EFFECTIVE MOTIVES WITH AND WITHOUT TRANSFERS IN CHARACTERISTIC p

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ABSTRACT. We prove the equivalence between the category $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(K,\mathbb{Q})$ of effective motives of rigid analytic varieties over a perfect complete non-archimedean field K and the category $\mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(K,\mathbb{Q})$ which is obtained by localizing the category of motives without transfers $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(K,\mathbb{Q})$ over purely inseparable maps. In particular, we obtain an equivalence between $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(K,\mathbb{Q})$ and $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(K,\mathbb{Q})$ in the characteristic 0 case and an equivalence between $\mathbf{DM}^{\text{eff}}_{\text{ét}}(K,\mathbb{Q})$ and $\mathbf{DA}^{\text{eff}}_{\text{Frobét}}(K,\mathbb{Q})$ of motives of algebraic varieties over a perfect field K. We also show a relative and a stable version of the main statement.

CONTENTS

1. INTRODUCTION

Morel and Voevodsky in [22] introduced the derived category of effective motives over a base B which, in the abelian context with coefficients in a ring Λ and with respect to the étale topology, is denoted by $\mathbf{DA}^{\text{eff}}_{\text{ét}}(B,\Lambda)$. It is obtained as the homotopy category of the model category $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{Sm}/B,\Lambda)$ of complexes of presheaves of Λ-modules over the category of smooth varieties over B, after a localization with respect to étale-local maps (giving rise to étale descent in homology) and projection maps $\mathbb{A}^1_X\to X$ (giving rise to the homotopy-invariance of homology). Voevodsky in [28], [20] also defined the category of *motives with transfers* $\mathbf{DM}^{\text{eff}}_{\text{ét}}(B,\Lambda)$ using analogous constructions starting from the category $\mathbf{Ch}\,\mathbf{PST}(\mathrm{Sm}/B,\Lambda)$ of complexes of presheaves *with transfers* over Sm/B i.e. with extra functoriality with respect to maps which are finite and surjective. Both categories of motives can be stabilized, by formally inverting the Tate twist functor $\Lambda(1)$ in a model-categorical sense, giving rise to the categories of stable motives with and without transfers $\mathbf{DM}_{\text{ét}}(B,\Lambda)$ and $\mathbf{DA}_{\text{ét}}(B,\Lambda)$ respectively.

There exists a natural adjoint pair between the category of motives without and with transfers which is induced by the functor a_{tr} of "adjoining transfers" and its right adjoint o_{tr} of "forgetting transfers". Different authors have proved interesting results on the comparison between the two categories $\mathbf{DA}^{\text{eff}}_{\text{ét}}(B,\Lambda)$ and $\mathbf{DM}^{\text{eff}}_{\text{ét}}(B,\Lambda)$ induced by this adjunction. Morel in [21] proved the equivalence between the stable categories $\mathbf{DA}_{\text{ét}}(B,\Lambda)$ and $\mathbf{DM}_{\text{ét}}(B,\Lambda)$ in case Λ is a $\mathbb{Q}$-algebra and B is the spectrum of a perfect field, by means of algebraic K-theory. Cisinski and Deglise in [9] generalized this fact to the case of a $\mathbb{Q}$-algebra Λ and a base B that is of finite dimension, noetherian, excellent and geometrically unibranch. Later, Ayoub (see [4, Theorem B.1]) gave a simplified proof of this equivalence for a normal basis B in characteristic 0 and a coefficient ring Λ over $\mathbb{Q}$ that also works for the effective categories. In [3] the same author proved the equivalence between the stable categories of motives with and without transfers for a more general ring of coefficients Λ, under some technical assumptions on the base B (see [3, Theorem B.1]).

The purpose of this paper is to give a generalization of the effective result of Ayoub [4, Theorem B.1]. We prove an equivalence between the effective categories of motives with rational coefficients for a normal base B over a perfect field K of arbitrary characteristic. Admittedly, in order to reach this equivalence in characteristic p we need to consider a perfect base B^{Perf} and invert extra maps in $\mathbf{DA}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}}, \mathbb{Q})$ namely the purely inseparable morphisms, or equivalently the relative Frobenius maps. This procedure can also be interpreted as a localization with respect to a finer topology, that we will call the $\mathrm{Frob\acute{e}t}$-topology. The associated homotopy category will be denoted by $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}, \mathbb{Q})$.

We remark that the approach *without transfers* is much more convenient when computing morphisms, and it is the most natural over a general base. On the other hand, Voevodsky proved a series of useful theorems for the category of motives *with transfers* over a field (say, the Cancellation theorem [29] or the homotopy invariance of cohomology [28, Proposition 3.1.11]) which are fundamental for developing the theory. Being able to switch between the two definitions via a canonical equivalence is then useful when dealing with motives, and has been used intensively in the literature (see [5] for an overview). This article shows that one can finally do so also for effective motives in positive characteristic.

All our statements will be given in the setting of rigid analytic varieties instead of algebraic varieties. The reason is twofold: on the one hand one can deduce immediately the statements on algebraic motives by considering a trivially valued field, on the other hand comparison theorems for motives of rigid analytic varieties $\mathbf{RigDA}^{\mathrm{eff}}_{\text{ét}}(B, \Lambda)$ and $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B, \Lambda)$ are equally useful for some purposes. For example, the result in characteristic 0 is mentioned and used in [7, Section 2.2]. Also, this equivalence in case B is the spectrum of a perfect field of arbitrary characteristic plays a crucial role in [26] and actually constitutes the main motivation of this work. For the theory of rigid analytic spaces over non-archimedean fields, we refer to [8].

The main theorem of the paper is the following (Theorem 4.1):

Theorem. *Let Λ be a $\mathbb{Q}$-algebra and let B be a normal rigid variety over a perfect, complete non-archimedean field K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}, \Lambda) \cong \mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}}, \Lambda).$$

The article is organized as follows. In Section 2 we introduce the $\mathrm{Frob\acute{e}t}$-topology on normal varieties and we prove some general properties it satisfies. In Section 3 we define the categories of motives that we are interested in, as well as other categories of motives which play an auxiliary role in the proof of the main result. In Section 4 we finally outline the proof of the equivalence above.

2. THE FROB-TOPOLOGY

We first define a topology on normal rigid analytic varieties over a field K. Along our work, we will always assume the following hypothesis.

Assumption 2.1. We let K be a perfect field which is complete with respect to a non-archimedean norm.

Unless otherwise stated, we will use the term "variety" to indicate a separated rigid analytic variety over K (see [8, Chapter 9]).

Definition 2.2. A map $f\colon Y \to X$ of varieties over K is called a Frob-*cover* if it is finite, surjective and for every affinoid U in X the affinoid inverse image $V = f^{-1}(U)$ is such that the induced map of rings $\mathcal{O}(U) \to \mathcal{O}(V)$ is radicial.

Remark 2.3. By [12, Corollary IV.18.12.11] a morphism of schemes is finite, surjective and radicial if and only if it is a finite universal homeomorphism. We can remark that the same holds true for rigid analytic varieties. That said, we will not use this characterization in this text.

If $\operatorname{char} K = p$ and X is a variety over K then the absolute n-th Frobenius map $X \to X$ given by the elevation to the p^n-th power, factors over a map $X \to X^{(n)}$ where we denote by $X^{(n)}$ the base change of X by the absolute n-th Frobenius map $K \to K$. We denote by $\Phi^{(n)}$ the map $X \to X^{(n)}$ and we call it the *relative n-th Frobenius*. Since K is perfect, $X^{(n)}$ is isomorphic to X endowed with the structure map $X \to \operatorname{Spa} K \overset{\Phi^{-n}}{\to} \operatorname{Spa} K$ and the relative n-th Frobenius is isomorphic to the absolute n-th Frobenius of X over $\mathbb{F}_p$. We can also define $X^{(n)}$ for negative n to be the base change of X over the map $\Phi^n\colon K \to K$ which is again isomorphic to X endowed with the structure map $X \to \operatorname{Spa} K \overset{\Phi^{-n}}{\to} \operatorname{Spa} K$. The Frobenius map induces a morphism $X^{(-1)} \to X$ and the collection of maps $\{X^{(-1)} \to X\}$ defines a coverage (see for example [18, Definition C.2.1.1]).

In case $\operatorname{char} K = 0$ we also define $X^{(n)}$ to be X and the maps $\Phi\colon X^{(n-1)} \to X^{(n)}$ to be the identity maps for all $n \in \mathbb{Z}$.

Proposition 2.4. *Let $Y \to X$ be a* Frob*-cover between normal quasi-compact varieties over K. There exists an integer n and a map $X^{(-n)} \to Y$ such that the composite map $X^{(-n)} \to Y \to X$ coincides with Φ^n and the composite map $Y \to X \to Y^{(n)}$ coincides with Φ^n.*

Proof. Let $f\colon Y \to X$ a Frob-cover of affinoid normal schemes over K. We can consider the induced map of K-algebras and apply [19, Proposition 6.6] to conclude that it exists an integer n and a map $h\colon X \to Y^{(n)}$ such that the composite map $Y \to X \to Y^{(n)}$ coincides with the relative n-th Frobenius. This factorization is also canonical, and therefore can be generalized to the situation in which X and Y are not necessarily affinoid.

We also remark that the map $Y \to X$ is an epimorphism (in the categorical sense) of normal varieties. From the equalities $fhf^{(n)} = \Phi^{(n)}_Y f^{(n)} = f\Phi^{(n)}_X$ we then conclude that the composite map $X \to Y^{(n)} \to X^{(n)}$ coincides with the n-relative Frobenius. This proves the claim. □

Definition 2.5. Let B be a normal variety over K. We define RigSm/B to be the category of quasi-compact varieties which are smooth over B. We denote by $\tau_{\text{ét}}$ the étale topology.

Definition 2.6. Let B be a normal variety over K. We define RigNor/B to be the category of quasi-compact normal varieties over B.

- We denote by τ_{Frob} the topology on RigNor/B induced by Frob-covers.
- We denote by $\tau_{\text{ét}}$ the étale topology.
- We denote by $\tau_{\text{Frobét}}$ the topology generated by τ_{Frob} and $\tau_{\text{ét}}$.
- We denote by τ_{fh} the topology generated by covering families $\{f_i\colon X_i \to X\}_{i\in I}$ such that I is finite, and the induced map $\bigsqcup f_i\colon \bigsqcup_{i\in I} X_i \to X$ is finite and surjective.
- We denote by $\tau_{\text{fhét}}$ the topology generated by τ_{fh} and $\tau_{\text{ét}}$.

Remark 2.7. The Frobét topology is denoted by quiet (quasi-étale) in [10, Section 5] and the fhét-topology is often denoted by qfh (see [27]). We stick to the notation fhét in order to be consistent with [4].

We are not imposing any additivity condition on the Frob-topology, i.e. the families $\{X_i \to \bigsqcup_{i\in I} X_i\}_{i\in I}$ are not Frob-covers. This does not interfere much with our theory since we will mostly be interested in the $\mathrm{Frob\acute{e}t}$-topology, with respect to which such families are covering families.

