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A nonexistence result for sign-changing solutions of the Brezis–Nirenberg problem in low dimensions ☆

Alessandro Iacopetti [a], Filomena Pacella [b]

[a] *Dipartimento di Matematica e Fisica, Università degli Studi di Roma Tre, L.go S. Leonardo Murialdo 1, 00146 Roma, Italy*

[b] *Dipartimento di Matematica, Università di Roma "La Sapienza", P.le Aldo Moro 5, 00185 Roma, Italy*

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Abstract

We consider the Brezis–Nirenberg problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\lambda > 0$ is a positive parameter.

The main result of the paper shows that if $N = 4, 5, 6$ and λ is close to zero, there are no sign-changing solutions of the form

$$u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda,$$

where PU_{δ_i} is the projection on $H_0^1(\Omega)$ of the regular positive solution of the critical problem in $\mathbb{R}^N$, centered at a point $\xi \in \Omega$ and w_λ is a remainder term.

Some additional results on norm estimates of w_λ and about the concentrations speeds of tower of bubbles in higher dimensions are also presented.

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E-mail addresses: iacopetti@mat.uniroma3.it (A. Iacopetti), pacella@mat.uniroma1.it (F. Pacella).

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1. Introduction

In this paper we study the semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, λ is a positive real parameter and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.

This problem is known as "the Brezis–Nirenberg problem" because the first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983 in the celebrated paper [6]. From their results it came out that the dimension was going to play a crucial role in the study of (1). Indeed, they proved that if $N \geq 4$ then there exists a positive solution of (1) for every $\lambda \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions, while if $N = 3$ then a positive solution exists only for λ away from zero. In particular, in the case of the ball B they showed that there are no positive solutions in the interval $(0, \frac{\lambda_1(B)}{4})$.

Since then several other interesting results were obtained for positive solutions, in particular, about the asymptotic behavior of solutions, mainly for $N \geq 5$ because also the case $N = 4$ presents more difficulties compared to the higher dimensional ones.

Concerning the case of sign-changing solutions, existence results hold if $N \geq 4$ both for $\lambda \in (0, \lambda_1(\Omega))$ and $\lambda > \lambda_1(\Omega)$ as shown in [3,9,7].

The case $N = 3$ presents even more difficulties than in the study of positive solutions. In particular, in the case of the ball it is not yet known what is the least value $\bar{\lambda}$ of the parameter λ for which sign-changing solutions exist, neither whether $\bar{\lambda}$ is larger or smaller than $\lambda_1(B)/4$. This question, posed by H. Brezis, has been given a partial answer in [5]. However, it is interesting to observe that in the study of sign-changing solutions even the "low dimensions" $N = 4, 5, 6$ exhibit some peculiarities. Indeed, it was first proved by Atkinson, Brezis and Peletier in [2] that if Ω is a ball, there exists $\lambda^* = \lambda^*(N)$ such that there are no radial sign-changing solutions of (1) for $\lambda \in (0, \lambda^*)$. Later this result was re-proved in [1] in a different way.

Moreover, for $N \geq 7$ a recent result of Schechter and Zou [13] shows that in any bounded smooth domain there exist infinitely many sign-changing solutions for any $\lambda > 0$. Instead, if $N = 4, 5, 6$ then only $N + 1$ pairs of solutions, for all $\lambda > 0$, have been proved to exist in [9] but it is not clear whether they change sign.

Coming back to the nonexistence result of [2] and [1], an interesting question would be to see whether and in which way it could be extended to other bounded smooth domains.

Since the result of [2] and [1] concerns nodal radial solutions in the ball the first issue is to understand what are, in general bounded domains, the sign-changing solutions which play the same role as the radial nodal solutions in the case of the ball. A main property of a radial nodal solution in the ball is that its nodal set does not touch the boundary, therefore, a class of solutions to consider, in general bounded domains, could be the one made of functions which have this property.

Moreover, in analyzing the asymptotic behavior of least energy nodal radial solutions u_λ in the ball, as $\lambda \to 0$, in dimension $N \geq 7$ (in which case they exist for all $\lambda \in (0, \lambda_1(B))$, see [8]) one can prove (see [10]) that their limit profile is that of a "tower of two bubbles". This terminology means that the positive part and the negative part of the solutions u_λ concentrate at the same point (which is obviously the center of the ball) as $\lambda \to 0$ and each one has the limit profile, after suitable rescaling, of a "standard" bubble in $\mathbb{R}^N$, i.e. of a positive solution of the critical exponent problem in $\mathbb{R}^N$. More precisely, the solutions u_λ can be written in the following way:

$$u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda, \tag{2}$$

where $PU_{\delta_i,\xi}$, $i = 1, 2$, is the projection on $H_0^1(\Omega)$ of the regular positive solution of the critical problem in $\mathbb{R}^N$, centered at $\xi = 0$, with rescaling parameter δ_i, and w_λ is a remainder term which converges to zero in $H_0^1(\Omega)$.

It is also interesting to observe that, thanks to a recent result of [11], sign-changing bubble-tower solutions exist also in bounded smooth symmetric domains in dimension $N \geq 7$ for λ close to zero, and they have the property that their nodal set does not touch the boundary of the domain.

In view of all these remarks we are entitled to assert that in general bounded domains sign-changing solutions which behave as the radial ones in the ball, at least for λ close to zero, are the ones which are of the form (2). Hence, a natural extension of the nonexistence result of [2] and [1] would be to show that, in dimension $N = 4, 5, 6$, sign-changing solutions of the form (2) do not exist in any bounded smooth domain.

This is indeed the main aim of this paper. Let us also note that in the 3-dimensional case a similar nonexistence result was already proved in [5]. Indeed, in studying the asymptotic behavior of low-energy nodal solutions it was shown in [5] that their positive and negative part cannot concentrate at the same point, as λ tends to a limit value $\bar{\lambda} > 0$. In the case $N \geq 4$ this question was left open in [4]. Therefore, our results also complete the analysis made in these last two papers.

To state precisely our result, let us recall that the functions

$$U_{\delta,\xi}(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{N-2}{2}}}, \quad \delta > 0,\ \xi \in \mathbb{R}^N, \tag{3}$$

$\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, describe all regular positive solutions of the problem

$$\begin{cases} -\Delta U = U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ U(x) \to 0, & \text{as } |x| \to +\infty. \end{cases}$$

Then, denoting by PU_δ their projection on $H_0^1(\Omega)$, and by $\|u\| := \int_\Omega |\nabla u|^2 \, dx$ for any $u \in H_0^1(\Omega)$, we have:

Theorem 1. *Let $N = 4, 5, 6$ and let ξ be a point in the domain Ω. Then, for λ close to zero, Problem (1) does not admit any sign-changing solution u_λ of the form (2) with $\delta_i = \delta_i(\lambda)$, $i = 1, 2$, such that $\delta_2 = o(\delta_1)$, $\|w_\lambda\| \to 0$ and $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$, $|\nabla w_\lambda| = o(\delta_1^{-\frac{N}{2}})$ uniformly in compact subsets of Ω, as $\lambda \to 0$.*

The previous notations mean that $\frac{|w_\lambda|}{\delta_1^{-\frac{N-2}{2}}}$, $\frac{|\nabla w_\lambda|}{\delta_1^{-\frac{N}{2}}}$ converge to zero as $\lambda \to 0$ uniformly in compact subsets of Ω.

The proof of the above theorem is based on a Pohozaev identity and fine estimates which are derived in a different way in the case $N = 4$ or $N = 5, 6$. We would like to point out that it cannot be deduced by the proof of Theorem 3.1 of [5] which holds only in dimension three.

Concerning the assumption on the C^1-norm in compact subsets of Ω of the remainder term w_λ, whose gradient is only required not to blow up too fast, in Section 4 we show that it is almost necessary.

Note that we do not even require that $w_\lambda \to 0$ uniformly in Ω neither that it remains bounded as $\lambda \to 0$, but only a control of possible blow-up of $|w_\lambda|$ and $|\nabla w_\lambda|$. We delay to the next sections some further comments and comparisons with the case $N \geq 7$.

Finally, in the last section we show that in dimension $N \geq 7$ if (u_λ) is a family of solutions of type (2) with $|w_\lambda|$, $|\nabla w_\lambda|$ as in Theorem 1 and $\delta_i = d_i \lambda^{\alpha_i}$, for some positive numbers $d_i = d_i(\lambda)$ with $0 < c_1 < d_i < c_2$, for all sufficiently small λ, and $0 < \alpha_1 < \alpha_2$, then necessarily:

$$\alpha_1 = \frac{1}{N-4}, \qquad \alpha_2 = \frac{3N-10}{(N-4)(N-6)}. \tag{4}$$

In other words, we prove that if the concentration speeds are powers of λ then necessarily the exponent must be as in (4). Note that these are exactly the type of speeds assumed in [11] to construct the tower of bubbles in higher dimensions.

2. Some preliminary results

Lemma 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and let $(\delta, \xi) \in \mathbb{R}^+ \times \Omega$. As $\delta \to 0$, it holds:*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \alpha_N \delta^{\frac{N-2}{2}} H(x,\xi) + o\big(\delta^{\frac{N-2}{2}}\big), \quad x \in \Omega$$

C^1-uniformly on compact subsets of Ω, where H is the regular part of the Green function for the Laplacian. Moreover, setting $\varphi_{\delta,\xi}(x) := U_{\delta,\xi}(x) - PU_{\delta,\xi}(x)$, the following uniform estimates hold:

(i) $0 \leq \varphi_{\delta,\xi} \leq U_{\delta,\xi}$,
(ii) $\|\varphi_{\delta,\xi}\|^2 = O((\frac{\delta}{d})^{N-2})$,

where $d = d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and the boundary of Ω.

Proof. See [12, Proposition 1] and its proof. □

Lemma 2. *Let $N \geq 4$ and let (u_λ) be a family of sign-changing solutions of (1) satisfying*

$$\|u_\lambda\|^2 \to 2S^{N/2}, \quad \text{as } \lambda \to 0.$$

Then, for all sufficiently small $\lambda > 0$, the set $\Omega \setminus \{x \in \Omega;\ u_\lambda(x) = 0\}$ has exactly two connected components.

