Quantitative Stability for Hypersurfaces with Almost Constant Mean Curvature in the Hyperbolic Space

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ABSTRACT. We provide sharp stability estimates for the Alexandrov Soap Bubble Theorem in the hyperbolic space. The closeness to a single sphere is quantified in terms of the dimension, the measure of the hypersurface and the radius of the touching ball condition. As consequence, we obtain a new pinching result for hypersurfaces in the hyperbolic space.

Our approach is based on the method of moving planes. In this context we carefully review the method and we provide the first quantitative study in the hyperbolic space.

CONTENTS

1. INTRODUCTION

In this paper we study compact embedded hypersurfaces in the hyperbolic space in relation to the mean curvature. The subject has been largely studied in literature (see, e.g., [5, 8, 10, 17–22, 25, 28, 29, 31–36] and the references therein).

Our starting point is the celebrated Alexandrov's theorem in the hyperbolic context.

Alexandrov's Theorem. *A connected closed C^2-regular hypersurface S embedded in the hyperbolic space has constant mean curvature if and only if it is a sphere.*

The theorem was proved by Alexandrov in [2] by using the method of moving planes and extends to the Euclidean space and the hemisphere [2–4]. The method uses maximum principles and consists in proving that the surface is symmetric in any direction. Then, the assertion follows by the following characterization of the sphere: a compact embedded hypersurface S in the hyperbolic space with center of mass $\mathcal{O}$ is a sphere if and only if, for every direction ω, there exists a hyperbolic hyperplane π_ω of symmetry of S orthogonal to ω at $\mathcal{O}$ (see Lemma 2.2).

In this paper, we study the method of moving planes in the hyperbolic space from a quantitative point of view, and we obtain sharp stability estimates for Alexandrov's theorem. We consider a C^2-regular, connected, closed hypersurface S embedded in the hyperbolic space. Since S is closed and embedded, there exists a bounded domain Ω such that $S = \partial\Omega$. We say that S (or equivalently Ω) satisfies a uniform touching ball condition of radius ρ if, for any point $p \in S$, there exist two balls B_ρ^- and B_ρ^+ of radius ρ, with B_ρ^- contained in Ω and B_ρ^+ outside Ω, which are tangent to S at p. Our main result is the following.

Theorem 1.1. *Let S be a C^2-regular, connected, closed hypersurface embedded in the n-dimensional hyperbolic space satisfying a uniform touching ball condition of radius ρ. There exist constants $\varepsilon, C > 0$ such that if the mean curvature H of S satisfies*

$$\operatorname{osc}(H) \le \varepsilon,$$

then there are two concentric balls B_r and B_R such that

$$S \subset \bar{\mathsf{B}}_R \setminus \mathsf{B}_r,$$

and

$$R - r \le C \operatorname{osc}(H). \tag{1.1}$$

The constants ε and C depend only on n, upper bounds on ρ^{-1}, and the area of S.

In Theorem 1.1, $\operatorname{osc}(H)$ is the oscillation of H, that is,

$$\operatorname{osc}(H) := \max_M H - \min_M H.$$

Note that the assumption $\operatorname{osc}(H) \le \varepsilon$ is equivalent to requiring that H be close to a constant in C^0-norm. We comment that the quantitative bound in (1.1) is sharp in the sense that no function of $\operatorname{osc}(H)$ converging to zero more than linearly can appear on the righthand side of (1.1), as can be seen by explicit calculations considering a small perturbation of the sphere. We prefer to state Theorem 1.1 by assuming that S is connected, but the theorem still holds if we just assume that Ω is connected (and the proof remains the same).

Theorem 1.1 has some remarkable consequences that we give in the following corollary.

Corollary 1.2. *Let $\rho_0, A_0 > 0$ and $n \in \mathbb{N}$ be fixed. There exists $\varepsilon > 0$, depending on n, ρ_0, and A_0, such that if S is a connected closed C^2 hypersurface embedded in the hyperbolic space having area bounded by A_0, satisfying a touching ball condition of radius $\rho \geq \rho_0$, and whose mean curvature H satisfies*

$$\operatorname{osc}(H) < \varepsilon,$$

then S is diffeomorphic to a sphere.

Moreover, S is $C^{1,\alpha}$-close to a sphere, that is, there exists a $C^{1,\alpha}$-map $\Psi\colon \partial\mathsf{B}_r \to \mathbb{R}$ such that

$$F(x) = \exp_x(\Psi(x)N_x)$$

defines a $C^{1,\alpha}$-diffeomorphism $F\colon \partial\mathsf{B}_r \to S$ and

$$\|\Psi\|_{C^{1,\alpha}(\partial B_r)} \leq C \operatorname{osc}(H), \tag{1.2}$$

for some $0 < \alpha < 1$ and where C depends only on n, ρ, and A_0.

Hence, the lower bound on ρ prevents any *bubbling* phenomenon, and Corollary 1.2 quantifies the proximity of S from a single bubble in a C^1 fashion.

As far as we know, our results are the first quantitative studies for almost constant mean curvature hypersurfaces in the hyperbolic space. We mention that, in the Euclidean space, almost constant mean curvature hypersurfaces have been recently studied in [9, 11, 12, 15, 26, 30]. In particular, Theorem 1.1 generalizes the results we obtained in [15] to the hyperbolic space. However, the generalization is not trivial. Indeed, even if a qualitative study of a problem via the method of moving planes in the hyperbolic space does not significantly differ from the Euclidean context, the quantitative study presents several technical differences which need to be tackled.

Now we describe the proof of Theorem 1.1. Here, we work in the half-space model

$$\mathbb{H}^n = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n \mid p_n > 0\}$$

equipped with the usual metric

$$g_p = \frac{1}{p_n^2} \sum_{k=1}^{n} \mathrm{d}p_k \otimes \mathrm{d}p_k.$$

Our approach consists in a quantitative study of the method of the moving planes (for the analogue approach in the Euclidean context see [1, 11, 13–15]). Our first crucial result is to prove approximate symmetry in one direction. Indeed, we fix a direction ω and we perform the moving plane method along the direction ω until we get a *critical* hyperplane π_ω (see Subsection 2.1 for a description of the method in the hyperbolic context). Possibly after applying an isometry we may assume π_ω to be the vertical hyperplane $\pi = \{p_1 = 0\}$. Hence, π intersects S, and the reflection of the righthand cap of S about π is contained in Ω and is tangent to S. More precisely, let $S_+ = S \cap \{p_1 \geq 0\}$ and $S_- = S \cap \{p_1 \leq 0\}$; then, the reflection of S_+ about π is contained in Ω and is tangent to S_- at a point p_0 (internally or at the boundary). If A is a set, we denote by A^π its reflection about π, and we use the following notation:

$$\hat{\Sigma} \text{ is the connected component of } S_- \text{ containing } p_0$$

and

$$\Sigma \text{ is the connected component of } S_+^\pi \text{ containing } p_0.$$

Furthermore, we denote by N the inward normal vector field on Σ. The inward normal vector field on $\hat{\Sigma}$ is still denoted by N, since no confusion arises. We prove the following theorem on the approximate symmetry in one direction.

Theorem 1.3. *There exists* $\varepsilon > 0$ *such that if*

$$\operatorname{osc}(H) \leq \varepsilon,$$

then for any $p \in \Sigma$ *there exists* $\hat{p} \in \hat{\Sigma}$ *such that*

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \leq C \operatorname{osc}(H).$$

Here, the constants ε *and* C *depend only on* n, ρ, *and the area of* S. *In particular,* ε *and* C *do not depend on the direction* ω.

Moreover, Ω *is contained in a neighborhood of radius* $C \operatorname{osc}(H)$ *of* $\Sigma \cup \Sigma^\pi$, *that is,*

$$d(p, \Sigma \cup \Sigma^\pi) \leq C \operatorname{osc}(H), \quad \text{for every } p \in \Omega.$$

In this last statement, $\tau_p^q \colon \mathbb{R}^n \to \mathbb{R}^n$ denotes the parallel transport along the unique geodesic path in $\mathbb{H}^n$ connecting p to q. We prove Theorem 1.3 by using quantitative tools for PDEs (like Harnack's inequality and quantitative versions of Carleson estimates and Hopf Lemma), as well as quantitative results for the parallel transport and graphs in the hyperbolic space.

In order to prove Theorem 1.1, we first define an approximate center of symmetry $\mathcal{O}$ by applying the moving planes procedure in n orthogonal directions. The argument here is not trivial, since n "orthogonal hyperplanes" do not necessarily intersect, and Theorem 1.3 come into play. Then, Theorem 1.3 is also used to prove that every critical hyperplane in the moving planes procedure is close to $\mathcal{O}$, and we finally prove estimates (1.1) by exploiting Theorem 1.3 again.

2. PRELIMINARIES

We recall some basic facts about the geometry of hypersurfaces in Riemannian manifolds. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇, and $i: S \hookrightarrow M$ be an embedded orientable hypersurface of class C^2. Fix a unitary normal vector field N on S. We recall that the *shape operator* of S at a point $p \in S$ is defined as

$$W_p(v) = -(\nabla_v \tilde{N}_p)^\perp \in T_pS$$

for $v \in T_pS$, where $\tilde{N}$ is an arbitrary extension of N in a neighborhood of p and the superscript "$\perp$" denotes the orthogonal projection onto T_pS. W_p is always symmetric with respect to g and the *principal curvatures* $\{\kappa_1(p), \dots, \kappa_{n-1}(p)\}$ of S at p are by definition eigenvalues of W_p. We recall that the lowest and the maximal principal curvature at p can be, respectively, obtained as the minimum and maximum of the map $\kappa_p : T_pS \setminus \{0\} \to \mathbb{R}$ defined as

$$\kappa_p(v) := -\frac{1}{|v|^2} g_p(W_p(v), v) = -\frac{1}{|v|^2} g_p(\nabla_v \tilde{N}_p, v).$$

Alternatively, $\kappa_p(v)$ can be defined by fixing a smooth curve $\alpha: (-\varepsilon, \varepsilon) \to S$ satisfying

$$\alpha(0) = p, \quad \dot{\alpha}(0) = v,$$

since in terms of α we can write

$$\kappa_p(v) = \frac{1}{|v|^2} g_p(N_p, \mathrm{D}_t \dot{\alpha}(0)),$$

where D_t denotes the covariant derivative on (M, g). The *main curvature* of S at p is then defined as

$$H(p) = \frac{\kappa_1(p) + \cdots + \kappa_{n-1}(p)}{n-1}.$$

From now on we focus on the hyperbolic space. Given a model of the hyperbolic space, we denote the hyperbolic metric by g, the hyperbolic distance by d, the hyperbolic norm at a point p by $|\cdot|_p$, and the ball of center p and radius r by $\mathsf{B}_r(p)$. The Euclidean inner product in $\mathbb{R}^n$ will be denoted by "$\cdot$" and the Euclidean norm by $|\cdot|$. The hyperbolic measure of a set A will be denoted by $|A|_g$.

We mainly work in the half-space model $\mathbb{H}^n$. In this model, hyperbolic balls and Euclidean balls coincide, but hyperbolic and Euclidean centers and the hyperbolic and Euclidean radii differ. Specifically, the Euclidean radius r_E of $\mathsf{B}_r(p)$ is

$$r_E = p_n \sinh r,$$

where $p = (p_1, \dots, p_n)$ are the coordinates of p in $\mathbb{R}^n$.

The Euclidean hyperplane $\{p_n = 0\} \subset \mathbb{R}^n$ will be denoted by π_∞ and the origin of π_∞ by O. Moreover, $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Given a point $p \in \mathbb{H}^n$, we denote by $\bar{p}$ its projection onto π_∞ and by $B_r(x)$ the (Euclidean) ball of π_∞ centered at $x \in \pi_\infty$ and having radius r. We omit to write the center of balls of π_∞ when they are centered at the origin, that is, $B_r(O) = B_r$.

Now we consider a closed C^2 hypersurface S embedded in $\mathbb{H}^n$. Given a point p in S we denote by T_pS its tangent space at p and by N_p the inward hyperbolic normal vector at p. Note that, according to our notation,

$$\nu_p := \frac{1}{p_n} N_p$$

is the Euclidean inward normal vector. We further denote by d_S the distance on S induced by the hyperbolic metric. Given a point $z_0 \in S$, we denote by $\mathcal{B}_r(z_0)$ the set of points on S with intrinsic distance from z_0 less than r, that is,

$$\mathcal{B}_r(z_0) = \{z \in S \mid d_S(z, z_0) < r\}.$$

We are going to prove several quantitative estimates by locally writing the hypersurface S as a Euclidean graph. Since this procedure is not invariant by isometries, we need to specify a "preferred" configuration in order to obtain uniform estimates. More precisely, such configuration is when $p = e_n \in S$ and $T_pS = \pi_\infty$; then, close to p, S is locally the Euclidean graph of a C^2-function $v\colon B_r \to \mathbb{R}$, and we denote by $\mathcal{U}_r(p)$ the graph of v. If p in S is an arbitrary point, then there exists an orientation-preserving isometry φ of $\mathbb{H}^n$ such that $\varphi(p) = e_n$ and $T_{\varphi(p)}\varphi(S) = \pi_\infty$. Hence, around $\varphi(p)$, $\varphi(S)$ is the graph of a C^2-map $v\colon B_r \to \mathbb{R}$, and we define $\mathcal{U}_r(p)$ as the preimage via φ of the graph of v. The definition of $\mathcal{U}_r(p)$ is well posed.

Lemma 2.1. *The definition of $\mathcal{U}_r(p)$ does not depend on the choice of φ.*

Proof. First, let $\mathcal{U}_r(p)$ be defined via an orientation-preserving isometry $\varphi\colon \mathbb{H}^n \to \mathbb{H}^n$ such that

$$\varphi(p) = e_n, \quad \varphi_{*|p}(T_pS) = \pi_\infty, \tag{2.1}$$

and let $\psi\colon \mathbb{H}^n \to \mathbb{H}^n$ be another orientation-preserving isometry satisfying (2.1). Then, $f = \psi \circ \varphi^{-1}$ is an orientation-preserving isometry of $\mathbb{H}^n$ satisfying

$$f(e_n) = e_n, \quad f_{|*}(\pi_\infty) = \pi_\infty,$$

and so it is a rotation about the e_n-axis. Therefore, $\psi(\mathcal{U}_r(p))$ is the graph of a C^2-map defined on a ball in π_∞ about the origin, and the claim follows. □

We denote by H the hyperbolic mean curvature of S. Note that H is related to the Euclidean mean curvature H_E by

$$H(p) = (\nu_p + pH_E(p)) \cdot e_n.$$

For instance, if S is the hyperbolic ball $\mathsf{B}_r(p)$ oriented by the inward normal, we have

$$H \equiv \frac{1}{\tanh r}, \quad H_E(p) = \frac{1}{p_n \sinh r}.$$

If S is locally the graph of a smooth function $v \colon B_r \to \mathbb{R}$, where B_r is a ball about the origin in π_∞, and $p = (x, v(x)) \in S$, then H at p takes the following expression:

$$H(p) = \frac{v(x)}{n-1} \operatorname{div}\left(\frac{\nabla v(x)}{\sqrt{1 + |\nabla v(x)|^2}}\right) + \frac{1}{\sqrt{1 + |\nabla v(x)|^2}}. \tag{2.2}$$

In the last expression, div and ∇ are the Euclidean divergence and gradient in π_∞, respectively. Moreover, we have

$$\nu_p = \frac{(-\nabla v(x), 1)}{\sqrt{|\nabla v(x)|^2 + 1}}.$$

Since S is compact and embedded, then it is the boundary of a bounded domain Ω in $\mathbb{H}^n$. Given p in S, we say that S *satisfies a touching ball condition of radius ρ at p* if there exist two hyperbolic balls of radius ρ tangent to S at p, one contained in Ω and one contained in the complement of Ω. Since S is compact, we have that S *satisfies a uniform touching ball condition of radius ρ* for some ρ, that is, it satisfies a touching ball condition of radius ρ at any point (see [16]).

