Regularized vortex approximation for 2D Euler equations with transport noise

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Abstract

We study a mean field approximation for the 2D Euler vorticity equation driven by a transport noise. We prove that the Euler equations can be approximated by interacting point vortices driven by a regularized Biot-Savart kernel and the same common noise. The approximation happens by sending the number of particles N to infinity and the regularization ϵ in the Biot-Savart kernel to 0, as a suitable function of N.

1 Introduction

In this paper we consider the stochastic Euler equations on the two-dimensional torus $\mathbb{T}^2$, in vorticity form, driven by transport noise, namely

$$\partial_t \xi + u \cdot \nabla \xi + \sum_k \sigma_k \cdot \nabla \xi \circ \dot{W}^k = 0, \qquad u = K \star \xi, \tag{1}$$

where $\xi = \xi(t,x,\omega)$ is the unknown vorticity, K is the Biot-Savart kernel, σ_k are given, divergence-free vector fields, satisfying certain assumptions, W^k are independent real Brownian motions and $\circ$ denotes Stratonovich integration. We prove convergence, with quantitative bounds, of a system of point vortices, with regularized kernel K^ϵ, to the bounded solution ξ to (1).

The deterministic Euler equations describe the motion of an incompressible, non-viscous fluid; in two dimensions one can use the equivalent vorticity formulation, that is (1), where $u = u(t,x)$ represents the velocity of the fluid at time t and space x and $\xi(t,x) = \operatorname{curl} u(t,x)$ is its vorticity. In 2D, well-posedness holds among bounded solution, as proved in [33], see also [30, Section 2.3] for an alternative proof. Concerning the stochastic Euler equations, there are various results depending on the type on noise; the transport noise in the vorticity, which we consider here in (1), is motivated by the transport nature of the vorticity equation. For equation (1) in 2D, existence and uniqueness of (probabilistically) strong, bounded solutions is proved in [9], see also [12] and [25] for resp. a 3D analogue and a rough path analogue of (1). Transport noise has also been used to show regularization by noise phenomena, mostly for the linear case ([16, 2] and several other works), though isolated nonlinear examples also exist (see e.g. [13, 17, 22, 18] and the recent review [7]).

The vortex approximation is an approximation of the solution to the Euler equations in vorticity form via the weighted empirical measure of a system of interacting diffusions. The idea is formally as follows: Take a weighted empirical measure $\frac{1}{N}\sum_{i=1}^N \xi^{i,N}\delta_{X_0^{i,N}}$ which approximates the initial condition ξ_0 and consider the following system of interacting diffusions:

$$dX_t^{i,N} = \frac{1}{N}\sum_{j\neq i} \xi^{j,N} K(X_t^{i,N} - X_t^{j,N})dt + \sum_k \sigma_k(X_t^{i,N}) \circ dW_t^k, \quad i = 1, \dots N. \tag{2}$$

Then, formally and ignoring self interaction (that is, assuming formally $K(0) = 0$), the empirical measure $\frac{1}{N}\sum_{i=1}^N \xi^{i,N}\delta_{X_t^{i,N}}$ is a solution to (1) in the distributional sense. Hence we might expect, by a continuity argument with respect to the initial condition, that $\frac{1}{N}\sum_{i=1}^N \xi^{i,N}\delta_{X_t^{i,N}}$ approximates the solution ξ_t. The

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system (2) describes the motion of interacting vortices and is similar to the system of interacting diffusions approximating a McKean-Vlasov SDE, see e.g. [32], with one important difference: the vortices in (2) are driven by the space correlated noise $\sum_k \sigma_k(x) \circ dW_t^k$, while in the McKean-Vlasov SDE approximation, the particles are driven by independent Brownian motions. In the case of independent Brownian motions as driving signals, the limit of the empirical measures is expected to solve the deterministic Navier-Stokes equations (in vorticity form) rather than the stochastic Euler equations, as noted by Chorin [10]. The vortex system can be also viewed as a discrete approximation of the Euler equations; other discrete models in stochastic fluid dynamics are the shell models and the dyadic models (see e.g. [5, 6, 1, 8]).

When coming to a rigorous proof of the above convergence argument, two difficulties arise: 1) the interaction kernel K is irregular, precisely we expect $K(x) \approx x^\perp/|x|^2$ close to 0; 2) the noise term prevents us from exploiting classical continuity arguments used in the deterministic context. In the deterministic context, for regular interaction kernel, convergence of the particle system is proved in [14]. The case of 2D Euler equations is considered in [28]: the authors consider a system of interacting vortices under a regularized kernel K^ϵ and prove the convergence of this system to the Euler equations, assuming the convergence of the initial positions at rate ζ_N and tuning the regularization parameter $\epsilon = \epsilon(N)$ as a suitable, double logarithmic function of ζ_N [28, Theorem 4.1]; see also [30, Section 5.3]. The convergence of the original vortex system (2) (without noise), with no regularization, is proved in [23] and also in [31], in the latter paper also for unbounded solutions to (1) (without noise), via a suitable randomization of the initial conditions of the vortex system. The paper [15] shows the approximation result for distributional solutions under the white noise invariant measure μ on $\mathbb{T}^2$: precisely, if the initial conditions $X_0^{i,N}$ are taken independent and identically distributed with uniform law and the intensities $\xi^{j,N}$ are taken i.i.d. $\mathcal{N}(0, N)$, then the vortex system converges a.s. to a random, stationary solution to the Euler equations with one-time marginals distributed as μ; the result is generalized also to solutions whose one-time marginals are absolutely continuous with respect to μ. For other convergence results in the deterministic case the reader can refer to [21, 19, 24].

