ASYMPTOTIC ANALYSIS FOR RADIAL SIGN-CHANGING SOLUTIONS OF THE BREZIS-NIRENBERG PROBLEM

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Abstract. We study the asymptotic behavior, as $\lambda \to 0$, of least energy radial sign-changing solutions u_λ, of the Brezis–Nirenberg problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $\lambda > 0$, $2^* = \frac{2n}{n-2}$ and B_1 is the unit ball of $\mathbb{R}^n$, $n \geq 7$.

We prove that both the positive and negative part u_λ^+ and u_λ^- concentrate at the same point (which is the center) of the ball with different concentration speeds. Moreover, we show that suitable rescalings of u_λ^+ and u_λ^- converge to the unique positive regular solution of the critical exponent problem in $\mathbb{R}^n$.

Precise estimates of the blow-up rate of $\|u_\lambda^\pm\|_\infty$ are given, as well as asymptotic relations between $\|u_\lambda^\pm\|_\infty$ and the nodal radius r_λ.

Finally, we prove that, up to constant, $\lambda^{-\frac{n-2}{2n-8}} u_\lambda$ converges in $C^1_{loc}(B_1 - \{0\})$ to $G(x,0)$, where $G(x,y)$ is the Green function of the Laplacian in the unit ball.

1. Introduction

Let $n \geq 3$, $\lambda > 0$ and Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We consider the Brezis–Nirenberg problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{1}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$. Problem (1) has been widely studied over the last decades, and many results for positive solutions have been obtained.

The first existence result for positive solutions of (1) has been given by Brezis and Nirenberg in their classical paper [10], where, in particular, the crucial role played by the dimension was enlightened. They proved that if $n \geq 4$ there exist positive solutions of (1) for every $\lambda \in (0, \lambda_1(\Omega))$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary condition. For the case $n = 3$, which is more delicate, Brezis and Nirenberg [10] proved that there exists $\lambda_*(\Omega) > 0$ such that positive solutions exist for every $\lambda \in (\lambda_*(\Omega), \lambda_1(\Omega))$. When $\Omega = B$ is a ball, they also proved that $\lambda_*(B) = \frac{\lambda_1(B)}{4}$ and a positive solution of (1) exists if and only if $\lambda \in (\frac{\lambda_1(B)}{4}, \lambda_1(B))$. Moreover, for more general bounded domains, they proved that if $\Omega \subset \mathbb{R}^3$ is strictly star-shaped about the origin, there are no positive solutions for λ close to zero. We point out that weak solutions of (1) are classical solution. This is a consequence of a well-known lemma of Brezis and Kato (see for instance Appendix B of [23]).

The asymptotic behavior for $n \geq 4$, as $\lambda \to 0$, of positive solutions of (1), minimizing the Sobolev quotient, has been studied by Han [18], Rey [21]. They showed, with different proofs, that such solutions blow up at exactly one point, and they also determined the exact blow-up rate as well as the location of the limit concentration points.

Concerning the case of sign-changing solutions of (1), several existence results have been obtained if $n \geq 4$. In this case, one can get sign-changing solutions for every $\lambda \in (0, \lambda_1(\Omega))$, or even

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$\lambda > \lambda_1(\Omega)$, as shown in the papers of Atkinson–Brezis–Peletier [4], Clapp–Weth [14], Capozzi–Fortunato–Palmieri [11]. The case $n = 3$ presents the same difficulties enlightened before for positive solutions and even more. In fact, differently from the case of positive solutions, it is not yet known, when $\Omega = B$ is a ball in $\mathbb{R}^3$, if there are sign-changing solutions of (1) when λ is smaller than $\lambda_*(B) = \lambda_1(B)/4$. A partial answer to this question posed by H. Brezis has been given in [8].

The blow-up analysis of low-energy sign-changing solutions of (1) has been done by Ben Ayed–El Mehdi–Pacella [7], [8]. In [8] the authors analyze the case $n = 3$. They introduce the number defined by

$$\bar{\lambda}(\Omega) := \inf\{\lambda \in \mathbb{R}^+;\ \text{Problem (1) has a sign-changing solution } u_\lambda, \text{with } \|u_\lambda\|_\Omega^2 - \lambda |u_\lambda|_{2,\Omega}^2 \leq 2S^{3/2}\},$$

where $\|u_\lambda\|_\Omega^2 = \int_\Omega |\nabla u_\lambda|^2\ dx$, $|u_\lambda|_{2,\Omega}^2 = \int_\Omega |u_\lambda|^2\ dx$ and S is the best Sobolev constant for the embedding $H_0^1(\Omega)$ into $L^{2*}(\Omega)$. To be precise, they study the behavior of sign-changing solutions of (1) which converge weakly to zero and whose energy converges to $2S^{3/2}$ as $\lambda \to \bar{\lambda}(\Omega)$. They prove that these solutions blow up at two different points $\bar{a}_1$, $\bar{a}_2$, which are the limit of the concentration points $a_{\lambda,1}$, $a_{\lambda,2}$ of the positive and negative part of the solutions. Moreover, the distance between $a_{\lambda,1}$ and $a_{\lambda,2}$ is bounded from below by a positive constant depending only on Ω and the concentration speeds of the positive and negative parts are comparable. This result shows that, in dimension 3, there cannot exist, in any bounded smooth domain Ω, sign-changing low-energy solutions whose positive and negative part concentrate at the same point.

In higher dimensions ($n \geq 4$), the same authors, in their paper [7], describe the asymptotic behavior, as $\lambda \to 0$, of sign-changing solutions of (1) whose energy converges to the value $2S^{n/2}$. Even in this case, they prove that the solutions concentrate and blow up at two separate points, but they need to assume the extra hypothesis that the concentration speeds of the two concentration points are comparable, while in dimension three, this was derived without any extra assumption (see Theorem 4.1 in [8]). They also describe in [7] the asymptotic behavior, as $\lambda \to 0$, of the solutions outside the limit concentration points proving that there exist positive constants m_1, m_2 such that

$$\lambda^{-\frac{n-2}{2n-8}} u_\lambda \to m_1 G(x, \bar{a}_1) - m_2 G(x, \bar{a}_2) \ \text{ in } C^2_{loc}(\Omega - \{\bar{a}_1, \bar{a}_2\}),\ \text{if } n \geq 5,$$
$$\|u_\lambda\|_\infty u_\lambda \to m_1 G(x, \bar{a}_1) - m_2 G(x, \bar{a}_2) \ \text{ in } C^2_{loc}(\Omega - \{\bar{a}_1, \bar{a}_2\}),\ \text{if } n = 4,$$

where $G(x, y)$ is the Green's function of the Laplace operator in Ω. So for $n \geq 4$ the question of proving the existence of sign-changing low-energy solutions (i.e., such that $\|u_\lambda\|_\Omega^2$ converges to $2S^{n/2}$ as $\lambda \to 0$) whose positive and negative part concentrate and blow up at the same point was left open.

To the aim to contribute to this question as well as to describe the precise asymptotic behavior of radial sign-changing solutions, we consider the Brezis–Nirenberg problem in the unit ball B_1, i.e.,

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{2}$$

It is important to recall that Atkinson–Brezis–Peletier [3], Adimurthi–Yadava [1] showed, with different proofs, that for $n = 3, 4, 5, 6$ there exists $\lambda^* = \lambda^*(n) > 0$ such that there is no radial sign-changing solution of (2) for $\lambda \in (0, \lambda^*)$. Instead, they do exist if $n \geq 7$, as shown by Cerami–Solimini–Struwe in their paper [13]. In Proposition 1 (see also Remark 1) we recall this existence result and get the limit energy of such solutions as $\lambda \to 0$.

In view of these results, we analyze the case $n \geq 7$ and $\lambda \to 0$. More precisely, we consider a family (u_λ) of least energy sign-changing solutions of (2). It is easy to see that u_λ has exactly two nodal regions. We denote by $r_\lambda \in (0, 1)$ the node of $u_\lambda = u_\lambda(r)$ and, without loss of generality, we assume $u_\lambda(0) > 0$, so that u_λ^+ is different from zero in B_{r_λ} and u_λ^- is different from zero in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^n;\ r_\lambda < |x| < 1\}$, where $u_\lambda^+ := \max(u_\lambda, 0)$, $u_\lambda^- := \max(0, -u_\lambda)$ are, respectively, the positive and the negative part of u_λ.

We set $M_{\lambda,+} := \|u_\lambda^+\|_\infty$, $M_{\lambda,-} := \|u_\lambda^-\|_\infty$, $\beta := \frac{2}{n-2}$, $\sigma_\lambda := M_{\lambda,+}^\beta r_\lambda$, $\rho_\lambda := M_{\lambda,-}^\beta r_\lambda$. Moreover, for $\mu > 0$, $x_0 \in \mathbb{R}^n$, let $\delta_{x_0,\mu}$ be the function $\delta_{x_0,\mu} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_{x_0,\mu}(x) := \frac{[n(n-2)\mu^2]^{(n-2)/4}}{[\mu^2 + |x - x_0|^2]^{(n-2)/2}}. \tag{3}$$

Proposition 3 states that both $M_{\lambda,+}$ and $M_{\lambda,-}$ diverge, u_λ weakly converge to 0 and $\|u_\lambda^\pm\|_{B_1}^2 \to S^{n/2}$, as $\lambda \to 0$. The results of this paper are contained in the following theorems.

Theorem 1. *Let $n \geq 7$ and (u_λ) be a family of least energy radial sign-changing solutions of (2) and $u_\lambda(0) > 0$. Consider the rescaled functions $\tilde{u}_\lambda^+(y) := \frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)$ in B_{σ_λ}, and $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^-\left(\frac{y}{M_{\lambda,-}^\beta}\right)$ in A_{ρ_λ}, where $B_{\sigma_\lambda} := M_{\lambda,+}^\beta B_{r_\lambda}$, $A_{\rho_\lambda} := M_{\lambda,-}^\beta A_{r_\lambda}$. Then:*

(i): *$\tilde{u}_\lambda^+ \to \delta_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^n)$ as $\lambda \to 0$, where $\delta_{0,\mu}$ is the function defined in (3) for $\mu = \sqrt{n(n-2)}$.*
(ii): *$\tilde{u}_\lambda^- \to \delta_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^n - \{0\})$ as $\lambda \to 0$, where $\delta_{0,\mu}$ is the same as in (i).*

From this theorem, we deduce that the positive and negative parts of u_λ concentrate at the origin. Moreover, as a consequence of the preliminary results for the proof of Theorem 1, we show that $M_{\lambda,+}$ and $M_{\lambda,-}$ are not comparable, i.e., $\frac{M_{\lambda,+}}{M_{\lambda,-}} \to +\infty$ as $\lambda \to 0$, which implies that the speed of concentration and blowup of u_λ^+ and u_λ^- are not the same, and hence, the asymptotic profile of u_λ is that of a tower of two "bubbles." Indeed, we are able to determine the exact rate of $M_{\lambda,-}$ and an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and the radius r_λ (see also Remark 6).

Theorem 2. *As $\lambda \to 0$ we have the following:*
(i): $M_{\lambda,+}^{2-2\beta} r_\lambda^{n-2} \lambda \to c(n)$,
(ii): $M_{\lambda,-}^{2-2\beta} \lambda \to c(n)$,
(iii): $\dfrac{M_{\lambda,-}^{2-2\beta}}{M_{\lambda,+}^{2-2\beta} r_\lambda^{n-2}} \to 1$,

where $c(n) := \frac{c_1^2(n)}{c_2(n)}$, $c_1(n) := \int_0^\infty \delta_{0,\mu}^{2^*-1}(s) s^{n-1} ds$, $c_2(n) := 2\int_0^\infty \delta_{0,\mu}^2(s) s^{n-1} ds$, $\mu = \sqrt{n(n-2)}$.

The last result we provide is about the asymptotic behavior of the functions u_λ in the ball B_1, outside the origin. We show that, up to a constant, $\lambda^{-\frac{n-2}{2n-8}} u_\lambda$ converges in $C^1_{loc}(B_1 - \{0\})$ to $G(x, 0)$, where $G(x, y)$ is the Green function of the Laplace operator in B_1.

Theorem 3. *As $\lambda \to 0$ we have*

$$\lambda^{-\frac{n-2}{2n-8}} u_\lambda \to \tilde{c}(n) G(x, 0) \quad \text{in } C^1_{loc}(B_1 - \{0\}),$$

where $G(x, y)$ is the Green function for the Laplacian in the unit ball, $\tilde{c}(n)$ is the constant defined by $\tilde{c}(n) := \omega_n \frac{c_2(n)^{\frac{n-2}{2n-8}}}{c_1(n)^{\frac{4}{2n-8}}}$, ω_n is the measure of the $(n-1)$-dimensional unit sphere S^{n-1} and $c_1(n), c_2(n)$ are the constants appearing in Theorem 2.

The proof of the above results is technically complicated and often rely on the radial character of the problem. We would like to stress that the presence of the lower-order term λu makes our analysis quite different from that performed in [9] for low-energy sign-changing solution of an almost critical problem.

Since we consider nodal solutions, our results cannot be obtained by following the proofs for the case of positive solutions ([5], [6],[18], [21]). In particular, in order to analyze the behavior of the negative part u_λ^-, which is defined in an annulus, we prove a new uniform estimate (Propositions 7, 11), which holds for any dimension $n \geq 3$ and is of its own interest (see Remark 3 and Proposition 8).

For the sake of completeness, let us mention that our results, as well as those of [9], show a big difference between the asymptotic behavior of radial sign-changing solutions in dimension $n > 2$ and $n = 2$. Indeed, in this last case, the limit problems as well as the limit energies of the positive and negative part of solutions are different (see [17]).

Finally, we point out, that in view of the above theorems, it is natural to ask whether solutions of (1) which behave like the radial ones exist in other bounded domains. More precisely, it would be interesting to show the existence of sign-changing solutions whose positive and negative part concentrate at the same point but with different speeds, each one carrying the same energy.

In [19] we answer positively this question at least in the case of some symmetric domains in $\mathbb{R}^n$, $n \geq 7$.

We point out that this type of bubble tower solutions have interest also for the associated parabolic problem, since, as proved in [20], [12], [15], they induce a peculiar blow-up phenomenon for the initial data close to them.

We conclude observing that with similar proofs, it is possible to extend our results to the case of radial sign-changing solutions of (2) with k nodal regions, $k > 2$, and such that $\|u_\lambda\|^2_{B_1} \to kS^{n/2}$, as $\lambda \to 0$. As expected, the limit profile will be that of a tower of k bubbles with alternating signs. Moreover, with the same methods applied here, we can deduce analogous asymptotic relations as those of Theorem 2.

The paper is divided into 6 sections. In Sect. 2, we give some preliminary results on radial sign-changing solutions. In Section 3, we prove estimates for solutions with two nodal regions and, in particular, prove the new uniform estimate of Proposition 11.

In Sect. 4, we analyze the asymptotic behavior of the rescaled solutions and prove Theorem 1. Section 5 is devoted to the study of the divergence rate of $\|u^\pm_\lambda\|_\infty$, as $\lambda \to 0$ and to the proof of Theorem 2. Finally, in Sect. 6, we prove Theorem 3.

2. Preliminary results on radial sign-changing solutions

In this section, we recall or prove some results about the existence and qualitative properties of radial sign-changing solutions of the Brezis–Nirenberg problem (2).

We start with the following:

Proposition 1. *Let $n \geq 7$, $k \in \mathbb{N}^+$ and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ in $H^1_0(B_1)$. Then, there exists a radial sign-changing solution $u_{k,\lambda}$ of (2) with the following properties:*

(i): *$u_{k,\lambda}(0) > 0$,*

(ii): *$u_{k,\lambda}$ has exactly k nodal regions in B_1,*

(iii): *$I_\lambda(u_{k,\lambda}) = \frac{1}{2}\left(\int_{B_1} |\nabla u_{k,\lambda}|^2 - \lambda |u_{k,\lambda}|^2 \, dx\right) - \frac{1}{2^*}\int_{B_1} |u_{k,\lambda}|^{2^*} \, dx \to \frac{k}{n} S^{n/2}$ as $\lambda \to 0$, where S is the best constant for the Sobolev embedding $H^1_0(B_1) \hookrightarrow L^{2^*}(B_1)$.*

Proof. The existence of radial solutions of (2) satisfying (i) and (ii) is proved in [13]. It remains only to prove (iii). To do this, we need to introduce some notations and recall some facts proved in [13] and [10]. Let $k \in \mathbb{N}^+$ and $0 = r_0 < r_1 < \ldots < r_k = 1$ any partition of the interval $[0, 1]$, we define the sets $\Omega_1 := B_{r_1} = \{x \in B_1; |x| < r_1\}$ and, if $k \geq 2$, $\Omega_j := \{x \in B_1; r_{j-1} < |x| < r_j\}$ for $j = 2, \ldots, k$.