Proof. Let us consider the nodal set $Z_\lambda := \{x \in \Omega;\ u_\lambda(x) = 0\}$ and let Ω_1 be a connected component of $\Omega \setminus Z_\lambda$. Multiplying (1) by u_λ and integrating on Ω_1, we get that

$$\int_{\Omega_1} |\nabla u_\lambda|^2\, dx \geq S^{N/2}\big(1 + o(1)\big),$$

where we have used the Sobolev embedding and the fact that $\lambda \to 0$ and $\lambda_1(\Omega_1)\int_{\Omega_1} u_\lambda^2\, dx \leq \int_{\Omega_1} |\nabla u_\lambda|^2\, dx$, where $\lambda_1(\Omega_1)$ is the first Dirichlet eigenvalue of $-\Delta$ on Ω_1.

Since $\|u_\lambda\|^2 \to 2S^{N/2}$, as $\lambda \to 0$, then for all sufficiently small $\lambda > 0$, we deduce that $\Omega \setminus Z_\lambda$ can have only two connected components. □

We recall now the Pohozaev identity for solutions of semilinear problems which are not necessarily zero on the boundary. Let D be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary and consider the equation

$$-\Delta u = f(u) \quad \text{in } D, \tag{5}$$

where $s \mapsto f(s)$ is a continuous function. Denoting $F(s) := \int_0^s f(t)\, dt$, we have:

Proposition 1. *Let u be a C^2-solution of (5), then*

$$\int_D \left\{NF(u) - \frac{N-2}{2} u f(u)\right\} dx$$
$$= \int_{\partial D} \left\{\sum_{i=1}^N x_i \nu_i \left(F(u) - \frac{1}{2}|\nabla u|^2\right) + \frac{\partial u}{\partial \nu}\sum_{i=1}^N x_i u_{x_i} + \frac{N-2}{2} u \frac{\partial u}{\partial \nu}\right\} d\sigma, \tag{6}$$

where ν denotes the outer normal to the boundary and u_{x_i} is the partial derivative with respect to x_i of u.

The following lemma gives information on the asymptotic behavior of the nodal set Z_λ of solutions of (1) as $\lambda \to 0$.

Lemma 3. *Let $N \geq 4$, $\xi \in \Omega$ and let (u_λ) be a family of solutions of (1), such that $u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda$, with $\delta_1 = \delta_1(\lambda)$ and $\delta_2 = \delta_2(\lambda)$ satisfying*

$$\delta_2 = o(\delta_1) \quad \textit{and} \quad \|w_\lambda\| \to 0,\ \textit{as } \lambda \to 0.$$

Moreover, assume that w_λ satisfies $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$ uniformly in compact subsets of Ω. Then, for all small $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ such that the nodal set Z_λ is contained in the annular region $A_{r_1,r_2}(\xi) := \{x \in \Omega;\ r_1 < |x - \xi| < r_2\}$, for all $\lambda \in (0, \lambda_\epsilon)$, where $r_1 := \delta_1^{\frac{1}{2}-\epsilon}\delta_2^{\frac{1}{2}+\epsilon}$, $r_2 := \delta_1^{\frac{1}{2}+\epsilon}\delta_2^{\frac{1}{2}-\epsilon}$.

Proof. Without loss of generality we assume that $\xi = 0$. In order to simplify the notation we write U_{δ_j} instead of $U_{\delta_j,0}$, for $j = 1, 2$. Let us fix a small $\epsilon > 0$ and a compact neighborhood of the origin K. Thanks to the assumptions and Lemma 1, we have the following expansion $u_\lambda(x) = U_{\delta_1}(x) - U_{\delta_2}(x) + o(\delta_1^{-\frac{N-2}{2}})$, which is uniform with respect to $x \in K$ and to all small $\lambda > 0$. By definition, for all sufficiently small $\lambda > 0$, we have that $A_{r_1,r_2}(0) \subset K$. For x such that $|x| = r_1$, we have:

$$U_{\delta_1}(x) = \alpha_N \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2 + \delta_1^{1-2\epsilon}\delta_2^{1+2\epsilon})^{\frac{N-2}{2}}} = \alpha_N \frac{\delta_1^{-\frac{N-2}{2}}}{[1 + (\frac{\delta_2}{\delta_1})^{1+2\epsilon}]^{\frac{N-2}{2}}}$$
$$= \alpha_N\, \delta_1^{-\frac{N-2}{2}} - \alpha_N \frac{N-2}{2} \delta_1^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{1+2\epsilon} + o\left(\delta_1^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{1+2\epsilon}\right),$$

and

$$U_{\delta_2}(x) = \alpha_N \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2 + \delta_1^{1-2\epsilon}\delta_2^{1+2\epsilon})^{\frac{N-2}{2}}} = \alpha_N \frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N-2}{2}+(N-2)\epsilon} \delta_2^{-\frac{N-2}{2}-(N-2)\epsilon}}{[1 + (\frac{\delta_2}{\delta_1})^{1-2\epsilon}]^{\frac{N-2}{2}}}$$
$$= \alpha_N \frac{\delta_1^{-\frac{N-2}{2}} (\frac{\delta_2}{\delta_1})^{-(N-2)\epsilon}}{[1 + (\frac{\delta_2}{\delta_1})^{1-2\epsilon}]^{\frac{N-2}{2}}}$$
$$= \alpha_N\, \delta_1^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{-(N-2)\epsilon} - \alpha_N \frac{N-2}{2} \delta_1^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{1-N\epsilon}$$
$$+ o\left(\delta_1^{-\frac{N-2}{2}} \left(\frac{\delta_2}{\delta_1}\right)^{1-N\epsilon}\right).$$

Hence, for $x \in K$, such that $|x| = r_1$, we have

$$u_\lambda(x) = \alpha_N\, \delta_1^{-\frac{N-2}{2}} \left(1 - \left(\frac{\delta_2}{\delta_1}\right)^{-(N-2)\epsilon}\right) + o\big(\delta_1^{-\frac{N-2}{2}}\big) < 0$$

for all sufficiently small $\lambda > 0$. On the other hand, by similar computations (just changing the sign of ϵ in every term of the previous equations), for x such that $|x| = r_2$, we have

$$u_\lambda(x) = \alpha_N\, \delta_1^{-\frac{N-2}{2}} \left(1 - \left(\frac{\delta_2}{\delta_1}\right)^{+(N-2)\epsilon}\right) + o\big(\delta_1^{-\frac{N-2}{2}}\big) > 0$$

for all sufficiently small $\lambda > 0$.

From Lemma 2 and since u_λ is a continuous function, we deduce that $Z_\lambda \subset A_{r_1,r_2}(0)$ for all sufficiently small $\lambda > 0$. □

3. Proof of the nonexistence result

We begin considering the case $N = 5, 6$ since the case $N = 4$ requires different estimates.

Proof of Theorem 1 for $N = 5, 6$. Arguing by contradiction let us assume that such a family of solutions exists and, without loss of generality set $\xi = 0$. Defining $r := \sqrt{\delta_1 \delta_2}$, we apply the Pohozaev formula (6) to u_λ in the ball $B_r = B_r(0)$. Since u_λ is a solution of (1), we set $f(u) := \lambda u + |u|^{p-1}u$, with $p = 2^* - 1$, and hence, using the notation of Proposition 1, we have $F(u) = \frac{\lambda}{2}u^2 + \frac{1}{p+1}|u|^{p+1}$. By elementary computations[1] (see the footnote), we get that the left-hand side of (6) reduces to

$$\lambda \int_{B_r} u_\lambda^2 \, dx.$$

For the right-hand side

$$\int_{\partial B_r} \left\{ \sum_{i=1}^N x_i \nu_i \left(F(u_\lambda) - \frac{1}{2}|\nabla u_\lambda|^2 \right) + \frac{\partial u_\lambda}{\partial \nu} \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} + \frac{N-2}{2} u_\lambda \frac{\partial u_\lambda}{\partial \nu} \right\} d\sigma,$$

since ∂B_r is a sphere, we have $\nu_i(x) = \frac{x_i}{|x|}$ for all $x \in \partial B_r$, $i = 1, \dots, N$, and hence, $\sum_{i=1}^N x_i \nu_i = |x|$. Furthermore, since $\frac{\partial u_\lambda}{\partial \nu} = \nabla u_\lambda \cdot \frac{x}{|x|}$ and $\sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = (\nabla u_\lambda \cdot \frac{x}{|x|})|x|$, we get that

$$\frac{\partial u_\lambda}{\partial \nu} \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = \left(\nabla u_\lambda \cdot \frac{x}{|x|} \right) \sum_{i=1}^N x_i \frac{\partial u_\lambda}{\partial x_i} = \left(\nabla u_\lambda \cdot \frac{x}{|x|} \right)^2 |x|,$$
$$u_\lambda \frac{\partial u_\lambda}{\partial \nu} = u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|} \right).$$

Thus, Eq. (6) rewrites as

$$\lambda \int_{B_r} u_\lambda^2 \, dx = \int_{\partial B_r} \left\{ |x| \left(F(u_\lambda) - \frac{1}{2}|\nabla u_\lambda|^2 \right) + \left(\nabla u_\lambda \cdot \frac{x}{|x|} \right)^2 |x| + \frac{N-2}{2} u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|} \right) \right\} d\sigma. \quad (7)$$

We estimate the left-hand side of (7). Let us fix a compact subset $K \subset \Omega$; for $\lambda > 0$ sufficiently small, we get that $B_r \subset K$. Thanks to Lemma 1, we have $PU_{\delta_j} = U_{\delta_j} - \varphi_{\delta_j}$, where $\varphi_{\delta_j} = O(\delta_j^{\frac{N-2}{2}})$, for $j = 1, 2$, and this estimate is uniform for $x \in K$, in particular, for $x \in B_r$. Thus, as $\lambda \to 0$, we get that

