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Special linear systems and syzygies

Alberto Alzati

Dipartimento di Matematica, Università di Milano
via C. Saldini 50, 20133-Milano (Italy)
E-mail: alzati@mat.unimi.it

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Abstract

Let X be the base locus of a linear system L of hypersurfaces in $\mathbb{P}^r(\mathbb{C})$. In this paper it is showed that the existence of linear syzygies for the ideal of X has strong consequences on the fibres of the rational map associated to L. The case of hyperquadrics is particularly addressed. The results are applied to the study of rational maps and to the Perazzo's map for cubic hypersurfaces.

1. Introduction

Let X be the base locus of a linear system of hypersurfaces in $\mathbb{P}^r(\mathbb{C})$. One can consider X as a projective scheme and the linear system is called special if X is smooth and irreducible. One also considers the rational map Φ induced by the linear system and studies the fibres of this map. It turns out that, in many cases, such fibres are linear. For instance when X has sufficiently independent linear syzygies. So that one can also consider the linear system special when this fact occurs.

Varieties with many "good" linear syzigies were recently considered by many authors (see [2, 6, 13, 14, 16]) for different goals. In this paper it is showed that the existence of linear syzigies has strong consequences on the fibre of Φ also in the cases in which the fibres are not linear (Theorem 1). On the other hand, when the Koszul syzygies of X are generated by the linear ones, Φ is completely and easily described (Proposition 3) as it was shown by Vermeire in [16]. In § 4 some applications of these results are given to the study of rational maps.

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In § 5 linear systems of hyperquadrics are considered. By using an old theorem of Degoli (see Theorem 2 and [3]) and a recent theorem of Landsberg (see [8]) it is possible to get other results about the fibres of Φ (Theorem 3). An application of Theorem 3 in Corollary 3 proves that the fibres of the Perazzo's map of a cubic hypersurface with vanishing Hessian, not a cone, are linear, yielding a shorter proof of the original result of Perazzo (see [9]).

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2. Notation

$\mathbb{P}^r$: r-dimensional projective space on $\mathbb{C}$

$\underline{x}$: column coordinates of a point in $\mathbb{P}^r$

M_t : transpose of the matrix M

$\mathcal{L}$: vector subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ for some $d \geq 2$

α : $\dim(\mathcal{L}) - 1$

$|\mathcal{L}| : \mathbb{P}(\mathcal{L})$

X : base locus of Φ

Φ : rational map from $\mathbb{P}^r$ to $\mathbb{P}^\alpha := \mathbb{P}(\mathcal{L}^*)$, induced by the linear system $|\mathcal{L}|$ in $\mathbb{P}^r$

$Z : Im(\Phi)$

$\Phi^{-1}(E)$: for any subset $E \subseteq \mathbb{P}^\alpha$ it is the closure in $\mathbb{P}^r$ of the set of points $P \in \mathbb{P}^r \backslash X$ such that $\Phi(P) \in E$

Φ_P : for any $P \in \mathbb{P}^r \backslash X$ it is $\Phi^{-1}(\Phi(P))$ i.e. the closure in $\mathbb{P}^r$ of the fibre of Φ which is contained in $(\mathbb{P}^r \backslash X)$

$|\mathcal{L}|$ is *homaloidal* if $\Phi^{-1}(Q)$ is a point for any generic point $Q \in Z$

$|\mathcal{L}|$ is *subhomaloidal* if $\Phi^{-1}(Q)$ is a linear space for any generic point $Q \in Z$

$|\mathcal{L}|$ is *completely subhomaloidal* if $\Phi^{-1}(Q)$ is a linear space for any point $Q \in Z$

$\rho(P)$: rank of the Jacobian matrix of $|\mathcal{L}|$ evaluated at $P \in \mathbb{P}^r$

ρ : rank of the Jacobian matrix of $|\mathcal{L}|$ evaluated at the generic point of $\mathbb{P}^r$

Λ : vector subspace of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2))$

$Con(P)$: linear space in $\mathbb{P}^r$ consisting of all points which are conjugated to $P \in \mathbb{P}^r$ with respect to all quadrics of Λ

L_P: linear space in $\mathbb{P}^r$ generated by $P \in \mathbb{P}^r$ and $Con(P)$

$Ann(P)$: linear subsystem of Λ given by quadrics which are singular at a point $P \in \mathbb{P}^r$

$Singloc(\Lambda')$: intersection of all singular loci of quadrics belonging to a subspace $\Lambda' \subseteq \Lambda$

$Baseloc\Lambda^*(F^\perp)$: subscheme of $\mathbb{P}^r$ defined by equations $f_{k+1} = \cdots = f_\alpha = 0$ where $\Lambda = \{f_0, ..., f_\alpha\}$ is a linear system of quadrics in $\mathbb{P}^r$ and F is the linear space in $\mathbb{P}^\alpha$ defined by: $y_{k+1} = \cdots = y_\alpha = 0$ where $y_0, ..., y_\alpha$ are the coordinates in $\mathbb{P}^\alpha$

$Sm(W)$: set of smooth points of a scheme W

W_{red} : reduced structure of a scheme W.

3. Linear systems and linear syzygies

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. We fix a base $f_0, f_1, ..., f_\alpha$ for $\mathcal{L}$ in order to define a rational map $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$. We always assume that Φ is not constant. Let X be the base locus of Φ. Let $M = (\varphi_{i,j}) = [M_1|M_2]$ be the $(\alpha+1, q')$ matrix of syzygies of the ideal I_X of X. M_1 is the $(\alpha+1, q)$ submatrix of the linear syzygies. We define $rk(M_1)$ as $rk[M_1(\underline{x})]$ for generic $\underline{x} \in \mathbb{P}^r$. In this section we assume that $M_1 \neq 0$ for any considered $|\mathcal{L}|$. By construction $\mathbb{P}^\alpha$ is the linear span of Z.

For any point of $\mathbb{P}^\alpha$ whose coordinates are $\underline{a}$ we put (see also [6]):

$$W_{\underline{a}} = \Big\{\underline{x} \in \mathbb{P}^r | \sum_{i=0}^{\alpha} a_i \varphi_{ij}(\underline{x}) = 0 \ \forall j = 1, ..., q'\Big\}$$

$$W^1_{\underline{a}} = \Big\{\underline{x} \in \mathbb{P}^r | \sum_{i=0}^{\alpha} a_i \varphi_{ij}(\underline{x}) = 0 \ \forall j = 1, ..., q\Big\}.$$

To any linear syzygy we can associate a $(\alpha+1, r+1)$ matrix B_j $j = 1, ..., q$ such that $(B_j\underline{x})_t \underline{F}(\underline{x}) = 0$ where $\underline{F}(\underline{x})$ is the column vector of coordinates: $f_0(\underline{x}), ..., f_\alpha(\underline{x})$. In this notation we have:

$$\begin{aligned} W^1_{\underline{a}} &= \{\underline{x} \in \mathbb{P}^r | \underline{a}_t B_j \underline{x} = 0 \ \forall j = 1, ..., q\} \quad \text{and we define:} \\ E(\underline{x}) &= \{\underline{y} \in \mathbb{P}^\alpha | \underline{y}_t B_j \underline{x} = \underline{0} \ \forall j = 1, ..., q\} \\ N &= \{\underline{y} \in \mathbb{P}^\alpha | \underline{y}_t B_j = \underline{0} \ \forall j = 1, ..., q\} \quad \text{and} \\ M_s &= \{\underline{x} \in \mathbb{P}^r | rk[M_1(\underline{x})] \leq s\}. \end{aligned}$$

Moreover for any point $P \equiv \underline{x} \in \mathbb{P}^r$ we can consider the linear subsystem $\mathcal{L}_P = \langle B_1\underline{x}, ..., B_q\underline{x}\rangle \subseteq \mathcal{L}$ such that $\dim(\mathcal{L}_P) = rk[M_1(\underline{x})]$. Note that $\mathcal{L} = \mathcal{L}_P \oplus \mathcal{L}'_P$ and $|\mathcal{L}_P| = E(\underline{x})^*$ is the dual space of $E(\underline{x})$ in the dual projective space of $\mathbb{P}^\alpha$.