2.1. Alexandrov's theorem and the method of moving planes in the hyperbolic space. In this paper, by *hyperplane* in the hyperbolic space we mean a totally geodesic hypersurface. In the half-space model $\mathbb{H}^n$, hyperplanes are either Euclidean half-spheres centered at a point in π_∞ or vertical planes orthogonal to π_∞, while in the ball model the hyperbolic hyperplanes are Euclidean spherical caps or planes orthogonal to the boundary of $\mathbb{B}^n$. Here, we recall that the *ball model* consists of $\mathbb{B}^n = \{p \in \mathbb{R}^n : |p| = 1\}$ equipped with the Riemannian metric

$$g_p = \frac{4}{(1 - |p|^2)^2} \sum_{k=1}^{n} \mathrm{d}p_k \otimes \mathrm{d}p_k.$$

If Ω is a bounded open set in the hyperbolic space, its *center of mass* is defined as the minimum point $\mathcal{O}$ of the map

$$P(p) = \frac{1}{2|\Omega|_g} \int_\Omega d(p, a)^2 \, \mathrm{d}a.$$

In view of [24], P is a convex function and the center of mass in unique. Furthermore, the gradient of P takes the expression

$$\nabla P(p) = -\frac{1}{|\Omega|_g}\int_\Omega \exp_p^{-1}(a)\,\mathrm{d}a. \tag{2.3}$$

Lemma 2.2. *Let Ω be a bounded open set in the hyperbolic space. Then, every hyperplane of symmetry of Ω contains the center of mass $\mathcal{O}$ of Ω.*

Proof. Even if the result is well known we give a proof for reader's convenience. We prove the statement in the ball model $\mathbb{B}^n$. Without loss of generality, we may assume that the center of mass $\mathcal{O}$ of Ω is the origin of $\mathbb{B}^n$. Assume by contradiction there exists a hyperplane π of symmetry for Ω not containing $\mathcal{O}$. Hence, π is a spherical cap which (up to a rotation) we may assume to be orthogonal to the line $(p_1, 0, \dots, 0)$ and lying in the half-space $p_1 > 0$. Let $\pi_1 = \{p_1 = 0\}$ be the vertical hyperplane orthogonal to e_1. Since π_1 and π are disjoint, they subdivide Ω in three subsets Ω_1, Ω_2, Ω_3, with $|\Omega_2|_g > 0$ (see Figure 2.1). Since Ω is

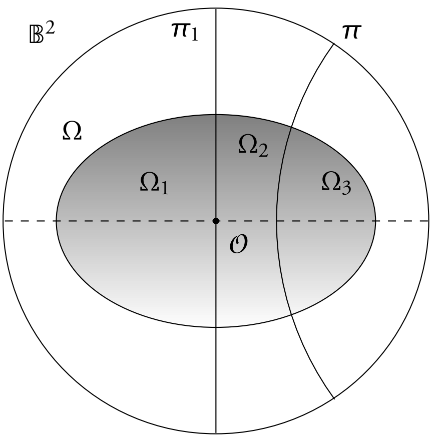

FIGURE 2.1.

symmetric about π, we have that $|\Omega_1|_g + |\Omega_2|_g = |\Omega_3|_g$. Moreover, since

$$\exp_{\mathcal{O}}^{-1}(p) = 2(\tanh^{-1}|p|)\frac{p}{|p|}, \quad \text{for every } p \in \mathbb{B}^n,$$

formula (2.3) implies

$$\int_{\Omega\cap\{p_1>0\}} (\tanh^{-1}|p|)\frac{p_1}{|p|}\,\mathrm{d}p = -\int_{\Omega\cap\{p_1<0\}} (\tanh^{-1}|p|)\frac{p_1}{|p|}\,\mathrm{d}p,$$

so that $|\Omega_1|_g = |\Omega_2|_g + |\Omega_3|_g$, which gives a contradiction. $\square$

Proposition 2.3. *Let $S = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in the n-dimensional hyperbolic space, where Ω is a bounded domain. Assume that for every direction $\omega \in \mathbb{R}^n$ there exists a hyperplane of symmetry of S orthogonal to ω at the center of mass $\mathcal{O}$ of Ω. Then, S is a hyperbolic sphere about $\mathcal{O}$.*

Proof. We prove the statement in the ball model $\mathbb{B}^n$, assuming that $\mathcal{O}$ is the origin of $\mathbb{B}^n$. In this case, the assumptions in the statement imply that S is symmetric about every Euclidean hyperplane passing through the origin. Thus, S is a Euclidean ball about $\mathcal{O}$ (see, e.g., [23, Lemma 2.2, Chapter VII]), and the claim follows. $\square$

Now, we give a description of the method of the moving planes in $\mathbb{H}^n$, declaring some notation we will use here and in Sections 6 and 7. The method consists in moving hyperbolic hyperplanes along a geodesic orthogonal to a fixed direction. Let ω be a fixed direction, and let $\gamma_\omega\colon (-\infty,\infty) \to \mathbb{H}^n$ be the maximal geodesic satisfying $\gamma(0) = e_n$, $\dot\gamma(0) = \omega$. For any $s \in \mathbb{R}$ we denote by $\pi_{\omega,s}$ the totally geodesic hyperplane passing through $\gamma_\omega(s)$ and orthogonal to $\dot\gamma_\omega(s)$.

The description of the method can be simplified by assuming $\omega = e_n$ (by using an isometry it is always possible to describe the method only for this direction). In this case, the hyperplane $\pi_{e_n,s}$ consists of a half-sphere

$$\pi_{e_n,s} = \{p \in \mathbb{H}^n : |p| = \mathrm{e}^s\}.$$

For s large enough, $S \subset \{|p| < \mathrm{e}^s\}$. We decrease the value of s until $\pi_{e_n,s}$ is tangent to S. Then, we continue to decrease s until the reflection $S^\pi_{e_n,s}$ of $S_{e_n,s} := S \cap \{|p| \geq \mathrm{e}^s\}$ about $\pi_{e_n,s}$ is contained in Ω, and we denote by π_{e_n} the hyperplane obtained at the limit configuration.

More precisely, for a general direction ω we define

$$m_\omega = \inf\{s \in \mathbb{R} \mid S^\pi_{\omega,s} \subset \Omega\},$$

and refer to $\pi_\omega := \pi_{\omega,m_\omega}$ and $S_\omega := S^\pi_{\omega,m_\omega}$ as to the *critical hyperplane* and *maximal cap* of S along the direction ω. Analogously, Ω_ω is addressed as the maximal cap of Ω in the direction ω. Note that by construction, the reflection S^π_ω of S_ω is tangent to S at a point p_0, and there are two possible configurations given by $p_0 \notin \pi_\omega$ and $p_0 \in \pi_\omega$.

Proof of Alexandrov's theorem. The proof is obtained by using the method of the moving planes described above and showing that, for every direction ω, we have that S is symmetric about π_ω. Once a direction ω is fixed, we may assume by using a suitable isometry that π_ω is the vertical hyperplane $\pi_\omega = \{x_1 = 0\}$ and $\omega = e_1$. We parametrize S and S^π_ω in a neighborhood of p_0 in $T_{p_0}S$ (which clearly coincides with $T_{p_0}S^\pi_\omega$) as graphs of two functions v and u, respectively. If $p_0 \notin \pi_\omega$ the functions v and u are defined on a ball B_r (case (i)); otherwise, they are defined in a half-ball $B_r \cap \{x_1 \le 0\}$ and $v = u$ on $B_r \cap \{p_1 = 0\}$ (case (ii)). In both cases the two functions v and u satisfy (2.2), and the difference $w = u - v$ is nonnegative and satisfies an elliptic equation $Lw = 0$, with $w(0) = 0$ in case (i) and $w = 0$ on $B_r \cap \{p_1 = 0\}$ in case (ii). The strong maximum principle in case (i) and Hopf's lemma in case (ii) yield $w \equiv 0$. This implies there exist two connected components of S_- and S_ω^π such that the set of tangency points between them is both closed and open. Since S is connected we also have that $S_\omega^\pi = S_-$, that is, S is symmetric about π_ω. The conclusion follows from Lemma 2.2 and Proposition 2.3. ◻

Remark 2.4. We mention that Alexandrov's theorem still holds by assuming that Ω is connected, and the proof given above can be easily modified accordingly.

Remark 2.5. In the defintion of the method of the moving planes one can replace e_n with an arbitrary point $p \in \mathbb{H}^n$ by replacing conditions $\gamma_\omega(0) = e_n$ and $\dot\gamma_\omega(0) = \omega$ with $\gamma_\omega(0) = p$ and $\dot\gamma_\omega(0) = \omega$, respectively.

Remark 2.6. The method of the moving planes described in this section differs from the method of moving planes described in [27], where the hyperplanes move along a horocycle instead of a geodesic. We comment that if one is interested in a qualitative result (such as Alexandrov's theorem), then the two methods are equivalent; instead, the method we adopt here is more suitable for a quantitative analysis of the problem.

3. Local Quantitative Estimates

In this section, we establish some local quantitative results that we need to prove Theorem 1.1. We will need to switch Euclidean and hyperbolic distances, and we need a preliminary lemma which quantifies their relation close to e_n. We recall that the hyperbolic distance d in the half-space model of $\mathbb{H}^n$ is given in terms of the Euclidean distance by the following formula:

$$d(p,q) = \operatorname{arccosh}\left(1 + \frac{|p-q|^2}{2p_nq_n}\right).$$

In particular,

$$d(e_n, te_n) = |\log t|, \quad \text{for any } t \in (0,\infty).$$

Lemma 3.1. *Let $R > 0$ be fixed, and let q in $\mathsf{B}_R(e_n)$. Then, there exist $c = c(R) > 0$ and $C = C(R) > 0$ such that*

$$c|q - e_n| \le d(q, e_n) \le C|q - e_n|. \tag{3.1}$$

Proof. Since $e^{-R} \le q_n \le e^R$, then

$$1 + \frac{e^{-R}}{2}|q-e_n|^2 \le 1 + \frac{|q-e_n|^2}{2q_n} \le 1 + \frac{e^R}{2}|q-e_n|^2,$$

and, since $|q - e_n| \leq e^R - 1$, then

$$1 + \frac{|q - e_n|^2}{2q_n} \leq A,$$

where $A = A(R)$. Let $\phi(t) = \operatorname{arccosh}(t)$, $t \in [1, +\infty)$. Since $1 \leq t \leq A$ then, keeping in mind that $\phi'(t) = (t^2 - 1)^{-1/2}$, we have

$$\frac{1}{\sqrt{A+1}} \frac{1}{\sqrt{t-1}} \leq \phi'(t) \leq \frac{1}{\sqrt{t-1}},$$

and hence

$$\frac{1}{2\sqrt{A+1}} \sqrt{t-1} \leq \phi(t) \leq \frac{1}{2} \sqrt{t-1} \quad t \in [1, A].$$

By letting

$$t = 1 + \frac{|q - e_n|^2}{2q_n},$$

and from

$$\frac{e^{-R/2}}{\sqrt{2}} |q - e_n| \leq \sqrt{t-1} \leq \frac{e^{-R/2}}{\sqrt{2}} |q - e_n|,$$

we conclude. □

3.1. Quantitative estimates for parallel transport. In this subsection, we prove quantitative estimates involving the parallel transport which will be useful in the proof of Theorem 1.3.

We recall that *the parallel transport* along a smooth curve $\alpha \colon [t_0, t_1] \to \mathbb{H}^n$ is the linear map $\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\tau(v) = X(t_1),$$

where $X \colon [t_0, t_1] \to \mathbb{H}^n$ is the solution to the linear ODE

$$\begin{cases} \dot{X}_k + \displaystyle\sum_{i,j=1}^{n} X_j \dot{\alpha}_i \Gamma_{ij}^k(\alpha) = 0, & k = 1, \dots, n, \\ X_k(t_0) = v_k, & k = 1, \dots, n, \end{cases}$$

and Γ_{ij}^k are the Christoffel symbols in $\mathbb{H}^n$. Here, we recall that the Γ_{ij}^k are all vanishing if either the three indexes i, j, k are distinct or one of them is different from n, while in the remaining cases they are given by

$$\Gamma_{in}^i = -\frac{1}{x_n}, \quad \Gamma_{ii}^n = \frac{1}{x_n}, \quad \Gamma_{ni}^i = -\frac{1}{x_n}, \quad \Gamma_{nn}^n = -\frac{1}{x_n}.$$

We adopt the following notation: given q and p in $\mathbb{H}^n$, we denote by

$$\tau_q^p : \mathbb{R}^n \to \mathbb{R}^n$$

the parallel transport along the unique geodesic path connecting q to p. Note that if q and p belong to the same vertical line (that is, if $\bar{q} = \bar{p}$ in our notation), then

$$\tau_q^p(v) = \frac{p_n}{q_n} v.$$

For the case $\bar{q} \neq \bar{p}$, we consider the following lemma where for simplicity we assume $p = e_n$.

Lemma 3.2. *Let $q \in \mathbb{H}^n$ be such that $q \in \langle e_{n-1}, e_n \rangle$, and let $v \in \mathbb{R}^n$. Assume $q_{n-1} \neq 0$; then,*

$$\tau_q^{e_n}(v) = \frac{1}{q_n}(v_1, \dots, v_{n-2}, \tilde{v}_{n-1}, \tilde{v}_n),$$

where

$$\begin{pmatrix} \tilde{v}_{n-1} \\ \tilde{v}_n \end{pmatrix} = \frac{1}{1+a^2} \begin{pmatrix} a(a-q_{n-1}) + q_n & a - q_{n-1} - aq_n \\ cq_n - a + q_{n-1} & a(a-q_{n-1}) + q_n \end{pmatrix} \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix}$$

and

$$a = \frac{|q|^2 - 1}{2q_{n-1}}.$$

Proof. Let $\alpha \colon [t_0, t_1] \to \mathbb{H}^n$ be defined as

$$\alpha(t) = (R\cos(t) + a)e_{n-1} + R\sin(t)e_n,$$

where

$$a = \frac{|q|^2 - 1}{2q_{n-1}}, \quad R = \sqrt{1 + a^2}$$

and

$$\alpha(t_0) = q, \quad \alpha(t_1) = e_n.$$

Then, α, up to being parametrized, is a geodesic path connecting q to e_n. The parallel transport equation along α yields

$$(\tau_q^{e_n}(v))_k = v_k, \quad k = 1, \dots, n-2,$$

while

$$(\tau_q^{e_n}(v))_{n-1} = X_{n-1}(t_1), \quad (\tau_q^{e_n}(v))_n = X_n(t_1),$$

where the pair (X_{n-1}, X_n) solves

$$\begin{pmatrix} \dot{X}_{n-1} \\ \dot{X}_n \end{pmatrix} = \begin{pmatrix} \operatorname{cotan} t & -1 \\ 1 & \operatorname{cotan} t \end{pmatrix} \begin{pmatrix} X_{n-1} \\ X_n \end{pmatrix}, \quad \begin{pmatrix} X_{n-1}(t_0) \\ X_n(t_0) \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} X_{n-1}(t) \\ X_n(t) \end{pmatrix} = A(t)A(t_0)^{-1} \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix}, \quad A(t) := \begin{pmatrix} \cos t \sin t & -\sin^2 t \\ \sin^2 t & \cos t \sin t \end{pmatrix},$$

and the claim follows. $\square$

The following two propositions give some quantitative estimates involving the map τ_q^p.

Proposition 3.3. *Let p and q in $\mathbb{H}^n$, and let ω be the global vector field $\omega_z = z_n e_1$. Then,*

$$|\omega_p - \tau_q^p(\omega_q)|_p \le C d(p, q),$$

where C depends on an upper bound on the distance between p and q.