[1]
$$\begin{aligned} NF(u) - \frac{N-2}{2} u f(u) &= N\left(\frac{\lambda}{2}u^2 + \frac{1}{p+1}|u|^{p+1} \right) - \frac{N-2}{2}(\lambda u^2 + |u|^{p+1}) \\ &= \left(\frac{N}{2} - \frac{N-2}{2} \right) \lambda u^2 + \left(\frac{N}{p+1} - \frac{N-2}{2} \right) |u|^{p+1} \\ &= \lambda u^2. \end{aligned}$$

$$
\begin{aligned}
\lambda \int_{B_r} u_\lambda^2 \, dx &= \lambda \int_{B_r} \big(PU_{\delta_1} - PU_{\delta_2} + o\big(\delta_1^{-\frac{N-2}{2}}\big)\big)^2 \, dx \\
&= \lambda \int_{B_r} \big(U_{\delta_1} - U_{\delta_2} - \varphi_{\delta_1} + \varphi_{\delta_2} + o\big(\delta_1^{-\frac{N-2}{2}}\big)\big)^2 \, dx \\
&= \lambda \int_{B_r} \big(U_{\delta_1} - U_{\delta_2} + o\big(\delta_1^{-\frac{N-2}{2}}\big)\big)^2 \, dx \\
&= \lambda \int_{B_r} \big(U_{\delta_1}^2 + U_{\delta_2}^2 - 2U_{\delta_1}U_{\delta_2} + o\big(\delta_1^{-\frac{N-2}{2}} U_{\delta_1}\big) + o\big(\delta_1^{-\frac{N-2}{2}} U_{\delta_2}\big) + o\big(\delta_1^{-\frac{N-2}{2}}\big)\big) \, dx \\
&= A + B + C + D + E + F. \qquad (8)
\end{aligned}
$$

We estimate every term of the previous decomposition.

$$
\begin{aligned}
A &= \lambda \int_{B_r} \alpha_N^2 \frac{\delta_1^{N-2}}{(\delta_1^2 + |x|^2)^{N-2}} \, dx = \alpha_N^2 \lambda \int_{B_r} \frac{\delta_1^{-(N-2)}}{(1 + |x/\delta_1|^2)^{N-2}} \, dx \\
&= \alpha_N^2 \lambda \delta_1^2 \int_{B_r/\delta_1} \frac{1}{(1+|y|^2)^{N-2}} \, dy \le \alpha_N^2 \lambda \delta_1^2 |B_r/\delta_1| \\
&= c_N \lambda \delta_1^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}},
\end{aligned}
$$

where we have set $c_N := \alpha_N^2 \frac{\omega_N}{N}$, ω_N is the measure of the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$.

$$
\begin{aligned}
B &= \lambda \int_{B_r} \alpha_N^2 \frac{\delta_2^{N-2}}{(\delta_2^2 + |x|^2)^{N-2}} \, dx = \alpha_N^2 \lambda \int_{B_r} \frac{\delta_2^{-(N-2)}}{(1 + |x/\delta_2|^2)^{N-2}} \, dx \\
&= \alpha_N^2 \lambda \delta_2^2 \int_{B_r/\delta_2} \frac{1}{(1+|y|^2)^{N-2}} \, dy \\
&= \alpha_N^2 \lambda \delta_2^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy + O\left(\lambda \delta_2^2 \int_{(\frac{\delta_1}{\delta_2})^{\frac{1}{2}}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}} \, dr\right) \\
&= a_1 \lambda \delta_2^2 + O\left(\lambda \delta_2^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-4}{2}}\right),
\end{aligned}
$$

where we have set $a_1 := \alpha_N^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}} \, dy$. We point out that since $N = 5$ or $N = 6$ the function $\frac{1}{(1+|y|^2)^{N-2}} \in L^1(\mathbb{R}^N)$, while this is not true when $N = 4$.

$$|C| = \lambda\, \alpha_N^2 \int_{B_r} \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}}\, dx$$

$$= \lambda\, \alpha_N^2 \int_{B_r/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+\delta_1^2|y|^2)^{\frac{N-2}{2}}}\, dy$$

$$= \lambda\, \alpha_N^2 \int_{B_r/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{((\frac{\delta_2}{\delta_1})^2+|y|^2)^{\frac{N-2}{2}}}\, dy$$

$$\le \lambda\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{B_r/\delta_1} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy$$

$$= O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_0^{(\frac{\delta_2}{\delta_1})^{1/2}} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}} r^{N-2}}\, dr\right)$$

$$= O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} \delta_1^2\right);$$

$$|D| = o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}}}\, dx\right)$$

$$\le o\left(\lambda \int_{B_r} \delta_1^{-(N-2)}\, dx\right)$$

$$= o\left(\lambda \delta_1^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right);$$

$$|E| = o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}}\, dx\right)$$

$$\le o\left(\lambda \delta_1^{-\frac{N-2}{2}} \int_{B_r} \frac{\delta_2^{\frac{N-2}{2}}}{|x|^{N-2}}\, dx\right)$$

$$= o\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right);$$

$$|F| = o\big(\lambda \delta_1^{-\frac{N-2}{2}} |B_r|\big)$$

$$= o\big(\lambda\, \delta_1\, \delta_2^{\frac{N}{2}}\big).$$

Now we estimate the right-hand side of (7). Remembering that $F(u_\lambda)=\frac{\lambda}{2}u_\lambda^2+\frac{1}{p+1}|u_\lambda|^{p+1}$, we get that the first term is equal to

$$\int_{\partial B_r} |x|\left(\frac{\lambda}{2}u_\lambda^2+\frac{1}{p+1}|u_\lambda|^{p+1}-\frac{1}{2}|\nabla u_\lambda|^2\right) d\sigma.$$

We observe that by definition of r it is immediate to see that

$$U_{\delta_1}(x)=U_{\delta_2}(x),$$

for all $x\in\partial B_r$, and hence, we have

$$\begin{aligned}\int_{\partial B_r} \frac{\lambda}{2}u_\lambda^2\,|x|\,d\sigma &= \frac{\lambda}{2}\int_{\partial B_r}\left(U_{\delta_1}-U_{\delta_2}+o\left(\delta_1^{-\frac{N-2}{2}}\right)\right)^2|x|\,d\sigma\\ &=\frac{\lambda}{2}\int_{\partial B_r}\left[o\left(\delta_1^{-\frac{N-2}{2}}\right)\right]^2|x|\,d\sigma\\ &=o\left(\lambda\delta_1^{-(N-2)}\int_{\partial B_r}|x|\,d\sigma\right)\\ &=o\left(\lambda\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\delta_1^2\right).\end{aligned}$$

As in the previous case, we have

$$\begin{aligned}\frac{1}{p+1}\int_{\partial B_r}|u_\lambda|^{p+1}|x|\,d\sigma &= \frac{1}{p+1}\int_{\partial B_r}\left|U_{\delta_1}-U_{\delta_2}+o\left(\delta_1^{-\frac{N-2}{2}}\right)\right|^{p+1}|x|\,d\sigma\\ &=\frac{1}{p+1}\int_{\partial B_r}\left|o\left(\delta_1^{-\frac{N-2}{2}}\right)\right|^{p+1}|x|\,d\sigma\\ &=o\left(\delta_1^{-N}\int_{\partial B_r}|x|\,d\sigma\right)\\ &=o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).\end{aligned}$$

To complete the estimate of the first term, it remains to analyze

$$-\frac{1}{2}\int_{\partial B_r}|\nabla u_\lambda|^2|x|\,d\sigma.$$

As before, writing $PU_{\delta_j}=U_{\delta_j}-\varphi_{\delta_j}$ for $j=1,2$, we have

$$|\nabla u_\lambda|^2 = |\nabla U_{\delta_1} - \nabla U_{\delta_2} - \nabla\varphi_{\delta_1} + \nabla\varphi_{\delta_2} + \nabla w_\lambda|^2 = |\nabla U_{\delta_1} - \nabla U_{\delta_2} + \nabla \Phi_\lambda|^2,$$

where we have set $\Phi_\lambda := -\varphi_{\delta_1} + \varphi_{\delta_2} + w_\lambda$. Hence, we get that

$$\begin{aligned}
&-\frac{1}{2}\int_{\partial B_r} |\nabla u_\lambda|^2 |x|\, d\sigma \\
&= -\frac{1}{2}\int_{\partial B_r} |\nabla U_{\delta_1}|^2\, |x|\, d\sigma - \frac{1}{2}\int_{\partial B_r} |\nabla U_{\delta_2}|^2\, |x|\, d\sigma + \int_{\partial B_r} \nabla U_{\delta_1}\cdot\nabla U_{\delta_2}\, |x|\, d\sigma \\
&\quad - \int_{\partial B_r} \nabla U_{\delta_1}\cdot\nabla \Phi_\lambda\, |x|\, d\sigma + \int_{\partial B_r} \nabla U_{\delta_2}\cdot\nabla \Phi_\lambda\, |x|\, d\sigma - \frac{1}{2}\int_{\partial B_r} |\nabla \Phi_\lambda|^2\, |x|\, d\sigma \\
&= A_1 + B_1 + C_1 + D_1 + E_1 + F_1. \qquad (9)
\end{aligned}$$

By elementary computations, for all $i = 1, \dots, N$, $j = 1, 2$, we have:

$$\begin{aligned}
\frac{\partial U_{\delta_j}}{\partial x_i}(x) &= -\alpha_N (N-2)\delta_j^{\frac{N-2}{2}} \frac{x_i}{(\delta_j^2 + |x|^2)^{\frac{N}{2}}}, \\
|\nabla U_{\delta_j}|^2 &= \alpha_N^2 (N-2)^2 \delta_j^{N-2} \frac{|x|^2}{(\delta_j^2 + |x|^2)^N}. \qquad (10)
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
A_1 &= -\alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_1^{-(N+2)}}{[1 + (\frac{\delta_2}{\delta_1})]^N} \int_{\partial B_r} |x|^3\, d\sigma \\
&= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \frac{\delta_1^{-(N+2)}}{[1 + (\frac{\delta_2}{\delta_1})]^N} \delta_1^{\frac{N+2}{2}} \delta_2^{\frac{N+2}{2}} \\
&= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+4}{2}}\right);
\end{aligned}$$

$$\begin{aligned}
B_1 &= -\alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_2^{N-2}\delta_1^{-N}\delta_2^{-N}}{[1 + (\frac{\delta_2}{\delta_1})]^N} \int_{\partial B_r} |x|^3\, d\sigma \\
&= -\alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).
\end{aligned}$$

$$C_1 = \alpha_N^2 (N-2)^2 \frac{\delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}} \delta_1^{-N} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{[1 + (\frac{\delta_2}{\delta_1})]^{\frac{N}{2}} [1 + (\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \int_{\partial B_r} |x|^3\, d\sigma$$

$$= \alpha_N^2 (N-2)^2 \omega_N \frac{(\frac{\delta_2}{\delta_1})^{\frac{N}{2}}}{[1+(\frac{\delta_2}{\delta_1})]^N}$$

$$= \alpha_N^2 (N-2)^2 \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} + O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}}\right).$$