Remark 1 For any $\underline{a} \in N$ $W^1_{\underline{a}} = \mathbb{P}^r$. For any $\underline{x} \in M_0$ $[E(\underline{x})] = \mathbb{P}^\alpha$. For any $\underline{x} \in \mathbb{P}^r$ $\dim[E(\underline{x})] = \alpha - rk[M_1(\underline{x})]$ and $N \subseteq E(\underline{x})$. For any $\underline{x} \in \mathbb{P}^r \backslash X$, $\underline{F}(\underline{x}) \in E(\underline{x})$ so that if $rk(M_1) = \alpha$ then $\underline{F}(\underline{x}) = E(\underline{x})$ for generic $\underline{x} \in \mathbb{P}^r \backslash X$.

The previous defined linear spaces in $\mathbb{P}^r$ and the matrix M are linked by the following propositions. First of all we have the

Lemma 1

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\mathcal{L}|$. Let Q be any point in Z, whose coordinates are $\underline{a}$, such that $Q = \Phi(P)$. Then $\Phi_P \subseteq W_{\underline{a}} \subseteq W^1_{\underline{a}}$.

Proof. We can always assume that $\underline{a} \equiv (1 : 0 : 0 : \cdots : 0)$.

$\Phi_P = \overline{\{\underline{x} \in \mathbb{P}^r | f_1(\underline{x}) = f_2(\underline{x}) = \cdots = 0, f_0(\underline{x}) \neq 0\}}$ while
$W_{\underline{a}} = \{\underline{x} \in \mathbb{P}^r | \varphi_{0j}(\underline{x}) = 0 \ \forall j = 1, ..., q'\}$. As :
$\sum_{i=0}^{\alpha} \varphi_{ij}(\underline{x}) f_i(\underline{x}) \equiv 0 \ \forall j = 1, ..., q'$ we have that for any $\underline{h}$ such that
$f_1(\underline{h}) = f_2(\underline{h}) = \cdots = 0, \ f_0(\underline{h}) \neq 0,$ we have : $\varphi_{0j}(\underline{h}) = 0 \ \forall j = 1, ..., q'$.

Hence $\{\underline{x} \in \mathbb{P}^r | f_1(\underline{x}) = f_2(\underline{x}) = \cdots = 0, f_0(\underline{x}) \neq 0\} \subseteq W_{\underline{a}}$ and $\Phi_P \subseteq \overline{W_{\underline{a}}} = W_{\underline{a}}$. The other inclusion is obvious. □

Then we can consider the $(\alpha+1, r+1)$ Jacobian matrix $J_{|\underline{x}}$ of Φ, evaluated at a point $P \equiv \underline{x} \in \mathbb{P}^r \backslash X$, the induced linear map and its dual in the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \ker(J_{|\underline{x}}) & \to & \mathbb{C}^{r+1} & \underset{J_{|\underline{x}}}{\to} & \mathbb{C}^{\alpha+1} & \to & coker(J_{|\underline{x}}) & \to & 0 \\ 0 & \to & \ker[(J_{|\underline{x}})_t] & \to & (\mathbb{C}^{\alpha+1})^* & \underset{(J_{|\underline{x}})_t}{\to} & (\mathbb{C}^{r+1})^* & \to & coker[(J_{|\underline{x}})_t] & \to & 0 \end{array}.$$

If we break down the sequences and we consider the linear maps induced by the matrices B_j, we can also consider the following non commutative diagram, for any $j = 1, ..., q$:

$$\begin{array}{ccccccccc} 0 & \to & \ker(J_{|\underline{x}}) & \to & \mathbb{C}^{r+1} & \to & Im(J_{|\underline{x}}) & \to & 0 \\ & & & & \downarrow B_j & & \downarrow (B_j)_t & & \\ 0 & \to & \ker[(J_{|\underline{x}})_t] & \to & (\mathbb{C}^{\alpha+1})^* & \to & Im[(J_{|\underline{x}})]_t & \to & 0\,. \end{array}$$

Now we have the following:

Proposition 1

Let $P \equiv \underline{x}$ be any point in $\mathbb{P}^r \backslash X$, let $P' = \Phi(P) \equiv \underline{a}$. Then the image of the vector subspace $\mathcal{L}_P = \langle B_1\underline{x}, ..., B_q\underline{x}\rangle \subseteq \mathcal{L}$ under $(J_{|\underline{x}})_t$ is such that $W^1_{\underline{a}}$ is the projectivization of its dual, moreover $W^1_{\underline{a}}$ is the tangent space to the base locus of $|\mathcal{L}_P|$ at P.

Proof. By the syzygies properties we have that $(B_j\underline{x})_t\underline{F}(\underline{x}) = 0$ for any j, where $\underline{F}(\underline{x})$ is the column vector of the coordinates $f_0(\underline{x}), ..., f_\alpha(\underline{x})$. By taking the partial derivatives we get:

$$(J_{|\underline{x}})_t B_j\underline{x} + (B_j)_t\underline{F}(\underline{x}) = 0 \ \forall j = 1, ..., q \qquad (*)$$

so that for any $j = 1, 2, ..., q$ we have: $(J_{|\underline{x}})_t B_j\underline{x} = -(B_j)_t\underline{a}$ by $(*)$. Hence the projectivization of the dual of the image of $\mathcal{L}_P$ is the linear space in $\mathbb{P}^r$defined by: $\underline{x}_t(B_j)_t\underline{a} = 0$, $j = 1, 2, ..., q$, i.e. it is $W^1_{\underline{a}}$.

Now let us recall that $\beta := \dim(\mathcal{L}_P) = rk[M_1(\underline{x})] \geq 1$ because we are assuming $M_1 \neq 0$. We can always choose a coordinate system in $\mathbb{P}^\alpha$ such that $\mathcal{L}_P = \langle f_0, ..., f_{\beta-1}\rangle$, hence $\langle B_1\underline{x}, ..., B_q\underline{x}\rangle$ is generated by the first β elements of the standard base of $\mathbb{C}^{\alpha+1} := \langle [1,0,...,0]_t, [0,1,0,...,0]_t, ..., [0,...,1,...,0]_t\rangle$. In this case the linear space $W^1_{\underline{a}}$ is defined by hyperplanes whose coefficients are the first β rows of $J_{|\underline{x}}$ and they define exactly the tangent space to the base locus of $|\mathcal{L}_P|$ at P. □

Corollary 1

If $rk(M_1) = \alpha$ then $|\mathcal{L}|$ is subhomaloidal.

Proof. In fact, for generic $P \equiv \underline{x} \in \mathbb{P}^r \backslash X$, we have that the base locus of $|\mathcal{L}_P|$ is reducible into X and Φ_P, hence it coincides with Φ_P locally near P and it is smooth at P. By the previous proposition we have $W^1_{\underline{a}} = T_P(\Phi_P)$, where $\Phi(P) \equiv \underline{a}$. Hence

$W^1_{\underline{a}} = \Phi_P$, because we always have $W^1_{\underline{a}} \supseteq \Phi_P$ by Lemma 1. This means that the generic fibre of Φ is a linear space of $\mathbb{P}^r$. □

From the above corollary we have that $rk(M_1) = \alpha$ implies that $|\mathcal{L}|$ is subhomaloidal. About the converse we have the following (essentially in [6]):

Proposition 2

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\mathcal{L}|$. Assume that $|\mathcal{L}|$ is subhomaloidal and that, for generic $P \in \mathbb{P}^r$, $W^1_{\underline{a}} = \Phi_P$ with $\Phi(P) \equiv \underline{a}$. Then $rk[M_1] = \alpha$.

