PAPER • OPEN ACCESS

The interplay between local and non-local master equations: exact and approximated dynamics

To cite this article: Nina Megier *et al* 2020 *New J. Phys.* **22** 083011

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

OPEN ACCESS

RECEIVED
31 January 2020

REVISED
13 May 2020

ACCEPTED FOR PUBLICATION
23 June 2020

PUBLISHED
6 August 2020

PAPER

The interplay between local and non-local master equations: exact and approximated dynamics

Nina Megier[1,2,3], **Andrea Smirne**[1,2] and **Bassano Vacchini**[1,2]

[1] Dipartimento di Fisica 'Aldo Pontremoli', Università degli Studi di Milano, via Celoria 16, 20133 Milan, Italy
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via Celoria 16, 20133 Milan, Italy
[3] Author to whom any correspondence should be addressed.

E-mail: nina.megier@mi.infn.it

Keywords: open quantum systems, master equations, memory kernels, quantum Markovianity

Abstract

Master equations are a useful tool to describe the evolution of open quantum systems. In order to characterize the mathematical features and the physical origin of the dynamics, it is often useful to consider different kinds of master equations for the same system. Here, we derive an exact connection between the time-local and the integro-differential descriptions, focusing on the class of commutative dynamics. The use of the damping-basis formalism allows us to devise a general procedure to go from one master equation to the other and vice versa, by working with functions of time and their Laplace transforms only. We further analyze the Lindbladian form of the time-local and the integro-differential master equations, where we account for the appearance of different sets of Lindbladian operators. In addition, we investigate a Redfield-like approximation, that transforms the exact integro-differential equation into a time-local one by means of a coarse graining in time. Besides relating the structure of the resulting master equation to those associated with the exact dynamics, we study the effects of the approximation on Markovianity. In particular, we show that, against expectation, the coarse graining in time can possibly introduce memory effects, leading to a violation of a divisibility property of the dynamics.

1. Introduction

Any realistic physical system is unavoidably coupled to some external degrees of freedom and should then be treated as an open system. This is especially relevant in the realm of quantum physics, which describes phenomena on small scales, typically fragile under the interaction with the environment. As a consequence, the theory of open quantum systems [1, 2] has a wide application area, e.g., in quantum chemistry [3], quantum information [4] and even biophysics [5, 6].

However, accounting for the interaction of the system of interest with the external degrees of freedom has its price. In particular, the equations of motion describing the evolution of open quantum systems are either fully characterized, but derived under very restrictive assumptions, or in principle appropriate for a wide range of dynamical evolutions, but computationally demanding and at least partially unexplored. The former situation refers to the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) master equation [7, 8], which is associated to quantum dynamical semigroups; the latter is based on more general time-local master equations or on integro-differential master equations, fixed by a memory kernel. The focus of this paper is precisely on the connection between the latter types of description.

Both the time-local and the integro-differential descriptions of the open-system dynamics are highly relevant. The local one is better suited to access some of the properties of the evolution and to conduct calculations (e.g. numerically), while the approach based on the use of memory kernels often gives a better insight into the physical processes underlying the dynamics. This is the case, for example, for quantum semi-Markov processes, where continuous in time quantum evolutions are randomly interrupted by jumps [9–14]. The structure of the memory kernels for these processes clearly reflects this origin and additionally guarantees the property of complete positivity (CP) of the corresponding dynamical maps. For a given

system it is in general of advantage to know both representations of its evolution, as each can extend one's knowledge and it is rarely possible to know a priori which is the most suitable one.

The first goal of this paper is to shed some light on the connection between the descriptions given by the time-local and the integro-differential approach, both treated in an exact way. We use a damping-basis representation of the generator and memory kernel, which proves to be a powerful tool to characterize their structural properties, as well as the connection between them [15]. Focusing on commutative dynamics, i.e., such that the dynamical maps at different times commute [16, 17], we define a general procedure to go directly from one form of the master equation to the other, and vice versa. In particular, we give an explicit link between the damping-basis description of the time-local and the integro-differential master equations, which only relies on transformations of functions of time and their Laplace transform. Furthermore, our analysis accounts for the differences and similarities of the structural properties of the time-local and integro-differential master equations, when expressed in the canonical Lindbladian form [1, 7, 8]. The possibly different sets of Lindblad operators of the two master equations are traced back to specific relations among the eigenvalues of the corresponding damping-basis representations. In this way, as shown in the examples, we provide a common theoretical framework to various models considered in the physical literature.

The second part of the paper concerns the study of a possible simplified treatment of the integro-differential master equation, which enforces a time-local structure, and can be seen as an approximated link between the two different descriptions of the open-system dynamics. Relying on the idea that the memory kernel is generally localized around the origin, with a width which can be interpreted as the memory time of the dynamics, one can define a general coarse-graining operation leading to a time-local master equation, along the lines of the seminal work of Redfield [18]. Here, we first show that the damping-basis description also enables us to connect this approximated time-local description and the two original, time-local and integro-differential, ones. Furthermore, we study how moving to a Redfield-like master equation modifies the (non-)Markovian nature of the dynamics.

In recent years, the investigation of memory effects in the open quantum system dynamics, summarized under the term 'non-Markovianity', has attracted a great interest, among others, in connection with quantum thermodynamics [19–21], quantum-control theory [22] and quantum metrology [23–26]. Till now, a wide range of non-equivalent characterizations of quantum non-Markovianity was introduced in the literature, see [27–29] for recent reviews. A lot of emphasis has been put on the divisibility properties of the dynamical maps as a possible signature of non-Markovian behavior. In particular, an open-system dynamics is called (C)P-divisible if the dynamical maps can be decomposed via (completely) positive propagators [30, 31]. CP-divisible dynamics correspond to so-called generalized GKSL master equations, which are time-local master equations with positive, but time dependent rates [32]; no equivalent general constraint is known for the memory-kernel master equations. The property of P-divisibility has a physical interpretation in terms of an information back-flow to the reduced system [33], and appears to be related to a continuous-measurement interpretation [34]. A characterization of P-divisible dynamics is only known for time-local generators [28], thanks to a result about positive semigroups [35], while the case of memory kernels remains undiscovered beyond some special cases.

From the structures of time-local and integro-differential equations one is tempted to conclude that the first description corresponds to 'memoryless dynamics' and the latter introduces some memory effects, but neither statement is in general true. In particular, one can easily show that the time-local representation is always possible for dynamics given by invertible maps [32], no matter how much non-Markovian the evolution is. The time-local and the integro-differential descriptions seem to be somehow complementary: if one has a well behaving form, the other one is often singular, however both can describe non-Markovian dynamics. Here, with the help of the damping-basis approach, we will also investigate the relation between the (C)P-divisibility of the exact and Redfield-like dynamics. Despite the intuition that the coarse graining in time might wash out the memory effects due to the interaction with the environment, we will show that imposing the Redfield-like master equation might actually result in going from a CP-divisible dynamics into a non-CP-divisible one.

The rest of the paper is organized as follows. Section 2 is concerned with the structure of the time-local generator and the memory kernel associated to the same open-system dynamics. After recalling the damping-basis formalism in section 2.1, we apply it to the class of commuting dynamics in section 2.2. Here, the main result of the paper is given: proposition 1, where a direct connection between the damping-basis representation of the time-local generator and the memory kernel is stated. In section 2.3 we further analyze the relation between the time-local and the integro-differential descriptions using a Lindbladian form of the master equation. In section 3, we investigate the main features of the related Redfield-like master equation. We first work out the structure of the time-local generator obtained by approximating the exact memory kernel master equation in section 3.1. Finally, in section 3.2 we show that

this approximation, somehow counter-intuitively, does not always improves the divisibility properties of the time evolution. Section 4 summarizes our findings.

2. Structure of the master equations

The properties of the open quantum system alone are fixed by its density operator $\rho(t)$, also referred to as reduced state, at a generic time t. The dynamics of the open system is thus fully characterized by a family of dynamical maps $\{\Lambda_t\}_{t\geqslant 0}$, which map the initial reduced state $\rho(0)$ to the reduced state at later times, according to $\rho(t) = \Lambda_t[\rho(0)]$ ($\Lambda_0 = \mathbb{1}$). In the time-local description of the dynamics, the family of dynamical maps satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t = \mathcal{K}_t^{\mathrm{TCL}}\Lambda_t, \tag{1}$$

where the superscript TCL stands for time-convolutionless [1]; instead, the integro-differential approach builds on the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t = \int_0^t \mathrm{d}\tau \mathcal{K}_{t-\tau}^{\mathrm{NZ}}\Lambda_\tau \equiv (\mathcal{K}^{\mathrm{NZ}} * \Lambda)_t, \tag{2}$$

where the symbol $*$ denotes the convolution in time, $\mathcal{K}_t^{\mathrm{NZ}}$ is the memory kernel and NZ stands for Nakajima–Zwanzig [1]. The conditions on the time-local generator $\mathcal{K}_t^{\mathrm{TCL}}$ and on the memory kernel $\mathcal{K}_t^{\mathrm{NZ}}$ guarantying that Λ_t is a proper dynamical map, i.e. completely positive[4] and trace preserving (CPTP), are in general unknown. Only in some limited cases definite statements in this respect have been obtained, see e.g. [9, 14, 36–46].

