EXISTENCE OF STABLE H-SURFACES IN CONES AND THEIR REPRESENTATION AS RADIAL GRAPHS

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Abstract. In this paper we study the Plateau problem for disk-type surfaces contained in conic regions of $\mathbb{R}^3$ and with prescribed mean curvature H. Assuming a suitable growth condition on H, we prove existence of a least energy H-surface X spanning an arbitrary Jordan curve Γ taken in the cone. Then we address the problem of describing such surface X as radial graph when the Jordan curve Γ admits a radial representation. Assuming a suitable monotonicity condition on the mapping $\lambda \mapsto \lambda H(\lambda p)$ and some strong convexity-type condition on the radial projection of the Jordan curve Γ, we show that the H-surface X can be represented as a radial graph.

1. Introduction

In the present paper we aim to investigate some aspects on the Plateau problem for disk-type surfaces with prescribed mean curvature in the directions described as follows. Fixing a cone of angular radius β

$$\mathfrak{C}_\beta := \{p = (x, y, z) \in \mathbb{R}^3 \mid z > |p| \cos \beta\},$$

a Jordan curve $\Gamma \subset \overline{\mathfrak{C}_\beta} \setminus \{0\}$, and a mapping $H : \overline{\mathfrak{C}_\beta} \to \mathbb{R}$, we are interested in finding conditions on H, possibly related to β, ensuring that stable surfaces in $\overline{\mathfrak{C}_\beta} \setminus \{0\}$ with mean curvature H, spanning $\lambda\Gamma$ do exist for every $\lambda > 0$. Moreover we address the problem of describing such surfaces as radial graphs when their boundaries admit a radial representation.

In order to state our main results, let us state the analytical formulation of the problem. Let $B = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ be the unit open disk. In general, the Plateau problem for a given Jordan curve Γ and a prescribed mean curvature function H consists in looking for maps $X : \overline{B} \to \mathbb{R}^3$ solving

$$\Delta X = 2H(X) X_u \wedge X_v \quad \text{in } B \tag{1.1}$$

$$|X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v \quad \text{in } B \tag{1.2}$$

$$X|_{\partial B} : \partial B \to \Gamma \text{ is an (oriented) parametrization of } \Gamma. \tag{1.3}$$

A map $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ satisfying (1.1)–(1.3) will be called *H-surface spanning* Γ (see [15]). It is known that that if X is an H-surface, then X has mean curvature $H(X)$ apart from branch points, i.e., points $(u, v) \in B$ where $\nabla X(u, v) = 0$.

Our first result can be stated as follows.

2010 *Mathematics Subject Classification.* 53A10, 35J66, 57R40.

Key words and phrases. Prescribed mean curvature, Plateau's problem, H-surfaces.

Acknowledgements. Research partially supported by the project ERC Advanced Grant 2013 n. 339958 Complex Patterns for Strongly Interacting Dynamical Systems COMPAT, and by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The first author is also supported by the PRIN-2012-74FYK7 Grant "Variational and perturbative aspects of nonlinear differential problems".

Theorem 1.1. *Let $\beta \in (0, \frac{\pi}{2})$ and let $H\colon \overline{\mathfrak{C}_\beta} \to \mathbb{R}$ be a mapping of class C^1, satisfying*

$$|H(p)||p| \leqslant \frac{\cos\beta}{2(1+\cos\beta)} \quad \forall p \in \overline{\mathfrak{C}_\beta}\,. \tag{1.4}$$

Then for every rectifiable Jordan curve $\Gamma \subset \overline{\mathfrak{C}_\beta} \setminus \{0\}$ there exists an H-surface $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ spanning Γ and contained in $\overline{\mathfrak{C}_\beta} \setminus \{0\}$. Moreover we have that $X(B) \subset \mathfrak{C}_\beta$.

We point out that the assumption (1.4) fixes a bound on the radial behaviour of H with respect to the angular diameter of the given Jordan curve Γ. Moreover, since (1.4) is asked to hold on a dilation-invariant domain and is independent of the curve Γ, the existence result stated by Theorem 1.1 remains true also taking $\lambda\Gamma$ instead of Γ, for every $\lambda > 0$. Note that the case of nonzero constant mean curvature is ruled-out.

In fact we can provide more information on the H-surface given by Theorem 1.1. More precisely, taking the variational character of the Plateau problem into account, such H-surface is characterized as a least energy surface, namely is a minimum point of the energy functional associated to system (1.1), in the class of admissible mappings satisfying (1.3). We refer to Sections 2 and 3 for more details about this aspect.

Our second result provides an answer to the issue of representing an H-surface as a radial graph, when its contour is a radial graph. To this purpose, we need a monotonicity condition on the mapping $\lambda \mapsto \lambda H(\lambda p)$ and some strong convexity-type condition on the radial projection of the Jordan curve Γ. In particular, we can show:

Theorem 1.2. *Let $\beta \in (0, \frac{\pi}{2})$ and let $H\colon \overline{\mathfrak{C}_\beta} \to \mathbb{R}$ be a mapping of class $C^{1,\alpha}$, satisfying (1.4) and*

$$H(p) + \nabla H(p) \cdot p \geqslant 0 \quad \forall p \in \overline{\mathfrak{C}_\beta}\,. \tag{1.5}$$

Let Γ be a regular Jordan curve of class $C^{3,\alpha}$ contained in $\overline{\mathfrak{C}_\beta} \setminus \{0\}$ and let X be the least energy H-surface spanning Γ, given by Theorem 1.1. Assume that:

(i) *Γ is a radial graph, i.e. there exists a domain $\Omega \subset \mathbb{S}^2$ and a map $g\colon \partial\Omega \to \mathbb{R}^+$ (with the same regularity of Γ) such that $\Gamma = \{g(p)p \mid p \in \partial\Omega\}$;*
(ii) *the domain Ω is β-convex (see Definition 4.2);*
(iii) *the radial projection of $X|_{\partial B}$ induces a positive orientation on $\partial\Omega$ (see Definition 5.3 and Remark 5.5).*

Then the radial projection of X is a diffeomorphism between $\overline{B}$ and $\overline{\Omega}$ and $X(\overline{B})$ can be represented as a radial graph. In particular X has no branch point.

We remark that Theorem 1.2 is a corollary of a more general result (Theorem 6.1) about the representation of stable H-surfaces as radial graphs. The meaning of *stable* H-surface is explained in Definition 2.7.

The study developed in the present paper is a natural counterpart of analogous issues on the Plateau problem for disk-type H-surfaces in a cylinder and their representation as cartesian graphs with respect to the direction of the axis of the cylinder. On this side some results are already known in in the literature: Radó proved in [10] that minimal surfaces, spanning a Jordan curve with one-one projection onto the boundary of a planar convex domain $D \subset \mathbb{R}^2$, can be represented as cartesian graphs of a function over D. Serrin in [13], Gulliver and Spruck in [8] proved the same result in the case of surfaces of constant mean curvature, but with different assumptions. Sauvigny in [11] studied the case of stable H-surfaces with H not necessarily constant. In particular, he proved that, under a suitably strong convexity condition on the planar domain D (which is the planar version of our β-convexity property), if H is monotone along the axial direction of the cylinder, then stable H-surfaces can be represented as cartesian graphs of a real valued function over D.

Clearly our results cannot be recovered by those obtained for the cylinder. Indeed we deal with a problem whose geometry exhibits a dilation invariance, in the sense that conditions (1.4) and (1.5) are regarded just as the radial behaviour of the prescribed mean curvature function H.

In the planar graph context, one can take advantage of the fact that the orthogonal projection onto the plane is linear: in the case of the plane $z = 0$ it is just the projection of the first two coordinates. As a consequence, nice expansions and differential equations can be deduced (see Sect. 7.1, [3]). Instead, considering cones and radial projections rather than cylinders and cartesian projections brings some non trivial complications. The reason is the presence of the singular point at the vertex of a cone and the nonlinear character of the radial projection. Let us display the main difficulties by highlighting the more delicate steps in our arguments.

Concerning the existence result stated by Theorem 1.1, we follow the standard procedure of minimizing the energy functional associated to the H-system (1.1) in the class of admissible functions. Assumption (1.4) guarantees that the energy functional is bounded from below and, by known results, one gets existence of a minimizer X. Actually, in principle, the minimizer could touch the obstacle, in particular the vertex of the cone $\mathfrak{C}_\beta$. To overcome this difficulty we smooth the cone at the origin in a suitable way and we use a deep result by Gulliver and Spruck (see [9]), together with the growth condition (1.4), in order to obtain that the minimizer does not touch the boundary of the smoothed cone and then stays far from the vertex of $\mathfrak{C}_\beta$. Thus, by well known regularity results (see for instance [4]), X is a classical solution of (1.1)–(1.3). Notice that our procedure needs more care than in the case of the analogous obstacle problem in a ball or in a cylinder (see Theorems 8 and 9 in Section 4.7 of [4]). We also observe that the minimizer X turns out to be stable in the sense of Definition 2.7, provided that Γ and H are regular enough (see also Proposition 2.3).

Now let us spend a few words about our second result, concerning the characterization of a stable H-surface X as radial graph, and let us shortly illustrate the strategy followed to show that the radial projection $PX = X/|X|$ is a homeomorphism between $\overline{B}$ and $\overline{\Omega} = PX(\overline{B})$. Under the assumptions on H and Γ as in the statement of Theorem 1.2, we prove that the radial component of the Gauss Map N is always positive in $\overline{B}$, namely

$$N \cdot X > 0 \quad \text{in } \overline{B}. \tag{1.6}$$

The maximum principle is the key tool to this aim. In fact, property (1.6) implies local invertibility of PX far from branch points. The issue of global invertibility is not tackled with the same strategy followed for the analogous problem of the projection along a fixed direction as in the papers [8] and [11], because the expansion about branch points, based on the Hartman-Wintener technique, does not fit well with the radial projection. Instead, we follow an argument which is mainly based on the degree theory, combined with a classical result about global invertibility (see [2]) and Jordan-Schönflies's Theorem (see [16]).

Finally we notice that the (non-parametric) Plateau problem for H-surfaces characterized as radial graphs was already discussed by Serrin in in [14]. Actually in that work a class of positively homogeneous prescribed mean curvature functions is considered and the existence of $(n-1)$-dimensional H-surfaces in $\mathbb{R}^n$ spanning a datum Γ is proved under the following assumptions: Γ is the radial graph of a positive mapping f defined on the boundary of a given smooth domain Ω contained in a hemisphere of $\mathbb{S}^n$, and

$$H_g(y) \geq \frac{n}{n-1} H(y) f(y) \quad \forall y \in \partial\Omega\,,$$

where H_g denotes the geodesic mean curvature of $\partial\Omega$. For spherical caps, this condition turns out to be less restrictive than (1.4). On the other hand, our results allow sign-changing and non-homogeneous mean curvature functions, which cannot be considered in [14].