Proof. Note that in the simple case where p and q belong to the same vertical line, the claim is trivial since $|\omega_p - \tau_q^p(\omega_q)|_p = 0$. We focus on the other case. Let $f\colon \mathbb{H}^n \to \mathbb{H}^n$ be

$$f(z) = \frac{1}{p_n}\mathcal{R}(z - \bar{p})$$

where $\mathcal{R}$ is a rotation around the e_n-axis such that

$$\mathcal{R}(q - \bar{p}) \in \langle e_{n-1}, e_n \rangle.$$

In this way, we have

$$f(p) = e_n, \quad f(q) \in \langle e_{n-1}, e_n \rangle, \quad f_{|*z}(\omega_z) = f(z)_n v \text{ for all } z \in \mathbb{H}^n,$$

where $v = \mathcal{R}(e_1)$. We set $f(q) = \hat{q}$ and we write $\hat{q} = \hat{q}_{n-1}e_{n-1} + \hat{q}_n e_n$. Now, $\hat{q}_{n-1} \neq 0$ and we can apply Lemma 3.2, obtaining

$$\tau_{\hat{q}}^{e_n}(\hat{q}_n v) = \Big(v_1, \dots, v_{n-2}, \frac{1}{1+a^2}(a(a - \hat{q}_{n-1}) + \hat{q}_n)v_{n-1}, \\ \frac{1}{1+a^2}(a\hat{q}_n - a + \hat{q}_{n-1})v_{n-1}\Big),$$

where

$$a = \frac{|\hat{q}|^2 - 1}{2\hat{q}_{n-1}}.$$

Furthermore, a direct computation gives

$$|v - \tau_{\hat{q}}^{e_n}(\hat{q}_n v)| = \frac{|v_{n-1}|}{\sqrt{1+a^2}}|\hat{q} - e_n|.$$

Since $|v| = 1$, keeping in mind Lemma 3.1, we have

$$\begin{aligned}|\omega_p - \tau_q^p(\omega_q)|_p = |v - \tau_{\hat{q}}^{e_n}(\hat{q}_n v)| &= \frac{|v_{n-1}|}{\sqrt{1+a^2}}|\hat{q} - e_n| \\ &\leq \frac{1}{c}d(e_n,\hat{q}) = \frac{1}{c}d(p,q),\end{aligned}$$

where c is a small constant depending on $d(e_n,\hat{q}) = d(p,q)$. Hence, the claim follows. □

Proposition 3.4. *Let q, $\hat{q}$, and z in $\mathbb{H}^n$ and $R > 0$ be such that*

$$q, \hat{q} \in \mathsf{B}_R(z).$$

Let $v, w \in \mathbb{R}^n$ be such that

$$|v|_q = |w|_{\hat{q}} = 1.$$

Then,

$$|\tau_q^z(v) - \tau_{\hat{q}}^z(w)|_z \leq C(d(z,q) + d(z,\hat{q}) + d(q,\hat{q}) + |v - \tau_{\hat{q}}^q(w)|_q),$$

where C is a constant depending only on R.

Proof. We first consider the case where the three points $q, \hat{q}, z$ belong to the same geodesic path. In this case, we may assume that $z = e_n$ and that q and $\hat{q}$ belong to the e_n axis, that is,

$$q = q_n e_n \quad \text{and} \quad \hat{q} = \hat{q}_n e_n.$$

Under these assumptions, we have

$$|\tau_q^z(v) - \tau_{\hat{q}}^z(w)|_z = \left|\frac{1}{q_n}v - \frac{1}{\hat{q}_n}w\right| = |v - \tau_{\hat{q}}^q(w)|_q$$

and the claim is trivial. Next, we focus on the case where the three points do not belong to the same geodesic path. Up to applying an isometry, we may assume: $z = e_n$, q, and $\hat{q}$ belong to the same vertical line and $z, q, \hat{q}$ belong to the plane $\langle e_{n-1}, e_n\rangle$. Note that $q_{n-1} = \hat{q}_{n-1} \neq 0$. In the next computation we denote by $\|\cdot\|$ the norm of linear operators $\mathbb{R}^n \to \mathbb{R}^n$ with respect to the Euclidean norm. Note that

$$\|\tau_q^z\| = \frac{1}{q_n}, \quad \|\tau_{\hat{q}}^z\| = \frac{1}{\hat{q}_n}, \quad |v - \tau_{\hat{q}}^q(w)|_q = \left|\frac{1}{q_n}v - \frac{1}{\hat{q}_n}w\right|.$$

Taking into account that $|v| = q_n$ and $|w| = \hat{q}_n$, we have

$$
\begin{aligned}
&|\tau^z_q(v) - \tau^z_{\hat{q}}(w)|_z \\
&\quad \le \left|1 - \frac{1}{q_n}\right| \, |\tau^z_q(v)| + \left|\tau^z_q\left(\frac{1}{q_n}v - \frac{1}{\hat{q}_n}w\right)\right| \\
&\qquad + \frac{1}{\hat{q}_n}|\tau^z_q(w) - \tau^z_{\hat{q}}(w)| + \left|\frac{1}{\hat{q}_n} - 1\right| \, |\tau^z_{\hat{q}}(w)| \\
&\quad \le |q_n - 1| \, \|\tau^z_q\| + \|\tau^z_q\| \left|\frac{1}{q_n}v - \frac{1}{\hat{q}_n}w\right| + \|\tau^z_q - \tau^z_{\hat{q}}\| + |\hat{q}_n - 1| \, \|\tau^z_{\hat{q}}\| \\
&= \qquad \frac{1}{q_n}(|q_n - 1| + |v - \tau^q_{\hat{q}}(w)|_q + \frac{|\hat{q}_n - 1|}{\hat{q}_n} + \|\tau^z_q - \tau^z_{\hat{q}}\|.
\end{aligned}
$$

From Lemma 3.2, we have that $\|\tau^z_q - \tau^z_{\hat{q}}\| \le C d(q, \hat{q})$, where C is a constant depending only on R, and from Lemma 3.1 we conclude. □

3.2. Local quantitative estimates for hypersurfaces. In this subsection we prove some quantitative estimates for hypersurfaces in the hyperbolic space.

Throughout this subsection, S denotes a C^2-regular closed hypersurface embedded in $\mathbb{H}^n$ satisfying a uniform touching ball condition of radius ρ. We notice that the hyperbolic ball of radius ρ centred at $q = (\bar{q}, q_n)$ of radius ρ is the Euclidean ball of radius $q_n \sinh(\rho)$ centred at $(\bar{q}, q_n \cosh \rho)$.

Furthermore, we set

$$\rho_0 = e^{-\rho} \sinh \rho, \tag{3.2}$$

$$\rho_1 = (1 - \rho_0)\rho_0. \tag{3.3}$$

Notice that ρ_0 is the Euclidean radius of a hyperbolic ball of radius ρ with center at $(0, \dots, 0, e^{-\rho})$. Therefore, if e_n belongs to S, then S satisfies an Euclidean touching ball condition of radius ρ_0 at e_n.

Note that, since S satisfies a uniform touching ball condition of radius ρ, every geodesic ball $\mathcal{B}_r(p)$ of radius $r \le \rho_0$ in S is such that

$$|\mathcal{B}_r(p)| \ge c r^{n-1}, \tag{3.4}$$

where c depends only on n. The inequality can be easily proved assuming $p = e_n$ and $T_pS = \pi_\infty$ and then applying Lemma 3.1.

Lemma 3.5. *Assume $e_n \in S$ and $T_{e_n}S = \pi_\infty$. Then, S can be locally written around e_n as the graph of a C^2-function $v : B_{\rho_1} \subset \pi_\infty \to \mathbb{R}$, satisfying*

$$v(O) = 1, \quad |v(x) - 1| \le \rho_1 - \sqrt{\rho_1^2 - |x|^2}, \quad |\nabla v(x)| \le \frac{|x|}{\sqrt{\rho_1^2 - |x|^2}} \tag{3.5}$$

for every $x \in B_{\rho_1}$.

Proof. Since S satisfies a touching ball condition of radius ρ, we have that any point $q \in S \cap (B_{\rho_0} \times (1-\rho_0, 1+\rho_0))$ satisfies a Euclidean touching ball condition of radius ρ_1. The claim then follows from [15, Lemma 2.1]. □

Note: according to the terminology introduced in the first part of Section 2, the graph of the map v in the statement above is denoted by $\mathcal{U}_{\rho_1}(e_n)$.

Proposition 3.6. *There is $\delta_0 = \delta_0(\rho)$ such that if $p, q \in S$ with $d_S(p,q) \leq \delta_0$, then*

$$
\begin{aligned}
g_p(N_p, \tau_q^p(N_q)) &\geq \sqrt{1 - C^2 d_S(p,q)^2}, \\
|N_p - \tau_q^p(N_q)|_p &\leq C d_S(p,q),
\end{aligned} \tag{3.6}
$$

where C is a constant depending only on ρ.

Proof. We will choose $\delta_0 = \min(r_2, 1/C)$ (see below for the definition of r_2 and C).

Possibly after applying an isometry, we can assume that $p = e_n$ and $q = te_n$. We notice that any point in S which is far from e_n less than ρ satisfies a Euclidean touching ball condition of radius r_1, where r_1 depends only on ρ. Moreover, from Lemma 3.1, there exists $0 < r_2 = r_2(\rho)$ such that if $d(e_n, q) \leq r_2$, then $|e_n - q| \leq r_1/2$; this implies that, since

$$
d(p,q) \leq d_S(p,q) \leq r_2,
$$

we have

$$
|1 - t| = |p - q| \leq \frac{r_1}{2}.
$$

Now we can apply the Euclidean estimates in [15, Lemma 2.1] to p and q (with r_1 in place of ρ), and we obtain

$$
\nu_p \cdot \nu_q \geq \sqrt{1 - \frac{|p-q|^2}{r_1^2}}.
$$

Since $d(p,q) \leq \rho$, from (3.1) we have that $|p - q| \leq C_1 d(p,q) \leq C_1 d_S(p,q)$ for some constant $C_1 = C_1(\rho)$, and hence

$$
\nu_p \cdot \nu_q \geq \sqrt{1 - C^2 d_S(p,q)^2}, \tag{3.7}
$$

where $C = C_1/r_1$ and provided that $d_S(p,q) < 1/C$. Since

$$
N_p = \nu_p, \quad \nu_q = \frac{1}{t} N_q = \tau_q^p(N_q),
$$

inequality (3.7) can be written as

$$
g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2 d_S(p,q)^2},
$$

which is the first inequality in (3.6). The second inequality in (3.6) follows by a direct computation. ◻

Lemma 3.7. *For any* $0 < \alpha < \frac{1}{2}\min(1, \rho_1^{-1})$, *there exists a universal constant* C *such that if* $q \in \mathcal{U}_{\alpha\rho_1}(p)$, *then*

$$d_S(p,q) \le \alpha C \rho_1 \tag{3.8}$$

and

$$d(p,q) \le d_S(p,q) \le C\cosh(\rho_1) d(p,q). \tag{3.9}$$

Proof. Possibly after applying an isometry, we can assume that $p = e_n$ and $\nu_p = e_n$. Lemma 3.5 implies that S is the graph of a C^2 function $v\colon B_{\rho_1} \to \mathbb{R}$. Let $q = (x, v(x))$ with $|x| < \rho_1$ (so that $q \in \mathcal{U}_{\rho_1}(p)$), and consider the curve $\gamma\colon [0,1] \to \mathcal{U}_{\rho_1}(p)$ joining p with q, defined by $\gamma(t) = (tx, v(tx))$. Then,

$$\dot\gamma(t) = (x, \nabla v(tx) \cdot x).$$

The Cauchy-Schwartz inequality implies

$$|\dot\gamma(t)| \le |x|\sqrt{1 + |\nabla v(tx)|^2}.$$

Therefore, inequality (3.5) in Lemma 3.5 implies

$$|\dot\gamma(t)| \le \frac{\rho_1 |x|}{\sqrt{\rho_1^2 - t^2|x|^2}} \le \frac{|x|}{\sqrt{1-\alpha^2}} \le \frac{2}{\sqrt{3}}|x|,$$

for $0 \le |x| \le \alpha\rho_1$. Since

$$d_S(p,q) \le \int_0^1 \frac{|\dot\gamma(t)|}{v(tx)}\,\mathrm{d}t,$$

and from (3.5), we obtain that

$$d_S(p,q) \le C|x|$$

for some universal constant C, which implies (3.8). Since

$$|x| \le |p - q|,$$

a careful analysis of the constant appearing in (3.1) gives (3.9). ◻

Lemma 3.8. *Assume* $p = te_n \in S$, *for some* $t \in [1,\infty)$, *and assume* ν_p *is such that*

$$\nu_p \cdot e_n > 0, \quad |\nu_p - e_n| \le \varepsilon,$$

for some $0 \le \varepsilon < 1$. *Then, in a neighborhood of* p, *there exists a* C^2*-function* $v : B_r \to \mathbb{R}$, *with* $r = \rho_1\sqrt{1-\varepsilon^2}$, *such that* $p = (0, v(0))$ *and* S *is locally the graph of* v.

Proof. Notice that if $d_S(p,q) \leq \log(1-\rho_0)$, then $q_n \geq 1-\rho_0$ and q satisfies a Euclidean touching ball condition of radius ρ_1. The claim then follows from the Euclidean case (see [15, Lemma 3.4]). □

4. CURVATURES OF PROJECTED SURFACES

In order to perform a quantitative study of the method of the moving planes, we need to handle the following situation: given a hypersurface U of class C^2 in $\mathbb{H}^n$, we consider its intersection U' with a hyperbolic hyperplane π. If π intersects U transversally, $U' = U \cap \pi$ is a hypersurface of class C^2 of π, and we consider its Euclidean orthogonal projection U'' onto π_∞ (see Figure 4.1 for an example in $\mathbb{H}^3$). The next propositions allow us to control the Euclidean principal curvature

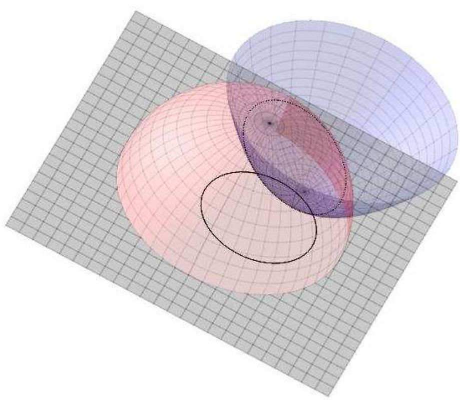

FIGURE 4.1. This figure provides an example of the statement of Proposition 4.1. According to the notation of Proposition 4.1, here U is the paraboloid in $\mathbb{H}^3$ parametrized by $\chi(u,v) = (v\cos(u), \frac{1}{2} - v\sin(u), v^2 + \frac{1}{2})$, and π is the half-sphere about the origin of radius one.

of U'' in terms of the hyperbolic principal curvature of U.

Proposition 4.1. *Let U be a C^2-regular embedded hypersurface in $\mathbb{H}^n$ oriented by a unitary normal vector field N. Let κ_j, $j = 1, \dots, n-1$, be the principal curvatures of U ordered increasingly, π be a hyperplane in $\mathbb{H}^n$ intersecting U transversally, and $U' = U \cap \pi$. Then, U' is an orientable hypersurface of class C^2 embedded in π and, once a unitary normal vector filed N' on U' in π is fixed, its principal curvatures κ_i' satisfy*

$$\frac{1}{g_q(N_q, N_q')}\kappa_1(q) \leq \kappa_i'(q) \leq \frac{1}{g_q(N_q, N_q')}\kappa_{n-1}(q) \tag{4.1}$$

for every $q \in U'$ and $i = 1, \dots, n-2$. Furthermore, once a unitary normal vector field ω on π is fixed, we have

$$\frac{1}{\sqrt{1 - g_q(\omega_q, N_q)^2}}\kappa_1(q) \le \kappa_i'(q) \le \frac{1}{\sqrt{1 - g_q(\omega_q, N_q)^2}}\kappa_{n-1}(q), \tag{4.2}$$

for every $q \in U'$ and a suitable choice of N'.