Taking into account the assumptions on the remainder term w_λ and thanks to Lemma 1, we have $|\nabla \Phi_\lambda| = o(\delta_1^{-\frac{N}{2}})$, uniformly on ∂B_r. Thus, we have the following:

$$\begin{aligned}
|D_1| &\le \int\limits_{\partial B_r} |\nabla U_{\delta_1}||\nabla \Phi_\lambda||x| \, d\sigma \\
&= o\left(\frac{\delta_1^{\frac{N-2}{2}}}{(\delta_1^2+\delta_1\delta_2)^{\frac{N}{2}}} \delta_1^{-\frac{N}{2}} \int\limits_{\partial B_r} |x|^2 \, d\sigma\right) \\
&= o\left(\frac{\delta_1^{\frac{N-2}{2}} \delta_1^{-N}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \delta_1^{-\frac{N}{2}} \int\limits_{\partial B_r} |x|^2 \, d\sigma\right) \\
&= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+1}{2}}\right);
\end{aligned}$$

$$\begin{aligned}
|E_1| &\le \int\limits_{\partial B_r} |\nabla U_{\delta_2}||\nabla \Phi_\lambda||x| \, d\sigma \\
&= o\left(\frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \delta_1^{-\frac{N}{2}} \int\limits_{\partial B_r} |x|^2 \, d\sigma\right) \\
&= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-1}{2}}\right).
\end{aligned}$$

And finally, the last term of (9) is trivial:

$$|F_1| = o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).$$

Now we analyze the term

$$\int\limits_{\partial B_r} \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)^2 |x| \, d\sigma. \tag{11}$$

As before, we write $u_\lambda = U_{\delta_1} - U_{\delta_2} + \Phi_\lambda$ and we have

$$\begin{aligned}\left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)^2 |x| &= \left(\nabla U_{\delta_1} \cdot \frac{x}{|x|}\right)^2 |x| + \left(\nabla U_{\delta_2} \cdot \frac{x}{|x|}\right)^2 |x| \\ &\quad - 2\left(\nabla U_{\delta_1} \cdot \frac{x}{|x|}\right)\left(\nabla U_{\delta_2} \cdot \frac{x}{|x|}\right)|x| \\ &\quad + 2\left(\nabla U_{\delta_1} \cdot \frac{x}{|x|}\right)\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)|x| \\ &\quad - 2\left(\nabla U_{\delta_2} \cdot \frac{x}{|x|}\right)\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)|x| \\ &\quad + \left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)^2 |x|. \qquad (12)\end{aligned}$$

By elementary computations we see that for $j = 1, 2$

$$\left(\nabla U_{\delta_j} \cdot \frac{x}{|x|}\right)^2 |x| = |\nabla U_{\delta_j}|^2 \, |x|,$$

$$-2\left(\nabla U_{\delta_1} \cdot \frac{x}{|x|}\right)\left(\nabla U_{\delta_2} \cdot \frac{x}{|x|}\right) |x| = -2(\nabla U_{\delta_1} \cdot \nabla U_{\delta_2}) \, |x|,$$

and for the remaining terms, we have

$$\left| \pm 2\left(\nabla U_{\delta_j} \cdot \frac{x}{|x|}\right)\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)|x| \right| \le 2|\nabla U_{\delta_j}||\nabla \Phi_\lambda||x|,$$

$$\left| \left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right)^2 |x| \right| \le |\nabla \Phi_\lambda|^2 |x|.$$

Thus, in order to estimate (11) it suffices to apply the estimates of the previous case, and hence, we get that

$$\int_{\partial B_r} \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right)^2 |x| \, d\sigma = \alpha_N^2 (N-2)^2 \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right).$$

To complete our analysis of (7) it remains only to study the term

$$\frac{N-2}{2} \int_{\partial B_r} u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right) d\sigma.$$

$$\begin{aligned}&\frac{N-2}{2} \int_{\partial B_r} u_\lambda \left(\nabla u_\lambda \cdot \frac{x}{|x|}\right) d\sigma \\ &= \frac{N-2}{2} \int_{\partial B_r} (U_{\delta_1} - U_{\delta_2} + \Phi_\lambda)\left[(\nabla U_{\delta_1} - \nabla U_{\delta_2} + \nabla \Phi_\lambda) \cdot \frac{x}{|x|}\right] d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \frac{N-2}{2} \int_{\partial B_r} \Phi_\lambda\left(\nabla U_{\delta_1} \cdot \frac{x}{|x|}\right) d\sigma - \frac{N-2}{2} \int_{\partial B_r} \Phi_\lambda\left(\nabla U_{\delta_2} \cdot \frac{x}{|x|}\right) d\sigma \\
&\quad + \frac{N-2}{2} \int_{\partial B_r} \Phi_\lambda\left(\nabla \Phi_\lambda \cdot \frac{x}{|x|}\right) d\sigma \\
&= A_2 + B_2 + C_2. \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
|A_2| &\le \alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_1^{\frac{N-2}{2}} \delta_1^{-N}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \int_{\partial B_r} |\Phi_\lambda|\, |x|\, d\sigma \\
&= o\left(\frac{\delta_1^{\frac{N-2}{2}} \delta_1^{-N}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \int_{\partial B_r} \delta_1^{-\frac{N-2}{2}}\, |x|\, d\sigma\right) \\
&= o\left(\frac{\delta_1^{-N}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \delta_1^{\frac{N}{2}} \delta_2^{\frac{N}{2}}\right) \\
&= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right).
\end{aligned}
$$

$$
\begin{aligned}
|B_2| &\le \alpha_N^2 \frac{(N-2)^2}{2} \frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \int_{\partial B_r} |\Phi_\lambda|\, |x|\, d\sigma \\
&= o\left(\frac{\delta_2^{\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_2^{-\frac{N}{2}}}{[1+(\frac{\delta_2}{\delta_1})]^{\frac{N}{2}}} \int_{\partial B_r} \delta_1^{-\frac{N-2}{2}}\, |x|\, d\sigma\right) \\
&= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right).
\end{aligned}
$$

$$
\begin{aligned}
|C_2| &\le \frac{(N-2)}{2} \int_{\partial B_r} |\Phi_\lambda||\nabla \Phi_\lambda|\, d\sigma \\
&= o\big(\delta_1^{-\frac{N-2}{2}} \delta_1^{-\frac{N}{2}} \delta_1^{\frac{N-1}{2}} \delta_2^{\frac{N-1}{2}}\big) \\
&= o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-1}{2}}\right).
\end{aligned}
$$

Summing up all the estimates, from (6), for all sufficiently small $\lambda > 0$, we deduce the following equation:

$$
a_1 \lambda \delta_2^2 + o\big(\lambda \delta_2^2\big) = \alpha_N^2 \frac{(N-2)^2}{2} \omega_N \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right). \qquad (14)
$$

From (14), we deduce that

$$a_1\lambda\delta_1^{\frac{N-2}{2}}\big(1+o(1)\big)=\alpha_N^2\frac{(N-2)^2}{2}\omega_N\delta_2^{\frac{N-6}{2}}\big(1+o(1)\big), \tag{15}$$

for all sufficiently small $\lambda > 0$. Since $N = 5, 6$, it is clear that (15) is contradictory; in fact, passing to the limit as $\lambda \to 0$, the left-hand side goes to zero while the right-hand side goes to a positive constant, when $N = 6$ and diverges to $+\infty$ when $N = 5$. The proof is complete. □

Now we turn to the case $N = 4$

Proof of Theorem 1 for $N = 4$. Again, without loss of generality we assume that $\xi = 0$. We repeat the scheme of the proof for the previous case, but some modification is needed. In fact, since $N = 4$, we have to change the estimate of the term B in (8):

$$B_* = \lambda\int_{B_r}\alpha_4^2\frac{\delta_2^2}{(\delta_2^2+|x|^2)^2}\,dx = \alpha_4^2\lambda\int_{B_r/\delta_2}\frac{\delta_2^{-2}}{(1+|y|^2)^2}\delta_2^4\,dy$$

$$= \alpha_4^2\lambda\delta_2^2\int_{B_r/\delta_2}\frac{1}{(1+|y|^2)^2}\,dy = \alpha_4^2\omega_4\lambda\delta_2^2\int_0^{(\frac{\delta_1}{\delta_2})}\frac{r^3}{(1+r^2)^2}\,dr$$

It is elementary to see that

$$\int_0^{(\frac{\delta_1}{\delta_2})}\frac{r^3}{(1+r^2)^2}\,dr = O\left(\log\left(\frac{\delta_1}{\delta_2}\right)\right),$$

and hence, we have that

$$B_* = O\left(\lambda\delta_2^2\log\left(\frac{\delta_1}{\delta_2}\right)\right). \tag{16}$$

Thus, summing up (16) with the other estimates made in the previous case (in which we take $N = 4$), from (6), we deduce the following asymptotic relation:

$$O\left(\lambda\delta_2^2\log\left(\frac{\delta_1}{\delta_2}\right)\right)+o\left(\lambda\delta_2^2\log\left(\frac{\delta_1}{\delta_2}\right)\right)=2\alpha_4^2\omega_4\left(\frac{\delta_2}{\delta_1}\right)+o\left(\frac{\delta_2}{\delta_1}\right). \tag{17}$$

It is clear that (17) gives a contradiction. In fact, dividing each side of (17) by $(\frac{\delta_2}{\delta_1})$, we have

$$O\left(\lambda\delta_1\delta_2\log\left(\frac{\delta_1}{\delta_2}\right)\right)+o\left(\lambda\delta_1\delta_2\log\left(\frac{\delta_1}{\delta_2}\right)\right)=2\alpha_4^2\omega_4+o(1). \tag{18}$$

Passing to the limit as $\lambda \to 0$ in (18), taking into account that $\delta_2 = o(\delta_1)$, we deduce that $0 = 2\alpha_4^2\omega_4$ which is a contradiction. □

Remark 1. In [4,5] sign-changing solutions u_λ of (1) with low energy were studied, namely, solutions such that

$$\int_\Omega |\nabla u_\lambda|^2\, dx \to 2S^{N/2}.$$

For this kind of solutions it is not difficult to show (see [4, Theorem 1.1]) that there exist two points $a_1 = a_1(\lambda)$, $a_2 = a_2(\lambda)$ in Ω (one of them is the global maximum point of $|u_\lambda|$) and two positive real numbers $\delta_1 = \delta_1(\lambda)$, $\delta_2 = \delta_2(\lambda)$, such that for $N \geq 4$, as $\lambda \to 0$, we have

$$\|u_\lambda - PU_{\delta_1,a_1} + PU_{\delta_2,a_2}\| \to 0, \qquad \delta_i^{-1} d(a_i, \partial\Omega) \to +\infty, \quad \text{for } i = 1, 2,$$

where $d(a_i, \partial\Omega)$ is the Euclidean distance between a_i and the boundary of Ω. Hence, these solutions are of the form (2) but with possibly different concentration points. In [4], assuming that the concentration speeds of u_λ^+ and u_λ^- were comparable, it was proved that the positive and the negative part of u_λ had to concentrate in two different points.