2.1. Damping bases

In this paper, we restrict to the case where the reduced system is associated with a Hilbert space $\mathcal{H}$ of finite dimension N. The set of linear operators on $\mathcal{H}$ is denoted as $\mathcal{B}(\mathcal{H})$ and it is an N^2 dimensional Hilbert space. Hence, given a linear map Ξ acting on $\mathcal{B}(\mathcal{H})$, often referred to as super-operator, and a basis $\{\sigma_\alpha\}_{\alpha=1,\ldots,N^2}$ of $\mathcal{B}(\mathcal{H})$ orthonormal with respect to the Hilbert–Schmidt scalar product,

$$\langle \sigma_\alpha, \sigma_\beta\rangle = \mathrm{Tr}[\,\sigma_\alpha^\dagger\,\sigma_\beta\,] = \delta_{\alpha\beta}, \tag{3}$$

we can write the action of Ξ on a generic element $\omega \in \mathcal{B}(\mathcal{H})$ as [16, 47]

$$\Xi(\omega) = \sum_{\alpha\beta=1}^{N^2} M_{\alpha\beta}^{\Xi}\mathrm{Tr}\left[\sigma_\beta^\dagger\omega\right]\sigma_\alpha, \quad M_{\alpha\beta}^{\Xi} = \mathrm{Tr}\left[\sigma_\alpha^\dagger\Xi(\sigma_\beta)\right]. \tag{4}$$

The matrix M^{Ξ} associated to the map Ξ is in general not hermitian and only allows for a spectral representation in Jordan form, leading to the expression

$$\Xi(\omega) = \sum_{\alpha=1}^{M}\left(\lambda_\alpha\sum_{\mu=1}^{k_\alpha}\mathrm{Tr}\left[\varsigma_{\alpha,\mu}^\dagger\,\omega\right]\tau_{\alpha,\mu} + \sum_{\mu=1}^{k_\alpha}\mathrm{Tr}\left[\varsigma_{\alpha,\mu}^\dagger\,\omega\right]\tau_{\alpha,\mu-1}\right) \quad \text{with} \quad \omega\in\mathcal{B}(\mathcal{H}), \tag{5}$$

where $M \leqslant N^2$ is the number of eigenvalues each counted with its geometric multiplicity, and k_α the algebraic multiplicity, reducing to one if the matrix can be diagonalized. We define $\tau_{\alpha,0} = 0$, and for $k_\alpha = 1$ we will consider the identifications $\tau_{\alpha,1} \equiv \tau_\alpha$ and $\varsigma_{\alpha,1} \equiv \varsigma_\alpha$. The families of N^2 operators $\{\tau_{\alpha,\mu}\}_{\alpha,\mu}$ and $\{\varsigma_{\alpha,\mu}\}_{\alpha,\mu}$ appearing in equation (5) are related according to

$$\langle \varsigma_{\alpha,\mu}, \tau_{\beta,\nu}\rangle = \delta_{\alpha\beta}\delta_{\mu\nu}. \tag{6}$$

The map Ξ is said to be diagonalizable iff the corresponding matrix M^{Ξ} is. In this case, according to the previous identifications we can write

$$\Xi(\omega) = \sum_{\alpha=1}^{N^2}\lambda_\alpha\mathrm{Tr}\left[\varsigma_\alpha^\dagger\,\omega\right]\,\tau_\alpha \quad \text{with } \omega\in\mathcal{B}(\mathcal{H}), \tag{7}$$

[4] Complete positivity generalizes the property of positivity and takes into account the fact that the open quantum system can be entangled with some inaccessible degrees of freedom.

and the two bases of $\mathcal{B}(\mathcal{H})$, $\{\tau_\alpha\}_{\alpha=1,\ldots,N^2}$ and $\{\varsigma_\alpha\}_{\alpha=1,\ldots,N^2}$, which are not necessarily orthogonal bases, are sometimes referred to as bi-orthogonal bases, obeying the orthogonality relation

$$\langle \varsigma_\alpha, \tau_\beta \rangle = \delta_{\alpha\beta}. \tag{8}$$

Within the context of open quantum systems, these bases were introduced in [48] and named damping bases; in particular, there they were associated with a GKSL generator. In the following we will focus on the case of diagonalizable maps, which is enough to cover all examples we will consider, corresponding to relevant physical examples typically considered in the literature, and in particular allows to consistently consider time dependence of the eigenvalues only.

The damping bases are also strictly connected to the relation between the map Ξ and its dual Ξ', where the latter is defined by

$$\langle \omega, \Xi(\rho) \rangle = \langle \Xi'(\omega), \rho \rangle \quad \forall\ \omega, \rho \in \mathcal{B}(\mathcal{H}). \tag{9}$$

Using the damping bases, one finds

$$\Xi'(\omega) = \sum_{\alpha=1}^{N^2} \lambda_\alpha^* \operatorname{Tr}\left[\tau_\alpha^\dagger\, \omega\right] \varsigma_\alpha \quad \omega \in \mathcal{B}(\mathcal{H}), \tag{10}$$

where c^* is the complex conjugate of c. From equations (7), (8) and (10) one can then see that the operators $\{\tau_\alpha\}_{\alpha=1,\ldots,N^2}$ and $\{\varsigma_\alpha\}_{\alpha=1,\ldots,N^2}$ are the eigenvectors, respectively, of the linear map Ξ and of its dual Ξ' with respect to complex conjugates eigenvalues, i.e.

$$\Xi(\tau_\alpha) = \lambda_\alpha\, \tau_\alpha, \qquad \Xi'(\varsigma_\alpha) = \lambda_\alpha^*\, \varsigma_\alpha \quad \alpha = 1, \ldots, N^2. \tag{11}$$

Indeed, if Ξ is a normal operator, i.e., $[\Xi, \Xi'] = 0$, then the damping bases both coincide with a single orthonormal basis; if Ξ is a Hermitian map, in addition the eigenvalues are real, $\lambda_\alpha = \lambda_\alpha^*$.

Let us now move to one-parameter families of maps, $\{\Xi_t\}_{t\geqslant 0}$, which are used to describe the dynamics of open quantum systems. In general, both the coefficients and the operators in equation (7) will depend on time. However, it can well happen that the time dependence is enclosed in the eigenvalues only, while the corresponding damping bases are time-independent, i.e., that one has

$$\Xi_t(\omega) = \sum_{\alpha=1}^{N^2} \lambda_\alpha(t) \operatorname{Tr}\left[\varsigma_\alpha^\dagger\, \omega\right]\ \tau_\alpha \quad \omega \in \mathcal{B}(\mathcal{H}). \tag{12}$$

Equation (12) implies that the maps at different times commute,

$$[\Xi_t, \Xi_s] = 0; \tag{13}$$

the converse implication holds, if we further assume that the maps Ξ_s and Ξ_t are diagonalizable. In particular, one can consider a one-parameter family of CPTP dynamical maps $\{\Lambda_t\}_{t\geqslant 0}$ which commute at different times (a situation which has been thoroughly investigated in [16, 17]), i.e., such that

$$[\Lambda_t, \Lambda_s] = 0 \quad \forall\ t, s \geqslant 0. \tag{14}$$

If the family $\{\Lambda_t\}_{t\geqslant 0}$ satisfies a time-local master equation of the form equation (1) then the dynamical maps satisfy equation (14) if and only if the time-local generator satisfies the analogous commutation relation

$$\left[\mathcal{K}_t^{\mathrm{TCL}}, \mathcal{K}_s^{\mathrm{TCL}}\right] = 0 \quad \forall\ t, s \geqslant 0. \tag{15}$$

If we further assume diagonalizability of the dynamical maps and the generator, they will share the same time-independent damping bases, see equation (12), according to:

$$\mathcal{K}_t^{\mathrm{TCL}} = \sum_{\alpha=1}^{N^2} m_\alpha^{\mathrm{TCL}}(t) \operatorname{Tr}\left[\varsigma_\alpha^\dagger\, \omega\right]\ \tau_\alpha, \tag{16}$$

$$\Lambda_t = \sum_{\alpha=1}^{N^2} m_\alpha(t) \operatorname{Tr}\left[\varsigma_\alpha^\dagger\, \omega\right]\ \tau_\alpha, \tag{17}$$

where

$$m_\alpha(t) = \mathrm{e}^{\int_0^t \mathrm{d}\tau\, m_\alpha^{\mathrm{TCL}}(\tau)}. \tag{18}$$

Let us stress that the dynamics satisfying equation (14) include, but are not restricted to, the case where equation (1) holds with $\mathcal{K}_t^{\mathrm{TCL}} = \gamma(t)\mathcal{L}$, which has been widely studied in the literature [36, 37, 49–51]; as relevant examples, let us mention pure dephasing, depolarization, spontaneous emission, two-level system in the presence of an infinite-temperature bosonic bath, photonic losses, as well as the composition of local evolutions made up of these dynamics.

2.2. Novel connection between time-local and integro-differential representation

Given a diagonalizable commutative dynamics, its damping-basis decomposition sets a representation of both the dynamical map and the time-local generator in which the time dependence is fully enclosed in the eigenvalues—i.e., in complex functions of time—but not in the operatorial structure, see equations (16) and (17). Here, we show that this feature is also shared by the memory kernel, which will allow us to derive some novel connections between the time-local and the integro-differential master equations associated to a given dynamics.

Proposition 1. *Consider a family of dynamical maps $\{\Lambda_t\}_{t\geqslant 0}$ with time-local generator $\mathcal{K}_t^{\mathrm{TCL}}$ and memory kernel $\mathcal{K}_t^{\mathrm{NZ}}$. Moreover, for any couple of operators τ_α and ς_α let $\mathcal{M}_\alpha$ be the linear map acting on $\mathcal{B}(\mathcal{H})$ defined as*

$$\mathcal{M}_\alpha(\omega) = \mathrm{Tr}\left[\varsigma_\alpha^\dagger\,\omega\right]\,\tau_\alpha \quad \omega \in \mathcal{B}(\mathcal{H}). \tag{19}$$

The following propositions are equivalent:

(*a*) *The time-local generator $\mathcal{K}_t^{\mathrm{TCL}}$ has the damping-basis diagonalization*

$$\mathcal{K}_t^{\mathrm{TCL}} = \sum_{\alpha=1}^{N^2} m_\alpha^{\mathrm{TCL}}(t)\mathcal{M}_\alpha. \tag{20}$$

(*b*) *The memory kernel $\mathcal{K}_t^{\mathrm{NZ}}$ has the damping-basis diagonalization*

$$\mathcal{K}_t^{\mathrm{NZ}} = \sum_{\alpha=1}^{N^2} m_\alpha^{\mathrm{NZ}}(t)\mathcal{M}_\alpha; \tag{21}$$

moreover, the corresponding eigenvalues are related by

$$m_\alpha^{\mathrm{NZ}}(t) = \mathfrak{I}\left(\frac{u\widetilde{G_\alpha}(u)}{1+\widetilde{G_\alpha}(u)}\right)(t), \tag{22}$$

$$m_\alpha^{\mathrm{TCL}}(t) = \frac{G_\alpha(t)}{1+\int_0^t \mathrm{d}\tau G_\alpha(\tau)}, \tag{23}$$

$$G_\alpha(t) = \mathfrak{I}\left(\frac{\widetilde{m_\alpha^{\mathrm{NZ}}}(u)}{u-\widetilde{m_\alpha^{\mathrm{NZ}}}(u)}\right)(t), \tag{24}$$

where we introduced the function

$$G_\alpha(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\int_0^t \mathrm{d}\tau m_\alpha^{\mathrm{TCL}}(\tau)}; \tag{25}$$

$\widetilde{f_t}(u) \equiv \widetilde{f}_u$ is the Laplace transform of f_t and $\mathfrak{I}\left(g(u)\right)(t)$ the inverse Laplace transform of $g(u)$.

