

General properties of $f(R)$ gravity vacuum solutions

Salvatore Capozziello,^{1,2,*} Carlo Alberto Mantica,^{3,†} and Luca Guido Molinari^{4,‡}

¹*Dipartimento di Fisica “E. Pancini”, Università di Napoli Federico II, Napoli,
and INFN Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,*
²*Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR),
634050 Tomsk (Russia).*

³*I.I.S. Lagrange, Via L. Modignani 65, I-20161, Milano, Italy
and INFN sez. di Milano, Via Celoria 16, I-20133 Milano, Italy*

⁴*Dipartimento di Fisica “A. Pontremoli”, Università degli Studi di Milano
and INFN sez. di Milano, Via Celoria 16, I-20133 Milano, Italy.*[‡]

(Dated: July 28, 2020)

General properties of vacuum solutions of $f(R)$ gravity are obtained by the condition that the divergence of the Weyl tensor is zero and $f'' \neq 0$. Specifically, a theorem states that the gradient of the curvature scalar, ∇R , is an eigenvector of the Ricci tensor and, if it is time-like, the space-time is a Generalized Friedman-Robertson-Walker metric; in dimension four, it is Friedman-Robertson-Walker.

PACS numbers: 98.80.-k, 95.35.+d, 95.36.+x

Keywords: Higher-order gravity; cosmology; perfect fluids

I. INTRODUCTION

The so called $f(R)$ gravity is a natural extension of Einstein’s gravity where the Hilbert-Einstein action of gravitational field, linear in the Ricci scalar R , is substituted with a generic function $f(R)$. The issues for this generalization mainly come from inflationary cosmology [1], late-time acceleration [2] and the possibility to unify late and early cosmic history [3, 4].

Furthermore, it is the subject of a vast research as a potential alternative to the so far undetected exotic fields that should account for dark matter and dark energy. This alternative is geometric: the further degrees of freedom of $f(R)$ gravity may produce observable effects at different astrophysical and cosmological scales that should be, otherwise, ascribed to exotic forms of matter. There are other motivations like quantum perturbative corrections on curved spacetimes and the natural question about the consequences of a straightforward generalization of the Hilbert-Einstein action to consider $f(R)$ gravity as the first logical step [5, 6].

Starting from a general $f(R)$ gravity action, the field equations can be written as

$$G[f]_{kl} = \kappa T_{kl}, \quad (1)$$

where $G[f]$ replaces the Einstein tensor:

$$G[f]_{kl} = f'(R)R_{kl} - f'''(R)\nabla_k R \nabla_l R - f''(R)\nabla_k \nabla_l R \\ + g_{kl}[f'''(R)\nabla_j R \nabla^j R + f''(R)\nabla^2 R - \frac{1}{2}f(R)]. \quad (2)$$

It becomes the standard expression, $G_{kl} = R_{kl} - \frac{1}{2}Rg_{kl}$, for $f(R) = R$. T_{kl} is the stress-energy tensor for standard matter and κ is the gravitational coupling. A main requirement is that the contracted Bianchi identity $\nabla^k G_{kl} = 0$ continues to hold in $f(R)$ gravity, that is:

$$\nabla^k G[f]_{kl} = f''(R)[R_{kl}\nabla^k R - \nabla^2 \nabla_l R + \nabla_l \nabla^2 R] = 0.$$

Therefore $f(R)$ theories are compatible with the physical requirement of energy-momentum conservation. In Ref.[7] we obtained a sufficient condition for $G[f]_{kl}$ to have the ‘perfect-fluid’ form

$$G[f]_{kl} = g_1(R)g_{kl} + g_2(R)u_k u_l, \quad (3)$$

*Electronic address: capozziello@na.infn.it

†Electronic address: carlo.mantica@mi.infn.it

‡Electronic address: luca.molinari@unimi.it

for any smooth $f(R)$ model, where u_k is some time-like unit vector field. The requirement that each tensor in the expression (2) of $G[f]_{kl}$ have the perfect fluid form, namely the Ricci tensor R_{kl} , $\nabla_k R \nabla_l R$ and $\nabla_k \nabla_l R$, can be stated as follows:

- C1) $\nabla_k u_j = \varphi(u_j u_k + g_{jk})$ with $\nabla_k \varphi = -\dot{\varphi} u_k$,
 C2) $\nabla_m C_{jkl}{}^m = 0$, where $C_{jkl}{}^m$ is the Weyl tensor.

Condition C1 characterizes the space-time as a generalized Friedman-Robertson-Walker (GFRW) space-time. The additional condition 2 implies that $u_k \propto \nabla_k R$. Then $\nabla_k R$ is eigenvector of the Ricci tensor.

In $n = 4$ dimensions, C1 and C2 imply that the space-time is Friedman-Robertson-Walker [22] Prop. 4.1. The unicity of the vector u_k realizing a GFRW space-time is discussed in [8].

The perfect-fluid structure of $G[f]_{kl}$ and T_{kl} , required in large-scale cosmology, put $f(R)$ geometric extension of General Relativity on the same footing as exotic modifications of standard matter. In other words, the question is whether cosmic phenomenology can be addressed by requiring exotic forms of matter/energy beyond the Standard Model of Particles, or gravity is not scale invariant and modifications are required at galactic scales and beyond [9, 10].