Lastly, let us sketch an outline of the present paper: Sect. 2 contains a collection of known facts and technical results which will be used in the sequel. In Sect. 3 we prove the existence result stated by Theorem 1.1. In Sect. 4 we discuss the notion of β-convex domain in $\mathbb{S}^2$. Finally Sections 5 and 6 contain the proof of (1.6) and of Theorem 1.2 (actually, a more general version), respectively.

2. Notation and preliminary results

In this section we fix some notation and we collect some known facts which will be useful in the rest of the paper.

We denote by B the unit open disk of $\mathbb{R}^2$ and by $\overline{B}$ its closure. We will use indistinctly both the real notation (u, v) or the complex notation z, w to denote a generic point of B or $\overline{B}$. In particular, it will be always understood that $z = e^{i\theta} \in \partial B$ stands for $(\cos\theta, \sin\theta)$. We denote by $\mathbb{S}^2$ the unit sphere of $\mathbb{R}^3$ and by $P\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ the radial projection map, defined by $P(X) := \frac{X}{|X|}$. We will use both the notation $P(Y)$ or PY to denote the composition $P \circ Y$, whenever Y is map with values in $\mathbb{R}^3 \setminus \{0\}$.

We begin with recalling some important facts about branch points and the normal N to an H-surface.

Theorem 2.1 (see Theorem 1, Sect. 2.10, [4] and also Remark 3, Sect. 5.1, [3])**.** *Let X be an H-surface of class $C^{2,\alpha}(B, \mathbb{R}^3)$ or $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ respectively. Then, for each point $w_0 \in B$ or $\overline{B}$, there is a vector $A = (A_1, A_2, A_3) \in \mathbb{C}^3 \setminus \{0\}$ with $A_1^2 + A_2^2 + A_3^2 = 0$, and a nonnegative integer $n = n(w_0)$ such that*

$$X_w(w) = A(w - w_0)^n + o(|w - w_0|^n), \quad \text{as } w \to w_0, \tag{2.1}$$

where $X_w := \frac{1}{2}(X_u - iX_v)$.

Remark 2.2. *The point w_0 in the above statement is a branch point of X if and only if $n(w_0) \geq 1$, and in this case $n(w_0)$ is called the order of the branch point $w_0 \in B$ (or $\overline{B}$ respectively). Obviously w_0 is regular point of X if and only if $n(w_0) = 0$. Thanks to (2.1) we deduce that branch points of an H-surface are isolated. In particular, if $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ then the set of branch points is finite.*

In order to get more analytic regularity on the solution X we have to ask more regularity on the function H and on the Jordan curve Γ. More precisely, we recall that:

Proposition 2.3 (see Chap. IX, Sect. 4, [12] and Sect. 2.3, [4])**.**

(i) *If $H \in C^{r,\alpha}(\mathbb{R}^3)$, for $r \in \mathbb{N}$, $\alpha \in (0, 1)$, then any solution $X \in C^2(B, \mathbb{R}^3)$ of (1.1) is of class $C^{r+2,\alpha}(B, \mathbb{R}^3)$.*

(ii) *If $H \in C^{0,\alpha}(\mathbb{R}^3)$, for some $\alpha \in (0, 1)$, and X is an H-surface such that $X(\partial B)$ lies on a regular Jordan curve of class $C^{2,\alpha}$ then $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$. More in general, if $H \in C^{r-2,\alpha}(\mathbb{R}^3)$ and $\Gamma \in C^{r,\alpha}$, for some $r \geq 2$, then $X \in C^{r,\alpha}(\overline{B}, \mathbb{R}^3)$.*

For $X \in C^{2,\alpha}(B, \mathbb{R}^3)$ we denote by B' the set of regular points. We recall that for an H-surface $X \in C^{2,\alpha}(B, \mathbb{R}^3)$ and $w \in B'$ the normal N at w is given by

$$N(w) = \frac{X_u(w) \wedge X_v(w)}{|X_u(w) \wedge X_v(w)|} = \frac{X_u(w) \wedge X_v(w)}{|X_u(w)|^2}. \tag{2.2}$$

Thanks to the expansion (2.1), writing $A = a - ib$, with $a, b \in \mathbb{R}^3$, from $A \neq 0$ and being $A_1^2 + A_2^2 + A_3^2 = 0$ it follows that $|a| = |b| \neq 0$, $a \cdot b = 0$. Hence, if w_0 is a branch point, then

$$N(w) \to \frac{a \wedge b}{|a|^2} \in \mathbb{S}^2, \quad \text{as } w \to w_0,\ w \in B'.$$

Therefore we deduce that the normal N can be extended to a continuous function $N \in C^0(\overline{B}, \mathbb{R}^3)$ with $N(\overline{B}) \subset \mathbb{S}^2$. Furthermore we have:

Theorem 2.4 (see Theorem 1, Sect. 5.1, [3])**.** *Assume that $H \in C^{1,\alpha}(\mathbb{R}^3)$ and that X is an H-surface of class $C^{3,\alpha}(B, \mathbb{R}^3)$. Then the normal N is of class $C^{2,\alpha}(B, \mathbb{R}^3)$ and satisfies the differential equation*

$$\Delta N + 2\rho N = -2E \nabla H(X), \tag{2.3}$$

where $E := |X_u|^2$, and where

$$\rho := E \left[2H^2(X) - K - (\nabla H(X) \cdot N)\right] \tag{2.4}$$

is the so-called "density function" associated to X and K is the Gaussian curvature of X. Moreover $\rho \in C^{0,\alpha}(B)$.

As remarked in the introduction, it is well known that H-surfaces are obtained as stationary points of the energy functional

$$\mathcal{F}(X) = \frac{1}{2} \int_B |\nabla X|^2 \, du \, dv + 2 \int_B Q(X) \cdot X_u \wedge X_v \, du \, dv, \tag{2.5}$$

where $Q \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field such that *div* $Q = H$. Let us also introduce the functional

$$\mathcal{G}(X) := \int_B |X_u \wedge X_v|^2 \, du \, dv + 2 \int_B Q(X) \cdot X_u \wedge X_v \, du \, dv.$$

Obviously we have $\mathcal{F}(X) \leq \mathcal{G}(X)$ and the equality $\mathcal{F}(X) = \mathcal{G}(X)$ holds if and only X satisfies the conformality relations (1.2).

Definition 2.5. *Let X an H-surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ and let $\varphi \in C_0^\infty(B)$ be a test function. We define the normal variation as the function $Z : \overline{B} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^3$, $\epsilon_0 > 0$, defined by*

$$Z(w,t) := X(w) + t\varphi(w)N(w).$$

We define the first variation and the second variation of $\mathcal{G}$, in the normal direction Z, respectively, as

$$\delta\mathcal{G}(X, \varphi N) := \frac{d}{dt}\mathcal{G}(Z)\Big|_{t=0} \quad \textit{and} \quad \delta^2\mathcal{G}(X, \varphi N) := \frac{d^2}{dt^2}\mathcal{G}(Z)\Big|_{t=0}.$$

The following result holds:

Theorem 2.6 (see Theorem 1, Sect. 5.3, [3])**.** *Let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ an H-surface and let $\varphi \in C_0^\infty(B)$ a test function. Then:*

(i) $\delta\mathcal{G}(X, \varphi N) = 0$,

(ii) $\delta^2\mathcal{G}(X, \varphi N) = \int_B |\nabla\varphi|^2 - 2\rho\varphi^2 \, du \, dv$, *where $\rho \colon B \to \mathbb{R}$ is given by* (2.4).

We recall now the fundamental notion of stability for H-surfaces.

Definition 2.7. *We say that an H-surface $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ is stable if it satisfies the following inequality*

$$\delta^2\mathcal{G}(X, \varphi N) \geq 0, \quad \textit{for all } \varphi \in C_0^\infty(B),$$

which, in view of Theorem 2.6, can be rewritten as

$$\int_B |\nabla\varphi|^2 - 2\rho\varphi^2 \, du \, dv \geq 0, \quad \textit{for all } \varphi \in C_0^\infty(B).$$

Remark 2.8. *We point out that global and local minimizers of $\mathcal{G}$ are stable. In particular if X satisfies the conformality relations and it is a minimizer of $\mathcal{F}$, then it is stable.*

The following result is a well known version of the maximum principle.

Proposition 2.9 (see Proposition 1, Sect. 5.3, [3])**.** *Assume that $q \in C^{0,\alpha}(B)$ satisfies the stability inequality*

$$\int_B |\nabla\varphi|^2 - 2q\varphi^2 \; du \; dv \geq 0, \quad \text{for all } \varphi \in C_0^\infty(B),$$

and let $f \in C^0(\overline{B}) \cap C^2(B)$ be a solution of the boundary value problem

$$\begin{cases} \Delta f + 2qf \leq 0 & \text{in } B \\ f(w) > 0 & \text{on } \partial B. \end{cases} \tag{2.6}$$

Then $f(w) > 0$ for all $w \in \overline{B}$.

Finally we state a classical result of global invertibility. We recall that a map $F \colon \mathcal{X} \to \mathcal{Y}$ between two topological spaces $\mathcal{X}$, $\mathcal{Y}$ is said to be proper if $F^{-1}(\mathcal{K})$ is compact in $\mathcal{X}$, for any compact subset $\mathcal{K}$ in $\mathcal{Y}$.

Theorem 2.10 (see Theorem 1.8, Sect. 3.1, [2])**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be two metric spaces and let $F \colon \mathcal{X} \to \mathcal{Y}$ be a continuous, surjective, proper map. Suppose that F is locally invertible, $\mathcal{X}$ arcwise connected and $\mathcal{Y}$ simply connected. Then F is a homeomorphism.*

3. Existence of H-surfaces in cones

In this section we prove Theorem 1.1. We divide the proof in several steps.

Step 1: Extension of H to $\overline{\mathfrak{C}_{\beta+\delta}}$, for $\delta > 0$ sufficiently small.
Let $\beta \in (0, \pi/2)$ and set

$$c_\beta := \frac{\cos\beta}{2(1+\cos\beta)} \, .$$

Let $\bar{\delta} > 0$ sufficiently small so that $\beta \pm \bar{\delta} \in (0, \pi/2)$ and $c_{\beta-\delta} < \frac{\operatorname{cotan}(\beta+\delta)}{2}$ for all $0 < \delta < \bar{\delta}$ (this choice will be useful in the sequel of the proof). We point out that there always exists a $\bar{\delta} = \bar{\delta}(\beta) > 0$ satisfying the previous inequality: in fact observe that since $\beta \in (0, \pi/2)$ the inequality $\frac{\cos\beta}{2(1+\cos\beta)} < \frac{\operatorname{cotan}(\beta)}{2}$ is equivalent to $\sin\beta < 1 + \cos\beta$, which holds true. Thus, by continuity of the function $\delta \mapsto \frac{\cos(\beta-\delta)}{2(1+\cos(\beta-\delta))} - \frac{\operatorname{cotan}(\beta+\delta)}{2}$ at 0 we get the desired assertion.