In the stochastic case, [11] proves the convergence of the particle system (2) for a regular kernel K and a non-negative initial distribution ξ_0. To our knowledge, the only paper dealing with approximation of stochastic Euler equations via vortices is [18], where the analogue result of [15] for the stochastic case is proved (the authors prove also an improved, compared to the deterministic case, regularity of the density with respect to μ). The paper [17] shows that, for any fixed N, the vortex system (2) is well-posed for every initial condition, at least for suitably non-degenerate σ_k, while the corresponding deterministic system can collapse for special (zero Lebesgue measure) initial conditions.

Note that, in the case of independent noises dW^i, that is, the case of deterministic Navier-Stokes as expected limiting equation, better results of convergences (in terms of rates and larger class of initial conditions) can be proved, see [20] and [26] as two remarkable examples. Note also that, in the 3D case, an analogue approximation has been proposed, for the deterministic Euler equations, in [3, 4], replacing vortex points by vortex filaments.

In this paper we show the vortex approximation for bounded solutions to the stochastic 2D Euler equation (1), using a vortex system with regularized kernel K^ϵ, namely

$$dX_t^{i,N} = \frac{1}{N} \sum_{j \neq i} \xi^{j,N} K^\epsilon(X_t^{i,N} - X_t^{j,N}) dt + \sum_k \sigma_k(X_t^{i,N}) \circ dW_t^k, \quad i = 1, \dots N, \tag{3}$$

for a regularization parameter $\epsilon = \epsilon(N)$. Our main result is

Theorem (see Theorem 18)**.** Assume that σ_k are sufficiently regular, and let ξ_0 be a bounded initial vorticity. Assume that $\frac{1}{N}\sum_{i=1}^N \xi^{i,N} \delta_{x^i}$ converges to ξ_0 with rate ζ_N, as $N \to \infty$. Let $\epsilon(N) \approx (-\log \zeta_N)^{-\delta}$ for a suitable $\delta \in \mathbb{R}_+$ and let $X^{i,N}$ be the solution to the regularized vortex system (3) with initial condition $(x^1, \dots, x^N)$. Then the path of empirical measures $(\frac{1}{N}\xi^{i,N} \sum_{i=1}^N \delta_{X_t^i})_t$ converges in $W^{1,\infty}(\mathbb{T}^2)^*$, as $N \to \infty$, to the (unique) bounded solution to the stochastic 2D Euler equations (1).

We use the strategy of [28] applied to the stochastic case. Note that, by a technical trick in the fixed point argument in Section 3 (see Remark 12), we can deal with $\epsilon(N)$ as logarithmic function of ζ_N, rather than double logarithmic as in [28], though we expect the result to be non-optimal, as for the deterministic case. We leave the investigation of the convergence of the true vortex system (2) for future research.

The paper is organized as follows. In Section 3, we deal with the convergence of (3) to the regularized version of the Euler equation (1) (replacing the Biot-Savart kernel K with K^ϵ, for fixed ϵ). We use the techniques in [11], showing in addition the convergence in L^p in the ω variable for every $p \geq 1$ and accounting for non positive measures as well. Then, in Section 4, we deal with the convergence of the solution of the regularized Euler equation as the regularization parameter ϵ tends to 0; we use the techniques in [9], showing in addition convergence in $L^p(\Omega; C([0,T]; L^1(\mathbb{T}^2)))$ for $p \geq 1$ (in [9], convergence is shown only in $C([0,T]; L^1(\Omega \times \mathbb{T}^2)))$. This is shown in Theorem 17. Finally, in Theorem 18 we prove that the empirical measure of the system (3) converges to the solution of the Euler equation (1).

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2 Preliminaries

2.1 Spaces of measures

We start with some notations used throughout the paper. Given a compact metric space (E,d) (in practice, $E = \mathbb{T}^2$ with the Euclidean distance), we call $\mathcal{M}(E)$ the space of finite, signed Borel measures on E. Given a measurable function $\varphi : E \to \mathbb{R}$ and a measure $\mu \in \mathcal{M}(E)$, we write

$$\mu(\varphi) := \int_E \varphi(x)\mu(dx).$$

We call $C_b(E)$ the space of continuous bounded functions on E, endowed with the supremum norm $\|\varphi\|_\infty = \sup_{x\in E} |\varphi(x)|$. The space $\mathcal{M}(E)$, being the dual of $C_b(E)$, is naturally endowed with the dual norm

$$\|\mu\| := \sup_{\|\varphi\|_\infty \leq 1} |\mu(\varphi)|.$$

Given a finite signed Borel measure μ, we denote by $|\mu|$ its variation measure (it holds $\|\mu\| = |\mu|(X)$).

The space of bounded Lipschitz continuous functions on E will be called $BL(E)$, while the unit ball in this space is

$$BL_1(E) := \{\varphi \in Lip(E) \quad | \quad \|\varphi\|_\infty + \mathrm{Lip}(\varphi) \leq 1\ \},$$

where $\mathrm{Lip}(\varphi) := \sup_{x,y\in E} \frac{|\varphi(x)-\varphi(y)|}{|x-y|}$.