Since here we assume that the concentration speeds are different, our result also completes the study made in [4].

4. About the estimate on the C^1-norm of w_λ

Here we show that the hypotheses of Theorem 1 on the C^1-norm of the remainder term w_λ are almost necessary. Indeed, we have:

Theorem 2. *Let Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary, $N \geq 4$, and let $\xi \in \Omega$. Let u_λ be a solution of (1) of the form*

$$u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda,$$

with $\delta_2 = o(\delta_1)$ as $\lambda \to 0$. Assume that the remainder term w_λ is uniformly bounded with respect to λ in compact subsets of Ω. Then for any open subset $\Omega'' \subset\subset \Omega$ such that $\xi \in \Omega''$ and for all sufficiently small $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, N, \Omega'')$ such that

$$\|w_\lambda\|_{C^1(\bar{\Omega}'')} \leq C\delta_1^{-\frac{N-2}{2}} \delta_2^{-1+O(\epsilon)},$$

for all sufficiently small $\lambda > 0$.

Proof. Without loss of generality we assume that $\xi = 0$. By definition w_λ satisfies the following:

$$\begin{cases} -\Delta w_\lambda = \lambda w_\lambda + \lambda(PU_{\delta_1} - PU_{\delta_2}) + U_{\delta_2}^p - U_{\delta_1}^p + |u_\lambda|^{2^*-2}u_\lambda & \text{in } \Omega, \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Let us set $f_\lambda := \lambda w_\lambda + \lambda(PU_{\delta_1} - PU_{\delta_2}) + U_{\delta_2}^p - U_{\delta_1}^p + |u_\lambda|^{2^*-2}u_\lambda$. Since w_λ and u_λ are smooth, applying the Calderón–Zygmund inequality, we deduce that for any $p \in (1, \infty)$, for any $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ it holds:

$$\|w_\lambda\|_{2,p,\Omega''} \leq C\big(|w_\lambda|_{p,\Omega'} + |f_\lambda|_{p,\Omega'}\big), \tag{20}$$

where C depends on Ω', N, p, Ω''. Thanks to the Sobolev embedding theorem, for any $\epsilon > 0$, if $p = N + \epsilon$, we have that $W^{2,p}(\Omega)$ is continuously embedded in $C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \frac{N}{N+\epsilon}$. Let us consider two open subsets Ω'', Ω' of Ω such that $0 \in \Omega''$ and $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Thanks to (19) and (20), in order to estimate $\|w_\lambda\|_{C^1(\bar{\Omega}'')}$, we have to estimate the following quantities: $|w_\lambda|_{N+\epsilon,\Omega'}$, $|f_\lambda|_{N+\epsilon,\Omega'}$.

Thanks to the assumptions on w_λ, we deduce immediately that $|w_\lambda|_{N+\epsilon,\Omega'} = O(1)$, uniformly with respect to λ. For the other term, we argue as it follows: we set $g(s) := |s|^{2^*-2}s$, $\Phi_\lambda := w_\lambda + \varphi_2 - \varphi_1$, where $\varphi_j := U_{\delta_j} - PU_{\delta_j}$, for $j = 1, 2$, and we write

$$\begin{aligned}
|f_\lambda|_{N+\epsilon,\Omega'} &\le \lambda|w_\lambda|_{N+\epsilon,\Omega'} + \lambda|PU_{\delta_1}|_{N+\epsilon,\Omega'} + \lambda|PU_{\delta_2}|_{N+\epsilon,\Omega'} + \left|U_{\delta_1}^p\right|_{N+\epsilon,\Omega'} \\
&\quad + \left|g(U_{\delta_1} - U_{\delta_2} + \Phi_\lambda) - g(-U_{\delta_2})\right|_{N+\epsilon,\Omega'} \\
&\le \lambda|w_\lambda|_{N+\epsilon,\Omega'} + \lambda|PU_{\delta_1}|_{N+\epsilon,\Omega'} + \lambda|PU_{\delta_2}|_{N+\epsilon,\Omega'} + \left|U_{\delta_1}^p\right|_{N+\epsilon,\Omega'} \\
&\quad + \left|g(U_{\delta_1} - U_{\delta_2} + \Phi_\lambda) - g(-U_{\delta_2}) - g'(-U_{\delta_2})(U_{\delta_1} + \Phi_\lambda)\right|_{N+\epsilon,\Omega'} \\
&\quad + \left|g'(-U_{\delta_2})(U_{\delta_1} + \Phi_\lambda)\right|_{N+\epsilon,\Omega'} \\
&= A + B + C + D + E + F.
\end{aligned}$$

The term A has been estimated before, and hence, $\lambda|w_\lambda|_{N+\epsilon,\Omega'} = O(\lambda)$. For B and C, we use the following estimates:

$$\begin{aligned}
&\int_{\Omega'} \alpha_N^{N+\epsilon} \frac{\delta_j^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_j^2 + |x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx \\
&\quad = \alpha_N^{N+\epsilon} \int_{\Omega'/\delta_j} \frac{\delta_j^{-\frac{N-2}{2}(N+\epsilon)+N}}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy \\
&\quad = \alpha_N^{N+\epsilon} \delta_j^{\frac{4-N}{2}N - \epsilon\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy \\
&\qquad + O\left(\delta_j^{\frac{4-N}{2}N - \epsilon\frac{N-2}{2}} \int_{1/\delta_j}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dr\right).
\end{aligned}$$

Thus, for all $\epsilon > 0$ sufficiently small, we have

$$\begin{aligned}
|PU_\delta|_{N+\epsilon,\Omega'} &\le \left(\int_{\Omega'} \alpha_N^{N+\epsilon} \frac{\delta_j^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_j^2+|x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dx\right)^{\frac{1}{N+\epsilon}} \\
&= \alpha_N \delta_j^{\frac{4-N}{2}+O(\epsilon)} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}(N+\epsilon)}}\, dy\right)^{\frac{1}{N+\epsilon}} + o\big(\delta_j^{\frac{4-N}{2}+O(\epsilon)}\big).
\end{aligned}$$

From this we deduce that $B = O(\lambda\delta_1^{\frac{4-N}{2}+O(\epsilon)})$, $C = O(\lambda\delta_2^{\frac{4-N}{2}+O(\epsilon)})$. Concerning the term D, with similar computations we see that

$$\left|PU_{\delta_1}^p\right|_{N+\epsilon,\Omega'} \le \left(\int_{\Omega'} \alpha_N^{\frac{N+2}{2}(N+\epsilon)} \frac{\delta_1^{\frac{N+2}{2}(N+\epsilon)}}{(\delta_1^2+|x|^2)^{\frac{N+2}{2}(N+\epsilon)}}\,dx\right)^{\frac{1}{N+\epsilon}}$$
$$= \alpha_N^p \delta_1^{-\frac{N}{2}+O(\epsilon)}\left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}(N+\epsilon)}}\,dy\right)^{\frac{1}{N+\epsilon}} + o\left(\delta_1^{-\frac{N}{2}+O(\epsilon)}\right),$$

and hence, $D = O(\delta_1^{-\frac{N}{2}+O(\epsilon)})$. In order to estimate E, we remember that by elementary inequalities, we have $|g(u+v) - g(u) - g'(u)v| \le c|v|^p$, for all $u, v \in \mathbb{R}$, for some constant depending only on p, and hence, we get that

$$E \le c\left||\Phi_\lambda|^p\right|_{N+\epsilon,\Omega'} = O(1).$$

For the last term, we have the following:

$$\left|g'(U_{\delta_2})U_{\delta_1}\right|_{N+\epsilon,\Omega'}^{N+\epsilon} = p^{N+\epsilon}\int_{\Omega'} \alpha_N^{\frac{N+2}{2}(N+\epsilon)} \frac{\delta_2^{\frac{4}{N-2}\frac{N-2}{2}(N+\epsilon)}}{(\delta_2^2+|x|^2)^{\frac{4}{N-2}\frac{N-2}{2}(N+\epsilon)}} \frac{\delta_1^{\frac{N-2}{2}(N+\epsilon)}}{(\delta_1^2+|x|^2)^{\frac{N-2}{2}(N+\epsilon)}}\,dx$$
$$= p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)}\int_{\Omega'} \frac{\delta_2^{-2(N+\epsilon)}}{(1+|x/\delta_2|^2)^{2(N+\epsilon)}} \frac{\delta_1^{-\frac{N-2}{2}(N+\epsilon)}}{(1+|x/\delta_1|^2)^{\frac{N-2}{2}(N+\epsilon)}}\,dx$$
$$\le p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)}\delta_1^{-\frac{N-2}{2}(N+\epsilon)}\delta_2^{-2(N+\epsilon)+N}\int_{\Omega'/\delta_2} \frac{1}{(1+|x/\delta_2|^2)^{2(N+\epsilon)}}dy$$
$$\le p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)}\delta_1^{-\frac{N-2}{2}(N+\epsilon)}\delta_2^{-N-2\epsilon}\int_{\Omega'/\delta_2} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}}dy$$
$$= p^{N+\epsilon}\alpha_N^{\frac{N+2}{2}(N+\epsilon)}\delta_1^{-\frac{N-2}{2}(N+\epsilon)}\delta_2^{-N-2\epsilon}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}}dy$$
$$+ O\left(\delta_1^{-\frac{N-2}{2}(N+\epsilon)}\delta_2^{-N-2\epsilon}\int_{1/\delta_2}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{2(N+\epsilon)}}\right).$$

Hence, we get that

$$\left|g'(U_{\delta_2})U_{\delta_1}\right|_{N+\epsilon,\Omega'} \le p\alpha_N^{\frac{N+2}{2}}\delta_1^{-\frac{N-2}{2}}\delta_2^{-1+O(\epsilon)}\left(\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{2(N+\epsilon)}}dy\right)^{\frac{1}{N+\epsilon}}$$
$$+ o\left(\delta_1^{-\frac{N-2}{2}}\delta_2^{-1+O(\epsilon)}\right).$$

By the same computations we see that

$$\left|g'(U_{\delta_2})\Phi_\lambda\right|_{N+\epsilon,\Omega'} = O\big(\delta_2^{-1+O(\epsilon)}\big).$$

Thus, we get that

$$|F| \le c(N,p)\delta_1^{-\frac{N-2}{2}}\delta_2^{-1+O(\epsilon)}.$$

Summing up all these estimates, from (20) and Sobolev embedding theorem, we deduce that

$$\|w_\lambda\|_{C^1(\bar{\Omega}'')} \le C\delta_1^{-\frac{N-2}{2}}\delta_2^{-1+O(\epsilon)},$$

where C is a positive constant depending on $\epsilon, N, \Omega'', \Omega'$. □

A straightforward consequence of the previous theorem is the following result:

Corollary 1. *Under the assumptions of Theorem 2, for all sufficiently small $\epsilon > 0$, we have*

$$\int_{\partial B_r} |\nabla w_\lambda|^2 |x|\, d\sigma \le C(\epsilon, N)\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-4}{2}} \delta_2^{O(\epsilon)},$$

for all sufficiently small $\lambda > 0$, where B_r is the ball centered at ξ having radius $r = \sqrt{\delta_1\delta_2}$.