Let $0 < \delta < \bar{\delta}$ sufficiently small so that H can be extended to a function $H \in C^1(\overline{\mathfrak{C}_{\beta+\delta}})$ with $|H(p)||p| \leq c_{\beta-\delta}$ (we observe that being $\gamma \mapsto c_\gamma$ a strictly decreasing function it holds $c_{\beta-\delta} > c_\beta$). Clearly Γ is strictly contained in $\mathfrak{C}_{\beta+\delta}$.

Step 2: Construction of a suitable smooth surface of revolution which approximates $\partial\mathfrak{C}_{\beta+\delta}$.
The cone $\partial\mathfrak{C}_{\beta+\delta}$ is a non-smooth surface of revolution obtained by rotating the half-line $\sigma(t) = (\sin(\beta+\delta)\; t, 0, \cos(\beta+\delta)\; t)$, $t \in \mathbb{R}^+ \cup \{0\}$, lying in the xz-plane, through the z-axis. We consider the following approximating surface of revolution: let $\epsilon > 0$ be a small parameter to be chosen later, and set $t_\epsilon := \frac{8}{3\sqrt{3}} \frac{1}{\cos(\beta+\delta)}\epsilon$, let $S_{\beta+\delta,\epsilon}$ be the surface obtained by rotating the curve $\sigma_\epsilon(t) := (\alpha_1(t), 0, \alpha_2(t))$ through the z-axis, parametrized by $\phi(t,\theta) = (\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_2(t))$,

where

$$\begin{aligned}\alpha_1(t) &:= \sin(\beta+\delta)t, \quad t \geq 0, \\ \alpha_2(t) &:= \begin{cases} a_\epsilon t^4 + b_\epsilon t^2 + c_\epsilon, & \text{if } t \in [0, t_\epsilon], \\ \cos(\beta+\delta)\ t, & \text{if } t \in]t_\epsilon, +\infty[. \end{cases}\end{aligned} \tag{3.1}$$

with a_ϵ, b_ϵ, c_ϵ chosen in a suitable way in order that:

(i) $S_{\beta+\delta,\epsilon}$ is of class C^2,

(ii) $0 \notin S_{\beta+\delta,\epsilon}$,

(iii) the component of $\mathbb{R}^3 \setminus S_{\beta+\delta,\epsilon}$ which does not contain the origin is convex,

(iv) the mean curvature (with respect to the inward normal) $H_{S_{\beta+\delta,\epsilon}}$ of $S_{\beta+\delta,\epsilon}$ satisfies

$$|H(p)| < H_{S_{\beta+\delta,\epsilon}}(p), \quad \text{for any } p \in S_{\beta+\delta,\epsilon}. \tag{3.2}$$

A good choice of the coefficients a_ϵ, b_ϵ, c_ϵ is

$$a_\epsilon := -\sqrt{3}\left(\frac{3}{8}\right)^4 \frac{\cos^4(\beta+\delta)}{\epsilon^3},\ b_\epsilon := 2\sqrt{3}\left(\frac{3}{8}\right)^2 \frac{\cos^2(\beta+\delta)}{\epsilon},\ c_\epsilon := \frac{\epsilon}{\sqrt{3}}.$$

We denote by $\mathfrak{S}_{\beta+\delta,\epsilon}$ the convex component of $\mathbb{R}^3 \setminus S_{\beta+\delta,\epsilon}$. Inequality (3.2) is checked at Step 8.

Step 3: Choice of a vector field $Q : \overline{\mathfrak{C}_{\beta+\delta}} \to \mathbb{R}^3$ such that $div\ Q = H$.
Let us set

$$Q(p) := \left(\int_0^1 H(tp)t^2\ dt\right) p, \quad p \in \overline{\mathfrak{C}_{\beta+\delta}}.$$

It is clear that $Q \in C^1(\overline{\mathfrak{C}_{\beta+\delta}}, \mathbb{R}^3)$ and by elementary computations we see that $div\ Q = H$ in $\overline{\mathfrak{C}_{\beta+\delta}}$. Moreover we observe that, since $|H(p)||p| \leq c_{\beta-\delta}$ for all $p \in \overline{\mathfrak{C}_{\beta+\delta}}$, we have that

$$\|Q\|_{\infty,\overline{\mathfrak{C}_{\beta+\delta}}} \leq \frac{c_{\beta-\delta}}{2} < \frac{1}{4}. \tag{3.3}$$

Step 4: Construction of a weak solution of (1.1) which satisfies (1.2) a.e. in B.
Let $\epsilon > 0$ sufficiently small such that $\Gamma \subset \mathfrak{S}_{\beta+\delta,\epsilon}$ and (3.2) holds. We consider the variational problem $\mathcal{P}(\Gamma, \overline{\mathfrak{S}_{\beta,\epsilon}})$ given by

$$\min_{X \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})} \mathcal{F}(X),$$

where $\mathcal{F}$ is the functional defined in (2.5) and $\mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$ is the class of the admissible functions, i.e., the set of the functions in $H^{1,2}(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ which map ∂B weakly monotonic onto Γ, satisfy a three point condition and have an image almost everywhere in $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$ (see also [3]).

Since Q verifies (3.3) and $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}} \subset \overline{\mathfrak{C}_{\beta+\delta}}$ we get that $\mathcal{F}$ is coercive. In fact, considering the associated Lagrangian

$$e(p,q) = \frac{1}{2}(q_1^2 + q_2^2) + 2Q(p) \cdot q_1 \wedge q_2,$$

where $p = (x,y,z) \in \overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$, $q = (q_1, q_2) \in \mathbb{R}^3 \times \mathbb{R}^3$, by elementary computations and using (3.3), we get that, for any $p \in \overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$, $q \in \mathbb{R}^3 \times \mathbb{R}^3$

$$\left(\frac{1}{2} - \frac{c_{\beta-\delta}}{2}\right)(|p_1|^2 + |p_2|^2) \leq e(p,q) \leq \left(\frac{1}{2} + \frac{c_{\beta-\delta}}{2}\right)(|p_1|^2 + |p_2|^2).$$

In order to minimize the energy functional we have to prove that the class of admissible functions is not empty, i.e., $\mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}}) \neq \varnothing$. To this end we recall that since Γ is rectifiable it is well known that the set $\mathcal{C}(\Gamma, \mathbb{R}^3)$ is not empty (see [3] pag. 255) and there exists a minimal surface $Y \in \mathcal{C}(\Gamma, \mathbb{R}^3)$ spanning Γ. Since $Y \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ is harmonic, by the Convex hull theorem (see Theorem 1, Section 4.1 of [4]) we have that $Y(\overline{B})$ is contained in the convex hull of Γ. In particular, being $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$ convex we get that $Y(\overline{B}) \subset \overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$. Hence $Y \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$.

By Theorem 3 in Section 4.7 of [4] we have that the variational problem $\mathcal{P}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$ has a weak solution $X \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$ and it satisfies the conformality relations

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \text{ a.e. in } B.$$

Step 6: The weak solution X found at Step 5 is a classical solution of (1.1) and maps homeomorphically ∂B onto Γ.
Since $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$ is a closed and convex set, and it is equal to the closure of its interior, we have that $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$ is a quasi-regular set (see Remark (i), pag 381 in [4]). Thanks to a well known regularity result about quasi-regular sets (see Theorem 4, pag 381 in [4]), it follows that X is continuous up to the boundary. In order to get more regularity and prove that X is a classical solution of (1.1), we show that X does not touch the boundary $\partial\mathfrak{S}_{\beta+\delta,\epsilon} = S_{\beta+\delta,\epsilon}$. To prove this, we will argue by contradiction and use an important result of Gulliver and Spruck, which is a sort of geometric maximum principle.

Assume by contradiction that X touches $S_{\beta+\delta,\epsilon}$. The idea is to show that in this case we can construct $Y \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$ such that $\mathcal{F}(Y) < \mathcal{F}(X)$, and hence, X being of least energy we get a contradiction. To this end we define a "truncation" map $T \colon \overline{\mathfrak{S}_{\beta+\delta,\epsilon}} \to \mathbb{R}^3$.

In order to define T we need some preliminary definitions: for $p \in \overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$ we define $r(p) := dist(p, S_{\beta+\delta,\epsilon})$, we observe that there exists a neighborhood V of $S_{\beta+\delta,\epsilon}$ such that for $p \in V$ there is a unique point $\pi(p) \in S_{\beta+\delta,\epsilon}$ with $|p - \pi(p)| = r(p)$. We observe that, in the definition of π it is fundamental that $S_{\beta+\delta,\epsilon}$ is smooth: in fact, in the case of a cone, for any neighborhood V of the cone, we have that any point $p \in V$ lying on the axis of the cone we have that p is equidistant from $S_{\beta+\delta,\epsilon}$, so $\pi(p)$ cannot be defined as in the previous way.

We also observe that $\pi : V \to S_{\beta+\delta,\epsilon}$ is a C^1 map. Finally, for $R > 0$ we define $T \colon \overline{\mathfrak{S}_{\beta+\delta,\epsilon}} \to \mathbb{R}^3$ by setting

$$T(p) := \begin{cases} \pi(p) + RN(\pi(p)) & \text{if } p \in V \text{ and } r(p) \le R, \\ p & \text{otherwise,} \end{cases}$$

where $N(q)$ is the inward normal at $q \in S_{\beta+\delta,\epsilon}$. In general T may be not continuous, but according to Theorem 3.1 of [9], since (3.2) holds, there exists $R_0 > 0$ such that if $0 < R \le R_0$ and $\inf_{z \in B} r(X(w)) < R$ we have $T \circ X \in C^0(B) \cap H^{1,2}(B, \mathbb{R}^3)$ and $\mathcal{F}(T \circ X) < \mathcal{F}(X)$. Since we are assuming that X touches $S_{\beta+\delta,\epsilon}$ we have that $\inf_{z \in B} r(X(w)) < R$, for any $0 < R < R_0$. Hence $\mathcal{F}(T \circ X) < \mathcal{F}(X)$. It remains to prove that $T \circ X \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$. From the proof of Theorem 3.1 in [9] we know that $T \circ X \in C^0(B) \cap H^{1,2}(B, \mathbb{R}^3)$, moreover since Γ is strictly in the interior of $\overline{\mathfrak{S}_{\beta+\delta,\epsilon}}$, for R sufficiently small, by definition of T we have that $T(p) = p$, for any $p \in \Gamma$. Hence, since X is a weakly monotonic map of ∂B onto Γ, and satisfies a three point condition, the same holds for $T \circ X$, and thus $T \circ X \in \mathcal{C}(\Gamma, \overline{\mathfrak{S}_{\beta+\delta,\epsilon}})$ and we get the contradiction.

Therefore we have that $X(\overline{B}) \cap S_{\beta+\delta,\epsilon} = \varnothing$, so from Theorem 7 in Section 4.7 of [4] we have that X is a classical solution of (1.1) and $X(\overline{B}) \subset \mathfrak{S}_{\beta+\delta,\epsilon}$. Moreover we observe that by construction, for all sufficiently small $\epsilon > 0$ we have that $\mathfrak{S}_{\beta+\delta,\epsilon} \subset \mathfrak{C}_{\beta+\delta}$.