Then, we consider the set

$$\begin{aligned}\mathcal{M}_{k,\lambda} := \quad & \big\{u \in H^1_{0,rad}(B_1);\ \text{there exists a partition } 0 = r_0 < r_1 < \ldots < r_k = 1 \\ & \quad \text{such that: } u(r_j) = 0, \text{for } 1 \leq j \leq k,\ (-1)^{j-1}u(x) \geq 0, u \not\equiv 0 \text{ in } \Omega_j, \text{ and} \\ & \quad \left.\int_{\Omega_j} \left(|\nabla u_j|^2 - u_j^2 - |u_j|^{2^*}\right) \, dx = 0, \text{ for } 1 \leq j \leq k \right\},\end{aligned}$$

where $H^1_{0,rad}(B_1)$ is the subspace of the radial functions in $H^1_0(B_1)$ and u_j is the function defined by $u_j := u\ \chi_{\Omega_j}$, where χ_{Ω_j} denotes the characteristic function of Ω_j. Note that for any $k \in \mathbb{N}^+$ we have $\mathcal{M}_{k,\lambda} \neq \emptyset$, so we define

$$c_k(\lambda) := \inf_{\mathcal{M}_{k,\lambda}} I_\lambda(u).$$

In [13] the authors prove, by induction on k, that for every $k \in \mathbb{N}^+$ there exists $u_{k,\lambda} \in \mathcal{M}_{k,\lambda}$ such that $I_\lambda(u_{k,\lambda}) = c_k(\lambda)$ and $u_{k,\lambda}$ solves (2) in B_1. Moreover, they prove that

$$c_{k+1}(\lambda) < c_k(\lambda) + \frac{1}{n} S^{n/2}. \tag{4}$$

Note that for $k = 1$ $u_{1,\lambda}$ is just the positive solution found in [10], since by the Gidas, Ni and Nirenberg symmetry result [16] every positive solution is radial, and from [2] or [22] we know that positive solutions of (2) are unique.

To prove (iii) we argue by induction. Since $c_1(0) = \frac{1}{n}S^{n/2}$, by continuity we get that $c_1(\lambda) \to \frac{1}{n}S^{n/2}$, as $\lambda \to 0$, so that (iii) holds for $k = 1$.

Now assume that $c_k(\lambda) \to \frac{k}{n}S^{n/2}$, and let us to prove that $c_{k+1}(\lambda) = I_\lambda(u_{k+1,\lambda}) \to \frac{k+1}{n}S^{n/2}$.

Let us observe that $c_{k+1}(\lambda) \geq (k+1)c_1(\lambda)$. In fact, $w := u_{k+1,\lambda}$ achieves the minimum for I_λ over $\mathcal{M}_{k+1,\lambda}$, so that, by definition, it has $k+1$ nodal regions and $w_j := w\chi_{\Omega_j}$ belongs to $H^1_{0,rad}(B_1)$ for all $j = 1, \dots, k+1$. Since $w \in \mathcal{M}_{k+1,\lambda}$ we have, depending on the parity of j, that one between w_j^+ and w_j^- is not zero and belongs to $\mathcal{M}_{1,\lambda}$, we denote it by $\tilde{w}_j$. Then, $I_\lambda(\tilde{w}_j) \geq c_1(\lambda)$ for all $j = 1, \dots, k+1$ and hence

$$c_{k+1}(\lambda) = I_\lambda(w) = \sum_{j=1}^{k+1} I_\lambda(w_j^{\pm}) \geq (k+1)c_1(\lambda).$$

Combining this with (4) we get

$$c_k(\lambda) + \frac{1}{n}S^{n/2} > c_{k+1}(\lambda) \geq (k+1)c_1(\lambda).$$

Since by induction hypothesis $c_k(\lambda) \to \frac{k}{n}S^{n/2}$ as $\lambda \to 0$ and we have proved that $c_1(\lambda) \to \frac{1}{n}S^{n/2}$ we get that $c_{k+1}(\lambda) \to \frac{k+1}{n}S^{n/2}$, and the proof is concluded. □

Remark 1. *Let $k \in \mathbb{N}^+$ and (u_λ) be a family of solutions of (2), satisfying (iii) of Proposition 1, then $\|u_\lambda\|^2_{B_1} = \int_{B_1} |\nabla u_\lambda|^2 \, dx \to kS^{n/2}$, as $\lambda \to 0$.*

This comes easily from Proposition 1, and the fact that u_λ belongs to the Nehari manifold $\mathcal{N}_\lambda$ associated with (2), which is defined by

$$\mathcal{N}_\lambda := \{u \in H^1_0(B_1);\ \|u\|^2_{B_1} - \lambda |u|^2_{2,B_1} = |u|^{2^*}_{2^*,B_1}\}.$$

The first qualitative property we state about any radial sign-changing solution u_λ of (2) is that the global maximum point of $|u_\lambda|$ is located at the origin, which is a well-known fact for positive solutions of (2), as consequence of [16].

Proposition 2. *Let u_λ be a radial solution of (2), then we have $|u_\lambda(0)| = \|u_\lambda\|_\infty$.*

Proof. Since $u_\lambda = u_\lambda(r)$ is a radial solution of (2), then it solves

$$\begin{cases} u_\lambda'' + \frac{n-1}{r}u_\lambda' + \lambda u_\lambda + |u_\lambda|^{2^*-2}u_\lambda = 0 & \text{in } (0,1) \\ u_\lambda'(0) = 0, \quad u_\lambda(1) = 0. \end{cases} \tag{5}$$

Multiplying the equation by u_λ' we get

$$u_\lambda'' u_\lambda' + \lambda u_\lambda u_\lambda' + |u_\lambda|^{2^*-2}u_\lambda u_\lambda' = -\frac{n-1}{r}(u_\lambda')^2 \leq 0.$$

We rewrite this as

$$\frac{d}{dr}\left[\frac{(u_\lambda')^2}{2} + \lambda\frac{u_\lambda^2}{2} + \frac{|u_\lambda|^{2^*}}{2^*}\right] \leq 0.$$

Which implies that the function

$$E(r) := \frac{(u_\lambda')^2}{2} + \lambda\frac{u_\lambda^2}{2} + \frac{|u_\lambda|^{2^*}}{2^*}$$

is not increasing. So $E(0) \geq E(r)$ for all $r \in (0,1)$, where $E(0) = \lambda\frac{(u_\lambda(0))^2}{2} + \frac{|u_\lambda(0)|^{2^*}}{2^*}$. Assume that $r_0 \in (0,1)$ is the global maximum for $|u_\lambda|$, so we have $u_\lambda'(r_0) = 0$, $|u_\lambda(r_0)| = \|u_\lambda\|_\infty$ and $E(r_0) = \lambda\frac{\|u_\lambda\|^2_\infty}{2} + \frac{\|u_\lambda\|^{2^*}_\infty}{2^*}$.

Now we observe that, for all $\lambda > 0$, the function $g(x) := \frac{\lambda}{2}x^2 + \frac{1}{2^*}x^{2^*}$, defined in $\mathbb{R}^+ \cup \{0\}$, is strictly increasing; thus, we have $E(r_0) \geq E(0)$ and hence, $E(r_0) = E(0)$. Since g is strictly increasing, we get $|u_\lambda(0)| = |u_\lambda(r_0)| = \|u_\lambda\|_\infty$ and we are done. □

A consequence of the previous proposition is the following:

Corollary 1. *Assume u_λ is a nontrivial radial solution of (2). If $0 \leq r_1 \leq r_2 < 1$ are two points in the same nodal region such that $|u_\lambda(r_1)| \leq |u_\lambda(r_2)|$, $u_\lambda'(r_1) = u_\lambda'(r_2) = 0$, then necessarily $r_1 = r_2$.*

Proof. Assume by contradiction $r_1 < r_2$. By the assumptions and since the function $g(x) := \frac{\lambda}{2}x^2 + \frac{1}{2^*}x^{2^*}$ is a strictly increasing function (in $\mathbb{R}^+ \cup \{0\}$), we have $E(r_1) = g(|u_\lambda(r_1)|) \leq g(|u_\lambda(r_2)|) = E(r_2)$. But, as proved in Proposition 2, $E(r)$ is a decreasing function, so necessarily $E(r_1) = g(|u_\lambda(r_1)|) = g(|u_\lambda(r_2)|) = E(r_2)$ from which we get $|u_\lambda(r_1)| = |u_\lambda(r_2)|$. Since r_1, r_2 are in the same nodal region from $|u_\lambda(r_1)| = |u_\lambda(r_2)|$ we have $u_\lambda(r_1) = u_\lambda(r_2)$, and thus, there exists $r_* \in (r_1, r_2)$ such that $u'_\lambda(r_*) = 0$, and, since $E(r)$ is a decreasing function, we have $E(r_1) \geq E(r_*) \geq E(r_2)$. From this, we deduce $g(|u_\lambda(r_1)|) \geq g(|u_\lambda(r_*)|) \geq g(|u_\lambda(r_2)|)$, and hence, $u_\lambda(r_1) = u_\lambda(r_*) = u_\lambda(r_2)$. Therefore, u_λ must be constant in the interval $[r_1, r_2]$ and, being a solution of (2), it must be zero in that interval. In fact, since (2) is invariant under a change of sign, we can assume that $u_\lambda \equiv c > 0$. Then, by the strong maximum principle, u_λ must be zero in the nodal region to which r_1, r_2 belong. This, in turn, implies that u_λ is a trivial solution of (2) which is a contradiction. $\square$

3. Asymptotic results for solutions with 2 nodal regions

3.1. **General results.** Let (u_λ) be a family of least energy radial, sign-changing solutions of (2) and such that $u_\lambda(0) > 0$.

We denote by $r_\lambda \in (0,1)$ the node, so we have $u_\lambda > 0$ in the ball B_{r_λ} and $u_\lambda < 0$ in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^n; r_\lambda < |x| < 1\}$. We write $u_\lambda^\pm$ to indicate that the statements hold both for the positive and negative part of u_λ.

Proposition 3. *We have:*
(i): $\|u_\lambda^\pm\|^2_{B_1} = \int_{B_1} |\nabla u_\lambda^\pm|^2 \, dx \to S^{n/2}$, *as* $\lambda \to 0$,
(ii): $|u_\lambda^\pm|^{2^*}_{2^*,B_1} = \int_{B_1} |u_\lambda^\pm|^{\frac{2n}{n-2}} \, dx \to S^{n/2}$, *as* $\lambda \to 0$,
(iii): $u_\lambda \rightharpoonup 0$, *as* $\lambda \to 0$,
(iv): $M_{\lambda,+} := \max\limits_{B_1} u_\lambda^+ \to +\infty$, $M_{\lambda,-} := \max\limits_{B_1} u_\lambda^- \to +\infty$, *as* $\lambda \to 0$.

Proof. This proposition is a special case of Lemma 2.1 in [7]. $\square$

Let's recall a classical result, due to Strauss, known as "radial lemma":

Lemma 1 (Strauss). *There exists a constant $c > 0$, depending only on n, such that for all $u \in H^1_{rad}(\mathbb{R}^n)$*

$$|u(x)| \leq c \, \frac{\|u\|^{1/2}_{1,2}}{|x|^{(n-1)/2}} \quad a.e. \ on \ \mathbb{R}^n, \tag{6}$$

where $\|\cdot\|_{1,2}$ is the standard H^1-norm.

Proof. For the proof of this result see for instance [24]. $\square$

We denote by $s_\lambda \in (0,1)$ the global minimum point of $u_\lambda = u_\lambda(r)$, so we have $0 < r_\lambda < s_\lambda$, $u_\lambda^-(s_\lambda) = M_{\lambda,-}$. The following proposition gives an information on the behavior of r_λ and s_λ as $\lambda \to 0$.

Proposition 4. *We have $s_\lambda \to 0$ (and so $r_\lambda \to 0$ as well), as $\lambda \to 0$.*

Proof. Assume by contradiction that $s_{\lambda_m} \geq s_0$ for a sequence $\lambda_m \to 0$ and for some $0 < s_0 < 1$. Then, by Lemma 1 we get

$$M_{\lambda_m,-} = |u_{\lambda_m}(s_{\lambda_m})| \leq c \frac{\|u_{\lambda_m}\|^{1/2}_{1,2,B_1}}{s_{\lambda_m}^{(n-1)/2}} \leq c \frac{\|u_{\lambda_m}\|^{1/2}_{1,2,B_1}}{s_0^{(n-1)/2}},$$

where c is a positive constant depending only on n. Since $|\nabla u_\lambda|^2_{2,B_1} \to 2S^{n/2}$ as $\lambda \to 0$ it follows that $M_{\lambda_m,-}$ is bounded, which is a contradiction. $\square$

We recall another well-known proposition:

Proposition 5. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & in \ \mathbb{R}^n \\ u \to 0 & as \ |y| \to +\infty. \end{cases} \tag{7}$$

Assume that u has a finite energy $I_0(u) := \frac{1}{2}|\nabla u|^2_{2,\mathbb{R}^n} - \frac{1}{2^*}|u|^{2^*}_{2^*,\mathbb{R}^n}$ and u satisfies one of these assumptions:
(i): *u is positive (negative) in $\mathbb{R}^n$,*
(ii): *u is spherically symmetric about some point.*
Then, there exist $\mu > 0$, $x_0 \in \mathbb{R}^n$ such that u is one of the functions $\delta_{x_0,\mu}$, defined in (3).

Proof. A sketch of the proof can be found in [13], Proposition 2.2. □

3.2. **An upper bound for u^+_λ, u^-_λ.** In this section, we recall an estimate for positive solutions of (2) in a ball and we generalize it to get an upper bound for u^-_λ, which is defined in the annulus $A_{r_\lambda} := \{x \in \mathbb{R}^n; r_\lambda < |x| < 1\}$.

Proposition 6. *Let $n \geq 3$ and u be a solution of*

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{n+2}{n-2}} & \text{in } B_R \\ u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{8}$$

for some positive λ. Then, $u(x) \leq w(x, u(0))$ in B_R, where

$$w(x, c) := c\left\{1 + \frac{c^{-1}f(c)}{n(n-2)}|x|^2\right\}^{-(n-2)/2},$$

and $f : [0, +\infty) \to [0, +\infty)$ is the function defined by $f(y) := \lambda y + y^{\frac{n+2}{n-2}}$.

Proof. The proof is based on the results contained in the papers of Atkinson and Peletier [5], [6]. Since the solutions of (8) are radial (see [16]) we consider the ordinary differential equation associated with (8) which, by some change of variable, can be turned into an Emden–Fowler equation. For it is easy to get the desired upper bound. All details are given in the next Proposition 7. □

Remark 2. *The previous proposition gives an upper bound for u^+_λ. In fact, taking into account that u^+_λ is defined and positive in the ball B_{r_λ} and $u^+_\lambda(0) = M_{\lambda,+}$, we have*

$$\begin{aligned} u^+_\lambda(x) &\leq M_{\lambda,+}\left\{1 + \frac{M^{-1}_{\lambda,+}\, f(M_{\lambda,+})}{n(n-2)}|x|^2\right\}^{-(n-2)/2} \\ &= M_{\lambda,+}\left\{1 + \frac{\lambda + M^{\frac{4}{n-2}}_{\lambda,+}}{n(n-2)}|x|^2\right\}^{-(n-2)/2}, \end{aligned} \tag{9}$$

for all $x \in B_{r_\lambda}$.

Proposition 7. *Let u_λ be as in Sect. 3.1 and $\epsilon \in (0, \frac{n-2}{2})$. There exist $\delta = \delta(\epsilon) \in (0, 1)$, $\delta(\epsilon) \to 1$ as $\epsilon \to 0$ and a positive constant $\overline{\lambda} = \overline{\lambda}(\epsilon)$, such that for all $\lambda \in (0, \overline{\lambda})$ we have*

$$u^-_\lambda(x) \leq M_{\lambda,-}\left\{1 + \frac{M^{-1}_{\lambda,-}\, f(M_{\lambda,-})}{n(n-2)}c(\epsilon)|x|^2\right\}^{-(n-2)/2}, \tag{10}$$

for all $x \in A_{\delta,\lambda}$, where $A_{\delta,\lambda} := \{x \in \mathbb{R}^n;\ \delta^{-1/N}s_\lambda < |x| < 1\}$, $c(\epsilon) = \frac{2}{n-2}\epsilon$, s_λ is the global minimum point of u_λ, $M_{\lambda,-} = u^-_\lambda(s_\lambda)$ and f is defined as in Proposition 6.

Remark 3. *The statement of the above proposition holds also for lower dimensions. More precisely, with small modification to the proof of Proposition 7 we have:*

Proposition 8. *Let $3 \leq n \leq 6$ and set*

$$\tilde{\lambda}(n) := \inf\{\lambda \in \mathbb{R}^+;\ \textit{Problem (1) has a radial sign-changing solution } u_\lambda\}.$$

There exists $\bar{\epsilon} \in (0, \frac{n-2}{2})$ such that for all $\epsilon \in (0, \bar{\epsilon})$ there exists $\delta = \delta(\epsilon) \in (0, 1)$, with $\delta(\epsilon) \to 1$ as $\epsilon \to 0$, such that, for all λ in a right neighborhood of $\tilde{\lambda}(n)$, (10) holds, where $M_{\lambda,-} = u^-_\lambda(s_\lambda)$, s_λ is the global minimum point of u_λ in the last nodal region. [1]

[1] We assume without loss of generality that u_λ is negative in that region.