Proof. Up to applying an isometry, we may assume that π is the vertical hyperplane $\{p_1 = 0\}$.

First, observe that U' is of class C^2 by the implicit function theorem, and is orientable since

$$N_q' = (-1)^n \frac{*(*(\nu_q \wedge \partial_{x_1}) \wedge \partial_{x_1})}{|*(*(\nu_q \wedge \partial_{x_1}) \wedge \partial_{x_1})|_q}$$

defines a unitary normal vector field on U', where $\nu_q = (1/q_n)N_q$ is the Euclidean normal vector filed on U and $*$ is the Euclidean Hodge star operator in $\mathbb{R}^n$.

In order to prove (4.1): fix $q \in U'$ and consider a vector $v \in T_qU'$ satisfying $|v|_q = 1$. Set

$$\kappa_q(v) = g_q(\nabla_v \tilde{N}, v),$$

where $\tilde{N}$ is an arbitrary extension of N in a neighborhood of q and ∇ is the Levi-Civita connection of g. Since N_q is orthogonal to T_qU', it belongs to the plane generated by ∂_{x_1} and N_q', and we can write

$$N = a\,\partial_{x_1} + bN', \quad \text{where } b = g(N, N').$$

Let $\tilde{a}$, $\tilde{b}$, and $\tilde{N}'$ be arbitrary extensions of a, b, and N' in the whole $\mathbb{H}^n$. Therefore,

$$\tilde{N} = \tilde{a}\,\partial_{x_1} + \tilde{b}\tilde{N}'$$

is an extension of N. We have

$$\begin{aligned}\kappa_q(v) = g_q(\nabla_v \tilde{N}, v) &= g_q(\nabla_v(\tilde{a}\,\partial_{x_1} + \tilde{b}\tilde{N}'), v)\\ &= v(\tilde{a})g_q(\partial_{x_1}, v) + v(\tilde{b})g_q(N_q', v)\\ &\quad + a(q)g_q(\nabla_v\,\partial_{x_1}, v) + b(q)g_q(\nabla_v \tilde{N}', v)\\ &= a(q)g_q(\nabla_v\,\partial_{x_1}, v) + b(q)g_q(\nabla_v \tilde{N}', v).\end{aligned}$$

Since π is a totally geodesic submanifold, $g_q(\nabla_v\,\partial_{x_1}, v) = 0$, and therefore

$$\kappa_q(v) = g_q(N_q, N_q')g_q(\nabla_v \tilde{N}', v),$$

which implies (4.1).

Now, we prove (4.2). Let $\nu'_q = (1/q_n)N'_q$. Then, ν' is a Euclidean unitary normal vector field on U', and a standard computation yields

$$\nu_q \cdot \nu'_q = 1 - (\nu_q \cdot e_1)^2$$

(see, e.g., [15, Section 2.3]). Therefore, if $\omega_q = q_n e_1$, then

$$g_q(N_q, N'_q) = \nu_q \cdot \nu'_q = 1 - (\nu_q \cdot e_1)^2 = 1 - g_q(N_q, \omega_q)^2,$$

and (4.2) follows. ◻

Note that in the statement of Proposition 4.1, the κ'_i are the curvatures of U' once it is considered a hypersurface of π and not when it is seen as hypersurface of U. A bound on the principal curvatures of U' as hypersurface in U is given by the following proposition.

Proposition 4.2. *Under the same assumptions of Proposition 4.1, the principal curvatures $\check{\kappa}'_i$ of U' seen as a hypersurface of U satisfy*

$$|\check{\kappa}'_i(q)| \le \frac{|g_q(\omega_q, N_q)|}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\},$$

where ω is a normal unitary vector field to π.

Proof. We prove the statement, assuming π to be the vertical hyperplane $\{p_1 = 0\}$ and $\omega_p = p_n e_1$, for $p \in \pi$. Let $q \in U'$, $v \in T_qU'$ such that $|v|_q = 1$, and let $\alpha\colon (-\delta, \delta) \to S$ be a unitary speed curve satisfying $\alpha(0) = q$, $\dot\alpha(0) = v$. Fix a unitary normal vector field $\check{N}'$ of U' in U near q. We may complete v with an orthonormal basis $\{v, v_2, \dots, v_{n-2}\}$ of T_qU' such that

$$\check{N}'_q = *_q(N_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2}),$$

where $*_q$ is the Hodge star operator at q in $\mathbb{H}^n$ with respect to g and the standard orientation. Set

$$\check{\kappa}'_q(v) = g_q(*_q(\check{N}_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2}), \mathrm{D}_t\dot\alpha_{|t=0}),$$

where D_t is the covariant derivative in $\mathbb{H}^n$. Since $\mathrm{D}_t\dot\alpha_{|t=0} \in \pi$, we have

$$\check{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(*_q(\omega_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2}), \mathrm{D}_t\dot\alpha_{|t=0}).$$

Now, $*_q(\omega_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2})$ is a normal vector to T_qU' in π, and so

$$\check{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(\nabla_v \check{N}, v),$$

where $\tilde{N}$ is an arbitrary extension of N in a neighborhood of q. Proposition 4.1 then implies

$$|\check{\kappa}'_q(v)| \leq \frac{|g_q(N_q, \omega_q)|}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\},$$

as required. $\square$

Before giving the last result of this section, we recall the following notation introduced in the first part of the paper: given a point $q \in \mathbb{H}^n$, we denote by $\bar{q}$ its orthogonal projection onto π_∞, that is,

$$q = (\bar{q}, q_n).$$

Proposition 4.3. *Let π be a non-vertical hyperplane in $\mathbb{H}^n$, and U' be a C^2 regular hypersurface of π oriented by a unitary normal vector field N' in π. Denote by κ'_i, for $i = 1, \dots, n-2$, the principal curvatures of U'. Then, the Euclidean orthogonal projection U'' of U' onto π_∞ is a C^2-regular hypersurface of π_∞ with a canonical orientation. Moreover, for any $q \in U'$ we have*

$$|\kappa''_i(\bar{q})| \leq \frac{1}{R}\left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-3/2} (\max\{|\kappa'_1(q)|, |\kappa'_{n-2}(q)|\} + 3), \tag{4.3}$$

for every $i = 1, \dots, n-2$, where $\{\kappa''_i\}$ are the principal curvatures of U'' with respect to the Euclidean metric and R is the Euclidean radius of π and $\nu'_q = (1/q_n)N'_q$.

Proof. By our assumptions, π is a half-sphere of radius R with center in π_∞. By considering a suitable isometry, we may assume that π has center at the origin of π_∞. If X is a local positive oriented parametrisation of U', then we have that $\bar{X} = X - (X \cdot e_n)e_n$ is a local parametrisation of U'', and we can orient U'' with

$$\nu'' \circ \bar{X} := \mathrm{vers}(*(\bar{X}_1 \wedge \bar{X}_2 \wedge \cdots \wedge \bar{X}_{n-2} \wedge e_n)), \tag{4.4}$$

where $\bar{X}_k$ is the k^{th} derivative of $\bar{X}$ with respect to the coordinates of its domain, and $*$ is the Hodge "star" operator in $\mathbb{R}^n$ with respect to the Euclidean metric and the standard orientation. Therefore, U'' is a C^2-regular hypersurface of π_∞ oriented by the map ν''.

Now, we prove inequalities (4.3). We fix a point $q = (\bar{q}, q_n) \in U'$ and let $\bar{v} \in T_{\bar{q}}U'$ be nonzero. Let $\beta\colon (-\delta, \delta) \to U''$ be an arbitrary regular curve contained in U'' such that

$$\beta(0) = \bar{q}, \quad \dot{\beta}(0) = \bar{v}.$$

Then,

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{|\bar{v}|^2}\nu''_{\bar{q}} \cdot \ddot{\beta}(0)$$

is the normal curvature of U'' at $(\bar{q}, \bar{v})$, viewed as hypersurface of π_∞ with the Euclidean metric. We can write

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{|\bar{v}|^2}\nu''_{\bar{q}} \cdot \ddot{\alpha}(0)$$

where $\alpha = (\beta, \alpha_n)$ is a regular curve in U' projecting onto β. From

$$\bar{X}_k = X_k - (X_k \cdot e_n)e_n,$$

and the definition of ν'' (4.4), we have

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{(*(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n)) \cdot \ddot{\alpha}(0)}{|\dot{\beta}|^2\,|X_1(\alpha) \wedge \cdots \wedge X_{n-2}(\alpha) \wedge e_n|}.$$

We may assume that $\{X_1(q), \dots, X_{n-2}(q)\}$ is an orthonormal basis of T_qU' with respect to the Euclidean metric. Therefore, $\{X_1(q), \dots, X_{n-2}(q), \nu'_q, q/R\}$ is a Euclidean orthonormal basis of $\mathbb{R}^n$, and we can split $\mathbb{R}^n$ in

$$\mathbb{R}^n = T_qU'' \oplus \langle \nu'_q \rangle \oplus \langle q/R \rangle.$$

Then, e_n splits accordingly into

$$e_n = e'_n + e''_n + e'''_n, and$$

therefore,

$$\begin{aligned} &* (X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) \\ &\qquad = *(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e'''_n) \cdot \ddot{\alpha}(0), \end{aligned}$$

that is,

$$\begin{aligned} &* (X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) = \\ &\qquad = \frac{q_n}{R} * \left(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \frac{q}{R}\right) \cdot \ddot{\alpha}(0). \end{aligned}$$

Since

$$\nu'_q = * \left(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \frac{q}{R}\right),$$

we obtain

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{q_n}{R|\dot{\beta}(0)|^2} \frac{\nu'_q \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|}.$$

We may assume that α is parametrized by arc length with respect to the hyperbolic metric, that is,

$$|\dot{\alpha}|^2 = \alpha_n^2,$$

and so

$$|\dot{\beta}|^2 = \alpha_n^2 - \dot{\alpha}_n^2,$$

which implies

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{q_n}{r(q_n^2 - v_n^2)} \frac{\nu'_q \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|}.$$

Finally,

$$\begin{aligned} X_1(q) \wedge \cdots \wedge X_{n-1}(q) \wedge e_{n+1} &= X_1(q) \wedge \cdots \wedge X_{n-1}(q) \wedge e''_{n+1} \\ &\quad + X_1(q) \wedge \cdots \wedge X_{n-1}(q) \wedge e'''_{n+1}, \end{aligned}$$

and

$$\begin{aligned} X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e''_n &= (\nu'_q \cdot e_n) X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \nu'_q, \\ X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e'''_n &= \frac{q_n}{R} X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \frac{q}{R}. \end{aligned}$$

Hence,

$$|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n| = \left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2} \right)^{1/2}.$$

Now, we set

$$\kappa'_q(v) = g_q(N'_q, \mathrm{D}_t \dot{\alpha}_{|t=0}),$$

where D_t is the covariant derivative in π. We have

$$\begin{aligned} \mathrm{D}_t \dot{\alpha} &= \ddot{\alpha} + \sum_{i,j,k=1}^{n} \Gamma_{ij}^k(\alpha) \dot{\alpha}_i \dot{\alpha}_j e_k \\ &= \ddot{\alpha} + \sum_{i=1}^{n} \left(-\frac{2}{\alpha_n} \dot{\alpha}_i \dot{\alpha}_n \right) e_i + \frac{1}{\alpha_n} \Big(\sum_{i=1}^{n} \dot{\alpha}_i^2 - \dot{\alpha}_n^2 \Big) e_n \end{aligned}$$

and

$$\mathrm{D}_t \dot{\alpha}_{|t=0} = \ddot{\alpha}(0) - 2 \frac{v_n}{q_n} v + \frac{1}{q_n} (q_n^2 - v_n^2) e_n.$$

Therefore,

$$\begin{aligned} \kappa'_q(v) &= g_q \left(N'_q, \ddot{\alpha}(0) - 2 \frac{v_n}{q_n} v + \frac{1}{q_n} (q_n^2 - v_n^2) e_n \right) \\ &= \frac{1}{q_n} \nu'_q \cdot \ddot{\alpha}(0) - 2 \frac{v_n}{q_n^2} \nu'_q \cdot v + \frac{q_n^2 - v_n^2}{q_n^2} \nu'_q \cdot e_n, \end{aligned}$$

and from

$$\nu'_q \cdot \ddot{\alpha}(0) = q_n \kappa'_q(v) + 2\frac{v_n}{q_n}\nu'_q \cdot v - \frac{q_n^2 - v_n^2}{q_n}\nu'_q \cdot e_n,$$

we get

$$\begin{aligned}\kappa''_{\bar{q}}(\bar{v}) = {} & \frac{q_n}{R(q_n^2 - v_n^2)}\left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-1/2} \\ & \times \left(q_n \kappa'_q(v) + 2\frac{v_n}{q_n}\nu'_q \cdot v - \frac{q_n^2 - v_n^2}{q_n}\nu'_q \cdot e_n\right)\end{aligned}$$

for every $v \in T_qU'$, $g_q(v,v) = 1$. Therefore,

$$\begin{aligned}\kappa''_1(\bar{q}) &= \frac{q_n^2}{R}\left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-1/2} \inf_{v \in \mathbb{S}_q^{n-2}} A_q(v), \\ \kappa''_{n-2}(\bar{q}) &= \frac{q_n^2}{R}\left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} A_q(v),\end{aligned}$$

where

$$A_q(v) = \frac{1}{(q_n^2 - v_n^2)}\left(\kappa'_q(v) + 2\frac{v_n}{q_n^2}\nu'_q \cdot v - \frac{q_n^2 - v_n^2}{q_n^2}\nu'_q \cdot e_n\right)$$

and $\mathbb{S}_q^{n-2} = \{v \in T_qU : |v|_q = 1\}$. Since

$$|\kappa''_i(\bar{q})| \le \max\{|\kappa''_1(\bar{q})|, |\kappa''_{n-2}(\bar{q})|\}, \quad i = 1, \dots, n-2,$$

we obtain

$$|\kappa''_i(\bar{q})| \le \frac{q_n^2}{R}\left((\nu'_q \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} |A_q(v)|. \tag{4.5}$$

We have

$$\begin{aligned}|A_q(v)| &\le \frac{1}{|q_n^2 - v_n^2|}\left(|\kappa'_q(v)| + 2\frac{v_n}{q_n} + \frac{q_n^2 - v_n^2}{q_n^2}\right) \\ &\le \frac{1}{|q_n^2 - v_n^2|}(|\kappa'_q(v)| + 3),\end{aligned}$$

where we have used $q_n^2 - v_n^2 > 0$, since $|v|_q = 1$. Since $\mathbb{R}^n = T_qU' \oplus \langle \nu'_q \rangle \oplus \langle q/R \rangle$, we have that

$$q_n^2 - v_n^2 \ge \left[\left(\frac{q_n}{R}\right)^2 + (\nu'_q \cdot e_n)^2\right] q_n^2,$$

and then from (4.5) we find

$$|\kappa_i''(\bar{q})| \leq \frac{1}{R}\left((\nu_q' \cdot e_n)^2 + \frac{q_n^2}{R^2}\right)^{-3/2} \Big(\sup_{v \in \mathbb{S}_q^{n-2}} |\kappa_q'(v)| + 3\Big),$$

which implies (4.3). □

Remark 4.4. We will use the previous proposition in the following way: if there exists a constant c such that $\nu_q' \cdot e_n \geq c$, then (4.3) implies

$$|\kappa_i''(\bar{q})| \leq \frac{1}{c^3 R} \max\{|\kappa_1'(q)|, |\kappa_{n-2}'(q)| + 2\}, \quad i = 1, \dots, n-2.$$

5. Proof of Theorem 1.3

The set-up is the following: let $S = \partial\Omega$ be a C^2-regular closed hypersurface embedded in $\mathbb{H}^n$, where Ω is a bounded open set. We assume that S satisfies a uniform touching ball condition of radius $\rho > 0$.

