leq \delta^2(9F + 6\,F).$$

If $m \geq 1$ we exploit the smallness condition $\delta K_0 < \frac{1}{2}$ to get $\sum_{i=0}^{m-1} \delta^i K_0^i < 2$ and

$$\sup_{\mathcal{B}_{\rho,s}(R_{m+1})} \|X_{\delta^{m+2}F^{(m+1)}}\|_{\mathcal{H}^{\rho,s}} \leq \delta^{m+2}\left(6F + 9\frac{F}{2^m} + 6\,\frac{F}{2^m}\right) \leq 15\,\delta^{m+2}F.$$

□

Proof of Lemma 3.6. The Hamiltonian (56) satisfies the assumptions of Lemma A.9 with $m = 0$, $F_{1,M}$ in place of $F^{(0)}$, $F = K_{1,s}^{(F)}\,M^{2+\gamma}$. So we apply Lemma A.9 with $d = R/4$, provided that

$$\delta < \frac{R}{12\,T\,F} = \frac{R}{12\,T\,K_{1,s}^{(F)} M^{2+\gamma}}$$

which is true due to (65). Hence there exists an analytic canonical transformation $\mathcal{T}^{(0)}_{\delta,M} : \mathcal{B}_{\rho,s}(3R/4) \to \mathcal{B}_{\rho,s}(R)$ with

$$\sup_{\mathcal{B}_{\rho,s}(3R/4)} \|\mathcal{T}^{(0)}_{\delta,M}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq 2T\,F\,\delta,$$

such that

step1

$$H_{1,M} \circ \mathcal{T}^{(0)}_{\delta,M} = h_0 + \delta Z^{(1)}_M + \delta^2 \mathcal{R}^{(1)}_M, \tag{227}$$

$$Z^{(1)}_M := \langle F_{1,M} \rangle, \tag{228}$$

$$\delta^2 \mathcal{R}^{(1)}_M := \delta^2 F^{(1)}$$

$$= \left(h_0 \circ \mathcal{T}^{(0)}_{\delta,M} - h_0 - \delta\{\chi_1, h_0\}\right) + \delta\left(Z^{(1)}_M \circ \mathcal{T}^{(0)}_{\delta,M} - Z^{(1)}_M\right) + \delta^2\left(F_{1,M} \circ \mathcal{T}^{(0)}_{\delta,M} - F_{1,M}\right), \tag{229}$$

$$\sup_{\mathcal{B}_{\rho,s}(3R/4)} \|X_{Z^{(1)}_M}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F, \tag{230}$$

$$\sup_{\mathcal{B}_{\rho,s}(3R/4)} \|X_{\mathcal{R}^{(1)}_N}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 15F. \tag{231}$$

and $K_0 = 15$, whence $\delta < \frac{1}{30}$. □

ApprEstSec11

Appendix B. Proof of Propositions 6.5 and 6.2

Proof of Proposition 6.5. In order to prove Proposition 6.5 we first discuss the specific energies associated to the high modes, and then the ones associated to the low modes.

First we remark that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ we have

EstFreqKdVr

$$\begin{aligned} \left|\frac{\omega_k^2}{\mu^2}\right| &\overset{(8)}{=} \frac{4}{\mu^2}\left[\sin^2\left(\frac{k_1\pi}{2N+1}\right) + \sin^2\left(\frac{k_2\pi}{2N+1}\right)\right] \\ &= \frac{4}{\mu^2}\left[\sin^2\left(\frac{\mu K_1\pi}{2}\right) + \sin^2\left(\frac{\mu^\sigma K_2\pi}{2}\right)\right] \\ &\leq \pi^2(K_1^2 + \mu^{2(\sigma-1)}K_2^2); \end{aligned} \tag{232}$$

moreover, for $K_1 \neq 0$

rmModeKdVEst1

$$\begin{aligned} \frac{|\hat{q}_K|^2 + \pi^2(K_1^2 + \mu^{2(\sigma-1)}K_2^2)|\hat{p}_K|^2}{2} &\leq \pi^2\, e^{-2\rho|K|}\, \frac{|\hat{q}_K|^2 + (K_1^2 + \mu^{2(\sigma-1)}K_2^2)|\hat{p}_K|^2}{2} e^{2\rho|K|} \\ &\leq \pi^2\, e^{-2\rho|K|}\, \left(1 + \mu^{2(\sigma-1)}\frac{K_2^2}{K_1^2}\right)\, \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}, \end{aligned} \tag{233}$$

while for $|K_2| \leq |K_1|$

rmModeKdVEst2

$$\frac{|\hat{q}_K|^2 + \pi^2(K_1^2 + \mu^{2(\sigma-1)}K_2^2)|\hat{p}_K|^2}{2} \overset{|K_2|\leq|K_1|}{\leq} 2\pi^2\, e^{-2\rho|K|}\, \|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}. \tag{234}$$

Hence, by (173) we obtain that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$

$$
\begin{aligned}
&\frac{\mathcal{E}_\kappa}{\mu^4} \\
&= \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|\le|K_1+L_1|}} \left(|\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2\right) + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|}} \left(|\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2\right) \\
&\overset{(232),(234),(86)}{\le} \pi^2\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\; 2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|\le|K_1+L_1|}} e^{-2\rho|K+L|} \\
&+ \pi^2\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0}} e^{-2\rho|K+L|}\left(1+\mu^{2(\sigma-1)}\frac{(K_2+L_2)^2}{(K_1+L_1)^2}\right) \\
&+ \pi^2\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1=0}} e^{-2\rho|K_2+L_2|}.
\end{aligned}
$$

Now,

HighFreqTerm1

$$
\begin{aligned}
(235)\quad &\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|} \\
&\le e^{-2\rho|K|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|} \\
(236)\quad &+ \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}.
\end{aligned}
$$