We observe that $X \colon \partial B \to \Gamma$ is a homeomorphism. This follows in a standard manner (see for instance the proof of Theorem 8 and the Remark at page 402 in [4]).

Step 7: $X(B) \subset \mathfrak{C}_\beta$.

We begin with proving that $X(\overline{B}) \subset \overline{\mathfrak{C}_\beta} \setminus \{0\}$. Let us set $\phi(u, v) := X(u, v) \cdot e_3 - |X(u, v)| \cos \beta$, where $e_3 = (0, 0, 1)$. We want to prove that $\phi \ge 0$ in $\overline{B}$. To this end we first show that $-\Delta\phi \ge 0$ in B, i.e., ϕ is super-harmonic in B.

By elementary computations we have that $\phi_u = X_u \cdot e_3 - \frac{X\cdot X_u}{|X|}\cos\beta$, and

$$\phi_{uu} = X_{uu}\cdot e_3 - \frac{X_u^2 + X\cdot X_{uu}}{|X|}\cos\beta + \frac{(X\cdot X_u)^2}{|X|^3}\cos\beta.$$

Hence we get that

$$-\Delta\phi = -\Delta X\cdot e_3 + \frac{2E + X\cdot\Delta X}{|X|}\cos\beta - \frac{(X\cdot X_u)^2 + (X\cdot X_u)^2}{|X|^3}\cos\beta,$$

where $E = |X_u|^2 = |X_v|^2$. Since $0 \notin X(\overline{B})$ we have that $\phi \in C^2(B)\cap C^0(\overline{B})$. Now, recalling that X is an H-surface, we deduce that in the subset $B' \subset B$ of regular points it holds

$$\begin{aligned}
-\Delta\phi &= -\Delta X\cdot e_3 + \frac{2E + X\cdot\Delta X}{|X|}\cos\beta - \frac{(X\cdot X_u)^2 + (X\cdot X_v)^2}{|X|^3}\cos\beta \\
&= -2H(X)(X_u\wedge X_v\cdot e_3) + \frac{2E}{|X|}\cos\beta + 2H(X)(P(X)\cdot X_u\wedge X_v)\cos\beta \\
&\quad - \frac{(P(X)\cdot P(X_u))^2 + (P(X)\cdot P(X_v))^2}{|X|}E\cos\beta \qquad (3.4)\\
&\geq -2|H(X)|E + \frac{2E}{|X|}\cos\beta - 2|H(X)|E\cos\beta - \frac{E}{|X|}\cos\beta \\
&= \frac{\cos\beta - 2(1+\cos\beta)|H(X)||X|}{|X|}E \geq 0\,.
\end{aligned}$$

We point out that the last inequality holds because H satisfies Assumption (1.4), whereas the previous inequality is a consequence of $(P(X)\cdot P(X_u))^2 + (P(X)\cdot P(X_u))^2 \leq 1$, which comes from the orthogonality of the versors $P(X_u)$, $P(X_v)$.

On the other hand, if $(u_0, v_0) \in B$ is a branch point then, from the first line of (3.4) we get that $-\Delta\phi(u_0, v_0) = 0$. Hence we have proved that $-\Delta\phi \geq 0$ in B and we are done.

Now, since X maps ∂B onto Γ and $\Gamma \subset \overline{\mathfrak{C}_\beta}\setminus\{0\}$ we have that $\phi \geq 0$ on ∂B. Therefore, by the maximum principle we get that $\phi \geq 0$ in $\overline{B}$, from which we get that $X(\overline{B}) \subset \overline{\mathfrak{S}_{\beta,\epsilon}} \subset \overline{\mathfrak{C}_\beta}\setminus\{0\}$.

Now, from Enclosure Theorem I (see Section 4.2 in [4]), in view of (3.2) (which holds for $\delta = 0$), we get that $X(B) \subset \mathfrak{S}_{\beta,\epsilon}$, from which we deduce that X maps B into $\mathfrak{C}_\beta$, and we are done.

Step 8: Proof of (3.2).
Using the parametrization $\phi(t,\theta) = (\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_2(t))$ we have that the mean curvature (with respect to the inward normal) of $S_{\beta+\delta,\epsilon}$ is given by

$$H_{S_{\beta+\delta,\epsilon}} = \frac{\alpha_1(\alpha_1'\alpha_2'' - \alpha_2'\alpha_1'') + \alpha_2'((\alpha_1')^2 + (\alpha_2')^2)}{2\alpha_1((\alpha_1')^2 + (\alpha_2')^2)^{3/2}}$$

(see for instance [1]). For $t > t_\epsilon$ we have that the rotation of $\sigma_\epsilon(t) = (\alpha_1(t), 0, \alpha_2(t))$ describes a portion of the cone $\partial\mathfrak{C}_{\beta+\delta}$ and it is elementary to see that

$$H_{S_{\beta+\delta,\epsilon}}(t) = \frac{\operatorname{cotan}(\beta+\delta)}{2t}.$$

On the other hand, if $p = \phi(t,\theta) \in S_{\beta+\delta,\epsilon}$, for $t > t_\epsilon$, $\theta \in [0, 2\pi[$ we have

$$|H(p)| \leq \frac{c_{\beta-\delta}}{|\phi(t,\theta)|} = \frac{c_{\beta-\delta}}{\sqrt{\alpha_1^2(t) + \alpha_2^2(t)}} = \frac{c_{\beta-\delta}}{t}.$$

Hence

$$|H(\phi(t,\theta))| < H_{S_{\beta+\delta,\epsilon}}(t), \text{ for any } t > t_\epsilon,\ \theta \in [0, 2\pi[\qquad (3.5)$$

if and only if

$$c_{\beta-\delta} < \frac{\operatorname{cotan}(\beta+\delta)}{2},$$

which holds true as displayed in Step 1.

For the remaining interval $[0, t_\epsilon]$ we have the following:

$$\lim_{\epsilon\to 0} \min_{t\in[0,t_\epsilon]} H_{S_{\beta+\delta,\epsilon}}(t) = +\infty. \tag{3.6}$$

Before proving (3.6) we observe that it implies that there exists a small $\bar{\epsilon} > 0$ such that $\|H\|_{\infty,\overline{\mathfrak{C}_{\beta+\delta}}\cap\{z\leq 1\}} < \min_{[0,t_\epsilon]} H_{S_{\beta+\delta,\epsilon}}(t)$ for all $0 < \epsilon < \bar{\epsilon}$. Hence for all sufficiently small $\epsilon > 0$ we have

$$H(\phi(t,\theta)) < H_{S_{\beta+\delta,\epsilon}}(t), \text{ for any } t\in[0,t_\epsilon],\ \theta\in[0,2\pi[. \tag{3.7}$$

At the end, combining (3.5), (3.7), we get (3.2).

Now we prove (3.6). First, for $\epsilon > 0$ sufficiently small, for any $t \in [0, t_\epsilon]$, we have that

$$H_{S_{\beta+\delta,\epsilon}}(t) \geq \frac{-4\sqrt{3}\left(\frac{3}{8}\right)^4 \frac{\cos^4(\beta+\delta)}{\epsilon^3}t^2 + 4\sqrt{3}\left(\frac{3}{8}\right)^2 \frac{\cos^2(\beta+\delta)}{\epsilon}}{2\sin(\beta+\delta)\left((\sin(\beta+\delta))^2 + (-4\sqrt{3}\left(\frac{3}{8}\right)^4 \frac{\cos^4(\beta+\delta)}{\epsilon^3}t^3 + 4\sqrt{3}\left(\frac{3}{8}\right)^2 \frac{\cos^2(\beta+\delta)}{\epsilon}t)^2\right)^{1/2}}. \tag{3.8}$$

In fact observe that $\alpha_1'' \equiv 0$, $\alpha_1 \geq 0$, $\alpha_1' > 0$, $\alpha_2'' > 0$ in $[0, t_\epsilon]$, hence

$$\begin{aligned} H_{S_{\beta+\delta,\epsilon}}(t) &= \frac{\alpha_1(\alpha_1'\alpha_2'' - \alpha_2'\alpha_1'') + \alpha_2'((\alpha_1')^2 + (\alpha_2')^2)}{2\alpha_1((\alpha_1')^2 + (\alpha_2')^2)^{3/2}} \\ &\geq \frac{\alpha_1(\alpha_1'\alpha_2'') + \alpha_2'((\alpha_1')^2 + (\alpha_2')^2)}{2\alpha_1((\alpha_1')^2 + (\alpha_2')^2)^{3/2}} \geq \frac{\alpha_2'}{2\alpha_1((\alpha_1')^2 + (\alpha_2')^2)^{1/2}} \end{aligned}$$

that is (3.8). Now, setting $s := \frac{t}{\epsilon}$, $g_\epsilon(s) := -4\sqrt{3}\left(\frac{3}{8}\right)^4 \cos^4(\beta+\delta)\frac{s^2}{\epsilon} + 4\sqrt{3}\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)\frac{1}{\epsilon}$, $h(s) := -4\sqrt{3}\left(\frac{3}{8}\right)^4 \cos^4(\beta+\delta)s^3 + 4\sqrt{3}\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)s$ by the previous estimate we deduce that

$$\min_{t\in[0,t_\epsilon]} H_{S_{\beta+\delta,\epsilon}}(t) \geq \min_{s\in\left[0,\frac{8}{3\sqrt{3}}\frac{1}{\cos(\beta+\delta)}\right]} \frac{g_\epsilon(s)}{2\sin(\beta+\delta)\left((\sin(\beta+\delta))^2 + (h(s))^2\right)^{1/2}}.$$

Since $s \in \left[0, \frac{8}{3\sqrt{3}}\frac{1}{\cos(\beta+\delta)}\right]$, $g_\epsilon(s) = 4\sqrt{3}\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)\left(-\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)\frac{s^2}{\epsilon} + \frac{1}{\epsilon}\right)$, it is elementary to see that

$$g_\epsilon(s) \geq 4\sqrt{3}\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)\frac{2}{3\epsilon}.$$

Moreover, since $s \mapsto \frac{1}{2\sin(\beta+\delta)((\sin(\beta+\delta))^2+(h(s))^2)^{1/2}}$ does not depend on ϵ, there exists a positive constant C_1 depending only on $\beta+\delta$ such that, for any $s \in \left[0, \frac{8}{3\sqrt{3}}\frac{1}{\cos(\beta+\delta)}\right]$, we have

$$\frac{1}{2\sin(\beta+\delta)\left((\sin(\beta+\delta))^2 + (h(s))^2\right)^{1/2}} > C_1.$$

Finally, putting together these estimates, we have

$$\min_{t\in[0,t_\epsilon]} H_{S_{\beta+\delta,\epsilon}}(t) \geq 4C_1\sqrt{3}\left(\frac{3}{8}\right)^2 \cos^2(\beta+\delta)\frac{2}{3\epsilon} \to +\infty, \quad \text{as } \epsilon \to 0.$$

Hence (3.6) is proved.