Let $\pi := \{p_1 = 0\}$ be the critical hyperplane in the method of moving planes along the direction e_1, and let $S_- = S \cap \{p_1 \leq 0\}$ and S_+^π be the reflection of $S_+ = S \cap \{p_1 \geq 0\}$ about π. From the method of moving planes we have that S_+^π is contained in Ω and tangent to S_- at a point p_0 (internally or at the boundary). Let Σ and $\hat{\Sigma}$ be the connected component of S_+^π and S_- containing p_0, respectively.

5.1. Preliminary lemmas Before giving the proof of Theorem 1.3, we need some preliminary results about the geometry of Σ.

For $t > 0$ we set

$$\Sigma_t = \{p \in \Sigma \mid d_\Sigma(p, \partial\Sigma) \geq t\}.$$

The following three lemmas show quantitatively that Σ_t is connected for t small enough.

Lemma 5.1. *Assume*

$$\nu_p \cdot e_1 \leq \mu \tag{5.1}$$

for every p on the boundary of Σ, for some $\mu \leq \frac{1}{2}$, and let $t_0 = \rho\sqrt{1-\mu^2}$. Then, Σ_t is connected for any $0 < t < t_0$.

Proof. Let $\mathrm{pr}\colon \Sigma \to \pi$ be the projection from Σ onto π. Given $p \in \Sigma$, $\mathrm{pr}(p)$ is defined as the closest point in π to p. Then, the boundary of $\mathrm{pr}(\Sigma)$ in π coincides with the boundary $\partial\Sigma$ of Σ in S. Proposition 4.1 implies

$$|\kappa_i'(p)| \leq \frac{1}{\sqrt{1-(\nu_p \cdot e_1)^2}} \max\{|\kappa_1(p)|, |\kappa_{n-1}(p)|\},$$

for any $p \in \partial\Sigma$ and $i = 1, \dots, n-1$, where κ_i' are the principal curvatures of $\partial\Sigma$ viewed as a hypersurface of π. The touching ball condition on S yields

$$|\kappa_i'(p)| \leq \frac{1}{\rho\sqrt{1-(\nu_p \cdot e_1)^2}}, \tag{5.2}$$

for $i = 1, \dots, n-1$. As any point $p \in \partial\Sigma$ satisfies a touching ball condition of radius ρ (considered as a point of S), the transversality condition (5.1) and (5.2) imply $\mathrm{pr}(\Sigma)$ enjoys a touching ball condition of radius $\rho' \geq \rho\sqrt{1-(\nu_p \cdot e_1)^2} \geq t_0$. Therefore, if $s < t_0$,

$$C_s = \{z \in \pi \mid d(z, \partial\Sigma) < s\}$$

is a collar neighborhood of $\partial\Sigma$ in $\mathrm{pr}(\Sigma)$ of radius s. Since π is a critical hyperplane in the method of the moving planes, if p belongs to the maximal cap S_+, then any point on the geodesic path connecting p to its projection onto π is contained in the closure of Ω. It follows that $\mathrm{pr}^{-1}(C_s)$ contains a collar neighborhood of $\partial\Sigma$ of radius s in Σ, and, for $t \leq s$, Σ can be retracted in Σ_t and the claim follows. □

Lemma 5.2. *There exists $\delta > 0$ depending only on ρ with the following property. Assume there exists a connected component Γ_δ of Σ_δ such that*

$$0 \leq \nu_q \cdot e_1 \leq \frac{1}{8}, \quad \text{for any } q \in \partial\Gamma_\delta. \tag{5.3}$$

Then, Σ_δ is connected.

Proof. Let $\delta \leq \delta_0(\rho)$, where δ_0 is the bound appearing in Proposition 3.6. In view of (5.3), we can choose a smaller δ (in terms of ρ) such that the interior and exterior touching balls at an arbitrary q in $\partial\Gamma_\delta$ intersect π, which implies that $\Sigma \setminus \Gamma_\delta$ is enclosed by π and the set obtained as the union of all the exterior and interior touching balls to S^π (recall that Σ is a subset of the reflection S^π of S about π) of radius ρ at the points on Γ_δ. Since δ is chosen small in terms of ρ, this implies that for any $p \in \Sigma \setminus \Gamma_\delta$, there exists $q \in \partial\Gamma_\delta$ such that $d_\Sigma(p,q) \leq \delta$, and from (3.6) we have that

$$|N_p - \tau_q^p(N_q)|_p \leq C\delta \quad \text{and} \quad g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2\delta^2},$$

where C depends on ρ. Therefore,

$$\begin{aligned} \nu_p \cdot e_1 &= g_p(N_p, \omega_p) \leq g_p(N_p - \tau_q^p(N_q), \omega_q) + g_p(\tau_q^p(N_q), \omega_p) \\ &\leq C\delta + g_p(\tau_q^p(N_q), \omega_p), \end{aligned}$$

and by using

$$g_p(\tau_q^p(N_q), \omega_p) = g_q(N_q, \tau_p^q(\omega_p))$$

and triangular inequality, we get

$$\begin{aligned} \nu_p \cdot e_1 &\le C\delta + g_q(N_q, \omega_q) + g_q(N_q, \tau_p^q(\omega_p) - \omega_q) \\ &\le C\delta + \nu_q \cdot e_1 + |\tau_p^q(\omega_p) - \omega_q|_q. \end{aligned}$$

In particular, the last bound holds for every $p \in \partial\Sigma$. From Proposition 3.3 and by choosing δ small enough in terms of ρ, we obtain $\nu_p \cdot e_1 \le \frac{1}{4}$, and Lemma 5.1 implies the statement. □

Lemma 5.3. *There exists $\delta > 0$ depending only on ρ with the following property. Assume there exists a connected component Γ_δ of Σ_δ such that, for any $q \in \partial\Gamma_\delta$, there exists $\hat{q} \in \hat{\Sigma}$ such that*

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \le \delta.$$

Then,

$$0 \le \nu_z \cdot e_1 \le \frac{1}{4} \quad \text{for any } z \in \partial\Sigma, \tag{5.4}$$

and Σ_δ is connected.

Proof. Let $q \in \partial\Gamma_\delta$. By construction, $\nu_q \cdot e_1 \ge 0$. Let q^π be the reflection of q with respect to π. By our assumptions, we have

$$d(q^\pi, \hat{q}) \le d(q^\pi, q) + d(q, \hat{q}) \le 3\delta.$$

We can choose δ small enough in terms of ρ and find $C = C(\rho)$ such that $d_S(q^\pi, \hat{q}) \le C\delta$ (as follows from (3.9)), $q^\pi \in \mathcal{U}_{\rho_1}(\hat{q})$, and

$$\begin{aligned} &g_{\hat{q}}(N_{\hat{q}}, \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})) \ge \sqrt{1 - C^2\delta^2}, \\ &|N_{\hat{q}} - \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})|_{\hat{q}} \le C\delta \end{aligned}$$

(see (3.6)). Since $N_{q^\pi} = (-(N_q)_1, (N_q)_2, \ldots, (N_q)_n)$ and q and q^π are symmetric about π, we have that

$$\nu_q \cdot e_1 = g_q(N_q, \omega_q) = -g_q(\tau_{q^\pi}^q(N_{q^\pi}), \omega_q),$$

and so

$$\begin{aligned} 2g_q(N_q, \omega_q) &= g_q(N_q - \tau_{q^\pi}^{\hat{q}}(N_{q^\pi}), \omega_q) \\ &= g_q(N_q - \tau_{\hat{q}}^q(N_{\hat{q}}), \omega_q) + g_q(\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi}), \omega_q). \end{aligned}$$

This implies that

$$0 \le 2g_q(N_q, \omega_q) \le |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q + |\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi})|_q,$$

and Lemma 3.4 together with our assumptions implies

$$0 \le 2\nu_q \cdot e_1 \le \frac{1}{8}.$$

From Lemma 5.2 we obtain that Σ_δ is connected.

Now fix $z \in \partial\Sigma$, and let q be such that $d_\Sigma(q, z) = \delta$ (so that $z \in \mathcal{U}_\rho(q)$). Since q and q^π are symmetric about π, then we have that

$$g_z(\tau_q^z(N_q), \omega_z) = -g_z(\tau_{q^\pi}^z(N_{q^\pi}), \omega_z),$$

and hence

$$2g_z(\tau_q^z(N_q), \omega_z) = g_z(\tau_q^z(N_q) - \tau_{q^\pi}^z(N_{q^\pi}), \omega_z).$$

We write

$$\begin{aligned} 2g_z(N_z, \omega_z) &= 2g_z(\tau_q^z(N_q), \omega_z) + 2g_z(N_z - \tau_q^z(N_q), \omega_z) \\ &= g_z(\tau_q^z(N_q) - \tau_{q^\pi}^z(N_{q^\pi}), \omega_z) + 2g_z(N_z - \tau_q^z(N_q), \omega_z) \\ &= g_z(\tau_q^z(N_q) - \tau_{\hat{q}}^z(N_{\hat{q}}), \omega_z) + g_z(\tau_{\hat{q}}^z(N_{\hat{q}}), \omega_z) \\ &\qquad - g_z(\tau_{q^\pi}^z(N_{q^\pi}), \omega_z) + 2g_z(N_z - \tau_q^z(N_q), \omega_z). \end{aligned}$$

By Cauchy-Schwarz and triangle inequalities, we have

$$\begin{aligned} |2g_z(N_z, \omega_z)| &\le |\tau_q^z(N_q) - \tau_{\hat{q}}^z(N_{\hat{q}})|_z + |\tau_{\hat{q}}^z(N_{\hat{q}}) - \tau_{q^\pi}^z(N_{q^\pi})|_z \\ &\qquad + 2|N_z - \tau_q^z(N_q)|_z \\ &\le |\tau_q^z(N_q) - \tau_{\hat{q}}^z(N_{\hat{q}})|_z + |\tau_{\hat{q}}^z(N_{\hat{q}}) - \hat{N}_z|_z + |\hat{N}_z - \tau_{q^\pi}^z(N_{q^\pi})|_z \\ &\qquad + 2|N_z - \tau_q^z(N_q)|_z, \end{aligned}$$

where N_z and $\hat{N}_z$ are the normal vectors to Σ and $\hat{\Sigma}$ at z, respectively. The first term can be bounded in terms of δ by Lemma 3.4. All the remaining terms on the righthand side can be estimated in terms of δ by using Proposition 3.6. This implies that

$$|2g_z(N_z, \omega_z)| \le C\delta.$$

By choosing δ small enough compared to C (and hence compared to ρ), we have that

$$0 \le \nu_z \cdot e_1 \le \frac{1}{4},$$

that is, Σ intersects π transversally. □

The following lemma will be used several times in the proof of Theorem 1.3.

Lemma 5.4. *Assume that $e_n \in \Sigma$ with $\nu_{e_n} = e_n$ and that there exist two local parametrizations $u, \hat{u} : B_r \to \mathbb{R}$ of Σ and $\hat{\Sigma}$, respectively, with $0 < r \le \rho_1$ and such that $u - \hat{u} \ge 0$, where ρ_1 is given by (3.3).*

Let $p_1 = (x_1, u(x_1))$ and $\hat{p}_1^* = (x_1, \hat{u}(x_1))$, with $x_1 \in \partial B_{r/4}$, and denote by γ the geodesic path starting from p_1 and tangent to ν_{p_1} at p_1. Assume that

$$d(p_1, \hat{p}_1^*) + |\nu_{p_1} - \nu_{\hat{p}_1^*}| \le \theta \tag{5.5}$$

for some $\theta \in [0, \frac{1}{2}]$. There exists $\bar{r}$ depending only on ρ such that if $r \le \bar{r}$ we have that $\gamma \cap \hat{\Sigma} \neq \emptyset$ and, if we denote by $\hat{p}_1$ the first intersection point between γ and $\hat{\Sigma}$, then

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \le C\theta,$$

where C is a constant depending only on n and ρ, and provided that $C\theta < \frac{1}{2}$.

Proof. We first notice that, by choosing r small enough in terms of ρ, from Lemma 3.5 we have that $|\nu_{p_1} - e_n| \le \frac{1}{4}$. By using the touching ball condition for $\hat{\Sigma}$ at $\hat{p}_1^*$, a simple geometric argument shows that the geodesic passing through p_1 and tangent to ν_{p_1} at p_1 intersects $\hat{\Sigma}$, so that $\hat{p}_1$ is well defined.

As shown in Figure 5.1, we estimate the distance between p_1 and $\hat{p}_1$ as follows. Let q be the unique point having distance 2ε from p_1 and lying on the geodesic path containing p_1 and $\hat{p}_1^*$. Let T be the geodesic right-angle triangle having vertices p_1 and q and hypotenuse contained in the geodesic passing through p_1 and $\hat{p}_1$. Since the angle α at the vertex p_1 is such that $|\sin\alpha| \le \frac{1}{4}$, then from the sine rule for hyperbolic triangles we have that

$$d(p_1, \hat{p}_1) \le C\theta. \tag{5.6}$$

Moreover, the cosine law formula in hyperbolic space gives that

$$d(\hat{p}_1^*, \hat{p}_1) \le C\theta \tag{5.7}$$

for some constant C, and from (3.6) we obtain that

$$\begin{aligned} |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \le\ & |N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} \\ & + |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}. \end{aligned} \tag{5.8}$$

Since p_1 and $\hat{p}_1^*$ lie on the same vertical line, we have that

$$|N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} = |\nu_{p_1} - \nu_{\hat{p}_1^*}| \le C\theta, \tag{5.9}$$

where the last inequality follows from (5.12). Moreover, from Proposition 3.4 we have

$$
\begin{aligned}
&|\tau_{\hat p_1^*}^{p_1}(N_{\hat p_1^*}) - \tau_{\hat p_1}^{p_1}(N_{\hat p_1})|_{p_1}\\
&\quad \le C(d(p_1,\hat p_1^*) + d(p_1,\hat p_1) + d(\hat p_1,\hat p_1^*) + |N_{\hat p_1} - \tau_{\hat p_1^*}^{\hat p_1}(N_{\hat p_1^*})|_{\hat p_1})\\
&\quad \le C\theta,
\end{aligned}
$$

where the last inequality follows from (5.12), (5.6), (5.7), and (3.6). This last inequality, (5.8), and (5.9) imply that

$$|N_{p_1} - \tau_{\hat p_1}^{p_1}(N_{\hat p_1})|_{p_1} \le C\theta,$$

and therefore from (5.6) we conclude. □

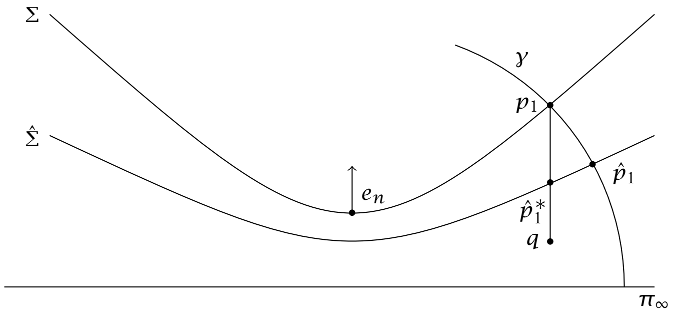

FIGURE 5.1.

5.2. Proof of the first part of Theorem 1.3. Now we can focus on the proof of the first part of Theorem 1.3, showing that there exist constants ε and C, depending only on n, ρ, and $|S|_g$, such that if

$$\operatorname{osc}(H) \le \varepsilon,$$

then for any p in Σ there exists $\hat p$ in $\hat\Sigma$ satisfying

$$d(p,\hat p) + |N_p - \tau_{\hat p}^{p}(N_{\hat p})|_p \le C\operatorname{osc}(H).$$

We will have to choose a number $\delta > 0$ sufficiently small in terms of ρ, n, and $|S|_g$, and subdivide the proof of the first part of the statement into four cases depending on the whether the distances of p_0 and p from $\partial\Sigma$ are greater or less than δ. A first requirement on δ is that it satisfies the assumptions of Lemmas 5.2 and 5.3; other restrictions on the value of δ will be done in the development of the proof.