Proof. We proceed by contradiction. Let us assume that, for generic P, $rk[M_1(P)] < \alpha$. As the union of $W^1_{\underline{a}}$ covers $\mathbb{P}^r$ we have also that, for fixed P and $\underline{a}$ and for generic $\underline{x}_0 \in W^1_{\underline{a}}$, $rk[M_1(\underline{x}_0)] < \alpha$. Then there are at least two rows in $M_1(\underline{x}_0)$ which are linear combination of the others. By choosing another base for $\mathcal{L}$ we can always suppose that the first two rows of $M_1(\underline{x}_0)$ are 0, i.e. $\varphi_{0j}(\underline{x}_0) = \varphi_{1j}(\underline{x}_0) = 0\ \forall j = 1, ..., q$.

Let us consider the points $\underline{a}(t) \equiv (1 : t : 0 : \cdots : 0)$ in $\mathbb{P}^\alpha$. Note that $\forall t \in \mathbb{C}$ $\underline{x}_0 \in W^1_{\underline{a}(t)}$. For generic $t \in \mathbb{C}$ we have that $\underline{a}(t)$ is generic in $\mathbb{P}^\alpha$. Hence the fibre over $\underline{a}(t)$ is $W^1_{\underline{a}(t)}$ for generic t and $\Phi(\underline{x}_0) = \underline{a}(t)$ for such t. But this is not possible. □

Remark 2 Note that the previous propositions tell us that $rk[M_1] = \alpha$ if and only if $M_1 \neq 0$ and the generic fibre of Φ is $W^1_{\underline{a}}$ (hence a linear space). However $|\mathcal{L}|$ might be subhomaloidal with $rk[M_1] < \alpha$, in this case $W^1_{\underline{a}} \supsetneq \Phi_P$.

Now we want to prove the following:

Theorem 1

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\mathcal{L}|$. Let $P \equiv \underline{x}$ be any point in $\mathbb{P}^r \backslash X$ and let us consider $W_{E(\underline{x})} := \{\underline{x}' \in \mathbb{P}^r |\ E(\underline{x}') \supseteq E(\underline{x})\}$. Then $W_{E(\underline{x})}$ is a linear space of $\mathbb{P}^r$, contained in $\Phi^{-1}(E(\underline{x}))$, such that the restriction of Φ to it is dominant over $E(\underline{x}) \cap Z$.

Proof. Let us consider a point $P \equiv \underline{x}$ in $\mathbb{P}^r \backslash X$ such that $s = rk[M_1(\underline{x})]$ is maximal. We have $\dim[E(\underline{x})] = \alpha - s$ and $\alpha \geq s$ because $E(\underline{x}) \supseteq \underline{F}(\underline{x})$. Let us consider $W_{E(\underline{x})} = \{\underline{x}' \in \mathbb{P}^r |\ E(\underline{x}') \supseteq E(\underline{x})\} = \{\underline{x}' \in \mathbb{P}^r |\ \underline{a}_t B_j \underline{x}' = 0\ \forall j = 1, ..., q$ for any $\underline{a}$ such that $\underline{a}_t B_j \underline{x} = 0\ \forall j = 1, ..., q\}$ which is the intersection of all $W^1_{\underline{a}}$ for $\underline{a} \in E(\underline{x})$, therefore it is a linear space, containing P. Moreover we can choose a coordinate system in $\mathbb{P}^\alpha$ such that $E(\underline{x})$ has equations: $y_{\alpha-s+1} = y_{\alpha-s+2} = \cdots = y_\alpha = 0$. In this case $W_{E(\underline{x})} = \{\underline{x}' \in \mathbb{P}^r |\ \underline{v}_t B_j \underline{x}' = 0\ \forall j = 1, ..., q$ for the first $\alpha+1-s$ vectors $\underline{v}_t = [1, 0, ..., 0]$, $[0, 1, 0, ..., 0], [0, ..., 0, 1, 0, ..., 0]$ of the standard base of $\mathbb{C}^{\alpha+1}\}$.

As $s = rk(M_1)$ we have that, for generic $\underline{x}' \in W_{E(\underline{x})}$, $s = rk[M_1(\underline{x}')]$ too and the first $\alpha + 1 - s$ rows of the matrix $M_1(\underline{x}')$ are null, because $\underline{x}' \in W_{E(\underline{x})}$. It follows that the (s, q) submatrix $M_1(\underline{x}')^\# := M_1(\underline{x}') \backslash$ (first $\alpha + 1 - s$ rows) has rank equal to $rk[M_1(\underline{x}')] = s$, i.e. it has maximal rank. Hence $\underline{w}_t M_1(\underline{x}')^\# = \underline{0}$ implies $\underline{w} = \underline{0}$ for any $\underline{w} \in \mathbb{C}^s$.

As $[\underline{F}(\underline{x}')]_t M_1(\underline{x}') = \underline{0}$ by the syzygies properties, we get $[\underline{F}(\underline{x}')^\#]_t M_1(\underline{x}')^\# = \underline{0}$ where $\underline{F}(\underline{x}')^\# := \underline{F}(\underline{x}') \backslash$ (first $\alpha+1-s$ coordinates). It follows that $\underline{F}(\underline{x}')^\# = \underline{0}$, but it means that $\underline{F}(\underline{x}') \in E(\underline{x})$, i.e. Φ maps the generic point of $W_{E(\underline{x})}$ into $E(\underline{x})$. Therefore in $\mathbb{P}^r \backslash X$ there exists an open set $A \subseteq W_{E(\underline{x})}$ such that $\overline{A} = W_{E(\underline{x})}$ and $\Phi(A) \subseteq E(\underline{x})$, so that: $\Phi(W_{E(\underline{x})}) = \Phi(\overline{A}) \subseteq \overline{\Phi(A)} \subseteq \overline{E(\underline{x})} = E(\underline{x})$. Hence: $W_{E(\underline{x})} \subseteq \Phi^{-1}(E(\underline{x}))$.

If the union U of all $W_{E(\underline{x})}$, for $rk[M_1(\underline{x})] = rk(M_1)$, covers all $\mathbb{P}^r \backslash X$ we are done. If not, we can choose a point $R \equiv \underline{k} \notin U$, $R \notin X$, such that $s' = rk[M_1(\underline{k})] < rk(M_1)$ is maximal among points outside U (note that at some points of U $rk[M_1(\underline{x})]$ may be less than s). We can repeat the previous argument for R and we have a $W_{E(\underline{k})} \subseteq \Phi^{-1}(E(\underline{k}))$. Now $W_{E(\underline{k})} \subseteq M_{s'}$, because for any $\underline{k}' \in W_{E(\underline{k})}$ we have that $E(\underline{k}') \supseteq E(\underline{k})$, hence $rk[M_1(\underline{k}')] \leq rk[M_1(\underline{k})]$. If $\dim(M_{s'}) = 0$ we are done, if not we can argue as in the previous case because, as $W_{E(\underline{k})} \subseteq M_{s'}$, all points $H \equiv \underline{h}$ of $W_{E(\underline{k})}$ are such that $rk[M_1(\underline{h})] \leq s'$ and because $rk[M_1(\underline{k})] = s'$ we have that $rk[M_1(\underline{h})] = s'$ for the generic point $H \equiv \underline{h}$ of $W_{E(\underline{k})}$. The conclusion is that $W_{E(\underline{k})} \subseteq \Phi^{-1}(E(\underline{k}))$ (and $W_{E(\underline{k})} \cap U = \emptyset$). Now if the union of all $W_{E(\underline{k})}$ is $\mathbb{P}^r \backslash X$ we are done, otherwise we can proceed by considering a point as R outside this union and so on.