Proof. The proof is given in appendix A. □

Proposition 1 is the central result of this paper. According to it, the time-local generator and the memory kernel have the same time-independent damping bases, while the time dependence appears in the eigenvalues only: to go from one master equation to the other, only functions of time (and their Laplace transforms) are involved, while the operatorial structure is unchanged; see the left part of figure 1.

With reference to the expression equation (17) of the associated time evolution as shown in the proof proposition 1 comes along with the relations

$$m_\alpha(t) = \mathrm{e}^{\int_0^t \mathrm{d}\tau m_\alpha^{\mathrm{TCL}}(\tau)}, \tag{26}$$

$$m_\alpha^{\mathrm{TCL}}(t) = \frac{1}{m_\alpha(t)}\frac{\mathrm{d}}{\mathrm{dt}}m_\alpha(t), \tag{27}$$

as well as

$$\widetilde{m_\alpha}(u) = \frac{1}{u-\widetilde{m_\alpha^{\mathrm{NZ}}}(u)}, \tag{28}$$

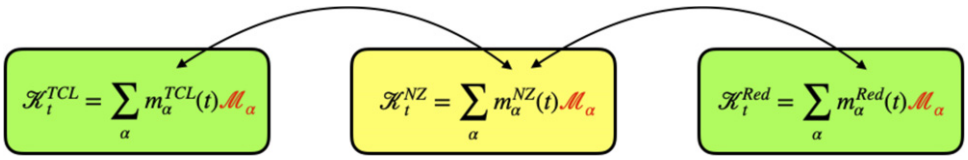

Figure 1. According to propositions 1 and 2, the time-local generator, the memory kernel and the Redfield-like generator of commutative dynamics have the same time-independent damping bases, while the time dependence appears in the eigenvalues only: to go from one to the other, only functions of time (and their Laplace transforms) are involved, while the operatorial structure is unchanged.

$$\widetilde{m_\alpha^{\mathrm{NZ}}}(u) = \frac{u\widetilde{m_\alpha}(u) - 1}{\widetilde{m_\alpha}(u)}. \tag{29}$$

As a consequence, proposition 1 is particularly useful when we want to compare the structures of the time-local generator and the memory kernel. As a general relevant example, consider the following corollary, which directly follows from the proposition above.

Corollary 1. *Under the same assumptions of proposition 1, consider a diagonalizable GKSL generator $\mathcal{L}$ with only one-nonzero eigenvalue ℓ, possibly degenerate with degeneracy d, i.e.,*

$$\mathcal{L} = \ell \sum_{\alpha=1}^{d} \mathcal{M}_\alpha; \tag{30}$$

then the following identities are equivalent

$$\mathcal{K}_t^{\mathrm{TCL}} = \gamma(t)\mathcal{L}; \tag{31}$$

$$\mathcal{K}_t^{\mathrm{NZ}} = \frac{m^{\mathrm{NZ}}(t)}{\ell}\mathcal{L}; \tag{32}$$

where $m^{\mathrm{NZ}}(t)$ and $G(t)$ are related as in equations (22) and (24) where now

$$G(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\ell \int_0^t \mathrm{d}\tau \gamma(\tau)}. \tag{33}$$

Proof. Simply note that $m^{\mathrm{TCL}}(t) = \gamma(t)\ell$ and $\sum_{\alpha=1}^{d} \mathcal{M}_\alpha = \mathcal{L}/\ell$, see equation (30); then apply proposition 1. □

In other terms, if $\mathcal{L}$ has only one non-zero eigenvalue, the super-operatorial part of the time-local generator and the memory kernel are exactly the same; the difference between the master equations is enclosed in one overall time-dependent factor. A physically relevant example of this situation will be given below, see example 3.

In proposition 1 we assumed commutativity of the dynamics. As mentioned at the end of section 2.1, and as we will show explicitly by means of example, such maps account for several dynamics of interest for open quantum systems. On the other hand, it is indeed natural to ask what happens if we relax this assumption. While a full-fledged extension of the result goes beyond the scope of this work, we show here that proposition 1 is indeed also valid for more general dynamics in modified form, in which a time-dependent damping basis, different for time-local generator and memory kernel, has to be considered.

A relevant class of non-commutative evolutions for which this is the case is provided by the phase-covariant qubit dynamics. Under such an evolution, the Bloch ball shrinks into a possibly rotated and shifted ellipsoid, such that the overall transformation commutes with the rotation about a fixed axis. Recently, phase-covariant dynamics have attracted considerable attention [25, 26, 44, 52–54], both because of their clear mathematical meaning and structure, and physical relevance e.g. for metrological tasks [26]. The expression of the time-local and integro-differential master equation for phase-covariant dynamics has been worked out in [44], and the result indeed complies with equations (22)–(25). One can observe that in this case the validity of these relations follows from

$$m_\alpha^{\mathrm{TCL}}(t)m_\beta^{\mathrm{TCL}}(t')\mathcal{M}_\alpha^{\mathrm{TCL}}(t)\mathcal{M}_\beta^{\mathrm{TCL}}(t') = \delta_{\alpha\beta}m_\alpha^{\mathrm{TCL}}(t)m_\alpha^{\mathrm{TCL}}(t')\mathcal{M}_\alpha^{\mathrm{TCL}}(t'), \tag{34}$$

which is indeed a strictly weaker requirement than commutativity whenever at least one of the eigenvalues $m_\alpha^{\mathrm{TCL}}(t)$ is equal to zero, e.g., if the dynamics has at least one steady state [2]. The linear map $\mathcal{M}_\alpha^{\mathrm{TCL}}(t)$ corresponds to the time-dependent damping basis of the time local generator: $\mathcal{K}_t^{\mathrm{TCL}} = \sum_\alpha m_\alpha^{\mathrm{TCL}}(t)\mathcal{M}_\alpha^{\mathrm{TCL}}(t)$. Equation (34) warrants that the dynamical map can be written as $\Lambda_t = \sum_\alpha m_\alpha(t)\mathcal{M}_\alpha(t)$ and the memory kernel as $\mathcal{K}_t^{\mathrm{NZ}} = \sum_\alpha m_\alpha^{\mathrm{NZ}}(t)\mathcal{M}_\alpha^{\mathrm{NZ}}(t)$, where the connection between $m_\alpha(t)$, $m_\alpha^{\mathrm{TCL}}(t)$ and $m_\alpha^{\mathrm{NZ}}(t)$ is still given by equations (16) and (17) and proposition 1, though of course now the respective damping bases will not coincide.

2.3. Lindbladian form

The time-local generator and the memory kernel in corollary 1 are directly proportional to a GKSL generator of quantum dynamical semigroups. Here we show that, starting from the damping-basis decomposition in equations (20) and (21), it is always possible to write the time-local generator and the memory kernel in a way which is directly related to the GKSL generator; we will refer to such form as Lindbladian. To do so, we essentially apply the general prescription given by lemma 2.3 in [7] to the situation of interest for us. Besides providing us with a canonical reference structure which eases the comparison between super-operators, as we show by different examples, the Lindbladian form allows us to infer the (C)P and the (C)P-divisibility of the dynamics in a more direct way.

Any linear map acting on $\mathcal{B}(\mathcal{H})$ can be represented in several ways. A relevant example is given by the matrix representation in equation (4), which is at the basis of the damping-basis decomposition, see equation (7). Alternatively, given an orthonormal basis $\{\sigma_\alpha\}_{\alpha=1,\ldots,N^2}$ of $\mathcal{B}(\mathcal{H})$, see equation (3), with $\sigma_{N^2} = \mathbb{1}/\sqrt{N}$, any linear map Ξ acting on $\mathcal{B}(\mathcal{H})$ can be uniquely written as

$$\Xi(\omega) = \sum_{\alpha,\beta=1}^{N^2} c_{\alpha,\beta}\ \sigma_\alpha\omega\sigma_\beta^\dagger \quad \text{with } \omega \in \mathcal{B}(\mathcal{H}). \tag{35}$$

Note that such a representation is strictly related to the CP of the map Ξ: in fact, the matrix of coefficients with elements $c_{\alpha\beta}$ is positive semidefinite if and only if Ξ is CP, in which case the decomposition in equation (35) directly leads to the Kraus decomposition of CP maps. Most importantly for us, the representation in equation (35) gives a general characterization also of the time-local and the integro-differential master equations associated with open-system dynamics. In fact, let us consider a time-local generator or a memory kernel $\mathcal{K}_t$. The dynamics $\{\Lambda_t\}_{t\geqslant 0}$ resulting from equations (1) and (2) is trace and Hermiticity preserving (where the latter means that any Hermitian operator $\omega = \omega^\dagger$ is mapped at any time t into an Hermitian operator $\Lambda_t(\omega) = (\Lambda_t(\omega))^\dagger$) if and only if the corresponding time-local generator and memory kernel satisfy[5]

$$\begin{aligned} \mathrm{Tr}\left[\mathcal{K}_t(\omega)\right] &= 0 \quad \forall\ \omega \in \mathcal{B}(\mathcal{H}), \\ \left(\mathcal{K}_t(\omega)\right)^\dagger &= \mathcal{K}_t(\omega^\dagger) \quad \forall\ \omega \in \mathcal{B}(\mathcal{H}). \end{aligned} \tag{36}$$

Restricting for the sake of convenience to diagonalizable super-operators, in the form

$$\mathcal{K}_t = \sum_{\alpha=1}^{N^2} m_\alpha(t)\mathcal{M}_\alpha. \tag{37}$$

Lemma 2.3 of [7] tells us that the two conditions in equation (36) hold if and only if

$$\mathcal{K}_t(\omega) = -i\left[H(t),\omega\right] + \sum_{\alpha,\beta=1}^{N^2-1} \kappa_{\alpha\beta}(t)\left(\sigma_\alpha\omega\sigma_\beta^\dagger - \frac{1}{2}\left\{\sigma_\beta^\dagger\sigma_\alpha,\omega\right\}\right), \tag{38}$$

where the coefficients $\kappa_{\alpha\beta}(t) = \kappa_{\beta\alpha}^*(t)$ are given by

$$\kappa_{\alpha\beta}(t) = \sum_{\gamma=1}^{N^2}\sum_{\chi=1}^{N^2} m_\gamma(t)\,\mathrm{Tr}\left[\sigma_\beta\sigma_\chi^\dagger\sigma_\alpha^\dagger\mathcal{M}_\gamma(\sigma_\chi)\right], \tag{39}$$