Remark 2.8. We are ultimately interested in considering the $\mathrm{Frob\acute{e}t}$-topology on $\mathrm{RigNor}\,/B$. As any object $X \in \mathrm{RigNor}\,/B$ is locally affinoid, we can restrict to considering the full subcategory $\mathrm{AffNor}\,/B$ of $\mathrm{RigNor}\,/B$ made of *affinoid* varieties that are smooth over B since it induces an equivalent étale (and $\mathrm{Frob\acute{e}t}$) topos. In proofs we will then, sometimes tacitly, assume that the objects of $\mathrm{RigNor}\,/B$ and $\mathrm{RigSm}\,/B$ are affinoid, without loss of generality. For the same reason, one can harmlessly drop the condition on quasi-compactness for objects in $\mathrm{RigNor}\,/B$ and $\mathrm{RigSm}\,/B$ without changing the associated topoi.

Remark 2.9. The fh-topology is obviously finer that the Frob-topology, which is the trivial topology in case $\mathrm{char}\, K = 0$.

Remark 2.10. The category of normal affinoid varieties is not closed under fiber products, and the fh-coverings do not define a Grothendieck pretopology. Nonetheless, they define a coverage which is enough to have a convenient description of the topology they generate (see for example [18, Section C.2.1]).

Remark 2.11. A particular example of fh-covers is given by *pseudo-Galois covers* which are finite, surjective maps $f\colon Y \to X$ of normal integral affinoid varieties such that the field extension $K(Y) \to K(X)$ is obtained as a composition of a Galois extension and a finite, purely inseparable extension. The Galois group G associated to the extension coincides with $\mathrm{Aut}(Y/X)$. As shown in [6, Corollary 2.2.5], a presheaf $\mathcal{F}$ on $\mathrm{AffNor}\,/B$ with values in a complete and cocomplete category is an fh-sheaf if and only if the two following conditions are satisfied.

(1) For every finite set $\{X_i\}_{i\in I}$ of objects in $\mathrm{RigNor}\,/B$ it holds $\mathcal{F}(\bigsqcup_{i\in I} X_i) \cong \prod_{i\in I} \mathcal{F}(X_i)$.
(2) For every pseudo-Galois covering $Y \to X$ with associated Galois group G the map $\mathcal{F}(X) \to \mathcal{F}(Y)^G$ is invertible.

Definition 2.12. Let B be a normal variety over K.

- We denote by $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ the 2-limit category $2-\varinjlim_n \mathrm{RigSm}\,/B^{(-n)}$ with respect to the functors $\mathrm{RigSm}\,/B^{(-n-1)} \to \mathrm{RigSm}\,/B^{(-n)}$ induced by the pullback along the map $B^{(-n-1)} \to B^{(-n)}$. More explicitly, it is equivalent to the category $\mathbf{C}_B[S^{-1}]$ where $\mathbf{C}_B$ is the category whose objects are pairs $(X, -n)$ with $n \in \mathbb{N}$ and $X \in \mathrm{RigSm}\,/B^{(-n)}$ and morphisms $\mathbf{C}_B((X, -n), (X', -n'))$ are maps $f\colon X \to X'$ forming commutative squares
$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ B^{(-n)} & \xrightarrow{\Phi} & B^{(-n')} \end{array}$$
and where S is the class of canonical maps $(X' \times_{B^{(-n')}} B^{(-n)}, -n) \to (X', -n')$ for each $X \in \mathrm{RigSm}\,/B^{(-n')}$ and $n \geq n'$ (see [11, Definition VI.6.3]).
- We say that a map $(X, -n) \to (X', -n')$ of $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ is a Frob-*cover* if the map $X \to X'$ is a Frob-cover. We denote by τ_{Frob} the topology on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ induced by Frob-covers.
- We denote by $\tau_{\text{ét}}$ the topology on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ generated by the étale coverings on each category $\mathrm{RigSm}\,/B^{(-n)}$. It defines the "inverse limit" topology on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$ according to [1, Definition VI.8.2.5].

- We denote by $\tau_{\text{Frobét}}$ the topology generated by τ_{Frob} and $\tau_{\text{ét}}$.

We now investigate some properties of the Frob-topology.

Proposition 2.13. *Let B be a normal variety over K.*

- *A presheaf $\mathcal{F}$ on RigNor/B is a Frob-sheaf if and only if $\mathcal{F}(X^{(-1)}) \cong \mathcal{F}(X)$ for all objects X in RigNor/B.*
- *A presheaf $\mathcal{F}$ on $\mathrm{RigSm}/B^{\mathrm{Perf}}$ is a Frob-sheaf if and only if $\mathcal{F}(X^{(-1)}, -n-1) \cong \mathcal{F}(X, -n)$ for all objects $(X, -n)$ in $\mathrm{RigSm}/B^{\mathrm{Perf}}$.*

Proof. The two statements are analogous and we only prove the claim for RigNor/B. By means of [18, Lemma C.2.1.6 and Lemma C.2.1.7] the topology generated by maps $f\colon Y \to X$ which factor a power of Frobenius $X^{(-n)} \to X$ is the same as the one generated by the coverage $X^{(-1)} \to X$. Using Proposition 2.4, we conclude that the Frob-topology coincides with the one generated by the coverage $\{X^{(-1)} \to X\}$. Since the Frobenius map is a monomorphism of normal varieties, the sheaf condition associated to the coverage $X^{(-1)} \to X$ is simply the one of the statement by [18, Lemma 2.1.3]. □

Corollary 2.14. *Let B be a normal variety over K.*

- *The class Φ of maps $\{X^{(-r)} \to X\}_{r\in\mathbb{N}, X\in\mathrm{RigNor}/B}$ admits calculus of fractions, and its saturation consists of Frob-covers. In particular, the continuous map*
$$(\mathrm{RigNor}/B, \mathrm{Frob}) \to \mathrm{RigNor}/B[\Phi^{-1}]$$
defines an equivalence of topoi.
- *The class Φ of maps $\{(X^{(-r)}, -n-r) \to (X, -n)\}_{r\in\mathbb{N},(X,n)\in\mathrm{RigSm}/B^{\mathrm{Perf}}}$ admits calculus of fractions, and its saturation consists of Frob-covers. In particular, the continuous map*
$$(\mathrm{RigSm}/B^{\mathrm{Perf}}, \mathrm{Frob}) \to \mathrm{RigSm}/B^{\mathrm{Perf}}[\Phi^{-1}]$$
defines an equivalence of topoi.

Proof. We only prove the first claim. The fact that Φ admits calculus of fractions is an easy check, and the characterization of its saturation follows from Proposition 2.4. The sheaf condition for a presheaf $\mathcal{F}$ with respect to the Frob-topology is simply $\mathcal{F}(X^{(-1)}) \cong \mathcal{F}(X)$ by Corollary 2.13 hence the last claim. □

Remark 2.15. We follow the notations introduced in Definition 2.12. Any pullback of a finite, surjective radicial map between normal algebraic varieties is also finite, surjective and radicial. This can be generalized to rigid analytic varieties, given the explicit description of the pull-back of a finite map (see for example [16, Lemma 1.4.5]). In particular, if B is a normal variety, the maps in the class S are invertible in $\mathrm{RigNor}/B[\Phi^{-1}]$. The functor $\mathbf{C}_B \to \mathrm{RigNor}/B[\Phi^{-1}]$ defined by mapping $(X, -n)$ to X factors through a functor $\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathrm{RigNor}/B[\Phi^{-1}]$. In particular, there is a functor $\mathrm{RigSm}/B^{\mathrm{Perf}}[\Phi^{-1}] \to \mathrm{RigNor}/B[\Phi^{-1}]$ defined by sending $(X, -n)$ to X hence, by Corollary 2.14, there is a functor $\mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B)$.

Remark 2.16. If $e\colon B' \to B$ is a finite map of normal varieties, any étale hypercover $\mathcal{U} \to B'$ has a refinement by a hypercover $\mathcal{U}'$ obtained by pullback from an étale hypercover $\mathcal{V}$ of B (see for example [24, Tag 04DL]). In particular, the functor $e_*\colon \mathbf{Psh}(\mathrm{RigSm}/B') \to \mathbf{Psh}(\mathrm{RigSm}/B)$ commutes with the functor $a_{\text{ét}}$ of ét-sheafification. The same holds true for the functor $e_*\colon \mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$.

From now on, we fix a commutative ring Λ and work with Λ-enriched categories. In particular, the term "presheaf" should be understood as "presheaf of Λ-modules" and similarly for the term "sheaf". It follows that the presheaf $\Lambda(X)$ represented by an object X of a category $\mathbf{C}$ sends an object Y of $\mathbf{C}$ to the free Λ-module $\Lambda\,\mathrm{Hom}(Y, X)$.

Assumption 2.17. Unless otherwise stated, we assume from now on that Λ is a $\mathbb{Q}$-algebra and we omit it from the notations.

The following facts are immediate, and will also be useful afterwards.

Proposition 2.18. *Let B be a normal variety over K.*

- *If $\mathcal{F}$ is an étale sheaf on $\mathrm{RigSm}/B^{\mathrm{Perf}}$ [resp. on RigNor/B] then $a_{\mathrm{Frob}}\mathcal{F}$ is a* Frobét-*sheaf.*
- *If $\mathcal{F}$ is a* Frob-*sheaf on $\mathrm{RigSm}/B^{\mathrm{Perf}}$ [resp. on RigNor/B] then $a_{\text{ét}}\mathcal{F}$ is a* Frobét-*sheaf.*

Proof. We only prove the claims for RigNor/B. First, suppose that $\mathcal{F}$ is an étale sheaf. By Proposition 2.4, we obtain that $a_{\mathrm{Frob}}\mathcal{F}(X) = \varinjlim_n \mathcal{F}(X^{(-n)})$. Whenever $U \to X$ is étale, then $U \times_X X^{(-n)} \cong U^{(-n)}$ and $U^{(-n)} \times_{X^{(-n)}} U^{(-n)} \cong (U \times_X U)^{(-n)}$ so that the following diagram is exact

$$0 \to \mathcal{F}(X^{(-n)}) \to \mathcal{F}(U^{(-n)}) \to \mathcal{F}((U \times_X U)^{(-n)}).$$

The first claim the follows by taking the limit over n.