Now let us consider the restriction of Φ to $W_{E(\underline{x})}$ and let us call it Ψ. There is an open set in $W_{E(\underline{x})}$ such that for any $\underline{x}'$ belonging to it $E(\underline{x}') = E(\underline{x})$ and there is an other open set such that for any $\underline{x}'$ belonging to it the dimension of the fibre of Φ passing through it is $r + 1 - \rho(\underline{x}')$. Hence, by changing the choice of P, if necessary, we can always assume that $\dim[\Phi^{-1}(E(\underline{x})] = \dim[E(x)] + r + 1 - \rho(P)$. We can always choose a coordinate system in $\mathbb{P}^\alpha$ such that $\mathcal{L}_P = \langle f_0, ..., f_{\beta-1} \rangle$ where $\beta := rk[M_1(\underline{x})]$; then Ψ is defined by $f_\beta, ..., f_\alpha$ and the Jacobian matrix of Ψ at P is given by the last $\alpha + 1 - \beta$ rows of $J_{|\underline{x}}$. The rank of the submatrix of $J_{|\underline{x}}$ given by the first β rows is $cod[W^1_{\underline{a}}]$ and, by Proposition 1, $\dim[W^1_{\underline{a}}] \geq \dim[\Phi^{-1}(E(\underline{x})]$, so that $cod[W^1_{\underline{a}}] \leq r - (\dim[E(\underline{x})] + r - \rho(P) + 1) = \rho(P) - \dim[E(\underline{x})] - 1$, because $\dim[\Phi^{-1}(E(\underline{x}))] = \dim[E(\underline{x})] + (r - \rho(P) + 1)$. It follows that the rank of the submatrix of $J_{|\underline{x}}$ given by the last $\alpha + 1 - \beta$ rows is $\rho(P) - (\rho(P) - \dim[E(\underline{x})] - 1) = \dim[E(\underline{x})] + 1$ at least. Hence Ψ is dominant onto $E(\underline{x}) \cap Z$. □

About the linear syzygies of a variety X we recall that X is said to have the K_d property if X is the schematically intersection of degree d forms and every Koszul syzygy of X is generated by the linear ones. Now we have the following proposition, which is essentially a translation in our language of some results contained in [16, 14]:

Proposition 3

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\mathcal{L}|$. Let X be the base locus of $|\mathcal{L}|$. Assume that X has the K_d property, then for any $P \in \mathbb{P}^r \backslash X$, Φ_P is a linear space of dimension $r + 1 - \rho(P)$ i.e. $|\mathcal{L}|$ is completely subhomaloidal. Moreover $\Phi_P = W^1_{\Phi(P)}$ and $rk[M_1(P)] = \alpha$ for any $P \in \mathbb{P}^r \backslash X$.

Proof. Let us consider $G := \{(\underline{x}, \underline{y}) \in \mathbb{P}^r \times \mathbb{P}^\alpha | \ f_i(\underline{x}) y_j - f_j(\underline{x}) y_i \ i, j = 0, ..., \alpha\}$ such that $G \cap [(\mathbb{P}^r \backslash X) \times \mathbb{P}^\alpha]$ is the graph of Φ. Let us consider $G' := \{(\underline{x}, \underline{y}) \in \mathbb{P}^r \times \mathbb{P}^\alpha | \ \underline{y}_t B_k \underline{x} = 0, \ k = 1, ..., q\}$. In our assumptions $\forall i, j = 0, ..., \alpha$, with $i \neq j$, we have that:

$$\begin{bmatrix} \dots \\ -f_j(\underline{x}) \\ \dots \\ f_i(\underline{x}) \\ \dots \end{bmatrix} = \sum_{k=1}^{q} \omega_{ij}^{k}(\underline{x}) \begin{bmatrix} l_k^0(\underline{x}) \\ \dots \\ \dots \\ \dots \\ l_k^\alpha(\underline{x}) \end{bmatrix}$$

where $\underline{l}_k(\underline{x})$ are the linear syzygies of X and $\omega_{ij}^k(\underline{x})$ are suitable forms of degree $d-1$. Now let us consider the product with $\underline{y}_t$. We get:

$$f_i(\underline{x})y_j - f_j(\underline{x})y_i = \sum_{k=1}^{q} \omega_{ij}^k(\underline{x})(\underline{y}_t B_k \underline{x}) \ \forall i,j = 0,...,\alpha \ , \ i \neq j,$$

therefore we have that $G' \subseteq G$.

Note that $G' \neq G$, because in general G' does not contain X (for instance if M_0 is empty). From the inclusion $G' \subseteq G$ it follows that $M_0 \subset X$ and $N = \emptyset$ as we are assuming that Φ is not constant. As $(\underline{x}, \underline{y}) \in [(\mathbb{P}^r \backslash X) \times Z]$ belongs to G' if and only if $\underline{y} = \Phi(\underline{x})$, we get that $G' \cap [(\mathbb{P}^r \backslash X) \times \mathbb{P}^\alpha] = G \cap [(\mathbb{P}^r \backslash X) \times \mathbb{P}^\alpha]$, i.e. an easy description of the graph of Φ (and of its closure in $\mathbb{P}^r \times \mathbb{P}^\alpha$). Now let Q be any point in Z, let $\underline{a}$ be its coordinates and let P be a point in $\mathbb{P}^r \backslash X$ such that $\Phi(P) = Q$. $\Phi_P \subseteq W_{\underline{a}}^1$, on the other hand now the equations of Φ_P are exactly those of $W_{\underline{a}}^1$. Hence $\Phi_P = W_{\underline{a}}^1$ and it has dimension $r + 1 - \rho(P)$. Moreover for any $P \equiv \underline{x} \in \mathbb{P}^r \backslash X$ we have only one $Q = \Phi(P) \equiv \underline{y}$ such that $(\underline{x}, \underline{y}) \in G'$, hence $0 = \dim[E(\underline{x})] = \alpha - rk[M_1(\underline{x})]$. □

Corollary 2

Let $|\mathcal{L}|$ be a linear system in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\mathcal{L}|$. Let X be the base locus of $|\mathcal{L}|$. Assume that X has the K_d property. If $\alpha \geq r + 1$, (or $\alpha \leq r$ with $q \geq r + 1$), then $Z := Im(\Phi)$ is a determinantal variety in $\mathbb{P}^\alpha$given by the vanishing of the $(r + 1, r + 1)$ minors of a matrix of linear forms. If $\alpha \leq r$ with $q \leq r$ then Φ is surjective. Moreover $N = \emptyset$.

Proof. By the Proposition 3 we know that the graph of Φ is

$$\{(\underline{x}, \underline{y}) \in \mathbb{P}^r \times \mathbb{P}^\alpha | \underline{y}_t B_j \underline{x} = 0, \quad j = 1, ..., q\} \cap [(\mathbb{P}^r \backslash X) \times \mathbb{P}^\alpha]$$

so that a point $Q \equiv \underline{a}$ belongs to Z only if the linear system: $\underline{a}_t B_j \underline{x} = \underline{0} \ \forall j = 1, ..., q$ has some solution. The matrix A of this linear system is of type $(q, r + 1)$. The K_d property implies that $q \geq \alpha$ by Propositions 2 and 3. If $\alpha \geq r + 1$ the system has solutions if and only if $rk(A) \leq r$ (see also [11]). The same is true if $q \geq r+1$. If $\alpha \leq r$ with $q \leq r$ then the system has always solutions and Φ is surjective. If $N \neq \emptyset$ then it would exists a point in $\mathbb{P}^\alpha$ whose fibre would be $\mathbb{P}^r \backslash X$ and this is not possible as we always assume that Φ is not constant. □

4. Some applications and examples

In this section we want to give some applications to the study of some rational and birational map.