Hereafter, we investigate vacuum solutions ($T_{kl} = 0$) in space-times of dimension n in order to derive some general properties of $f(R)$ gravity.

Vacuum solutions have been studied based on special forms of $f(R)$ or assumptions about symmetries of the metric [11]. Perturbations of models like $f(R) = R^{1+\delta}$ in spherical symmetry are studied in [12, 13]. The non-linear equations for f' for spherical-symmetric vacuum solutions are studied in [14], and solved for some cases. Constant curvature solutions, i.e. $f'(R)R - 2f(R) = 0$, were found with cylindrical symmetry [15, 16], plane-symmetry [17], and local rotational symmetry [18]. Spherically symmetric solutions in connection to black-holes and the rotation curve of galaxies are studied in [19].

In this work we want to characterize the vacuum solutions of any smooth $f(R)$ theory with null divergence of the Weyl tensor. The results can be summarized in the following theorem:

Theorem 1. *For any smooth function $f(R)$ with $f'' \neq 0$, the vacuum solution with null divergence of the Weyl tensor ($\nabla_m C_{jkl}{}^m = 0$), has the properties:*

- 1) $\nabla_k R$ is an eigenvector of the Ricci tensor: $R_j{}^k \nabla_k R = \xi \nabla_j R$.
 2) If $\nabla_k R = \alpha u_k$ with $u^k u_k = -1$, the Ricci tensor has the perfect fluid (quasi-Einstein) form:

$$R_{kl} = \frac{R - n\xi}{n - 1} u_k u_l + \frac{R - \xi}{n - 1} g_{kl}. \quad (4)$$

- 3) The vector u_k is vorticity-free, acceleration-free and shear-free, and satisfies the relation:

$$\nabla_j u_k = \varphi(u_j u_k + g_{jk}), \quad (5)$$

where $\nabla_k \varphi = -\dot{\varphi} u_k$.

The property (5) characterizes the Lorentzian space-time as a generalised Robertson-Walker space-time. It is usually described as a time-warped space-time, i.e. there is a coordinate frame where the metric tensor is:

$$ds^2 = -dt^2 + a^2(t) g_{\mu\nu}^*(x) dx^\mu dx^\nu, \quad (6)$$

and g^* is a Riemannian metric [20, 21]. For such space-times we proved the special property $C_{jkl}{}^m u_m = 0$ iff $\nabla_m C_{jkl}{}^m = 0$ [22]. In $n = 4$ it implies that the Weyl tensor is zero if the divergence vanishes, i.e. the space-time is Robertson-Walker.

The proof of the above Theorem is given in the next sections. In Sect.2, we discuss $\nabla_k R$ which is an eigenvector of the Ricci tensor. In Sect.3 the related form of Ricci tensor is obtained. Sect.4 is devoted to the vector u_k which is vorticity and acceleration-free; its shear is proportional to the electric component of the Weyl tensor $\sigma_{kl} \propto C_{jklm} u^j u^m$. In Sect.5 an equation for the time evolution of the shear is obtained. A vorticity and acceleration free velocity field restricts the metrics to the form $ds^2 = -dt^2 + a(t, x)_{\mu\nu} dx^\mu dx^\nu$. By the equations in Appendix we show, in Sect.6, that the special form of the shear implies that it is zero. The consequences are discussed in the conclusions (Sect.7). All results are then simplified because the Ricci tensor is quasi-Einstein, the Ricci tensor of the space-submanifold is Einstein, the space-time is a GFRW. In Appendix are reported all the quantities used for the derivations developed in the paper. We indicate with a dot the directional derivative $u^k \nabla_k$ and with a prime the derivative with respect to R .

II. $\nabla_k R$ IS AN EIGENVECTOR OF THE RICCI TENSOR

Let us rewrite the field equations $G[f]_{kl} = 0$, with $G[f]$ given in Eq.(2), as:

$$R_{kl} = A\nabla_k R \nabla_l R + B\nabla_k \nabla_l R - Cg_{kl}, \quad (7)$$

where $A = \frac{f'''}{f'}$, $B = \frac{f''}{f'}$, $C = A\nabla_j R \nabla^j R + B\nabla^2 R - \frac{f}{2f'}$.

The trace is:

$$R = -(n-1)[A\nabla_k R \nabla^k R + B\nabla^2 R] + n\frac{f}{2f'}. \quad (8)$$

Then

$$(n-1)C = -R + \frac{f}{2f'}. \quad (9)$$

Proof: let us evaluate

$$\begin{aligned} \nabla_j R_{kl} &= (A'\nabla_j R)\nabla_k R \nabla_l R + A(\nabla_j \nabla_k R)\nabla_l R + A\nabla_k R(\nabla_j \nabla_l R) \\ &\quad + (B'\nabla_j R)\nabla_k \nabla_l R + B\nabla_j \nabla_k \nabla_l R - C'g_{kl}\nabla_j R. \end{aligned}$$