Now we endow $\mathcal{M}(E)$ with the Kantorovich-Rubinstein (or 1-Wasserstein) metric

$$W_1(\mu,\nu) := \sup_{\varphi \in BL_1(E)} |\mu(\varphi) - \nu(\varphi)|\,.$$

The space $\mathcal{M}(E)$ is not complete with respect to this metric. However, for every $M > 0$, the closed ball in the total variation norm $\mathcal{M}_M(E) := \{\mu \in \mathcal{M} \mid \|\mu\| \leq M\}$ is complete with respect to W_1.

We call $\mathcal{P}(E)$ the space of probability measures on E.

Remark 1. The fact that $\mathcal{M}_M(E)$ is closed under W^1 is classical, we give here a short proof. Let $(\mu^n)_{n\in\mathbb{N}} \in \mathcal{M}_M(E)$ be a sequence converging to μ in W^1. For every $\varphi \in BL(E)$, we have $\lim_{n\to\infty} \mu^n(\varphi) - \mu(\varphi) \leq \lim_{n\to\infty} \|\varphi\|_{BL} W_1(\mu^n,\mu) = 0$. Hence, since the Lipschitz functions are dense in the continuous functions, we have

$$\sup_{\varphi\in C_b(E), \|\varphi\|_\infty\leq 1} |\mu(\varphi)| \leq \sup_{\varphi\in BL(E), \|\varphi\|_\infty\leq 1} |\mu(\varphi)| \leq \sup_{\varphi\in BL(E), \|\varphi\|_\infty\leq 1} \sup_n |\mu^n(\varphi)| \leq M.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, satisfying the standard assumption (that is, completeness and right-continuity). Fix a time horizon $T > 0$ and a real number $p \in [1,\infty)$, we define the space $V_M^{p,T} := L^p(\Omega; C([0,T], \mathcal{M}_M(\mathbb{T}^2)))$ of $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable stochastic processes endowed with the distance

$$d_p(\mu,\nu) := \mathbb{E}\left[\sup_{t\in[0,T]} W_1(\mu_t,\nu_t)^p\right]^{\frac{1}{p}}.$$

Remark 2. The space $V_M^{p,T}$ with the distance d_p is complete, we give a short proof for completeness. Indeed, given a Cauchy sequence $(\mu_n)_{n\in\mathbb{N}} \subset V_M^{p,T}$ there is a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ which is almost surely a Cauchy sequence in $C([0,T],\mathcal{M}_M(\mathbb{T}^2))$. Since $C([0,T],\mathcal{M}_M(\mathbb{T}^2))$ is complete, there exists a null set $N \subset \Omega$, such that, for all $\omega \in N^c$, there exists $\mu(\omega) \in C([0,T],\mathcal{M}_M(\mathbb{T}^2))$ such that $\sup_{t\in[0,T]} W_1(\mu_{n_k}(\omega),\mu(\omega)) \to 0$ as $k \to \infty$. Adaptedness of μ follows from adaptedness of μ_{n_k}. Since the distance is bounded, dominated convergence concludes the argument.

For later convenience, given a positive constant $c > 0$, we define the distance

$$d_p^c(\mu,\nu) = \mathbb{E}\left[\sup_{t\in[0,T]} \left(e^{-ct} W_1(\mu_t,\nu_t)^p\right)\right]^{\frac{1}{p}}. \tag{4}$$

Note that, for every $p \in [1,\infty)$ and $c > 0$, the two distances d_p and d_p^c are equivalent. We will sometimes use the short notation L_x^p to mean $L^p(\mathbb{T}^2)$.

Remark 3. The distance W^1 has the following property: for any μ in $\mathcal{M}_M(E)$, for every two Borel maps $f,g : E \to E$, it holds

$$W_1(f_\#\mu, g_\#\mu) \le \|\mu\| \|f-g\|_\infty.$$

Indeed, for every φ in $BL_1(E)$, we have

$$|f_\#\mu(\varphi) - g_\#\mu(\varphi)| = |\mu(\varphi(f) - \varphi(g))| \le \|\mu\| \|f-g\|_\infty.$$

For the distance d_p^c a similar property holds: for any μ in $\mathcal{M}_M(\mathbb{T}^2)$, for every two measurable maps $f, g : [0,T]\times\mathbb{T}^2\times\Omega \to \mathbb{T}^2$, it holds

$$d_p^c(f_\#\mu, g_\#\mu) \le \|\mu\| \sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\sup_{t\in[0,T]} \left(e^{-ct}|f_t(x) - g_t(x)|\right)^p\right]^{\frac{1}{p}}.$$