DecompHighFreqTerm1

We now estimate the last sum in (236); we point out that for $L_1, L_2 \neq 0$ we have

$$|L| \ge \frac{2}{\mu} + \frac{2}{\mu^\sigma},$$

hence

BoundKL

$$2|K| \le |L|. \tag{237}$$

Therefore, for any k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| \geq \frac{(2+\delta)|\log\mu|}{\rho}$

1HighFreqTerm1

$$\begin{aligned}
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|} &\leq \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho\,||K|-|L||}\\
&\leq \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{2\rho|K|}e^{-2\rho|L|}\\
&\leq e^{2\rho|K|}\,2\pi\int_{2|K|}^{+\infty} Re^{-2\rho R}\mathrm{d}R\\
&= 2\pi\, e^{2\rho|K|}\left(-\frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_{2|K|}^{+\infty} e^{-2\rho R}\mathrm{d}R\right]\\
&= -\pi\, e^{2\rho|K|}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{e^{-4\rho|K|}}{2\rho}\right)\\
&= -\pi e^{2\rho|K|}\left(-\frac{1}{2\rho^2}e^{-4\rho|K|} - 2|K|\,e^{-4\rho|K|}\right)\\
&= \frac{\pi}{2\rho}\left(\frac{1}{\rho}+4\right)e^{-2\rho|K|} \qquad (238)
\end{aligned}$$

Next we estimate the second sum in (236); we have

2HighFreqTerm1

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|} \leq e^{-2\rho\,(|K_1|+|K_2|)}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu}, \tag{239}$$

which is exponentially small with respect to μ. Similarly,

3HighFreqTerm1

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|} \leq e^{-2\rho\,(|K_1|+|K_2|)}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}. \tag{240}$$

Then,

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}$$
$$\le e^{-2\rho|K|}\left(\frac{K_2}{K_1}\right)^2$$
$$+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}$$
$$+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}. \tag{241}$$

DecompHighFreqTerm2

First we estimate the last term in (241): we have that $|L+K|\ge|K|$, hence

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}$$
$$=\int_{|K|}^{+\infty}\int_0^{\pi/4} e^{-2\rho\,\xi}\,\xi\tan^2\phi\,\mathrm{d}\phi\,\mathrm{d}\xi$$
$$=\left(1-\frac{\pi}{4}\right)e^{-2\rho|K|}\,\frac{1+2\rho|K|}{4\rho^2}$$
$$\le\left(1-\frac{\pi}{4}\right)\mu^4\,e^{-2\rho\left[|K|-\frac{2|\log\mu|}{\rho}-\frac{1}{2\rho}\log(2\rho|K|)\right]}$$
$$\overset{\delta<1-1/e}{\le}\left(1-\frac{\pi}{4}\right)\mu^4\,e^{-2\rho\left[\delta|K|-\frac{2|\log\mu|}{\rho}\right]} \tag{242}$$
$$=\left(1-\frac{\pi}{4}\right)\mu^8 e^{-2\rho\delta|K|} \tag{243}$$

Est1HighFreqTerm2

Now we bound the other two nontrivial terms in (241); on the one hand, we notice that

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}L_2^2 \tag{244}$$

DecompHighFreqTerm22

vanishes, while on the other hand

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_2^2 \leq e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}$$

$$\leq 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}\,\mathrm{d}\ell, \tag{245}$$

where the last integral is exponentially small with respect to μ.

On the other hand, for any k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|\leq\frac{(2+\delta)|\log\mu|}{\rho}$

$$\begin{aligned}
&\left|\frac{\mathcal{E}_\kappa}{\mu^4}-\frac{|\hat{\xi}_K|^2+|\hat{\eta}_K|^2}{2}\right|\\
&\leq\left|\frac{\omega_k^2-\mu^2\pi^2K_1^2}{2\mu^2}\right||\hat{p}_K|^2+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat{q}_{K+L}|^2+\omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2,\\
&\overset{(232)}{\leq}(\mu^2\pi^4K_1^4+\pi^2\mu^{2(\sigma-1)}K_2^2)|\hat{p}_K|^2\\
&\quad+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat{q}_{K+L}|^2+\pi^2[(K_1+L_1)^2+\mu^{2(\sigma-1)}(K_2+L_2)^2]|\hat{p}_{K+L}|^2,\\
&\leq\left(\pi^4\mu^2K_1^4+\pi^2\mu^{2(\sigma-1)}\,\frac{9|\log\mu|^2}{\rho^2}\right)|\hat{p}_K|^2\\
&\quad+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat{q}_{K+L}|^2+\pi^2[(K_1+L_1)^2+\mu^{2(\sigma-1)}(K_2+L_2)^2]|\hat{p}_{K+L}|^2,
\end{aligned}$$

$$\leq\left(\pi^4\mu^2+\pi^2\mu^{2(\sigma-1)}\right)\frac{9|\log\mu|^2}{\rho^2}\,2\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{246}$$

$$+\frac{\pi^2}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{2\rho|K+L|}\,(|\hat{\xi}_{K+L}|^2+|\hat{\eta}_{K+L}|^2)\left(1+2\mu^{2(\sigma-1)}\,\frac{K_2^2+L_2^2}{(K_1+L_1)^2}\right)e^{-2\rho|K+L|}, \tag{247}$$

and we can conclude by estimating (246) by exploiting the fact that $|\log\mu|\leq\mu^{-2/5}$, while we can estimate (247) by

$$\begin{aligned}
&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}\left(1+2\mu^{2(\sigma-1)}\,\frac{K_2^2+L_2^2}{(K_1+L_1)^2}\right)e^{-2\rho|K+L|}\\
&\leq\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}\left(1+2\,\mu^{2(\sigma-1)}\,K_2^2+2\,\mu^{2(\sigma-1)}\,L_2^2\right)e^{-2\rho|K+L|}\\
&\leq\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\left[(1+2\,\mu^{2(\sigma-1)}\,K_2^2)2\pi\int_{2/\mu}^{+\infty}e^{-2\rho\ell}\,\ell\mathrm{d}\ell+4\pi\int_{2/\mu}^{+\infty}e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\
&=\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\times
\end{aligned}$$

$$\left[2\pi\left(1+2\,\mu^{2(\sigma-1)}\,\frac{9|\log\mu|^2}{\rho^2}\right)e^{-4\rho/\mu}\,\frac{\mu+4\rho}{4\mu\rho^2}+4\pi\,e^{-4\rho/\mu}\,\frac{3\mu^3+12\rho\mu^2+24\rho^2\mu+32\rho^3}{8\mu^3\rho^4}\right]. \tag{248}$$