As said, the prime indicates the derivative in R and we antisymmetrize in two indices:

$$\begin{aligned} \nabla_j R_{kl} - \nabla_k R_{jl} &= (A - B')[(\nabla_k R)\nabla_j \nabla_l R - (\nabla_j R)\nabla_k \nabla_l R] \\ &\quad + BR_{jkl}{}^m \nabla_m R - C'(g_{kl}\nabla_j R - g_{jl}\nabla_k R). \end{aligned}$$

The divergence of the Weyl tensor is

$$\nabla_m C_{jkl}{}^m = -\frac{n-3}{n-2} \left[\nabla_j R_{kl} - \nabla_k R_{jl} - \frac{g_{kl}\nabla_j R - g_{jl}\nabla_k R}{2(n-1)} \right].$$

If $\nabla_m C_{jkl}{}^m = 0$ then:

$$\begin{aligned} \frac{g_{kl}\nabla_j R - g_{jl}\nabla_k R}{2(n-1)} &= (A - B')[(\nabla_k R)\nabla_j \nabla_l R - (\nabla_j R)\nabla_k \nabla_l R] \\ &\quad + BR_{jkl}{}^m \nabla_m R - C'(g_{kl}\nabla_j R - g_{jl}\nabla_k R). \end{aligned}$$

Note that $A - B' = B^2$ and $C' = -\frac{1}{2(n-1)} - \frac{1}{2(n-1)}\frac{f}{f'}B$

$$-B\frac{g_{kl}\nabla_j R - g_{jl}\nabla_k R}{2(n-1)}\frac{f}{f'} = B^2[(\nabla_k R)\nabla_j \nabla_l R - (\nabla_j R)\nabla_k \nabla_l R] + BR_{jkl}{}^m \nabla_m R.$$

We can factor out $B \neq 0$. The term $B\nabla_j \nabla_l R$ is obtained from the expression (7) of the Ricci tensor. In this substitution, terms with A cancel out:

$$-\frac{g_{kl}\nabla_j R - g_{jl}\nabla_k R}{2(n-1)}\frac{f}{f'} = (\nabla_k R)(R_{jl} + Cg_{jl}) - (\nabla_j R)(R_{kl} + Cg_{kl}) + R_{jkl}{}^m \nabla_m R.$$

A simplification occurs with Eq.(9):

$$0 = (\nabla_k R)(R_{jl} - \frac{R}{n-1}g_{jl}) - (\nabla_j R)(R_{kl} - \frac{R}{n-1}g_{kl}) + R_{jkl}{}^m \nabla_m R. \quad (10)$$

The contraction with $\nabla^l R$ cancels out terms, leaving $(\nabla_k R)R_{jl}\nabla^l R = (\nabla_j R)R_{kl}\nabla^l R$. The equation is solved by:

$$R_{jl}\nabla^l R = \xi\nabla_j R, \quad (11)$$

for some eigenvalue ξ . This completes the proof of point 1 of the Theorem 1. \square

III. THE RICCI TENSOR

Let us obtain the structure of the Ricci tensor with the approach used in [22]. The result will be simplified after showing that the shear of u_k is zero.

In the following, we refer to a time-like unit vector: $\nabla_k R = \alpha u_k$, where $u^k u_k = -1$. u^k is an eigenvector of the Ricci tensor, $R_{kj} u^j = \xi u_k$. Eq.(10) is

$$R_{jklm} u^m = -u_k (R_{jl} - \frac{R}{n-1} g_{jl}) + u_j (R_{kl} - \frac{R}{n-1} g_{kl}). \quad (12)$$

Contracting with u^j , gives:

$$R_{jklm} u^j u^m = -\xi u_k u_l - R_{kl} + \frac{R}{n-1} (g_{kl} + u_k u_l). \quad (13)$$

The Weyl tensor

$$C_{jklm} = R_{jklm} + \frac{g_{jm} R_{kl} - g_{km} R_{jl} + g_{kl} R_{jm} - g_{jl} R_{km}}{n-2} - R \frac{g_{jm} g_{kl} - g_{km} g_{jl}}{(n-1)(n-2)},$$

is contracted with $u^j u^m$ to obtain the Ricci tensor, and Eq.(13) is used. It is

$$\begin{aligned} (n-2) C_{jklm} u^j u^m &= (n-2) R_{jklm} u^j u^m - R_{kl} - \xi (2u_k u_l + g_{kl}) + R \frac{g_{kl} + u_k u_l}{n-1} \\ &= (R - n\xi) u_k u_l - (n-1) R_{kl} + (R - \xi) g_{kl} \end{aligned}$$

The resulting Ricci tensor has a quasi-Einstein term and a Weyl term:

$$R_{kl} = \frac{R - n\xi}{n-1} u_k u_l + \frac{R - \xi}{n-1} g_{kl} - \frac{n-2}{n-1} C_{jklm} u^j u^m \quad (14)$$

The Weyl term will be shown to be zero.