Indeed, recalling that $|\mu(\psi)|^p \le \|\mu\|^{p-1} \int |\psi|^p d|\mu|$ for every ψ,

$$\begin{aligned}
d_p^c(f_\#\mu, g_\#\mu)^p &= \mathbb{E}\left[\sup_{t\in[0,T]} \sup_{\varphi\in BL_1(\mathbb{T}^2)} \left(e^{-ct}|\mu(\varphi(f_t) - \varphi(g_t))|\right)^p\right] \\
&\le \|\mu\|^{p-1} \int_{\mathbb{T}^2} \mathbb{E}\left[\sup_{t\in[0,T]} \sup_{\varphi\in BL_1(\mathbb{T}^2)} \left(e^{-ct}|\varphi(f_t) - \varphi(g_t)|\right)^p\right] d|\mu|(dx) \\
&\le \|\mu\|^{p-1} \int_{\mathbb{T}^2} \mathbb{E}\left[\sup_{t\in[0,T]} \left(e^{-ct}|f_t - g_t|\right)^p\right] d|\mu|(dx) \\
&\le \|\mu\|^p \sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\sup_{t\in[0,T]} \left(e^{-ct}|f_t(x) - g_t(x)|\right)^p\right].
\end{aligned}$$

2.2 The noise

Here we give the assumptions on the noise. In the following, $\sigma_k : \mathbb{T}^2 \to \mathbb{R}^2$ is a vector field, for every $k \in \mathbb{N}$ and $Q : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}^{2\times 2}$ is the space covariance (matrix-valued) function defined by

$$Q^{ij}(x,y) := \sum_{k=1}^{\infty} \sigma_k^i(x)\,\sigma_k^j(y).$$

Assumptions 4.

i) $\sigma_k : \mathbb{T}^2 \to \mathbb{R}^2$ are C^2 functions satisfying $\sum_{k=1}^{\infty} \|\sigma_k\|_{C^2} < \infty$.

ii) σ_k are divergence free vector fields, i.e. $\operatorname{div} \sigma_k = 0, \quad \forall k \ge 1$.

iii) The covariance function : $\mathbb{T}^2 \to \mathbb{R}^{2\times 2}$ satisfies

(a) $Q(x,y) = Q(x-y)$ (space homogeneity of the random field $\sum_{k=1}^{\infty} \sigma_k(x) B_t^k$);

(b) $Q(0) = aI$ for some $a \geq 0$ (where I is the 2×2 identity matrix).

One can find examples of this model (or its analougue in the full space) in several references, e.g. [13], [27] and [11].

Example 5. We present here a family of σ_k which satisfies Assumptions 4.

For every $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ we define

$$\sigma_k(x) = (\cos(k \cdot x) + \sin(k \cdot x)) \frac{k^\perp}{|k|^\beta}.$$

Now we verify the Assumptions 4 for $\beta > 4$.

We have $\|\sigma_k\|_{C^h} \leq C|k|^{-\beta+1+h}$, hence assumption $i)$ is satisfied for $\beta > 4$. The Jacobian matrix is

$$D\sigma_k(x) = \frac{1}{|k|^\beta}(\cos(k \cdot x) - \sin(k \cdot x)) \begin{pmatrix} -k_1 k_2 & -k_2^2 \\ k_1^2 & k_1 k_2 \end{pmatrix}.$$

The trace of this matrix is equal to 0 and so assumption $ii)$ is also satisfied.

The covariance matrix Q is equal to

$$Q(x,y) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|k|^{2\beta}} \left[\cos(k \cdot (x-y)) + \sin(k \cdot (x+y)) \right] \begin{pmatrix} k_2^2 & -k_2 k_1 \\ -k_2 k_1 & k_1^2 \end{pmatrix}.$$

Now we group together the terms with k and $-k$: $\sin(k \cdot (x+y))$ disappears and we get (calling $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}_+$)

$$Q(x,y) = 2 \sum_{k \in \mathbb{Z}_+^2} \frac{1}{|k|^{2\beta}} \cos(k \cdot (x-y)) \begin{pmatrix} k_2^2 & -k_2 k_1 \\ -k_2 k_1 & k_1^2 \end{pmatrix}.$$

Thus Q depends only on the difference $x - y$ and assumption $iii) - a)$ is satisfied. To verify assumption $iii) - b)$ we look at $Q(0)$: here the terms with k and $k^\perp$ sum up to a diagonal matrix, precisely (calling $\mathbb{Z}_{++}^2 = \{k \in \mathbb{Z} \mid k_1 \geq 0, k_2 > 0\}$)

$$Q(0) = 2 \sum_{k \in \mathbb{Z}_+^2} \frac{1}{|k|^{2\beta}} \begin{pmatrix} k_2^2 & -k_2 k_1 \\ -k_2 k_1 & k_1^2 \end{pmatrix} = 2 \sum_{k \in \mathbb{Z}_{++}^2} \frac{1}{|k|^{2\beta}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that $Q(0) = aI$ for some a, that is assumption $iii) - b)$.

2.3 The Biot-Savart kernel

We recall here the needed properties of the 2-dimensional Biot-Savart kernel. The following results are standard for the Green function and can be found, among others, in [29] and [9].