Case 1: $d_\Sigma(p_0, \partial\Sigma) > \delta$ and $d_\Sigma(p, \partial\Sigma) \geq \delta$. In this first case, we assume p_0 and p are interior points of Σ, which are far from $\partial\Sigma$ more than δ. We first assume p_0 and p are in the same connected component of Σ_δ; then, Lemma 5.2 will be used in order to show that Σ_δ is in fact connected.

From Lemma 3.7 we can choose $\alpha \in (0, \frac{1}{2}\min(1, \rho_1^{-1}))$ such that $\alpha C \rho_1 \leq \delta/4$, where C is the constant appearing in (3.8), and we set

$$r_0 = \min(\bar{r}, \alpha\rho_1),$$

where $\bar{r}$ is given by Lemma 5.4. Accordingly to this definition of r_0, from (3.8) we have that if $p_i \in \Sigma_\delta$ then $\mathcal{U}_{r_0}(p_i) \subset \Sigma$.

Lemma 5.5. *Let $\varepsilon_0 \in [0, \frac{1}{2}]$, p_0, and p be in a connected component of Σ_δ and $r_i = (1-\varepsilon_0^2)^i r_0$. There exist an integer $J \leq J_\delta$, where*

$$J_\delta := \max\left(4, \frac{2^{n-1}|S|_g}{\delta^{n-1}}\right), \tag{5.10}$$

and a sequence of points $\{p_1, \ldots, p_J\}$ in $\Sigma_{\delta/2}$ such that

$$\begin{aligned} &p_0, p \in \bigcup_{i=0}^{J} \bar{\mathcal{U}}_{r_i/4}(p_i), \\ &\mathcal{U}_{r_0}(p_i) \subseteq \Sigma, \quad i = 0, \ldots, J, \\ &p_{i+1} \in \bar{\mathcal{U}}_{r_i/4}(p_i), \quad i = 0, \ldots, J-1. \end{aligned}$$

Proof. For every z in Σ and $r \leq \rho_0$, the geodesic ball $\mathcal{B}_r(z)$ in Σ satisfies

$$|\mathcal{B}_r(z)|_\Sigma \geq c r^{n-1},$$

where c is a constant depending only on n (see formula (3.4)). A general result for Riemannian manifolds with boundary (see, e.g., Proposition 8.1) implies there exists a piecewise geodesic path parametrized by arc length $\gamma\colon [0, L] \to \Sigma_{\delta/2}$ connecting p_0 to p and of length L bounded by δJ_δ, where J_δ is given by (5.10).

We define $p_i = \gamma(r_i/4)$, for $i = 1, \ldots, J-1$ and $p_J = p$. Our choice of r_0 guarantees that $\mathcal{U}_{r_0}(p_i) \subseteq \Sigma$, for every $i = 0, \ldots, J$, and the other required properties are satisfied by construction. ◻

Since p and p_0 are in a connected component of Σ_δ, there exist $\{p_1, \ldots, p_J\}$ in the connected component of $\Sigma_{\delta/2}$ containing p_0 and a chain $\{\mathcal{U}_{r_0}(p_i)\}_{\{i=0,\ldots,J\}}$ of subsets of Σ as in Lemma 5.5. We notice that Σ and $\hat{\Sigma}$ are tangent at p_0 and that, in particular, the two normal vectors to Σ and $\hat{\Sigma}$ at p_0 coincide. Up to an isometry we can assume $p_0 = e_n$ and $\nu_{p_0} = e_n$, and then Σ and $\hat{\Sigma}$ can be locally represented near p_0 as the graphs of two functions $u_0, \hat{u}_0\colon B_{r_0} \subset \pi_\infty \to \mathbb{R}$. Lemma 3.5 implies that $|\nabla u_0|, |\nabla \hat{u}_0| \le M$ in B_{r_0}, where M is some constant which depends only on r_0, that is, only on ρ. Since u_0 and $\hat{u}_0$ satisfy (2.2) and $|\nabla u_0|, |\nabla \hat{u}_0| \le M$, then the difference $u_0 - \hat{u}_0$ solves a second-order linear uniformly elliptic equation of the form

$$\mathcal{L}(u_0 - \hat{u}_0)(x) = H(x, u(x)) - \hat{H}(x, \hat{u}(x))$$

with ellipticity constants uniformly bounded by a constant depending only on n and ρ. Since $u_0(0) = \hat{u}_0(0)$ and $u_0 \ge \hat{u}_0$, the Harnack inequality (see Theorems 8.17 and 8.18 in [16]) yields

$$\sup_{B_{r_0/2}} (u_0 - \hat{u}_0) \le C \operatorname{osc}(H),$$

and from interior regularity estimates (see, e.g., [16, Theorem 8.32]) we obtain

$$\|u_0 - \hat{u}_0\|_{C^1(B_{r_0/4})} \le C \operatorname{osc}(H), \tag{5.11}$$

where C depends only on ρ and n.

Since $p_1 \in \partial \mathcal{U}_{r_0/4}(p_0)$, we can write $p_1 = (x_1, u_0(x_1))$, with $x_1 \in \partial B_{r_0/4}$, and define $\hat{p}_1^*$ and $\hat{p}_1$ as in Lemma 5.4. We notice that (5.11) yields

$$d(p_1, \hat{p}_1^*) + |\nu_{p_1} - \nu_{\hat{p}_1^*}| \le C \operatorname{osc}(H), \tag{5.12}$$

so that (5.5) in Lemma 5.4 is fulfilled. From Lemma 5.4 we find

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \le C \operatorname{osc}(H). \tag{5.13}$$

Now we apply an isometry in such a way that $p_1 = e_n$ and $\nu_{p_1} = e_n$. We notice that by construction $\hat{p}_1$ becomes of the form $\hat{p}_1 = te_n$, with $t \ge 1$ (notice that $t = 1 + d(p_1, \hat{p}_1)$). From the Euclidean point of view, in this configuration $\mathcal{U}_{r_0}(p_1) \subset \Sigma$ satisfies an Euclidean touching ball condition of radius ρ_1. Moreover, since $\hat{p}_1 = te_n$ with $t \ge 1$, also $\hat{\mathcal{U}}_{r_0}(p_1) \subset \hat{\Sigma}$ satisfies the Euclidean touching ball condition of radius ρ_1. Since in this configuration we have that

$$|\nu_{p_1} - \nu_{\hat{p}_1}| = |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1},$$

from (5.13) we find

$$|\nu_{p_1} - \nu_{\hat{p}_1}| \le C \operatorname{osc}(H),$$

where C is a constant that depends only on ρ and n. A suitable choice of ε in the statement of Theorem 1.1 (i.e., such that $C\varepsilon < 1$) guarantees that we can apply Lemma 3.8 (recall that $\operatorname{osc}(H) \le \varepsilon$), and we obtain that Σ and $\hat{\Sigma}$ are locally graphs of two functions

$$u_1, \hat{u}_1 : B_{r_1} \to \mathbb{R}^+,$$

such that $u_1(0) = p_1$ and $\hat{u}_1(0) = \hat{p}_1$ and where $r_1 = (1-\varepsilon^2)r$. Now, we can iterate the argument before. Indeed, since

$$0 \le \inf_{B_{r_1/2}} (u_1 - \hat{u}_1) \le u_1(0) - \hat{u}_1(0) \le C \operatorname{osc}(H),$$

by applying Harnack's inequality we obtain that

$$\sup_{B_{r_1/2}} (u_1 - \hat{u}_1) \le C \operatorname{osc}(H),$$

and from interior regularity estimates we find

$$\|u_1 - \hat{u}_1\|_{C^1(B_{r_1/4})} \le C \operatorname{osc}(H), \tag{5.14}$$

where C depends only on ρ and n. Hence, (5.14) is the analogue of (5.11), and we can iterate the argument. The iteration goes on until we arrive at $p_N = p$ and obtain a point $\hat{p}_N \in \hat{\Sigma}$ such that

$$d(p, \hat{p}_N) + |N_p - \tau^p_{\hat{p}_N}(N_{\hat{p}_N})|_p \le C \operatorname{osc}(H).$$

In view of Lemma 5.3, we have that Σ_δ is connected and the claim follows.

Case 2: $d_\Sigma(p_0, \partial\Sigma) \ge \delta$ and $d_\Sigma(p, \partial\Sigma) < \delta$. Here, we extend the estimates found at Case 1 to a point p which is far less than δ from the boundary of Σ. Let $q \in \Sigma$ and $p_{\min} \in \partial\Sigma$ be such that

$$d_\Sigma(q, \partial\Sigma) = \delta, \quad d_\Sigma(p, q) + d_\Sigma(p, \partial\Sigma) = \delta \quad \text{and} \quad d_\Sigma(p, p_{\min}) = d_\Sigma(p, \partial\Sigma).$$

From Case 1, we have that there exists $\hat{q}$ in $\hat{\Sigma}$ such that

$$d(q, \hat{q}) + |N_q - \tau^q_{\hat{q}}(N_{\hat{q}})|_q \le C \operatorname{osc}(H).$$

Lemma 5.3 yields that

$$0 \le g_z(N_z, \omega_z) \le \frac{1}{4}, \tag{5.15}$$

for any $z \in \partial\Sigma$ and where Σ_δ is connected.

For $r \le \rho_1$, with ρ_1 given by (3.3), we define $U_r(q)$ as the reflection of $\mathcal{U}_r(q^\pi) \cap \{x_1 \ge 0\}$ with respect to π and $U' = \mathcal{U}_r(q^\pi) \cap \{x_1 = 0\}$. From Proposition 4.1, U' is a hypersurface of π with a natural orientation and its principal curvatures κ_i' satisfy

$$\frac{1}{\sqrt{1 - g_z(N_z, \omega_z)^2}} \kappa_1(z) \le \kappa_i'(z) \le \frac{1}{\sqrt{1 - g_z(N_z, \omega_z)^2}} \kappa_{n-1}(z),$$

for every $z \in U'$ and $i = 1, \dots, n-1$. From (5.15) and since $|\kappa_i(z)| \le \rho^{-1}$ for any $z \in S$ (this follows from the touching sphere condition), we have

$$|\kappa_i'(z)| \le \frac{2}{\rho}. \tag{5.16}$$

Now, we apply an isometry $f : \mathbb{H}^n \to \mathbb{H}^n$ such that $f(q) = e_n$ and the normal vector to $f(S)$ at $f(q)$ is e_n (i.e., $f_{*|q}(N_q) = e_n$).

Let U'' be the Euclidean orthogonal projection of $f(U')$ onto π_∞. Our goal is to estimate the curvatures of U''. It is clear that $f(\pi)$ is either a vertical hyperplane or a half-sphere intersecting $f(S)$. In the first case, we immediately conclude since the curvatures of U'' vanish.

Thus, we assume that $f(\pi)$ is a half-sphere. A straightforward computation yields that the radius of $f(\pi)$ is

$$R = \frac{q_n(\Theta^2 + 1)}{2|\Theta|\,|a\Theta + q_n|},$$

where

$$\Theta = -\frac{\sin\theta}{1 + \cos\theta}, \quad \cos\theta = \nu_q \cdot e_n$$

and a is the Euclidean distance of q from π. It is easy to see $a \le q_n \sinh(\delta)$, and so

$$\frac{1}{R} \le \frac{2|\Theta|(\sinh(\delta)|\Theta| + 1)}{\Theta^2 + 1},$$

which implies

$$\frac{1}{R} \le 1 + 2\sinh(\delta). \tag{5.17}$$

We notice that the last estimate can be alternatively found by noticing that an isometry that fixes e_n maps a vertical hyperplane into a half sphere, where the radius can be estimated by using the distance of e_n from the vertical hyperplane.

We still denote by ν' the Euclidean normal vector field to $f(U')$. We denote by κ_i'' the principal curvatures of U'' with respect to the Euclidean metric on π_∞ and a chosen orientation. Now, we want to find an upper bound on the curvatures of U'' which will allow us to use Carleson-type estimates. Proposition 4.3 and formula (5.17) imply

$$\begin{aligned}
|\kappa_i''(\bar\xi)| &\le \frac{1}{R}\left((\nu_\xi' \cdot e_n)^2 + \frac{\xi_n^2}{R^2}\right)^{-3/2} \\
&\qquad \times (2 + \max\{|\kappa_1'(f^{-1}(\xi))|, |\kappa_{n-1}'(f^{-1}(\xi))|\}) \\
&\le \frac{1 + 2\sinh\delta}{|\nu_\xi' \cdot e_n|^3}(2 + \max\{|\kappa_1'(f^{-1}(\xi))|, |\kappa_{n-1}'(f^{-1}(\xi))|\})
\end{aligned}$$

for every $\xi = (\bar{\xi}, \xi_n)$ in $f(U')$ and $i = 1, \dots, n-2$. Then, (5.16) yields that

$$|\kappa_i''(\bar{\xi})| \leq \frac{2(1+\rho)(1+2\sinh\delta)}{\rho|\nu_\xi' \cdot e_n|^3}.$$

Next, we show

$$\nu_\xi' \cdot e_n \geq \frac{1}{2}. \tag{5.18}$$

We write

$$\nu_\xi' \cdot e_n = \nu_\xi' \cdot (e_n - \nu_\xi) + \nu_\xi' \cdot \nu_\xi,$$

where we still denote by ν the normal vector field to $f(S)$. Since $f_{*|q}(\nu_q) = e_n$, from Lemma 2.1 in [15] we have that $|e_n - \nu_\xi| \leq \frac{1}{4}$ by choosing r small enough in terms of ρ_1 and hence of ρ. Moreover, since

$$\nu_\xi' \cdot \nu_\xi = \nu_{f^{-1}(\xi)}' \cdot \nu_{f^{-1}(\xi)},$$

[15, formula (2.29)] implies

$$\nu_{f^{-1}(\xi)}' \cdot \nu_{f^{-1}(\xi)} = \sqrt{1 - (\nu_{f^{-1}(\xi)} \cdot e_1)^2}$$

and (5.15) gives (5.18). Therefore,

$$|\kappa_i''(\bar{\xi})| \leq C, \tag{5.19}$$

for some constant $C = C(\rho)$.