The proof is now complete.

4. On β-convex domains and related results

In this section we introduce the definition of β-convexity and prove some geometric results about β-convex subsets of $\mathbb{S}^2$ as well as geometric results about H-surfaces having support in a cone, whose boundary datum radially projects onto the boundary of a smooth β-convex subset.

Let Ω be a open subset of the unit sphere $\mathbb{S}^2$ such that $\partial\Omega$ is a Jordan curve. We denote by $\mathfrak{C}_\Omega$ the conic region in $\mathbb{R}^3$ spanned by Ω.

Definition 4.1. *We say that Ω is convex if $\mathfrak{C}_\Omega$ is a convex subset of $\mathbb{R}^3$.*

In order to get our results we need of a stronger convexity notion. For $\hat{p}_0 \in \mathbb{S}^2$ and $\beta \in (0, \frac{\pi}{2})$ we set

$$\mathfrak{C}_{\hat{p}_0,\beta} := \{p \in \mathbb{R}^3;\ p \cdot \hat{p}_0 - |p| \cos\beta > 0\}.$$

We introduce the following definition:

Definition 4.2. *Let $\beta \in (0, \frac{\pi}{2})$. We say that Ω verifies a β-cone condition at a given $p \in \partial\Omega$ if there exists $\hat{p}_0 \in \mathbb{S}^2$ such that $p \in \partial\mathfrak{C}_{\hat{p}_0,\beta}$ and $\overline{\Omega} \subset \overline{\mathfrak{C}_{\hat{p}_0,\beta}}$. We say that Ω is β-convex if, for any $p \in \partial\Omega$, Ω verifies a β-cone condition at p.*

We observe that, by definition, if Ω is β-convex, then, it is strictly contained in a hemisphere.

At first sight, one could think that for any $p \in \partial\Omega$ there could be many $\hat{p}_0 \in \mathbb{S}^2$ satisfying the β-cone condition at p, but this is not the case:

Proposition 4.3. *Assume that Ω verifies a β-cone condition at $p \in \partial\Omega$ and that $\partial\Omega$ is a regular Jordan curve of class C^1. Then there exists only one $\hat{p}_0 \in \mathbb{S}^2$ such that $p \in \partial\mathfrak{C}_{\hat{p}_0,\beta}$ and $\overline{\Omega} \subset \overline{\mathfrak{C}_{\hat{p}_0,\beta}}$. Moreover the mapping $p \mapsto \hat{p}_0$ is continuous from $\partial\Omega$ into $\mathbb{S}^2$.*

Proof. Let $\sigma : (-\delta, \delta) \to \partial\Omega$ be a C^1-parametrization of a portion of $\partial\Omega$, centered at p. Since $\partial\Omega$ is a regular curve we can assume that $\sigma'(t) \neq 0$ in $(-\delta, \delta)$. Since Ω verifies a β-cone condition at p, then, all possible $\hat{p}_0 = \hat{p}_0(p, \beta)$ lie in $\partial\mathfrak{C}_{p,\beta} \cap \mathbb{S}^2$. Now observe that for any admissible $\hat{p}_0$, since $\sigma(0) \cdot \hat{p}_0 = \cos\beta$, $\sigma(t) \cdot \hat{p}_0 \geq \cos\beta$ in $(-\delta, \delta)$, then, the function $h(t) := \sigma(t) \cdot \hat{p}_0$ must have null derivative at 0. Hence $\sigma'(0) \cdot \hat{p}_0 = 0$, which means that all possible $\hat{p}_0(p, \beta)$ must lie in the plane $\{\sigma'(0)\}^\perp$. We also observe that since $|\sigma| \equiv 1$, then, by deriving this relation, we get that $p \in \{\sigma'(0)\}^\perp$.

Thus all possible $\hat{p}_0$ are given by the intersection $\partial\mathfrak{C}_{p,\beta} \cap \{\sigma'(0)\}^\perp \cap \mathbb{S}^2$ which consists of two vectors $\hat{p}_{0,1}$, $\hat{p}_{0,2}$. By construction we observe that they generate two cones $\partial\mathfrak{C}_{\hat{p}_{0,1},\beta}$, $\partial\mathfrak{C}_{\hat{p}_{0,2},\beta}$ such that $\partial\mathfrak{C}_{\hat{p}_{0,1},\beta} \cap \partial\mathfrak{C}_{\hat{p}_{0,2},\beta} = \{\lambda p,\ \lambda \in \mathbb{R}^+\}$. Hence, since $\overline{\Omega}$ must be entirely contained in one of the regions $\overline{\mathfrak{C}_{p_{0,1},\beta}}$, $\overline{\mathfrak{C}_{p_{0,2},\beta}}$, we have that only one of the two vectors $\hat{p}_{0,1}$, $\hat{p}_{0,2}$ is admissible. The first part of proof is then complete.

We prove now the continuity of the map $p \mapsto \hat{p}_0$, from $\partial\Omega$ into $\mathbb{S}^2$. If $\sigma : (-\delta, \delta) \to \partial\Omega$ is a local parametrization centered at $p \in \partial\Omega$, then, as seen in the first part of the proof we have $\hat{p}_0(\sigma(t)) = \partial\mathfrak{C}_{\sigma(t),\beta} \cap \{\sigma'(t)\}^\perp \cap \mathbb{S}^2 \cap \overline{\Omega}$. Hence, it is clear that $\hat{p}_0(\sigma(t))$ depends continuously on t and we are done.

The proof is then complete. □

Next proposition states that β-convexity is actually a convexity property.

Proposition 4.4. *If Ω is β-convex then Ω is convex.*

Proof. Assume by contradiction that Ω is not convex. Then, there exist two distinct points $p_1, p_2 \in \mathfrak{C}_\Omega$ such that the segment $\sigma(t)$ joining p_1 and p_2 is not entirely contained in $\mathfrak{C}_\Omega$. Let us set $\hat{p}_1 := P(p_1)$, $\hat{p}_2 := P(p_2)$, $\hat{\sigma} := P \circ \sigma$. Then, there exists $t_0 \in (0, 1)$ such that $\hat{\sigma}(t_0) \in \partial\Omega$.

Since Ω is β-convex, choosing $p := \hat{\sigma}(t_0)$ in the definition, we get that there exists $\hat{p}_0$ such that $\overline{\Omega}$ is contained in the region $\overline{\mathfrak{C}_{\hat{p}_0,\beta}}$, and $p \in \partial\Omega \cap \partial\mathfrak{C}_{\hat{p}_0,\beta}$. We observe that since $\hat{p}_1, \hat{p}_2 \in \Omega$ then $\hat{p}_1, \hat{p}_2 \in \mathfrak{C}_{\hat{p}_0,\beta}$ (they cannot lie on its boundary $\partial\mathfrak{C}_{\hat{p}_0,\beta}$, otherwise they would belong to $\partial\Omega$).

Hence we have that $p_1, p_2 \in \mathfrak{C}_{\hat{p}_0,\beta}$ but $\sigma(t_0) \notin \mathfrak{C}_{\hat{p}_0,\beta}$ which contradicts the convexity of $\mathfrak{C}_{\hat{p}_0,\beta}$. □

Now let us examine the relationship between the notion of β-convexity and some geometrical properties of H-surfaces. We begin with the following preliminary result:

Proposition 4.5. *Let $\beta \in (0, \frac{\pi}{2})$, let $\Omega \subset \mathbb{S}^2$ be a β-convex domain and let Γ be a smooth regular Jordan curve such that $P(\Gamma) \subset \partial\Omega$. Assume that H satisfies (1.4), and let $X \in C^2(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ be an H-surface, with $X(\overline{B}) \subset \overline{\mathfrak{C}_\beta} \setminus \{0\}$. Then, for any $p \in \partial\Omega$, the associated function $\phi_p(u,v) := X(u,v) \cdot \hat{p}_0 - |X(u,v)| \cos\beta$ is strictly positive in B, where $\hat{p}_0 = \hat{p}_0(p, \beta) \in \mathbb{S}^2$ is given by the definition of β-convexity.*

Proof. Let us fix $p \in \partial\Omega$. Since Ω is β-convex there exists $\hat{p}_0 \in \mathbb{S}^2$ such that $p \in \partial\mathfrak{C}_{\hat{p}_0,\beta}$ and $\overline{\Omega}$ is contained in $\overline{\mathfrak{C}_{\hat{p}_0,\beta}}$. Hence, setting $\phi_p(u,v) := X(u,v) \cdot \hat{p}_0 - |X(u,v)| \cos\beta$, we have that $\phi_p \geq 0$ in ∂B. By replacing e_3 with $\hat{p}_0$ in the proof of Step 7 we have that ϕ_p is super-harmonic in B, and by the maximum principle we get that $\phi_p \geq 0$ in B. From the strong maximum principle it follows that $\phi_p > 0$ in B or $\phi_p \equiv 0$ in B. To complete the proof we have to show that the latter possibility cannot occur.

Assume by contradiction that $\phi_p \equiv 0$ in B, then, by definition and since X is smooth we have that $X(\overline{B}) \subset \partial\mathfrak{C}_{\hat{p}_0,\beta} \setminus \{0\}$. Without loss of generality we can assume that $\hat{p}_0 = e_3$ so that $X(\overline{B})$ is entirely contained in the surface $\partial\mathfrak{C}_\beta \setminus \{0\}$ which is the surface of revolution generated by the rotation, with respect of the z-axis, of the curve σ, lying in the xz-plane, given by $\sigma(t) := (\alpha_1(t), 0, \alpha_2(t))$, where $\alpha_1(t) = \sin(\beta)t$, $\alpha_2(t) = \cos(\beta)t$, $t > 0$. As seen in the proof of Theorem 1.1, using the parametrization $\phi(t,\theta) = (\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_2(t))$, we have that the mean curvature of $\partial\mathfrak{C}_\beta \setminus \{0\}$ (with respect to the inward normal) is given by $H_{\partial\mathfrak{C}_\beta\setminus\{0\}}(t) = \frac{1}{t}\frac{\text{cotan}(\beta)}{2}$, $t > 0$, moreover $|H(\phi(t,\theta))| < H_{\partial\mathfrak{C}_\beta\setminus\{0\}}(t)$ for all $t > 0$, $\theta \in [0, 2\pi]$. In fact, since H satisfies (1.4), then for all $p = \phi(t,\theta) \in \partial\mathfrak{C}_\beta \setminus \{0\}$ we have

$$|H(\phi(t,\theta))| \leq \frac{c_\beta}{|\phi(t,\theta)|} = \frac{c_\beta}{t} < \frac{1}{2t}\,\text{cotan}(\beta) = H_{\partial\mathfrak{C}_\beta\setminus\{0\}}(t), \tag{4.1}$$

because $c_\beta = \frac{\cos\beta}{2(1+\cos\beta)} < \frac{\text{cotan}\beta}{2}$. Thanks to (4.1), Theorem 2 and Corollary 3 in Section 4.4 of [4] (or by Enclosure Theorem I in Section 4.2 of [4]) it follows that $X(B) \cap (\partial\mathfrak{C}_\beta \setminus \{0\}) = \varnothing$ which gives a contradiction. The proof is then concluded. □

Corollary 4.6. *Under the same assumptions of the previous proposition we have that, for any $(u,v) \in \partial B$, the normal derivative, with respect to the exterior normal ν of the function ϕ_p, corresponding to $p = PX(u,v) \in \partial\Omega$, is strictly negative at (u,v), i.e.*

$$\frac{\partial}{\partial\nu}\phi_p(u,v) < 0.$$

In particular, if $X \in C^1(\overline{B}, \mathbb{R}^3)$ then X has no boundary branch points.