Proof of Proposition 7. Let v_λ the function defined by $v_\lambda(s) := u_\lambda^-(s+s_\lambda)$, $s \in (0, 1-s_\lambda)$. Since u_λ^- is a positive radial solution of (2) then v_λ is a solution of

$$\begin{cases} v_\lambda'' + \frac{n-1}{s+s_\lambda} v_\lambda' + \lambda v_\lambda + v_\lambda^{2^*-1} = 0 \quad \text{in} \quad (0, 1-s_\lambda) \\ v_\lambda'(0) = 0, \quad v_\lambda(1-s_\lambda) = 0. \end{cases} \tag{11}$$

To eliminate λ from the equation, we make the following change of variable, $\rho := \sqrt{\lambda}\ (s+s_\lambda)$, and we define $w_\lambda(\rho) := \lambda^{-\frac{n-2}{4}} v_\lambda(\frac{\rho}{\sqrt{\lambda}} - s_\lambda) = \lambda^{-\frac{n-2}{4}} u_\lambda^-(\frac{\rho}{\sqrt{\lambda}})$. By elementary computation, we see that w_λ solves

$$\begin{cases} w_\lambda'' + \frac{n-1}{\rho} w_\lambda' + w_\lambda + w_\lambda^{2^*-1} = 0 \quad \text{in} \quad (\sqrt{\lambda}\ s_\lambda, \sqrt{\lambda}) \\ w_\lambda'(\sqrt{\lambda}\ s_\lambda) = 0, \quad w_\lambda(\sqrt{\lambda}) = 0. \end{cases} \tag{12}$$

Making another change of variable, precisely $t := \left(\frac{n-2}{\rho}\right)^{n-2}$, and setting $y_\lambda(t) := w_\lambda\left(\frac{n-2}{t^{\frac{1}{n-2}}}\right)$ we eliminate the first derivative in (12). Thus, we get

$$\begin{cases} y_\lambda''\ t^k + y_\lambda + y_\lambda^{2^*-1} = 0 & \text{in} \quad \left(\frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}}, \frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}\ s_\lambda^{n-2}}\right), \\ y_\lambda'\left(\frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}\ s_\lambda^{n-2}}\right) = 0, \quad y_\lambda\left(\frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}}\right) = 0. \end{cases} \tag{13}$$

where $k = 2\dfrac{n-1}{n-2} > 2$. To simplify the notation, we set $t_{1,\lambda} := \frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}}$, $t_{2,\lambda} := \frac{(n-2)^{n-2}}{\lambda^{\frac{n-2}{2}}\ s_\lambda^{n-2}}$, $I_\lambda = (t_{1,\lambda}, t_{2,\lambda})$ and $\gamma_\lambda := y_\lambda(t_{2,\lambda}) = \lambda^{-\frac{n-2}{4}} M_{\lambda,-}$. Observe also that $2^* - 1 = 2k-3$.

We write the equation in (13) as $y_\lambda'' + t^{-k}(y_\lambda + y_\lambda^{2k-3}) = 0$, which is an Emden–Fowler type equation $y'' + t^{-k} h(y) = 0$ with $h(y) := y + y^{2k-3}$. The first step to prove (10) is the following inequality:

$$(y_\lambda' t^{k-1} y_\lambda^{1-k})' + t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda) \leq 0, \quad \text{for all } t \in I_\lambda. \tag{14}$$

To prove (14) we differentiate $y_\lambda' t^{k-1} y_\lambda^{1-k}$. Since $y_\lambda'' + t^{-k} h(y_\lambda) = 0$ we get

$$\begin{aligned} & y_\lambda'' t^{k-1} y_\lambda^{1-k} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \quad & -t^{-k}(y_\lambda + y_\lambda^{2k-3}) t^{k-1} y_\lambda^{1-k} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \quad & -t^{-1} y_\lambda^{2-k} - t^{-1} y_\lambda^{k-2} + y_\lambda'(k-1) t^{k-2} y_\lambda^{1-k} - (k-1)(y_\lambda')^2 t^{k-1} y_\lambda^{-k} \\ = \quad & -2(k-1) t^{k-2} y_\lambda^{-k} \left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda^2 + \tfrac{1}{2(k-1)} t^{1-k} y_\lambda^{2k-2} - \tfrac{1}{2} y_\lambda y_\lambda' + \tfrac{1}{2} t (y_\lambda')^2\right) \\ = \quad & -2(k-1) t^{k-2} y_\lambda^{-k} \left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \tfrac{1}{2} y_\lambda y_\lambda' + \tfrac{1}{2} t (y_\lambda')^2\right). \end{aligned}$$

Now, we add and subtract the number $\frac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)$ inside the parenthesis, so we have

$$\begin{aligned} & (y_\lambda' t^{k-1} y_\lambda^{1-k})' \\ = \quad & -2(k-1) t^{k-2} y_\lambda^{-k} \left(\tfrac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \tfrac{1}{2} y_\lambda y_\lambda' + \tfrac{1}{2} t (y_\lambda')^2 - \tfrac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)\right) \\ & - t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda). \end{aligned}$$

Setting $L_\lambda(t) := \frac{1}{2(k-1)} t^{1-k} y_\lambda h(y_\lambda) - \frac{1}{2} y_\lambda y_\lambda' + \frac{1}{2} t (y_\lambda')^2 - \frac{1}{2(k-1)} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda)$ we get

$$(y_\lambda' t^{k-1} y_\lambda^{1-k})' + t^{k-2} y_\lambda^{-k} t_{2,\lambda}^{1-k} \gamma_\lambda h(\gamma_\lambda) = -2(k-1) t^{k-2} y_\lambda^{-k} L_\lambda(t).$$

If we show that $L_\lambda(t) \geq 0$ for all $t \in I_\lambda$ we get (14). By definition it's immediate to verify that $L_\lambda(t_{2,\lambda}) = 0$, also by direct calculation, we have $L_\lambda'(t) = \frac{1}{2(k-1)} t^{1-k} y_\lambda' [y_\lambda h'(y_\lambda) - (2k-3) h(y_\lambda)] =$

$\frac{1}{2(k-1)}t^{1-k}y'_\lambda[(4-2k)y_\lambda]$. Since $y_\lambda > 0$, $y'_\lambda \geq 0$ in I_λ [2] and $k > 2$ we have $L'_\lambda(t) \leq 0$ in I_λ, and from $L_\lambda(t_{2,\lambda}) = 0$ it follows $L_\lambda(t) \geq 0$ for all $t \in I_\lambda$.

As second step, we integrate (14) between t and $t_{2,\lambda}$, for all $t \in I_\lambda$. Then, since $y'_\lambda(t_{2,\lambda}) = 0$ we get

$$-y'_\lambda(t)t^{k-1}y_\lambda^{1-k}(t) + \int_t^{t_{2,\lambda}} s^{k-2}y_\lambda^{-k}(s)\ t_{2,\lambda}^{1-k}\gamma_\lambda h(\gamma_\lambda)\ ds \leq 0.$$

We rewrite this last inequality as

$$y'_\lambda(t)t^{k-1}y_\lambda^{1-k}(t) \geq t_{2,\lambda}^{1-k}\gamma_\lambda h(\gamma_\lambda)\ \int_t^{t_{2,\lambda}} s^{k-2}y_\lambda^{-k}(s)\ ds.$$

Since $u_\lambda^- \leq M_{\lambda,-}$ by definition, it follows $y_\lambda^{-k} \geq \gamma_\lambda^{-k}$, so

$$\begin{aligned} y'_\lambda(t)t^{k-1}y_\lambda^{1-k}(t) &\geq t_{2,\lambda}^{1-k}\gamma_\lambda^{1-k}h(\gamma_\lambda)\ \int_t^{t_{2,\lambda}} s^{k-2}\ ds \\ &= \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\ \frac{t_{2,\lambda}^{k-1}-t^{k-1}}{t_{2,\lambda}^{k-1}} \\ &= \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\left[1-\left(\frac{t}{t_{2,\lambda}}\right)^{k-1}\right]. \end{aligned}$$

Multiplying the first and the last term of the above inequality by t^{1-k} we get

$$\frac{1}{2-k}(y_\lambda^{2-k})'(t) = y'_\lambda(t)\ y_\lambda^{1-k}(t) \geq \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\left(t^{1-k} - \frac{1}{t_{2,\lambda}^{k-1}}\right),$$

for all $t \in I_\lambda$. Integrating this inequality between t and $t_{2,\lambda}$ we have

$$\begin{aligned} \frac{\gamma_\lambda^{2-k}}{2-k} - \frac{y_\lambda^{2-k}(t)}{2-k} &\geq \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\int_t^{t_{2,\lambda}}\left(s^{1-k} - \frac{1}{t_{2,\lambda}^{k-1}}\right)ds \\ &= \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\left(\frac{t_{2,\lambda}^{2-k}}{2-k} - \frac{t^{2-k}}{2-k} - \frac{1}{t_{2,\lambda}^{k-2}} + \frac{t}{t_{2,\lambda}^{k-1}}\right). \end{aligned}$$

We rewrite this last inequality as

$$\begin{aligned} \frac{y_\lambda^{2-k}(t)}{k-2} - \frac{\gamma_\lambda^{2-k}}{k-2} &\geq \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\left(\frac{t^{2-k}}{k-2} + \frac{t}{t_{2,\lambda}^{k-1}} - \frac{k-1}{k-2}\frac{1}{t_{2,\lambda}^{k-2}}\right) \\ &\geq \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\ t^{2-k}\left[\frac{1}{k-2} + \left(\frac{t}{t_{2,\lambda}}\right)^{k-1} - \frac{k-1}{k-2}\left(\frac{t}{t_{2,\lambda}}\right)^{k-2}\right]. \end{aligned} \tag{15}$$

To the aim of estimating the last term in (15) we set $s := \left(\frac{t}{t_{2,\lambda}}\right)^{k-1}$ and study the function $g(s) := \frac{1}{k-2} + s - \frac{k-1}{k-2}s^{\frac{k-2}{k-1}}$ in the interval $[0,1]$. Clearly, $g(0) = \frac{1}{k-2} = \frac{n-2}{2} > 0$, $g(1) = 0$ and g is a decreasing function because $g'(s) = 1 - s^{-\frac{1}{k-1}} < 0$ in $(0,1)$. In particular, we have $g(s) > 0$ in $(0,1)$. Let's fix $\epsilon \in (0, \frac{n-2}{2})$, by the monotonicity of g we deduce that there exists only one $\delta = \delta(\epsilon) \in (0,1)$ such that $g(s) > \epsilon$ for all $0 \leq s < \delta$, $g(\delta) = \epsilon$ and $\delta \to 1$ as $\epsilon \to 0$. Now remembering that $s = \left(\frac{t}{t_{2,\lambda}}\right)^{k-1}$, we have $\left(\frac{t}{t_{2,\lambda}}\right)^{k-1} < \delta$ if and only if $t < \delta^{\frac{1}{k-1}}t_{2,\lambda}$ and $t_{1,\lambda} < \delta^{\frac{1}{k-1}}t_{2,\lambda}$ if and only if $s_\lambda^{n-2} < \delta^{\frac{1}{k-1}}$ which is true for all $0 < \lambda < \overline{\lambda}$, for some positive number $\overline{\lambda} = \overline{\lambda}(\epsilon)$. Setting $c(\epsilon) := (k-2)\epsilon$, from (15) and the previous discussion, we have

$$y_\lambda^{2-k}(t) - \gamma_\lambda^{2-k} \geq \frac{\gamma_\lambda^{1-k}h(\gamma_\lambda)}{k-1}\ t^{2-k}c(\epsilon), \tag{16}$$

---

[2] $y'_\lambda \geq 0$ because $(u_\lambda^-)'(r) \leq 0$ for $s_\lambda < r < 1$ as we can easily deduce from Corollary 1.

for all $t \in (t_{1,\lambda},\ \delta^{\frac{1}{k-1}} t_{2,\lambda})$, $0 < \lambda < \overline{\lambda}$. Now from (16) we deduce the desired bound for u_λ^-. In fact, we have

$$y_\lambda^{2-k}(t) \geq \gamma_\lambda^{2-k} + \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1}\ t^{2-k} c(\epsilon),$$

from which, since $k > 2$, we get

$$\begin{aligned} y_\lambda(t) &\leq \left(\gamma_\lambda^{2-k} + \frac{\gamma_\lambda^{1-k} h(\gamma_\lambda)}{k-1}\ t^{2-k} c(\epsilon)\right)^{-\frac{1}{k-2}} \\ &= \gamma_\lambda \left(1 + \frac{\gamma_\lambda^{-1} h(\gamma_\lambda)}{k-1}\ t^{2-k} c(\epsilon)\right)^{-\frac{1}{k-2}} \end{aligned} \tag{17}$$

Now, by definition, we have $y_\lambda(t) = \lambda^{-\frac{n-2}{4}} u_\lambda^-\left(\frac{\rho}{\sqrt{\lambda}}\right) = \lambda^{-\frac{n-2}{4}} u_\lambda^-(s + s_\lambda)$, $\gamma_\lambda = \lambda^{-\frac{n-2}{4}} M_{\lambda,-}$, $k-2 = \frac{2}{n-2}$, $k-1 = \frac{n}{n-2}$, $t = \left(\frac{n-2}{\rho}\right)^{n-2} = \left(\frac{n-2}{\sqrt{\lambda}(s+s_\lambda)}\right)^{n-2}$, in particular $t^{2-k} = t^{-\frac{2}{n-2}} = \left(\frac{\sqrt{\lambda}(s+s_\lambda)}{n-2}\right)^2 = \frac{\lambda(s+s_\lambda)^2}{(n-2)^2}$. Thus, we get

$$\begin{aligned} \frac{\gamma_\lambda^{-1} h(\gamma_\lambda)}{k-1}\ t^{2-k} c(\epsilon) &= \frac{\lambda^{\frac{n-2}{4}} M_{\lambda,-}^{-1}\left(\lambda^{-\frac{n-2}{4}} M_{\lambda,-} + \lambda^{-\frac{n+2}{4}} M_{\lambda,-}^{\frac{n+2}{n-2}}\right)}{\frac{n}{n-2}} c(\epsilon) \frac{\lambda(s+s_\lambda)^2}{(n-2)^2} \\ &= \frac{M_{\lambda,-}^{-1}\left(\lambda M_{\lambda,-} + M_{\lambda,-}^{2^*-1}\right)}{n(n-2)} c(\epsilon)(s+s_\lambda)^2 \\ &= \frac{M_{\lambda,-}^{-1}\ f\left(M_{\lambda,-}\right)}{n(n-2)} c(\epsilon)(s+s_\lambda)^2, \end{aligned}$$

where $f(z) := \lambda z + z^{2^*-1}$. Also, by direct computation, we see that the interval $(t_{1,\lambda},\ \delta^{\frac{1}{k-1}} t_{2,\lambda})$, corresponds to the interval $(\delta^{-\frac{1}{n}} s_\lambda, 1)$ for $s + s_\lambda = \frac{\rho}{\sqrt{\lambda}} = \frac{n-2}{\sqrt{\lambda}\ t^{\frac{1}{n-2}}}$. Thus, from the previous computations and (17) we have

$$\lambda^{-\frac{n-2}{4}} u_\lambda^-(s+s_\lambda) \leq \lambda^{-\frac{n-2}{4}} M_{\lambda,-}\left(1 + \frac{M_{\lambda,-}^{-1}\ f\left(\lambda M_{\lambda,-}\right)}{n(n-2)} c(\epsilon)(s+s_\lambda)^2\right)^{-\frac{n-2}{2}}.$$

Finally, dividing each term by $\lambda^{-\frac{n-2}{4}}$ and setting $r := s + s_\lambda$ we have

$$u_\lambda^-(r) \leq \left(1 + \frac{M_{\lambda,-}^{-1}\ f\left(\lambda M_{\lambda,-}\right)}{n(n-2)} c(\epsilon) r^2\right)^{-\frac{n-2}{2}},$$

for all $r \in (\delta^{-\frac{1}{n}} s_\lambda, 1)$, which is the desired inequality since u_λ^- is a radial function. □

4. Asymptotic analysis of the rescaled solutions

4.1. **Rescaling the positive part.** As in Sect. 3, we consider a family (u_λ) of least energy radial, sign-changing solutions of (2) with $u_\lambda(0) > 0$. Let us define $\beta := \frac{2}{n-2}$, $\sigma_\lambda := M_{\lambda,+}^\beta \cdot r_\lambda$; consider the rescaled function $\tilde{u}_\lambda^+(y) = \frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right)$ in B_{σ_λ}. The following lemma is elementary but crucial.

Lemma 2. *We have:*

(i): $\|u_\lambda^+\|_{B_{r_\lambda}}^2 = \|\tilde{u}_\lambda^+\|_{B_{\sigma_\lambda}}^2$,

(ii): $|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*} = |\tilde{u}_\lambda^+|_{2^*,B_{\sigma_\lambda}}^{2^*}$,

(iii): $|u_\lambda^+|_{2,B_{r_\lambda}}^2 = \frac{1}{M_{\lambda,+}^{2^*-2}} |\tilde{u}_\lambda^+|_{2,B_{\sigma_\lambda}}^2$

Proof. To prove (i) we have only to remember the definition of $\tilde{u}_\lambda$ and make the change of variable $x \to \frac{y}{M^\beta_{\lambda,+}}$. Taking into account that by definition $\nabla_y \tilde{u}^+_\lambda(y) = \frac{1}{M^{1+\beta}_{\lambda,+}}(\nabla_x u^+_\lambda)(\frac{y}{M^\beta_{\lambda,+}})$ and $2+2\beta = 2 + \frac{4}{n-2} = n\frac{2}{n-2} = n\beta = 2^*$, we get

$$\begin{aligned}
\|u^+_\lambda\|^2_{B_{r_\lambda}} &= \int_{B_{r_\lambda}} |\nabla_x u^+_\lambda(x)|^2 dx &&= \frac{1}{M^{n\beta}_{\lambda,+}} \int_{B_{\sigma_\lambda}} \left|\nabla_x u^+_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)\right|^2 dy \\
&= \frac{M^{2+2\beta}_{\lambda,+}}{M^{n\beta}_{\lambda,+}} \int_{B_{\sigma_\lambda}} |\nabla_y \tilde{u}_\lambda(y)|^2\, dy &&= \|\tilde{u}^+_\lambda\|^2_{B_{\sigma_\lambda}}.
\end{aligned}$$

The proof of (ii) is simpler:

$$\begin{aligned}
\int_{B_{r_\lambda}} |u^+_\lambda(x)|^{2^*} dx &= \int_{B_{\sigma_\lambda}} \frac{1}{M^{n\beta}_{\lambda,+}} \left|u^+_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)\right|^{2^*} dy \\
&= \int_{B_{\sigma_\lambda}} |\tilde{u}^+_\lambda(y)|^{2^*} dy.
\end{aligned}$$

The proof of (iii) is similar:

$$\begin{aligned}
\int_{B_{r_\lambda}} |u^+_\lambda(x)|^2 dx &= \int_{B_{\sigma_\lambda}} \frac{1}{M^{n\beta}_{\lambda,+}} \left|u^+_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)\right|^2 dy \\
&= \int_{B_{\sigma_\lambda}} \frac{1}{M^{n\beta-2}_{\lambda,+}} \left|\frac{1}{M_{\lambda,+}} u^+_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)\right|^2 dy \\
&= \frac{1}{M^{2^*-2}_{\lambda,+}} \int_{B_{\sigma_\lambda}} |\tilde{u}^+_\lambda(y)|^2 dy.
\end{aligned}$$

□

Remark 4. *Obviously, the previous lemma is still true if we consider any radial function $u \in H^1_{rad}(D)$, where D is a radially symmetric domain in $\mathbb{R}^n$, and for any rescaling of the kind $\tilde{u}(y) := \frac{1}{M} u\left(\frac{y}{M^\beta}\right)$, where $M > 0$ is a constant.*

The first qualitative result concerns the asymptotic behavior, as $\lambda \to 0$, of the radius $\sigma_\lambda = M^\beta_{\lambda,+} \cdot r_\lambda$ of the rescaled ball B_{σ_λ}. From Proposition 4 we know that $r_\lambda \to 0$ as $\lambda \to 0$, so this result gives also information on the growth of $M_{\lambda,+}$ compared to the decay of r_λ.