We now prove the second claim. Suppose $\mathcal{F}$ is a Frob-sheaf. For any étale covering $\mathcal{U} \to X$ we indicate with $\mathcal{U}'$ the associated covering of $X^{(-1)}$ obtained by pullback. From Remark 2.16 one can compute the sections of $a_{\text{ét}}\mathcal{F}(X^{(-1)})$ with the formula

$$a_{\text{ét}}\mathcal{F}(X^{(-1)}) = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}'_0) \to \mathcal{F}(\mathcal{U}'_1)\right)$$

where $\mathcal{U} \to X$ varies among Čech covers of X. Since $\mathcal{F}$ is a Frob-sheaf, then $\mathcal{F}(U'_0) \cong \mathcal{F}(U_0)$ and $\mathcal{F}(U'_1) \cong \mathcal{F}(U_1)$. The formula above then implies

$$a_{\text{ét}}\mathcal{F}(X^{(-1)}) = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}_0) \to \mathcal{F}(\mathcal{U}_1)\right) = a_{\text{ét}}\mathcal{F}(X)$$

proving the claim. ◻

Proposition 2.19. *Let B be a normal variety over K. If $\mathcal{F}$ is a* fh-*sheaf on RigNor/B then $a_{\text{ét}}\mathcal{F}$ is a* fhét-*sheaf.*

Proof. Let $f\colon X' \to X$ be a pseudo-Galois cover in AffNor/B with associated group G. In light of Remark 2.11, we need to show that $a_{\text{ét}}\mathcal{F}(X) \cong a_{\text{ét}}\mathcal{F}(X')^G$. For any étale covering $\mathcal{U} \to X$ we indicate with $\mathcal{U}'$ the associated covering of X' obtained by pullback. From Remark 2.16 one can compute the sections of $a_{\text{ét}}\mathcal{F}(X')$ with the formula

$$a_{\text{ét}}\mathcal{F}(X') = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}'_0) \to \mathcal{F}(\mathcal{U}'_1)\right)$$

where $\mathcal{U} \to X$ varies among Čech covers of X. Taking the G-invariants is an exact functor as Λ is a $\mathbb{Q}$-algebra and when applied to the formula above it yields

$$a_{\text{ét}}\mathcal{F}(X')^G = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}'_0)^G \to \mathcal{F}(\mathcal{U}'_1)^G\right) = \varinjlim_{\mathcal{U}\to X} \ker\left(\mathcal{F}(\mathcal{U}_0) \to \mathcal{F}(\mathcal{U}_1)\right) = a_{\text{ét}}\mathcal{F}(X)$$

as wanted. ◻

Proposition 2.20. *Let B be a normal variety over K. The canonical inclusions*

$$o_{\mathrm{Frob}}\colon\ \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B) \to \mathbf{Psh}(\mathrm{RigNor}/B)$$
$$o_{\mathrm{Frob}}\colon\ \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$
$$o_{\mathrm{fh}}\colon\ \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B) \to \mathbf{Psh}(\mathrm{RigNor}/B)$$

are exact.

Proof. In light of Proposition 2.13 the statements about o_{Frob} are obvious. Since Λ is a $\mathbb{Q}$-algebra, the functor of G-invariants from $\Lambda[G]$-modules to Λ-modules is exact. The third claim then follows from Remark 2.11. □

We now investigate the functors of the topoi introduced above induced by a map of varieties $B' \to B$.

Proposition 2.21. *Let $f\colon B' \to B$ be a map of normal varieties over K.*

- *Composition with f defines a functor $f_\sharp$ from normal varieties over B' to normal varieties over B which induces the following adjoint pair*

$$f_\sharp\colon \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigNor}\,/B') \rightleftarrows \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigNor}\,/B) : f^*$$

- *The base change over f defines functors $f^{(-n)*}$ from smooth varieties over $B^{(n)}$ to smooth varieties over $B'^{(n)}$ which induce the following adjoint pair*

$$f^*\colon \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigSm}\,/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigSm}\,/B'^{\mathrm{Perf}}) : f_*$$

- *If f is a* Frob*-cover, the functors above are equivalences of categories.*
- *If f is a smooth map, the composition with f defines functors $f_\sharp^{(-n)}$ from smooth varieties over $B'^{(-n)}$ to smooth varieties over $B^{(-n)}$ which induce the following adjoint pair*

$$f_\sharp\colon \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigSm}\,/B'^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{Frob\acute{e}t}}(\mathrm{RigSm}\,/B^{\mathrm{Perf}}) : f^*$$

Proof. We initially remark that the functors $f^{(-n)*}$ induce a functor $f^*\colon \mathbf{C}_B \to \mathbf{C}_{B'}$ where $\mathbf{C}_B$ is the fibered category introduced in Definition 2.12 where we drop the condition of being quasi-compact (see Remark 2.8). As cartesian squares are mapped to cartesian squares, they also induce a functor from smooth varieties over B^{Perf} to smooth varieties over B'^{Perf}.

The existence of the first two adjoint pairs is then a formal consequence of the continuity of the functors $f_\sharp$ and f^*.

Let now f be a Frob-cover. The functors $f^*\colon \mathrm{RigSm}\,/B^{\mathrm{Perf}}[\Phi^{-1}] \to \mathrm{RigSm}\,/B'^{\mathrm{Perf}}[\Phi^{-1}]$ and $f_\sharp\colon \mathrm{RigNor}\,/B'[\Phi^{-1}] \to \mathrm{RigNor}\,/B[\Phi^{-1}]$ are equivalences, and we conclude the third claim by what proved above and Corollary 2.14.

For the fourth claim, we use a different model for the $\mathrm{Frob\acute{e}t}$-topos on $\mathrm{RigSm}\,/B^{\mathrm{Perf}}$. The fibered category $\mathbf{C}_B$ can be endowed with the Frob-topology and the $\mathrm{Frob\acute{e}t}$-topology. Following the proof of Corollary 2.14, the map $(\mathbf{C}_B, \mathrm{Frob}) \to \mathbf{C}_B[\Phi^{-1}]$ induces an equivalence of topoi. Moreover, the canonical functor $\mathbf{C}_B[\Phi^{-1}] \to \mathrm{RigSm}\,/B^{\mathrm{Perf}}[\Phi^{-1}]$ induces an equivalence of categories.

The existence of the last Quillen functor is therefore a formal consequence of the continuity of the functor $f_\sharp\colon (\mathbf{C}_{B'}[\Phi^{-1}], \acute{\mathrm{e}}\mathrm{t}) \to (\mathbf{C}_B[\Phi^{-1}], \acute{\mathrm{e}}\mathrm{t})$. □

Remark 2.22. Let $f\colon B' \to B$ be a map of normal varieties. The image via f^* of the presheaf represented by $(X, -n)$ is the presheaf represented by $(X \times_B B'^{(-n)}, -n)$ and if f is smooth, the image via $f_\sharp$ of the presheaf represented by $(X', -n)$ is the sheaf represented by $(X', -n)$.

3. Rigid motives and Frob-motives

We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra, and that presheaves and sheaves take values in the category of Λ-modules.

We make extensive use of the theory of model categories and localization, following the approach of Ayoub in [2] and [6]. Fix a site $(\mathbf{C}, \tau)$. The category of complexes of presheaves $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ can be endowed with the *projective model structure* for which weak equivalences are quasi-isomorphisms (maps inducing isomorphisms of homology presheaves) and fibrations are maps $\mathcal{F} \to \mathcal{F}'$ such that $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a surjection for all X in $\mathbf{C}$ (cfr [14, Section 2.3] and [2, Proposition 4.4.16]).

Remark 3.1. If we take $\mathbf{C} = \{*\}$ we obtain in particular the usual projective model category structure on $\mathbf{Ch}(\Lambda)$ which is cellular and left proper (see for example [2, Example 4.4.24(2)] and [14, Proposition 2.3.22]). For any $\mathbf{C}$ the category $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ is equivalent to the category of presheaves on $\mathbf{C}$ with values in $\mathbf{Ch}(\Lambda)$. With this respect, the projective model structure described above coincides with the one induced by defining weak-equivalences and fibrations point-wise, starting from the projective model structure on $\mathbf{Ch}(\Lambda)$. One could alternatively consider the (Quillen equivalent) *injective model structure* on $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ obtained by defining weak-equivalences and cofibrations point-wise (see [2, Definition 4.4.15]).

Also the category of complexes of sheaves $\mathbf{Ch}(\mathbf{Sh}_\tau(\mathbf{C}))$ can be endowed with the *projective model structure* defined in [2, Proposition 4.4.41]. In this structure, weak equivalences are quasi-isomorphisms of complexes of sheaves (maps inducing isomorphisms on the sheaves associated to the homology presheaves).

Just as in [17], [20], [22] or [23], we consider the left Bousfield localization of $\mathbf{Ch}(\mathbf{Psh}(\mathbf{C}))$ with respect to the topology we select, and a chosen "contractible object". We recall that left Bousfield localizations with respect to a class of maps S (see [13, Chapter 3]) is the universal model categories in which the maps in S become weak equivalences. The existence of such structures is granted only under some technical hypothesis, as shown in [13, Theorem 4.1.1] and [2, Theorem 4.2.71].

Proposition 3.2. *Let $(\mathbf{C}, \tau)$ be a site with finite direct products and let $\mathbf{C}'$ be a full subcategory of $\mathbf{C}$ such that every object of $\mathbf{C}$ has a covering by objects of $\mathbf{C}'$. Let also I be an object of $\mathbf{C}'$.*

(1) The projective model category $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ admits a left Bousfield localization $\mathbf{Ch}_I\,\mathbf{Psh}(\mathbf{C})$ with respect to the set S_I of all maps $\Lambda(I \times X)[i] \to \Lambda(X)[i]$ as X varies in $\mathbf{C}$ and i varies in $\mathbb{Z}$.

(2) The projective model categories $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C}')$ admit left Bousfield localizations $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C}')$ with respect to the class S_τ of maps $\mathcal{F} \to \mathcal{F}'$ inducing isomorphisms on the ét*-sheaves associated to $H_i(\mathcal{F})$ and $H_i(\mathcal{F}')$ for all $i \in \mathbb{Z}$. Moreover, the two localized model categories are Quillen equivalent and the sheafification functor induces a Quillen equivalence to the projective model category $\mathbf{Ch}\,\mathbf{Sh}_\tau(\mathbf{C})$.*

(3) The model categories $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C}')$ admit left Bousfield localizations $\mathbf{Ch}_{\tau,I}\,\mathbf{Psh}(\mathbf{C})$ and $\mathbf{Ch}_{\tau,I}\,\mathbf{Psh}(\mathbf{C}')$ with respect to the set S_I defined above. Moreover, the two localized model categories are Quillen equivalent.

Proof. By [2, Proposition 4.4.16] and Remark 3.1 the projective model structures in the statement are left proper and cellular. Any such model category admits a left Bousfield localization with respect to a set of maps ([13, Theorem 4.1.1]) hence the first claim.

For the first part of second claim, it suffices to apply [2, Proposition 4.4.32, Lemma 4.4.35] showing that the localization over S_τ is equivalent to a localization over a set of maps. The second part is a restatement of [2, Corollary 4.4.43, Proposition 4.4.56].

Since by [2, Proposition 4.4.32] the τ-localization coincides with the Bousfield localization with respect to a set, we conclude by [2, Theorem 4.2.71] that the model category $\mathbf{Ch}_\tau\,\mathbf{Psh}(\mathbf{C})$ is still left proper and cellular. The last statement then follows from [13, Theorem 4.1.1] and the second claim. □

In the situation above, we will denote by $S_{(\tau,I)}$ the union of the class S_τ and the set S_I.