Proof. Let us fix $(u,v) \in \partial B$ and let $p = PX(u,v) \in \partial\Omega$. Consider the associated function ϕ_p. As seen in the proof of Proposition 4.5 we have that ϕ_p is super-harmonic in B and $\phi_p > 0$ in B. Hence, since $\phi_p(u,v) = 0$, by Hopf's Lemma, we get that $\frac{\partial}{\partial\nu}\phi_p(u,v) < 0$, where $\nu = (\nu^1, \nu^2)$ denotes the exterior normal at $(u,v) \in \partial B$. The first part is then proved.

For the second part we observe that since $\frac{\partial \phi_p}{\partial u} = X_u \cdot \hat{p}_0 - \frac{X \cdot X_u}{|X|} \cos\beta$, we have

$$\frac{\partial}{\partial \nu}\phi_p(u,v) = (X_u \cdot \hat{p}_0)\nu^1 + (X_v \cdot \hat{p}_0)\nu^2 - \frac{(X \cdot X_u)\nu^1 + (X \cdot X_v)\nu^2}{|X|}\cos\beta < 0.$$

Since (u,v) is arbitrary we get that X cannot have branch points on ∂B. $\square$

Another important and immediate consequence of Proposition 4.5 is the following:

Proposition 4.7. *Under the same assumptions of Proposition 4.5 we have that*

$$PX(B) \subset \Omega.$$

In particular X has support in the cone spanned by Ω, i.e., $X(\overline{B}) \subset \overline{\mathfrak{C}_\Omega} \setminus \{0\}$.

Proof. Assume by contradiction that there exists some $(u_0, v_0) \in B$ such that $PX(u_0, v_0) \in \mathbb{S}^2 \backslash \Omega$, then, necessarily, there exists $(u_1, v_1) \in B$ such that $X(u_1, v_1) \in \partial\mathfrak{C}_\Omega \setminus \{0\}$.

In fact, on the contrary, we would have that $X(B) \cap (\partial\mathfrak{C}_\Omega \setminus \{0\}) = \varnothing$, and hence we would have

$$X(B) = (X(B) \cap \mathfrak{C}_\Omega) \cup [X(B) \cap (\mathbb{R}^3 \setminus \overline{\mathfrak{C}_\Omega})].$$

Since we are assuming that $PX(u_0, v_0) \in \mathbb{S}^2 \backslash \Omega$, we have that both the open sets in the right-hand side are nonempty, hence, since they are disjoint and $X(B)$ is connected we get a contradiction.

Hence there exists $(u_1, v_1) \in B$ such that $X(u_1, v_1) \in \partial\mathfrak{C}_\Omega \setminus \{0\}$, and taking $p_1 = PX(u_1, v_1) \in \partial\Omega$, by the definition of β-convexity and applying Proposition 4.5 to the function ϕ_{p_1}, we get a contradiction since $\phi_{p_1}(u_1, v_1) = 0$. $\square$

5. Stable H-surfaces with one-one radial projection onto a β-convex subset

In this section we analyze the geometrical properties of stable H-surfaces whose boundary is a Jordan curve Γ that projects bijectively onto the boundary of a smooth β-convex domain Ω of the unit sphere $\mathbb{S}^2$. It will be understood, if not specified, that Γ is a Jordan curve of class $C^{3,\alpha}$ and $H \in C^{1,\alpha}$, for some $\alpha \in (0,1)$, so that the solution found in Theorem 1.1 is of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ (see also Proposition 2.3).

We begin with a preliminary proposition:

Proposition 5.1. *Let X be a stable H-surface of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$, with H satisfying* (1.5). *Assume that $N \cdot X > 0$ on ∂B, then $N \cdot X > 0$ in $\overline{B}$.*

Proof. Let us set $f := N \cdot X$. By elementary computations we have $f_u = N_u \cdot X + N \cdot X_u = N_u \cdot X$, and thus $f_{uu} = N_{uu} \cdot X + N_u \cdot X_u$. Deriving the relation $N \cdot X_u \equiv 0$ we also get that $N_u \cdot X_u = -N \cdot X_{uu}$. Hence $f_{uu} = N_{uu} \cdot X - N \cdot X_{uu}$ and thus $\Delta f = \Delta N \cdot X - N \cdot \Delta X$. Now, thanks to Theorem 2.4, in the subset $B' \subset B$ of regular points, we get that

$$\begin{aligned} \Delta f + 2\rho f &= -2E\nabla H(X) \cdot X - 2H(X)[N \cdot (X_u \wedge X_v)] \\ &= -2E(\nabla H(X) \cdot X + H(X)). \end{aligned}$$

Since we are assuming (1.5) we have $-2E(\nabla H(X) \cdot X + H(X)) \le 0$ in B'.
On the other hand in the subset of branch points of X we have $\Delta f + 2\rho f = -2E\nabla H(X) \cdot X - 2H(X)[N \cdot (X_u \wedge X_v)] = 0$. Now applying Proposition 2.9 (we recall that $\rho \in C^{0,\alpha}(B)$) we get that $f > 0$ in $\overline{B}$ and we are done. $\square$

It remains to study the sign of $N \cdot X$ on the boundary ∂B. The next proposition ensures that $N \cdot X$ never vanishes on ∂B.

Proposition 5.2. *Let Ω be a β-convex domain of class $C^{3,\alpha}$ and let Γ be a Jordan curve of class $C^{3,\alpha}$ which radially projects onto $\partial\Omega$. Assume that X is an H-surface of class $C^{3,\alpha}(\overline{B},\mathbb{R}^3)$ with H satisfying* (1.4)*. Then the function $N\cdot X$ never vanishes on ∂B, hence $N\cdot X$ has a constant sign on ∂B.*

Proof. Let us set $f := N\cdot X$. Assume by contradiction that there exists $z_0\in\partial B$ such that $f(z_0)=0$. In particular, since X has no boundary branch points (see Corollary 4.6) then we have $X(z_0)\cdot X_u(z_0)\wedge X_v(z_0)=0$. This means that $X(z_0)\in Span\{X_u(z_0),\ X_v(z_0)\}:=\Pi$. Hence it follows that

$$(PX)_u(z_0)=\frac{X_u(z_0)}{|X(z_0)|}-\frac{X(z_0)\cdot X_u(z_0)}{|X(z_0)|^3}X(z_0)\in\Pi,$$

and the same happens for $(PX)_v(z_0)$. Moreover, by deriving $|PX|\equiv 1$, we get that $PX\cdot(PX)_u\equiv 0$, $PX\cdot(PX)_v\equiv 0$.

Let us set $v_1 := PX(z_0)$, $v_2 := (PX)_u(z_0)$, $v_3 := (PX)_v(z_0)$ and let us observe that $v_1, v_2, v_3\in\Pi$ and $v_1\cdot v_2 = v_1\cdot v_3 = 0$. In particular, since $v_1\neq 0$ we deduce that

$$v_2\wedge v_3 = 0. \tag{5.1}$$

Let $\phi = X\cdot\hat{p}_0 - |X|\cos\beta$ be the associated function to $v_1\in\partial\Omega$. In particular we have $\phi(z_0)=0$, $\phi\geq 0$ in $\overline{B}$ and $\frac{\partial\phi}{\partial\nu}(z_0)<0$ (see Corollary 4.6). Let us also introduce the function $\psi\colon\mathbb{R}\to\mathbb{R}$, defined by $\psi(\theta) := PX(\cos\theta,\sin\theta)\cdot\hat{p}_0-\cos\beta$ and let $\theta_0\in[0,2\pi[$ such that $z_0=(\cos\theta_0,\sin\theta_0)$. Since $\psi\geq 0$, $\psi(\theta_0)=0$ and $\psi\in C^1(\mathbb{R})$ we have that $\psi'(\theta_0)=0$. This means that

$$v_2\cdot\hat{p}_0(-\nu^2)+v_3\cdot\hat{p}_0(\nu^1)=0, \tag{5.2}$$

where $\nu_1=\cos\theta_0$, $\nu_2=\sin\theta_0$. Moreover, since $\phi(z_0)=0$ we observe that $\frac{\partial\phi}{\partial\nu}(z_0)<0$ can be rewrited as

$$v_2\cdot\hat{p}_0(\nu^1)+v_3\cdot\hat{p}_0(\nu^2)<0. \tag{5.3}$$

We show that (5.1), (5.2) and (5.3) lead to a contradiction.

If $v_2\neq 0$ and $v_3\neq 0$ then, setting $a := v_2\cdot\hat{p}_0$, $b := v_3\cdot\hat{p}_0$ we rewrite (5.2), (5.3) as

$$\begin{cases} -\nu^2 a+\nu^1 b &= & 0\\ \nu^1 a+\nu^2 b &= & -k,\end{cases}$$

for some $k>0$. Then, by elementary computations it follows that $(a,b)=-k(\nu^1,\nu^2)$. Hence we have that

$$\begin{cases} v_2\cdot\hat{p}_0 &= & -k\nu^1,\\ v_3\cdot\hat{p}_0 &= & -k\nu^2.\end{cases} \tag{5.4}$$

On the other hand $v_2\wedge v_3=0$ implies that $v_2=\lambda v_3$, for some $\lambda\neq 0$, and hence from (5.4) we have

$$-k\nu^1=\lambda v_3\cdot\hat{p}_0=\lambda(-k\nu^2). \tag{5.5}$$

Remembering that (1.1) is invariant under conformal transformations of the unit disk into itself, up to a rotation of angle $2\pi-\theta_0$, we can assume that $z_0=(1,0)$, in particular $\nu^1=1$, $\nu^2=0$ and so, since $k\neq 0$, we contradicts (5.5).