5. Concentration speeds for $N \ge 7$

We consider, as in the previous sections, sign-changing solutions of Problem (1) which are of the form $u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda$, with $\delta_1 = \delta_1(\lambda)$, $\delta_2 = \delta_2(\lambda)$ satisfying $\delta_2 = o(\delta_1)$ as $\lambda \to 0$. In addition, we assume that δ_i, for $i = 1, 2$, is of the form

$$\delta_i = d_i \lambda^{\alpha_i}, \tag{21}$$

where $d_i = d_i(\lambda)$ is a strictly positive function such that $d_i \to \bar{d}_i > 0$, as $\lambda \to 0$, and the exponents α_i satisfy $0 < \alpha_1 < \alpha_2$. Following the ideas contained in [12] and applying the asymptotic relation (14), found in the proof of Theorem 1, we determine precisely the exponents α_1, α_2 in the case $N \ge 7$. We observe that these speeds are exactly the same used in [11] to construct solutions of (1) of the form (2).

Theorem 3. *Let Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary, $N \ge 7$, and let $\xi \in \Omega$. Let u_λ be a solution of (1) such that u_λ is of the form $u_\lambda = PU_{\delta_1,\xi} - PU_{\delta_2,\xi} + w_\lambda$, where δ_i, for $i = 1, 2$, is of the form (21) with $\alpha_2 > \alpha_1 > 0$, $w_\lambda \in V_{\lambda,\xi}$, $V_{\lambda,\xi}$ is the subspace of $H_0^1(\Omega)$:*

$$V_{\lambda,\xi} := \left\{ v \in H_0^1(\Omega);\ (v, PU_{\delta_i,\xi})_{H_0^1(\Omega)} = \left(v, P\frac{\partial U_{\delta_i,\xi}}{\partial \delta_i}\right)_{H_0^1(\Omega)} = 0,\ i = 1, 2 \right\}.$$

Moreover, assume that $|w_\lambda| = o(\delta_1^{-\frac{N-2}{2}})$, $|\nabla w_\lambda| = o(\delta_1^{-\frac{N}{2}})$, uniformly in compact subsets of Ω. Then $\alpha_1 = \frac{1}{N-4}$, $\alpha_2 = \frac{3N-10}{(N-4)(N-6)}$.

In order to prove Theorem 3 we need some preliminary lemmas. Without loss of generality we assume that $\xi = 0$. The first one is the following:

Lemma 4. *Let Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary and assume that $0 \in \Omega$, $N \geq 5$. Then, as $\delta \to 0$, we have*

$$\int_{\partial\Omega} \left(\frac{\partial PU_\delta}{\partial \nu}\right)^2 (x \cdot \nu)\, d\sigma = a_2 \delta^{N-2} + o\big(\delta^{N-2}\big),$$

for some positive real number a_2, depending only on N and Ω.

Proof. We multiply the equation $-\Delta PU_\delta = U_\delta^p$ by $\sum_{i=1}^N x_i \frac{\partial PU_\delta}{\partial x_i}$ and we integrate on Ω. On one hand, integrating by parts we obtain

$$\begin{aligned}
&\int_\Omega -\Delta PU_\delta \sum_{i=1}^N x_i \frac{\partial PU_\delta}{\partial x_i}\, dx \\
&\quad = \left(1 - \frac{N}{2}\right) \int_\Omega |\nabla PU_\delta|^2\, dx - \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial PU_\delta}{\partial \nu}\right)^2 (x \cdot \nu)\, d\sigma \\
&\quad = \left(1 - \frac{N}{2}\right) \int_\Omega U_\delta^p PU_\delta\, dx - \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial PU_\delta}{\partial \nu}\right)^2 (x \cdot \nu)\, d\sigma. \qquad (22)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega U_\delta^p \sum_{i=1}^N x_i \frac{\partial PU_\delta}{\partial x_i}\, dx &= -\sum_{i=1}^N \int_\Omega \left(U_\delta^p + p x_i U_\delta^{p-1} \frac{\partial U_\delta}{\partial x_i}\right) PU_\delta\, dx \\
&= -N \int_\Omega U_\delta^p PU_\delta\, dx - p \sum_{i=1}^N \int_\Omega x_i U_\delta^{p-1} \frac{\partial U_\delta}{\partial x_i} PU_\delta\, dx. \qquad (23)
\end{aligned}$$

By elementary computations we see that

$$-\sum_{i=1}^N x_i U_\delta^{p-1} \frac{\partial U_\delta}{\partial x_i} = \frac{N-2}{2} U_\delta + \delta \frac{\partial U_\delta}{\partial \delta},$$

and hence, from (23), we get that

$$\begin{aligned}
&\int_\Omega U_\delta^p \sum_{i=1}^N x_i \frac{\partial PU_\delta}{\partial x_i}\, dx \\
&\quad = -N \int_\Omega U_\delta^p PU_\delta\, dx + p \frac{N-2}{2} \int_\Omega U_\delta^p PU_\delta\, dx + p\delta \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} PU_\delta\, dx
\end{aligned}$$

$$= \left(1 - \frac{N}{2}\right) \int_\Omega U_\delta^p PU_\delta \, dx + p\delta \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} PU_\delta \, dx. \tag{24}$$

We analyze the last term of (24). Applying Lemma 1 and since it is well known that

$$\int_{\mathbb{R}^N} U_\delta^p \frac{\partial U_\delta}{\partial \delta} \, dx = 0,$$

we have

$$\begin{aligned} p\delta \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} PU_\delta \, dx &= p\delta \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} U_\delta \, dx - p\alpha_N \delta^{\frac{N}{2}} \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} H(x,0) \, dx \\ &\quad + o\left(\delta^{\frac{N}{2}} \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} H(x,0) \, dx\right) \\ &= -p\delta \int_{\mathbb{R}^N \setminus \Omega} U_\delta^p \frac{\partial U_\delta}{\partial \delta} \, dx - p\alpha_N \delta^{\frac{N}{2}} \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} H(x,0) \, dx \\ &\quad + o\left(\delta^{\frac{N}{2}} \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} H(x,0) \, dx\right), \end{aligned} \tag{25}$$

where H denotes, the regular part of the Green function for the Laplacian. By definition it is easy to see that

$$\begin{aligned} \left| -p\delta \int_{\mathbb{R}^N \setminus \Omega} U_\delta^p \frac{\partial U_\delta}{\partial \delta} \, dx \right| &\leq \alpha_N^{p+1} \frac{N+2}{2} \delta \int_{\mathbb{R}^N \setminus \Omega} \frac{\delta^{\frac{N+2}{2}}}{(\delta^2 + |x|^2)^{\frac{N+2}{2}}} \frac{\delta^{\frac{N-2}{2}} ||x|^2 - \delta^2|}{(\delta^2 + |x|^2)^{\frac{N}{2}}} \, dx \\ &\leq \alpha_N^{p+1} \frac{N+2}{2} \int_{\mathbb{R}^N \setminus \Omega} \frac{\delta^{N+1}}{|x|^{N+2}} \frac{||x|^2 - \delta^2|}{|x|^N} \, dx \\ &= O\big(\delta^{N+1}\big). \end{aligned} \tag{26}$$

Moreover, by the usual change of variable and applying the mean value theorem, we have

$$\begin{aligned} &p\alpha_N \delta^{\frac{N}{2}} \int_\Omega U_\delta^{p-1} \frac{\partial U_\delta}{\partial \delta} H(x,0) \, dx \\ &= p\alpha_N^{p+1} \delta^{\frac{N-2}{2}} \int_\Omega \frac{\delta^2}{(\delta^2 + |x|^2)^2} \frac{\delta^{\frac{N-2}{2}} (|x|^2 - \delta^2)}{(\delta^2 + |x|^2)^{\frac{N}{2}}} H(x,0) \, dx \\ &= p\alpha_N^{p+1} \delta^{\frac{N-2}{2}} \int_\Omega \frac{\delta^2}{\delta^4 (1 + |\frac{x}{\delta}|^2)^2} \frac{\delta^{\frac{N-2}{2}} \delta^2 (|\frac{x}{\delta}|^2 - 1)}{\delta^N (1 + |\frac{x}{\delta}|^2)^{\frac{N}{2}}} H(x,0) \, dx \end{aligned}$$

$$
\begin{aligned}
&= p\alpha_N^{p+1}\delta^{N-2}\int_{\Omega/\delta}\frac{1}{(1+|y|^2)^2}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N}{2}}}H(\delta y,0)\,dy\\
&= p\alpha_N^{p+1}\delta^{N-2}\int_{\Omega/\delta}\frac{1}{(1+|y|^2)^2}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N}{2}}}H(0,0)\,dy\\
&\quad + O\bigg(\delta^{N-1}\int_{\Omega/\delta}\frac{1}{(1+|y|^2)^2}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N}{2}}}\big(\nabla H(\eta y,0)\cdot y\big)\,dy\bigg)\\
&= p\alpha_N^{p+1}\delta^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^2}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N}{2}}}H(0,0)\,dy\\
&\quad + O\bigg(\delta^{N-2}\int_{1/\delta}^{+\infty}\frac{r^{N-1}}{(1+r^2)^2}\frac{(r^2-1)}{(1+r^2)^{\frac{N}{2}}}H(0,0)\,dr\bigg)\\
&\quad + O\bigg(\delta^{N-1}\int_{\Omega/\delta}\frac{1}{(1+|y|^2)^2}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N}{2}}}\big(\nabla H(\eta y,0)\cdot y\big)\,dy\bigg)\\
&= p\alpha_N^{p+1}H(0,0)\delta^{N-2}\int_{\mathbb{R}^N}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N+4}{2}}}\,dy + O\big(\delta^{N-1}\big).
\end{aligned}
\tag{27}
$$