[5] The conditions in equation (36) have to be satisfied by any time-local generator, if we assume that Λ_t^{-1} exists: this can be checked by applying equation (1) to a generic initial state ρ_0 and then taking the trace, for the first condition, and the Hermitian conjugate, for the second one, on both sides. Analogously, they have to be satisfied by any memory kernel, if we assume that $\tilde{\Lambda}_u^{-1}$ exists, as can be checked by applying equation (2) to a generic initial state and taking the Laplace transform.

while the Hamiltonian $H(t) = H^\dagger(t)$ is given by

$$H(t) = \frac{1}{2i}(\sigma^\dagger(t) - \sigma(t)), \quad \sigma(t) = \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N^2-1} \kappa_{\alpha N^2}(t)\sigma_\alpha. \tag{40}$$

In addition, since the matrix with elements $\kappa_{\alpha\beta}(t)$ is Hermitian, it can be diagonalized by a unitary matrix $V(t)$ with elements $V_{\alpha\beta}(t)$, so that equation (38) can be rewritten as

$$\mathcal{K}_t(\omega) = -i\,[H(t), \omega] + \sum_{\alpha=1}^{N^2-1} r_\alpha(t) \left(L_\alpha(t)\omega L_\alpha^\dagger(t) - \frac{1}{2}\left\{L_\alpha^\dagger(t)L_\alpha(t), \omega\right\}\right), \tag{41}$$

with

$$\begin{aligned} r_\alpha(t) &= \sum_{\gamma,\gamma'=1}^{N^2-1} V^\dagger_{\alpha\gamma}(t)\kappa_{\gamma\gamma'}(t)V_{\gamma'\alpha}(t) \quad r_\alpha(t) \in \mathbb{R} \\ L_\alpha(t) &= \sum_{\beta=1}^{N^2-1} V_{\beta\alpha}(t)\,\sigma_\beta. \end{aligned} \tag{42}$$

Equations (39)–(42) provide us with the wanted recipe to get the Lindbladian form, starting from the damping-basis representation, equations (20) and (21). The main difficulty is that the different non-zero eigenvalues $m_\alpha(t)$ will 'mix' in a non-trivial way, so that there is not a direct connection between them and the coefficients $r_\alpha(t)$ in the Lindbladian structure. As a first consequence, one looses the correspondence between the super-operatorial structures of, respectively, the time-local generator and the memory kernel, which is guaranteed by corollary 1 in the case of one non-zero eigenvalue; this is shown by the examples below.

Example 1. Consider the time-local generator

$$\mathcal{K}_t^{\mathrm{TCL}}(\omega) = \gamma_-(t) \left(\sigma_-\omega\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-, \omega\right\}\right), \tag{43}$$

which describes, for example, the reduced dynamics of a two-level system interacting with a zero-temperature bosonic bath via a Jaynes–Cummings interaction term [47, 52], neglecting for the sake of simplicity the free Hamiltonian term. This is a special case of the generator treated (for constant coefficients) in [48]. The dual generator is

$$\left(\mathcal{K}_t^{\mathrm{TCL}}\right)'(\omega) = \gamma_-(t) \left(\sigma_+\omega\sigma_- - \frac{1}{2}\left\{\sigma_+\sigma_-, \omega\right\}\right), \tag{44}$$

and the resulting eigenvalues and damping bases are [see equation (11)][6]

$$\left\{m_\alpha^{\mathrm{TCL}}(t)\right\}_{\alpha=1,\ldots,4} = \left\{0, -\gamma_-(t), -\frac{1}{2}\gamma_-(t), -\frac{1}{2}\gamma_-(t)\right\} \tag{45}$$

$$\left\{\tau_\alpha\right\}_{\alpha=1,\ldots,4} = \left\{\frac{\mathbb{1} - \sigma_z}{2}, \sigma_z, \sigma_+, \sigma_-\right\} \tag{46}$$

$$\left\{\varsigma_\alpha\right\}_{\alpha=1,\ldots,4} = \left\{\mathbb{1}, \frac{\mathbb{1} + \sigma_z}{2}, \sigma_+, \sigma_-\right\}. \tag{47}$$

Indeed, the relations in equation (8) are satisfied; moreover, we note that we have now two eigenvalues different from zero (one two-fold degenerate) and the damping bases are not made of self-adjoint operators.

The time dependence is enclosed in the eigenvalues only, so that we are in the case of commuting dynamics treated in the previous sections. In particular, by applying proposition 1, we find that the integro-differential generator $\mathcal{K}_t^{\mathrm{NZ}}$ has the same damping bases, equations (46) and (47), with eigenvalues

[6] Note that in [48] they used a different duality relation, without the Hermitian conjugate, so that there is a different dual damping basis with respect to ours.

given by equations (22) and (25) with respect to the functions in equation (45). In other terms, both the time-local generator and the memory kernel can be written in the form [see equation (7)]

$$\mathcal{K}_t^X(\omega) = \frac{m_2^X(t)}{2}\,\mathrm{Tr}\left[(\mathbb{1}+\sigma_z)\omega\right]\sigma_z + m_3^X(t)\left(\mathrm{Tr}\left[\sigma_-\omega\right]\sigma_+ + \mathrm{Tr}\left[\sigma_+\omega\right]\sigma_-\right), \quad X = \mathrm{TCL}, \mathrm{NZ} \tag{48}$$

with $m_i^X(t)$ the corresponding eigenvalues. But, by using equations (39)–(42), one can see that the generator as in equation (48) corresponds to a Lindbladian form

$$\mathcal{K}_t^X(\omega) = -m_2^X(t)\left(\sigma_-\omega\sigma_+ - \frac{1}{2}\left\{\sigma_+\sigma_-,\omega\right\}\right) + \frac{1}{2}(m_2^X(t) - 2\,m_3^X(t))\,(\sigma_z\omega\sigma_z - \omega)\,. \tag{49}$$

Crucially, while for $\mathcal{K}_t^{\mathrm{TCL}}$ one has $m_2^{\mathrm{TCL}}(t) = 2\,m_3^{\mathrm{TCL}}(t)$ [see equation (45)], so that the pure dephasing term cancels out, this is generally not the case for $\mathcal{K}_t^{\mathrm{NZ}}$, which will then present a pure-dephasing term, in addition to the spontaneous-emission term; an explicit example of this is given in [47, 52]. This implies in particular that at variance with the case of the standard GKSL generator, a direct physical interpretation of the operatorial contribution is not available.

As anticipated, even though the time-local master equation we started from, equation (43), is in the form given by equation (31), now the corresponding GKSL generator has more than one eigenvalue different from 0. As a consequence, the transformation to the memory kernel no longer preserves the super-operatorial part of the Lindbladian structure (compare with corollary 1), but rather generates one more term.

Example 2. A somehow opposite example is obtained by starting with a memory kernel of the form

$$\mathcal{K}_t^{\mathrm{NZ}}(\omega) = k(t)\,(\sigma_-\omega\sigma_+ + \sigma_+\omega\sigma_- - \omega)\,, \tag{50}$$

with $k(t)$ such that the described dynamics is CPTP, as e.g. in [31]. First, we note that such a generator can be written in a 'manifest' Lindbladian structure as

$$\mathcal{K}_t^{\mathrm{NZ}}(\omega) = k(t)\left(\sigma_-\omega\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\omega + \omega\sigma_+\sigma_-) + \sigma_+\omega\sigma_- - \frac{1}{2}(\sigma_-\sigma_+\omega + \omega\sigma_-\sigma_+)\right), \tag{51}$$

so that this time we start from a non-local generator in the form $\mathcal{K}_t^{\mathrm{NZ}} = k(t)\mathcal{L}$ [compare with equation (31)]. Such a generator is self-adjoint, $\left(\mathcal{K}_t^{\mathrm{NZ}}\right)' = \mathcal{K}_t^{\mathrm{NZ}}$, so that the eigenvalues are real, and the damping bases coincide, yielding an orthonormal basis:

$$\left\{m_\alpha^{\mathrm{NZ}}(t)\right\}_{\alpha=1,\ldots,4} = \{0, -k(t), -k(t), -2k(t)\} \tag{52}$$

$$\left\{\tau_\alpha\right\}_{\alpha=1,\ldots,4} = \left\{\frac{\mathbb{1}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\right\}. \tag{53}$$

Also in this case, since we have two eigenvalues different from 0 we cannot apply corollary 1, which would guarantee that the time-local equation would have the same Lindbladian structure. Instead, we can apply proposition 1, so that both the time-local generator and the memory kernel have the form

$$\mathcal{K}_t^X(\omega) = m_2^X(t)\left(\mathrm{Tr}\left[\sigma_x\omega\right]\sigma_x + \mathrm{Tr}\left[\sigma_y\omega\right]\sigma_y\right) + m_4^X(t)\,\mathrm{Tr}\left[\sigma_z\omega\right]\sigma_z, \quad X = \mathrm{TCL}, \mathrm{NZ} \tag{54}$$

where for the memory kernel the eigenvalues are given by equation (52), while for the time-local generator they are obtained from the latter via equation (24) and (25). Using equations (39)–(42), equation (54) can be written in Lindbladian form as

$$\begin{aligned}\mathcal{K}_t^X(\omega) = &-\frac{m_4^X(t)}{2}\left(\sigma_-\omega\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\omega + \omega\sigma_+\sigma_-) + \sigma_+\omega\sigma_- - \frac{1}{2}(\sigma_-\sigma_+\omega + \omega\sigma_-\sigma_+)\right)\\ &+ \frac{1}{2}(m_4^X(t) - 2\,m_2^X(t))\,(\sigma_z\omega\sigma_z - \omega)\,.\end{aligned} \tag{55}$$

For the memory kernel $m_4^{\mathrm{NZ}}(t) = 2\,m_2^{\mathrm{NZ}}(t)$, so that the pure-dephasing term cancels out, while this will not generally be the case for the time-local generator. An example is given in [31]. Once again, the multiple eigenvalues in the damping-basis decomposition generate further terms, this time when going from the integro-differential to the time-local master equation. Of course, the situation is symmetrical, so that we could obtain further examples by simply inverting the starting points in the examples above; the only

difference when going, respectively, from the time-local to the integro-differential master equation or vice versa is the connection between the eigenvalues of the damping-basis decomposition, i.e., whether one should use equations (22) or (24).