It remains to examine the case in which at least one between v_2, v_3 is zero. Assume by contradiction that $v_2=0$. Then, in view of (5.2), (5.3) we get that

$$\begin{cases} v_3\cdot\hat{p}_0(\nu^1) &= & 0\\ v_3\cdot\hat{p}_0(\nu^2) &< & 0,\end{cases} \tag{5.6}$$

Up to a rotation we can assume that $\nu^1\neq 0$, $\nu^2\neq 0$, and hence (5.6) gives a contradiction. The same argument shows that $v_3=0$ cannot happen. The proof is complete. ☐

It remains to prove that $N\cdot X > 0$ on ∂B. To this end we we introduce the following definition:

Definition 5.3. *Let $\Omega \subset \mathbb{S}^2$ be a β-convex domain, such that $\partial\Omega$ is a regular Jordan curve of class C^1, i.e., there exists a parametrization $\gamma\colon \partial B \to \partial\Omega$ of class C^1 which is a homeomorphism and satisfies $\gamma'(z) \neq 0$ for all $z = e^{i\theta} \in \partial B$, where $\gamma'(z) = \frac{d}{d\theta}\gamma(e^{i\theta})$. We say that $\partial\Omega$ is positively oriented by γ if we have $(\gamma'(z) \wedge \gamma(z)) \cdot \hat{p}_0(z) < 0$, for all $z = e^{i\theta}$, $\theta \in [0, 2\pi[$, where $\hat{p}_0(z)$ is the versor associated to $\gamma(z)$, given by the definition of β-convexity.*

Remark 5.4. *We point out that the sign of $(\gamma'(z) \wedge \gamma(z)) \cdot \hat{p}_0(z)$ is well defined since, as proved in Proposition 4.3, there is only one $\hat{p}_0(z) \in \mathbb{S}^2$ satisfying the β-convexity condition at $\gamma(z) \in \partial\Omega$. Moreover, for any $z \in \partial B$, it cannot happen that $(\gamma'(z) \wedge \gamma(z)) \cdot \hat{p}_0(z) = 0$. In fact, if we consider the scalar function $\theta \mapsto h(\theta) := \gamma(e^{i\theta}) \cdot \hat{p}_0$, since h has a minimum at θ_0 corresponding to z, we get that $\gamma'(z) \cdot \hat{p}_0 = 0$ and hence if by contradiction $\hat{p}_0(z) \in Span\{\gamma'(z), \gamma(z)\}$, then $\hat{p}_0$ would be proportional to $\gamma(z)$ which is not possible. Hence we must have $Det[\gamma'(z), \gamma(z), \hat{p}_0(z)] \neq 0$ on ∂B. Furthermore, by the second part of Proposition 4.3, we deduce that $Det[\gamma'(z), \gamma(z), \hat{p}_0(z)]$ is continuous on ∂B. Hence Definition 5.3 well defines an orientation on $\partial\Omega$.*

Now we have all the instruments to state our assumption, which will be crucial for getting our next results.

Given an H-surface X of class $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ spanning a regular Jordan curve Γ of class $C^{3,\alpha}$ we introduce the following:

Assumption (I):

(i) Γ is a radial graph, i.e. there exists a domain $\Omega \subset \mathbb{S}^2$ and a map $g\colon \partial\Omega \to \mathbb{R}^+$ (with the same regularity of Γ) such that $\Gamma = \{g(p)p \mid p \in \partial\Omega\}$;

(ii) the domain Ω is β-convex;

(iii) the radial projection of $X|_{\partial B}$ induces a positive orientation on $\partial\Omega$.

Remark 5.5. *We observe that, in our context, Assumption (iii) makes sense. In fact, by definition of H-surface we have that $X|_{\partial B}\colon \partial B \to \Gamma$ is an homeomorphism and by Corollary 4.6 we know that X has no boundary branch points.*

Proposition 5.6. *Let Γ be a regular Jordan curve of class $C^{3,\alpha}$ contained in $\overline{\mathfrak{C}_\beta} \setminus \{0\}$ and let $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ be an H-surface spanning Γ. Suppose that Assumption (I) is satisfied. Then $N \cdot X > 0$ on ∂B.*

Proof. Let $z_0 = (u_0, z_0) \in \partial B$ the point in which $|X|^2$ achieves its maximum and set $M_0 := \sup_{p\in\Gamma} |p|^2$. Let $\hat{p}_0 \in \mathbb{S}^2$ be the vector associated to $PX(z_0)$ by the definition of β-convexity. Up to a rotation of angle $\theta_0 \in [0, 2\pi[$ we can assume that $z_0 = (1, 0)$. We point out that this does not change the induced orientation on $\partial\Omega$. Thanks to Corollary 4.6, since $\nu = (1, 0)$, we have that

$$X_u(z_0) \cdot \hat{p}_0 < \frac{X(z_0) \cdot X_u(z_0)}{|X(z_0)|} \cos\beta. \tag{5.7}$$

On the other hand if we consider the map $\eta\colon \mathbb{R} \to \mathbb{R}$ given by $\eta(\theta) := |X(\cos\theta, \sin\theta)|^2$, since $\theta_0 = 0$ is a maximum point and X is smooth up to the boundary, then $\psi'(0) = 0$ and hence we get that

$$X(z_0) \cdot X_v(z_0) = 0. \tag{5.8}$$

Now consider the function $\psi\colon \mathbb{R} \to \mathbb{R}$, given by $\psi(\theta) := X(\cos\theta, \sin\theta) \cdot \hat{p}_0 - |X(\cos\theta, \sin\theta)| \cos\beta$. Since $\theta = 0$ is a minimum point for ψ, and X is smooth up to the boundary, we get that $\psi'(0) = 0$, and taking into account of (5.8), we deduce that

$$X_v(z_0) \cdot \hat{p}_0 = 0. \tag{5.9}$$

Equations (5.8), (5.9) mean that $X_v(z_0)$ is orthogonal to both $X(z_0)$ and $\hat{p}_0$. Thus, for some $\lambda \in \mathbb{R}\setminus\{0\}$, it holds $X_v(z_0) = \lambda\hat{p}_0 \wedge X(z_0)$. Thanks to Assumption (I), $\frac{X_v(z_0)}{|X_v(z_0)|}$ being the tangent versor to Γ at $X(z_0)$ (we recall that, by Corollary 4.6, X has no boundary branch points) we have that $\lambda > 0$, in particular $X_v(z_0)$ has the same direction and verse of $\hat{p}_0 \wedge X(z_0)$. To prove this, we first observe that due to the definition of β-convexity $\hat{p}_0$ and $X(z_0)$ must be linearly independent, we set Π to be the plane spanned by them. Moreover, taking into account of Assumption (I) and Remark 5.4, PX induces a positive orientation on $\partial\Omega$. Hence, by Definition 5.3, since $(PX)'(0) = \frac{X_v(z_0)}{|X(z_0)|} - \frac{X(z_0)\cdot X_v(z_0)}{|X(z_0)|^3}X(z_0)$, we must have

$$\left(\frac{X_v(z_0)}{|X(z_0)|} \wedge X(z_0)\right)\cdot \hat{p}_0 = Det\left[\frac{X_v(z_0)}{|X(z_0)|}, X(z_0), \hat{p}_0\right] < 0.$$

Hence, given $X_v(z_0) = \lambda\ \hat{p}_0 \wedge X(z_0)$, by the elementary properties of the determinant we get that $\lambda > 0$.

Now let us consider the map $|X|^2 \colon \overline{B} \to \mathbb{R}$. Since X is an H-surface, and H satisfies (1.4) we have that $|X|^2$ is subharmonic. In fact, by elementary computations, we have $(|X|^2)_u = 2X\cdot X_u$, $(|X|^2)_{uu} = 2X_u \cdot X_u + 2X\cdot X_{uu}$ and hence

$$\begin{aligned} -\Delta|X|^2 &= -4E - 4X\cdot H(X)(X_u \wedge X_v) \\ &\leq -4E + 4|X||H(X)||X_u \wedge X_v| \\ &\leq -(4-4c_\beta)E \leq 0. \end{aligned}$$

In particular $|X|^2 - M_0$ is subharmonic and $|X|^2 - M_0 \leq 0$. Hence, by Hopf's lemma, since $|X|^2 - M_0 \not\equiv 0$ (otherwise X would be a portion of a sphere, and hence $|H(X)| \equiv \frac{1}{\sqrt{M_0}}$, which contradicts (1.4)), we get that

$$X(z_0)\cdot X_u(z_0) > 0\,. \tag{5.10}$$

Now, let us observe that by construction and since $X_u \cdot X_v \equiv 0$ we have $X_u(z_0) \in \Pi$. We want to understand where X_u is located with respect to $\hat{p}_0$ and $X(z_0)$. By construction the two vectors $\hat{p}_0$ and $X(z_0)$ determine an angle of amplitude β. Let us denote by R_1 the angular region in Π generated by $\hat{p}_0$ and $X(z_0)$, and by R_2 its complementary in Π.

We show that $X_u(z_0) \notin R_1$. In fact if $X_u(z_0) \in R_1$, then, denoting by $\alpha \in]0,\beta]$ the angle between $X_u(z_0)$ and $X(z_0)$ ($\alpha \neq 0$ in view of Proposition 5.2, or by (5.7)) we have that $\beta - \alpha$ is the angle between $\hat{p}_0$ and $X_u(z_0)$. Then, by dividing by $|X_u(z_0)|$ each side of (5.7), we get that

$$\cos(\beta-\alpha) < \cos(\alpha)\cos(\beta), \tag{5.11}$$

and, by elementary trigonometry, the last inequality is impossible for $\alpha, \beta \in]0, \pi/2[$.

Hence, we have that $X_u(z_0) \in R_2$ and according to (5.10) $X_u(z_0)$ must also lie in the half-plane $T := \{p \in \Pi;\ p\cdot X(z_0) > 0\}$. Thus, $X_u(z_0) \in R_2 \cap T$, and let us consider the two subregions in which $R_2 \cap T$ splits: $R_{2,1}$, $R_{2,2}$. $R_{2,1}$ is defined as the subset of $R_2 \cap T$ such that $\hat{p}_0 \in \partial R_{2,1}$. Arguing as in the previous case we see that $X_u(z_0) \notin R_{2,1}$. In fact, if $X_u(z_0) \in R_{2,1}$, denoting by $\alpha \in]\beta, \pi/2[$ the angle between $X_u(z_0)$ and $X(z_0)$ we have that $\alpha - \beta$ is the angle between $\hat{p}_0$ and $X_u(z_0)$ and as before we have

$$\cos(\alpha-\beta) < \cos(\alpha)\cos(\beta),$$

which is again impossible.

At the end the only possibility is $X_u(z_0) \in R_{2,2}$. Now, since $X_v(z_0) = \lambda\ \hat{p}_0 \wedge X(z_0)$, by the elementary properties of the determinant we get that

$$X(z_0)\cdot(X_u(z_0) \wedge X_v(z_0)) = \lambda^3 Det\,[X_u, X, X \wedge \hat{p}_0]$$

and since $X_u \in R_{2,2}$ we have that $\{X_u, X, X \wedge \hat{p}_0\}$ is a positively oriented base of $\mathbb{R}^3$. Hence we get that $X(z_0) \cdot X_u(z_0) \wedge X_v(z_0) > 0$. Now, due to Proposition 5.2, we have that the function $X \cdot X_u \wedge X_v$ has a constant sign on ∂B. Hence $N \cdot X > 0$ on ∂B and the proof is complete. $\square$

From Proposition 5.1 and Proposition 5.6 we finally get the following:

Proposition 5.7. *Let* Γ *be a regular Jordan curve of class* $C^{3,\alpha}$ *contained in* $\overline{\mathfrak{C}_\beta} \setminus \{0\}$ *and let* $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ *be an* H*-surface spanning* Γ*. Suppose that Assumption (I) is satisfied. Then* $N \cdot X > 0$ *in* $\overline{B}$*.*

6. Global invertibility of the radial projection

In this section we prove that under our assumptions the radial projection of an H-surface is a homeomorphism, in particular it can be represented as a radial graph. At the end of this section we prove Theorem 1.2.