For an $r > 0$, we define

$$\gamma(r) = \begin{cases} r(1 - \log(r)) & \text{if } 0 < r < 1/e \\ r + 1/e & \text{if } r \geq 1/e. \end{cases} \tag{5}$$

For this function γ and the Biot-Savart kernel K, the following properties hold:

(i) for every $0 < \epsilon < \frac{1}{e}$,

$$\gamma(r) \leq -\log(\epsilon) r + \epsilon.$$

(ii) K is a divergence free vector field and

$$\int_{\mathbb{T}^2} |K(x-y) - K(x'-y)| dy \lesssim \gamma(|x - x'|), \quad \forall x, x' \in \mathbb{T}^2.$$

(iii) for every $\xi \in L^\infty$,

$$|(K * \xi_t)(x) - (K * \xi_t)(x')| \lesssim \|\xi_t\|_{L^\infty} \gamma(|x - x'|), \quad \forall x, x' \in \mathbb{T}^2.$$

3 Vortex approximation for regularized Euler equations

In this section, we work with a regularized kernel and we show the convergence of the particle system to the regularized Euler equation (Corollary 16). The idea is the following. First, we show the existence and uniqueness for the regularized Euler equation by expressing any solution as fixed point of a certain operator Ψ on the space $V_M^{1,T}$ and proving the contraction property for this operator. Then we note that both the weighted empirical measure $S_t^{N,\epsilon}$ and the desired limit ξ_t^ϵ are solutions to the regularized Euler equation, with different initial data and, with the previous representation in mind, we prove the convergence theorem as a continuity theorem with respect to the initial data.

For $\epsilon > 0$ we take the mollifier $\rho^\epsilon(x) = \epsilon^{-2}\rho(\epsilon^{-1}x)$, for $x \in \mathbb{R}^2$, where $\rho \in C_0^\infty(\mathbb{R}^2)$, $\rho \geq 0$, $\rho(-x) = \rho(x)$ and $\|\rho\|_{L^1} = 1$. We define

$$K^\epsilon(x) := \int_{\mathbb{R}^2} K(x-y)\rho^\epsilon(y)dy, \qquad x \in \mathbb{T}^2.$$

This function has the following properties.

Lemma 6. *Let $\epsilon > 0$, the following holds:*

(i) $\|K^\epsilon - K\|_{L^1(\mathbb{T}^2)} \to 0$, *as* $\epsilon \to 0$.

(ii) $K^\epsilon \in C_b^\infty(\mathbb{T}^2; \mathbb{R}^2)$.

(iii) For any $\delta > 0$ and $k \in \mathbb{N}$ there exists $C = C(\rho, k, \delta) > 0$ such that

$$\|DK^\epsilon\|_{C^k} \leq C\|K\|_{L^{2/(1+\delta)}}\epsilon^{-(k+1+\delta)}.$$

Proof. (i) and (ii) are standard properties of the mollification, we only show (iii). Using Hölder inequality, with $q = 2/(1-\delta)$, and the change of variables $y' = \epsilon^{-1}y$, we get

$$\begin{aligned}|D^k K^\epsilon(x)| \leq & \|K\|_{L^{2/(1+\delta)}}\|D^k\rho^\epsilon\|_{L^q} = \|K\|_{L^{2/(1+\delta)}}\left(\int_{\mathbb{R}^2} |\epsilon^{-2}\epsilon^{-k}D^k\rho(\epsilon^{-1}y)|^q dy\right)^{\frac{1}{q}} \\ \leq & \|K\|_{L^{2/(1+\delta)}}\epsilon^{(-(2+k)q+2)/q}\left(\int_{\mathbb{R}^2} |D^k(y)|^q \rho dy\right)^{\frac{1}{q}}.\end{aligned}$$

This concludes the proof. ☐

Having the regularized kernel, for every $\mu \in \mathcal{M}(\mathbb{T}^2)$ we define (omitting the ϵ dependence in the notation)

$$b(x, \mu) := \int_{\mathbb{T}^2} K^\epsilon(x-y)\mu(dy), \qquad x \in \mathbb{T}^2.$$

Remark 7. The function b is locally uniformly Lipschitz continuous in both arguments, precisely:

- For every $x, x' \in \mathbb{T}^2$ and $\mu \in \mathcal{M}(\mathbb{T}^2)$, $|b(x,\mu) - b(x',\mu)| \leq \mathrm{Lip}(K^\epsilon)\|\mu\||x - x'|$.
- For every$x \in \mathbb{T}^2$ and $\mu, \mu' \in \mathcal{M}(\mathbb{T}^2)$, $|b(x,\mu) - b(x,\mu')| \leq \mathrm{Lip}(K^\epsilon)W_1(\mu,\mu')$.

We introduce the regularized Euler equation in vorticity form, which takes the form

$$\partial_t\mu + \mathrm{div}(b(\mu)\mu) + \sum_{k=1} \mathrm{div}(\sigma_k\mu) \circ dW_t^k = 0. \tag{6}$$

For the rigorous definition, we consider the Itô formulation of the above equation. Note that, for the assumptions on the noise (see e.g. [11, Section 2.2]), the Itô formulation reads formally:

$$\partial_t\mu + \mathrm{div}(b(\mu)\mu) + \sum_{k=1} \mathrm{div}(\sigma_k\mu)dW_t^k = \frac{1}{2}\Delta\mu.$$

We study distributional solutions of this equation in the following sense.

Definition 8. Let $\mu_0 \in \mathcal{M}_M(\mathbb{T}^2)$. We say that $\mu \in V_M^{p,T}$ is a solution to equation (6) if, for every $\varphi \in C^2(\mathbb{T}^2)$,

$$\mu_s(\varphi) = \mu_0(\varphi) + \int_0^t \left[\mu_s(b(\mu_s)\nabla\varphi) + \frac{1}{2}\mu_s(\Delta\varphi) \right] ds + \sum_{k\geq 1} \int_0^t \mu_s(\sigma_k \nabla\varphi) dW_s^k, \qquad \mathbb{P} - a.s. \quad \forall t \in [0,T] \quad (7)$$

(the $\mathbb{P}$-exceptional set being independent of t).