Proposition 9. *Up to a subsequence, $\sigma_\lambda \to +\infty$ as $\lambda \to 0$.*

Proof. Up to a subsequence, as $\lambda \to 0$, we have three alternatives:
(i): $\sigma_\lambda \to 0$,
(ii): $\sigma_\lambda \to l > 0$, $l \in \mathbb{R}$,
(iii): $\sigma_\lambda \to +\infty$.
We will show that (i) and (ii) cannot occur. Assume, by contradiction, that (i) holds then writing $|u^+_\lambda|^{2^*}_{2^*,B_{r_\lambda}}$ in polar coordinates we have

$$\begin{aligned}
|u^+_\lambda|^{2^*}_{2^*,B_{r_\lambda}} &= \omega_n \int_0^{r_\lambda} [u^+_\lambda(r)]^{2^*} r^{n-1} dr \\
&\le \omega_n\, M^{2^*}_{\lambda,+} \int_0^{r_\lambda} r^{n-1} dr \\
&= \omega_n\, (M^\beta_{\lambda,+})^n\, \frac{r^n_\lambda}{n} \\
&= \frac{\omega_n}{n}\, (M^\beta_{\lambda,+}\, r_\lambda)^n \to 0 \quad \text{as } \lambda \to 0.
\end{aligned}$$

But from Proposition 3 we know that $|u_\lambda^+|^{2^*}_{2^*,B_{r_\lambda}} \to S^{n/2}$ as $\lambda \to 0$, so we get a contradiction.

Next, assume, by contradiction, that (ii) holds. Since the rescaled functions $\tilde{u}_\lambda^+$ are solutions of

$$\begin{cases} -\Delta u = \frac{\lambda}{M_\lambda^{2\beta}} u + u^{2^*-1} & \text{in } B_{\sigma_\lambda} \\ u > 0 & \text{in } B_{\sigma_\lambda} \\ u = 0 & \text{on } \partial B_{\sigma_\lambda}. \end{cases} \tag{18}$$

and $(\tilde{u}_\lambda^+)$ is uniformly bounded, then by standard elliptic theory, $\tilde{u}_\lambda^+ \to \tilde{u}$ in $C^2_{loc}(B_l)$, where B_l is the limit domain of B_{σ_λ} and $\tilde{u}$ solves

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in } B_l \\ u > 0 & \text{in } B_l. \end{cases} \tag{19}$$

Let us show that the boundary condition $\tilde{u} = 0$ on ∂B_l holds. Since $M_{\lambda,+}$ is the global maximum of u_λ (see Proposition 2) then the rescaling $\tilde{u}_\lambda(y) := \frac{1}{M_{\lambda,+}} u_\lambda\left(\frac{y}{M_{\lambda,+}^\beta}\right)$ of the whole function u_λ is a bounded solution of

$$\begin{cases} -\Delta u = \frac{\lambda}{M_\lambda^{2\beta}} u + |u|^{2^*-2}u & \text{in } B_{M_{\lambda,+}^\beta} \\ u = 0 & \text{on } \partial B_{M_{\lambda,+}^\beta}. \end{cases}$$

So as before we get that $\tilde{u}_\lambda \to \tilde{u}_0$ in $C^2_{loc}(\mathbb{R}^n)$, where $\tilde{u}_0$ is a solution of $-\Delta u = |u|^{2^*-2}u$ in $\mathbb{R}^n$. Obviously, by definition, we have $\tilde{u}_\lambda(y) = \tilde{u}_\lambda^+(y)$ for all $y \in B_{\sigma_\lambda}$, $\tilde{u}_\lambda(y) = 0$ for all $y \in \partial B_{\sigma_\lambda}$ and $\tilde{u}_\lambda(y) < 0$ for all $y \in B_{M_{\lambda,+}^\beta} - \overline{B_{\sigma_\lambda}}$. Passing to the limit as $\lambda \to 0$, since $\overline{B_l}$ is a compact set of $\mathbb{R}^n$ we have $\tilde{u}_\lambda \to \tilde{u}_0$ in $C^2(\overline{B_l})$, now since $\tilde{u} = \tilde{u}_0 > 0$ in B_l and $\tilde{u}_0 = 0$ on ∂B_l, it follows $\tilde{u} = 0$ on ∂B_l. Since B_l is a ball, by Pohozaev's identity, we know that the only possibility is $\tilde{u} \equiv 0$ which is a contradiction since $\tilde{u}(0) = 1$. So the assertion is proved. □

Proposition 10. *We have:*

$$\tilde{u}_\lambda^+(y) \le \left\{1 + \frac{1}{n(n-2)}|y|^2\right\}^{-(n-2)/2}, \tag{20}$$

for all $y \in \mathbb{R}^n$.

Proof. From (9) for all $x \in B_{r_\lambda}$ we have

$$u_\lambda^+(x) \le M_{\lambda,+}\left\{1 + \frac{\lambda + M_{\lambda,+}^{\frac{4}{n-2}}}{n(n-2)}|x|^2\right\}^{-(n-2)/2}.$$

Dividing each side by $M_{\lambda,+}$ and setting $x = \frac{y}{M_{\lambda,+}^\beta} = \frac{y}{M_{\lambda,+}^{\frac{2}{n-2}}}$ we get

$$\begin{aligned} \frac{1}{M_{\lambda,+}} u_\lambda^+\left(\frac{y}{M_{\lambda,+}^\beta}\right) &\le \left\{1 + \frac{\lambda + M_{\lambda,+}^{\frac{4}{n-2}}}{M_{\lambda,+}^{\frac{4}{n-2}}\, n(n-2)}|y|^2\right\}^{-(n-2)/2} \\ &= \left\{1 + \frac{\lambda}{M_{\lambda,+}^{\frac{4}{n-2}}\, n(n-2)}|y|^2 + \frac{1}{n(n-2)}|y|^2\right\}^{-(n-2)/2} \\ &\le \left\{1 + \frac{1}{n(n-2)}|y|^2\right\}^{-(n-2)/2}, \end{aligned}$$

for all $y \in B_{\sigma_\lambda}$. Thus, we have proved (20) for all $y \in B_{\sigma_\lambda}$. Since $\tilde{u}_\lambda^+$ is zero outside the ball B_{σ_λ} and the second term in (20) is independent of λ, this bound holds in the whole $\mathbb{R}^n$. □

4.2. **An estimate on the first derivative at the node.** In this subsection, we prove an inequality concerning $(u_\lambda^+)'(r_\lambda)$ (or $(u_\lambda^-)'(r_\lambda)$) that will be useful in the next sections.

Lemma 3. *There exists a constant c_1, depending only on n, such that*

$$|(u_\lambda^+)'(r_\lambda)r_\lambda^{n-1}| \leq c_1\ r_\lambda^{\frac{n-2}{2}} \tag{21}$$

for all sufficiently small $\lambda > 0$. Since $(u_\lambda^-)'(r_\lambda) = -(u_\lambda^+)'(r_\lambda)$ the same inequality holds for $(u_\lambda^-)'(r_\lambda)$.

Proof. Since $u_\lambda^+ = u_\lambda^+(r)$ is a solution of $-[(u_\lambda^+)'r^{n-1}]' = \lambda u_\lambda^+ r^{n-1} + (u_\lambda^+)^{2^*-1} r^{n-1}$ in $(0, r_\lambda)$ and $(u_\lambda^+)'(0) = 0$ by integration, we get

$$\begin{aligned}(u_\lambda^+)'(r_\lambda)r_\lambda^{n-1} &= -\left[\int_0^{r_\lambda} \lambda u_\lambda^+ r^{n-1} dr + \int_0^{r_\lambda} (u_\lambda^+)^{2^*-1} r^{n-1} dr\right] \\ &= -\left[\frac{\lambda}{\omega_n}\int_{B_{r_\lambda}} u_\lambda^+(x)\ dx + \frac{1}{\omega_n}\int_{B_{r_\lambda}} [u_\lambda^+(x)]^{2^*-1}\ dx\right],\end{aligned}$$

where, as before, ω_n denotes the measure of the $(n-1)$-dimensional unit sphere S^{n-1}. Using Hölder's inequality and observing that $meas(B_{r_\lambda}) = \frac{\omega_n}{n} r_\lambda^n$ we deduce

$$\left|(u_\lambda^+)'(r_\lambda)r_\lambda^{n-1}\right| \leq \frac{\lambda}{(n\ \omega_n)^{\frac{1}{2}}} r_\lambda^{\frac{n}{2}} |u_\lambda^+|_{2,B_{r_\lambda}} + \frac{1}{n^{\frac{n-2}{2n}}\ \omega_n^{\frac{n+2}{2n}}} r_\lambda^{\frac{n-2}{2}} \left[|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*}\right]^{\frac{2^*-1}{2^*}}.$$

From Proposition 3 we know that both $|u_\lambda^+|_{2,B_{r_\lambda}}$, $|u_\lambda^+|_{2^*,B_{r_\lambda}}^{2^*}$ are bounded, moreover from Proposition 4 we have $r_\lambda \to 0$ as $\lambda \to 0$. So there exists a constant $c_1 = c_1(n)$ such that for all sufficiently small $\lambda > 0$ (21) holds. ☐

4.3. **Rescaling the negative part.** Now, we study the rescaled function $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^-\left(\frac{y}{M_{\lambda,-}^\beta}\right)$ in the annulus $A_{\rho_\lambda} := \{y \in \mathbb{R}^n; M_{\lambda,-}^\beta r_\lambda < |y| < M_{\lambda,-}^\beta\}$, where $\rho_\lambda := M_{\lambda,-}^\beta r_\lambda$. This case is more delicate than the previous one since the radius s_λ, where the minimum is achieved, depends on λ. Thus, roughly speaking, we have to understand how r_λ and s_λ behave with respect to the scaling parameter $M_{\lambda,-}^\beta$. This means that we have to study the asymptotic behavior of $M_{\lambda,-}^\beta r_\lambda$ and $M_{\lambda,-}^\beta s_\lambda$ as $\lambda \to 0$. It will be convenient to consider also the one-dimensional rescaling

$$z_\lambda(s) := \frac{1}{M_{\lambda,-}} u_\lambda^-\left(s_\lambda + \frac{s}{M_{\lambda,-}^\beta}\right),$$

which satisfies

$$\begin{cases} z_\lambda'' + \frac{n-1}{s + M_{\lambda,-}^\beta s_\lambda} z_\lambda' + \frac{\lambda}{M_{\lambda,-}^{2\beta}} z_\lambda + z_\lambda^{2^*-1} = 0 & \text{in } (a_\lambda, b_\lambda) \\ z_\lambda'(0) = 0, \quad z_\lambda(0) = 1, \end{cases} \tag{22}$$

where $a_\lambda := M_{\lambda,-}^\beta \cdot (r_\lambda - s_\lambda) < 0$, $b_\lambda := M_{\lambda,-}^\beta \cdot (1 - s_\lambda) > 0$. We define $\gamma_\lambda := M_{\lambda,-}^\beta s_\lambda$.

Since $s_\lambda \to 0$ as $\lambda \to 0$, we have $b_\lambda \to +\infty$; for the remaining parameters $a_\lambda, \gamma_\lambda$ it will suffice to study the asymptotic behavior of γ_λ as $\lambda \to 0$.

Up to a subsequence, we have three alternatives:

(a): $\gamma_\lambda \to +\infty$,
(b): $\gamma_\lambda \to \gamma_0 > 0$,
(c): $\gamma_\lambda \to 0$.

Lemma 4. *$\gamma_\lambda \to +\infty$ cannot happen.*

Proof. Assume $\gamma_\lambda \to +\infty$; up to a subsequence, we have $a_\lambda \to \bar{a} \leq 0$, as $\lambda \to 0$, where $\bar{a} \in \mathbb{R} \cup \{-\infty\}$.

If $\bar{a} < 0$ or $\bar{a} = -\infty$ then passing to the limit in (22) as $\gamma_\lambda = M_{\lambda,-}^\beta \cdot s_\lambda \to +\infty$ we have that $z_\lambda \to z$ in $C^1_{loc}(\bar{a}, +\infty)$, where z solves the limit problem

$$\begin{cases} z'' + z^{2^*-1} = 0 & \text{in } (\bar{a}, +\infty) \\ z'(0) = 0, \quad z(0) = 1. \end{cases} \tag{23}$$

Since $z_\lambda \to z$ in $C^1_{loc}(\bar a, +\infty)$ and being $z_\lambda > 0$, then by Fatou's lemma we have

$$\liminf_{\lambda \to 0} \int_{a_\lambda}^{b_\lambda} [z_\lambda(s)]^{2^*} ds \geq \int_{\bar a}^{+\infty} [z(s)]^{2^*} ds \geq c_1 > 0.$$

In particular, being $a_\lambda < 0$, by the same argument it follows that for all small $\lambda > 0$

$$\int_{0}^{b_\lambda} [z_\lambda(s)]^{2^*} ds \geq \int_{0}^{+\infty} [z(s)]^{2^*} ds \geq c_2 > 0.$$

Now, we have the following estimate:

$$\begin{aligned}
|u_\lambda^-|^{2^*}_{2^*, A_{r_\lambda}} &= \omega_n \int_{r_\lambda}^1 [u_\lambda^-(r)]^{2^*} r^{n-1} dr &&\geq \omega_n s_\lambda^{n-1} \int_{s_\lambda}^1 [u_\lambda^-(r)]^{2^*} dr \\
&= \omega_n s_\lambda^{n-1} M_{\lambda,-}^{2^*} \int_{s_\lambda}^1 \left[\frac{1}{M_{\lambda,-}} u_\lambda^-(r)\right]^{2^*} dr &&= \omega_n s_\lambda^{n-1} M_{\lambda,-}^{2^*-\beta} \int_0^{b_\lambda} [z_\lambda(s)]^{2^*} ds \\
&= \omega_n \gamma_\lambda^{n-1} \int_0^{b_\lambda} [z_\lambda(s)]^{2^*} ds &&\geq \omega_n \gamma_\lambda^{n-1} c_2,
\end{aligned}$$

having used the change of variable $r = s_\lambda + \frac{s}{M_{\lambda,-}^{\beta}}$. Since $|u_\lambda^-|^{2^*}_{2^*, A_{r_\lambda}} \to S^{n/2}$ while $\gamma_\lambda \to +\infty$, as $\lambda \to 0$, we get a contradiction.

If instead $\bar a = 0$ we consider the rescaled function $\tilde u_\lambda^-$ which solves

$$\begin{cases} -\Delta \tilde u_\lambda = \frac{\lambda}{M_{\lambda,-}^{2\beta}} \tilde u_\lambda + \tilde u_\lambda^{2^*-1} & \text{in } A_{\rho_\lambda} \\ \tilde u = 0 & \text{on } \partial A_{\rho_\lambda}, \end{cases} \tag{24}$$

and is uniformly bounded. We observe that since $a_\lambda \to 0$ then $\rho_\lambda = a_\lambda + \gamma_\lambda \to +\infty$. By definition, we have $\tilde u_\lambda^-(\rho_\lambda) = 0$, $\tilde u_\lambda^-(\gamma_\lambda) = 1$, for all $\lambda \in (0, \lambda_1)$. Thus, we have

$$\frac{|\tilde u_\lambda^-(\rho_\lambda) - \tilde u_\lambda^-(\gamma_\lambda)|}{|\rho_\lambda - \gamma_\lambda|} = \frac{1}{|a_\lambda|} \to +\infty \quad \text{as } \lambda \to 0.$$

From standard elliptic regularity theory, we know that $\tilde u_\lambda^-$ is a classical solution, so by the mean value theorem,

$$\frac{|\tilde u_\lambda^-(\rho_\lambda) - \tilde u_\lambda^-(\gamma_\lambda)|}{|\rho_\lambda - \gamma_\lambda|} = |(\tilde u_\lambda^-)'(\xi_\lambda)|,$$

for some $\xi_\lambda \in (\rho_\lambda, \gamma_\lambda)$; thus, $|(\tilde u_\lambda^-)'(\xi_\lambda)| \to +\infty$ as $\lambda \to 0$. From Corollary 1 it follows that $(\tilde u_\lambda^-)' > 0$ in $(\rho_\lambda, \gamma_\lambda)$ for all $\lambda > 0$.