We conclude that proposition 1 and equations (39)–(42) yield a systematic procedure to obtain the integro-differential master equation from the time-local one and vice versa, whenever we are able to diagonalize them and the resulting damping bases are time-independent. An example of application for a higher dimensional system is given in appendix C.

3. Redfield-like master equation

Until now, we compared the time-local and integro-differential master equations in an exact way, i.e., without introducing any approximation. On the other hand, as recalled in the introduction, it is useful to consider situations in which an integro-differential master equation is transformed into a time-local one by means of some approximations. Here, we focus on a master equation which is obtained via a coarse graining in time analogous to the one introduced by Redfield in [18].

More precisely, if τ_{R} is the relaxation time of the open-system dynamics and $\mathcal{K}_t^{\mathrm{NZ}}$ is appreciably different from zero only on a time scale much shorter than τ_{R}, one might approximate the dynamical maps Λ_t with Λ_t^{Red}, where the latter is obtained by replacing equation (2) with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_t^{\mathrm{Red}} = \mathcal{K}_t^{\mathrm{Red}}\Lambda_t, \quad \mathcal{K}_t^{\mathrm{Red}} = \int_0^t \mathrm{d}\tau \mathcal{K}_\tau^{\mathrm{NZ}}. \tag{56}$$

When this approximation is used in the presence of a weak-coupling interaction between the open system and its environment, the resulting equation is often called Redfield equation [1, 18]. We will refer to equation (56) as Redfield-like master equation, in order to emphasize that we take it into account without necessarily restricting to the weak-coupling regime. The Redfield equation is commonly exploited in several different contexts, such as the study of transport processes in condensed-matter or biophysical systems [55–59]. Importantly, the Redfield equation might lead to a not well-defined evolution, as studied by now extensively in the literature [60–62]. We will show how the damping-basis representation enables us to determine the structural properties of the resulting time-local generator $\mathcal{K}_t^{\mathrm{Red}}$, as well as to investigate the Markovian nature of the dynamics $\left\{\Lambda_t^{\mathrm{Red}}\right\}_{t\geqslant 0}$.

3.1. Structure of the time-local generator

We first consider the following simple connection between the time-local generator, the memory kernel and the Redfield-like generator, in the case of commutative dynamics.

Proposition 2. *Under the same assumptions of proposition 1, for time-local generator and memory kernel in the form equations (20) and (21) respectively, the Redfield-like generator $\mathcal{K}_t^{\mathrm{Red}}$ has the damping-basis diagonalization*

$$\mathcal{K}_t^{\mathrm{Red}} = \sum_{\alpha=1}^{N^2} m_\alpha^{\mathrm{Red}}(t)\mathcal{M}_\alpha, \tag{57}$$

with

$$m_\alpha^{\mathrm{Red}}(t) = \int_0^t \mathrm{d}\tau m_\alpha^{\mathrm{NZ}}(\tau). \tag{58}$$

Moreover, under the same assumptions of corollary 1, the Redfield-like generator $\mathcal{K}_t^{\mathrm{Red}}$ reads

$$\mathcal{K}_t^{\mathrm{Red}} = \frac{m^{\mathrm{Red}}(t)}{\ell}\mathcal{L}. \tag{59}$$

Accordingly, the Redfield-like dynamics has the same operational structure as the exact one, with time-dependent eigenvalues obtained from the corresponding original ones. This situation is depicted in figure 1. What is more, the eigenvalues $m_\alpha^{\mathrm{Red}}(t)$ can be written in a compact form in terms of the functions $G_\alpha(t)$ defined in equation (25).

Proposition 3. *Under the same assumptions of proposition 1, consider a diagonal time-local generator as in equation (20). The eigenvalues of the Redfield-like time-local generator satisfy the integral equation*

$$m_\alpha^{\mathrm{Red}}(t) = G_\alpha(t) - \int_0^t \mathrm{d}\tau G_\alpha(t-\tau) m_\alpha^{\mathrm{Red}}(\tau), \tag{60}$$

whose solution, provided

$$|\widetilde{G_\alpha}(u)| < 1, \tag{61}$$

can be written as

$$m_\alpha^{\mathrm{Red}}(t) = \sum_{j=1}^{\infty} (-)^{j+1} \underbrace{G_\alpha * \cdots * G_\alpha}_{\text{j times}}(t). \tag{62}$$

Proof. The proof is given in appendix B. □

3.2. Markovianity of the dynamics

As mentioned in the introduction, a highly relevant property of the dynamics of an open quantum system is its (non-)Markovianity. Quantum non-Markovianity is often defined in terms of the divisibility property, positive or completely positive, of the corresponding dynamical maps. As we will see below, the damping-basis representation enables us to make some statements about the preservation of (C)P-divisibility under the approximation leading to the Redfield-like master equation.

First, we note that the relation in equation (62) directly allows us to infer that (C)P-divisibility is preserved in the Redfield-like master equation, whenever $\mathcal{L}$ has only one non-zero eigenvalue.

Corollary 2. *Under the same assumptions of proposition 3, consider a time-local generator of the form* $\mathcal{K}_t^{\mathrm{TCL}} = \gamma(t)\mathcal{L}$*, such that* $\mathcal{L}$ *has only one eigenvalue* ℓ *different from zero, so that* $\mathcal{L} = \ell \sum_{\alpha=1}^{d} \mathcal{M}_\alpha$*, and assume equation (61) holds. If* $\{\Lambda_t\}_{t\geqslant 0}$ *is (C)P-divisible, then* $\left\{\Lambda_t^{\mathrm{Red}}\right\}_{t\geqslant 0}$ *is (C)P-divisible.*

Proof. First, we note that the time-local generators, $\mathcal{K}_t^{\mathrm{TCL}}$ and $\mathcal{K}_t^{\mathrm{Red}}$, are of the form

$$\mathcal{K}_t^{\mathrm{TCL}} = \gamma(t)\mathcal{L}, \quad \mathcal{K}_t^{\mathrm{Red}} = \gamma^{\mathrm{Red}}(t)\mathcal{L},$$

where $\gamma^{\mathrm{Red}}(t) = \frac{m^{\mathrm{Red}}(t)}{l}$. The dynamics $\{\Lambda_t\}_{t\geqslant 0}$ ($\left\{\Lambda_t^{\mathrm{Red}}\right\}_{t\geqslant 0}$) is (C)P-divisible if and only if $\gamma(t) \geqslant 0$ ($\gamma^{\mathrm{Red}}(t) \geqslant 0$). Thus, since we assume $\{\Lambda_t\}_{t\geqslant 0}$ (C)P-divisible, we have $\gamma(t) \geqslant 0$. Moreover, ℓ has to be real and negative, since it is the only non-zero eigenvalue of $\mathcal{L}$. It follows that the function $G(t)$, see equation (33),

$$G(t) = \ell\gamma(t)\mathrm{e}^{\ell\Gamma(t)}$$

is negative. But then, from equation (62) we have that $m^{\mathrm{Red}}(t)/\ell$ is positive and thus $\left\{\Lambda_t^{\mathrm{Red}}\right\}_{t\geqslant 0}$ is (C)P-divisible. □

Note that the reverse statement, that is that (C)P-divisibility of the Redfield-like dynamics originates from (C)P-divisibility of the original dynamics, is in general not true (even in the case of a single non-zero eigenvalue). Indeed, in the next example we will consider two dynamics, which both have a (C)P-divisible approximated evolution, but only one of them has this property originally.

Example 3. Let us consider the case of a pure dephasing dynamics for a two-level system, characterized by a monotonic reduction of coherences described by the decoherence function $\varphi(t)$. Starting from the expression of the dynamics we can easily work out the corresponding time-local generator, which takes the form

$$\mathcal{K}_t^{\mathrm{TCL}} = \gamma(t)\mathcal{L}, \tag{63}$$

with

$$\gamma(t) = -\frac{\dot{\varphi}(t)}{\varphi(t)} \tag{64}$$

and

$$\mathcal{L} = \sigma_+\sigma_- \cdot \sigma_+\sigma_- + \sigma_-\sigma_+ \cdot \sigma_-\sigma_+ - \mathbb{1}. \tag{65}$$

It can be easily checked that $\mathcal{L}$ has only one non-zero two-fold degenerate eigenvalue, so that relying on corollary 1 we get the corresponding memory kernel,

$$\mathcal{K}_t^{\mathrm{NZ}} = k(t)\mathcal{L}, \tag{66}$$

with

$$\tilde{k}(u) = -\frac{\widetilde{\dot{\varphi}}(u)}{\tilde{\varphi}(u)}. \tag{67}$$

Since equation (66) exactly corresponds to equation (63), this memory kernel describes the very same CPTP dynamics. In particular, it describes a CP-divisible dynamics since $\varphi(t)$ is monotonically decreasing. According to proposition 2, the Redfield-like generator is given by

$$\mathcal{K}_t^{\mathrm{Red}} = \int_0^t \mathrm{d}\tau k(\tau)\mathcal{L}, \tag{68}$$

where the function $K(t) = \int_0^t \mathrm{d}\tau k(\tau)$ is indeed positive since $\tilde{K}(u) = -\widetilde{\dot{\varphi}}(u)/(1+\widetilde{\dot{\varphi}}(u))$ and $\varphi(t)$ is monotonically decreasing. The Redfield-like generator therefore also describes a CP-divisible dynamics. Let us now start from equation (66) changing the operatorial structure in the memory kernel via the replacement $\mathcal{L} \to \overline{\mathcal{L}}$ with

$$\overline{\mathcal{L}} = \sigma_z \cdot \sigma_z - \mathbb{1}, \tag{69}$$

so that we define

$$\overline{\mathcal{K}}_t^{\mathrm{NZ}} = k(t)\overline{\mathcal{L}}. \tag{70}$$

The associated Redfield-like generator reads

$$\overline{\mathcal{K}}_t^{\mathrm{Red}} = \int_0^t \mathrm{d}\tau k(\tau)\overline{\mathcal{L}}, \tag{71}$$

and again corresponds to a CP-divisible dynamics. Making reference to corollary 1 (also $\overline{\mathcal{L}}$ has only one non-zero eigenvalue), we can obtain via the damping-basis approach the time-local generator exactly corresponding to equation (70), namely

$$\overline{\mathcal{K}}_t^{\mathrm{TCL}} = \bar{\gamma}(t)\overline{\mathcal{L}}, \tag{72}$$

where now

$$\bar{\gamma}(t) = -\frac{1}{2}\frac{1}{|\bar{\varphi}(t)|}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\varphi}(t)|, \tag{73}$$

with

$$\bar{\varphi}(t) = \mathfrak{I}\left(\frac{1}{u}\frac{1+\tilde{\dot{\varphi}}(u)}{1-\tilde{\dot{\varphi}}(u)}\right)(t). \tag{74}$$

The sign of $\bar{\gamma}(t)$ is now not necessarily fixed, so that the dynamics is in general not C(P)-divisible. The evolution is however always well defined, that is CPTP, since $\int_0^t \mathrm{d}\tau\bar{\gamma}(\tau) \geqslant 0$ [28]. The same therefore holds for the evolution described by $\overline{\mathcal{K}}_t^{\mathrm{NZ}}$, due to the exact correspondence between equations (72) and (70). We have therefore shown an example of two dynamics whose associated Redfield-like master equations both describe a CP-divisible collection of maps, despite the fact that only one of the two was originally CP-divisible, while the other can even break P-divisibility.