EXAMPLE 1 Let X be the non minimal $K3$ surface of degree 7 in $\mathbb{P}^4$ whose ideal sheaf has the following resolution (see [4, p. 224]):

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \oplus \mathcal{O}_{\mathbb{P}^4}(-4) \to \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 3} \to \mathcal{I}_X \to 0.$$

The vector bundle map is given by the transpose of the following matrix:

$$\begin{bmatrix} q_1 & q_2 & q_3 \\ l_1 & l_1 & l_3 \end{bmatrix}$$

where q_i and l_i are general forms of degree 2 and 1 respectively.

It is always possible to choose a coordinate system $(x : y : z : w : u)$ in $\mathbb{P}^4$ such that the three linear forms are: x, y, z, so that the ideal of X is generated by these cubics: $f_0 = xq_2 - yq_1$, $f_1 = xq_3 - zq_1$, $f_2 = yq_3 - zq_2$. In this case the matrix M is the transpose of the following one:

$$\begin{bmatrix} z & -y & x \\ q_3 & -q_2 & q_1 \end{bmatrix}$$

and we have only one linear syzygy. Now if we consider the rational map $\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ we have that it is surjective and the general fibre has dimension 2. By choosing coordinates $(a : b : c)$ in $\mathbb{P}^2$, we get the only bilinear relation: $az - by + cx = 0$. It follows that N is empty and for any $P \equiv (\overline{x} : \overline{y} : \overline{z} : \overline{w} : \overline{u}) \in \mathbb{P}^4 \backslash X$, $rk[M_1(P)] = 1 = rk(M_1)$, hence $E(P)$ is always a line: $\overline{z}a - \overline{y}b + \overline{x}c = 0$, $W_{E(P)} = \{P' \in \mathbb{P}^4 | E(P') = E(P)\}$ is always a plane: $\overline{x}y - \overline{y}x = \overline{x}z - \overline{z}x = \overline{y}z - \overline{z}y = 0$ and $W^1_{\Phi(P)}$ is always a linear space of dimension 3. Note that the only cubic in $|\mathcal{L}_P|$ is given by: $\overline{z}f_0 - \overline{y}f_1 + \overline{x}f_2 = (\overline{y}z - \overline{z}y)q_1 - (\overline{x}z - \overline{z}x)q_2 + (\overline{x}y - \overline{y}x)q_3 = 0$ so that it coincides with its base locus, moreover this cubic is also $\Phi^{-1}[E(P)]$ and it obviously contains $W_{E(P)}$.

If we consider the restriction Ψ of Φ to $W_{E(P)} \simeq \mathbb{P}^2$ we get a rational surjective map $\Psi : \mathbb{P}^2 \dashrightarrow E(P) \simeq \mathbb{P}^1$ given by a pencil of conics. In fact $W_{E(P)}$ cuts X along the fixed line $L : x = y = z = 0$ (independent from P) and the base locus of the following pencil of quadrics: $\langle \overline{x}q_2 - \overline{y}q_1, \overline{x}q_3 - \overline{z}q_1, \overline{y}q_3 - \overline{z}q_2 \rangle$. L is the exceptional divisor of the blowing up at one point of the minimal model, in fact X is the projection of a minimal degree 8 $K3$ surface in $\mathbb{P}^5$ from one of its points.

If we consider any point $(a : b : c)$ in the target $\mathbb{P}^2$ the union of the fibre over it and X is given by the intersection of the following 3 non independent quadrics:

$$\begin{aligned} &(az - by)q_1 + bxq_2 - axq_3 = 0 \\ &-cyq_1 + (cx + az)q_2 - ayq_3 = 0 \\ &-czq_1 + bzq_2 + (cx - by)q_3 = 0. \end{aligned}$$

On the other hand the fibre is contained in the hyperplane $W^1_{(a:b:c)}$ of equation: $cx - by + az = 0$. So that, by intersecting it with the quadrics and by eliminating L, which is contained in X, we get that the fibre is the surface quadric whose equations are:

$$\begin{aligned} &cx - by + az = 0 \\ &cq_1 - bq_2 + aq_3 = 0. \end{aligned}$$

Note that this quadric is exactly $W_{(a:b:c)}$.

EXAMPLE 2 Let us consider the vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1$. The tautological line bundle T in $\mathbb{P}(\mathcal{E})$ gives rise to a morphism whose image is a cone C in $\mathbb{P}^6$ over the Segre product $\mathbb{P}^1 \times \mathbb{P}^2$. We can choose a coordinate system $(a : b : c : d : e : f : g)$ in $\mathbb{P}^6$ such that C is given by: $df - gc = bg - de = ce - bf = 0$. Let F be the linear class of a fibre in $\mathbb{P}(\mathcal{E})$, then the intersection of two generic elements belonging to $|2T - F|$ is a smooth degree 8 and genus 3 surface X in $\mathbb{P}^6$. X can be also considered as the blow up of $\mathbb{P}^2$ at 8 points in general position, embedded in $\mathbb{P}^6$ by the linear systems of plane quartics passing through the 8 points (see [15, 1], see also [7])

We can always assume that two fixed fibres of $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}^6$ have equations: F_1) $e = f = g = 0$ and F_2) $b = c = d = 0$. So that the intersection of two generic elements of $|2T - F|$ linearly equivalent to $2T - F_1$ and $2T - F_2$ is given by the following equations:

$$\begin{aligned} &eL + fM + gN = bL + cM + dN = 0 \\ &eL' + fM' + gN' = bL' + cM' + dN' = 0 \\ &df - gc = bg - de = ce - bf = 0. \end{aligned}$$

where L, M, N, L', M', N' are general linear forms in $\mathbb{P}^6$ (see [15]).

It is easy to see that the matrix M_1 is the following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -b & c & -d \\ 0 & 0 & 0 & 0 & 0 & e & -f & g \\ 0 & 0 & -b & c & -d & 0 & 0 & 0 \\ 0 & 0 & e & -f & g & 0 & 0 & 0 \\ b & -e & 0 & N' & M' & 0 & N & M \\ -c & f & -N' & 0 & L' & -N & 0 & L \\ d & -g & -M' & -L' & 0 & -M & -L & 0 \end{bmatrix}$$

see also [6] and that $rk(M_1) = 6$. Now let us consider the rational map $\Phi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ where the generic point of the target $\mathbb{P}^6$ has coordinates $(u : r : t : w : z : y : x)$. It is well known that Φ is a birational map and its fibres are described in [15] (see also [1] for the point of view of Mori's theory): there is a singular degree 8, genus 5 and codimension 2 variety Y in the target $\mathbb{P}^6$ such that for any generic point of Y the fibre is a line; there are 28 isolated triple point in Y such that the fibre over them is a plane; there is a quadric $Q = \Phi(C)$ whose equations are: $x = y = z = uw - rt = 0$ such that any point of Q is double for Y and its fibre is a quadric.

Here we are interested by the information given by M_1. By any computer algebra system it is easy to see that for any $P \in \mathbb{P}^6 \backslash X$ $rk[M_1(P)] \geq 5$. More precisely we have that $E(P)$ is a line if and only if $P \in C \backslash X$, otherwise $E(P) = \Phi(P)$. Hence $\Phi_P = W^1_{\Phi(P)}$ is a linear space for any $P \in \mathbb{P}^6 \backslash C$ whose dimension is 1 with the 28 quoted exceptions (they correspond to 28 conics on X, becoming from the 28 lines joining the 8 blowed up points). For any $P \in C \backslash X$ $E(P)$ is a line of one of the two rulings of Q and $W^1_{\Phi(P)}$ is a 3-dimensional linear space containing the fibre. This space coincides with $W_{E(P)}$ because W^1_A is constant for any point $A \in E(P)$ and the restriction of the rational map Φ from $W_{E(P)} \simeq \mathbb{P}^3$ to $E(P) \simeq \mathbb{P}^1$ is induced by a pencil of quadrics in $\mathbb{P}^3$ whose base locus is an elliptic quartic in X. To get the fibre over a point A we can use the other three degree 2 syzygies: the fibre is exacly W_A.