Remark 3.3. A geometrically relevant situation is induced when I is endowed with a multiplication map $\mu\colon I \times I \to I$ and maps i_0 and i_1 from the terminal object to I satisfying the relations of a monoidal object with 0 as in the definition of an interval object (see [22, Section 2.3]). Under these hypotheses, we say that the triple $(\mathbf{C}, \tau, I)$ is a *site with an interval*.

Example 3.4. The affinoid rigid variety $\mathbb{B}^1 = \operatorname{Spa} K\langle\chi\rangle$ is an interval object with respect to the natural multiplication μ and maps i_0 and i_1 induced by the substitution $\chi \mapsto 0$ and $\chi \mapsto 1$ respectively.

Definition 3.5. Let B be a normal variety over K.

- The triangulated homotopy category of $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\operatorname{RigSm}/B)$ will be denoted by $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(B,\Lambda)$.
- The triangulated homotopy category of $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{Psh}(\operatorname{Rig}\operatorname{Sm}/B^{\text{Perf}})$ will be denoted by $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}},\Lambda)$ and the one of $\mathbf{Ch}_{\text{Frobét},\mathbb{B}^1}\,\mathbf{Psh}(\operatorname{Rig}\operatorname{Sm}/B^{\text{Perf}})$ will be denoted by $\mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}},\Lambda)$.
- The triangulated homotopy category of $\mathbf{Ch}_{\text{Frobét},\mathbb{B}^1}\,\mathbf{Psh}(\operatorname{RigNor}/B)$ will be denoted by $\mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\operatorname{RigNor}/B,\Lambda)$ and the one of $\mathbf{Ch}_{\text{fhét},\mathbb{B}^1}\,\mathbf{Psh}(\operatorname{RigNor}/B)$ will be denoted by $\mathbf{D}^{\text{fh}}_{\text{ét},\mathbb{B}^1}(\operatorname{RigNor}/B,\Lambda)$.
- If $\mathbf{C}$ is one of the categories RigSm/B, $\operatorname{Rig}\operatorname{Sm}/B^{\text{Perf}}$ and RigNor/B and $\eta \in \{\text{ét},\text{Frob},\text{fh},\text{Frobét},\text{fhét},\mathbb{B}^1,(\text{ét},\mathbb{B}^1),(\text{Frobét},\mathbb{B}^1),(\text{fhét},\mathbb{B}^1)\}$ we say that a map in $\mathbf{Ch}\,\mathbf{Psh}(\mathbf{C})$ is a η-*weak equivalence* if it is a weak equivalence in the model structure $\mathbf{Ch}_\eta\,\mathbf{Psh}(\mathbf{C})$ whenever this makes sense.
- We will omit Λ from the notation whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda(X)$.

We now want to introduce the analogue of the previous definitions for motives with transfers. By Remark 2.15 the mapping $(X,-n) \mapsto X$ induces a functor $\mathbf{Sh}_{\text{Frob}}(\operatorname{RigSm}/B^{\text{Perf}}) \to \mathbf{Sh}_{\text{Frob}}(\operatorname{RigNor}/B)$. If we compose it with the Yoneda embedding and the functor a_{fh} of fh-sheafification we obtain a functor

$$\operatorname{RigSm}/B^{\text{Perf}} \to \mathbf{Sh}_{\text{Frob}}(\operatorname{RigSm}/B^{\text{Perf}}) \to \mathbf{Sh}_{\text{fh}}(\operatorname{RigNor}/B).$$

Definition 3.6. Let B be a normal variety over K.

- We define the category RigCor/B as the category whose objects are those of RigSm/B and whose morphisms $\operatorname{Hom}(X,Y)$ are computed in $\mathbf{Sh}_{\text{fh}}(\operatorname{RigNor}/B)$. The category $\mathbf{Psh}(\operatorname{RigCor}/B)$ will be denoted by $\mathbf{PST}(\operatorname{RigSm}/B)$.
- We define the category $\operatorname{RigCor}/B^{\text{Perf}}$ as the category whose objects are those of $\operatorname{RigSm}/B^{\text{Perf}}$ and whose morphisms $\operatorname{Hom}(X,Y)$ are computed in $\mathbf{Sh}_{\text{fh}}(\operatorname{RigNor}/B)$. The category $\mathbf{Psh}(\operatorname{RigCor}/B^{\text{Perf}})$ will be denoted by $\mathbf{PST}(\operatorname{RigSm}/B^{\text{Perf}})$.

We remark that, as Λ is a $\mathbb{Q}$-algebra, morphisms $X \to Y$ of RigCor admit a more concrete description in terms of *correspondences* defined in [6, Noltation 2.2.22] and denoted in [6] by $\operatorname{Cor}(X,Y)$. We also remark that the inclusions of categories $\operatorname{RigSm}/B \to \operatorname{RigCor}/B$ and $\operatorname{RigSm}/B^{\text{Perf}} \to \operatorname{RigCor}/B^{\text{Perf}}$ induce the following adjunctions:

$$a_{\text{tr}}\colon \mathbf{Ch}\,\mathbf{Psh}(\operatorname{RigSm}/B) \rightleftarrows \mathbf{Ch}\,\mathbf{PST}(\operatorname{RigSm}/B) : o_{\text{tr}}.$$

$$a_{\text{tr}}\colon \mathbf{Ch}\,\mathbf{Psh}(\operatorname{RigSm}/B^{\text{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{PST}(\operatorname{RigSm}/B^{\text{Perf}}) : o_{\text{tr}}.$$

We now define the category of motives with transfers.

Proposition 3.7. *Let B be a normal variety and $\mathbf{C}$ be either RigSm/B or $\operatorname{RigSm}/B^{\text{Perf}}$. The projective model category $\mathbf{Ch}\,\mathbf{PST}(\mathbf{C})$ admits a left Bousfield localization $\mathbf{Ch}_{\text{ét}}\,\mathbf{PST}(\mathbf{C})$ with respect to $S_{\text{ét}}$, the class of of maps f such that $o_{\text{tr}}(f)$ is a ét-weak equivalence. It also admits a further Bousfield localization $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\mathbf{C})$ with respect to the set formed by all maps $\Lambda(\mathbb{B}^1_X)[i] \to \Lambda(X)[i]$ by letting X vary in $\mathbf{C}$ and i vary in $\mathbb{Z}$.*

Proof. The proof of [6, Theorem 2.5.7] also applies in our situation. For the second statement, it suffices to apply [13, Theorem 4.1.1]. □

Remark 3.8. By means of an étale version of [6, Corollary 2.5.3], if $\mathcal{F}$ is a presheaf with transfers then the associated étale sheaf $a_{\text{ét}}\mathcal{F}$ can be endowed with a unique structure of presheaf with transfers such that $\mathcal{F} \to a_{\text{ét}}\mathcal{F}$ is a map of presheaves with transfers. The class $S_{\text{ét}}$ can then be defined intrinsically, as the class of maps $\mathcal{F} \to \mathcal{F}'$ inducing isomorphisms of étale sheaves with transfers $a_{\text{ét}}H_i\mathcal{F} \to a_{\text{ét}}H_i\mathcal{F}'$.

Definition 3.9. Let B be a normal variety over K.

- The triangulated homotopy category of $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\operatorname{RigSm}/B)$ will be denoted by $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B,\Lambda)$.
- The triangulated homotopy category of $\mathbf{Ch}_{\text{ét},\mathbb{B}^1}\,\mathbf{PST}(\operatorname{RigSm}/B^{\text{Perf}})$ will be denoted by $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}},\Lambda)$.
- We will omit Λ from the notation whenever the context allows it. The image of a variety X in one of these categories will be denoted by $\Lambda_{\text{tr}}(X)$.

We remark that if $\operatorname{char} K = 0$ the two definitions above coincide. Also, if B is the spectrum of the perfect field K the category $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})$ coincides with $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(K)$. In this case, the definition of $\mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}})$ also coincides with the one of $\mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(K)$ given in the introduction as the following fact shows.

Proposition 3.10. *Let B be a normal variety over K. The category $\mathbf{Ch}_{\text{Frobét}}(\operatorname{RigSm}/B^{\text{Perf}})$ is Quillen equivalent to the left Bousfield localization of $\mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\operatorname{RigSm}/B^{\text{Perf}})$ over the set of all shifts of maps $\Lambda(X^{(-1)},-n-1) \to \Lambda(X,-n)$ as $(X,-n)$ varies in $\operatorname{RigSm}/B^{\text{Perf}}$.*

Proof. From Lemmas 2.18, 2.20 and 3.11 we conclude that Frobét-local objects are those which are Frob-local and ét-local. We can then conclude using Lemma 3.12. □

Lemma 3.11. *Let $\mathbf{C}$ be a category endowed with two Grothendieck topologies τ_1, τ_2 and let τ_3 be the topology generated by τ_1 and τ_2. We denote by a_{τ_i} the associated sheafification functor and with o_{τ_i} their right adjoint functors. If o_{τ_1} is exact and $a_{\tau_3} = a_{\tau_2}a_{\tau_1}$ then the following categories are canonically equivalent:*

(1) The homotopy category of $\mathbf{Ch}_{\tau_3}\,\mathbf{Psh}(\mathbf{C})$.
(2) The full triangulated subcategory of $\mathbf{D}(\mathbf{Psh}(\mathbf{C}))$ formed by objects which are τ_3-local.
(3) The full triangulated subcategory of $\mathbf{D}(\mathbf{Psh}(\mathbf{C}))$ formed by objects which are τ_1-local and τ_2-local.

Proof. The equivalence between the first and the second category follows by definition of the Bousfield localization. We are left to prove the equivalence between the second and the third. We remark that τ_3-local objects are in particular (τ_1,τ_2)-local.

Since o_{τ_1} is exact, the category of τ_1-local objects coincides with the category of complexes quasi-isomorphic to complexes of τ_1-sheaves. Consider the model category $\mathbf{Ch}_{\tau_3}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ which is the Bousfield localization of $\mathbf{Ch}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ over the class of maps of complexes inducing isomorphisms on the τ_3-sheaves associated to the homology presheaves, that we will call τ_3-equivalences. From the assumption $a_{\tau_3} = a_{\tau_2}a_{\tau_1}$ the class of τ_3-equivalences coincides with the class of maps S_{τ_2} of complexes inducing isomorphisms on the τ_2-sheaves associated to the homology τ_1-sheaves. Hence $\mathbf{Ch}_{\tau_3}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ coincides with $\mathbf{Ch}_{\tau_2}(\mathbf{Sh}_{\tau_1}(\mathbf{C}))$ and its derived category is equivalent to the category of (τ_1,τ_2)-local complexes.