Remark 9. The stochastic integral in (7) is well-defined, because $\mathbb{P}-$a.s.

$$\sum_{k\geq 1} \mu_t(\sigma_k \nabla\varphi)^2 \leq M \|\varphi\|_{C^2}^2 \sup_{x\in\mathbb{T}^2} \sum_{k\geq 1} |\sigma_k(x)| < \infty, \quad \forall t \in [0,T].$$

To handle the nonlinearity in equation (6), we fix a positive constant $M > 0$ and define the following auxiliary stochastic differential equation, where the random measure $\mu \in V_M^{p,T}$ is fixed:

$$\begin{cases} dX_t = b(X_t, \mu_t)dt + \sum_{k\geq 1} \sigma_k(X_t) dW_t^k, \\ X_0 = x \in \mathbb{T}^2. \end{cases} \quad (8)$$

Remark 10. Since the coefficients are Lipschitz continuous and bounded, equation (8) admits a unique strong solution for $t \in [0,T]$. Moreover, there exists a version of the solution map $\Phi(t,x,\omega)$ which is Lipschitz continuous in the initial datum x, uniformly in t, and Hölder continuous in t, uniformly in x (see [27, Theorem 4.6.5]).

We call $\Phi^\mu(t,x)$ the flow associated with equation (8), to stress the dependence on μ of the drift. For every measure $\mu_0 \in \mathcal{M}(\mathbb{T}^2)$, with $\|\mu_0\| \leq M$, we define the operator

$$\begin{array}{rccc} \Psi^{\mu_0}: & V_M^{p,T} & \to & V_M^{p,T} \\ & \mu & \mapsto & \Phi^\mu_\# \mu_0. \end{array}$$

Note that the map $t \mapsto (\Phi_t^\mu)_\# \mu_0$ is a.s. continuous, indeed, by Remarks 3 and 10, for every $s < t$,

$$W_1((\Phi_t^\mu)_\#\mu_0, (\Phi_s^\mu)_\#\mu_0) \leq M \|\Phi_t^\mu - \Phi_s^\mu\|_\infty \leq C_\omega M |t-s|^\alpha.$$

Moreover the total variation norm satisfies $\|\Psi^{\mu_0}\| \leq \|\mu_0\|$, $\mathbb{P}$-a.s., for every $t \in [0,T]$. Hence the operator Ψ^{μ_0} is well-defined.

Now we show that the operator is a contraction in the norm d_p^c, for a suitable c.

Lemma 11. *Let $T > 0$ and $p > 2$ be fixed. Assume $\|\mu_0\| \leq M$. There exists a constant $c > 0$, depending on ϵ (and on a, T, p, M) such that*

$$d_p^c(\Phi^\mu_\#\mu_0, \Phi^{\mu'}_\#\mu_0) \leq \frac{1}{2} d_p^c(\mu, \mu'),$$

for every μ, μ' in $V_M^{p,T}$. Moreover,

$$c = c(\sigma, p, M, \epsilon) \sim \|DK^\epsilon\|_{C^0}^{p^2/(2(p-2))}, \qquad \text{as } \epsilon \to 0. \quad (9)$$

Remark 12. Here we see the main reason to introduce the distance d_p^c in (4) with the e^{-ct} factor: by a suitable choice of c, the map Ψ^{μ_0} is a contraction on $V_M^{p,T}$ with the distance d_p^c, without any need to take T small. Beside being technically convenient, this choice allows also to avoid the double exponential rate in [28, Theorem 4.1].

Proof of Lemma 11. We estimate the difference of the two images in terms of the differences of the two flows, namely

$$d_p^c(\Phi^\mu_\#\mu_0, \Phi^{\mu'}_\#\mu_0)^p \leq \|\mu_0\|^p \sup_{x\in\mathbb{T}^2} \mathbb{E}\left[\sup_{t\in[0,T]} e^{-pct} |\Phi_t^\mu(x) - \Phi_t^{\mu'}(x)|^p \right]. \quad (10)$$

For two $\mu, \mu' \in V_M^{p,T}$ and $x \in \mathbb{T}^2$, we apply Itô formula to $f_\eta(x) = (|x|^2 + \eta)^{\frac{q}{2}}$. Here $q = \frac{p}{2}$ and we choose $\eta = 0$ if $q \geq 2$, $\eta > 0$ if $1 < q < 2$. For this choice of η and q, it holds

$$|\nabla f_\eta(x)| \leq q|x|^{q-1} \quad |D^2 f_\eta(x)| \leq q(q-1)|x|^{q-2}. \tag{11}$$

Notice that we endow the space of matrices with Hilbert-Schmidt norm.