By writing (24) in polar coordinates, we get:

$$(\tilde u_\lambda^-)'' + \frac{n-1}{r} (\tilde u_\lambda^-)' + \frac{\lambda}{M_{\lambda,-}^{2\beta}} \tilde u_\lambda^- + (\tilde u_\lambda^-)^{2^*-1} = 0.$$

From this, since $\tilde u_\lambda^- > 0$ and $(\tilde u_\lambda^-)' > 0$ in $(\rho_\lambda, \gamma_\lambda)$, we get $(\tilde u_\lambda^-)'' < 0$ in $(\rho_\lambda, \gamma_\lambda)$. Thus, $(\tilde u_\lambda^-)'(\rho_\lambda) > (\tilde u_\lambda^-)'(\xi_\lambda) > 0$, for all $\lambda > 0$. In particular, $(\tilde u_\lambda^-)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$.

Since, by elementary computation, we have $(\tilde u_\lambda^-)'(\rho_\lambda) = \frac{1}{M_{\lambda,-}^{1+\beta}} (u_\lambda^-)'(r_\lambda)$, by Lemma 3 we get

$$|(\tilde u_\lambda^-)'(\rho_\lambda)| \leq c \frac{1}{M_{\lambda,-}^{1+\beta}\ r_\lambda^{n/2}}$$

for a constant c independent from λ. Remembering that $1+\beta = 1 + \frac{2}{n-2} = \beta \cdot \frac{n}{2}$, and the definition of ρ_λ we have the following estimate

$$|(\tilde u_\lambda^-)'(\rho_\lambda)| \leq c \frac{1}{\rho_\lambda^{n/2}}.$$

Since $\rho_\lambda \to +\infty$, as $\lambda \to 0$, we deduce that $(\tilde u_\lambda^-)'(\rho_\lambda)$ is uniformly bounded, against $(\tilde u_\lambda^-)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$. Thus, we get a contradiction. $\square$

Thanks to Lemma 4 we deduce that (γ_λ) is a bounded sequence. The following proposition states an uniform upper bound for $\tilde u_\lambda^-$.

Proposition 11. *Let's fix $\epsilon \in (0, \frac{n-2}{2})$, and set $\bar{M} := \sup_\lambda \gamma_\lambda$. There exist $h = h(\epsilon)$ and $\bar{\lambda} = \bar{\lambda}(\epsilon) > 0$ such that*

$$\tilde{u}_\lambda^-(y) \leq U_h(y) \tag{25}$$

for all $y \in \mathbb{R}^n$, $0 < \lambda < \bar{\lambda}$, where

$$U_h(y) := \begin{cases} 1 & \text{if } |y| \leq h \\ \left[1 + \frac{1}{n(n-2)} c(\epsilon)|y|^2\right]^{-(n-2)/2} & \text{if } |y| > h, \end{cases} \tag{26}$$

with $c(\epsilon) = \frac{2}{n-2}\epsilon$.

Proof. We fix $\epsilon \in (0, \frac{n-2}{2})$, so by Proposition 7 there exist $\delta = \delta(\epsilon) \in (0,1)$ and $\overline{\lambda}(\epsilon) > 0$ such that

$$u_\lambda^-(x) \leq M_{\lambda,-} \left\{1 + \frac{M_{\lambda,-}^{-1}\, f(M_{\lambda,-})}{n(n-2)} c(\epsilon)|x|^2\right\}^{-(n-2)/2},$$

for all $x \in A_{\delta,\lambda} = \{x \in \mathbb{R}^n;\ \delta^{-1/N} s_\lambda < |x| < 1\}$, for all $\lambda \in (0, \overline{\lambda})$, where $c(\epsilon) = \frac{2}{n-2}\epsilon$. The same proof of Proposition 10 shows that

$$\tilde{u}_\lambda^-(y) \leq \left\{1 + \frac{1}{n(n-2)} c(\epsilon)|y|^2\right\}^{-(n-2)/2},$$

for all $y \in \tilde{A}_{\delta,\lambda} = \{y \in \mathbb{R}^n;\ M_{\lambda,-}^\beta \delta^{-1/N} s_\lambda < |y| < M_{\lambda,-}^\beta\}$. Now, since by definition $\tilde{u}_\lambda^-$ is uniformly bounded by 1, we get an upper bound defined in the whole annulus $\tilde{A}_{\rho_\lambda} = \{y \in \mathbb{R}^n;\ M_{\lambda,-}^\beta r_\lambda < |y| < M_{\lambda,-}^\beta\}$; to be more precise $\tilde{u}_\lambda^-(y) \leq U_\lambda(y)$, where

$$U_\lambda(y) := \begin{cases} 1 & \text{if } M_{\lambda,-}^\beta r_\lambda < |y| \leq M_{\lambda,-}^\beta \delta^{-1/N} s_\lambda \\ \left[1 + \frac{1}{n(n-2)} c(\epsilon)|y|^2\right]^{-(n-2)/2} & \text{if } M_{\lambda,-}^\beta \delta^{-1/N} s_\lambda < |y| < M_{\lambda,-}^\beta. \end{cases} \tag{27}$$

Since $\gamma_\lambda = M_{\lambda,-}^\beta s_\lambda \leq \bar{M}$, then setting $h := \delta^{-1/N} \bar{M}$ we get that $\delta^{-1/N} M_{\lambda,-}^\beta s_\lambda \leq h$. Therefore, from (27), since $\tilde{u}_\lambda^-$ is zero outside $\tilde{A}_{\rho_\lambda}$, we deduce (25). □

Lemma 5. *$\gamma_\lambda \to \gamma_0 > 0$, $\gamma_0 \in \mathbb{R}$, cannot happen.*

Proof. Assume that $\gamma_\lambda \to \gamma_0 > 0$, $\gamma_0 \in \mathbb{R}$. Since $0 < r_\lambda < s_\lambda$ there are only two possibilities for a_λ. To be precise, up to a subsequence we can have:

(i): $a_\lambda \to 0$,

(ii): $a_\lambda \to \bar{a} < 0$, $\bar{a} \in \mathbb{R}$.

We will show that both (i) and (ii) lead to a contradiction.

If we assume (i) the same proof of Lemma 4 gives a contradiction. We point out that now $\rho_\lambda \to \gamma_0$, as $\lambda \to 0$, so as before we get a contradiction since $(\tilde{u}_\lambda^-)'(\rho_\lambda)$ is uniformly bounded, against $(\tilde{u}_\lambda^-)'(\rho_\lambda) \to +\infty$ as $\lambda \to 0$.

Assuming (ii) we have $a_\lambda \to \bar{a} < 0$ and $\gamma_\lambda \to \gamma_0 > 0$. We define $m := \bar{a} + \gamma_0$. Clearly, we have $0 \leq m < \gamma_0$ and $\rho_\lambda \to m$ as $\lambda \to 0$. Assume $m > 0$ and consider the rescaling $\tilde{u}_\lambda^-$ in the annulus A_{ρ_λ} defined as before. Since $\tilde{u}_\lambda^-$ satisfies (24) and $(\tilde{u}_\lambda^-)$ is uniformly bounded then passing to the limit as $\lambda \to 0$ we get $\tilde{u}_\lambda^- \to \tilde{u}$ in $C_{loc}^2(\Pi)$, where Π is the limit domain $\Pi := \{y \in \mathbb{R}^n; |y| > m\}$ and $\tilde{u}$ is a positive radial solution of

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \Pi \tag{28}$$

By definition $\tilde{u}_\lambda^-(\gamma_\lambda) = 1$, $(\tilde{u}_\lambda^-)'(\gamma_\lambda) = 0$ for all λ, so as $\lambda \to 0$ we get $\tilde{u}(\gamma_0) = 1$, $\tilde{u}'(\gamma_0) = 0$ because of the convergence of $\tilde{u}_\lambda^- \to \tilde{u}$ in $C^2(K)$, for all compact subsets K in Π, and $\gamma_0 > m$. In particular, we deduce that $\tilde{u} \not\equiv 0$. We now show that $\tilde{u}$ can be extended to zero on $\partial\Pi = \{y \in \mathbb{R}^n; |y| = m\}$. Thanks to Lemma 3 and since we are assuming $m > 0$, which is the limit of ρ_λ as $\lambda \to 0$, we get that $(\tilde{u}_\lambda^-)'(\rho_\lambda)$ is uniformly bounded by a constant M, and by the monotonicity of $(\tilde{u}_\lambda^-)'$ the same bound holds for $(\tilde{u}_\lambda^-)'(s)$ for all $s \in (\rho_\lambda, \gamma_\lambda)$. It follows that in that interval $\tilde{u}_\lambda^-(s) \leq M(s - \rho_\lambda)$. Passing to the limit as $\lambda \to 0$ we have $\tilde{u}(s) \leq M(s - m)$ for all $s \in (m, \gamma_0)$ which implies $\tilde{u}$ can be extended by continuity to zero on $\partial\Pi$. We use the same notation $\tilde{u}$ to denote this extension.

Observe that $\tilde{u}$ has finite energy, in particular, using Fatou's lemma and thanks to Lemma 2, Remark 4, Proposition 3, we get

$$\int_{\Pi} |\nabla \tilde{u}|^2 dy \leq \liminf_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\nabla \tilde{u}_\lambda^-|^2 dy = \liminf_{\lambda \to 0} \int_{A_{r_\lambda}} |\nabla u_\lambda^-|^2 dx = S^{n/2}, \tag{29}$$

$$\int_{\Pi} |\tilde{u}|^{2*} dy \leq \liminf_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\tilde{u}_\lambda^-|^{2*} dy = \liminf_{\lambda \to 0} \int_{A_{r_\lambda}} |u_\lambda^-|^{2*} dx = S^{n/2}. \tag{30}$$

Moreover, since $\tilde{u}_\lambda^- \to \tilde{u}$ in $C^2_{loc}(\Pi)$ and thanks to the uniform upper bound given by Proposition 11, by Lebesgue's theorem, we have

$$\int_{\Pi} |\tilde{u}|^{2*} dy = \lim_{\lambda \to 0} \int_{A_{r_\lambda}} |u_\lambda^-|^{2*} dx = S^{n/2}. \tag{31}$$

Since $\tilde{u} \in H^1(\Pi) \cap C^0(\bar{\Pi})$ and is zero on $\partial \Pi$, then $\tilde{u} \in H^1_0(\Pi)$ and thanks to (29), (31) it follows that $\tilde{u}$ achieves the best constant in the Sobolev embedding on Π, which is impossible (see for instance [23], Theorem III.1.2). This ends the proof for the case $m > 0$.

Assume now $m = 0$, then $\tilde{u}_\lambda^-$ converges in $C^2_{loc}(\mathbb{R}^n - \{0\})$ to a radial function $\tilde{u}$ which is a positive bounded solution of

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \mathbb{R}^n - \{0\} \tag{32}$$

Since $\tilde{u}$ is a radial solution of (32), then integrating $-(\tilde{u}'(r) r^{n-1})' = \tilde{u}^{2^*-1}(r) r^{n-1}$ between $\delta > 0$ sufficiently small and γ_0 we get

$$\tilde{u}'(\delta)\delta^{n-1} = \int_\delta^{\gamma_0} \tilde{u}^{2^*-1} r^{n-1} dr.$$

Since the right-hand side is a positive and decreasing function of δ, we get $\tilde{u}'(\delta)\delta^{n-1} \to \tilde{l} > 0$ as $\delta \to 0$. Thus, $\tilde{u}'(\delta)$ behaves as δ^{1-n} near the origin, and this is a contradiction since $\int_{\mathbb{R}^n} |\nabla \tilde{u}|^2 dy = \omega_n \int_0^{+\infty} |\tilde{u}'(r)|^2 r^{n-1} dr$ is finite, and the proof is complete. □

As a consequence of Lemma 4 and Lemma 5 we have proved:

Proposition 12. *Up to a subsequence, we have* $\gamma_\lambda \to 0$ *as* $\lambda \to 0$.

4.4. **Final estimates and proof of Theorem 1.** From Proposition 12 we know that, up to a subsequence, $\gamma_\lambda = M^\beta_{-,\lambda} s_\lambda \to 0$ as $\lambda \to 0$. The rescaled function $\tilde{u}_\lambda^-(y) := \frac{1}{M_{\lambda,-}} u_\lambda^- \left(\frac{y}{M^\beta_{\lambda,-}} \right)$ in the annulus $A_{\rho_\lambda} := \{ y \in \mathbb{R}^n; M^\beta_{\lambda,-} r_\lambda < |y| < M^\beta_{\lambda,-} \}$ solves (24) and the functions $(\tilde{u}_\lambda^-)$ are uniformly bounded. Since $\gamma_\lambda \to 0$ as $\lambda \to 0$, in particular the limit domain of A_{ρ_λ} is $\mathbb{R}^n - \{0\}$ and by standard elliptic theory $\tilde{u}_\lambda^- \to \tilde{u}$ in $C^2_{loc}(\mathbb{R}^n - \{0\})$, where $\tilde{u}$ is positive, radial and solves

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \mathbb{R}^n - \{0\} \tag{33}$$

As in the proof of Lemma 5 by Fatou's Lemma, it follows that $\tilde{u}$ has finite energy $I_0(\tilde{u}) = \frac{1}{2}|\nabla \tilde{u}|^2_{2,\mathbb{R}^n} - \frac{1}{2^*}|\tilde{u}|^{2^*}_{2^*,\mathbb{R}^n}$. Moreover, thanks to the uniform upper bound (25), by Lebesgue's theorem, we have

$$\lim_{\lambda \to 0} \int_{A_{\rho_\lambda}} |\tilde{u}_\lambda^-|^{2*} dy = \int_{\mathbb{R}^n} |\tilde{u}|^{2*} dy,$$

so, by Lemma 2, Remark 4 and Proposition 3 we get

$$\int_{\mathbb{R}^n} |\tilde{u}|^{2*} dy = S^{n/2}.$$

The next two lemmas show that the function $\tilde{u} = \tilde{u}(s)$ can be extended to a $C^1([0, +\infty))$ function if we set $\tilde{u}(0) := 1$ and $\tilde{u}'(0) := 0$.

Lemma 6. *We have*

$$\lim_{s \to 0} \tilde{u}(s) = 1.$$

Proof. Since $\tilde{u}_\lambda^-$ is a radial solution of (24) and $\tilde{u}_\lambda^- \leq 1$, then

$$\begin{aligned}[(\tilde{u}_\lambda^-)' s^{n-1}]' &= -\tfrac{\lambda}{M_{\lambda,-}^{2\beta}} \tilde{u}_\lambda^-(s) s^{n-1} - [\tilde{u}_\lambda^-(s)]^{2^*-1} s^{n-1} \\ &\geq -\tfrac{\lambda}{M_{\lambda,-}^{2\beta}} s^{n-1} - s^{n-1} \\ &\geq -2s^{n-1}.\end{aligned}$$

Integrating between γ_λ and $s > \gamma_\lambda$ (with $s < M_{\lambda,-}^\beta$) we get

$$(\tilde{u}_\lambda^-)'(s) s^{n-1} \geq -2 \int_{\gamma_\lambda}^s t^{n-1} dt \geq -\frac{2}{n} s^n.$$

Hence, $(\tilde{u}_\lambda^-)'(s) \geq -\frac{2}{n} s$ for all $s \in (\gamma_\lambda, M_{\lambda,-}^\beta)$. Integrating again between γ_λ and s we have

$$\tilde{u}_\lambda^-(s) - 1 \geq -\frac{1}{n}(s^2 - \gamma_\lambda^2) \geq -\frac{1}{n} s^2.$$

Hence, $\tilde{u}_\lambda^-(s) \geq 1 - \frac{1}{n} s^n$ for all $s \in (\gamma_\lambda, M_{\lambda,-}^\beta)$. Since $\gamma_\lambda \to 0$ and $M_{\lambda,-}^\beta \to +\infty$, then, passing to the limit as $\lambda \to 0$, we get $\tilde{u}(s) \geq 1 - \frac{1}{n} s^2$, for all $s > 0$. From this inequality and since $\tilde{u} \leq 1$ we deduce $\lim_{s \to 0} \tilde{u}(s) = 1$. □

Lemma 7. *We have*

$$\lim_{s \to 0} \tilde{u}'(s) = 0.$$

Proof. As before, from the radial equation satisfied by $\tilde{u}_\lambda^-$, integrating between γ_λ and $s > \gamma_\lambda$ (with $s < M_{\lambda,-}^\beta$) we get

$$-(\tilde{u}_\lambda^-)'(s) s^{n-1} = \frac{\lambda}{M_{\lambda,-}^{2\beta}} \int_{\gamma_\lambda}^s \tilde{u}_\lambda^- t^{n-1} dt + \int_{\gamma_\lambda}^s (\tilde{u}_\lambda^-)^{2^*-1} t^{n-1} dt.$$

Since $\tilde{u} \leq 1$, and $\gamma_\lambda \to 0$ it follows that for all $\lambda > 0$ sufficiently small

$$|(\tilde{u}_\lambda^-)'(s) s^{n-1}| \leq \frac{\lambda}{M_{\lambda,-}^{2\beta}} \int_{\gamma_\lambda}^s t^{n-1} dt + \int_{\gamma_\lambda}^s t^{n-1} dt \leq 2 \frac{s^n}{n}.$$

Passing to the limit, as $\lambda \to 0$, we get $|\tilde{u}'(s)| \leq 2\dfrac{s}{n}$ for all $s > 0$, hence $\lim_{s \to 0} \tilde{u}'(s) = 0$. □

From Lemma 6 and Lemma 7 it follows that the radial function $\tilde{u}(y) = \tilde{u}(|y|)$ can be extended to a $C^1(\mathbb{R}^n)$ function. From now on, we denote by $\tilde{u}$ this extension. Next lemma shows that $\tilde{u}$ is a weak solution of (33) in the whole $\mathbb{R}^n$.