Because of the following Quillen adjunction

$$\mathbb{L}a_{\tau_1} = a_{\tau_1} \colon \operatorname{Ho}(\mathbf{Ch}_{\tau_3}\,\mathbf{Psh}(\mathbf{C}) \rightleftarrows \operatorname{Ho}(\mathbf{Ch}_{\tau_3}\,\mathbf{Sh}_{\tau_1}(\mathbf{C})) : \mathbb{R}o_{\tau_1} = o_{\tau_1}.$$

we conclude that the image via o_{τ_1} of a τ_2-local complex of sheaves i.e. a $(\tau_1.\tau_2)$-local complex, is τ_3-local, as wanted. □

Lemma 3.12. *Let B be a normal variety over K. A fibrant object of* $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ *is* Frob*-local if and only if it is local with respect to the set of all shifts of maps* $\Lambda(X^{(-1)},-n-1)\to\Lambda(X,-n)$ *as* $(X,-n)$ *varies in* $\mathrm{RigSm}/B^{\mathrm{Perf}}$.

Proof. We initially remark that a fibrant complex $\mathcal{F}$ is local with respect to the set of maps in the claim if and only if $(H_i\mathcal{F})(X,-n)\cong(H_i\mathcal{F})(X^{(-1)},-n-1)$ for all X and i. By Proposition 2.4, this amounts to say that $H_i\mathcal{F}$ is a Frob-sheaf for all i.

Suppose now that $\mathcal{F}$ is fibrant and Frob-local. Since the map of presheaves $\Lambda(X^{(-1)},-n-1)\to\Lambda(X,-n)$ induces an isomorphism on the associated Frob-sheaves, we deduce that $(H_i\mathcal{F})(X^{(-1)},-n-1)\cong(H_i\mathcal{F})(X,-n)$. This implies that $H_i\mathcal{F}$ is a Frob-sheaf and hence $\mathcal{F}$ is local with respect to the maps of the claim, as wanted.

Suppose now that $\mathcal{F}$ is fibrant and local with respect to the maps of the claim. Let $\mathcal{F}\to C^{\mathrm{Frob}}\mathcal{F}$ a Frob-weak equivalence to a fibrant Frob-local object. By definition, we deduce that the Frob-sheaves associated to $H_i\mathcal{F}$ and to $H_iC^{\mathrm{Frob}}\mathcal{F}$ are isomorphic. On the other hand, we know that these presheaves are already Frob-sheaves, and hence the map $\mathcal{F}\to C^{\mathrm{Frob}}\mathcal{F}$ is a quasi-isomorphism of presheaves and $\mathcal{F}$ is Frob-local. □

We now want to find another model for the category $\mathbf{D}^{\mathrm{fh}}_{\text{ét},\mathbb{B}^1}(\mathrm{RigNor}/B)$. This is possible by means of the model-categorical machinery developed above.

By Remark 2.11 an object $\mathcal{F}$ in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigNor}/B)$ is fh-local if and only if it is additive and

$$\mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)(\Lambda(X),\mathcal{F})\to\mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)(\Lambda(X'),\mathcal{F})^{\mathrm{Aut}(X'/X)}$$

is an isomorphism, for all pseudo-Galois coverings $X'\to X$ in AffNor/B. Therefore, if we consider $\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)$ as the subcategory of $(\mathbb{B}^1,\mathrm{Frobét})$-local objects in $\mathbf{D}\,\mathbf{Psh}(\mathrm{RigNor}/B)$ we say that an object $\mathcal{F}$ of $\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)$ is fh-local if and only if

$$\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X),\mathcal{F})\to\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X'),\mathcal{F})^{\mathrm{Aut}(X'/X)}$$

is an isomorphism, for all pseudo-Galois coverings $X'\to X$.

Proposition 3.13. *Let B be a normal variety over K. The category* $\mathbf{D}^{\mathrm{fh}}_{\text{ét},\mathbb{B}^1}(\mathrm{RigNor}/B)$ *is canonically isomorphic to the category of* fh*-local objects in* $\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)$.

Proof. It suffices to prove the claim before performing the $\mathbb{B}^1$-localization on each category. The statement then follows from Propositions 2.18 and 2.19 together with Lemmas 2.20 and 3.11. □

We now study some functoriality properties of the categories just defined, and later prove a fundamental fact: the locality axiom (see [22, Theorem 3.2.21]).

Proposition 3.14. *Let $f\colon B'\to B$ be a map of normal varieties over K. The first two adjoint pairs of Proposition 2.21 induce the following Quillen pairs:*

$$\mathbb{L}f_\sharp\colon\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B')\rightleftarrows\mathbf{D}_{\mathrm{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B):\mathbb{R}f^*$$

$$\mathbb{L}f^*\colon\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét}}(B^{\mathrm{Perf}})\rightleftarrows\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét}}(B'^{\mathrm{Perf}}):\mathbb{R}f_*$$

which are equivalences whenever f is a Frob*-covering. Moreover, if f is a smooth map, the third adjoint pair of Proposition 2.21 induces a Quillen pair:*

$$\mathbb{L}f_\sharp\colon\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét}}(B'^{\mathrm{Perf}})\rightleftarrows\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frobét}}(B^{\mathrm{Perf}}):\mathbb{L}f^*$$

Proof. The statement is a formal consequence of Proposition 2.21, [2, Theorem 4.4.61] and the formulas $f^*(\mathbb{B}^1_X)=\mathbb{B}^1_{f^*(X)}$ and $f_\sharp(\mathbb{B}^1_X)=\mathbb{B}^1_X$. □

Proposition 3.15. *Let* $e\colon B' \to B$ *be a finite map of normal varieties over* K. *The functor*

$$e_*\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$*-equivalences.*

Proof. Let $e\colon B' \to B$ be a finite map of normal varieties. The functor e_* is induced by the map $\mathrm{RigSm}/B^{\mathrm{Perf}} \to \mathrm{RigSm}/B'^{\mathrm{Perf}}$ sending $(X, -n)$ to $(X \times_{B^{(-n)}} B'^{(-n)}, -n)$. From Remark 2.16 it commutes with ét-sheafification. As the image of $(X^{(-1)}, -n-1)$ is isomorphic to $((X \times_{B^{(-n)}} B'^{(-n)})^{(-1)}, -n-1)$ we deduce from Corollary 2.14 that e_* commutes with Frob-sheafification. Therefore by Proposition 2.18 we deduce that $e_*\colon \mathbf{Psh}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) \to \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ commutes with the functor $a_{\mathrm{Frob\acute{e}t}}$ of Frobét-sheafification, hence it preserves Frobét-equivalences.

We now prove that it also preserves $\mathbb{B}^1$-equivalences. By [2, Proposition 4.2.74] it suffices to show that $e_*(\Lambda(\mathbb{B}^1_V) \to \Lambda(V))$ is a $\mathbb{B}^1$-weak equivalence for any V in $\mathrm{RigSm}/X'^{\mathrm{Perf}}$. This follows from the explicit homotopy between the identity and the zero map on $e_*(\Lambda(\mathbb{B}^1_V))$ (see the argument of [6, Theorem 2.5.24]). □

The following property is an extension of [6, Theorem 1.4.20] and referred to as the *locality axiom.*

Theorem 3.16. *Let* $i\colon Z \hookrightarrow B$ *be a closed immersion of normal varieties over* K *and let* $j\colon U \hookrightarrow B$ *be the open complement. For every object* M *in* $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ *there is an distinguished triangle*

$$\mathbb{L}j_\sharp \mathbb{L}j^* M \to M \to \mathbb{R}i_* \mathbb{L}i^* M \to$$

In particular, the pair $(\mathbb{L}j^*, \mathbb{L}i^*)$ *is conservative.*

Proof. First of all, we remark that by Proposition 3.15 one has $\mathbb{R}i_* = i_*$. In particular it suffices to prove the claim before performing the localization over the shifts of maps $\Lambda(X^{(-1)}, -n-1) \to \Lambda(X, -n)$ i.e. in the category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$.

The functors $\mathbb{L}j_\sharp\ \mathbb{L}j^*$ and $\mathbb{L}i^*$ commute with small sums because they admit right adjoint functors. Also $\mathbb{R}i_*$ does, since it holds $\mathbb{R}i_* = i_*$. We conclude that the full subcategory of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ of objects M such that

$$\mathbb{L}j_\sharp \mathbb{L}j^* M \to M \to \mathbb{R}i_* \mathbb{L}i^* M \to$$

is an distinguished triangle is closed under cones, and under small sums. We can then equivalently prove the claim in the subcategory $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ of compact objects, since these motives generate $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ as a triangulated category with small sums.

By means of Lemma 3.17 and Proposition 3.15, it suffices to prove the statement for each category $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$. It is then enough to prove the claim for the categories $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Nis}}(B^{(-n)})$ as defined in [6, Definition 1.4.12] since $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ is a further localization of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Nis}}(B^{(-n)})$. In this case, the statement is proved in [6, Theorem 1.4.20]. □

Lemma 3.17. *Let* B *be a normal variety over* K. *The canonical functors* $\mathrm{RigSm}/B^{(-n)} \to \mathrm{RigSm}/B^{\mathrm{Perf}}$ *induce a triangulated equivalence of categories*

$$\varinjlim_n \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)}) \cong \mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$$

where we denote by $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ *[resp. with* $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$*] the subcategory of compact objects of* $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ *[resp. of* $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$*].*

Proof. The functor $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)}) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$ is triangulated and sends the objects $\Lambda(X)[i]$ which are compact generators of the first category, to compact objects of the second. It then induces an exact functor between the two subcategories of compact objects.

Moreover, by letting n vary, the images of the objects in $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{(-n)})$ generate the category $\mathbf{RigDA}^{\mathrm{ct}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$.

Up to shifting indices, it therefore suffices to show that for X, Y in RigSm/B one has

$$\varinjlim_n \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{(-n)})(\Lambda(X\times_B B^{(-n)}),\Lambda(Y\times_B B^{(-n)}))\cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})(\Lambda(\bar{X}),\Lambda(\bar{Y}))$$

where we denote by $\bar{X}=(X,0)$ and $\bar{Y}=(Y,0)$ the object of $\mathrm{RigSm}/B^{\mathrm{Perf}}$ associated to X resp. Y. To this aim, we simply follow the proof of [6, Proposition 1.A.1]. For the convenience of the reader, we reproduce it here.

Step 1: We consider the directed diagram $\mathcal{B}$ formed the maps $B^{(-n-1)}\to B^{(-n)}$ and we let $\mathrm{RigSm}/\mathcal{B}$ be the the category of rigid smooth varieties over it as defined in [6, Section 1.4.2]. We can endow the category $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B})$ with the $(\mathrm{\acute{e}t},\mathbb{B}^1)$-local model structure, and consider the Quillen adjunctions induced by the map of diagrams $\alpha_n\colon B^{(-n)}\to\mathcal{B}$, $f_{nm}\colon B^{(-n)}\to B^{(-m)}$:

$$\alpha_n^*\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B})\rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}):\alpha_{n*}$$
$$\alpha_{n\sharp}\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)})\rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/\mathcal{B}):\alpha_n^*$$
$$f_{nm}^*\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-m)})\rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)}):f_{nm*}$$

We also remark that the canonical map $\mathrm{RigSm}/B^{(-n)}\to\mathrm{RigSm}/B^{\mathrm{Perf}}$ induces a Quillen adjunction

$$f_{\infty n}^*\colon \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{(-n)})\rightleftarrows \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}):f_{\infty n*}.$$

Consider a trivial cofibration $\alpha_{0*}\Lambda(Y)\to R$ with target R that is $(\mathrm{\acute{e}t},\mathbb{B}^1)$-fibrant. Since α_n^* is a left and right Quillen functor and $\alpha_n^*\alpha_{0*}=f_{n0}^*$ we deduce that the map $\Lambda(Y\times_B B^{(-n)})=f_{n0}^*\Lambda(Y)\to\alpha_n^*R$ is also an $(\mathrm{\acute{e}t},\mathbb{B}^1)$-trivial cofibration with an $(\mathrm{\acute{e}t},\mathbb{B}^1)$-fibrant target.