Let $\bar{c} = \frac{pc}{2}$. Using Itô Formula and Assumption 4 we obtain that for each $x \in \mathbb{T}^2$, it holds $\mathbb{P}$-a.s.: for every t,

$$\begin{aligned} e^{-\bar{c}t} f_\eta(\Phi_t^\mu(x) - \Phi_t^{\mu'}(x)) \leq & (C(q,\sigma)^2 - \bar{c}) \int_0^t e^{-\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^q \mathrm{d}s \\ & + q \int_0^t e^{-\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^{q-1} |b_s(\Phi_s^\mu(x), \mu) - b_s(\Phi_s^{\mu'}(x), \mu')| \mathrm{d}s \quad (12) \\ & + \left| \sum_{k \geq 1} \int_0^t e^{-\bar{c}s} \nabla f_\eta(\Phi_s^\mu(x) - \Phi_s^{\mu'}(x))(\sigma_k(\Phi_s^\mu(x)) - \sigma_k(\Phi_s^{\mu'}(x))) dW_s^k \right|, \quad (13) \end{aligned}$$

where $C(q,\sigma)$ is a positive constant (depending on q and $(\sigma_k)_k$ and possibly changing from one line to another).

To estimate term (12), we use a triangular inequality and the Lipschitz property of $b(x,\mu)$, both in x and μ (remember $\|\mu_t\| \leq M$). Hence, term (12) is bounded by the following

$$q\mathrm{Lip}^q(K^\epsilon) \left[M \int_0^t e^{-\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^q \mathrm{d}s + \int_0^t e^{-\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^{q-1} W_1(\mu_s, \mu_s') \mathrm{d}s \right].$$

We apply Young inequality with $\frac{q}{q-1}$ and q to the second term to obtain, for every $s \in [0,T]$ and $\delta > 0$ (to be determined later),

$$|\Phi_t^\mu(x) - \Phi_t^{\mu'}(x)|^{q-1} W_1(\mu_s, \mu_s') \leq \delta^{-1/(q-1)} |\Phi_t^\mu(x) - \Phi_t^{\mu'}(x)|^q + \frac{\delta}{q} W_1(\mu_s, \mu_s')^q.$$

Substituting into (12) we obtain

$$e^{-\bar{c}t} f_\eta(\Phi_t^\mu(x) - \Phi_t^{\mu'}(x)) \leq L \int_0^t e^{-\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^q \mathrm{d}s + \delta \mathrm{Lip}^q(K^\epsilon) \int_0^t e^{-\bar{c}s} W_1(\mu_s, \mu_s')^q \mathrm{d}s + (13), \tag{14}$$

where $L = C(q,\sigma) + q\mathrm{Lip}^q(K^\epsilon)(M + \delta^{-1/(q-1)}) - \bar{c}$. We can choose $\bar{c}$ as a function of δ, M, $Lip(K^\epsilon)$ and q such that

$$L = C(q,\sigma) + q\mathrm{Lip}^q(K^\epsilon)(M + \delta^{-1/(q-1)}) - \bar{c} = 0,$$

so that we can remove the corresponding term from the estimates. We estimate now the expectation of the square of (13). First we use the Burkholder-Davis-Gundy inequality, then the Lipschitz assumption on σ and (11) to obtain

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0,t]} \left| \sum_{k \geq 1} \int_0^s e^{-\bar{c}r} \nabla f_\eta(\Phi_r^\mu(x) - \Phi_r^{\mu'}(x))(\sigma_k(\Phi_r^\mu(x)) - \sigma_k(\Phi_r^{\mu'}(x))) dW_r^k \right|^2 \\ & \leq C\mathbb{E} \sum_{k \geq 1} \int_0^t e^{-2\bar{c}r} |\nabla f_\eta(\Phi_r^\mu(x) - \Phi_r^{\mu'}(x))|^2 |\sigma_k(\Phi_r^\mu(x)) - \sigma_k(\Phi_r^{\mu'}(x))|^2 dr \\ & \leq C(q,\sigma)\mathbb{E} \int_0^t \sup_{r \in [0,s]} e^{-2\bar{c}r} |\Phi_r^\mu(x) - \Phi_r^{\mu'}(x)|^{2q} \mathrm{d}s. \end{aligned} \tag{15}$$

Now we estimate the expectation of the square of (14): using (15) and Jensen inequality we have (remember that $f_\eta(x) \geq |x|^q$ and $L = 0$)

$$\begin{aligned} \mathbb{E} \sup_{s \in [0,t]} e^{-2\bar{c}s} |\Phi_s^\mu(x) - \Phi_s^{\mu'}(x)|^{2q} \leq & T\delta^2 \mathrm{Lip}^{2q}(K^\epsilon) \int_0^t e^{-2\bar{c}s} W_1(\mu_s, \mu_s')^{2q} \mathrm{d}s \\ & + C(q,\sigma)\mathbb{E} \int_0^t \sup_{r \in [0,s]} e^{-2\bar{c}r} |\Phi_r^\mu(x) - \Phi_r^{\mu'}(x)|^{2q} \mathrm{d}s. \end{aligned}$$

Applying Gronwall's Lemma, we obtain (remember that $2q = p$ and $2\bar{c} = pc$)

$$\mathbb{E} \sup_{s\in[0,t]} e^{-pcs}|\Phi_s^{\mu}(x) - \Phi_s^{\mu'}(x)|^p \leq T\delta^2 \mathrm{Lip}^p(K^\epsilon)e^{C(q,\sigma)T}\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} e^{-pcr}W_1(\mu_r,\mu_r')^p\right]\mathrm{d}s.$$