Lemma 8. *The function $\tilde{u}$ is a weak solution of*

$$-\Delta \tilde{u} = \tilde{u}^{2^*-1} \quad \text{in } \mathbb{R}^n \tag{34}$$

Proof. Let's fix a test function $\phi \in C_0^\infty(\mathbb{R}^n)$. If $0 \notin supp(\phi)$ the proof is trivial so from now on we assume $0 \in supp(\phi)$. Let $B(\delta)$ be the ball centered at the origin having radius $\delta > 0$, with δ sufficiently small such that $supp(\phi) \subset\subset B(1/\delta)$. Applying Green's formula to $\Omega(\delta) := B(1/\delta) - B(\delta)$, since $\tilde{u}$ is a $C_{loc}^2(\mathbb{R}^n - \{0\})$ solution of (33) and $\phi \equiv 0$ on $\partial B(1/\delta)$, we have

$$\int_{\Omega(\delta)} \nabla \tilde{u} \cdot \nabla \phi \, dy = \int_{\Omega(\delta)} \phi \, \tilde{u}^{2^*-1} \, dy + \int_{\partial B(\delta)} \phi \left(\frac{\partial \tilde{u}}{\partial \nu} \right) \, d\sigma. \tag{35}$$

We show now that $\int_{\partial B(\delta)} \phi \left(\frac{\partial \tilde{u}}{\partial \nu} \right) \, d\sigma \to 0$ as $\delta \to 0$. In fact since $\tilde{u}$ is a radial function, we have $\frac{\partial \tilde{u}}{\partial \nu}(y) = \tilde{u}'(\delta)$ for all $y \in \partial B(\delta)$, and from this relation, we get

$$\begin{aligned}\left| \int_{\partial B(\delta)} \phi \left(\frac{\partial \tilde{u}}{\partial \nu} \right) \, d\sigma \right| &\leq |\tilde{u}'(\delta)| \int_{\partial B(\delta)} |\phi| \, d\sigma \\ &\leq \omega_n |\tilde{u}'(\delta)| \delta^{n-1} ||\phi||_\infty.\end{aligned}$$

Thanks to Lemma 7 we have $|\tilde{u}'(\delta)|\delta^{n-1} \to 0$ as $\delta \to 0$. To complete the proof, we pass to the limit in (35) as $\delta \to 0$. We observe that

$$\begin{aligned} |\nabla\tilde{u}\cdot\nabla\phi|\ \chi_{\Omega(\delta)} &\leq |\nabla\tilde{u}|^2\ \chi_{\{|\nabla\tilde{u}|>1\}}|\nabla\phi| + |\nabla\tilde{u}|\ \chi_{\{|\nabla\tilde{u}|\leq 1\}}|\nabla\phi| \\ &\leq |\nabla\tilde{u}|^2\ \chi_{\{|\nabla\tilde{u}|>1\}}|\nabla\phi| + \ \chi_{\{|\nabla\tilde{u}|\leq 1\}}|\nabla\phi|. \end{aligned} \tag{36}$$

Since $\int_{\mathbb{R}^n}|\nabla\tilde{u}|^2dy \leq S^{n/2}$ and ϕ has compact support, the right-hand side of (36) belongs to $L^1(\mathbb{R}^n)$. Hence, from Lebesgue's theorem, we have

$$\lim_{\delta\to 0}\int_{\Omega(\delta)}\nabla\tilde{u}\cdot\nabla\phi\ dy = \int_{\mathbb{R}^n}\nabla\tilde{u}\cdot\nabla\phi\ dy. \tag{37}$$

Since ϕ has compact support by Lebesgue's theorem, we have

$$\lim_{\delta\to 0}\int_{\Omega(\delta)}\phi\ \tilde{u}^{2^*-1}\ dy = \int_{\mathbb{R}^n}\phi\ \tilde{u}^{2^*-1}\ dy. \tag{38}$$

From (35), (37), (38) and since we have proved $\int_{\partial B(\delta)}\phi\left(\frac{\partial\tilde{u}}{\partial\nu}\right)\ d\sigma \to 0$ as $\delta \to 0$ it follows that

$$\int_{\mathbb{R}^n}\nabla\tilde{u}\cdot\nabla\phi\ dy = \int_{\mathbb{R}^n}\phi\ \tilde{u}^{2^*-1}\ dy,$$

which completes the proof. ☐

Now, we have all the tools to prove Theorem 1.

Proof of Theorem 1. We start proving (i). By Proposition 9, arguing as in the previous proofs, we know that $(\tilde{u}_\lambda^+)$ is an equi-bounded family of radial solutions of (18) and converges in $C^2_{loc}(\mathbb{R}^n)$ to a function $\tilde{u}$ which solves $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^n$. From (20) we deduce that $\tilde{u} \to 0$ as $|y| \to +\infty$. To apply Proposition 5 we have to check that $\tilde{u}$ has finite energy, but this is an immediate consequence of Fatou's lemma and the assumption that u_λ has finite energy (for the details see (29) and (30)). Thus, $\tilde{u} = \delta_{x_0,\mu}$ for some $x_0 \in \mathbb{R}^n$, $\mu > 0$. Since $\tilde{u}$ is a radial function, we have $x_0 = 0$. Moreover, since $\tilde{u}(0) = 1$, by an elementary computation, we see that $\mu = \sqrt{n(n-2)}$.

Now we prove (ii). As we have seen at the beginning of this section, the equi-bounded family $(\tilde{u}_\lambda^-)$ converges in $C^2_{loc}(\mathbb{R}^n - \{0\})$ to a function $\tilde{u}$ which solves (33). From Lemma 6 and Lemma 7 we have that $\tilde{u}$ can be extended to a $C^1(\mathbb{R}^n)$ function such that $\tilde{u}(0) = 1$, $\nabla\tilde{u}(0) = 0$. Moreover, from Lemma 8 we know that $\tilde{u}$ is a weak solution of (34) and from Fatou's lemma, as seen in (29), (30), we have that $\tilde{u}$ has finite energy. Also, from Proposition 11 we deduce that $\tilde{u} \to 0$ as $|y| \to +\infty$.

By elliptic regularity (see for instance Appendix B of [23]) since $\tilde{u}$ is a weak solution of (34) we deduce that $\tilde{u} \in C^2(\mathbb{R}^n)$. Thanks to Proposition 5, since $\tilde{u}$ is a radial function and $\tilde{u}(0) = 1$, we have $\tilde{u} = \delta_{0,\mu}$, where $\mu > 0$ is the same as in (i). ☐

5. Asymptotic behavior of $M_{\lambda,+}$, $M_{\lambda,-}$ and proof of Theorem 2

We know from Proposition 3 that $M_{\lambda,+}, M_{\lambda,-} \to +\infty$ as $\lambda \to 0$, in addition in the last two sections we have proved that $M_{\lambda,+}^\beta r_\lambda \to +\infty$ while $M_{\lambda,-}^\beta r_\lambda \to 0$, as $\lambda \to 0$. Thus, $\frac{M_{\lambda,+}}{M_{\lambda,-}} \to +\infty$ as $\lambda \to 0$; in other words $M_{\lambda,+}$ goes to infinity faster than $M_{\lambda,-}$. In this section, we determine the order of infinity of $M_{\lambda,-}$ as negative power of λ and also an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and the node r_λ.

Proposition 13. *As $\lambda \to 0$ we have*

(i): $M_{\lambda,+}|(u_\lambda^+)'(r_\lambda)|r_\lambda^{n-1} \to c_1(n)$;

(ii): $\lambda^{-1}M_{\lambda,+}^{2\beta}r_\lambda^n|(u_\lambda^+)'(r_\lambda)|^2 \to c_2(n)$;

(iii): $M_{\lambda,+}^{2-2\beta}r_\lambda^{n-2}\lambda \to c_3(n)$,

where $c_1(n) = \int_0^\infty \delta_{0,\mu}^{2^*-1}(s)s^{n-1}ds$, $c_2(n) = 2\int_0^\infty \delta_{0,\mu}^2(s)s^{n-1}ds$, $c_3(n) = \frac{c_1^2(n)}{c_2(n)}$.

Proof. To prove (i) we integrate the equation $-[(u_\lambda^+)' r^{n-1}]' = \lambda u_\lambda^+ r^{n-1} + (u_\lambda^+)^{2^*-1} r^{n-1}$ between 0 and r_λ and multiply both sides by $M_{\lambda,+}$. Since $(u_\lambda^+)'(0) = 0$ we have

$$M_{\lambda,+} |(u_\lambda^+)'(r_\lambda)| r_\lambda^{n-1} = \lambda M_{\lambda,+} \int_0^{r_\lambda} u_\lambda^+ r^{n-1}\, dr + M_{\lambda,+} \int_0^{r_\lambda} (u_\lambda^+)^{2^*-1} r^{n-1}\, dr. \tag{39}$$

We first prove that $\lambda M_{\lambda,+} \int_0^{r_\lambda} u_\lambda^+ r^{n-1}\, dr \to 0$ as $\lambda \to 0$. In fact by the usual change of variable $r = \frac{s}{M_{\lambda,+}^\beta}$ we have

$$\begin{aligned}
\lambda M_{\lambda,+} \int_0^{r_\lambda} u_\lambda^+(r)\, r^{n-1}\, dr &= \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \int_0^{M_{\lambda,+}^\beta r_\lambda} \frac{1}{M_{\lambda,+}} u_\lambda^+ \left(\frac{s}{M_{\lambda,+}^\beta} \right) s^{n-1}\, ds \\
&= \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \int_0^{M_{\lambda,+}^\beta r_\lambda} \tilde{u}_\lambda^+(s) s^{n-1}\, ds
\end{aligned}$$

Thanks to the uniform upper bound (20) we have

$$\begin{aligned}
\lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \int_0^{M_{\lambda,+}^\beta r_\lambda} \tilde{u}_\lambda^+\, s^{n-1}\, ds &\leq \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \int_0^{M_{\lambda,+}^\beta r_\lambda} \left\{ 1 + \frac{1}{n(n-2)} s^2 \right\}^{-(n-2)/2} s^{n-1} ds \\
&\leq \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \int_0^1 s^{n-1} ds \\
&+ \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} [n(n-2)]^{(n-2)/2} \int_1^{M_{\lambda,+}^\beta r_\lambda} s^{-(n-2)} s^{n-1} ds \\
&= I_{\lambda,1} + I_{\lambda,2}.
\end{aligned}$$

Since $M_{\lambda,+} \to +\infty$ and $\int_0^1 s^{n-1} ds = \frac{1}{n}$ it's obvious that $I_{\lambda,1} \to 0$, as $\lambda \to 0$. Now, we show that the same holds for $I_{\lambda,2}$. In fact, setting $C_1(n) := [n(n-2)]^{(n-2)/2}$ we have

$$\begin{aligned}
I_{\lambda,2} &= \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} C_1(n) \int_1^{M_{\lambda,+}^\beta r_\lambda} s\, ds \\
&= \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} C_1(n) \left(\frac{M_{\lambda,+}^{2\beta} r_\lambda^2}{2} - \frac{1}{2} \right) \\
&= \lambda r_\lambda^2 \frac{C_1(n)}{2} - \lambda \frac{1}{M_{\lambda,+}^{2^*-2}} \frac{C_1(n)}{2} \to 0, \text{ as } \lambda \to 0,
\end{aligned}$$

since by definition, $2\beta = \frac{4}{n-2} = 2^* - 2$. To complete the proof of (i) we show that $M_{\lambda,+} \int_0^{r_\lambda} (u_\lambda^+)^{2^*-1} r^{n-1}\, dr \to \int_0^\infty \delta_{0,\mu}^{2^*-1}(s) s^{n-1} ds$ as $\lambda \to 0$. In fact, as before, by the change of variable $r = \frac{s}{M_{\lambda,+}^\beta}$ we have

$$\begin{aligned}
M_{\lambda,+} \int_0^{r_\lambda} [u_\lambda^+(r)]^{2^*-1}\, r^{n-1}\, dr &= \frac{1}{M_{\lambda,+}^{2^*-1}} \int_0^{M_{\lambda,+}^\beta r_\lambda} \left[u_\lambda^+ \left(\frac{s}{M_{\lambda,+}^\beta} \right) \right]^{2^*-1} s^{n-1}\, ds \\
&= \int_0^{M_{\lambda,+}^\beta r_\lambda} [\tilde{u}_\lambda^+(s)]^{2^*-1} s^{n-1}\, ds.
\end{aligned}$$

Since $\tilde{u}_\lambda^+ \to \delta_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^n)$, in particular we have $[\tilde{u}_\lambda^+(s)]^{2^*-1} \to [\delta_{0,\mu}(s)]^{2^*-1}$ as $\lambda \to 0$, for all $s \geq 0$, and thanks to the uniform upper bound (20), by Lebesgue's dominated convergence theorem, it follows that $\int_0^{M_{\lambda,+}^\beta r_\lambda} [\tilde{u}_\lambda^+(s)]^{2^*-1} s^{n-1}\, ds \to \int_0^\infty \delta_{0,\mu}^{2^*-1}(s) s^{n-1} ds$ so by (39) the proof of (i) is complete.

Now, we prove (ii). Applying Pohozaev's identity to u_λ^+, which solves $-\Delta u = \lambda u + u^{2^*-1}$ in B_{r_λ}, we have

$$\lambda \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\, dx = \frac{1}{2} \int_{\partial B_{r_\lambda}} (x \cdot \nu) \left(\frac{\partial u_\lambda^+}{\partial \nu} \right)^2\, d\sigma,$$

where ν is the exterior unit normal vector to ∂B_{r_λ}. Since u_λ^+ is radial, we have also $\left(\frac{\partial u_\lambda^+}{\partial \nu}\right)^2 = \left[(u_\lambda^+)'(r_\lambda)\right]^2$ so, passing to the unit sphere S^{n-1}, we get

$$\begin{aligned}\lambda \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\ dx &= \frac{1}{2} r_\lambda^{n-1} \int_{S^{n-1}} r_\lambda \left[(u_\lambda^+)'(r_\lambda)\right]^2\ d\omega \\ &= \frac{1}{2}\omega_n r_\lambda^n \left[(u_\lambda^+)'(r_\lambda)\right]^2.\end{aligned}$$

Thus, we have

$$\lambda^{-1} r_\lambda^n \left[(u_\lambda^+)'(r_\lambda)\right]^2 = 2\ \omega_n^{-1} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\ dx. \tag{40}$$

Now, performing the same change of variable as in (i) we have

$$\begin{aligned}\int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\ dx &= \frac{1}{M_{\lambda,+}^{2^*-2}} \int_{B_{\sigma_\lambda}} \left[\frac{1}{M_{\lambda,+}} u_\lambda^+ \left(\frac{y}{M_{\lambda,+}^\beta}\right)\right]^2\ dy \\ &= \frac{1}{M_{\lambda,+}^{2^*-2}} \int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2\ dy,\end{aligned}$$

Thus, we get

$$M_{\lambda,+}^{2\beta} \int_{B_{r_\lambda}} [u_\lambda^+(x)]^2\ dx = \int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2\ dy. \tag{41}$$

As in (i) since $\tilde{u}_\lambda^+ \to \delta_{0,\mu}$ in $C^2_{loc}(\mathbb{R}^n)$ and thanks to the uniform upper bound (20) we have

$$\int_{B_{\sigma_\lambda}} \left[\tilde{u}_\lambda^+(y)\right]^2\ dy \to \int_{\mathbb{R}^n} [\delta_{0,\mu}(y)]^2\ dy = \omega_n \int_0^{+\infty} [\delta_{0,\mu}(r)]^2 r^{n-1}\ dr.$$

From this, (40) and (41) we deduce that $\lambda^{-1} M_{\lambda,+}^{2\beta} r_\lambda^n \left[(u_\lambda^+)'(r_\lambda)\right]^2 \to 2\int_0^{+\infty} [\delta_{0,\mu}(r)]^2 r^{n-1}\ dr$, and (ii) is proved.

The proof of (iii) is a trivial consequence of (i) and (ii). □

Now, we state a similar result for $M_{\lambda,-}$.

Proposition 14. *As $\lambda \to 0$ we have the following:*

(i): $M_{\lambda,-}|(u_\lambda^-)'(1)| \to c_1(n)$;

(ii): $\lambda^{-1} M_{\lambda,-}^{2\beta} \left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n\right\} \to c_2(n)$;

(iii): $\lambda^{-1} M_{\lambda,-}^{2\beta} [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n \to 0$;

(iv): $M_{\lambda,-}^{2-2\beta}\lambda \to c_3(n)$,

where $c_1(n)$, $c_2(n)$ and $c_3(n)$ are the constants defined in Proposition 13.

Proof. The proof of (i) is similar to the proof of (i) of Proposition 13. Here, we integrate the equation $-[(u_\lambda^-)' r^{n-1}]' = \lambda u_\lambda^- r^{n-1} + (u_\lambda^-)^{2^*-1} r^{n-1}$ between s_λ and 1. Since $(u_\lambda^-)'(s_\lambda) = 0$ we have

$$(u_\lambda^-)'(1) = \lambda \int_{s_\lambda}^1 u_\lambda^-\ r^{n-1}\ dr + \int_{s_\lambda}^1 (u_\lambda^-)^{2^*-1} r^{n-1}\ dr.$$

By $M_\lambda^\beta s_\lambda \to 0$ and thanks to the uniform upper bound (25), arguing like in the proof of (i) of Proposition 13, we have

$$M_{\lambda,-}\ \lambda \int_{s_\lambda}^1 u_\lambda^-\ r^{n-1}\ dr \to 0$$

and

$$M_{\lambda,-} \int_{s_\lambda}^1 (u_\lambda^-)^{2^*-1} r^{n-1}\ dr = \int_{M_{\lambda,-}^\beta s_\lambda}^{M_{\lambda,-}^\beta} (\tilde{u}_\lambda^-)^{2^*-1} s^{n-1}\ ds \to \int_0^{+\infty} \delta_{0,\mu}^{2^*-1} s^{n-1}\ ds,$$

as $\lambda \to 0$. The proof of (i) is complete.