Step 2: By applying the left Quillen functors f_{nm}^* and $f_{\infty m}^*$ we also obtain that $f_{n0}^*\Lambda(Y)=f_{nm}^*f_{m0}^*\Lambda(Y)\to f_{nm}^*\alpha_m^*R$ and $f_{\infty 0}^*\Lambda(Y)=f_{\infty m}^*f_{m0}^*\Lambda(Y)\to f_{\infty m}^*\alpha_m^*R$ are $(\mathrm{\acute{e}t},\mathbb{B}^1)$-trivial cofibrations. By the 2-out-of-3 property of weak equivalences applied to the composite map

$$f_{n0}^*\Lambda(Y)\to f_{nm}^*\alpha_m^*R\to\alpha_n^*R$$

we then deduce that the map $f_{nm}^*\alpha_m^*R\to\alpha_n^*R$ is an $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalence.

Step 3: We now claim that the natural map $\Lambda(\bar{Y})\to\hat{R}$ with $\hat{R}:=\mathrm{colim}_n\,f_{\infty n}^*\alpha_i^*R$ is an $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalence in $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$. By what shown in Step 2, it suffices to prove that the functor

$$\mathrm{colim}\colon\ \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})^{\mathbb{N}}\to\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalences. First of all, we remark that it is a Quillen left functor with respect to the projective model structure on the diagram category $\mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})^{\mathbb{N}}$ induced by the point-wise $(\mathrm{\acute{e}t},\mathbb{B}^1)$-structure. Hence, it preserves $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalences between cofibrant objects. On the other hand, as directed colimits commute with homology, it also preserves weak equivalences of presheaves. Since any complex is quasi-isomorphic to a cofibrant one, we deduce the claim.

Step 4: We now prove that $\hat{R}$ is $\mathbb{B}^1$-local. Consider a variety U smooth over $B^{(-n)}$. From the formula

$$\hat{R}(\bar{U})=\mathrm{colim}_{m\geq n}\,\alpha_m^*R(U\times_{B^{(-n)}}B^{(-m)})$$

and the fact that α_m^*R is $\mathbb{B}^1$-local, we deduce a quasi-isomorphism $\hat{R}(U)\cong\hat{R}(\mathbb{B}^1_U)$ as wanted.

Step 5: We now prove that $\hat{R}$ is ét-local. It suffices to show that for any U smooth over $B^{(-n)}$ one has $\mathbb{H}^k_{\mathrm{\acute{e}t}}(\bar{U},\hat{R})\cong H_{-k}\hat{R}(\bar{U})$. The topos associated to Et/U is equivalent to the one of $\varinjlim\mathrm{Et}/(U\times_{B^{(-n)}}B^{(-m)})$ and all these sites have a bounded cohomological dimension since Λ

is a $\mathbb{Q}$-algebra. By applying [1, Theorem VI.8.7.3] together with a spectral sequence argument given by [25, Theorem 0.3], we then deduce the formula

$$\mathbb{H}^k_{\text{ét}}(\bar{U},\hat{R})\cong \operatorname{colim}_m \mathbb{H}^k_{\text{ét}}(U\times_{B^{(-n)}} B^{(-m)},\alpha_m^*R).$$

On the other hand, as α_m^*R is ét-local, we conclude that

$$\operatorname{colim}_m \mathbb{H}^k_{\text{ét}}(U\times_{B^{(-n)}} B^{(-m)},\alpha_i^*R)\cong \operatorname{colim}_m H_{-k}(\alpha_m^*R)(U\times_{B^{(-n)}} B^{(-m)})\cong H_{-k}\hat{R}(\bar{U})$$

proving the claim.

Step 6: From Steps 3-5, we conclude that we can compute $\mathbf{RigDA}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})(\Lambda(\bar{X}),\Lambda(\bar{Y}))$ as $\hat{R}(\bar{X})$ which coincides with $\operatorname{colim}_n(\alpha_n^*R)(X\times_B B^{(-n)})$. By what is proved in Step 1, we also deduce that α_n^*R is a $(\text{ét},\mathbb{B}^1)$-fibrant replacement of $\Lambda(Y\times_B B^{(-n)})$ and hence the last group coincides with $\operatorname{colim}_n \mathbf{RigDA}^{\text{eff}}_{\text{ét}}(B^{(-n)})(\Lambda(X\times_B B^{(-n)}),\Lambda(Y\times_B B^{(-n)}))$ proving the statement. □

4. The equivalence between motives with and without transfers

We can finally present the main result of this paper. We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra.

Theorem 4.1. *Let B be a normal variety over K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}})\cong \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}}).$$

As a corollary, we obtain the two following results, which are indeed our main motivation.

Theorem 4.2. *The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(K)\cong \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(K).$$

Theorem 4.3. *Let B be a normal variety over a field K of characteristic 0. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}^{\text{eff}}_{\text{ét}}(B)\cong \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B).$$

The proof of Theorem 4.1 is divided into the following steps.

1. We first produce a functor $\mathbb{L}a_{\text{tr}}\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}})\to \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})$ commuting with sums, triangulated, sending a set of compact generators of the first category into a set of compact generators of the second.
2. We define a fully faithful functor $\mathbb{L}i^*\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}})\to \mathbf{D}^{\text{fh}}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B)$.
3. We define a fully faithful functor $\mathbb{L}j^*\colon \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})\to \mathbf{D}^{\text{fh}}_{\text{Frobét},\mathbb{B}^1}(\text{RigNor}/B)$.
4. We check that $\mathbb{L}j^*\circ\mathbb{L}a_{\text{tr}}$ is isomorphic to $\mathbb{L}i^*$ proving that $\mathbb{L}a_{\text{tr}}$ is also fully faithful.

We now prove the first step.

Proposition 4.4. *Let B be a normal variety over K. The functor a_{tr} induces a triangulated functor*

$$\mathbb{L}a_{\text{tr}}\colon \mathbf{RigDA}^{\text{eff}}_{\text{Frobét}}(B^{\text{Perf}})\to \mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})$$

commuting with sums, sending a set of compact generators of the first category into a set of compact generators of the second.

Proof. The functor a_{tr} induces a Quillen functor

$$\mathbb{L}a_{\text{tr}}\colon\ \mathbf{Ch}_{\text{ét}}\,\mathbf{Psh}(\text{RigSm}/B^{\text{Perf}})\to \mathbf{Ch}_{\text{ét}}\,\mathbf{PST}(\text{RigSm}/B^{\text{Perf}})$$

sending $\Lambda(X,-n)$ to $\Lambda_{\text{tr}}(X)$. We are left to prove that it factors over the Frob-localization, i.e. that the map $\Lambda_{\text{tr}}(X^{(-1)})\to\Lambda_{\text{tr}}(X)$ is an isomorphism in $\mathbf{RigDM}^{\text{eff}}_{\text{ét}}(B^{\text{Perf}})$ for all $X\in$

$\mathrm{RigSm}/B^{(-n)}$. Actually, since the map $X^{(-1)} \to X$ induces an isomorphism of fh-sheaves, we deduce that it is an isomorphism in the category $\mathrm{RigCor}/B^{\mathrm{Perf}}$ hence also in $\mathbf{RigDM}^{\mathrm{eff}}_{\text{ét}}(B^{\mathrm{Perf}})$. □

We are now ready to prove the second step.

Proposition 4.5. *Let B be a normal variety over K. The functors $\mathrm{RigSm}/B^{(-n)} \to \mathrm{RigNor}/B$ induce a fully faithful functor*

$$\mathbb{L}i_B^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B).$$

Proof. We let $\mathbf{C}_B$ be the category introduced in Definition 2.12. As already remarked in the proof of Proposition 2.21 we can endow it with the Frobét-topology and the topos associated to it is equivalent to the Frobét-topos on $\mathrm{RigSm}/B^{\mathrm{Perf}}$. In particular, the continuous functor $i_B\colon \mathbf{C}_B \to \mathrm{RigNor}/B$ induces an adjunction

$$\mathbb{L}i_B^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \rightleftarrows \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B) :\mathbb{R}i_{B*}.$$

As $i_{B*}i_B^*$ is isomorphic to the identity, it suffices to show that $\mathbb{R}i_{B*} = i_{B*}$ so that $\mathbb{R}i_{B*}\mathbb{L}i_B^*$ is isomorphic to the identity as well.

The functor i_{B*} commutes with Frobét-sheafification, and hence it preserves Frobét-weak equivalences, and since $i_{B*}(\Lambda(\mathbb{B}^1_V)) \cong \Lambda(\mathbb{B}^1_B) \otimes i_{B*}(\Lambda(V))$ is weakly equivalent to $i_{B*}(\Lambda(V))$ for every V in RigNor/B we also conclude that it preserves $\mathbb{B}^1$-weak equivalences, as wanted. □

Remark 4.6. As a corollary of the proof of Proposition 4.5 we obtain that the functor i_{B*} preserves $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences.

We remark that the previous result does not yet prove our claim. This is reached by the following crucial fact. Its proof will demand a series of technical lemmas that are proven right below it.

Proposition 4.7. *Let B be a normal variety over K. The image of $\mathbb{L}i_B^*$ is contained in the subcategory of* fh*-local objects.*

Proof. Let M be an object of $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ let $f\colon X \to B$ be a normal irreducible variety over B and let $r\colon X' \to X$ be a pseudo-Galois covering in AffNor/B with $G = \mathrm{Aut}(X'/X)$. We are left to prove that

$$\mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X), \mathbb{L}i^*M) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(X'), \mathbb{L}i^*M)^G$$

is an isomorphism. Using Lemma 4.8 we can equally prove that

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(\Lambda, \mathbb{L}f^*M) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X'^{\mathrm{Perf}})(\Lambda, \mathbb{L}r^*\mathbb{L}f^*M)^G$$

is an isomorphism. Using the notation of Lemma 4.11, it suffices to prove that the natural transformation $\mathrm{id} \to (\mathbb{R}r_*\mathbb{L}r^*)^G$ is invertible.