□

Proof of Proposition 6.2. Proposition 6.2 is obtained as a Corollary of Proposition 6.5 by setting $\sigma = 2$. □

ApprEstSec12

Appendix C. Proof of Proposition 6.6 and Proposition 6.3

Proof of Proposition 6.6. The argument follows along the lines of Appendix C in [?] (bambusi2006metastability).

Exploiting the canonical transformation found in Theorem 3.3, we also define

$$\zeta_a := (\xi_a, \eta_a) = \mathcal{T}_{\mu^2}(\tilde{\xi_a}, \tilde{\eta_a}) = \tilde{\zeta}_a + \psi_a(\tilde{\zeta}_a), \tag{249}$$

where $\psi_a(\tilde{\zeta}_a) := (\psi_\xi(\tilde{\zeta}_a), \psi_\eta(\tilde{\zeta}_a))$; by (54) we have

estRemThm
$$\sup_{\zeta \in \mathcal{B}_{\rho,n}(R)} \|\psi_a(\zeta)\|_{\mathcal{H}^{\rho,m}} \le C_n' \mu^2 R. \tag{250}$$

For convenience we define

qappr
$$q_a(\tau, y) := \frac{1}{\sqrt{2}} \left[\xi_a(\mu^2\tau, y_1 - \tau, y_2) + \eta_a(\mu^2\tau, y_1 + \tau, y_2) \right] \tag{251}$$

pappr
$$\partial_{y_1} p_a(\tau, y) := \frac{1}{\sqrt{2}} \left[\xi_a(\mu^2\tau, y_1 - \tau, y_2) - \eta_a(\mu^2\tau, y_1 + \tau, y_2) \right], \tag{252}$$

We observe that the pair (q_a, p_a) satisfies

ApprEq1
$$\mu^2 (q_a)_t = -\Delta_1 \, \mu p_a + \mu^6 \mathcal{R}_q \tag{253}$$

ApprEq2
$$\mu (p_a)_t = -\mu^2 q_a - \mu^4 \, \alpha \, \pi_0 q_a^2 + \mu^5 \mathcal{R}_p, \tag{254}$$

where the operator Δ_1 acts on the variable x, π_0 is the projector on the space of the functions with zero average, and the remainders are functions of the rescaled variables τ and y which satisfy

$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_q\|_{\ell^2_{\rho,0}} \le C,$$
$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_p\|_{\ell^2_{\rho,1}} \le C.$$

We now restrict the space variables to integer values; keeping in mind that q_a and p_a are periodic, we assume that $j \in \mathbb{Z}^2_{N,N^\sigma}$.

For a finite sequence $Q = (Q_j)_{j \in \mathbb{Z}^2_{N,N^\sigma}}$ we define the norm

NormSeq
$$\|Q\|^2_{\ell^2_{N,N^\sigma}} := \sum_{j \in \mathbb{Z}^2_{N,N^\sigma}} |Q_j|^2. \tag{255}$$

Now we consider the discrete model (5): we rewrite in the following form,

DiscrEq1
$$\dot{Q}_j = -(\Delta_1 P)_j \tag{256}$$

DiscrEq2
$$\dot{P}_j = -Q_j - \alpha \, \pi_0 Q_j^2 \tag{257}$$

and we want to show that there exist two sequences $E = (E_j)_{j \in \mathbb{Z}^2_{N,N^\sigma}}$ and $F = (F_j)_{j \in \mathbb{Z}^2_{N,N^\sigma}}$ such that

$$Q = \mu^2 \, q_a + \mu^{2+\gamma} E, \;\; P = \mu p_a + \mu^{2+\gamma} F$$

fulfills (256)-(257), where $\gamma > 0$ is a parameter we will fix later in the proof. Therefore, we have that

EqSeq1
$$\dot{E} = -\Delta_1 \, F - \mu^{6-2-\gamma} \mathcal{R}_q \tag{258}$$

EqSeq2
$$\dot{F} = -E - \alpha \pi_0 \left(\mu^2 \, 2 q_a E + \mu^{2+\gamma} E^2 \right) - \mu^{5-2-\gamma} \mathcal{R}_p, \tag{259}$$

where we impose initial conditions on (E, F) such that $(\tilde{q}, \tilde{p})$ has initial conditions corresponding to the ones of the true initial datum,

$$\mu^2 q_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{2+\gamma} E_{0,j} = Q_{0,j},$$
$$\mu p_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{2+\gamma} F_{0,j} = P_{0,j}.$$

We now define the operator ∂_i, $i = 1, 2$, by $(\partial_i f)_j := f_j - f_{j-e_i}$ for each $f \in \ell^2_{N,N^\sigma}$.