Let $x = \overline{f(p_{\min})}$ and $y = \overline{f(p)}$ be the projections of $f(p_{\min})$ and $f(p)$ onto π_∞, respectively, and let E_r be the projection of $f(U_r(q))$ onto π_∞. From (3.1), we have that $|x - y| \leq C\delta$, with $C \geq 1$ which depends only on ρ. We can choose δ small enough (compared to ρ) such that $B_{8C\delta}(x) \cap \partial E_r \subset U''$, apply Theorem 1.3 in [7] and Corollary 8.36 in [16], and find

$$\sup_{B_{2C\delta}(x) \cap E_r} (u - \hat{u}) \leq C_1 (u - \hat{u})(z) + \operatorname{osc}(H), \tag{5.20}$$

with $z = x + 4C\delta\nu_x''$, where ν_x'' is the interior normal to U'' at x. By choosing δ small enough in terms of ρ, the bound on the curvatures of U'' implies that the point z has distance $4C\delta$ from the boundary of E_r. Since $d_\Sigma(q, U') = \delta$, the distance (in π_∞) of O from the boundary of E_r is at least $c\delta$ (as follows from (3.1)), where $c < C$ depends only on ρ. From Harnack's inequality,

$$C_1(u - \hat{u})(z) + \operatorname{osc}(H) \leq C_2(u(0) - \hat{u}(0) + \operatorname{osc}(H)),$$

and from (5.20) we obtain that

$$0 \le \sup_{B_{2C\delta}(x)\cap E_r} (u - \hat{u}) \le C_2(u(0) - \hat{u}(0) + \operatorname{osc}(H)).$$

Boundary regularity estimates (see, e.g., [16, Corollary 8.36]) yield

$$0 \le \|u - \hat{u}\|_{C^1(B_{C\delta}(x)\cap E_r)} \le C_3((u(0) - \hat{u}(0)) + \operatorname{osc}(H)). \tag{5.21}$$

Since $d_\Sigma(q, \partial\Sigma) = \delta$, from Case 1 we know that

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \le C \operatorname{osc}(H),$$

where $\hat{q}$ is the first intersecting point between $\hat{\Sigma}$ and the geodesic path starting form q and tangent to ν_q at q (recall that $f(q) = e_n$ and $N_q = e_n$). From (5.21) we obtain that

$$0 \le \|u - \hat{u}\|_{C^1(B_{C\delta}(x)\cap E_r)} \le C \operatorname{osc}(H). \tag{5.22}$$

We define $\hat{p}^*$ so that $\hat{p}^* = f(y, \hat{u}(y))$. Since $y \in B_{C\delta}(x)$, (5.22) implies

$$d(f(p), f(\hat{p}^*)) + |\nu_{f(p)} - \nu_{f(\hat{p}^*)}| \le C \operatorname{osc}(H).$$

Since $f(p)$ and $f(\hat{p}^*)$ are on the same vertical line, we can write

$$d(f(p), f(\hat{p}^*)) + |N_{f(p)} - \tau(N_{f(\hat{p}^*)})|_{f(p)} \le C \operatorname{osc}(H),$$

where τ is the parallel transport along the vertical segment connecting $f(\hat{p}^*)$ with $f(p)$. Lemma 5.4 yields

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \le C \operatorname{osc}(H),$$

as required.

Case 3: $0 < d_\Sigma(p_0, \partial\Sigma) < \delta$. We first show that the center of the interior touching ball of radius ρ to S at p_0, say $\mathsf{B}_\rho(a)$, lies on the left of π, that is, $a \cdot e_1 \le 0$. Indeed, since p_0 is a tangency point, $p_0^\pi \in S$, and hence p_0^π does not lie in $\mathsf{B}_\rho(a)$. This implies

$$d(p_0, a) = \rho \le d(p_0^\pi, a),$$

and since p_0 and p_0^π have the same height we have

$$|p_0 - a|^2 \le |p_0^\pi - a|^2,$$

which implies that $a \cdot e_1 \le 0$.

Now, we prove that Σ and π intersect transversally at p_0 (see (5.24) below). Since $d(p_0,\pi) \le d_\Sigma(p_0,\partial\Sigma) \le \delta$, we have $d(p_0,p_0^\pi) \le 2\delta$. We can choose δ small in terms of ρ so that $p_0^\pi \in \mathcal{U}_{\rho_1}(p_0)$. From (3.6), we have that

$$g_{p_0}(N_{p_0},\tau_{p_0^\pi}^{p^\pi}(N_{p_0^\pi})) \ge \sqrt{1-C^2\delta^2}, \qquad (5.23)$$
$$|N_{p_0}-\tau_{p_0^\pi}^{p_0}(N_{p_0^\pi})|_{p_0} \le C\delta.$$

Since

$$g_{p_0}(N_{p_0},\omega_{p_0}) = -g_{p_0^\pi}(N_{p_0},\omega_{p_0^\pi}),$$

and $g_{p_0}(N_{p_0},\omega_{p_0}) \ge 0$ by construction,

$$\begin{aligned} 0 \le 2g_{p_0}(N_{p_0},\omega_{p_0}) &= g_{p_0}(N_{p_0}-\tau_{p_0^\pi}^{p_0}(N_{p_0^\pi}),\omega_{p_0}) \\ &\le |N_{p_0}-\tau_{p_0^\pi}^{p_0}(N_{p_0^\pi})|_{p_0} \le C\delta, \end{aligned}$$

where the last inequality follows from (5.23). By choosing δ small compared to C (in terms of ρ), we have

$$0 \le g_{p_0}(N_{p_0},\omega_{p_0}) \le \frac{1}{4}. \qquad (5.24)$$

Now, we apply an isometry $f\colon \mathbb{H}^n \to \mathbb{H}^n$ such that

$$f(p_0) = e_n \quad \text{and} \quad f_{*|p_0}(N_{p_0}) = e_n.$$

As for Case 2 (with q replaced by p_0), we locally write $f(\Sigma)$ and $f(\hat{\Sigma})$ as graphs of function $u,\hat{u}\colon E_r \to \mathbb{R}$, respectively. Moreover, we denote by U'' the portion of ∂E_r which is obtained by projecting $f(U_r(p_0)\cap\pi)$ onto π_∞. We comment that $u=\hat{u}$ on U'' and, again by arguing as in Case 2, that the principal curvatures of U'' can be bounded by a constant $\mathcal{K}$ depending only on ρ.

Let $\bar{x} \in U''$ be a point such that

$$|\bar{x}| = \min_{x\in U''} |x|.$$

Notice that $|\bar{x}| \le C d_\Sigma(p_0,\partial\Sigma) < C\delta$, where C is the constant appearing in (3.1). Let $\nu''_{\bar{x}}$ be the interior normal to U'' at $\bar{x}$, and set

$$y = \bar{x} + 2C\delta\nu''_{\bar{x}}$$

(see Figure 5.2). We notice that the principal curvatures of U'' are bounded by $\mathcal{K}$ and, by choosing δ small compared to ρ, we have $2C\delta \le \mathcal{K}^{-1}$ and the ball $B_{2C\delta}(y)$ is contained in E_r and tangent to U'' at $\bar{x}$, with $\nu''_{\bar{x}} = -\bar{x}/|\bar{x}|$. Since

$u(O) = \hat{u}(O)$ and from [15, Lemma 2.5] (where we set $x_0 = \bar{x}$, $c = y$, and $r = 2C\delta$), we find that

$$\|u - \hat{u}\|_{C^1(B_{C\delta/2}(y))} \le C \operatorname{osc}(H). \tag{5.25}$$

Let $q = (y, u(y))$ and $\hat{q}^* = (y, \hat{u}(y))$ so that (5.25) gives

$$d(q, \hat{q}^*) + |\nu_q - \nu_{\hat{q}^*}| \le C \operatorname{osc}(H).$$

Up to choosing a smaller δ, we can assume that $r = 2C\delta \le \bar{r}$, so that Lemma 5.4 yields

$$d(q, \hat{q}) + |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q \le C \operatorname{osc}(H),$$

where $\hat{q}$ is defined as $\hat{p}_1$ in Lemma 5.4. Next, we observe from our construction that

$$d_\Sigma(q, \partial\Sigma) \ge \delta.$$

Indeed, if we denote by z the point on $\partial U_r(p_0)$ which realizes $d(q, \partial U_r(p_0))$, then

$$d_\Sigma(q, \partial\Sigma) \ge d(q, z) = \operatorname{arccosh}\left(1 + \frac{|\bar{q} - \bar{z}|^2}{2q_n z_n}\right) \ge \operatorname{arccosh}\left(1 + \frac{2C^2\delta^2}{q_n z_n}\right).$$

Moreover, since $|y|, |\bar{z}| \le 2C\delta$, from (3.5) we have that $q_n \ge 1 - C_1(\rho)\delta^2$ and $z_n \ge 1 - C_1(\rho)\delta^2$ so that we can obtain $d_\Sigma(q, \partial\Sigma) \ge \delta$ by choosing δ small enough in terms of ρ. Since $d_\Sigma(q, \partial\Sigma) \ge \delta$ we can apply Case 1 and Case 2 to conclude.

Case 4: $p_0 \in \partial\Sigma$. This case follows from Case 3 when $d_\Sigma(p_0, \partial\Sigma) \to 0$. Indeed, in this case E_r is a half-ball on π_∞ and the argument used in Case 3 can be easily adapted (see also the corresponding case in [15]). This completes the proof of the first part of Theorem 1.3.

5.3. Proof of the second part of Theorem 1.3. Now, we focus on the second part of the statement of Theorem 1.3, showing that Ω is contained in a neighborhood of radius $C \operatorname{osc}(H)$ of $\Sigma \cup \Sigma^\pi$.

Assume by contradiction that

$$\exists\, x \in \Omega \quad \text{such that } d(x, \Sigma \cup \Sigma^\pi) > C \operatorname{osc}(H).$$

By construction, we can assume that $x \cdot e_1 < 0$, and hence from the connectedness of Ω we can find a point $y \in \Omega$, with $y \cdot e_1 < 0$, such that

$$C \operatorname{osc}(H) < d(y, \Sigma) \le 2C \operatorname{osc}(H).$$

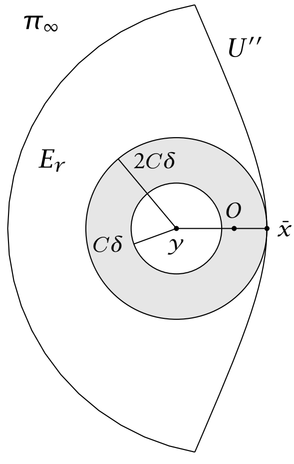

FIGURE 5.2. Case 3 in the proof of Theorem 1.3.

Let p be a projection of y over Σ. First, assume $p \cdot e_1 \neq 0$. From the first part of Theorem 1.3, we have that there is a point $\hat{p} \in S$ such that $\hat{p} = \gamma(t)$ where γ is the geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = -N_p$ and such that $0 \le t \le C\,\mathrm{osc}(H)$ and $|N_p - \tau_{\hat{p}}^{p}(N_{\hat{p}})|_p \le C\,\mathrm{osc}(H)$. Moreover, we notice that by construction $\hat{p}$ is on the geodesic γ connecting y and p. Since $C\,\mathrm{osc}(H)$ is small (less than ρ is enough), this implies that y belongs to the exterior touching ball of radius ρ at p, that is, $y \notin \Omega$, which is a contradiction. If $p \cdot e_1 = 0$ we obtain again a contradiction from the exterior touching ball condition, since from (5.15) we have that $g_p(N_p, p_n e_1) \le \frac{1}{4}$. Hence, the claim follows.

6. PROOF OF THEOREM 1.1

Let $\varepsilon > 0$ be the constant given by Theorem 1.3. Let S be a connected closed C^2-hypersurface embedded in the hyperbolic half-space $\mathbb{H}^n$ satisfying a touching ball condition of radius ρ and such that $\mathrm{osc}(H) \le \varepsilon$, as in the statement of Theorem 1.1. Given a direction ω, let Ω_ω be the maximal cap of Ω in the direction ω, according to the notation introduced in Subsection 2.1. As a consequence of the second part of Theorem 1.3, we have that

$$|\Omega_\omega|_g \ge \frac{|\Omega|_g}{2} - C\,\mathrm{osc}(H), \tag{6.1}$$

for some constant C depending only on n, ρ and $|S|_g$. Moreover, the reflection Ω^π of Ω about π satisfies

$$|\Omega \triangle \Omega^\pi|_g = 2(|\Omega|_g - 2|\Omega_\omega|_g) \le 4C\,\mathrm{osc}(H), \tag{6.2}$$

where $\Omega \triangle \Omega^\pi$ denotes the symmetric difference between Ω and Ω^π.

Now the problem consists in defining an approximate center of mass $\mathcal{O}$ and quantifying the reflection about it. In the Euclidean case this step is obtained by applying the method of the moving planes in n orthogonal directions and defining $\mathcal{O}$ as the intersection of the corresponding n critical hyperplanes (see, e.g., [15]). In the hyperbolic context, the situation is different since the critical hyperplanes corresponding to n orthogonal directions do not necessarily intersects. However, when Theorem 1.3 is in force we can prove that they always intersect.

Lemma 6.1. *Let S satisfy the assumptions of Theorem 1.3 and let the critical hyperplanes corresponding to $\{e_1,\dots,e_n\}$ be $\{\pi_{e_1},\dots,\pi_{e_n}\}$. Then,*

$$\bigcap_{i=1}^{n} \pi_{e_i} = \mathcal{O} \quad \text{for some } \mathcal{O} \in \mathbb{H}^n.$$

Proof. It is enough to show that $\pi_{e_i} \cap \pi_{e_j} \neq \emptyset$ for every $i,j = 1,\dots,n$. We may assume that $e_n \in S$. Let $i \neq j$. To simplify the notation we set

$$\pi_k^s = \pi_{e_k, m_{e_k}+s}, \quad k \in \{1,\dots,n\},\ s \in \mathbb{R},$$

so that the critical hyperplane in the direction e_k is denoted by π_k^0.

We prove the assertion by contradiction. Assume that $\pi_i^0 \cap \pi_j^0 = \emptyset$ for some $i \neq j$. Then, π_i^0 and π_j^0 divide Ω into three disjoint sets which we denote by $\Omega_1, \Omega_2, \Omega_3$, and we may assume that Ω_1 is the maximal cap in the direction e_i and $\Omega_1 \cup \Omega_2$ is the maximal cap in the direction e_j (see Figure 6.1). Moreover, in view of (6.1) we have that

$$|\Omega_1|_g \geq \frac{|\Omega|_g}{2} - C \operatorname{osc}(H),$$

and

$$|\Omega_1|_g + |\Omega_2|_g \geq \frac{|\Omega|_g}{2} - C \operatorname{osc}(H).$$

From this, and since the reflection of Ω_1 about π_i^0 is contained in $\Omega_2 \cup \Omega_3$ and the reflection of $\Omega_1 \cup \Omega_2$ about π_j^0 is contained in Ω_3, we have that

$$|\Omega_2|_g \leq 2C \operatorname{osc}(H).$$

We notice that for every $k = 1,\dots,n$, we have that π_k^{s+t} and π_k^{s-t} are the two connected components of the set of points which are t-far from π_k^s. We define

$$\ell = \min\{d(\pi_i^0 \cap \Omega, \pi_j^0 \cap \Omega) \mid i,j = 1,\dots,n \text{ and } i \neq j\}.$$

Since π_i^0 and π_j^0 do not intersect and $S \subset \mathsf{B}_{\operatorname{diam}(S)}(e_n)$, we have that $\ell > 0$ and Proposition 8.2 implies that ℓ depends only on n, ρ, and $|S|_g$. Therefore,

$$\Omega_2 \supseteq \mathcal{E}_1 := \bigcup_{s \in (0,\ell)} \Omega \cap \pi_j^s,$$

and hence $|\mathcal{E}_1|_g \le 2C\,\mathrm{osc}(H)$. By reflecting $\mathcal{E}_1$ about π_i^0 we obtain that most of the mass of Ω_1 must be at distance more than ℓ from π_i^0; that is, the set $\Omega_{e_i,\ell} := \bigcup_{s\in(\ell,+\infty)} \Omega \cap \pi_i^s$ is such that

$$|\Omega_{e_i,\ell}|_g \ge \frac{|\Omega|_g}{2} - 2C\,\mathrm{osc}(H).$$

Since $d(\Omega_{e_i,\ell}, \pi_j^0 \cap \Omega) \ge 2\ell$, we have that most of the mass of Ω_3 is at distance 2ℓ from π_j^0. This implies that the set

$$\mathcal{E}_2 = \bigcup_{s\in(-2\ell,\ell)} \Omega \cap \pi_i^s$$

is such that $|\mathcal{E}_2|_g \le 4C\,\mathrm{osc}(H)$. By iterating this argument above we find $m \in \mathbb{N}$ such that $m\ell > \mathrm{diam}(S)$ and

$$0 = |\Omega_{e_i,m\ell}|_g \ge \frac{|\Omega|_g}{2} - (m+1)C\,\mathrm{osc}(H).$$

This leads to a contradiction provided that $C\,\mathrm{osc}(H)$ is small in terms of n, ρ, and $|S|_g$. Therefore, $\pi_{e_i} \cap \pi_{e_j} \ne \emptyset$. ◻

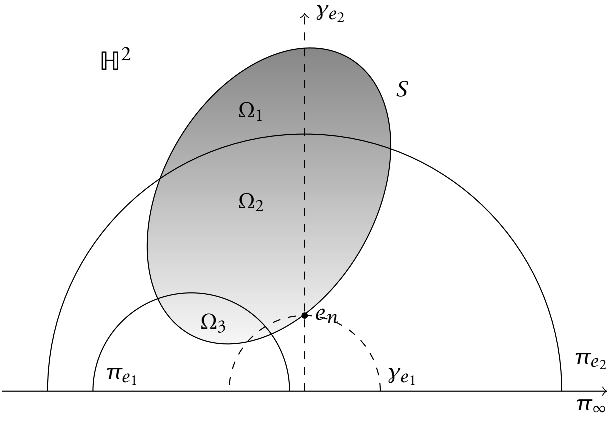

FIGURE 6.1. A picture of the proof of Lemma 6.1 in $\mathbb{H}^2$. Here, $e_j = e_1$ and $e_i = e_2$.