IV. THE VECTOR FIELD u_k

Let us obtain now the properties of the vector field u_k ($\nabla_k R = \alpha u_k$). We rewrite the Ricci tensor (7) in terms of u_k :

$$R_{kl} = A\alpha^2 u_k u_l + B(\nabla_k \alpha) u_l + B\alpha \nabla_k u_l - C g_{kl}.$$

The contraction with u^l gives: $\xi u_k = -A\alpha^2 u_k - B(\nabla_k \alpha) - C u_k$. Then $\nabla_k \alpha$ is proportional to u_k :

$$\nabla_k \alpha = -\dot{\alpha} u_k \quad (15)$$

and the Ricci tensor is:

$$R_{kl} = (A\alpha^2 - B\dot{\alpha}) u_k u_l + B\alpha (\nabla_k u_l) - C g_{kl}. \quad (16)$$

Lemma IV.1. *The vector field u_k is vorticity-free and acceleration-free.*

Proof. Eq.(15) and the identity $\nabla_k \nabla_j R = \nabla_j \nabla_k R$, i.e. $\nabla_k (\alpha u_j) = \nabla_j (\alpha u_k)$, give

$$\nabla_j u_k - \nabla_k u_j = 0. \quad (17)$$

Contraction with u^j gives zero acceleration: $u^j \nabla_j u_k = 0$ because $u^j u_j = -1$. \square

This Lemma has the consequence that the gradient of u_k has the structure

$$\nabla_j u_k = \varphi (u_j u_k + g_{jk}) + \sigma_{jk}, \quad (18)$$

with $\varphi = \frac{\nabla_i u^i}{n-1}$, and shear tensor σ_{jk} (traceless, symmetric and $\sigma_{jk}u^k = 0$). The Ricci tensor becomes:

$$R_{kl} = (A\alpha^2 - B\dot{\alpha} + B\alpha\varphi)u_k u_l + (B\alpha\varphi - C)g_{kl} + B\alpha\sigma_{kl}. \quad (19)$$

Comparison with the expression (14) gives

$$B\alpha\sigma_{kl} = -\frac{n-2}{n-1}C_{jklm}u^j u^m,$$

and the relations

$$\begin{aligned} R - n\xi &= (n-1)(A\alpha^2 - B\dot{\alpha} + B\alpha\varphi) \\ R - \xi &= (n-1)(B\alpha\varphi - C). \end{aligned}$$

The second one simplifies with Eq.(9), and it is used in the first one:

$$(n-1)B\alpha\varphi = \frac{f}{2f'} - \xi \quad (20)$$

$$(n-1)(A\alpha^2 - B\dot{\alpha} + \xi) = R - \frac{f}{2f'}. \quad (21)$$

Remark IV.2. A consequence of Eq.(12) is that u^k is Riemann-compatible [23], that is:

$$u_i R_{jklm} u^m + u_j R_{kilm} u^m + u_k R_{ijlm} u^m = 0.$$

It has been proven [24] that this property also implies that u^k is Weyl compatible:

$$u_i C_{jklm} u^m + u_j C_{kilm} u^m + u_k C_{ijlm} u^m = 0.$$

It follows that $C_{jklm} u^m = u_k E_{jl} - u_j E_{kl}$, where $E_{kl} = C_{jklm} u^j u^m$ is the electric part of the Weyl tensor.

V. THE SHEAR σ_{kl}

Lemma V.1. Let us prove that

$$\nabla^k \sigma_{kl} = u_l (\sigma_{km} \sigma^{km}), \quad (22)$$

$$\nabla_k \varphi = -\dot{\varphi} u_k. \quad (23)$$

Proof. Since $\nabla^k C_{jklm} = 0$, $\nabla_k (B\alpha) = (B'\nabla_k R)\alpha - B\dot{\alpha}u_k = (B'\alpha^2 - B\dot{\alpha})u^k$ and $u^k \sigma_{jk} = 0$ one evaluates:

$$B\alpha \nabla^k \sigma_{kl} = -\frac{n-2}{n-1} C_{jklm} \nabla^k (u^j u^m) = -\frac{n-2}{n-1} u^j C_{jklm} \sigma^{km}.$$

Now, let us use the property (see the above Remark) $u^j C_{jklm} = u_l E_{km} - u_m E_{lk}$:

$$B\alpha \nabla^k \sigma_{kl} = -\frac{n-2}{n-1} u_l E_{km} \sigma^{km} = u_l B\alpha \sigma_{km} \sigma^{km}.$$

The second statement results from the identity $R_{jk} u^j = \nabla^2 u_k - \nabla_k \nabla^j u_j$, i.e.

$$\begin{aligned} \xi u_k &= \nabla^l (\varphi (u_l u_k + g_{lk} + \sigma_{lk}) - (n-1) \nabla_k \varphi) \\ &= \dot{\varphi} u_k + (n-1) \varphi^2 u_k + \nabla_k \varphi + \nabla^l \sigma_{kl} - (n-1) \nabla_k \varphi \\ &= [\dot{\varphi} + (n-1) \varphi^2 + \sigma_{ij} \sigma^{ij}] u_k - (n-2) \nabla_k \varphi. \end{aligned}$$

The contraction with u^k gives:

$$\xi = (n-1)(\varphi^2 + \dot{\varphi}) + \sigma_{kl} \sigma^{kl}, \quad (24)$$

and the previous equation simplifies to $(n-2)(\nabla_k \varphi + u_k \dot{\varphi}) = 0$. \square