If we go beyond the case of only one non-zero eigenvalue, the situation gets more involved. Focusing in particular on Pauli maps, one can see that P-divisibility is more robust under the Redfield approximation than CP-divisibility.

Corollary 3. *Consider a Pauli dynamical map in the form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho_t\sigma_k - \rho_t). \tag{75}$$

Assume that the dynamical map is P-divisible and equation (61) holds. Then the associated Redfield-like map is also P-divisible.

Proof. For Pauli dynamical maps the eigenvalues of the time-local generator read: $m_k^{\mathrm{TCL}}(t) = -(\gamma_i(t)+\gamma_j(t))$, $k \neq i \neq j$. A map in such a form is P-divisible iff [63, 64]

$$\gamma_i(t)+\gamma_j(t) \geqslant 0\ , \quad i \neq j \Leftrightarrow m_\alpha^{\mathrm{TCL}}(t) \leqslant 0\ , \quad \forall\ \alpha. \tag{76}$$

Accordingly, $G_\alpha(t) \leqslant 0, \forall\,\alpha$. Using this result, from equation (62) we conclude that $m_\alpha^{\mathrm{Red}}(t) \leqslant 0, \quad \forall\ \alpha$, and the Redfield-like map is then also P-divisible. □

On the other hand, CP-divisibility can be lost in the approximation leading to the Redfield-like master equation, as shown in the following example; indeed, this is a consequence of the presence of more than one non-zero eigenvalues.

Example 4. We analyze the dephasing dynamics in random directions first introduced in [65]:

$$\Lambda_t = x_1 e^{\mathcal{L}_1 t} + x_2 e^{\mathcal{L}_2 t} + x_3 e^{\mathcal{L}_3 t}, \tag{77}$$

where $\mathcal{L}_i[\rho] = \sigma_i \rho \sigma_i - \rho$, describing dephasing along the ith direction, and the x_i's are the corresponding probabilities constrained by $x_1 + x_2 + x_3 = 1$. The time evolution obeys the following time-local master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{1}{2}\sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho(t) \sigma_k - \rho(t)), \tag{78}$$

where the time dependent decoherence rates can be expressed as

$$\begin{aligned} \gamma_1(t) &= \mu_1(t) - \mu_2(t) - \mu_3(t) \\ \gamma_2(t) &= -\mu_1(t) + \mu_2(t) - \mu_3(t) \\ \gamma_3(t) &= -\mu_1(t) - \mu_2(t) + \mu_3(t), \end{aligned} \tag{79}$$

with

$$\mu_1(t) = -\frac{x_2 + x_3}{x_2 + x_3 + e^{2t} x_1}, \quad \mu_2(t) = -\frac{x_3 + x_1}{x_3 + x_1 + e^{2t} x_2}, \quad \mu_3(t) = -\frac{x_1 + x_2}{x_1 + x_2 + e^{2t} x_3}.$$

The dynamics given by equation (78) is always P-divisible, since equation (76) are satisfied. However, depending on the choice of the x_i parameters one of the decay rates can become negative so that the evolution looses its CP-divisibility, see figure 2(a). In particular, the well-known eternally non-Markovian master equation belongs to this family [32]. It corresponds to the choice $x_1 = x_2 = \frac{1}{2}$, $x_3 = 0$ and generates an evolution which for all $t > 0$ is non-CP-divisible, as in this case $\gamma_3(t) = -\tanh(t) < 0, \forall\, t > 0$. The evolution resulting from the Redfield-like master equation equation (56) has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{1}{2}\sum_{k=1}^{3} \gamma_k^{\mathrm{Red}}(t)(\sigma_k \rho(t) \sigma_k - \rho(t)), \tag{80}$$

with

$$\gamma_1^{\mathrm{Red}}(t) = Y_1(t) - Y_2(t) - Y_3(t), \tag{81}$$

$$\gamma_2^{\mathrm{Red}}(t) = -Y_1(t) + Y_2(t) - Y_3(t), \tag{82}$$

$$\gamma_3^{\mathrm{Red}}(t) = -Y_1(t) - Y_2(t) + Y_3(t), \tag{83}$$

and $Y_k(t) = \exp(-2x_k t)(x_k - 1)$. This is still a proper CPTP quantum dynamics, as it fulfills the relevant constraints given in [63]. The dynamics (80) is P-divisible iff the condition (76) for $\gamma_k^{\mathrm{Red}}(t)$ is satisfied. As

$$\gamma_i^{\mathrm{Red}}(t) + \gamma_j^{\mathrm{Red}}(t) = 2(1 - x_k)e^{-2x_k t}, \quad i \neq j \neq k, \tag{84}$$

is always positive, by conducting our approximation the dynamics keeps its P-divisibility (in accordance with corollary 3). However, almost all of the dynamics which were CP-divisible for the original map, are no more CP-divisible for the Redfield-like master equation, see figure 2(a), for the original dynamics and figure 2(b), for the approximated evolution. Initially, for $t = 0$, all $\gamma(t)$'s and $\gamma^{\mathrm{Red}}(t)$'s are positive. After a transient behavior, positivity of $\gamma^{\mathrm{Red}}(t)$ implies positivity of $\gamma(t)$, since the parameter range warranting positivity of the time dependent coefficients in the Redfield-like master equation is smaller. The Redfield-like dynamical maps are CP-divisible only for the choices $x_i = x_j \leqslant x_k$, with $i \neq j \neq k$ (all $\gamma^{\mathrm{Red}}(t)$'s are positive for all times). This is in contrast to the original evolution, where a significant fraction of the dynamics are CP-divisible [65]. Note that there are only four choices of parameters x_1, x_2, x_3 (black dots in figure 2), for which the resulting original generator has only one eigenvalue and the corresponding dynamical map is CP-divisible. In accordance with corollaries 1 and 2 these properties are also present in the approximated dynamics.

These results put into question the idea that the Redfield-like approximation represents a coarse graining in time of the open-system dynamics leading toward a memoryless evolution. Indeed, our analysis is complementary with respect to the investigation of the validity of the approximation itself given the overall system-environment microscopic description [60–62].

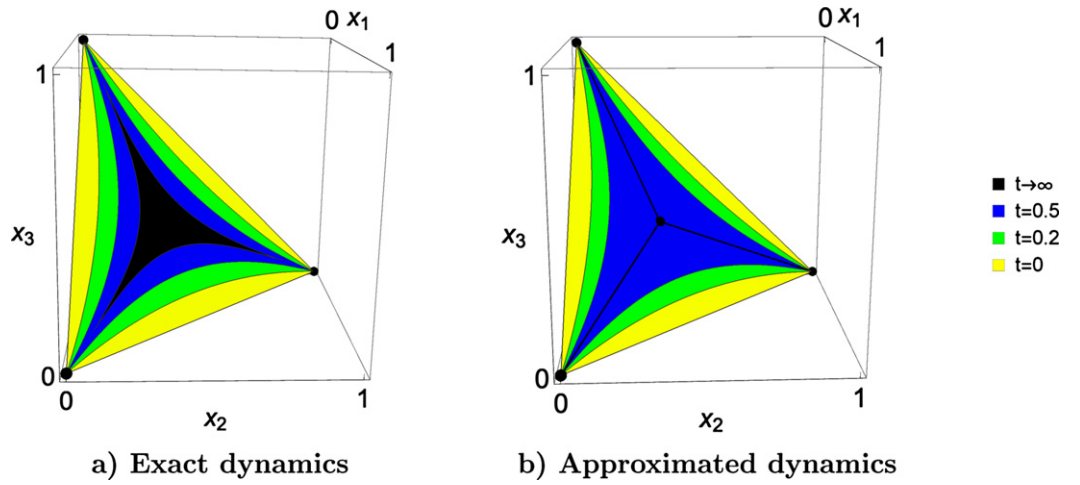

Figure 2. The areas depict the range of parameters x_1, x_2, x_3, for which all of the $\gamma_k(t)$ (panel (a) the exact dynamics) and $\gamma_k^{\mathrm{Red}}(t)$ (panel (b) the approximated dynamics) are positive, that is the dynamics is CP-divisible, up to the given time; different colors of the areas correspond to different times (in arbitrary units). Initially ($t = 0$, yellow triangle) the only constraint is in both cases the condition $x_1 + x_2 + x_3 = 1$. For long enough times $\gamma_k^{\mathrm{Red}}(t) > 0$ implies $\gamma_k(t) > 0$. The black color corresponds to $t \to \infty$, so that for these choices of x_1, x_2, x_3 the corresponding dynamical map always remains CP-divisible. Only the choices $x_i = x_j \leqslant x_k$, with $i \neq j \neq k$ lead to CP-divisible Redfield-like dynamics (panel (b) tripod configuration consisting of three lines), in contrast to the exact dynamics (panel (a) black star shape). The four black dots denote choices of parameters x_1, x_2, x_3 for which both the original and the approximated generator have only one non-zero eigenvalue and the corresponding dynamical map is CP-divisible (in panel (a) the dot in the middle is not visible).

Table 1. Connections between C(P)-divisibility property for exact and Redfield-like approximated dynamics. Arrows $\Rightarrow$ and barred arrows $\not\Rightarrow$ express the implications shown by means of proofs and counterexamples (in the case of a single non-zero eigenvalue by corollary 2 and example 3, and in the case of a Pauli channel by corollary 3 and example 4, respectively).

	Exact dynamics		Redfield-like
Single eigenvalue	C(P)-div	$\Rightarrow$	C(P)-div
		$\not\Leftarrow$	
Pauli channel	P-div	$\Rightarrow$	P-div
	CP-div	$\not\Rightarrow$	CP-div

The results of this section are summarized in table 1. When going to higher dimensional systems the situation gets more involved and no easy connections between the (C)P-divisibility of the exact and approximated dynamics were found. However, in a case of the generalized Pauli channels [66] some statements are still possible, as elaborated in appendix D.