The proof of (ii) is similar to the corresponding one of Proposition 13. This time we apply Pohozaev's identity to u_λ^- in the annulus $A_{r_\lambda} = \{x \in \mathbb{R}^n;\ r_\lambda < |x| < 1\}$ whose boundary has two connected components, namely $\{x \in \mathbb{R}^n;\ |x| = r_\lambda\}$ and the unit sphere S^{n-1}. Thus, we have

$$\begin{aligned}\lambda \int_{A_{r_\lambda}} [u_\lambda^-(x)]^2 dx &= \frac{1}{2}\int_{\partial A_{r_\lambda}} (x\cdot\nu)\left(\frac{\partial u_\lambda^-}{\partial \nu}\right)^2 d\sigma\\ &= \frac{1}{2}\omega_n\left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n\right\}.\end{aligned}$$

Thus, multiplying each member by $M_{\lambda,-}^{2\beta}$ and rewriting the previous equation, we have

$$\begin{aligned}M_{\lambda,-}^{2\beta}\lambda^{-1}\left\{[(u_\lambda^-)'(1)]^2 - [(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n\right\} &= 2\omega_n^{-1} M_{\lambda,-}^{2\beta}\int_{A_{r_\lambda}} [u_\lambda^-(x)]^2 dx\\ &= 2\omega_n^{-1} M_{\lambda,-}^{2\beta}\frac{1}{M_{\lambda,-}^{n\beta}}\int_{A_{\sigma_\lambda}}\left[u_\lambda^-\left(\frac{y}{M_{\lambda,-}^{\beta}}\right)\right]^2 dy\\ &= 2\int_{M_{\lambda,-}^{\beta} r_\lambda}^{M_{\lambda,-}^{\beta}} \left[\tilde{u}_\lambda^-(s)\right]^2\ s^{n-1} ds.\end{aligned}$$

Since $2\int_{M_{\lambda,-}^{\beta} r_\lambda}^{M_{\lambda,-}^{\beta}} \left[\tilde{u}_\lambda^-(s)\right]^2\ s^{n-1}ds \to 2\int_0^\infty \delta_{0,\mu}^2(s)s^{n-1}ds$ as $\lambda \to 0$ we are done.

To prove (iii) we write

$$\begin{aligned}\lambda^{-1}M_{\lambda,-}^{2\beta}[(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n &= \frac{\lambda^{-1}M_{\lambda,-}^{2\beta}[(u_\lambda^-)'(r_\lambda)]^2 r_\lambda^n}{\lambda^{-1}M_{\lambda,+}^{2\beta}[(u_\lambda^+)'(r_\lambda)]^2 r_\lambda^n}\cdot \lambda^{-1}M_{\lambda,+}^{2\beta}[(u_\lambda^+)'(r_\lambda)]^2 r_\lambda^n\\ &= \frac{M_{\lambda,-}^{2\beta}}{M_{\lambda,+}^{2\beta}}\cdot\ \lambda^{-1}M_{\lambda,+}^{2\beta}[(u_\lambda^+)'(r_\lambda)]^2 r_\lambda^n \to 0\end{aligned}$$

since $\frac{M_{\lambda,-}}{M_{\lambda,+}} \to 0$ and $\lambda^{-1}M_{\lambda,+}^{2\beta}[(u_\lambda^+)'(r_\lambda)]^2 r_\lambda^n \to c_2(n)$ as $\lambda \to 0$ (by (ii) of Proposition 13).

Finally, the proof of (iv) is trivial. In fact from (ii) and (iii) it immediately follows that

$$\lambda^{-1}M_{\lambda,-}^{2\beta}[(u_\lambda^-)'(1)]^2 \to c_2(n).$$

From this and (i), we get (iv). □

Remark 5. *By elementary computation* $2 - 2\beta = 2 - \frac{4}{n-2} = \frac{2n-8}{n-2}$ *so by (iv) of Proposition 14 we have that* $M_{\lambda,-}$ *is an infinite of the same order as* $\lambda^{-\frac{n-2}{2n-8}}$.

From (iii) of Proposition 13 and (iv) of Proposition 14 we deduce the following result which gives an asymptotic relation between $M_{\lambda,+}$, $M_{\lambda,-}$ and r_λ.

Proposition 15. $\dfrac{M_{\lambda,-}^{2-2\beta}}{M_{\lambda,+}^{2-2\beta} r_\lambda^{n-2}} \to 1,$ *as* $\lambda \to 0$.

Proof of Theorem 2. It suffices to sum up the results contained in Proposition 13, Proposition 14 and Proposition 15. □

Remark 6. *We point out that in order to determine the explicit rate of* $M_{\lambda,+}$ *or, equivalently, that of* r_λ*, some difficulties arise. The techniques used in the previous proofs of integrating the equation and using the Pohozaev's identity do not seem to be sufficient to this purpose. Nevertheless, as a consequence of the methods applied in [19] we get, for* $n \geq 7$ *and for all sufficiently small* λ*, the existence of radial sign-changing solutions of (1) with the shape of a tower of two bubbles, and the parameters* μ_1, μ_2 *of these two bubbles are given. The lowest order bubble diverges as* $\lambda^{-\frac{n-2}{2n-8}}$*, which is the same order of* $M_{\lambda,-}$*, while the other diverges as* $\lambda^{-\frac{(3n-10)(n-2)}{(2n-8)(n-6)}}$*. Moreover, in a paper in preparation, we show, under some additional hypotheses, that the previous speeds are the only possible ones, for* $n \geq 7$*. Hence, we conjecture that* $M_{\lambda,+} \simeq \lambda^{-\frac{(3n-10)(n-2)}{(2n-8)(n-6)}}$.

6. Proof of Theorem 3

This section is entirely devoted to the proof of Theorem 3.

Proof of Theorem 3. We want to prove that $\lambda^{-\frac{n-2}{2n-8}}u_\lambda \to \tilde{c}(n)G(x,0)$ in $C^1_{loc}(B_1-\{0\})$. We begin from the local uniform convergence of $\lambda^{-\frac{n-2}{2n-8}}u_\lambda$. The same argument with some modifications will work for the local uniform convergence of its derivatives. Thanks to the representation formula, since $-\Delta u_\lambda = \lambda u_\lambda + |u_\lambda|^{2^*-2}u_\lambda$ in B_1, we have

$$\lambda^{-\frac{n-2}{2n-8}}u_\lambda(x) = -\lambda^{-\frac{n-2}{2n-8}}\lambda\int_{B_1} G(x,y)u_\lambda(y)\ dy - \lambda^{-\frac{n-2}{2n-8}}\int_{B_1} G(x,y)|u_\lambda|^{2^*-2}u_\lambda(y)\ dy. \tag{42}$$

Since $\lambda^{-\frac{n-2}{2n-8}}\lambda = \lambda^{\frac{n-6}{2n-8}}$, splitting the integrals we have

$$\begin{aligned}
\lambda^{-\frac{n-2}{2n-8}}u_\lambda(x) &= -\lambda^{\frac{n-6}{2n-8}}\int_{B_{r_\lambda}} G(x,y)u^+_\lambda(y)\ dy + \lambda^{\frac{n-6}{2n-8}}\int_{A_{r_\lambda}} G(x,y)u^-_\lambda(y)\ dy \\
&\quad -\lambda^{-\frac{n-2}{2n-8}}\int_{B_{r_\lambda}} G(x,y)[u^+_\lambda(y)]^{2^*-1}\ dy + \lambda^{-\frac{n-2}{2n-8}}\int_{A_{r_\lambda}} G(x,y)[u^-_\lambda(y)]^{2^*-1}\ dy \\
&= I_{1,\lambda}+I_{2,\lambda}+I_{3,\lambda}+I_{4,\lambda}.
\end{aligned}$$

Let K be a compact subset of $B_1-\{0\}$. We are going to prove that $I_{1,\lambda}, I_{2,\lambda}, I_{3,\lambda} \to 0$ uniformly in K, as $\lambda \to 0$. We begin with $I_{1,\lambda}$. For all $x \in K$ we have

$$\begin{aligned}
|I_{1,\lambda}| &\leq \left|\lambda^{\frac{n-6}{2n-8}}\int_{B_{r_\lambda}} G(x,y)u^+_\lambda(y)\ dy\right| \\
&= \left|\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{n\beta}_{\lambda,+}}\int_{B_{M^\beta_{\lambda,+}r_\lambda}} G\left(x,\frac{y}{M^\beta_{\lambda,+}}\right)u^+_\lambda\left(\frac{y}{M^\beta_{\lambda,+}}\right)\ dy\right| \\
&\leq \lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\int_{B_{M^\beta_{\lambda,+}r_\lambda}}\left|G\left(x,\frac{y}{M^\beta_{\lambda,+}}\right)\right|\tilde{u}^+_\lambda(y)\ dy.
\end{aligned}$$

Since K is a compact subset of $B_1-\{0\}$ and $|\frac{y}{M^\beta_{\lambda,+}}| < r_\lambda$ by an elementary computation, we see that for all $x \in K$, for all $\lambda > 0$ sufficiently small $\left|G\left(x,\frac{y}{M^\beta_{\lambda,+}}\right)\right| \leq c(K)$ for all $y \in B_{M^\beta_{\lambda,+}r_\lambda}$, where $c = c(K)$ is a positive constant depending only on K and n. Now, thanks to the uniform upper bound (20) we have

$$\begin{aligned}
&\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\int_{B_{M^\beta_{\lambda,+}r_\lambda}}\left|G\left(x,\frac{y}{M^\beta_{\lambda,+}}\right)\right|\tilde{u}^+_\lambda(y)\ dy \\
\leq\ & c(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\int_{B_{M^\beta_{\lambda,+}r_\lambda}}\left\{1+\frac{1}{n(n-2)}|y|^2\right\}^{-(n-2)/2} dy \\
=\ & c(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\omega_n\int_0^{M^\beta_{\lambda,+}r_\lambda}\left\{1+\frac{1}{n(n-2)}s^2\right\}^{-(n-2)/2} s^{n-1}\ ds \\
\leq\ & c_1(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\int_0^{M^\beta_{\lambda,+}r_\lambda} s^{-(n-2)}s^{n-1}\ ds = c_1(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\int_0^{M^\beta_{\lambda,+}r_\lambda} s\ ds \\
=\ & c_2(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M^{2^*-1}_{\lambda,+}}\ M^{2\beta}_{\lambda,+}r^2_\lambda = c_2(K)\lambda^{\frac{n-6}{2n-8}}\frac{1}{M_{\lambda,+}}r^2_\lambda \to 0,\ \text{as } \lambda\to 0.
\end{aligned}$$

Since this inequality is uniform respect to $x \in K$ we have $\|I_{1,\lambda}\|_{\infty,K} \to 0$ as $\lambda \to 0$. The proof that $\|I_{3,\lambda}\|_{\infty,K} \to 0$ is quite similar to the previous one, in fact with small modifications we get the following uniform estimate:

$$\begin{aligned}
|I_{3,\lambda}| &\leq \lambda^{\frac{n-6}{2n-8}} \frac{1}{M_{\lambda,+}} \int_{B_{M^{\beta}_{\lambda,+} r_\lambda}} \left| G\left(x, \frac{y}{M^{\beta}_{\lambda,+}}\right)\right| [\tilde{u}^{+}_{\lambda}(y)]^{2^*-1}\, dy \\
&\leq c(K)\lambda^{\frac{n-6}{2n-8}} \frac{1}{M_{\lambda,+}} \int_{B_{M^{\beta}_{\lambda,+} r_\lambda}} \left\{1 + \frac{1}{n(n-2)}|y|^2\right\}^{-(n+2)/2} dy \\
&\leq c(K)\lambda^{\frac{n-6}{2n-8}} \frac{1}{M_{\lambda,+}} \int_{\mathbb{R}^n} \left\{1 + \frac{1}{n(n-2)}|y|^2\right\}^{-(n+2)/2} dy \\
&= c_1(K)\, \lambda^{\frac{n-6}{2n-8}} \frac{1}{M_{\lambda,+}}, \quad \text{as } \lambda \to 0.
\end{aligned}$$

The proof for $I_{2,\lambda}$ is more delicate since for all small $\lambda > 0$ the Green function is not bounded when $x \in K$, $y \in A_{r_\lambda}$. We split the Green function in the singular part and the regular part so that

$$I_{2,\lambda} = \lambda^{\frac{n-6}{2n-8}} \int_{A_{r_\lambda}} G_{sing}(x,y) u^{-}_{\lambda}(y)\, dy + \lambda^{\frac{n-6}{2n-8}} \int_{A_{r_\lambda}} G_{reg}(x,y) u^{-}_{\lambda}(y)\, dy.$$

The singular part of the Green function is given by $\frac{1}{(2-n)\omega_n}\frac{1}{|x-y|^{n-2}}$, we want to show that

$$\lambda^{\frac{n-6}{2n-8}} \frac{1}{(2-n)\omega_n} \int_{A_{r_\lambda}} \frac{1}{|x-y|^{n-2}} u^{-}_{\lambda}(y)\, dy \to 0$$

uniformly for $x \in K$. The usual change of variable gives

$$\begin{aligned}
&\lambda^{\frac{n-6}{2n-8}} \frac{1}{(2-n)\omega_n} \int_{A_{r_\lambda}} \frac{1}{|x-y|^{n-2}} u^{-}_{\lambda}(y)\, dy \\
= &\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{\tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} u^{-}_{\lambda}\left(\frac{w}{M^{\beta}_{\lambda,-}}\right) dw.
\end{aligned}$$

Let η be a positive real number such that $\eta < \min\{\frac{d(0,K)}{2}; \frac{d(K,\partial B_1)}{2}\}$, where $d(\cdot,\cdot)$ denotes the Euclidean distance. It is clear that for all $\lambda > 0$ sufficiently small, we have $B(x,\eta) \subset\subset A_{r_\lambda}$, for all $x \in K$. Thus, $B(M^{\beta}_{\lambda,-}x, M^{\beta}_{\lambda,-}\eta) \subset\subset \tilde{A}_{r_\lambda}$, for all $x \in K$, and we split the last integral in two parts as indicated below:

$$\begin{aligned}
&\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{\tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} u^{-}_{\lambda}\left(\frac{w}{M^{\beta}_{\lambda,-}}\right) dw \\
= &\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{|M^{\beta}_{\lambda,-}x - w| < M^{\beta}_{\lambda,-}\eta} \frac{1}{|x - \frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} u^{-}_{\lambda}\left(\frac{w}{M^{\beta}_{\lambda,-}}\right) dw \\
+ &\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{\{|M^{\beta}_{\lambda,-}x - w| \geq M^{\beta}_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{|x - \frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} u^{-}_{\lambda}\left(\frac{w}{M^{\beta}_{\lambda,-}}\right) dw \\
= &\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{|M^{\beta}_{\lambda,-}x - w| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{|M^{\beta}_{\lambda,-}x - w|^{n-2}} \tilde{u}^{-}_{\lambda}(w)\, dw \\
+ &\frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{\{|M^{\beta}_{\lambda,-}x - w| \geq M^{\beta}_{\lambda,-}\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{M^{(n-2)\beta}_{\lambda,-}}{|M^{\beta}_{\lambda,-}x - w|^{n-2}} \tilde{u}^{-}_{\lambda}(w)\, dw := \tilde{I}_{A,\lambda} + \tilde{I}_{B,\lambda}.
\end{aligned}$$

Let's show that $\tilde{I}_{A,\lambda} \to 0$, uniformly for $x \in K$, as $\lambda \to 0$. First, by making the change of variable $z := w - M^{\beta}_{\lambda,-}x$ we have

$$\tilde{I}_{A,\lambda} = \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{|z| < M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}} \tilde{u}^{-}_{\lambda}\left(z + M^{\beta}_{\lambda,-}x\right) dz.$$

Let us fix $\epsilon \in (0, \frac{n-2}{2})$ and set $C = \frac{2}{n-2}\epsilon$. Thanks to the uniform upper bound (25), since

$$|M^{\beta}_{\lambda,-}x + z| \geq |M^{\beta}_{\lambda,-}|x| - |z|| = M^{\beta}_{\lambda,-}|x| - |z| \geq M^{\beta}_{\lambda,-}(|x| - \eta) > M^{\beta}_{\lambda,-}\frac{d(0,K)}{2} \geq M^{\beta}_{\lambda,-}\eta, \quad (43)$$

for all $x \in K$, for all z such that $|z| < \eta M^{\beta}_{\lambda,-}$, then for all sufficiently small λ we have

$$\begin{aligned}
|\tilde{I}_{A,\lambda}| &\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}\frac{1}{(n-2)\omega_n}\int_{|z|<M^{\beta}_{\lambda,-}\eta}\frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}}\left[1+\frac{1}{n(n-2)}C|z+M^{\beta}_{\lambda,-}x|^2\right]^{-(n-2)/2} dz \\
&\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_1\int_{|z|<M^{\beta}_{\lambda,-}\eta}\frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}}\left[M^{2\beta}_{\lambda,-}\eta^2\right]^{-(n-2)/2} dz \\
&= \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_2(K)\omega_n\int_0^{M^{\beta}_{\lambda,-}\eta} r\ dr = \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_2(K)\omega_n\frac{M^{2\beta}_{\lambda,-}\eta^2}{2} \\
&= c_3(K)\frac{\lambda^{\frac{n-6}{2n-8}}}{M_{\lambda,-}} \to 0, \quad \text{as } \lambda \to 0.
\end{aligned}$$

Thus, $\tilde{I}_{A,\lambda} \to 0$, uniformly for $x \in K$, as $\lambda \to 0$. Now, we prove that the same holds for $\tilde{I}_{B,\lambda}$.