- Claim 1: Let $\sigma > 2$ and $\gamma > 0$, we have

$$\begin{aligned}
\|E_0\|_{\ell^2_{N,N^\sigma}} &\leq C' \mu^{(3-2\gamma-\sigma)/2},\\
\|\partial_1 F_0\|_{\ell^2_{N,N^\sigma}} &\leq C' \mu^{(3-2\gamma-\sigma)/2},\\
\|\partial_2 F_0\|_{\ell^2_{N,N^\sigma}} &\leq C' \mu^{(1-2\gamma+\sigma)/2}.
\end{aligned}$$

To prove Claim 1 we observe that

$$\begin{aligned}
E_0 &= \mu^2 \frac{\xi_a + \eta_a - (\widetilde{\xi}_a + \widetilde{\eta}_a)}{\sqrt{2}\mu^{2+\gamma}} \;=\; \mu^{-\gamma}\, \frac{\psi_\xi + \psi_\eta}{\sqrt{2}},\\
F_0 &= \mu \frac{\partial_{y_1}^{-1}[\xi_a - \eta_a - (\widetilde{\xi}_a - \widetilde{\eta}_a)]}{\sqrt{2}\mu^{2+\gamma}} \;=\; \mu^{-1-\gamma} \frac{\partial_{y_1}^{-1}(\psi_\xi - \psi_\eta)}{\sqrt{2}},
\end{aligned}$$

from which we can deduce

$$\begin{aligned}
\|E_0\|^2_{\ell^2_{N,N^\sigma}} &\leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |E_{0,j}|^2 \;\leq\; C\, 4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \;=\; C\,\mu^{3-2\gamma-\sigma},\\
\|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} &\leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 F_{0,j}|^2 \;\leq\; C\, 4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \;\leq\; C\,\mu^{3-2\gamma-\sigma}\\
\|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} &\leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 F_{0,j}|^2 \;\leq\; C\, 4N^{\sigma+1}\,(\mu^{1+\sigma-\gamma})^2 \;=\; C\,\mu^{1-2\gamma+\sigma}
\end{aligned}$$

and this leads to the thesis.

- Claim 2: Fix $n \geq 1$, $T_0 > 0$ and $K_* > 0$, then for any $\mu < \mu_s$ and for any $\sigma > 2$ and $\gamma \geq 1$ such that $\sigma + 2\gamma < 7$ we have

$$\|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F\|^2_{\ell^2_{N,N^\sigma}} \leq K_*, \quad |t| < \frac{T_0}{\mu^3}. \tag{260}$$

To prove the claim, we define

auxFuncClaim2

$$\mathcal{F}(E,F) := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} \frac{E_j^2 + F_j(-\Delta_1 F)_j}{2} + \frac{2\mu^2\alpha q_{a,j} E_j^2}{2}, \tag{261}$$

and we remark that, using the boundedness of $q_{a,j}$,

$$\frac{1}{2}\mathcal{F}(E,F) \;\leq\; \|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \;\leq\; 4\mathcal{F}(E,F).$$

Now we compute the time derivative of $\mathcal{F}$. Exploiting (258)-(259)

imeDerAuxFunc1, imeDerAuxFunc2, imeDerAuxFunc3, imeDerAuxFunc4, neDerAuxFunc21, neDerAuxFunc22

$$\begin{aligned}
\dot{\mathcal{F}} = \sum_j E_j \left[-(\Delta_1 F)_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] && (262)\\
+ \sum_j (-\Delta_1 F)_j \left[-E_j - \alpha(\mu^2 2 q_{a,j} E_j + \mu^{2+\gamma} E_j^2) - \mu^{3-\gamma}(\mathcal{R}_p)_j\right] && (263)\\
+ \sum_j 2\mu^2\,\alpha\, q_{a,j} E_j \left[-(\Delta_1 F)_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] && (264)\\
+ \sum_j \mu^2 \alpha E_j^2\, \mu\, \frac{\partial q_{a,j}}{\partial \tau} && (265)\\
= \sum_j -E_j\, \mu^{4-\gamma}(\mathcal{R}_q)_j + \sum_j (-\Delta_1 F)_j \left[-\alpha\,\mu^{2+\gamma} E_j^2 - \mu^{3-\gamma}(\mathcal{R}_p)_j\right] && (266)\\
- \sum_j 2\mu^2\,\alpha\, q_{a,j} E_j\, \mu^{4-\gamma}(\mathcal{R}_q)_j + \sum_j \mu^2 \alpha E_j^2\, \mu\, \frac{\partial q_{a,j}}{\partial \tau} && (267)
\end{aligned}$$

In order to estimate (266)-(267), we notice that

$$\sup_j |(\Delta_1 F)_j| \le 2\sup_j |(\partial_1 F)_j| + |(\partial_2 F)_j| \le 4\sqrt{\mathcal{F}},$$
$$\|\mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \le \sum_j |(\mathcal{R}_q)_j|^2 \le 4N^{\sigma+1}\sup_y |\mathcal{R}_q(y)|^2 \le C\mu^{-1-\sigma},$$

and that $|(\partial_i\mathcal{R}_p)_j| \le \mu\sup_y \left|\frac{\partial\mathcal{R}_p}{\partial y}(y)\right|$, which implies

$$\|\partial_i\mathcal{R}_p\|^2_{\ell^2_{N,N^\sigma}} \le C\mu^{1-\sigma}.$$

Now, the first sum in (266) is estimated by $C\mathcal{F}^{1/2}\mu^{(7-2\gamma-\sigma)/2}$; the second sum in (266) can be bounded by

$$C(\mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{(7-2\gamma-\sigma)/2}\mathcal{F}^{1/2}).$$

Recalling that $q_{a,j}$ is bounded, the first sum in (267) can be bounded by $C\mathcal{F}^{1/2}\mu^{(11-2\gamma-\sigma)/2}$, while the second one is estimated by $C\mu^3\mathcal{F}$. Hence, as long as $\mathcal{F} < 2K_*$ we have

$$\left|\dot{\mathcal{F}}\right| \le C\left|\mathcal{F}^{1/2}\mu^{(7-2\gamma-\sigma)/2} + \mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{(7-2\gamma-\sigma)/2}\mathcal{F}^{1/2} + \mathcal{F}^{1/2}\mu^{(11-2\gamma-\sigma)/2} + \mu^3\mathcal{F}\right|$$