It is now possible to obtain an equation for the shear. Eq.(12) simplifies with the expression (14) of the Ricci tensor:

$$[\nabla_j, \nabla_k]u_l = u_k \left[\frac{\xi}{n-1} g_{jl} - B\alpha\sigma_{jl} \right] - u_j \left[\frac{\xi}{n-1} g_{kl} - B\alpha\sigma_{kl} \right].$$

The left-hand side, with the aid of the expression for $\nabla_j u_k$, is:

$$(\dot{\varphi} + \varphi^2)(u_k g_{jl} - u_j g_{kl}) + \varphi(u_k \sigma_{jl} - u_j \sigma_{kl}) + \nabla_j \sigma_{kl} - \nabla_k \sigma_{jl}.$$

We then obtain, with (24):

$$\nabla_j \sigma_{kl} - \nabla_k \sigma_{jl} = (\varphi + B\alpha)(u_j \sigma_{kl} - u_k \sigma_{jl}) - \frac{\sigma_{rs} \sigma^{rs}}{n-1} (u_j g_{kl} - u_k g_{jl}). \quad (25)$$

The contraction with u^j and Eq. (22) gives:

$$\dot{\sigma}_{kl} + \sigma_{kl}^2 + (2\varphi + B\alpha)\sigma_{kl} = \frac{\sigma_{rs} \sigma^{rs}}{n-1} (g_{kl} + u_k u_l). \quad (26)$$

This equation, considered in a useful coordinate frame, will imply that $\sigma_{jk} = 0$.

VI. THE COMOVING FRAME

Since u^k is vorticity-free and acceleration-free, in the coordinates (t, x^1, \dots, x^{n-1}) where $u_0 = -1$ and $u_\mu = 0$, the metric of the Lorentzian manifold has the block structure [25] eq.2.19:

$$g_{ij} dx^i dx^j = -dt^2 + a_{\mu\nu}(t, x) dx^\mu dx^\nu,$$

where, at any time, the metric $a_{\mu\nu}$ is Riemannian. With the formulae in Appendix, the relation $(n-1)\varphi = \nabla_k u^k$ becomes:

$$(n-1)\varphi = \Gamma_{\mu 0}^\mu = \frac{1}{2} a^{\mu\nu} \dot{a}_{\mu\nu}. \quad (27)$$

The relation $\nabla_k u_j = \varphi(u_i u_j + g_{ij}) + \sigma_{ij}$ gives $\sigma_{00} = 0$, $\sigma_{0\mu} = 0$ and

$$\sigma_{\mu\nu} = \Gamma_{\mu\nu}^0 - \varphi a_{\mu\nu} = \frac{1}{2} \dot{a}_{\mu\nu} - \varphi a_{\mu\nu}, \quad (28)$$

Proposition VI.1. *It is possible to show that $\sigma_{\mu\nu} = 0$.*

Proof. The shear is a purely spatial tensor and $\sigma_{\mu\nu} a^{\mu\nu} = 0$. Let us evaluate:

$$\begin{aligned} \sigma_{\mu\nu} \sigma^{\mu\nu} &= \left(\frac{1}{2} \dot{a}_{\mu\nu} - \varphi a_{\mu\nu} \right) a^{\mu\tau} a^{\nu\sigma} \left(\frac{1}{2} \dot{a}_{\tau\sigma} - \varphi a_{\tau\sigma} \right) \\ &= \frac{1}{4} \dot{a}_{\mu\nu} a^{\mu\tau} a^{\nu\sigma} \dot{a}_{\tau\sigma} - \varphi \dot{a}_{\mu\nu} a^{\mu\nu} + \varphi^2 (n-1) \\ &= -\frac{1}{4} \dot{a}_{\mu\nu} \dot{a}^{\mu\tau} a^{\nu\sigma} a_{\tau\sigma} - 2\varphi^2 (n-1) + \varphi^2 (n-1) \\ &= -\frac{1}{4} \dot{a}_{\mu\nu} \dot{a}^{\mu\nu} - (n-1)\varphi^2. \end{aligned}$$

We used $0 = \dot{a}_{\mu\nu} a^{\nu\rho} + \dot{a}^{\nu\rho} a_{\mu\nu}$.

The equation for the shear tensor is

$$0 = \dot{\sigma}_{\mu\nu} + \sigma_{\mu\nu}^2 + (2\varphi + B\alpha)\sigma_{\mu\nu} - \frac{\sigma_{rs} \sigma^{rs}}{n-1} a_{\mu\nu}.$$

The trace is

$$\begin{aligned} 0 &= \dot{\sigma}_{\mu\nu} a^{\mu\nu} + \sigma_{\mu\nu}^2 a^{\mu\nu} + (2\varphi + B\alpha)\sigma_{\mu\nu} a^{\mu\nu} - \sigma_{rs} \sigma^{rs} \\ &= -\sigma_{\mu\nu} \dot{a}^{\mu\nu} \\ &= -\frac{1}{2} \dot{a}_{\mu\nu} \dot{a}^{\mu\nu} + \varphi a_{\mu\nu} \dot{a}^{\mu\nu} \\ &= -\frac{1}{2} \dot{a}_{\mu\nu} \dot{a}^{\mu\nu} - \varphi \dot{a}_{\mu\nu} a^{\mu\nu} \\ &= -\frac{1}{2} \dot{a}_{\mu\nu} \dot{a}^{\mu\nu} - 2(n-1)\varphi^2 \\ &= 2\sigma_{\mu\nu} \sigma^{\mu\nu}. \end{aligned}$$