Theorem 6.1. *Let* Γ *be a regular Jordan curve of class* $C^{3,\alpha}$ *contained in* $\overline{\mathfrak{C}_\beta} \setminus \{0\}$ *and let* X *be a stable* H*-surface of class* $C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ *spanning* Γ*, with* H *satisfying* (1.4), (1.5)*. Suppose that Assumption (I) is satisfied. Then* $PX : \overline{B} \to \overline{\Omega}$ *is a homeomorphism.*

Proof. The idea is to apply a classical result of global invertibility (see Theorem 2.10). We divide the proof in four steps.

Step 1: PX is a surjective map from $\overline{B}$ to $\overline{\Omega}$.

Since X maps homeomorphically ∂B onto Γ, and Γ satisfies Assumption (I) then PX maps homeomorphically ∂B onto $\partial\Omega$ (it is a composition of a homeomorphism and a continuous bijective map from a compact space into a Hausdorff space which is a homeomorphism too).

Without loss of generality, assume that $PX(\overline{B})$ is contained in the upper hemisphere $\mathbb{S}^+ := \mathbb{S} \cap \{z > 0\}$ and let us denote by $\pi : \mathbb{S}^2 \setminus P_S \to \mathbb{R}^2$ the stereographic projection from the south pole $P_S = (0, 0, -1)$. Since $\pi(PX)$ maps homeomorphically ∂B onto $\pi(\partial\Omega)$ it follows that for $deg(\pi(PX), q) \equiv 1$ or $deg(\pi(PX), q) \equiv -1$, where $q \in \pi(PX(B))$.

In fact $q \notin \pi(PX(\partial B))$ and by the basic properties of the degree (see for instance [5]) we know that $q \mapsto deg(\pi(PX), q)$ is constant in each connected component of $\mathbb{R}^2 \setminus \pi(PX(\partial B))$ (we recall that since $\pi(PX(\partial B))$ is a Jordan curve then $\mathbb{R}^2 \setminus \pi(PX(\partial B))$ has only two connected components), in particular it is constant for $q \in \pi(\Omega)$. Hence, being $\pi(PX) : \partial B \to \partial\Omega$ a homeomorphism there are only two possibilities: $deg(\pi(PX), q) \equiv 1$ or $deg(\pi(PX), q) \equiv -1$.

Now we know that $PX \in C^0(\overline{B}, \mathbb{R}^3)$, $P(B) \subset \Omega$ and being $deg(\pi(PX), q) \neq 0$ for any $q \in \pi(\Omega)$, it follows that $\pi(PX)(B) = \pi(\Omega)$ (see [5]). As π is a diffeomorphism, it follows that $PX(B) = \Omega$. At the end, since PX maps ∂B onto $\partial\Omega$, we have $PX(\overline{B}) = \overline{\Omega}$. Hence PX is a surjective map from $\overline{B}$ to $\overline{\Omega}$.

Step 2: $PX : \overline{B} \to \overline{\Omega}$ is locally invertible.

We begin with the local invertibility of $PX : B \to \Omega$. Given $(PX)_u = \frac{X_u}{|X|} - \frac{X \cdot X_u}{|X|^3} X$, by elementary computations we have

$$(PX)_u \wedge (PX)_v \cdot PX = \frac{X_u \wedge X_v \cdot X}{|X|^3}. \tag{6.1}$$

Hence, from Proposition 5.7, since $N \cdot X > 0$ it follows that in the set B' of regular points of X it holds

$$X_u \wedge X_v \cdot X > 0,$$

which, in view of (6.1), implies that $(PX)_u(z)$ and $(PX)_v(z)$ are linearly independent in B'. Thanks to a standard argument based on the the inverse function theorem it follows that PX

is a local diffeomorphism except on a discrete set of critical points (given by the branch points of X). Hence, from the standard properties of the degree (see for instance Theorem 2.9 in [5]), from Proposition 5.7 and since $deg(\pi(PX), q) \equiv \pm 1$ for $q \in \pi(\Omega)$, it follows that each regular value has exactly one pre-image. In fact let $q \in \pi(\Omega)$ be a regular value, then the set of pre-images of q is discrete and hence, being $\overline{B}$ compact, it is finite, and assuming for instance that $deg(\pi(PX), q) = 1$ (see the proof of Step 1), by the index formula (see Theorem 2.9-(1) in [5]) we get that

$$1 = deg(\pi(PX), q) = \sum_{p \in [\pi(PX)]^{-1}(q)} i(\pi(PX), p). \tag{6.2}$$

Now, being q a regular value we have that PX is local diffeomorphism at any $p \in [\pi(PX)]^{-1}(q)$, and $i(\pi(PX), p) = \pm 1$. From Proposition 5.7 and (6.1) it follows that $(PX)_u \wedge (PX)_v \cdot PX > 0$ in the set of regular points, in particular this holds near each $p \in [\pi(PX)]^{-1}(q)$. Hence, near each pre-image of q, PX has the same orientation, so it follows that $i(\pi(PX), p) \equiv 1$ or $i(\pi(PX), p) \equiv -1$, for $p \in [\pi(PX)]^{-1}(q)$. Thus, from (6.2), we deduce that $i(\pi(PX), p) \equiv 1$ and

$$1 = deg(\pi(PX), q) = \sum_{p \in [\pi(PX)]^{-1}(q)} i(\pi(PX), p) = k,$$

where $k \in \mathbb{N}^+$ is the cardinality of the set $[\pi(PX)]^{-1}(q)$. Hence the only possibility is $k = 1$, i.e. q has only one pre-image.

It remains to prove the local invertibility in the finite set of branch points. Let z_0 be a branch point and assume, on the contrary, that PX is not invertible at z_0 and set $p_0 := PX(z_0)$, then, for any neighborhood V of z_0 we have that PX is not injective in V. Since branch points are isolated we can assume without loss of generality that V contains only z_0 as branch point. Then, there exist $z_1, z_2 \in B$, $z_1 \neq z_2$ such that $PX(z_1) = PX(z_2)$ and necessarily one of them (for instance z_1) is not a branch point. It cannot happen that $PX(z_1)$ is a regular value, since we have proved that each regular value has exactly one pre-image. Hence $PX(z_1) = PX(z_0) = p_0$. By induction, repeating this argument we can construct a sequence of regular points $(z_n) \subset V$, $z_n \to z_0$ and such that $z_i \neq z_j$ for any $i \neq j$. In particular $S := PX^{-1}(p_0)$ is not finite. Now, up to an isometry we can assume that $PX(z_0) = e_3$ and $N(z_0) \cdot e_3 > 0$. Let us denote by $\Pi_{e_3} : \mathbb{R}^3 \to \mathbb{R}^2$ the projection of the first two coordinates. We observe that for any $z \in S$ we have $\Pi_{e_3}(X(z)) = 0$. On the other hand, arguing as in the proof of Theorem 1, Section 7.1 in [3] (in particular, see (25)), since $N(z_0) \cdot e_3 > 0$, using the complex notation we can expand $\Pi_{e_3}(X(z))$ near z_0 as $\Pi_{e_3}(X(z)) = l((z - z_0)^{n+1}) + o(|z - z_0|^{n+1})$, where $n = n(z_0) \in \mathbb{N}$ is given by Theorem 2.1 and $l : \mathbb{C} \to \mathbb{C}$ is the map associated to a nonsingular real matrix (see (24), Sect. 7.1, [3]). From this expansion we deduce that 0 must have a finite set of pre-images near z_0, and hence we get a contradiction. Hence $PX : B \to \Omega$ is locally invertible.

Now we show that PX is locally invertible as a map from $\overline{B}$ onto $\overline{\Omega}$. In fact, as proved in Corollary 4.6, PX has no boundary branch points, so, considering a suitable C^1-extension of PX, to some open neighborhood V of ∂B, we can assume that X has no branch points in V. Now, from (6.1), Proposition 5.7, we have

$$X_u \wedge X_v \cdot X > 0 \quad \text{in } V$$

which, in view of (6.1), implies that $(PX)_u(z)$ and $(PX)_v(z)$ are linearly independent in V. Hence, as before by an application of the inverse function theorem it follows that PX is a locally invertible for any $z \in V$ and we are done.

Step 3: $PX : \overline{B} \to \overline{\Omega}$ is proper.

For any compact subset $K \subset \overline{\Omega}$ we have that K is closed and being PX continuous we have $(PX)^{-1}(K)$ is a closed subset of $\overline{B}$. Being $\overline{B}$ compact it follows that $(PX)^{-1}(K)$ is compact.

Step 4: $\overline{\Omega}$ is simply connected.

Thanks to an important result of differential geometry, known as Schönflies's Theorem or also Jordan-Schönflies's Theorem (for the proof see for instance [16]) we know that the closure of the complement of the bounded region determined by a planar Jordan curve is homeomorphic to a closed ball, in particular it is simply connected. Hence, taking the stereographic projection of $\overline{\Omega}$, since $\partial\Omega$ is mapped onto a plane Jordan curve, we get that $\overline{\Omega}$ is simply connected.

From Step 1-Step 4, and $\overline{B}$ being arcwise connected, we have that the hypotheses of Theorem 2.10 are satisfied, so we get that PX is a homeomorphism. The proof is complete. □

Remark 6.2. *An immediate consequence of the previous theorem is that the H-surface X can be expressed as a radial graph. In fact, let $(PX)^{-1} : \overline{\Omega} \to \overline{B}$ the inverse function of PX, and set $F(p) := (PX)^{-1}(p)$. Then, being $X(F(p)) = PX(F(p))|X(F(p))| = p|X(F(p))|$, we get that*

$$X(\overline{B}) = \{q \in \mathbb{R}^3; \quad q = \lambda(p)p, \ p \ \in \overline{\Omega}\},$$

where $\lambda \colon \overline{\Omega} \to \mathbb{R}^+$ is the function defined by $\lambda(p) := |X(F(p))|$.

Proof of Theorem 1.2. Let X be the H-surface given by Theorem 1.1. Under our assumptions we have that $X \in C^{3,\alpha}(\overline{B}, \mathbb{R}^3)$ (see Proposition 2.3) and X is stable (see Remark 2.8). From a remarkable result of Gulliver (see Theorem 8.1 and Theorem 8.2 in [6]) we have that X is free of interior branch points and Corollary 4.6 excludes also boundary branch points. Hence, from the proof of Theorem 6.1 it follows that the radial projection of X is actually a global diffeomorphism. □

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