EstTimeDer1 (268)
$$\le C(\mu^{2+\gamma}\sqrt{2}\,K_*^{1/2} + \mu^3)\mathcal{F} + C(2\mu^{(7-2\gamma-\sigma)/2} + \mu^{(11-2\gamma-\sigma)/2})\sqrt{2}\,K_*^{1/2},$$

EstTimeDer2 (269)
$$\overset{\gamma\ge 1}{\le} C\,\mu^3\,2\sqrt{2}\,K_*^{1/2}\mathcal{F} + C\,3\mu^{(7-2\gamma-\sigma)/2}\sqrt{2}\,K_*^{1/2},$$

and by applying Gronwall's lemma we get

Gronwall (270)
$$\mathcal{F}(t) \le \mathcal{F}(0)e^{C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t} + e^{C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t}\,C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t\,C\,3\mu^{(7-2\gamma-\sigma)/2}\sqrt{2}\,K_*^{1/2},$$

from which we can deduce the thesis. □

Proof of Proposition 6.3. Proposition 6.3 can be obtained from Proposition 6.6 by setting $\sigma = 2$. □

ApprEstSec21

Appendix D. Proof of Proposition 6.9

We argue as in the proof of Proposition (6.5).

First we remark that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ we have

$$|\omega_k^2| \overset{(24)}{=} 1 + 4\left[\sin^2\left(\frac{k_1\pi}{2N+1}\right) + \sin^2\left(\frac{k_2\pi}{2N+1}\right)\right]$$
$$= 1 + 4\left[\sin^2\left(\frac{\mu K_1\pi}{2}\right) + \sin^2\left(\frac{\mu^\sigma K_2\pi}{2}\right)\right]$$
$$\le 1 + \pi^2(\mu^2 K_1^2 + \mu^{2\sigma}K_2^2),$$

EstFreq1DNLSr (271)
$$\le \pi^2(1 + \mu^2 K_1^2 + \mu^{2\sigma}K_2^2),$$

hence

$$\frac{|\hat{p}_K|^2 + \pi^2(1+\mu^2K_1^2+\mu^{2\sigma}K_2^2)|\hat{q}_K|^2}{2} \le \pi^2\,e^{-2\rho|K|}\,\frac{|\hat{p}_K|^2 + (1+\mu^2K_1^2+\mu^{2\sigma}K_2^2)|\hat{q}_K|^2}{2}e^{2\rho|K|}$$

NormMode1DNLSEst (272)
$$\le \pi^2\,e^{-2\rho|K|}\,\left(1+\mu^2 K_1^2+\mu^{2\sigma}K_2^2\right)\,\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}.$$

Hence, by (190) we obtain that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$

ighModes1DNLS

$$
\begin{aligned}
\frac{\mathcal{E}_\kappa}{\mu^2} &\\
\leq \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} & \left(|\hat{p}_{K+L}|^2 + \omega_k^2\, |\hat{q}_{K+L}|^2\right)\\
\stackrel{(271),(272)}{\leq} \pi^2\, \|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\; 2 & \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}\left[1+\mu^2\,(K_1+L_1)^2+\mu^{2\sigma}\,(K_2+L_2)^2\right], \qquad (273)
\end{aligned}
$$

where the sum in (273) can be rewritten as follows,

pecHighModes1 / pecHighModes2 / pecHighModes3

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|} \qquad (274)
$$

$$
+\,\mu^2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}(K_1+L_1)^2 \qquad (275)
$$

$$
+\,\mu^{2\sigma} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}(K_2+L_2)^2. \qquad (276)
$$

Now,

ighModesTerm1

$$
\begin{aligned}
&\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}\\
&\leq e^{-2\rho|K|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}\\
&\quad + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}, \qquad (277)
\end{aligned}
$$

and we can estimate the above terms as for (236) in Proposition 6.5; indeed, by (238), (239) and (240) we have that (277) is bounded by

ighModesTerm1

$$
\begin{aligned}
&e^{-2\rho|K|} + \pi\left(\frac{1}{2\rho^2}+2|K|\right)e^{-2\rho|K|} + e^{-2\rho\,(|K_1|+|K_2|)}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu}\\
&+ e^{-2\rho\,(|K_1|+|K_2|)}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}. \qquad (278)
\end{aligned}
$$

Now we estimate (275). We have

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}\,(K_1+L_1)^2$$
$$\le e^{-2\rho|K|}\,K_1^2$$
$$+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}\,K_1^2$$

DecompHighModesTerm2

$$+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2. \tag{279}$$

First we estimate the last term in (279): we have that $|L+K|\ge|K|$, hence

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2$$
$$=\int_{|K|}^{+\infty}\int_0^{2\pi} e^{-2\rho\,\xi}\,\xi\cos^2\phi\,\mathrm{d}\phi\,\mathrm{d}\xi$$
$$=\pi\,e^{-2\rho|K|}\,\frac{1+2\rho|K|}{4\rho^2}$$
$$\le\pi\,\mu^4\,e^{-2\rho\left[|K|-\frac{2|\log\mu|}{\rho}-\frac{1}{2\rho}\log(2\rho|K|)\right]}$$

Est1HighModesTerm2

$$\overset{\delta<1-1/e}{\le}\quad\pi\,\mu^4\,e^{-2\rho\left[\delta|K|-\frac{2|\log\mu|}{\rho}\right]} \tag{280}$$

Now we bound the other two nontrivial terms in (279); on the one hand, we notice that

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}(K_1+L_1)^2$$
$$\le 2\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}\,K_1^2$$

DecompHighModesTerm22

$$+2\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}\,L_1^2, \tag{281}$$

where the first sum can be bounded as the second term in (277), while

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_1^2\le e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu}\,\frac{\ell^2}{\mu^2}$$

DecompHighModesTerm23

$$\le 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu}\,\frac{\ell^2}{\mu^2}\,\mathrm{d}l, \tag{282}$$

where the last integral is exponentially small with respect to μ.