Since $\sigma_{\mu\nu}$ is symmetric, the eigenvalues are real. The vanishing of the sum of squared eigenvalues means $\sigma_{ij} = 0$. \square

Proposition VI.2. *The property $a_{\mu\nu}(t, x) = a^2(t)g_{\mu\nu}^*(x)$ holds.*

Proof. In the comoving frame $u_\mu = 0$, then Eq.(12) gives $R_{\mu\nu\rho 0} = 0$. Its expression in Appendix shows that $\dot{a}_{\mu\nu}$ is the Codazzi tensor: $D_\mu \dot{a}_{\nu\rho} = D_\nu \dot{a}_{\mu\rho}$. Since $\sigma_{\mu\nu} = 0$, we obtain

$$\dot{a}_{\mu\nu}(t, x) = 2\varphi a_{\mu\nu}(t, x).$$

Then $a_{\nu\rho}D_\mu\varphi = a_{\mu\rho}D_\nu\varphi$; this is true if $\partial_\nu\varphi = 0$ i.e. φ only depends on time. Integration gives the warped expression $a_{\mu\nu}(t, x) = a^2(t)g_{\mu\nu}^*(x)$, where

$$\varphi(t) = \frac{\dot{a}}{a},$$

which is nothing else but the Hubble parameter of Friedman cosmology. □

VII. DISCUSSION AND CONCLUSIONS

In this paper, we discussed some properties of vacuum solutions of $f(R)$ gravity. Specifically, if the shear σ_{jk} vanishes, several formulae result simplified. In particular, the Weyl term in the Ricci tensor (14) cancels and the Ricci tensor becomes quasi-Einstein as in Eq. (4). Finally, the velocity has the simplest expression for the gradient (5). Furthermore, the vanishing of the Weyl tensor divergence implies $C_{jklm}u^m = 0$ and $E_{kl} \equiv C_{jklm}u^j u^m = 0$. In the comoving frame, $E_{kl} = 0$ and $a_{\mu\nu}(t, x) = a^2(t)g_{\mu\nu}^*(x)$ give a space-submanifold that is Einstein:

$$r_{\mu\nu} = \frac{r^*}{n-1}g_{\mu\nu}^*,$$

with $r^* = ra^2$ constant and

$$R = \frac{r^*}{a^2} + 2\xi + \varphi^2(n-1)(n-2). \quad (29)$$

The eigenvalue is $\xi = (n-1)(\varphi^2 + \dot{\varphi}) = (n-1)\frac{\ddot{a}}{a}$. The parameter α is evaluated as $\alpha = -\dot{R}$. The parameters α , R , ξ and φ can all be expressed in terms of the constant r^* and of derivatives of the scale parameter a^2 . The $f(R)$ gravity enters in the relations (20) and (21), that contain f and its derivatives in R up to f''' :

$$(n-1)f''\dot{R}\varphi = \xi f' - \frac{f}{2} \quad (30)$$

$$f''' \dot{R}^2 + f'' \ddot{R} + \xi f' = \frac{1}{n-1}(Rf' - \frac{f}{2}) \quad (31)$$

The first equation reproduces Eq.10 in [7] in vacuo ($\mu = 0$); the second equation is Eq.9 (with $p = 0$) plus $\frac{n-2}{n-1}$ times Eq.10 in [7].

According to these considerations, if the divergence of the Weyl tensor is zero, the vacuum solutions of $f(R)$ gravity are Generalized Friedman-Robertson-Walker space-times which, in 4-dimensions, reduce to the standard Friedman-Robertson-Walker solutions. This result holds, in particular, for $f'' \neq 0$ and then it generalizes the fact that $f(R)$ gravity in vacuum can be reduced to General Relativity plus a cosmological constant as often stated. In the present perspective, the properties of the Weyl tensor and the geodesic structure determine the solutions.

In 4-dimensions, where the FRW metric is obtained, the difference with ordinary gravity is in the equations for the evolution of the scale function: the Friedmann equations for $f(R) = R$ and equations (30), (31) for $f(R)$ with null divergence of the Weyl tensor. Therefore, the effect of $f(R)$ with Weyl constraint is to dress the original Friedmann equations with new geometric effects that yield the same form of metric and manifest in a different scale function.

There are cosmological spaces that elude this analysis, because of physical processes that do not allow the stringent requirement on the Weyl tensor to hold. For example, this is the case of de Sitter solutions derived in $f(R)$ gravity motivated by the inflation and the dark energy issues. In Ref.[26], the one-loop quantization approach is developed for a family of $f(R)$ gravity models and de Sitter universes are investigated, extending a similar earlier program for Einstein gravity. The authors adopt a generalized zeta regularization, and the one-loop effective action is obtained off-shell. In this framework, the (de)stabilization of the de Sitter background is obtained by quantum effects. In such

a context, the requirements of Theorem 1 are violated by quantum effects. In fact, they consider small fluctuations around a (de Sitter) maximally symmetric space with Riemann tensor

$$R_{ijklm}^{(0)} = \frac{R^{(0)}}{12} (g_{jl}^{(0)} g_{km}^{(0)} - g_{jm}^{(0)} g_{kl}^{(0)})$$

and metric $g_{ij}^{(0)}$. In this metric the Weyl tensor is zero. However, in a perturbed metric $g_{ij} = g_{ij}^{(0)} + h_{ij}$ the shape of the Ricci tensor (eq. 2.11 in [27]) in general prevents the validity of $\nabla_m C_{jkl}^m = 0$.