Finally, from (22)–(27) we get that

$$
\int_{\partial\Omega}\left(\frac{\partial PU_\delta}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma = 2p\alpha_N^{p+1}H(0,0)\delta^{N-2}\int_{\mathbb{R}^N}\frac{(|y|^2-1)}{(1+|y|^2)^{\frac{N+4}{2}}}\,dy + O\big(\delta^{N-1}\big),
$$

and the proof is complete. □

Another preliminary lemma is the following:

Lemma 5. *Under the assumptions of Theorem 3, as $\lambda \to 0$, we have*

$$
\left|\int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma\right| = O\big(\lambda^2\delta_1^4\big) + o\big(\delta_1^{N-2}\big).
$$

Proof. The first step is the following:

$$
\begin{aligned}
&\left|\int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma\right|\\
&\quad \le \int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2|x\cdot\nu|\,d\sigma \le \int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2|x|\,d\sigma \le c(\Omega)\int_{\partial\Omega}\left(\frac{\partial w_\lambda}{\partial\nu}\right)^2 d\sigma.
\end{aligned}
$$

Thus, we need to estimate $\int_{\partial\Omega}(\frac{\partial w_\lambda}{\partial\nu})^2\,d\sigma$. Let us consider a smooth function $\zeta:\mathbb{R}^N\to\mathbb{R}$ such that $0\le\zeta\le 1$, $\zeta(x)=0$ for $|x|\le\frac{1}{2}$ and $\zeta(x)=1$ for $|x|\ge 1$. We set $\eta(x):=\zeta(\frac{x}{d(0,\partial\Omega)})$. It is elementary to see that ηw_λ is a solution of the following problem:

$$\begin{cases}-\Delta(\eta w_\lambda)=\lambda\eta w_\lambda+g_\lambda & \text{in } \Omega,\\ \eta w_\lambda=0 & \text{on } \partial\Omega,\end{cases}\tag{28}$$

where $g_\lambda=\eta(\lambda PU_{\delta_1}-\lambda PU_{\delta_2}-U^p_{\delta_1}+U^p_{\delta_2}+|u_\lambda|^{2^*-2}u_\lambda)-2\nabla\eta\cdot\nabla w_\lambda-w_\lambda\Delta\eta$. Since ηw_λ is a solution of (28), the following inequality holds (see Appendix C in [12]):

$$\left|\frac{\partial}{\partial\nu}(\eta w_\lambda)\right|^2_{2,\partial\Omega}=\left|\frac{\partial w_\lambda}{\partial\nu}\right|^2_{2,\partial\Omega}\le C|g_\lambda|^2_{\frac{2N}{N+1},\Omega},\tag{29}$$

where C is a positive constant depending only on Ω and N. Hence, in order to complete the proof, it suffices to estimate the $L^{\frac{2N}{N+1}}(\Omega)$-norm of g_λ. We point out that, thanks to the multiplication by the cut-off function η, what occurs around the origin does not count anymore and this will make the boundary estimate sharper. By elementary inequalities, we get that

$$|g_\lambda|\le c(p)\eta\big(\lambda U_{\delta_1}+\lambda U_{\delta_2}+U^p_{\delta_1}+U^p_{\delta_2}+|w_\lambda|^p\big)+2|\nabla\eta||\nabla w_\lambda|+|\Delta\eta||w_\lambda|.$$

Thus, we have to estimate the following quantities:

$$\lambda|\eta U_{\delta_j}|_{\frac{2N}{N+1},\Omega},\qquad \big|\eta U^p_{\delta_j}\big|_{\frac{2N}{N+1},\Omega},\qquad \text{for } j=1,2,\quad\text{and}$$
$$\big|\eta|w_\lambda|^p\big|_{\frac{2N}{N+1},\Omega},\qquad \big||\nabla\eta||\nabla w_\lambda|\big|_{\frac{2N}{N+1},\Omega},\qquad \big||\Delta\eta||w_\lambda|\big|_{\frac{2N}{N+1},\Omega}.$$

This is a long computation already made by O. Rey (see Appendix C of [12]), in the case of positive solutions of the form $u_\lambda=PU_\delta+w_\lambda$. In that paper it is shown that

$$\begin{aligned}&\big|\eta U^p_{\delta_j}\big|^2_{\frac{2N}{N+1},\Omega}=o\big(\delta_j^{N-2}\big),\qquad |\eta\lambda U_{\delta_j}|^2_{\frac{2N}{N+1},\Omega}=O\big(\lambda^2\delta_j^{N-2}\big),\\ &\big||\nabla\eta||\nabla w_\lambda|\big|^2_{\frac{2N}{N+1},\Omega}=O\big(\|w_\lambda\|^2\big),\qquad \big||\Delta\eta||w_\lambda|\big|^2_{\frac{2N}{N+1},\Omega}=O\big(\|w_\lambda\|^2\big).\end{aligned}\tag{30}$$

Moreover, by the same computations of Appendix C in [12] we see that

$$\big|\eta|w_\lambda|^p\big|^2_{\frac{2N}{N+1},\Omega}=o\big(\delta_1^{N-2}\big).$$

In order to complete the proof we need to estimate the quantities in (30), and hence, we have to study the asymptotic behavior of $\|w_\lambda\|$. An estimate for $\|w_\lambda\|$ is contained in [4]; in particular, by the proof of Lemma 3.3 of [4] we see that

$$\|w_\lambda\|\le c\left[\sum_i\big(\lambda\delta_i^{(N-2)/2}+\delta_i^{N-2}\big)+\epsilon_{12}\big(\log\epsilon_{12}^{-1}\big)^{(N-2)/N}\right],\tag{31}$$

where ϵ_{12} is defined by $\epsilon_{12} := (\frac{\delta_1}{\delta_2} + \frac{\delta_2}{\delta_1})^{(2-N)/2}$. Since $\frac{\delta_2}{\delta_1} \to 0$ as $\lambda \to 0$ we see that

$$\epsilon_{12} = \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} + o\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}.$$

Moreover, by the assumptions on the growth of ∇w_λ and w_λ, and thanks to (14), we get that ϵ_{12} is of the same order as $\lambda \delta_2^2$, hence, since $\delta_2 = o(\delta_1)$ as $\lambda \to 0$, we have that

$$\epsilon_{12}\left(\log \epsilon_{12}^{-1}\right)^{(N-2)/N} = o\left(\lambda \delta_1^2\right).$$

Thus, from (31), and since $N \geq 7$, we deduce that for all sufficiently small λ it holds

$$\|w_\lambda\| \leq c\left(\delta_1^{N-2} + \lambda \delta_1^2\right). \tag{32}$$

Summing up all these estimates, we deduce the desired relation. □

Lemma 6. *Let Ω be a bounded open set of $\mathbb{R}^N$ with smooth boundary and assume that $0 \in \Omega$, $N \geq 5$. Then, as $\delta \to 0$, we have*

$$\int_{\partial\Omega} \left(\frac{\partial PU_\delta}{\partial \nu}\right)^2 d\sigma = O\left(\delta^{N-2}\right).$$

Proof. We consider a smooth function $\eta : \mathbb{R}^N \to \mathbb{R}$ having the same properties as the one considered in the previous proof. By elementary computation we see that ηPU_δ satisfies

$$\begin{cases} -\Delta(\eta PU_\delta) = -(\Delta\eta)PU_\delta - \nabla\eta \cdot \nabla PU_\delta + \eta U_\delta^p & \text{in } \Omega, \\ \eta PU_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Since ηPU_δ is a solution of (33), the following inequality holds:

$$\left|\frac{\partial}{\partial \nu}(\eta PU_\delta)\right|_{2,\partial\Omega}^2 = \left|\frac{\partial PU_\delta}{\partial \nu}\right|_{2,\partial\Omega}^2 \leq C\left||\Delta\eta|PU_\delta + |\nabla\eta \cdot \nabla PU_\delta| + \eta U_\delta^p\right|_{\frac{2N}{N+1},\Omega}^2, \tag{34}$$

where C is a positive constant depending only on Ω and N. In order to complete the proof, we have to estimate the quantities: $|(\Delta\eta)PU_\delta|_{\frac{2N}{N+1},\Omega}^{2}$, $|\nabla\eta \cdot \nabla PU_\delta|_{\frac{2N}{N+1},\Omega}^2$, $|\eta U_\delta^p|_{\frac{2N}{N+1},\Omega}^2$. Using the same computations made by O. Rey in [12], and since $\eta \equiv 0$ in a neighborhood of the origin, we get that

$$\left|\eta U_\delta^p\right|_{\frac{2N}{N+1},\Omega}^2 = o\left(\delta^{N-2}\right), \qquad \left||\nabla\eta||\nabla PU_\delta|\right|_{\frac{2N}{N+1},\Omega}^2 = O\left(\|PU_\delta\|_{\Omega\cap \mathrm{supp}(\nabla\eta)}^2\right),$$
$$\left||\Delta\eta||PU_\delta|\right|_{\frac{2N}{N+1},\Omega}^2 = O\left(\|PU_\delta\|_{\Omega\cap \mathrm{supp}(\nabla\eta)}^2\right). \tag{35}$$

Applying Lemma 1 and taking account of (10), since $\nabla\eta \equiv 0$ in an open neighborhood of the origin, we have

$$\|PU_\delta\|^2_{\Omega\cap \mathrm{supp}(\nabla\eta)} = \int\limits_{\Omega\cap \mathrm{supp}(\nabla\eta)} \left|\nabla(U_\delta - \varphi_\delta)\right|^2 dx$$
$$\leq \int\limits_{\Omega\cap \mathrm{supp}(\nabla\eta)} |\nabla U_\delta|^2 dx + 2 \int\limits_{\Omega\cap \mathrm{supp}(\nabla\eta)} |\nabla U_\delta||\nabla \varphi_\delta| dx$$
$$+ \int\limits_{\Omega\cap \mathrm{supp}(\nabla\eta)} |\nabla \varphi_\delta|^2 dx$$
$$= O\big(\delta^{N-2}\big). \tag{36}$$