Similarly,

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_2^2 \leq e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}$$

$$\leq 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}\,\mathrm{d}\ell, \tag{283}$$

where the last integral is exponentially small with respect to μ.

On the other hand, for any k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|\leq\frac{(2+\delta)|\log\mu|}{\rho}$

$$\left|\frac{\mathcal{E}_\kappa}{\mu^2}-\frac{|\hat\psi_K|^2}{2}\right|$$

$$\leq\left|\omega_k^2-1\right||\hat q_K|^2+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat p_{K+L}|^2+\omega_k^2\,|\hat q_{K+L}|^2,$$

$$\overset{(271)}{\leq}(\mu^2\pi^2K_1^2+\pi^2\mu^{2\sigma}K_2^2)|\hat p_K|^2$$

$$+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat p_{K+L}|^2+|\hat q_{K+L}|^2+\pi^2[\mu^2(K_1+L_1)^2+\mu^{2\sigma}(K_2+L_2)^2]|\hat q_{K+L}|^2,$$

$$\leq\left(\pi^2\,\mu^2K_1^2+\pi^2\mu^{2\sigma}\,K_2^2\right)|\hat p_K|^2$$

$$+\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}[1+\pi^2\mu^2(K_1+L_1)^2+\pi^2\mu^{2\sigma}(K_2+L_2)^2]$$

$$\leq\pi^2\,\mu^2\left(1+\mu^{2(\sigma-1)}\right)\frac{9|\log\mu|^2}{\rho^2}\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}} \tag{284}$$

$$+\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|} \tag{285}$$

$$+\pi^2\mu^2\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}(K_1+L_1)^2 \tag{286}$$

$$+\pi^2\mu^{2\sigma}\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}(K_2+L_2)^2 \tag{287}$$

and we can conclude by estimating (284) by exploiting the fact that $|\log\mu|\le\mu^{-2/5}$, while we can bound (285)-(286) by

$$
\begin{aligned}
&\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma}L_2\in 2\mathbb{Z}}} \left[1+\mu^2(K_1+L_1)^2\right] e^{-2\rho|K+L|}\\
&\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma}L_2\in 2\mathbb{Z}}} (1+2\,\mu^2\,K_1^2+2\,\mu^2\,L_1^2)\, e^{-2\rho|K+L|}\\
&\le\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\left[(1+2\,\mu^2\,K_1^2)\,2\pi\int_{2/\mu}^{+\infty} e^{-2\rho\ell}\,\ell\,\mathrm{d}\ell+4\pi\,\mu^2\int_{2/\mu}^{+\infty}e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\
&=\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\times
\end{aligned}
$$

Est1LowModesTerm3
$$
\left[2\pi\left(1+2\mu^2\frac{9|\log\mu|^2}{\rho^2}\right)e^{-4\rho/\mu}\,\frac{\mu+4\rho}{4\mu\rho^2}+4\pi\,\mu^2 e^{-4\rho/\mu}\,\frac{3\mu^3+12\rho\mu^2+24\rho^2\mu+32\rho^3}{8\mu^3\rho^4}\right], \tag{288}
$$

and we can estimate (287) by

$$
\begin{aligned}
&\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma}L_2\in 2\mathbb{Z}}} (K_2+L_2)^2\, e^{-2\rho|K+L|}\\
&\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma}L_2\in 2\mathbb{Z}}} (2K_2^2+2L_2^2)\, e^{-2\rho|K+L|}\\
&\le\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)}\left[2K_1^2\,2\pi\int_{2/\mu^\sigma}^{+\infty} e^{-2\rho\ell}\,\ell\,\mathrm{d}\ell+4\pi\int_{2/\mu^\sigma}^{+\infty}e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\
&=\frac{\pi^2}{2}\,\|(\psi,\bar\psi)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)}\times
\end{aligned}
$$

Est1LowModesTerm4
$$
\left[2\pi\,2\frac{9|\log\mu|^2}{\rho^2}\,e^{-4\rho/\mu^\sigma}\,\frac{\mu^\sigma+4\rho}{4\mu^\sigma\rho^2}+4\pi\,e^{-4\rho/\mu^\sigma}\,\frac{3\mu^{3\sigma}+12\rho\mu^{2\sigma}+24\rho^2\mu^\sigma+32\rho^3}{8\mu^{3\sigma}\rho^4}\right]. \tag{289}
$$

Appendix E. Proof of Proposition 6.10

ApprEstSec22

The argument follows along the lines of Appendix C in [?] (bambusi2006metastability).

Exploiting the canonical transformation found in Theorem 3.3, we also define

$$
\zeta_a := (\psi_a,\bar\psi_a)=\mathcal{T}_{\mu^2}(\widetilde{\psi_a},\widetilde{\bar\psi_a}) = \tilde\zeta_a+\phi_a(\tilde\zeta_a), \tag{290}
$$

where $\phi_a(\tilde\zeta_a):=(\phi_\xi(\tilde\zeta_a),\phi_\eta(\tilde\zeta_a))$; by (54) we have

estRemThm3
$$
\sup_{\zeta\in\mathcal{B}_{\rho,n}(R)}\|\phi_a(\zeta)\|_{\mathcal{H}^{\rho,n}}\le C_n'\mu^2\,R. \tag{291}
$$

For convenience we define

qappr3
$$
q_a(\tau,y):=\frac{1}{\sqrt2}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2)+e^{-i\tau}\widetilde{\bar\psi_a}(\tau,y_1,y_2)\right] \tag{292}
$$

pappr3
$$
p_a(\tau,y):=\frac{1}{\sqrt2 i}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2)-e^{-i\tau}\widetilde{\bar\psi_a}(\tau,y_1,y_2)\right], \tag{293}
$$