4. Summary

In this paper, we investigated the connection between the time-local and the memory-kernel master equations associated with the dynamics of an open quantum system. Focusing on the class of commutative dynamics and making use of the damping-basis approach, we formulated a general strategy to obtain each kind of master equation from the other one. Only transformations among functions of time and their Laplace transforms are involved, as the operational structure of the two master equations coincide in the damping-basis picture. In the presence of a single non-zero eigenvalue also the Lindbladian structure of the time-local generator and the memory kernel is exactly the same. Instead, when more eigenvalues are present, new Lindbladian operators can be generated in going from the time-local master equation to the integro-differential one, or vice versa.

Furthermore, we analyzed the impact of the approximation leading to the Redfield-like master equation on both the operatorial structure of the master equation and the Markovianity of the dynamics. In the case of a single non-zero eigenvalue both CP-divisibility and P-divisibility are preserved, but this is no longer guaranteed for more general dynamics. In particular, restricting to Pauli dynamical maps, we showed that, while P-divisibility is still preserved, CP-divisibility can be lost as a result of the Redfield-like coarse graining in time.

Our results highlight the relevance of describing open system dynamics with different representations—such as the time-local or the integro-differential master equations, as well as the damping-basis or the Lindbladian picture—and the possibility to find direct connections among them. Indeed, it will be of interest to push forward such an analysis and deal with more general types of evolution in order to shed light on their relevance and usefulness for the description and characterization of non-Markovianity.

Acknowledgments

NM acknowledges funding by the Alexander von Humboldt Foundation in form of a Feodor-Lynen Fellowship, AS and BV acknowledge support from the FFABR project of MIUR. BV acknowledges support from the Joint Project 'Quantum Information Processing in Non-Markovian Quantum Complex Systems' funded by FRIAS, University of Freiburg and IAR, Nagoya University. All authors acknowledge support from the UniMi Transition Grant H2020.

Appendix A. Proof of proposition 1

Proof. First, we recall that, assuming equation (20), equation (1) is solved by (see equation (17))

$$\Lambda_t = \sum_{\alpha=1}^{N^2} m_\alpha(t)\mathcal{M}_\alpha, \tag{A.1}$$

where $m_\alpha(t)$ is defined via

$$\log\ m_\alpha(t) = \int_0^t \mathrm{d}\tau m_\alpha^{\mathrm{TCL}}(\tau) \tag{A.2}$$

and we have used

$$\sum_{\alpha=1}^{N^2} \mathcal{M}_\alpha = \mathbb{1}, \quad \mathcal{M}_\alpha\mathcal{M}_\beta = \delta_{\alpha,\beta}\mathcal{M}_\alpha.$$

Moving to the Laplace domain due to linearity we get

$$\widetilde{\Lambda}_u = \sum_{\alpha=1}^{N^2} \widetilde{m_\alpha}(u)\mathcal{M}_\alpha; \tag{A.3}$$

and for the memory kernel

$$\widetilde{\mathcal{K}^{\mathrm{NZ}}}_u = u\mathbb{1} - \left(\widetilde{\Lambda}_u\right)^{-1}. \tag{A.4}$$

Equation (A.3) implies

$$\left(\widetilde{\Lambda}_u\right)^{-1} = \sum_{\alpha=1}^{N^2} \left(\widetilde{m_\alpha}(u)\right)^{-1}\mathcal{M}_\alpha, \tag{A.5}$$

which, replaced in equation (A.4), yields equation (21) with (using also $\sum_{\alpha=1}^{N^2}\mathcal{M}_\alpha = \mathbb{1}$)

$$m_\alpha^{\mathrm{NZ}}(t) = \mathfrak{I}\left(u - \left(\widetilde{m_\alpha}(u)\right)^{-1}\right).$$

Equation (22) follows from the property of the Laplace transform of a derivative $\widetilde{\dot{f}}(u) = u\widetilde{f}(u) - f(0)$. We further have, from the definition of $G_\alpha(t)$ in equation (25)

$$m_\alpha(t) = 1 + \int_0^t \mathrm{d}\tau G_\alpha(\tau) \tag{A.6}$$

so that

$$m_\alpha^{\mathrm{TCL}}(t) = \frac{\dot{m}_\alpha(t)}{m_\alpha(t)} \tag{A.7}$$

implying equation (23). □

Appendix B. Proof of proposition 3

Proof. Using equations (22) and (58) and going to the Laplace domain, we have

$$\widetilde{m_\alpha^{\mathrm{Red}}}(u) = \frac{1}{u}\widetilde{m_\alpha^{\mathrm{NZ}}}(u) = \frac{\widetilde{G_\alpha}(u)}{1+\widetilde{G_\alpha}(u)},$$

from which

$$\widetilde{m_\alpha^{\mathrm{Red}}}(u) = \widetilde{G_\alpha}(u) - \widetilde{G_\alpha}(u)\widetilde{m_\alpha^{\mathrm{Red}}}(u),$$

which directly implies equation (60) when going back to the time domain. Equation (62) is obtained from equation (60) by iteration or, equivalently, via the geometric series, whose convergence is guaranteed by equation (61). □

Appendix C. Lindbladian form: higher dimensional example

We will consider now an example in higher dimensions, $\mathcal{H} = \mathbb{C}^3$, showing how the dimensionality strongly enhances the lack of correspondence in the operatorial structures of the time-local generator and memory kernel.

Consider the (normalized) Gell–Mann matrices

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$
$$\sigma_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad \sigma_5 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \quad \sigma_6 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$
$$\sigma_7 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \quad \sigma_8 = \frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

along with

$$S_+ = \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}, \quad S_- = \begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}.$$

The latter are the ladder operators in $\mathbb{C}^3$, and then the example 2 is generalized to three-level systems by looking at the memory kernel [compare with equation (51)]

$$\mathcal{K}_t^{\mathrm{NZ}}(\omega) = k(t)\left(S_-\omega S_+ - \frac{1}{2}(S_+S_-\omega + \omega S_+S_-) + S_+\omega S_- - \frac{1}{2}(S_-S_+\omega + \omega S_-S_+)\right). \tag{C.1}$$

This is still a self-adjoint generator, with now four non-zero eigenvalues, and the corresponding damping-basis decomposition reads

$$\left\{m_\alpha^{\mathrm{NZ}}(t)\right\}_\alpha = \left\{0, -3k(t), -\frac{5}{2}k(t), -\frac{5}{2}k(t), -k(t), -k(t), -k(t), -\frac{1}{2}k(t), -\frac{1}{2}k(t)\right\} \tag{C.2}$$

$$\{\tau_\alpha\}_\alpha = \left\{\frac{\mathbb{1}}{\sqrt{3}}, -\frac{\sqrt{3}\sigma_3}{2} + \frac{\sigma_8}{2}, -\frac{\sigma_2}{\sqrt{2}} + \frac{\sigma_7}{\sqrt{2}}, -\frac{\sigma_1}{\sqrt{2}} + \frac{\sigma_6}{\sqrt{2}}, \frac{\sigma_3}{2} + \frac{\sqrt{3}\sigma_8}{2}, \sigma_4, \sigma_5, , \frac{\sigma_2}{\sqrt{2}} + \frac{\sigma_7}{\sqrt{2}}, \frac{\sigma_1}{\sqrt{2}} + \frac{\sigma_6}{\sqrt{2}}\right\}. \tag{C.3}$$

Once again, applying proposition 1 both the time local generator and the memory kernel can be written as

$$\mathcal{K}_t^X(\omega) = \sum_{\alpha=0}^{8} m_\alpha^X(t)\,\mathrm{Tr}\left[\tau_\alpha \omega\right]\tau_\alpha \quad X = \mathrm{TCL}, \mathrm{NZ}, \tag{C.4}$$

where the memory-kernel eigenvalues are given by equation (C.2), while the time-local-generator ones are obtained via equations (24) and (25); the eigenbasis is indeed given in both cases by equation (C.3). By means of equations (39)–(42), the generator above can be written in Lindbladian form as

$$\begin{aligned}\mathcal{K}_t^X = {} & \frac{1}{6}(m_1^X(t) - 3\,m_3^X(t))\left(\mathcal{D}_{(\sigma_3+\sqrt{3}\sigma_8)/2} + \mathcal{D}_{\sigma_4} + \mathcal{D}_{\sigma_5}\right) + \frac{1}{6}(3\,m_1^X(t) - 4\,m_2^X(t) + 3\,m_3^X(t) - 4\,m_4^X(t)) \\ & \times \mathcal{D}_{(-\sqrt{3}\sigma_3+\sigma_8)/2} - \frac{1}{6}(2\,m_1^X(t) - 3\,m_2^X(t) + 3\,m_4^X(t))\left(\mathcal{D}_{(-\sigma_2+\sigma_7)/\sqrt{2}} + \mathcal{D}_{(-\sigma_1+\sigma_6)/\sqrt{2}}\right) \\ & - \frac{1}{12}(2\,m_1^X(t) + 3\,m_2^X(t) - 3\,m_4^X(t)\left(\mathcal{D}_{S_+} + \mathcal{D}_{S_-}\right),\end{aligned} \tag{C.5}$$

where we used $\mathcal{D}_A$ to denote the Lindblad dissipator with respect to the operator A. The previous expression reduces to equation (C.1) for eigenvalues as in equation (C.2), so that one recovers the Lindblad-operator structure of the memory kernel with two contributions only, while the time-local generator will have in general 8 Lindblad operators.

Appendix D. P-divisibility in higher dimensions

For generalised Pauli dynamical maps, no sufficient and necessary conditions for P-divisibility are known. However, with the necessary condition as given in [66], one can obtain the following corollary.