Note that Q contains another ruling of lines but none of them can be $E(P)$ for some $P \in C\backslash X$; in fact, let l be one of these lines, the intersection of all the $W^1_B \simeq \mathbb{P}^3$, such that $B \in l$, is the vertex of the cone C, hence it is empty in $\mathbb{P}^6\backslash X$ and this is a contradiction with Theorem 2. Each one of the W^1_B contains one of the elliptic quartic quoted above. This situation is described from another point of wiew in [15, 1].

EXAMPLE 3 It is known that the $K3$ smooth surface X of degree 9 and sectional genus 8 in $\mathbb{P}^4$ has the K_4 property (see [4, p. 225]) and $M_1 = M$. I_X is generated by 6 quartics and $\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$. It is also known that Z is a smooth cubic hypersurface (see [5]) and that Φ is birational onto Z. By Propositions 2 and 3 $rk[M_1(P)] = 5$ for any $P \in \mathbb{P}^4\backslash X$, G' is given by 6 bilinear forms $\underline{y}_t B_j \underline{x} = \underline{0}$ $j = 1, ..., 6$ and for any $\underline{a} \in \mathbb{P}^5$ $\{\underline{a}_t B_j \underline{x} = \underline{0}\ \forall j = 1, ..., 6\}$ is a homogeneous linear system of 6 equations in 5 indeterminates. It has solutions if and only if the 5 maximal minors of the related matrix are null. The variety defined by these 5 quintics in $\mathbb{P}^5$ is reducible into Z and into another smooth variety Y which is exactly the locus of points in $\mathbb{P}^5$ for which the fibre of Φ has positive dimension. Y is a Del Pezzo surface of degree 5 in $\mathbb{P}^5$ (see [5]) and the above 5 quintics in fact are broken into the product of the cubic equation of Z and 5 quadratic forms which define exactly Y. You can verify such computations with any computer algebra system, as Macaulay for instance.

EXAMPLE 4 The above situation can also be studied by reversing Φ. Let us consider the Del Pezzo surface Y in $\mathbb{P}^5$ whose ideal is generated by 5 quadratic forms so we get a rational map $\Psi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$. Y has the K_2 property: the 5 forms have only 5 linear syzygies and, by Propositions 2 and 3, $rk[M_1(P)] = 4$ for any $P \in \mathbb{P}^5\backslash X$. We can consider G' which is given by 5 bilinear forms and we can study the fibres of Ψ by considering a homogeneous linear systems of 5 equations in 6 indeterminates, as in the previous example. It is easy to see that Ψ is surjective and that every fibre over a point $Q \in \mathbb{P}^4$ is a secant line for Y but for the points belonging to 5 disjoint lines in $\mathbb{P}^4$: in these cases the fibre is a plane cutting a conic on Y.

It is very interesting that, by using linear syzygies, we can also study the restriction of Ψ to any cubic hypersurface contaning Y. Let H be any smooth generic cubic containing Y. Its equation gives rise to another bilinear equation in $\mathbb{P}^5 \times \mathbb{P}^4$ so that there exists a fibre of $\Psi_{|H}$ over a point $Q \in \mathbb{P}^4$ only if there exist solutions for the corresponding linear system. Now this homogeneous system has 6 equations in 6 indeterminates and the related matrix has generic rank 5, i.e. the generic fibre of $\Psi_{|H}$ is a point, in fact this map is birational (see [5]). We can also study the exceptional fibres by looking for points for which the rank of the related matrix is strictly less than 5. In this way we get a determinantal ideal in $\mathbb{P}^4$ and it is easy to see, by a computer algebra system (see also [5]), that this ideal define exactly a smooth $K3$ surface X of degree 9 and sectional genus 8 which is the intersection of 6 quartics. The surface contains the 5 lines in $\mathbb{P}^4$ $L_1, ..., L_5$ such that for any point of these lines the fibres of Ψ are planes. The fibres of $\Psi_{|H}$ are the intersections of the fibres of Ψ with H, out of $H \cap Y$. For any point the fibre is a point (as the fibres of Ψ were secant lines to Y) but for points belonging to X. For any $Q \in X$ the fibre over Q is a line. This is obvious when $Q \notin L_i$, if $Q \in L_i$ the fibre of Ψ over Q is a plane but the fibre of $\Psi_{|H}$ is a line too (recall that Y cuts a conic on these planes) unless H contains some of these planes, but for generic H

it is not possible. The conclusion is that $\Psi_{|H}$ is a birational map for generic H (see also [5]).

EXAMPLE 5 The final part of the previous example can be generalized as follows. Assume that X in $\mathbb{P}^r$ is the base locus of a $r-$dimensional linear system of degree d hypersurfaces with $\rho = r$ and assume also that X has the K_d property, then the generic degree $d+1$ hypersurface H of $\mathbb{P}^r$ containing X is birational to $\mathbb{P}^{r-1}$ via the restriction of Φ to H, where Φ is the rational map induced by the linear system. In fact, by assumptions, Φ is surjective and, by Proposition 3, the generic fibres of Φ are lines. These lines are $d-$secant lines for X so they cut H at one point out of X, hence $\Phi_{|H}$ is birational. We remark that for any generic point of $\mathbb{P}^r$there passes only one $d-$secant line for X.

The above assumptions are verified, for instance, when X is the determinantal variety given by the vanishing of the maximal minors of a $(r, r-1)$ matrix of generic linear forms of $\mathbb{P}^r$. In this case the columns of the matrix give the linear syzygies and condition K_d holds with $d = r - 1$ (see also [12, § 5.4]).

5. Linear systems of quadrics

In this section we consider linear systems $|\Lambda|$ of quadrics in $\mathbb{P}^r$ and we want to give some results about the fibres of $\Phi_{|\Lambda|}$.

First of all we consider linear systems with degenerate Jacobian, i.e. such that $\rho < \min(r+1, \alpha+1)$. Note that, in this case, the rational map $\Phi_{|\Lambda|}$ is never surjective because $\dim(Z) = \rho - 1$. We give the following

DEFINITION 1 Let $|\Lambda|$ be a α-dimensional linear systems in $\mathbb{P}^r$. We say that a linear subsystem $|\Lambda'| \subseteq |\Lambda|$ is *essential* if:
i) $\dim|\Lambda'| < \dim|\Lambda|$
ii) $\rho' < \rho$
iii) $\rho' \leq \dim|\Lambda'| := \alpha'$
where ρ and ρ' are the ranks of the Jacobian matrix of the linear systems in $\mathbb{P}^r$.

Obviously we have $\rho > \rho' \geq 2$, in fact if $\rho' = 1$ the Jacobian matrix of Λ would have two proportional rows, and this is not possible. Hence $\alpha' \geq 2$, in fact if $\alpha' = 1$ then $\rho' = 1$. If Λ has some essential subsystem Λ' then the rational map $\Phi_{|\Lambda'|}$ is the composition of the rational map $\Phi_{|\Lambda|}$ and the projection π: $\mathbb{P}^\alpha \dashrightarrow \mathbb{P}^{\alpha'}$ from the linear space V, whose equations are: $y_0 = y_1 = \cdots = y_{\alpha'} = 0$ if $\Lambda' = \langle f_0, f_1, ..., f_{\alpha'}\rangle$. Z is contained in the cone of $\mathbb{P}^\alpha$ whose base is $Z' := Im(\Phi_{|\Lambda'|})$ and whose vertex is V. The fibres of the restriction of π to Z have positive dimension by ii) and $|\Lambda'|$ has degenerated Jacobian because $\rho' < \min(r+1, \alpha'+1)$. If $|\Lambda|$ has an essential subsystem then $\Phi_{|\Lambda|}$ is not surjective, otherwise $\Phi_{|\Lambda'|}$ would be surjective too and this is not possible as $\dim(Z') = \rho' - 1 < \alpha'$.