We observe that the pair (q_a,p_a) satisfies

ApprEq31
$$
\mu(q_a)_t=\mu p_a+\mu^5\mathcal{R}_q \tag{294}
$$

ApprEq32
$$
\mu(p_a)_t=-\mu q_a+\mu\Delta_1 q_a-\mu^3\,\beta\,\pi_0 q_a^3+\mu^5\mathcal{R}_p, \tag{295}
$$

where the operator Δ_1 acts on the variable x, π_0 is the projector on the space of the functions with zero average, and the remainders are functions of the rescaled variables τ and y which satisfy

$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_q\|_{\ell^2_{\rho,0}} \le C,$$
$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_p\|_{\ell^2_{\rho,1}} \le C.$$

We now restrict the space variables to integer values; keeping in mind that q_a and p_a are periodic, we assume that $j \in \mathbb{Z}^2_{N,N^\sigma}$.

For a finite sequence $Q = (Q_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ we define the norm

NormSeq3 (296)
$$\|Q\|^2_{\ell^2_{N,N^\sigma}} := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |Q_j|^2.$$

Now we consider the discrete model (5): we rewrite in the following form,

DiscrEq31 (297)
$$\dot{Q}_j = P_j$$
DiscrEq32 (298)
$$\dot{P}_j = -Q_j + (\Delta_1 Q)_j - \beta\,\pi_0 Q_j^3$$

and we want to show that there exist two sequences $E = (E_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ and $F = (F_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ such that

$$Q \;=\; \mu\, q_a + \mu^{1+\gamma} E,\;\; P \;=\; \mu p_a + \mu^{1+\gamma} F$$

fulfills (297)-(298), where $\gamma > 0$ is a parameter we will fix later in the proof. Therefore, we have that

EqSeq31 (299)
$$\dot{E} = F - \mu^{5-1-\gamma}\mathcal{R}_q$$
EqSeq32 (300)
$$\dot{F} = -E + \Delta_1 E - \beta\pi_0\,(3\mu^{3+\gamma-1-\gamma}\, q_a^2 E + 3\mu^{1+2+2\gamma-1-\gamma}\, q_a E^2 + \mu^{3+3\gamma-1-\gamma} E^3) - \mu^{5-1-\gamma}\mathcal{R}_p,$$

where we impose initial conditions on (E, F) such that $(\tilde{q}, \tilde{p})$ has initial conditions corresponding to the ones of the true initial datum,

$$\mu q_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{1+\gamma} E_{0,j} = Q_{0,j},$$
$$\mu p_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{1+\gamma} F_{0,j} = P_{0,j}.$$

We now define the operator ∂_i, $i = 1, 2$, by $(\partial_i f)_j := f_j - f_{j-e_i}$ for each $f \in \ell^2_{N,N^\sigma}$.

- Claim 1: Let $\sigma > 1$ and $\gamma > 0$, we have

$$\|E_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(3-2\gamma-\sigma)/2},$$
$$\|F_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(3-2\gamma-\sigma)/2},$$
$$\|\partial_1 E_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(5-2\gamma-\sigma)/2},$$
$$\|\partial_2 E_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(3-2\gamma+\sigma)/2},$$
$$\|\partial_1 F_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(5-2\gamma-\sigma)/2},$$
$$\|\partial_2 F_0\|_{\ell^2_{N,N^\sigma}} \le C'\mu^{(3-2\gamma+\sigma)/2}.$$

To prove Claim 1 we observe that

$$E_0 = \mu\frac{\psi_a + \bar{\psi}_a - (\widetilde{\psi}_a + \bar{\widetilde{\psi}}_a)}{\sqrt{2}\mu^{1+\gamma}} \;=\; \mu^{-\gamma}\,\frac{\phi_\xi + \phi_\eta}{\sqrt{2}},$$
$$F_0 = \mu\frac{\psi_a - \bar{\psi}_a - (\widetilde{\psi}_a - \bar{\widetilde{\psi}}_a)]}{\sqrt{2}i\,\mu^{1+\gamma}} \;=\; \mu^{-\gamma}\,\frac{\phi_\xi - \phi_\eta}{\sqrt{2}i},$$

from which we can deduce

$$\|E_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |E_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 = C\,\mu^{3-2\gamma-\sigma},$$

$$\|F_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |F_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 = C\,\mu^{3-2\gamma-\sigma},$$

$$\|\partial_1 E_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 E_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2+1-\gamma})^2 \leq C\,\mu^{5-2\gamma-\sigma},$$

$$\|\partial_2 E_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 E_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2+\sigma-\gamma})^2 = C\,\mu^{3-2\gamma+\sigma},$$

$$\|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 F_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2+1-\gamma})^2 \leq C\,\mu^{5-2\gamma-\sigma},$$

$$\|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \leq \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 F_{0,j}|^2 \leq C\,4N^{\sigma+1}\,(\mu^{2+\sigma-\gamma})^2 = C\,\mu^{3-2\gamma+\sigma},$$

and this leads to the thesis.