We refer to the point $\mathcal{O} = \bigcap_{i=1}^n \pi_{e_i}$ as to the *the approximate center of symmetry*. Note that the reflection $\mathcal{R}$ about $\mathcal{O}$ can be written as

$$\mathcal{R}(p) = \pi_{e_1} \circ \cdots \circ \pi_{e_n}(p),$$

where we identify π_{e_i} with the reflection about the corresponding hyperplane.

Next, we show that if $\operatorname{osc}(H)$ is small enough, then π_ω is close to $\mathcal{O}$, for every direction ω.

Lemma 6.2. *There exist $\varepsilon, C > 0$ depending on ρ, n and $|S|_g$ such that if the mean curvature of S satisfies $\operatorname{osc}(H) \le \varepsilon$, then*

$$d(\mathcal{O}, \pi_\omega) \le C \operatorname{osc}(H).$$

Proof. We may assume $\mathcal{O} \in \pi_{\omega, m_\omega - \mu}$, for some $\mu > 0$ (otherwise, we switch ω and $-\omega$). Now, we argue as in [11, Lemma 4.1]. We define

$$\mathcal{R}(\Omega) = \{\mathcal{R}(p) \mid p \in \Omega\}.$$

By choosing ε as the one given by Theorem 1.3, from (6.1) and since $\mathcal{R}$ is the composition of n reflections, we have that

$$|\Omega \triangle \mathcal{R}(\Omega)|_g \le C \operatorname{osc}(H),$$

where C is a constant depending on n, ρ, and $|S|_g$. Clearly, $d(\mathcal{O}, \pi_\omega) \le \operatorname{diam}(S)$. We denote by Ω^{π_ω} the reflection of Ω about π_ω, and from (6.2) we have that

$$|\Omega \triangle \Omega^{\pi_\omega}|_g \le C \operatorname{osc}(H).$$

Then, the maximal cap Ω_ω satisfies

$$|\Omega \cap \mathcal{R}(\Omega_\omega)|_g = |\mathcal{R}(\Omega) \cap \Omega_\omega|_g \ge |\Omega_\omega|_g - |\Omega \triangle \mathcal{R}(\Omega)|_g \ge \frac{|\Omega|_g}{2} - C \operatorname{osc}(H),$$

and from

$$\mathcal{R}(\Omega_\omega) \subset \bigcup_{s<0} \pi_{\omega, m_\omega - s},$$

we obtain that

$$\mu_0 := |\{\Omega \cap \pi_{\omega,s} : m_\omega - \mu < s < m_\omega\}|_g \le C \operatorname{osc}(H).$$

Let

$$\mu_k = |\{p \in \Omega \cap \pi_{\omega,s} : m_\omega + (k-1)\mu < s < m_\omega + k\mu\}|_g$$

for $k \in \mathbb{N}$. We notice that by construction of the method of the moving planes we have that μ_k is decreasing, and hence

$$\mu_k \le \mu_0 \le C \operatorname{osc}(H).$$

Let $\Lambda = \sup\{s \in \mathbb{R} \mid \Omega \cap \pi_{\omega, m_\omega - \mu + s} \neq \varnothing\}$. It is clear that $\Lambda \le \operatorname{diam}(\Omega)$. Define k_0 as the smallest integer such that

$$k_0 m_\omega \le \operatorname{diam}(\Omega) \le (k_0 + 1) m_\omega.$$

From (6.1) we have

$$\frac{|\Omega|_g}{2} - C\operatorname{osc}(H) \le |\Omega_\omega|_g \le \sum_{k=0}^{k_0} \mu_k \le k_0\mu_0 \le \frac{\operatorname{diam}(\Omega)}{m_\omega} C\operatorname{osc}(H).$$

Since $\operatorname{diam}(\Omega) \le \operatorname{diam}(S)$, from Proposition 8.2 and assuming that $\operatorname{osc}(H)$ is less than a small constant depending on n, ρ, and $|S|_g$, we have that

$$m_\omega \le C\operatorname{osc}(H),$$

where C depends on n, ρ, and $|S|_g$. ◻

Proof of Theorem 1.1. We are ready to complete the proof of our main theorem. Let ε be as in Lemma 6.2, and assume that the mean curvature of S satisfies $\operatorname{osc}(H) \le \varepsilon$. Let

$$r = \sup\{s > 0 \mid \mathsf{B}_s(\mathcal{O}) \subset \Omega\}$$

and

$$R = \inf\{s > 0 \mid \mathsf{B}_s(\mathcal{O}) \supset \Omega\},$$

so that $S \subset \bar{\mathsf{B}}_R \setminus \mathsf{B}_r$. We aim to prove that

$$R - r \le C\operatorname{osc}(H),$$

for some C depending only on n, ρ, and $|S|_g$.

Let $p, q \in S$ be such that $d(p,\mathcal{O}) = r$ and $d(q,\mathcal{O}) = R$. We can assume that $p \neq q$ (otherwise the assertion is trivial). Let $t = d(p,q)$,

$$\omega := \frac{1}{t}\tau_p^{e_n}(\exp_p^{-1}(q)),$$

and consider π_ω. Let $\gamma : (-\infty,+\infty) \to \mathbb{H}^n$ be the geodesic such that $\gamma(s_p) = p$ and $\gamma(s_q) = q$. We denote by z the point on π_ω which realizes the distance of $\mathcal{O}$ from π_ω. By construction, $p \in \pi_{\omega,s_p}$ and $q \in \pi_{\omega,s_q}$ with $s_q = s_p + t$. We first prove that $d(q,z) \le d(p,z)$. By contradiction, assume that $d(q,z) > d(p,z)$. Since q and p belong to a geodesic orthogonal to the hyperplanes $\pi_{\omega,s}$ and $s_p < s_q$, we have $s_q > m_\omega$. Since $\pi_\omega = \pi_{\omega,m_\omega}$ corresponds to the critical position on the method of moving planes in the direction ω, we have that $\gamma(s) \in \Omega$ for any $s \in (m_\omega, s_q)$. Since $s_p < s_q$ we have that $|s_p - m_\omega| \ge |s_q - m_\omega|$, and with γ orthogonal to π_ω, we obtain $d(q,z) \le d(p,z)$, which gives a contradiction.

From $d(q,z) \le d(p,z)$ and by triangular inequality, we find

$$r \ge R - d(\mathcal{O},z) = R - d(\mathcal{O},\pi_m),$$

and Lemma 6.2 implies $R - r \le C\operatorname{osc}(H)$ and the proof is complete. ◻

7. PROOF OF COROLLARY 1.2

The proof is analogous to the proof of [11, Theorems 1.2 and 1.5]. We first prove an intermediate result, which proves that S is a graph over B_r, and moreover it gives a first (non-optimal) bound on $\|\Psi\|_{C^1(\partial B_r)}$; that is, it gives that $\|\Psi\|_{C^1(\partial B_r)} \leq C(\operatorname{osc}(H))^{1/2}$). Then, we obtain the sharp estimate (1.2) by using elliptic regularity theory.

We let $\mathsf{B}_r(\mathcal{O})$ and $\mathsf{B}_R(\mathcal{O})$ be such that $0 \leq R - r \leq C \operatorname{osc}(H)$, and let also $0 < t < r - C \operatorname{osc}(H)$. For any point $p \in S$ we consider the set $\mathcal{E}^-(p)$ consisting of points of $\mathbb{H}^n$ belonging to some geodesic path connecting p to the boundary of $\mathsf{B}_t(\mathcal{O})$ tangentially. Then, we denote by $C^-(\mathcal{O})$ the geodesic cone enclosed by $\mathcal{E}^-(p)$ and the hyperplane containing $\mathcal{E}^-(p) \cap \mathsf{B}_t(\mathcal{O})$. Moreover, we define $C^+(p)$ as the reflection of $C^-(p)$ with respect to p.

We first show that for any $p \in S$ we have that $C^-(p)$ and $C^+(p)$ are contained in the closure of Ω and in the complement of Ω, respectively. Moreover, the axis of $C^-(p)$ is part of the geodesic path connecting p to $\mathcal{O}$, and this fact will allow us to define a diffeomorphism between S and $\partial\mathsf{B}_r$. We will prove that the interior of $C^-(p)$ is contained in Ω. An analogous argument shows that $C^+(p)$ is contained in the complement of Ω.

We argue by contradiction. Assume $p \notin \mathsf{B}_r(\mathcal{O})$ (otherwise the claim is trivial) and that there exists a point $q \in C^-(p) \cap \partial\mathsf{B}_t(\mathcal{O})$ such that the geodesic path γ connecting q to p is not contained in Ω. Let z be a point on γ which does not belong to the closure of Ω. Let

$$\omega := \frac{1}{d(p,q)} \tau^q_{e_n}(\exp_q^{-1}(p))$$

and consider the critical hyperplane π_ω in the direction ω. Since z does not belong to the closure of Ω, the method of the moving planes “stops” before reaching z, and therefore $z \in \pi_{\omega,s_z}$ for some $s_z \leq m_\omega$. Moreover, by construction $q \in \pi_{\omega,s_q}$ with $s_q \geq s_0$, where s_0 is such that $\mathcal{O} \in \pi_{\omega,s_0}$. Since $s_z - s_q = d(z,q)$ and $d(z,\mathcal{O}) \geq r$, we have

$$\begin{aligned} d(\mathcal{O},\pi_\omega) &= m_\omega - s_0 \geq s_z - s_0 \geq s_z - s_q = d(z,q) \\ &\geq d(z,\mathcal{O}) - d(\mathcal{O},q) = d(z,\mathcal{O}) - t \geq r - t; \end{aligned}$$

since $0 < t < r - C \operatorname{osc}(H)$ and from Lemma 6.2, we obtain

$$C \operatorname{osc}(H) < r - t \leq d(\mathcal{O},\pi_\omega) \leq C \operatorname{osc}(H),$$

which gives a contradiction.

We notice that by fixing any $t = r - \varepsilon/2$, from the argument above we have that for any $p \in S$ the geodesic path connecting p to $\mathcal{O}$ is contained in Ω. This implies there exists a C^2-regular map $\Psi : \partial\mathsf{B}_r(\mathcal{O}) \to \mathbb{R}$ such that

$$F(p) = \exp_x(\Psi(p)N_p),$$

which defines a C^2-diffeomorphism from B_r to S.

Now, we make a suitable choice of t in order to prove that

$$\|\Psi\|_{C^1} \le C(\operatorname{osc}(H))^{1/2}. \tag{7.1}$$

Indeed, by choosing $t = r - \sqrt{C \operatorname{osc}(H)}$ we have that for any $p \in S$ there exists a uniform cone of opening $\pi - \sqrt{C \operatorname{osc}(H)}$ with vertex at p and axis on the geodesic connecting p to $\mathcal{O}$. This implies that Ψ is locally Lipschitz and the bound (7.1) on $\|\Psi\|_{C^1}$ follows (see also [11, Theorem 1.2]).

Finally, we prove the optimal linear bound $\|\Psi\|_{C^{1,\alpha}} \le C \operatorname{osc}(H)$ by using elliptic regularity. Let $\phi\colon U \to \partial\mathsf{B}_r$ be a local parametrization of $\partial\mathsf{B}_r$, with U an open set of $\mathbb{R}^{n-1}$. By the first part of the proof, $F \circ \phi$ gives a local parametrization of S. A standard computation yields that we can write

$$L(\Psi \circ \phi) = H(F \circ \phi) - H_{\mathsf{B}_r},$$

where H_{B_r} is the mean curvature of $\partial\mathsf{B}_r$ and L is an elliptic operator which, thanks to the bounds on Ψ above, can be seen as a second-order linear operator acting on $\Psi \circ \phi$. Then, [16, Theorem 8.32] implies the bound on the $C^{1,\alpha}$-norm of Ψ, as required.

8. Appendix. A General Result on Riemannian Manifolds with Boundary

Let (M, g_M) be a κ-dimensional orientable compact Riemannian C^2-manifold with boundary. For $\delta, r \in \mathbb{R}^+$, $z \in M$, we denote

$$\begin{aligned} M^\delta &= \{p \in M \mid d_M(p, \partial M) > \delta\}, \\ \mathcal{B}_r(z) &= \{p \in M \mid d_M(z,p) < r\}, \end{aligned}$$

where d_M is the geodesic distance on M induced by g.

Proposition 8.1. *Assume there exist positive constants* c *and* δ_0 *such that*

$$|\mathcal{B}_r(z)|_{g_M} \ge c r^\kappa, \tag{8.1}$$

and assume $\mathcal{B}_r(z)$ *belongs to the image of the exponential map, for every* $z \in M^\delta$ *and* $0 < r \le \delta < \delta_0$. *Fix* p *and* q *in a connected component of* M^δ. *Then, there exists a piecewise geodesic path* $\gamma\colon [0,1] \to M^{\delta/2}$ *connecting* p *and* q, *of length bounded by* δN_δ *where*

$$N_\delta := \max\left(4, \frac{2^\kappa |M|_{g_M}}{c\delta^\kappa}\right).$$

Proof. Let $\tilde\gamma = \tilde\gamma(t)$ be a continuous path connecting p and q in M^δ. Following the approach in [15, Lemma 3.2], we can construct a chain of *pairwise* *disjoint* geodesic balls $\{\mathcal{B}_1, \dots, \mathcal{B}_I\}$ of radius $\delta/2$ such that $\mathcal{B}_1$ is centered at p, $\mathcal{B}_i$ is centered at $c_i = \tilde{\gamma}(t_i)$, the sequence t_i is increasing, $\mathcal{B}_I$ contains q, and $\mathcal{B}_i$ is tangent to $\mathcal{B}_{i+1}$ for any $i = 1, \dots, I-1$. Since

$$\Big| \bigcup_{i=1}^{I} \mathcal{B}_i \Big|_{g_M} \leq |M|_{g_M},$$

from (8.1) we get $I \leq N_\delta$. For every i we choose a tangency point p_i between $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$. The piecewise geodesic path γ is then constructed by connecting c_i with p_i and p_i with c_{i+1} by using geodesic radii, for $i = 1, \dots, I-2$, and connecting c_{I-1} with q by using a geodesic path contained in $\mathcal{B}_I$. Hence,

$$\operatorname{length}(\gamma) \leq I\delta \leq \delta N_\delta,$$

as required. □

In the next proposition we give an upper bound of the diameter of M when $\partial M = \emptyset$. The proof of the next proposition is analogous to the one of Proposition 8.1, and is omitted.

Proposition 8.2. *Assume $\partial M = \emptyset$ and that there exists a constant $c, \delta > 0$ such that*

$$|\mathcal{B}_r(z)|_{g_M} \geq c r^\kappa, \tag{8.2}$$

for every $z \in M$ and $0 < r \leq \delta$. Let p and q in M. Then, there exists a piecewise geodesic path $\gamma\colon [0,1] \to M$ connecting p and q, of length bounded by δN_δ where

$$N_\delta := \max\left(4, \frac{2^\kappa |M|_{g_M}}{c\delta^\kappa}\right).$$

In particular, the diameter of M is bounded by δN_δ.

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