The meaning of the non existence of some essential subsystem $|\Lambda'|$ of $|\Lambda|$ relies in the following theorem of Degoli (see [3, Theorem A and Theorem D])

Theorem 2

Let $|\Lambda|$ be a α-dimensional linear systems of quadrics in $\mathbb{P}^r$, $r \geq 3$, with degenerate Jacobian such that $\rho = r - t \leq \alpha$, $t \geq 0$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\Lambda|$. Assume that $|\Lambda|$ does not have essential subsystems, then $|\Lambda|$ is subhomaloidal, the generic fibre of Φ is a $\mathbb{P}^{t+1}$and for the generic point of $\mathbb{P}^r$ there passes a t-dimensional family of secant lines to the base locus of $|\Lambda|$.

Theorem 2 describes very explicitly the fibres of Φ only in some particular case because, in general, some essential subsystem does exist. In this case we have the following

Proposition 4

Let $|\Lambda|$ be a α-dimensional linear systems of quadrics in $\mathbb{P}^r$, $r \geq 3$, with degenerate Jacobian. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the rational map induced by $|\Lambda|$. Assume that $|\Lambda|$ has an essential, α'-dimensional, minimal, subsystem $|\Lambda'|$. Let $\Psi : \mathbb{P}^r \dashrightarrow \mathbb{P}^{\alpha'}$ be the rational map induced by $|\Lambda'|$. Then there exists a surjective rational map $\pi : Z := Im(\Phi) \to Z' := Im(\Psi)$ such that $\Phi^{-1}[\pi^{-1}(Q)]$ is a linear space for any generic point $Q \in Z'$.

Proof. As we saw, the existence of an essential subsystem $|\Lambda'| \subset |\Lambda|$ means that Z is contained in a cone in $\mathbb{P}^\alpha$. Let ρ' and ρ be the respective ranks of the Jacobian matrices. By assumption $\rho' \leq \alpha'$ and $\rho' < \rho \leq r$, so that $|\Lambda'|$ has degenerate Jacobian. Moreover, as $|\Lambda'|$ is minimal among the essential subsystems of $|\Lambda|$, $|\Lambda'|$ has no essential subsystems and by the previous theorem the generic fibre of Ψ is a linear space.

Let π be the projection from the vertex V of the cone in $\mathbb{P}^\alpha$ containing Z. Obviously $\Psi = \pi \circ \Phi$ as rational map so that the claim follows. □

Note that if, in the assumptions of Proposition 4, Z is not a cone, then, for any generic point $Q \in Z'$, we can consider the restriction of Φ among $\Psi^{-1}(Q)$ and $\langle V, Q\rangle$. This restriction is not surjective because its target is $\pi^{-1}(Q)$ which is strictly contained in $\langle V, Q\rangle$ as Z is not a cone. Hence the restriction has degenerate Jacobian, so that we can apply the previous results in this situation and so on. On the contrary, if Z is a cone, the restriction of Φ among $\Psi^{-1}(Q)$ and $\langle V, Q\rangle$ is surjective and we can say nothing else.

Now let us show that we get the same situation, more generally, when Z has degenerated Gauss map (the following theorem is a consequence of [8, Lemma 6.16]).

Theorem 3

Let Λ be a α-dimensional linear systems of quadrics in $\mathbb{P}^r$. Let $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^\alpha$ be the induced rational map. Assume that for any generic point $P' \in Z$ there exists a linear space $L' \subset Z$, $P' \in L'$, such that for any point $H' \in L'$, $T_{P'}(Z) = T_{H'}(Z)$. Assume also that L' is the maximal linear space in Z with the above property. Then $L := \Phi^{-1}(L')$ is a linear space in $\mathbb{P}^r$.

Proof. Let P be a generic point in $\mathbb{P}^r$ such that $\Phi(P) = P'$. Let ρ be the rank of the Jacobian of Λ at the generic point of $\mathbb{P}^r$, so that $\dim(Z) = \rho - 1 = \dim(T_{P'}(Z))$ because $P' \in Sm(Z)$. Let us consider the $r+1$ columns of J_P as coordinates of vectors in $\mathbb{C}^{\alpha+1}$. They generates a $(\rho-1)$-dimensional linear space in $\mathbb{P}^\alpha$ which is the embedded $T_{P'}(Z)$. Its dual vector space, which has dimension $\alpha - \rho$, and corresponds to a linear subspace of Λ, is exactly $Ann(P)$. In fact $Ann(P)$, as a subvector space of $(\mathbb{C}^{\alpha+1})^*$, is the kernel of the linear map associated to the transpose of J_P.

More precisely: $T_{P'}(Z)$ is a linear space of dimension $\rho-1$ in $\mathbb{P}^\alpha = \mathbb{P}(\Lambda^*)$ given by (at least) $\alpha+1-\rho$ independent linear equations. Its dual space $[T_{P'}(Z)]^* \subset \mathbb{P}^{\alpha*} = \mathbb{P}(\Lambda)$ has dimension $\alpha - 1 - (\rho - 1) = \alpha - \rho$ and it is given by $r + 1$ linear equations whose coefficients are the columns of J_P. So that the points of $[T_{P'}(Z)]^*$ correspond exactly to the elements of the kernel of the linear map associated to the transpose of J_P, i.e. $Ann(P)$.

Now if we take another generic point H in $\mathbb{P}^r$ such that $\Phi(H) = H' \in L'$ then $T_{H'}(Z) = T_{P'}(Z)$ and therefore $Ann(H) = Ann(P)$ as they are the dual spaces of the same linear space. This fact shows that $Ann(P) = Ann(H)$ for any couple of generic $P, H \in L$, so that there exists a linear subsystem Λ_L in Λ such that any quadric $Q \in \Lambda_L$ is singular at all generic points of L. As the singular locus of a quadric is a linear space we have that any quadric $Q \in \Lambda_L$ is singular at all points of the linear span $\langle L\rangle$ of L. Moreover $\langle L\rangle$ is contained in $Singloc(Ann(P))$ for any generic $P \in L$.

Let us consider a fixed L' in Z. Let k be its dimension. Let us take a generic point $P' \in L'$ and a generic point $P \in L$. We can choose a base for Λ: $f_0, ..., f_\alpha$ such that the equations of L' in $\mathbb{P}^\alpha$ are $y_{k+1} = \cdots = y_\alpha = 0$. According to Landsberg's notation we have that $baseloc\Lambda^*(L'^{\perp})$ is given by the equations: $f_{k+1} = \cdots = f_\alpha = 0$. On the other hand, by [8, Lemma 6.16], (which is true also when $\alpha \geq r$ because Landsberg's results are true for all linear systems of quadrics with degenerate Jacobian, see [8, § 11]) we have that $Singloc(Ann(P)) \subset baseloc\Lambda^*(L'^{\perp})$, hence $\langle L\rangle \subset baseloc\Lambda^*(L'^{\perp})$ which is $X \cup L$ in $\mathbb{P}^r$. As $\langle L\rangle$ is a linear space, not contained in X, we get $\langle L\rangle \subseteq L$, i.e. $\langle L\rangle = L$. □

Remark 3 Note that the previous theorem is trivial when $Z = \mathbb{P}^\alpha$. In this case $L' = \mathbb{P}^\alpha$ and $L = \mathbb{P}^r$. Note also that, when $L' = P'$ is a point, L is the linear span of P and $Con(P)$. In fact this linear space is contained in $Singloc(Ann(P))$ and it has dimension $1 + r - \rho$.

Now let V be a degree $d \geq 3$ irreducible, reduced, hypersurface in $\mathbb{P}^r$, $r \geq 2$, and let $v(x_0 : \cdots : x_r) = 0$ its equation. Let us assume that V is not a cone. Let us consider the first polar map $\Phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^{r*}$ given by the $r + 1$ first partial derivatives of v. Let X be the base locus of the linear system given by those $r + 1$ forms of degree $d - 1$, obviously $X = Sing(V)$. From now on let us assume that the determinant of the Hessian matrix $\mathcal{H}$ of v is zero, i.e. that $\rho \leq r$, hence $\dim(Z) = \rho - 1 < r$ because $\mathcal{H} = J_\Phi$. Note that Z can be contained in a linear subspace of $\mathbb{P}^{r*}$ if and only if V is a cone, as we are assuming that V is not a cone then Z is nondegenerated.