- Claim 2: Fix $n \geq 0$, $T_0 > 0$ and $K_* > 0$, then for any $\mu < \mu_s$ and for any $\sigma > 1$ and $\gamma > 0$ such that $\sigma + 2\gamma < 7$ we have

$$\|E\|^2_{\ell^2_{N,N^\sigma}} + \|F\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 E_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 E_0\|^2_{\ell^2_{N,N^\sigma}} \leq K_*, \quad |t| < \frac{T_0}{\mu^2}. \tag{301}$$

To prove the claim, we define

auxFuncClaim32
$$\mathcal{F}(E,F) := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} \frac{F_j^2 + E_j^2 + E_j(-\Delta_1 E)_j}{2} + \frac{3\mu^2\beta q_a^2 E_j^2 + 3\mu^{2+\gamma}\beta q_a E_j^3}{2}, \tag{302}$$

and we remark that

$$\frac{1}{2}\mathcal{F}(E,F) \leq \|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \leq 2\mathcal{F}(E,F).$$

Now we compute the time derivative of $\mathcal{F}$. Exploiting (258)-(259)

TimeDerAuxFunc31
$$\dot{\mathcal{F}} = \sum_j F_j\left[-E_j + (\Delta_1 E)_j - \beta\pi_0\left(3\mu^2\, q_a^2 E_j + 3\mu^{2+\gamma}\, q_a E_j^2 + \mu^{2+2\gamma} E_j^3\right) - \mu^{4-\gamma}(\mathcal{R}_p)_j\right] \tag{303}$$

TimeDerAuxFunc32
$$+ \sum_j (E_j - (\Delta_1 E)_j)\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] \tag{304}$$

TimeDerAuxFunc33
$$+ \sum_j 3\mu^2\,\beta\, q_a^2 E_j\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + 3\mu^2\beta E_j^2 q_a\,\mu\frac{\partial q_a}{\partial\tau} \tag{305}$$

TimeDerAuxFunc34
$$+ \sum_j \frac{9}{2}\mu^{2+\gamma}\beta E_j^2\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + \frac{3}{2}\mu^{2+\gamma}\beta E_j^3\,\mu\frac{\partial q_a}{\partial\tau} \tag{306}$$

TimeDerAuxFunc41
$$= \sum_j F_j\left[-\beta\pi_0\left(3\mu^{2+\gamma}\, q_a E_j^2 + \mu^{2+2\gamma} E_j^3\right) - \mu^{4-\gamma}(\mathcal{R}_p)_j\right] \tag{307}$$

TimeDerAuxFunc42
$$+ \sum_j E_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] - (\Delta_1 E)_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] \tag{308}$$

TimeDerAuxFunc43
$$+ \sum_j 3\mu^2\,\beta\, q_a^2 E_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] + 3\mu^2\beta E_j^2 q_a\,\mu\frac{\partial q_a}{\partial\tau} \tag{309}$$

TimeDerAuxFunc44
$$+ \sum_j \frac{9}{2}\mu^{2+\gamma}\beta E_j^2\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + \frac{3}{2}\mu^{2+\gamma}\beta E_j^3\,\mu\frac{\partial q_a}{\partial\tau} \tag{310}$$

In order to estimate (307)-(310), we notice that

$$\sup_j |(\Delta_1 E)_j| \le 2 \sup_j |(\partial_1 E)_j| + |(\partial_2 E)_j| \le 4\sqrt{\mathcal{F}},$$

$$\|\mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \le \sum_j |(\mathcal{R}_q)_j|^2 \le 4N^{\sigma+1} \sup_y |\mathcal{R}_q(y)|^2 \le C\mu^{-1-\sigma},$$

$$\|\mathcal{R}_p\|^2_{\ell^2_{N,N^\sigma}} \le C\mu^{-1-\sigma},$$

and that $|(\partial_i \mathcal{R}_q)_j| \le \mu \sup_y \left|\frac{\partial \mathcal{R}_q}{\partial y}(y)\right|$, which implies

$$\|\partial_i \mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \le C\mu^{1-\sigma}.$$

Now, we can estimate (307) by

neDerAuxFunc41 (311) $$C\left(\mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{2+2\gamma}\,\mathcal{F}^2 + \mu^{4-\gamma}\,\mu^{-(1+\sigma)/2}\mathcal{F}^{1/2}\right).$$

Then, (308) can be bounded by

neDerAuxFunc42 (312) $$C\left(\mu^{4-\gamma-(1+\sigma)/2}\,\mathcal{F}^{1/2} + \mu^{4-\gamma+(1-\sigma)/2}\,\mathcal{F}^{1/2}\right);$$

next, we can estimate (309) by

neDerAuxFunc43 (313) $$C\left(\mu^{6-\gamma-(1+\sigma)/2}\mathcal{F}^{1/2} + \mu^3\mathcal{F}\right),$$

while (310) can be bounded by

neDerAuxFunc44 (314) $$C\left(\mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{6-(1+\sigma)/2}\mathcal{F} + \mu^{2+\gamma}\mathcal{F}^{3/2}\right).$$

Hence, as long as $\mathcal{F} < 2K_*$ we have

EstTimeDer31 (315) $$\left|\dot{\mathcal{F}}\right| \le C\left[\mu^{2+\gamma}\,K_*^{1/2} + \mu^{2+2\gamma}\,K_* + \mu^3 + \mu^{2+\gamma}\,K_*^{1/2} + \mu^{6-(1+\sigma)/2} + \mu^{2+\gamma}\,K_*^{1/2}\right]\mathcal{F}$$

EstTimeDer32 (316) $$+\,C\left[\mu^{4-\gamma}\,\mu^{-(1+\sigma)/2} + \mu^{4-\gamma-(1+\sigma)/2} + \mu^{4-\gamma+(1-\sigma)/2} + \mu^{6-\gamma-(1+\sigma)/2}\right]K_*^{1/2}$$

EstTimeDer33 (317) $$\overset{\sigma+2\gamma<7}{\le} C\,\mu^2\,(1+K_*^{1/2})\,\mathcal{F} + C\,\mu^{(7-2\gamma-\sigma)/2}\,K_*^{1/2}$$

and by applying Gronwall's lemma we get

Gronwall3 (318) $$\mathcal{F}(t) \le \mathcal{F}(0)e^{C\,(1+K_*^{1/2})\,\mu^2 t} + e^{C\,(1+K_*^{1/2})\,\mu^2 t}\,C\,(1+K_*^{1/2})\,\mu^2 t\,C\,\mu^{(7-2\gamma-\sigma)/2}\,K_*^{1/2},$$

from which we can deduce the thesis.

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