Corollary 4. *Consider a generalized Pauli dynamical map for which equation (61) holds. If at least one eigenvalue of the Redfield-like time-local generator takes on positive values for some time $t \geqslant 0$, then the original dynamical map is not P-divisible.*

Proof. Assume that the original dynamics is P-divisible. As elaborated in [66], from this it follows that $m_\alpha^{\mathrm{TCL}}(t) \leqslant 0$ for any $t \geqslant 0$. However from (62) we conclude then that $m_\alpha^{\mathrm{Red}}(t) \leqslant 0\ , \quad \forall\ \alpha$. Accordingly, if for some α and $t \geqslant 0$ $m_\alpha^{\mathrm{Red}}(t) > 0$, then the original dynamical map is not P-divisible. □

ORCID iDs

Nina Megier https://orcid.org/0000-0002-3149-9655
Andrea Smirne https://orcid.org/0000-0003-4698-9304
Bassano Vacchini https://orcid.org/0000-0002-7574-9951

References

[1] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)
[2] Rivas A and Huelga S F 2012 *Open Quantum Systems: An Introduction* (Berlin: Springer)
[3] Nitzan A 2007 *Chemical Dynamics in Condensed Phases Relaxation, Transfer and Reactions in Condensed Molecular Systems* (Oxford: Oxford University Press)
[4] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[5] May V and Kühn O 2011 *Charge and Energy Transfer Dynamics in Molecular Systems* (Weinheim: Wiley)
[6] Huelga S F and Plenio M B 2013 Vibrations, quanta and biology *Contemp. Phys.* **54** 181
[7] Gorini V, Kossakowski A and Sudarshan E C G 1976 Completely positive dynamical semigroups of N-level systems *J. Math. Phys.* **17** 821
[8] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119
[9] Budini A A 2004 Stochastic representation of a class of non-Markovian completely positive evolutions *Phys. Rev.* A **69** 042107
[10] Breuer H-P and Vacchini B 2008 Quantum semi-Markov processes *Phys. Rev. Lett.* **101** 140402
[11] Breuer H-P and Vacchini B 2009 Structure of completely positive quantum master equations with memory kernel *Phys. Rev.* E **79** 041147
[12] Vacchini B 2012 A classical appraisal of quantum definitions of non-Markovian dynamics *J. Phys. B: At. Mol. Opt. Phys.* **45** 154007
[13] Chruściński D and Kossakowski A 2016 Sufficient conditions for a memory-kernel master equation *Phys. Rev.* A **94** 020103
[14] Vacchini B 2016 Generalized master equations leading to completely positive dynamics *Phys. Rev. Lett.* **117** 230401
[15] Chruściński D and Kossakowski A 2010 Non-Markovian quantum dynamics: local versus nonlocal *Phys. Rev. Lett.* **104** 070406
[16] Chruściński D, Kossakowski A, Aniello P, Marmo G and Ventriglia F 2010 A class of commutative dynamics of open quantum systems *Open Syst. Inf. Dynam.* **17** 255
[17] Chruściński D 2014 On time-local generators of quantum evolution *Open Syst. Inf. Dynam.* **21** 1440004
[18] Redfield A G 1957 On the theory of relaxation processes *IBM J. Res. Dev.* **1** 19
[19] Erez N, Gordon G, Nest M and Kurizki G 2008 Thermodynamic control by frequent quantum measurements *Nature* **452** 724

[20] Strasberg P, Schaller G, Lambert N and Brandes T 2016 Nonequilibrium thermodynamics in the strong coupling and non-Markovian regime based on a reaction coordinate mapping *New J. Phys.* **18** 073007
[21] Pezzutto M, Paternostro M and Yasser O 2016 Implications of non-Markovian quantum dynamics for the Landauer bound *New J. Phys.* **18** 123018
[22] Mukherjee V, Giovannetti V, Fazio R, Huelga S F, Calarco T and Montangero S 2015 Efficiency of quantum controlled non-Markovian thermalization *New J. Phys.* **17** 063031
[23] Matsuzaki Y, Benjamin S C and Fitzsimons J 2011 Magnetic field sensing beyond the standard quantum limit under the effect of decoherence *Phys. Rev.* A **84** 012103
[24] Chin A W, Huelga S F and Plenio M B 2012 Quantum metrology in non-Markovian environments *Phys. Rev. Lett.* **109** 233601
[25] Smirne A, Kołodyński J, Huelga S F and Demkowicz-Dobrzański R 2016 Ultimate precision limits for noisy frequency estimation *Phys. Rev. Lett.* **116** 120801
[26] Haase J F, Smirne A, Kołodyński J, Demkowicz-Dobrzański R and Huelga S F 2018 Fundamental limits to frequency estimation: a comprehensive microscopic perspective *New J. Phys.* **20** 053009
[27] Rivas Á, Huelga S F and Plenio M B 2014 Quantum non-Markovianity: characterization, quantification and detection *Rep. Prog. Phys.* **77** 094001
[28] Breuer H-P, Laine E-M, Piilo J and Vacchini B 2016 *Colloquium*: Non-Markovian dynamics in open quantum systems *Rev. Mod. Phys.* **88** 021002
[29] Li L, Hall M and Wiseman H 2017 Concepts of quantum non-Markovianity: a hierarchy *Phys. Rep.* **759** 1
[30] Rivas Á, Huelga S F and Plenio M B 2010 Entanglement and non-Markovianity of quantum evolutions *Phys. Rev. Lett.* **105** 050403
[31] Vacchini B, Smirne A, Laine E-M, Piilo J and Breuer H-P 2011 Markovianity and non-Markovianity in quantum and classical systems *New J. Phys.* **13** 093004
[32] Hall M J W, Cresser J D, Li L and Andersson E 2014 Canonical form of master equations and characterization of non-Markovianity *Phys. Rev.* A **89** 042120
[33] Wißmann S, Breuer H-P and Vacchini B 2015 Generalized trace distance measure connecting quantum and classical non-Markovianity *Phys. Rev.* A **92** 042108
[34] Smirne A, Caiaffa M and Piilo J Rate operator unraveling for open quantum system dynamics *Phys. Rev. Lett.* **124** 190402
[35] Kossakowski A 1972 On quantum statistical mechanics of non-Hamiltonian systems *Rep. Math. Phys.* **3** 247
[36] Barnett S M and Stenholm S 2001 Hazards of reservoir memory *Phys. Rev.* A **64** 033808
[37] Daffer S, Wódkiewicz K, Cresser J D and McIver J K 2004 Depolarizing channel as a completely positive map with memory *Phys. Rev.* A **70** 010304
[38] Maniscalco S 2007 Complete positivity of a spin-1/2 master equation with memory *Phys. Rev.* A **75** 062103
[39] Vacchini B 2013 Non-markovian master equations from piecewise dynamics *Phys. Rev.* A **87** 030101
[40] Vacchini B 2014 General structure of quantum collisional models *Int. J. Quantum Inf.* **12** 1461011
[41] Witt B, Rudnicki L, Tanimura Y and Mintert F 2017 Exploring complete positivity in hierarchy equations of motion *New J. Phys.* **19** 013007
[42] Siudzińska K and Chruściński D 2017 Memory kernel approach to generalized Pauli channels: Markovian, semi-Markov, and beyond *Phys. Rev.* A **96** 022129
[43] Reimer V and Wegewijs M R 2019 Density-operator evolution: complete positivity and the Keldysh real-time expansion *SciPost Phys* **7** 12
[44] Filippov S N, Glinov A N and Leppäjärvi L 2020 Phase covariant qubit dynamics and divisibility *Lobachevskii J. Math.* **41** 617
[45] Vacchini B 2020 Quantum renewal processes *Sci. Rep.* **10** 5592
[46] Nestmann K, Bruch V and Wegewijs M R 2020 How quantum evolution with memory is generated in a time-local way (arXiv:2002.07232)
[47] Smirne A and Vacchini B 2010 Nakajima–Zwanzig versus time-convolutionless master equation for the non-Markovian dynamics of a two-level system *Phys. Rev.* A **82** 022110
[48] Briegel H-J and Englert B-G 1993 Quantum optical master equations: the use of damping bases *Phys. Rev.* A **47** 3311
[49] Shabani A and Lidar D A 2005 Completely positive post-Markovian master equation via a measurement approach *Phys. Rev.* A **71** 020101
[50] Maniscalco S and Petruccione F 2006 Non-Markovian dynamics of a qubit *Phys. Rev.* A **73** 012111
[51] Mazzola L, Laine E-M, Breuer H-P, Maniscalco S and Piilo J 2010 Phenomenological memory-kernel master equations and time-dependent Markovian processes *Phys. Rev.* A **81** 062120
[52] Vacchini B and Breuer H-P 2010 Exact master equations for the non-Markovian decay of a qubit *Phys. Rev.* A **81** 042103
[53] Lostaglio M, Korzekwa K and Milne A 2017 Markovian evolution of quantum coherence under symmetric dynamics *Phys. Rev.* A **96** 032109
[54] Teittinen J, Lyyra H, Sokolov B and Maniscalco S 2018 Revealing memory effects in phase-covariant quantum master equations *New J. Phys.* **20** 073012
[55] Kondov I, Kleinekathöfer U and Schreiber M 2001 Efficiency of different numerical methods for solving redfield equations *J. Chem. Phys.* **114** 1497
[56] Egorova D, Thoss M, Domcke W and Wang H 2003 Modeling of ultrafast electron-transfer processes: validity of multilevel Redfield theory *J. Chem. Phys.* **119** 2761
[57] Timm C 2008 Tunneling through molecules and quantum dots: master-equation approaches *Phys. Rev.* B **77** 195416
[58] Jeske J, Ing D J, Plenio M B, Huelga S F and Cole J H 2015 Bloch-redfield equations for modeling light-harvesting complexes *J. Chem. Phys.* **142** 064104
[59] Oviedo-Casado S, Prior J, Chin A W, Rosenbach R, Huelga S F and Plenio M B 2016 Phase-dependent exciton transport and energy harvesting from thermal environments *Phys. Rev.* A **93** 020102
[60] Benatti F and Floreanini R 2005 Open quantum dynamics: complete positivity and entanglement *Int. J. Mod. Phys.* B **19** 3063
[61] Whitney R S 2008 Staying positive: going beyond Lindblad with perturbative master equations *J. Phys. A: Math. Theor.* **41** 175304
[62] Hartmann R and Strunz W T 2020 Accuracy assessment of perturbative master equations: embracing nonpositivity *Phys. Rev.* A **101** 012103
[63] Chruściński D and Wudarski F A 2015 Non-Markovianity degree for random unitary evolution *Phys. Rev.* A **91** 012104

[64] Chruściński D and Wudarski F A 2013 Non-Markovian random unitary qubit dynamics *Phys. Lett.* A **377** 1425
[65] Megier N, Chruściński D, Piilo J and Strunz W T 2017 Eternal non-Markovianity: from random unitary to Markov chain realisations *Sci. Rep.* **7** 6379
[66] Chruściński D and Siudzińska K 2016 Generalized Pauli channels and a class of non-Markovian quantum evolution *Phys. Rev.* A **94** 022118