Using Lemma 4.12, we can define a stratification $(X_i)_{0\leq i\leq n}$ of X made of locally closed connected normal subvarieties of X such that $r_i\colon X'_i \to X_i$ is a composition of an étale cover and a Frob-cover of normal varieties, by letting X'_i be the reduction of the subvariety $X_i \times_X X' \subset X'$. Using the locality axiom (Theorem 3.16) for $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}$ applied to the inclusions $u_i\colon X_i \to X$ we can then restrict to proving that each transformation $\mathbb{L}u_i^* \to \mathbb{L}u_i^*(\mathbb{R}r_*\mathbb{L}r^*)^G \cong (\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G\mathbb{L}u_i^*$ is invertible, where the last isomorphism follows from Lemma 4.11. It suffices then to prove that $\mathrm{id} \to (\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G$ is invertible. If $s\colon Z \to T$ is a Frob-cover, the functors $(\mathbb{L}s^*, \mathbb{R}s_*)$ define an equivalence of categories $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(T^{\mathrm{Perf}}) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(Z^{\mathrm{Perf}})$ by Proposition 3.14 hence we can assume that the maps r_i are étale covers. Moreover, since $\mathbb{L}r_i^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X_i^{\mathrm{Perf}}) \to \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X_i'^{\mathrm{Perf}})$ is conservative by Lemma 4.10, we can equivalently prove that $\mathbb{L}r_i^* \to \mathbb{L}r_i^*(\mathbb{R}r_{i*}\mathbb{L}r_i^*)^G \cong (\mathbb{R}r'_{i*}\mathbb{L}r_i'^*)^G\mathbb{L}r_i^*$ is invertible, where r'_i is the base change of r_i over itself (see Lemma 4.11). By the assumptions on r_i we conclude that r'_i is a projection $\bigsqcup X'_i \to X'_i$ with G acting transitively on the fibers, so that the functor $(\mathbb{R}r'_{i*}\mathbb{L}r_i'^*)^G$ is the identity, proving the claim. □

The following lemmas were used in the proof of the previous proposition.

Lemma 4.8. *Let $f\colon B' \to B$ be a map of normal rigid varieties over K. For any $M \in \mathbf{RigDA}_{\text{Frobét}}(B)$ there is a canonical isomorphism*

$$\mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(B'), \mathbb{L}i_B^*M) \cong \mathbf{RigDA}_{\text{Frobét}}(B')(\Lambda, \mathbb{L}f^*M).$$

Proof. Consider the following diagram of functors:

$$\begin{array}{ccc} \mathbf{Psh}(\mathbf{C}_B[\Phi^{-1}]) & \xrightarrow{i_B^*} & \mathbf{Psh}(\mathrm{RigNor}/B[\Phi^{-1}]) \\ \downarrow f^* & & \downarrow f^* \\ \mathbf{Psh}(\mathbf{C}_{B'}[\Phi^{-1}]) & \xrightarrow{i_{B'}^*} & \mathbf{Psh}(\mathrm{RigNor}/B'[\Phi^{-1}]) \end{array}$$

Let $\mathcal{F}$ be in $\mathbf{Psh}(\mathbf{C}_B[\Phi^{-1}])$ and X' be in RigNor/B'. One has $(i_{B'}^*f^*)(\mathcal{F})(X') = \operatorname{colim}\mathcal{F}(V)$ where the colimit is taken over the maps $X' \to V \times_{B^{(-n)}} B'^{(-n)}$ in $\mathrm{RigNor}/B'[\Phi^{-1}]$ by letting V vary among varieties which are smooth over some $B^{(-n)}$. On the other hand, one has $(f^*i_B^*)(\mathcal{F})(X') = \operatorname{colim}\mathcal{F}(V)$ where the colimit is taken over the maps $X' \to V$ in $\mathrm{RigNor}/B[\Phi^{-1}]$ by letting V vary among varieties which are smooth over some $B^{(-n)}$. Since $V \times_{B^{(-n)}} B'^{(-n)} \cong (V \times_B B')_{\mathrm{red}}$ in $\mathrm{RigSm}/B'[\Phi^{-1}]$ we deduce that the indexing categories are equivalent, hence the diagram above is commutative and therefore by Corollary 2.14 and what shown in the proof of Proposition 2.21 also the following one is:

$$\begin{array}{ccc} \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{i_B^*} & \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\mathrm{RigNor}/B) \\ \downarrow f^* & & \downarrow f^* \\ \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\mathrm{RigSm}/B'^{\mathrm{Perf}}) & \xrightarrow{i_{B'}^*} & \mathbf{Ch}\,\mathbf{Sh}_{\text{Frobét}}(\mathrm{RigNor}/B') \end{array}$$

This fact together with Lemma 4.9 implies $f^*\mathbb{L}i_B^* \cong \mathbb{L}i_{B'}^*\mathbb{L}f^*$. By Propositions 3.14 and 4.5 we then deduce

$$\begin{aligned} &\mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)(\Lambda(B'), \mathbb{L}i_B^*M) = \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B)(\mathbb{L}f_\sharp(\Lambda), \mathbb{L}i_B^*M) \cong \\ &\cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B')(\Lambda, f^*\mathbb{L}i_B^*M) \cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B')(\Lambda, \mathbb{L}i_{B'}^*\mathbb{L}f^*M) \cong \\ &\cong \mathbf{D}_{\text{Frobét},\mathbb{B}^1}(\mathrm{RigNor}/B')(\mathbb{L}i_{B'}^*\Lambda, \mathbb{L}i_{B'}^*\mathbb{L}f^*M) \cong \mathbf{RigDA}_{\text{Frobét}}(B')(\Lambda, \mathbb{L}f^*M) \end{aligned}$$

as claimed. □

Lemma 4.9. *Let $f\colon B' \to B$ be a map of normal varieties over K. The functor*

$$f^*\colon\ \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigNor}/B) \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigNor}/B')$$

preserves the $(\text{Frobét}, \mathbb{B}^1)$*-equivalences.*

Proof. Since f^* commutes with Frobét-sheafification and with colimits, it preserves Frobét-equivalences. Since $f^*(\Lambda(\mathbb{B}^1_V)) \cong \mathbb{B}^1_B \otimes f^*(\Lambda(V))$ is weakly equivalent to $f^*(\Lambda(V))$ for every V in RigNor/B we also conclude that f^* preserves $\mathbb{B}^1$-weak equivalences, hence the claim. □

Lemma 4.10. *Let B be a normal variety over K and let $f\colon X \to Y$ be a composition of* Frob*-coverings and* ét*-coverings in* RigNor/B*. The functor $\mathbb{L}f^*\colon \mathbf{RigDA}^{\mathrm{eff}}_{\text{Frobét}}(Y^{\mathrm{Perf}}) \to \mathbf{RigDA}^{\mathrm{eff}}_{\text{Frobét}}(X^{\mathrm{Perf}})$ is conservative.*

Proof. If f is a Frob-cover, then $\mathbb{L}f^*$ is an equivalence by Proposition 3.14. We are left to prove the claim in case f is an ét-covering. In this case, we can use the proof of the analogous statement in algebraic geometry [3, Lemma 3.4]. □

Lemma 4.11. *Let $e\colon X' \to X$ be a finite morphism of normal varieties over K and let G be a finite group acting on $\mathbb{R}e_*\mathbb{L}e^*$. There exists a subfunctor $(\mathbb{R}e_*\mathbb{L}e^*)^G$ of $\mathbb{R}e_*\mathbb{L}e^*$ such that for all M, N in $\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})$ one has*

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(M, (\mathbb{R}e_*\mathbb{L}e^*)^G N) \cong \mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(X^{\mathrm{Perf}})(M, \mathbb{R}e_*\mathbb{L}e^* N)^G.$$

Moreover for any map $f\colon Y \to X$ of normal rigid varieties factoring into a closed embedding followed by a smooth map, and any diagram of normal varieties

$$\begin{array}{ccc} (Y \times_X X')_{\mathrm{red}} & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ Y & \xrightarrow{f} & X \end{array}$$

there is an induced action of G on $\mathbb{R}e'_*\mathbb{L}e'^*$ and an invertible transformation $\mathbb{L}f^*(\mathbb{R}e_*\mathbb{L}e^*)^G \xrightarrow{\sim} (\mathbb{R}e'_*\mathbb{L}e'^*)^G\mathbb{L}f^*$.

Proof. We define $(\mathbb{R}e_*\mathbb{L}e^*)^G$ to be subfunctor obtained as the image of the projector $\frac{1}{|G|}\sum g$ acting on $\mathbb{R}e_*\mathbb{L}e^*$.

In order to prove the second claim, it suffices to prove that $\mathbb{L}f^*\mathbb{R}e_*\mathbb{L}e^* \cong \mathbb{R}e'_*\mathbb{L}e'^*\mathbb{L}f^*$. As the latter term coincides with $\mathbb{R}e'_*\mathbb{L}(fe')^* = \mathbb{R}e'_*\mathbb{L}(ef')^* = \mathbb{R}e'_*\mathbb{L}f'^*\mathbb{L}e^*$ it suffices to show that the base change transformation $\mathbb{L}f^*\mathbb{R}e_* \to \mathbb{R}e'_*\mathbb{L}f'^*$ is invertible. We can consider individually the case in which f is smooth, and the case in which f is a closed embedding.

Step 1: Suppose that f is smooth. Then f^* has a left adjoint $f_\sharp$. We can equally prove that the natural transformation $\mathbb{L}f'_\sharp\mathbb{L}e'^* \to \mathbb{L}e^*\mathbb{L}f_\sharp$ is invertible. This follows from the isomorphism between the functors $f'_\sharp e'^*$ and $e^*f_\sharp$ from $\mathbf{Psh}(\mathrm{RigSm}/X'^{\mathrm{Perf}})$ to $\mathbf{Psh}(\mathrm{RigSm}/Y^{\mathrm{Perf}})$ obtained by direct inspection.

Step 2: Suppose that f is a closed immersion. Let $j\colon U \to X$ be the open immersion complementary to f and j' be the open immersion complementary to f'. By the locality axiom (Theorem 3.16) we can equally prove that $\mathbb{L}j_\sharp\mathbb{R}e'_* \to \mathbb{R}e_*\mathbb{L}j'_\sharp$ is invertible.

Step 3: It is easy to prove that the transformation $\mathbb{L}j_\sharp\mathbb{R}e'_* \to \mathbb{R}e_*\mathbb{L}j'_\sharp$ is invertible once we know that e_*, e'_*, $j_\sharp$ and $j'_\sharp$ preserve the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences. Indeed, if this is the case, the functors derive trivially and it suffices to prove that for any $\mathrm{Frob\acute{e}t}$-sheaf $\mathcal{F}$ the map $(j_\sharp e'_*)(\mathcal{F}) \to (e_*j'_\sharp)(\mathcal{F})$ is invertible. This follows from the very definitions.