It is important to stress that one-loop effective actions are useful also for studying black hole nucleation rates and for providing reliable mechanisms capable of solving the cosmological constant problem.

In conclusion, the mathematical results reported here highlight the decisive effect of the Weyl condition in restricting the form of the vacuum solutions, and this should be of guidance in the study of realistic situations where physical processes are acting in the observable universe.

Acknowledgments

SC acknowledges the Istituto Nazionale di Fisica Nucleare (INFN), sezione di Napoli, *iniziative specifiche* MOON-LIGHT2 and QGSKY.

APPENDIX

Given the space-time metric

$$ds^2 = -dt^2 + a_{\mu\nu}(t, x) dx^\mu dx^\nu$$

let $\gamma_{\mu\nu}^\rho$, $r_{\mu\nu\rho}^\sigma$, $r_{\mu\nu}$ and r be the Christoffel symbols, the Riemann tensor, the Ricci tensor and the Ricci scalar of the space-submanifold at fixed t , and let D_μ be the covariant derivative with symbols $\gamma_{\mu\nu}^\rho$, and define $a^{\mu\nu} a_{\mu\rho} = \delta^\mu{}_\rho$. The related space-time quantities are:

- Christoffel symbols: $\Gamma_{ij}^k = \Gamma_{ji}^k = \frac{1}{2} g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij})$

$$\Gamma_{00}^0 = 0, \quad \Gamma_{\mu 0}^0 = 0, \quad \Gamma_{00}^\mu = 0,$$

$$\Gamma_{\mu\nu}^0 = \frac{1}{2} \dot{a}_{\mu\nu}, \quad \Gamma_{\mu 0}^\nu = \frac{1}{2} a^{\nu\rho} \dot{a}_{\mu\rho}, \quad \Gamma_{\mu\nu}^\rho = \gamma_{\mu\nu}^\rho$$

- Riemann tensor: $R_{ijkl}{}^m = \partial_k \Gamma_{jl}^m - \partial_j \Gamma_{kl}^m + \Gamma_{jl}^i \Gamma_{ik}^m - \Gamma_{kl}^i \Gamma_{ij}^m$

$$R_{\mu\nu\rho}{}^\sigma = r_{\mu\nu\rho}{}^\sigma + \frac{1}{4} a^{\sigma\lambda} (\dot{a}_{\mu\rho} \dot{a}_{\lambda\nu} - \dot{a}_{\nu\rho} \dot{a}_{\lambda\mu})$$

$$R_{\mu\nu\rho}{}^0 = \frac{1}{2} (\partial_\nu \dot{a}_{\mu\rho} - \partial_\mu \dot{a}_{\nu\rho}) + \frac{1}{2} (\gamma_{\mu\rho}^\sigma \dot{a}_{\nu\sigma} - \gamma_{\nu\rho}^\sigma \dot{a}_{\mu\sigma}) = \frac{1}{2} (D_\nu \dot{a}_{\mu\rho} - D_\mu \dot{a}_{\nu\rho})$$

$$R_{0\mu 0}{}^\nu = -\frac{1}{2} a^{\nu\rho} \ddot{a}_{\mu\rho} - \frac{1}{4} \dot{a}^{\nu\rho} \dot{a}_{\mu\rho}$$

$$R_{\mu 0\nu}{}^0 = \frac{1}{2} \ddot{a}_{\mu\nu} - \frac{1}{4} a^{\rho\sigma} \dot{a}_{\nu\sigma} \dot{a}_{\mu\rho}$$

- Ricci tensor: $R_{ij} = R_{ikj}{}^k$

$$R_{00} = -\frac{1}{2} a^{\mu\nu} \ddot{a}_{\mu\nu} - \frac{1}{4} \dot{a}^{\mu\nu} \dot{a}_{\mu\nu}$$

$$R_{\mu\nu} = r_{\mu\nu} - \frac{1}{2} \dot{a}_{\mu\rho} a^{\rho\sigma} \dot{a}_{\nu\sigma} + \frac{1}{2} \ddot{a}_{\mu\nu} + \frac{1}{4} \dot{a}_{\mu\nu} (a^{\rho\sigma} \dot{a}_{\rho\sigma})$$

- Curvature scalar $R = -R_{00} + a^{\mu\nu} R_{\mu\nu}$

$$R = r + a^{\mu\nu} \ddot{a}_{\mu\nu} + \frac{3}{4} \dot{a}^{\mu\nu} \dot{a}_{\mu\nu} + \frac{1}{4} (a^{\mu\nu} \dot{a}_{\mu\nu})^2$$