$$\begin{aligned}
|\tilde{I}_{B,\lambda}| &\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}\frac{1}{(n-2)\omega_n}\int_{\{|M^{\beta}_{\lambda,-}x-w|\geq M^{\beta}_{\lambda,-}\eta\}\ \cap\ \tilde{A}_{r_\lambda}}\frac{1}{|\eta|^{n-2}}\tilde{u}^-_{\lambda}(w)\ dw \\
&\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c(K)\int_{\tilde{A}_{r_\lambda}}\tilde{u}^-_{\lambda}(w)\ dw \\
&\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c(K)\int_{|w|\leq h}1\ dw + \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c(K)\int_{h<|w|<M^{\beta}_{\lambda,-}}\left[1+\frac{1}{n(n-2)}C|w|^2\right]^{-(n-2)/2} dw \\
&\leq \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_1(K) + \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_2(K)\int_h^{M^{\beta}_{\lambda,-}} r\ dr \\
&= \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_1(K) + \frac{\lambda^{\frac{n-6}{2n-8}}}{M^{2^*-1}_{\lambda,-}}c_2(K)\left(\frac{M^{2\beta}_{\lambda,-}}{2} - \frac{h^2}{2}\right) \to 0, \text{ as } \lambda \to 0,
\end{aligned}$$

having used again (25). Since this estimate is uniform for $x \in K$ we have proved that $\tilde{I}_{B,\lambda} \to 0$ in $C^0(K)$ and from this and the analogous result for $\tilde{I}_{A,\lambda}$ we have $\lambda^{\frac{n-6}{2n-8}}\int_{A_{r_\lambda}} G_{sing}(x,y)u^-_{\lambda}(y)\ dy \to 0$ in $C^0(K)$. To complete the proof of $I_{2,\lambda} \to 0$ in $C^0(K)$ it remains to prove that $\lambda^{\frac{n-6}{2n-8}}\int_{A_{r_\lambda}} G_{reg}(x,y)u^-_{\lambda}(y)\ dy \to 0$ in $C^0(K)$. This is easy because the regular part of the Green function for the ball is uniformly bounded, to be precise let $l(K) := \sup\{d(0,x), x \in K\}$, clearly, being K a compact subset of $B_1 - \{0\}$, we have $l(K) < 1$ and since it is well known that

$$G_{reg}(x,y) = \frac{1}{(2-n)\omega_n}\frac{1}{|(|x||y|)^2 + 1 - 2x\cdot y|^{\frac{n-2}{2}}},$$

we have for all $x \in K$, $y \in A_{r_\lambda}$

$$\begin{aligned}
\frac{1}{|(|x||y|)^2 + 1 - 2x\cdot y|^{\frac{n-2}{2}}} &\leq \frac{1}{|(1-|x||y|)^2|^{\frac{n-2}{2}}} \\
&\leq \frac{1}{|1-l(K)|^{n-2}}\ .
\end{aligned} \quad (44)$$

Thus we have

$$\begin{aligned}
\left|\lambda^{\frac{n-6}{2n-8}}\int_{A_{r_\lambda}} G_{reg}(x,y)u_\lambda^-(y)\;dy\right| &\le c(K)\lambda^{\frac{n-6}{2n-8}}\int_{A_{r_\lambda}}|u_\lambda^-(y)|\;dy\\
&= c(K)\frac{\lambda^{\frac{n-6}{2n-8}}}{M_{\lambda,-}^{2^*}}\int_{\tilde A_{r_\lambda}}\left|u_\lambda^-\left(\frac{w}{M_{\lambda,-}^{\beta}}\right)\right|\;dw\\
&= c(K)\frac{\lambda^{\frac{n-6}{2n-8}}}{M_{\lambda,-}^{2^*-1}}\int_{\tilde A_{r_\lambda}}\left|\tilde u_\lambda^-(w)\right|\;dw.
\end{aligned}$$

As in the previous case, we see that $c(K)\frac{\lambda^{\frac{n-6}{2n-8}}}{M_{\lambda,-}^{2^*-1}}\int_{\tilde A_{r_\lambda}}\left|\tilde u_\lambda^-(w)\right|\;dw \to 0$ and the proof of $I_{2,\lambda}\to 0$ in $C^0(K)$ is complete.

Now to end the proof, we need to show that $I_{4,\lambda}\to \tilde c(n)G(x,0)$ in $C^0(K)$. We start making the usual change of variable

$$I_{4,\lambda}=\lambda^{-\frac{n-2}{2n-8}}\frac{1}{M_{\lambda,-}}\int_{\tilde A_{r_\lambda}} G\left(x,\frac{w}{M_{\lambda,-}^{\beta}}\right)[\tilde u_\lambda^-(w)]^{2^*-1}\;dw.$$

We split the Green function in the singular and the regular part, so that

$$\begin{aligned}
I_{4,\lambda} &= \frac{1}{(2-n)\omega_n}\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\int_{\tilde A_{r_\lambda}}\frac{1}{|x-\frac{w}{M_{\lambda,-}^{\beta}}|^{n-2}}[\tilde u_\lambda^-(w)]^{2^*-1}\;dw\\
&+\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\int_{\tilde A_{r_\lambda}} G_{reg}\left(x,\frac{w}{M_{\lambda,-}^{\beta}}\right)[\tilde u_\lambda^-(w)]^{2^*-1}\;dw
\end{aligned}$$

We begin with the singular integral which is more delicate. We want to show that

$$\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\frac{1}{(2-n)\omega_n}\int_{\tilde A_{r_\lambda}}\frac{1}{|x-\frac{w}{M_{\lambda,-}^{\beta}}|^{n-2}}[\tilde u_\lambda^-(w)]^{2^*-1}\;dw\to \tilde c(n)G_{sing}(x,0)\;\text{ in } C^0(K). \tag{45}$$

As in the previous case, we consider the ball $B(M_{\lambda,-}^{\beta}x, M_{\lambda,-}^{\beta}\eta)\subset\subset \tilde A_{r_\lambda}$, where $\eta>0$ is the same as before. Thus, we have

$$\begin{aligned}
&\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\frac{1}{(2-n)\omega_n}\int_{\tilde A_{r_\lambda}}\frac{1}{|x-\frac{w}{M_{\lambda,-}^{\beta}}|^{n-2}}[\tilde u_\lambda^-(w)]^{2^*-1}\;dw\\
=\;&\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\frac{1}{(2-n)\omega_n}\int_{|M_{\lambda,-}^{\beta}x-w|<M_{\lambda,-}^{\beta}\eta}\frac{M_{\lambda,-}^{(n-2)\beta}}{|M_{\lambda,-}^{\beta}x-w|^{n-2}}[\tilde u_\lambda^-(w)]^{2^*-1}\;dw\\
+\;&\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}\frac{1}{(2-n)\omega_n}\int_{\{|M_{\lambda,-}^{\beta}x-w|\ge M_{\lambda,-}^{\beta}\eta\}\,\cap\,\tilde A_{r_\lambda}}\frac{M_{\lambda,-}^{(n-2)\beta}}{|M_{\lambda,-}^{\beta}x-w|^{n-2}}[\tilde u_\lambda^-(w)]^{2^*-1}\;dw\\
:=\;&\tilde I_{C,\lambda}+\tilde I_{D,\lambda}.
\end{aligned}$$

We show that $\tilde I_{C,\lambda}\to 0$ in $C^0(K)$. As before, using the uniform upper bound (25) and (43) we get

$$
\begin{aligned}
|\tilde{I}_{C,\lambda}| &= \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} \frac{1}{(n-2)\omega_n} \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}} \left[\tilde{u}^{-}_{\lambda}\left(z+M^{\beta}_{\lambda,-}x\right)\right]^{2^*-1} dz \\
&\leq \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} \frac{1}{(n-2)\omega_n} \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}} \left[1+\frac{1}{n(n-2)}C|z+M^{\beta}_{\lambda,-}x|^2\right]^{-(n+2)/2} dz \\
&\leq \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} c_1 \int_{|z|<M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{|z|^{n-2}} \left[M^{2\beta}_{\lambda,-}\eta^2\right]^{-(n+2)/2} dz \\
&= \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} c_2(K) \int_0^{M^{\beta}_{\lambda,-}\eta} \frac{M^{(n-2)\beta}_{\lambda,-}}{r^{n-2}} M^{-(n+2)\beta}_{\lambda,-} r^{n-1}\, dr \\
&= \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} c_2(K) \frac{1}{M^{4\beta}_{\lambda,-}} \int_0^{M^{\beta}_{\lambda,-}\eta} r\, dr = \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} c_2(K) \frac{1}{M^{4\beta}_{\lambda,-}} \frac{M^{2\beta}_{\lambda,-}\eta^2}{2} \\
&= c_3(K) \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} \frac{1}{M^{2\beta}_{\lambda,-}}.
\end{aligned}
$$

Since $\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}$ is bounded (see Proposition 14 (iv) and Remark 5) then $\tilde{I}_{C,\lambda} \to 0$ uniformly for $x \in K$. Now, we show that $\tilde{I}_{D,\lambda} \to \tilde{c}(n)G_{sing}(x,0)$ in $C^0(K)$. We have

$$
\tilde{I}_{D,\lambda} = \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} \frac{1}{(2-n)\omega_n} \int_{\{|x-\frac{w}{M^{\beta}_{\lambda,-}}|\geq\eta\} \,\cap\, \tilde{A}_{r_\lambda}} \frac{1}{|x-\frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} [\tilde{u}^{-}_{\lambda}(w)]^{2^*-1}\, dw
$$

The first step is to prove that for all $w \in \mathbb{R}^n - \{0\}$

$$
\chi(w)_{\left\{\{|x-\frac{w}{M^{\beta}_{\lambda,-}}|\geq\eta\} \,\cap\, \tilde{A}_{r_\lambda}\right\}} \frac{1}{(2-n)\omega_n} \frac{1}{|x-\frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} [\tilde{u}^{-}_{\lambda}(w)]^{2^*-1} \to G_{sing}(x,0)\delta^{2^*-1}_{0,\mu}(w), \tag{46}
$$

uniformly for $x \in K$. First, observe that we need only to show that

$$
\frac{1}{|x-\frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} [\tilde{u}^{-}_{\lambda}(w)]^{2^*-1} \to \frac{1}{|x|^{n-2}} \delta^{2^*-1}_{0,\mu}(w) \quad \text{in } C^0(K). \tag{47}
$$

In fact, if we fix $w \in \mathbb{R}^n - \{0\}$, and $\lambda > 0$ is sufficiently small so that $w \in \tilde{A}_{r_\lambda}$ and $\frac{w}{M^{\beta}_{\lambda,-}} < \frac{d(0,K)}{2}$ then we have $|x - \frac{w}{M^{\beta}_{\lambda,-}}| \geq \eta$, for all $x \in K$. Hence we get

$$
\left| \chi(w)_{\left\{\{|x-\frac{w}{M^{\beta}_{\lambda,-}}|\geq\eta\} \,\cap\, \tilde{A}_{r_\lambda}\right\}} - 1 \right| = \chi(w)_{\left\{\{|x-\frac{w}{M^{\beta}_{\lambda,-}}|<\eta\} \cup \tilde{A}^c_{r_\lambda}\right\}} = 0,
$$

for all $x \in K$, for all $\lambda > 0$ sufficiently small, from which we deduce that

$$
\chi(w)_{\left\{\{|x-\frac{w}{M^{\beta}_{\lambda,-}}|\geq\eta\} \,\cap\, \tilde{A}_{r_\lambda}\right\}} \to 1 \quad \text{in } C^0(K).
$$

Now, the proof of (47) is trivial if we show that, for any fixed $w \in \mathbb{R}^n - \{0\}$

$$
\left| \frac{1}{\left|x - \frac{w}{M^{\beta}_{\lambda,-}}\right|^{n-2}} - \frac{1}{|x|^{n-2}} \right| \leq c(K) \left| \frac{w}{M^{\beta}_{\lambda,-}} \right| \tag{48}
$$

for all $x \in K$ and for all $\lambda > 0$ sufficiently small. This is an elementary computation but for the sake of completeness, we give the proof. We observe that the segment $\sigma\left(x, x - \frac{w}{M^{\beta}_{\lambda,-}}\right)$ joining x

and $x - \frac{w}{M^{\beta}_{\lambda,-}}$ is an uniformly bounded set and stays away from the origin. In fact for all $x \in K$, $t \in [0,1]$ and for all $\lambda > 0$ sufficiently small, we have

$$\left| x - t\frac{w}{M^{\beta}_{\lambda,-}} \right| \leq |x| + |t| \left| \frac{w}{M^{\beta}_{\lambda,-}} \right| < 1 + \frac{d(0,K)}{2} \tag{49}$$

$$\left| x - t\frac{w}{M^{\beta}_{\lambda,-}} \right| \geq \left| |x| - |t| \frac{|w|}{M^{\beta}_{\lambda,-}} \right| \geq d(0,K) - t\frac{d(0,K)}{2} \geq \frac{d(0,K)}{2}. \tag{50}$$

Thus, setting $g(x) := \frac{1}{|x|^{n-2}}$, by the mean value theorem, we have

$$g\left(x - \frac{w}{M^{\beta}_{\lambda,-}} \right) - g(x) = \nabla g(\xi_{\lambda,x}) \cdot \left(-\frac{w}{M^{\beta}_{\lambda,-}} \right),$$

where $\xi_{\lambda,x}$ lies on $\sigma\left(x, x - \frac{w}{M^{\beta}_{\lambda,-}} \right)$. By (49) and (50) we deduce that $|\nabla g(\xi_{\lambda,x})|$ is uniformly bounded[3] and (48) is proved.

To complete the first part of the proof, we apply Lebesgue's theorem. For all $x \in K$, $w \in \mathbb{R}^n - \{0\}$ we have

$$\begin{aligned}
&\left| \chi_{\left\{ \{|x - \frac{w}{M^{\beta}_{\lambda,-}}| \geq \eta\} \,\cap\, \tilde{A}_{r_\lambda} \right\}} \frac{1}{(2-n)\omega_n} \frac{1}{|x - \frac{w}{M^{\beta}_{\lambda,-}}|^{n-2}} [\tilde{u}^-_\lambda(w)]^{2^*-1} \right| \\
\leq\ & \eta^{-(n-2)} \frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}} [U_h(w)]^{2^*-1} \\
=\ & c_1(K)[U_h(w)]^{2^*-1},
\end{aligned}$$

where U_h is the function defined in (26). Since $(U_h)^{2^*-1} \in L^1(\mathbb{R}^n)$ and thanks to (46), (iv) of Proposition 14, by Lebesgue's theorem we deduce (45), where $G_{sing}(x,0) = \frac{1}{(2-n)\omega_n} \frac{1}{|x|^{n-2}}$, $\tilde{c}(n) = (\lim_{\lambda \to 0} \frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}}) \int_{\mathbb{R}^n} \delta^{2^*-1}_{0,\mu}(w)\, dw$. It's an elementary computation to see that $\tilde{c}(n)$ equals the expected constant $\omega_n \frac{c_2(n)^{\frac{n-2}{2n-8}}}{c_1(n)^{\frac{4}{2n-8}}}$, where $c_1(n), c_2(n)$ are the constants defined in Proposition 13. And the proof of (45) is done.

Finally, we prove that

$$\frac{\lambda^{-\frac{n-2}{2n-8}}}{M_{\lambda,-}} \int_{\tilde{A}_{r_\lambda}} G_{reg}\left(x, \frac{w}{M^{\beta}_{\lambda,-}} \right) [\tilde{u}^-_\lambda(w)]^{2^*-1}\, dw \to \tilde{c}(n) G_{reg}(x,0) \ \text{ in } C^0(K). \tag{51}$$

Since

$$G_{reg}\left(x, \frac{w}{M^{\beta}_{\lambda,-}} \right) = \frac{1}{(2-n)\omega_n} \frac{1}{\left| |x|^2 \frac{|w|^2}{M^{2\beta}_{\lambda,-}} + 1 - 2x \cdot \frac{w}{M^{\beta}_{\lambda,-}} \right|^{\frac{n-2}{2}}}$$

by the mean value theorem, repeating a similar argument as in the proof of (48), we deduce that for any fixed $w \in \mathbb{R}^n - \{0\}$

$$G_{reg}\left(x, \frac{w}{M^{\beta}_{\lambda,-}} \right) \to G_{reg}(x,0) \quad \text{in } C^0(K).$$

Thus, for any $w \in \mathbb{R}^n - \{0\}$ we have

$$G_{reg}\left(x, \frac{w}{M^{\beta}_{\lambda,-}} \right) [\tilde{u}^-_\lambda(w)]^{2^*-1} \to G_{reg}(x,0) \delta^{2^*-1}_{0,\mu}(w) \ \text{ in } C^0(K).$$

---

[3]by $\|\nabla g\|_{\infty, R(K)}$, where $R(K)$ is the compact annulus $R(K) := \{x \in \mathbb{R}^n;\ \frac{d(0,K)}{2} \leq |x| \leq 1 + \frac{d(0,K)}{2}\}$

Thanks to (44) we know that $G_{reg}\left(x, \frac{w}{M^{\beta}_{\lambda,-}}\right)$ is uniformly bounded, moreover, as we have done in the proof of (45), thanks to the upper bound (25), Proposition 14 we deduce (51).

To prove the local uniform convergence of $\lambda^{-\frac{n-2}{2n-8}}\nabla u_\lambda$ to $\tilde{c}(n)\nabla G(x,0)$ we simply derive (42) and repeat the previous proof, taking into account that for $i = 1, \dots, n$ we have

$$\partial_{x_i} G_{sing}(x,y) = \frac{1}{\omega_n}\frac{x_i - y_i}{|x-y|^n}.$$

□

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