The Perazzo's map associated to X (see [9]) is the rational map : $\mathcal{P}_X : \mathbb{P}^r \dashrightarrow G(r - \rho, r)$, where $G(r - \rho, r)$ is the Grassmannian of the $(r - \rho)-$dimensional linear subspaces of $\mathbb{P}^r$, such that, for any generic $P \in \mathbb{P}^r \backslash X$, $\mathcal{P}_X(P) = [T_{P'}(Z)]^*$. In fact, for such $P, T_{P'}(Z)$ is a $\rho - 1$ dimensional liner space of $\mathbb{P}^{r*}$, and its dual is a $(r - \rho)-$dimensional linear subspace of $\mathbb{P}^r$. By recalling that in this case $(J_{|P})_t = J_{|P} = \mathcal{H}_{|P}$, we also get that $\mathcal{P}_X(P) = Sing(\mathcal{Q}_P)$, where $\mathcal{Q}_P$ is the (polar) quadric of $\mathbb{P}^r$ associated to the matrix $\mathcal{H}_{|P}$. This fact means that the generic fibre of $\mathcal{P}_X$ is supported over the set of points P such that the corresponding $\mathcal{Q}_P$ have the same singular locus.

Now we can prove the following:

Corollary 3

With the previous notation, let us assume that $d = 3$. Then the generic fibre of $\mathcal{P}_X$ is a linear space of $\mathbb{P}^r$.

Proof. If $d = 3$, X is the base locus of a linear systems $|\mathcal{L}|$ of quadrics and $\mathcal{P}_X$ is the composition of $\Phi_{|\mathcal{L}|}$ with the dual of the Gauss map for $Z := Im(\Phi)$. Then we have only to apply Theorem 3.

Now we want to show that Theorem 1 can give a description of the fibres of $\mathcal{P}_X$ in many examples with $r \geq 4$.

Let us choose an integer s such that $2 \leq s \leq r-2$ and a rational map $\Psi : \mathbb{P}^{s-1} \dashrightarrow \mathbb{P}^{r-s}$ given by $r-s+1$ forms of degree $\delta \geq 2$ $M_j = M_j(x_0 : \cdots : x_{s-1})$, $j = s, ..., r$. Let Z' be $Im(\Psi)$ and let Π be the linear space in $\mathbb{P}^r$ having equations: $x_0 = x_1 = \cdots = x_s = 0$. Let us put $Q := x_s M_s + \cdots + x_r M_r$ and let P_l be any element of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(l))$. Let $(y_0 : \cdots : y_r)$ be the coordinates in $\mathbb{P}^{r*}$. It is known that a degree n hypersurface whose equation is:

$$v = P_n + P_{n-(\delta+1)}Q + \cdots + P_{n-k(\delta+1)}Q^k,\ n - k(\delta+1) \geq 1,\ k \geq 1,$$

is such that $\det(\mathcal{H}) = 0$, moreover the assumption about s yields that V is not a cone (see [10]). For sake of simplicity let us assume that Z and Z' are hypersurfaces.

We can identify the points $(x_0 : \cdots : x_{s-1})$ of $\mathbb{P}^{s-1}$ with the $(r-s+1)$-dimensional linear spaces containing Π and we can identify the target of Ψ with a suitable subspace of $\mathbb{P}^{r*}$. In fact the partial derivatives of v with respect of $x_s, ..., x_r$ are: $\frac{\partial v}{\partial Q}M_s, ..., \frac{\partial v}{\partial Q}M_r$, hence the only equation $g(M_s : \cdots : M_r) = 0$ of Z' is also the only equation of Z. Therefore Z is a cone over the hypersurface $Z' \subset \mathbb{P}^{r-s} \subset \mathbb{P}^{r*}$, having a $\mathbb{P}^{s-1}$ as vertex, and every $(r-s+1)$-dimensional linear space containing Π is sent by Φ onto the $\mathbb{P}^s$ spanned by the vertex of Z and the point

$$(M_s(x_0 : \cdots : x_{s-1}) : \cdots : M_r(x_0 : \cdots : x_{s-1})) \in \mathbb{P}^{r-s}.$$

In this case the dual of the Gauss map sends Z into the dual of Z' which is a variety Z'^* contained in the linear space $\Pi \simeq \mathbb{P}^{r-s} \subset \mathbb{P}^r$. Obviously $Z'^* = Im(\mathcal{P}_X)$.

If Ψ is generically injective then $s-1 = r-s-1$, i.e. $r = 2s$, and the generic fibre of $\mathcal{P}_X$ is a $\mathbb{P}^{s+1}$ containing Π. If the generic fibre of Ψ if finite then $r = 2s$ and the generic fibre of $\mathcal{P}_X$ is the union of a finite number of $\mathbb{P}^{s+1}$. If the generic fibre of Ψ is a linear space of dimension $t \leq s-2$ then $s-1-t = r-s-1$, i.e. $r = 2s-t$, and the generic fibre of $\mathcal{P}_X$ is a $\mathbb{P}^{s+1}$. By Theorem 3, for instance, it happens when $\delta = 2$ and Z' has a nondegenerate Gauss map. Note that, on the contrary, the generic fibre of Φ is always the union of a finite number of lines by [17, Propositon 4.9 (ii)].

EXAMPLE 6 Let us consider the following degree 5 polynomial:

$$v = x^5 + y^5 + (x^2 - 3xy + y^2)(x^2z + xyw + y^2u),$$

here $r = 4$, $s = \delta = 2$, $k = 1$, $\Psi(x : y) = (x^2 : xy : y^2)$ and the condition K_2 holds for such degree 2 forms. If we choose $(a : b : c : d : e)$ as coordinates in $\mathbb{P}^{4*}$ then Z is the quadric cone: $d^2 - ec = 0$ and $Z'^* = Im(\mathcal{P}_X)$ is the conic: $4uz - w^2 = 0$ on the plane $\Pi : x = y = 0$. The fibres of $\mathcal{P}_X$ are the hyperplanes of $\mathbb{P}^4$ containing Π. As they are a pencil, $\mathcal{P}_X$ is nothing else than an embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a smooth conic.

The transpose of the matrix M_1 is the following:

$$\begin{bmatrix} 0 & 0 & 0 & -y & x \\ 0 & 0 & -y & x & 0 \end{bmatrix}.$$

For any generic point $P \equiv (x : y : z : w : u) \in \mathbb{P}^4 \backslash X$, in this case, $E(P)$ is the plane of the quadric cone Z containing $\Phi(P) : -yd + xe = -yc + xd = 0$, spanned by $\Phi(P)$ and the vertex of Z. $W_{E(P)}$ is the fibre of $\mathcal{P}_X$ passing through P. The generic fibre of Φ is a line.

If we change the polynomial v in the following way:

$$v = x^7 + y^7 + (x^2 - 3xy + y^2)(x^4 z + x^2 y^2 w + y^4 u)$$

we get an example with $r = 4$, $s = 2, \delta = 4, k = 1, n = 7$. Here Z, Z' and Z'^* have the same equations as in the previous case, but Ψ is generically $(2 : 1)$ and there are no linear syzygies neither for Ψ nor for Φ. The generic fibre of $\mathcal{P}_X$ consists of a couple of hyperplanes of $\mathbb{P}^4$ containing the plane $\Pi : x = y = 0$; the points of these hyperplanes are sent by Φ onto a plane of the cone Z. The generic fibre of Φ is a couple of lines intersecting at a point of Z'^*.

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