- Electric tensor $E_{\mu\nu} = C_{0\mu\nu 0}$

$$(n-2)E_{\mu\nu} = -r_{\mu\nu} + \frac{1}{2}(n-3)\ddot{a}_{\mu\nu} - \frac{1}{4}(n-4)a^{\rho\sigma}\dot{a}_{\rho\mu}\dot{a}_{\sigma\nu} - \frac{1}{4}\dot{a}_{\mu\nu}(a^{\rho\sigma}\dot{a}_{\rho\sigma}) + a_{\mu\nu}(R_{00} + \frac{R}{n-1})$$

-
- [1] A. A. Starobinsky, A New Type of Isotropic Cosmological Models Without Singularity, *Phys. Lett.* **91B** (1980) 99
- [2] S. Capozziello, Curvature quintessence, *Int. J. Mod. Phys. D* **11** (2002) 483
- [3] S. Nojiri and S. D. Odintsov, “Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models,” *Phys. Rept.* **505** (2011) 59
- [4] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution, *Phys. Rept.* **692** (2017) 1
- [5] S. Capozziello and V. Faraoni, “Beyond Einstein gravity”, *Fundamental theories of Physics*, Vol.170 (Springer, 2011)
- [6] S. Capozziello and M. De Laurentis, *Extended theories of gravity*, *Phys. Rep.* **509** n.4–5 (2011) 167–321.
- [7] S. Capozziello, C. A. Mantica and L. G. Molinari, Cosmological perfect fluids in $f(R)$ gravity, *Int. J. Geom. Meth. Mod. Phys.* **16** (2019), 1950008 (14 pp).
- [8] L. G. Molinari, A. Tacchini and C. A. Mantica, *On the uniqueness of a shear-vorticity-acceleration-free velocity field in space-times*, *Gen. Relativ. Gravit.* **51** (2019) 217 (13pp.)
- [9] S. Capozziello, P. Jovanovic, V. B. Jovanovic and D. Borka, Addressing the missing matter problem in galaxies through a new fundamental gravitational radius, *JCAP* **1706** (2017) 044
- [10] S. Capozziello, V. B. Jovanovic, D. Borka and P. Jovanovic, Constraining theories of gravity by fundamental plane of elliptical galaxies, *Phys. Dark Univ.* **29** (2020) 100573
- [11] S. Capozziello, A. Stabile and A. Troisi, Spherical symmetry in $f(R)$ -gravity, *Class. Quantum Grav.* **25** (2008) 085004
- [12] T. Clifton, *Spherically symmetric solutions to fourth-order theories of gravity*, *Class. Quantum Grav.* **23** n.24 (2006) 7445
- [13] V. Faraoni, *Cliftons spherical solution in $f(R)$ vacuo harbours a naked singularity*, *Class. Quantum Grav.* **26** (2009) 195013.
- [14] T. R. P. Caramés and E. R. B. de Mello, *Spherically symmetric vacuum solutions of modified gravity theory in higher dimensions*, *Eur. Phys. J. C* **64** (2009) 113.
- [15] A. Azadi, D. Momeni, and M. Nouri-Zonoz, *Cylindrical solutions in metric $f(R)$ gravity*, *Phys. Lett. B* **670** (2008) 210–214.
- [16] D. Momeni and H. Gholizade, *A note on constant curvature solutions in cylindrically symmetric metric $f(R)$ gravity*, *Int. J. Mod. Phys. D* **18** (2009) 1.
- [17] M. Sharif and M. Farasat Shamir, *Plane symmetric solutions in $f(R)$ gravity*, *Mod. Phys. Lett. A* **25** n.15 (2010) 1281–1288.
- [18] M. J. Amir and S. Sattar, *Locally rotationally symmetric vacuum solutions in $f(R)$ gravity*, *Int. J. Theor. Phys.* **53** (2014) 773–787.
- [19] M. Calzà, M. Rinaldi and L. Sebastiani, *A special class of solutions in $F(R)$ -gravity*, *Eur. Phys. J. C* **78** (2018) 178.
- [20] B.-Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, *Gen. Relativ. Gravit.* **46** 1833 (2014).
- [21] C. A. Mantica and L. G. Molinari, *Generalized RobertsonWalker spacetimes - a survey*, *Int. J. Geom. Meth. Mod. Phys.* **14** n.3 (2017) 1730001 (27 pp.).
- [22] C. A. Mantica and L. G. Molinari, *On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times*, *J. Math. Phys.* **57** 102502 (2016) (6 pp.).
- [23] C. A. Mantica and L. G. Molinari, *Jordan algebras of Riemann, Weyl and curvature compatible tensors*, arXiv:1910.03929 [math.dg]
- [24] C. A. Mantica and L. G. Molinari, *Extended Derdziński-Shen theorem for curvature tensors*, *Colloq. Math.* **128** n. 1 (2012) 1-6.
- [25] A. A. Coley and D. McManus, *On spacetimes admitting shear-free, irrotational, geodesic timelike congruences*, *Class. Quantum Grav.* **11** (1994) 1261–1282.
- [26] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, S. Zerbini, *One-loop $f(R)$ gravity in de Sitter universe*, *JCAP* **0502** 010, (2005).
- [27] P. Ginsparg and M. J. Perry. *Semiclassical perdurance of De Sitter space*, *Nucl. Phys. B* **222** (1983) 245.