From (34), (35) and (36), we deduce that

$$\left|\frac{\partial PU_\delta}{\partial \nu}\right|^2_{2,\partial\Omega} = O\big(\delta^{N-2}\big),$$

and the proof is complete. □

Proof of Theorem 3. We apply the Pohozaev's identity to $u_\lambda = PU_{\delta_1} - PU_{\delta_2} + w_\lambda$. Since u_λ is a solution of Problem (1), we have

$$\lambda \int\limits_\Omega u_\lambda^2 \, dx = \frac{1}{2} \int\limits_{\partial\Omega} \left(\frac{\partial u_\lambda}{\partial \nu}\right)^2 (x\cdot\nu)\, d\sigma. \tag{37}$$

For the left-hand side of (37), as in the previous proofs, we set $\Phi_\lambda := w_\lambda - \varphi_{\delta_1} + \varphi_{\delta_2}$, where $\varphi_{\delta_j} = U_{\delta_j} - PU_{\delta_j}$ for $j = 1, 2$, and we have

$$\lambda \int\limits_\Omega u_\lambda^2 \, dx = \lambda \int\limits_\Omega (PU_{\delta_1} - PU_{\delta_2} + w_\lambda)^2 \, dx = \lambda \int\limits_\Omega (U_{\delta_1} - U_{\delta_2} + \Phi_\lambda)^2 \, dx$$
$$= \lambda \int\limits_\Omega \big(U_{\delta_1}^2 + U_{\delta_2}^2 - 2U_{\delta_1}U_{\delta_2} + 2U_{\delta_1}\Phi_\lambda - 2U_{\delta_2}\Phi_\lambda + \Phi_\lambda^2\big)\, dx$$
$$= A + B + C + D + E + F. \tag{38}$$

In order to estimate A and B, we use the following:

$$\lambda \int\limits_\Omega U_{\delta_j}^2 \, dx = \lambda\, \alpha_N^2 \int\limits_\Omega \frac{\delta_j^{-(N-2)}}{(1+|x/\delta_j|^2)^{N-2}}\, dx = \lambda\, \alpha_N^2 \int\limits_{\Omega/\delta_j} \frac{\delta_j^{-(N-2)}}{(1+|y|^2)^{N-2}} \delta_j^N \, dy$$
$$= \lambda\, \alpha_N^2 \delta_j^2 \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy + O\left(\lambda \delta_j^2 \int\limits_{1/\delta_j}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-2}}\, dr\right)$$
$$= \lambda\, \alpha_N^2 \delta_j^2 \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2}}\, dy + O\big(\lambda \delta_j^{N-2}\big). \tag{39}$$

We point out that since we are assuming that $N \geq 5$, the first integral in the last line of (39) converges. To estimate C we apply the following:

$$\begin{aligned}
\lambda \int_{\Omega} U_{\delta_1} U_{\delta_2}\, dx &= \lambda\, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+\delta_1^2|y|^2)^{\frac{N-2}{2}}}\, dy \\
&= \lambda\, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{((\frac{\delta_2}{\delta_1})^2+|y|^2)^{\frac{N-2}{2}}}\, dy \\
&\leq \lambda\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\Omega/\delta_1} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&= \lambda\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&\quad + O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{1/\delta_1}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}} r^{N-2}}\, dr\right) \\
&= \lambda\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&\quad + O\left(\lambda \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^{N-2}\right). \qquad (40)
\end{aligned}$$

In order to estimate D, E, F, thanks to (32), Hölder's inequality and Poincaré's inequality, we get that

$$\int_{\Omega} w_\lambda^2 \leq c_1 \|w_\lambda\|^2 \leq c_2 \big(\delta_1^{N-2} + \lambda \delta_1^2\big)^2. \qquad (41)$$

We observe that, by Lemma 1 and since $N \geq 5$, we have $|\varphi_{\delta_j}|_{2,\Omega} = O(\delta_j^{\frac{N-2}{2}}) = o(\delta_j)$. Thus, by definition of Φ_λ and (41), we deduce that

$$\int_{\Omega} \Phi_\lambda^2\, dx = \int_{\Omega} (w_\lambda + \varphi_{\delta_2} - \varphi_{\delta_1})^2\, dx = o\big(\delta_1^2\big), \qquad (42)$$

and hence,

$$F = o\big(\lambda \delta_1^2\big). \qquad (43)$$

Moreover, by the same computations of (39), we have $\int_\Omega U_{\delta_j}^2 = a_1\delta_j^2 + o(\delta_j^2)$, for some positive constant a_1. Hence, by Hölder's inequality and (42), we get that

$$|D| = o\big(\lambda\delta_1^2\big),$$

and

$$|E| = o(\lambda\delta_1\delta_2) = o\big(\lambda\delta_1^2\big).$$

We analyze now the right-hand side of (37): by definition, we have

$$\begin{aligned}
\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u_\lambda}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma &= \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial PU_{\delta_1}}{\partial\nu} - \frac{\partial PU_{\delta_2}}{\partial\nu} + \frac{\partial w_\lambda}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma \\
&= \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial PU_{\delta_1}}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial PU_{\delta_2}}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma \\
&\quad - \int_{\partial\Omega}\frac{\partial PU_{\delta_1}}{\partial\nu}\frac{\partial PU_{\delta_2}}{\partial\nu}(x\cdot\nu)\,d\sigma + \int_{\partial\Omega}\frac{\partial PU_{\delta_1}}{\partial\nu}\frac{\partial w_\lambda}{\partial\nu}(x\cdot\nu)\,d\sigma \\
&\quad - \int_{\partial\Omega}\frac{\partial PU_{\delta_2}}{\partial\nu}\frac{\partial w_\lambda}{\partial\nu}(x\cdot\nu)\,d\sigma + \frac{1}{2}\int_{\partial\Omega}\left(\frac{w_\lambda}{\partial\nu}\right)^2(x\cdot\nu)\,d\sigma \\
&= A_1 + B_1 + C_1 + D_1 + E_1 + F_1. \qquad (44)
\end{aligned}$$

Thanks to Lemma 4, we have:

$$\begin{aligned}
A_1 &= \frac{a_2}{2}\delta_1^{N-2} + o\big(\delta_1^{N-2}\big), \\
B_1 &= \frac{a_2}{2}\delta_2^{N-2} + o\big(\delta_2^{N-2}\big). \qquad (45)
\end{aligned}$$

Thanks to Lemma 6 and applying Hölder inequality, we get that

$$\begin{aligned}
|C_1| &\le \int_{\partial\Omega}\left|\frac{\partial PU_{\delta_1}}{\partial\nu}\right|\left|\frac{\partial PU_{\delta_2}}{\partial\nu}\right||x\cdot\nu|\,d\sigma \\
&\le \operatorname{diam}(\partial\Omega)\bigg(\int_{\partial\Omega}\left|\frac{\partial PU_{\delta_1}}{\partial\nu}\right|^2 d\sigma\bigg)^{\frac{1}{2}}\bigg(\int_{\partial\Omega}\left|\frac{\partial PU_{\delta_2}}{\partial\nu}\right|^2 d\sigma\bigg)^{\frac{1}{2}} \\
&= O\big(\delta_1^{\frac{N-2}{2}}\delta_2^{\frac{N-2}{2}}\big). \qquad (46)
\end{aligned}$$

Thanks to (29), Lemma 5, Lemma 6 and applying Hölder inequality, we get that

$$|D_1| \le \int\limits_{\partial\Omega} \left|\frac{\partial PU_{\delta_1}}{\partial \nu}\right| \left|\frac{\partial w_\lambda}{\partial \nu}\right| |x \cdot \nu| \, d\sigma$$

$$\le \operatorname{diam}(\partial\Omega) \left(\int\limits_{\partial\Omega} \left|\frac{\partial PU_{\delta_1}}{\partial \nu}\right|^2 d\sigma \right)^{\frac{1}{2}} \left(\int\limits_{\partial\Omega} \left|\frac{\partial w_\lambda}{\partial \nu}\right|^2 d\sigma \right)^{\frac{1}{2}}$$

$$= o\big(\lambda\delta_1^2\big) + o\big(\delta_1^{N-2}\big); \tag{47}$$

$$|E_1| \le \int\limits_{\partial\Omega} \left|\frac{\partial PU_{\delta_2}}{\partial \nu}\right| \left|\frac{\partial w_\lambda}{\partial \nu}\right| |x \cdot \nu| \, d\sigma$$

$$\le \operatorname{diam}(\partial\Omega) \left(\int\limits_{\partial\Omega} \left|\frac{\partial PU_{\delta_2}}{\partial \nu}\right|^2 d\sigma \right)^{\frac{1}{2}} \left(\int\limits_{\partial\Omega} \left|\frac{\partial w_\lambda}{\partial \nu}\right|^2 d\sigma \right)^{\frac{1}{2}}$$

$$= o\big(\lambda\delta_1^2\big) + o\big(\delta_1^{N-2}\big); \tag{48}$$

$$|F_1| = \frac{1}{2} \int\limits_{\partial\Omega} \left(\frac{\partial w_\lambda}{\partial \nu}\right)^2 (x \cdot \nu) \, d\sigma = o\big(\lambda\delta_1^2\big) + o\big(\delta_1^{N-2}\big). \tag{49}$$

Summing up all the estimates, from (37) and since $\delta_2 = o(\delta_1)$ as $\lambda \to 0$, we deduce the following equality:

$$a_1\lambda\delta_1^2 + o\big(\lambda\delta_1^2\big) = a_2\delta_1^{N-2} + o\big(\delta_1^{N-2}\big). \tag{50}$$

Since δ_j is of the form (21), we deduce that α_1 must satisfy the equation

$$1 + 2\alpha_1 = (N-2)\alpha_1,$$

and hence, we get that $\alpha_1 = \frac{1}{N-4}$. Moreover, from (14), we deduce that α_1, α_2 must satisfy the following algebraic equation:

$$1 + 2\alpha_2 = \frac{N-2}{2}(\alpha_2 - \alpha_1). \tag{51}$$

Thus, combining this result with (51), we get that $\alpha_2 = \frac{3N-10}{(N-4)(N-6)}$ and the proof is complete. □

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