Step 4: The fact that $j_\sharp$ (and similarly $j'_\sharp$) preserves the $(\mathrm{Frob\acute{e}t})$-weak equivalences follows from the fact that it respects quasi-isomorphisms of complexes of $\mathrm{Frob\acute{e}t}$-sheaves, since it is the functor of extension by 0. In order to prove that it preserves the $\mathbb{B}^1$-equivalences, by [2, Proposition 4.2.74] we can prove that $j_\sharp(\Lambda(\mathbb{B}^1_V) \to \Lambda(V))$ is a $\mathbb{B}^1$-weak equivalence for all V in $\mathrm{RigSm}/U^{\mathrm{Perf}}$ and this is clear. The fact that e_* (and similarly e'_*) preserves the $(\mathrm{Frob\acute{e}t}, \mathbb{B}^1)$-equivalences is proved in Proposition 3.15. We then conclude the claim in case f is a closed immersion. □

Lemma 4.12. *Let $f\colon X' \to X$ be a pseudo-Galois map of normal varieties over K. There exists a finite stratification $(X_i)_{1\leq i\leq n}$ of locally closed normal subvarieties of X such that each induced map $f_i\colon (X' \times_X X_i)_{\mathrm{red}} \to X_i$ is a composition of an étale cover and a* Frob*-cover of normal rigid varieties.*

Proof. For every affinoid rigid variety $\mathrm{Spa}\, R$ there is a map of ringed spaces $\mathrm{Spa}\, R \to \mathrm{Spec}\, R$ which is surjective on points, and such that the pullback of a finite étale map $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$

[resp. of an open inclusion $U \to \operatorname{Spec} R$] over $\operatorname{Spa} R \to \operatorname{Spec} R$ exists (following the notation of [15, Lemma 3.8]) and is finite étale [resp. an open inclusion]. The claim then follows from the analogous statement valid for schemes over K. $\square$

Remark 4.13. In the proof of Proposition 4.7, we made use of the fact that Λ is a $\mathbb{Q}$-algebra in a crucial way, for instance, in order to define the functor $(\mathbb{R}e_*\mathbb{L}e^*)^G$.

The following result proves the second step.

Corollary 4.14. *Let B be a normal variety over K. The composite functor*

$$\mathbf{RigDA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \to \mathbf{D}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$$

is fully faithful.

Proof. This follows at once from Proposition 3.13 and Proposition 4.7. $\square$

We now move to the third step. We recall that the category $\mathrm{RigCor}(B^{\mathrm{Perf}})$ is a subcategory of $\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$. We denote by j this inclusion of categories.

Proposition 4.15. *Let B be a normal variety over K. The functor j induces a fully faithful functor $\mathbb{L}j^*\colon \mathbf{RigDM}^{\mathrm{eff}}(B^{\mathrm{Perf}}) \to \mathbf{D}^{\mathrm{fh}}_{\mathrm{\acute{e}t},\mathbb{B}^1}(\mathrm{RigNor}/B)$.*

Proof. The functor j extends to a functor $\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)$ and induces a Quillen pair $j^*\colon \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}\,\mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B) :j_*$ with respect to the projective model structures. We prove that it is a Quillen adjunction also with respect to the $(\mathrm{\acute{e}t},\mathbb{B}^1)$-model structure on the two categories by showing that j_* preserves $(\mathrm{\acute{e}t},\mathbb{B}^1)$-local objects.

From the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{RigSm}/B^{\mathrm{Perf}} & \longrightarrow & \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{i} & \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B) \\
\downarrow & & \downarrow{\scriptstyle a_{\mathrm{tr}}} & & \downarrow{\scriptstyle a_{\mathrm{fh}}} \\
\mathrm{RigCor}/B^{\mathrm{Perf}} & \longrightarrow & \mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{j} & \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B)
\end{array}$$

we deduce that $o_{\mathrm{tr}}j_* = i_*o_{\mathrm{fh}}$ which is a right Quillen functor. It therefore suffices to show that if $o_{\mathrm{tr}}\mathcal{F}$ is $(\mathrm{\acute{e}t},\mathbb{B}^1)$-local then also $\mathcal{F}$ is, for every fibrant object $\mathcal{F}$. Let $\mathcal{F} \to \mathcal{F}'$ be a $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalence to a $(\mathrm{\acute{e}t},\mathbb{B}^1)$-fibrant object of $\mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$. By Lemma 4.16, we deduce that $o_{\mathrm{tr}}\mathcal{F} \to o_{\mathrm{tr}}\mathcal{F}'$ is a $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalence between $(\mathrm{\acute{e}t},\mathbb{B}^1)$-fibrant objects, hence it is a quasi-isomorphism. As o_{tr} reflects quasi-isomorphisms, we conclude that $\mathcal{F}$ is quasi-isomorphic to $\mathcal{F}'$ hence $(\mathrm{\acute{e}t},\mathbb{B}^1)$-local.

We now prove that $\mathbb{L}j^*$ is fully faithful by proving that $\mathbb{R}j_*\mathbb{L}j^*$ is isomorphic to the identity. As j_*j^* is isomorphic to the identity, it suffices to show that $\mathbb{R}j_* = j_*$. We start by proving that j_* preserves $\mathrm{Frob\acute{e}t}$-weak equivalences. As shown in Remark 4.6, the functor i_* preserves $\mathrm{Frob\acute{e}t}$-equivalences. It is also clear that o_{fh} does. Since o_{tr} reflects $\mathrm{Frob\acute{e}t}$-weak equivalences, the claim follows from the equality $o_{\mathrm{tr}}j_* = i_*o_{\mathrm{fh}}$. Since $j_*(\Lambda(\mathbb{B}^1_V)) \cong \Lambda(\mathbb{B}^1_B) \otimes j_*(\Lambda(V))$ is weakly equivalent to $j_*(\Lambda(V))$ for every V in RigNor/B, we also conclude that j_* preserves $\mathbb{B}^1$-weak equivalences, hence the claim. $\square$

Lemma 4.16. *Let B be a normal variety over K. The functor*

$$o_{\mathrm{tr}}\colon \mathbf{Ch}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \to \mathbf{Ch}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$$

preserves $(\mathrm{\acute{e}t},\mathbb{B}^1)$-weak equivalences.

Proof. The argument of [4, Lemma 2.111] easily generalizes to our context. We point out that in the proof, the the class of *injective* trivial cofibrations in the category of complexes of presheaves is used (see Remark 3.1). $\square$

The fourth step is just an easy check, as the next proposition shows.

Proposition 4.17. *Let B be a normal variety over K. The composite functor $\mathbb{L}j^* \circ \mathbb{L}a_{\mathrm{tr}}$ is isomorphic to $\mathbb{L}i^*$. In particular $\mathbb{L}a_{\mathrm{tr}}$ is fully faithful.*

Proof. It suffices to check that the following square is quasi-commutative.

$$\begin{array}{ccc} \mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) & \xrightarrow{a_{\mathrm{tr}}} & \mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathbf{Sh}_{\mathrm{Frob}}(\mathrm{RigNor}/B) & \xrightarrow{a_{\mathrm{fh}}} & \mathbf{Sh}_{\mathrm{fh}}(\mathrm{RigNor}/B) \end{array}$$

This can be done by inspecting the two composite right adjoints, which are canonically isomorphic. □

This also ends the proof of Theorem 4.1.

We remark that in case K is endowed with the trivial norm, we obtain a result on the category of motives constructed from schemes over K. It is the natural generalization of [4, Theorem B.1] in positive characteristic. We recall that the ring of coefficients Λ is assumed to be a $\mathbb{Q}$-algebra.

Theorem 4.18. *Let B be a normal algebraic variety over a perfect field K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{DA}^{\mathrm{eff}}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \cong \mathbf{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}).$$

We now define the stable version of the categories of motives introduced so far, and remark that Theorem 4.3 extends formally to the stable case providing a generalization of the result [9, Theorem 15.2.16].

Definition 4.19. We denote by $\mathbf{RigDA}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}})$ [resp. by $\mathbf{RigDM}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}})$] the homotopy category associated to the model category of symmetric spectra (see [2, Section 4.3.2]) $\mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}})$ [resp. $\mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}})$] where T is the cokernel of the unit map $\Lambda(B) \to \Lambda(\mathbb{T}^1_B)$ [resp $\Lambda_{\mathrm{tr}}(B) \to \Lambda_{\mathrm{tr}}(\mathbb{T}^1_B)$].

Corollary 4.20. *Let B be a normal variety over K. The functor a_{tr} induces an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}_{\mathrm{Frob\acute{e}t}}(B^{\mathrm{Perf}}) \cong \mathbf{RigDM}_{\mathrm{\acute{e}t}}(B^{\mathrm{Perf}}).$$

Proof. Theorem 4.3 states that the adjunction

$$a_{tr}\colon \mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) :o_{\mathrm{tr}}$$

is a Quillen equivalence. It therefore induces a Quillen equivalence on the categories of symmetric spectra

$$a_{tr}\colon \mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{Psh}(\mathrm{RigSm}/B^{\mathrm{Perf}}) \rightleftarrows \mathrm{Sp}^{\Sigma}_{T}\,\mathbf{Ch}_{\mathrm{Frob\acute{e}t},\mathbb{B}^1}\,\mathbf{PST}(\mathrm{RigSm}/B^{\mathrm{Perf}}) :o_{\mathrm{tr}}$$

by means of [2, Proposition 4.3.35]. □

We now assume that Λ equals $\mathbb{Z}$ if $\mathrm{char}\,K = 0$ and equals $\mathbb{Z}[1/p]$ if $\mathrm{char}\,K = p$. In analogy with the statement $\mathbf{DA}_{\mathrm{\acute{e}t}}(B,\Lambda) \cong \mathbf{DM}_{\mathrm{\acute{e}t}}(B,\Lambda)$ proved for motives associated to schemes (see [3, Appendix B]) it is expected that the following result also holds.

Conjecture 4.21. *Let B be a normal variety over K. The functors (a_{tr}, o_{tr}) induce an equivalence of triangulated categories:*

$$\mathbb{L}a_{tr}\colon \mathbf{RigDA}_{\mathrm{\acute{e}t}}(B,\Lambda) \cong \mathbf{RigDM}_{\mathrm{\acute{e}t}}(B,\Lambda).$$

We remark that in the above statement differs from Corollary 4.20 for two main reasons: the ring of coefficients is no longer assumed to be a $\mathbb{Q}$-algebra, and the class of maps with respect to which we localize are the ét-local maps and no longer the Frobét-local maps.

In order to reach this twofold generalization, using the techniques developed in [3], it would suffice to show the two following formal properties of the 2-functor $\mathbf{RigDA}_{\text{ét}}$:

- *Separateness*: for any Frob-cover $B' \to B$ the functor

$$\mathbf{RigDA}_{\text{ét}}(B, \Lambda) \to \mathbf{RigDA}_{\text{ét}}(B', \Lambda)$$

  is an equivalence of categories.
- *Rigidity*: if $\operatorname{char} K \nmid N$ the functor

$$\mathbf{D}\,\mathbf{Sh}_{\text{ét}}(\operatorname{Et}/B, \mathbb{Z}/N\mathbb{Z}) \to \mathbf{RigDA}_{\text{ét}}(B, \mathbb{Z}/N\mathbb{Z})$$

  is an equivalence of categories, where Et/B is the small étale site over B.

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