We can choose now $\delta = e^{-\frac{1}{2}C(q,\sigma)T}(2^pM^pT\,\mathrm{Lip}^p(K^\epsilon))^{-\frac{1}{2}}$. With this choice, we get

$$c = c(\sigma,p,M,\epsilon) \sim \|DK^\epsilon\|_{C^0}^{p/2}\|DK^\epsilon\|_{C^0}^{p/2\cdot 2/(p-2)} = \|DK^\epsilon\|_{C^0}^{p^2/(2(p-2))}, \qquad \text{as } \epsilon\to 0.$$

With the constants chosen in this way we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{-pct}|\Phi_t^{\mu}(x) - \Phi_t^{\mu'}(x)|^p\right] \leq \frac{1}{2^pM^p}\mathbb{E}\left[\sup_{t\in[0,T]} e^{-ct}W_1(\mu_t,\mu_t')^p\right]. \tag{16}$$

Estimates (16) and (10) allow to conclude the proof of the lemma. □

Theorem 13. *Let $T, M > 0$ and $p > 2$. Let $\mu_0 \in \mathcal{M}_M(E)$ Then (6) has a solution in the space $V_M^{p,T}$ starting from μ_0. Precisely, the unique fixed point for the operator Ψ^{μ_0} is a solution to (6).*

Proof. From Lemma 11 follows that there exists $c > 0$ such that the operator Ψ^{μ_0} is a contraction $(V_M^{p,T}, d_p^c)$. Hence it has a unique fixed point. As a straight forward application of Itô formula one can show that every fixed point satisfies (7). The proof is complete. □

Now we investigate the continuity of the fixed point of Ψ^{μ_0} with respect to the initial condition μ_0. We need a preliminary estimate on the derivative of the flow Φ associated to (8).

Lemma 14. *Let Φ_t be the stochastic flow of equation (8), we denote by $D\Phi_t(x)$ its derivative in space. For every every $p > 3$, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in\mathbb{T}^2}|D\Phi_t(x)|^p\right] \leq \Lambda(p),$$

where $\Lambda(p) = C\left(1 + M^p\|DK^\epsilon\|_{C^1}^p\right)\exp\left(C(1 + M^{2p}\|DK^\epsilon\|_{C^0}^{2p})\right)$ and $C = C(p,T,\|\sigma\|_{C^2})$.

Proof. Let $\eta_t(x) = D\Phi_t(x)$ be the space derivative of the flow. By formal computation η satisfies the following stochastic differential equation, for every $x \in \mathbb{T}^2$,

$$\eta_t(x) = I + \int_0^t Db(\Phi_u(x))\eta_u(x)\mathrm{d}u + \int_0^t\sum_k D\sigma_k(\Phi_u(x))\eta_u(x)\mathrm{d}W_u^k, \qquad \forall t\in[0,T], \quad \mathbb{P}-a.s.$$

where I is the 2×2 identity matrix.

We estimate the $L^\gamma(\Omega)$ norm of η, for fixed $\gamma > 2$ and fixed $x \in \mathbb{T}^2$, $t \in [0,T]$. By a standard computation, which we do not repeat here, we obtain, for every $t \in [0,T]$,

$$\mathbb{E}\left[|\eta_t(x)|^\gamma\right] \leq c_\gamma \exp\left(C(1 + M^\gamma\|DK^\epsilon\|_{C^0}^\gamma)\right), \tag{17}$$

where c_γ, C are positive constants depending respectively on γ and on $\gamma, T, \|D\sigma\|_{C^0}$. In view of Kolmogorov criterion, we would like to control, for fixed $x, y \in \mathbb{T}^2$, $t > s \in [0,T]$ and $\gamma \geq 2$,

$$\mathbb{E}\left|\eta_t(x) - \eta_s(y)\right|^\gamma \leq \mathbb{E}\left|\eta_s(x) - \eta_s(y)\right|^\gamma + \mathbb{E}\left|\eta_t(x) - \eta_s(x)\right|^\gamma. \tag{18}$$

For the first addend in the right hand side of (18), we have

$$\mathbb{E}\left|\eta_s(x) - \eta_s(y)\right|^\gamma \leq c_\gamma\mathbb{E}\left|\int_0^s\left(Db(\Phi_u(x))\eta_u(x) - Db(\Phi_u(y))\eta_u(y)\right)\mathrm{d}u\right|^\gamma \tag{19}$$

$$+ c_\gamma\mathbb{E}\left|\int_0^s\sum_k\left(D\sigma_k(\Phi_u(x))\eta_u(x) - D\sigma_k(\Phi_s(y))\eta_u(y)\right)\mathrm{d}W_u^k\right|^\gamma. \tag{20}$$

We first estimate term (19). Using Cauchy-Schwarz inequality, we obtain for any $0 < \alpha < 1$,

$$\begin{aligned}
c_\gamma \mathbb{E}\left|\int_0^s (Db(\Phi_u(x))\eta_u(x) - Db(\Phi_u(y))\eta_u(y))\,\mathrm{d}u\right|^\gamma \\
&\le c_\gamma T^{\gamma-1}\mathbb{E}\left[\int_0^s (|Db(\Phi_u(x)) - Db(\Phi_u(y))|^\gamma |\eta_u(x)|^\gamma + |Db(\Phi_u(y))|^\gamma |\eta_u(x) - \eta_u(y)|^\gamma)\,\mathrm{d}u\right] \\
&\le c_\gamma T^{\gamma-1}\|DK^\epsilon\|_{C^\alpha}^\gamma M^\gamma \left(\int_0^s \mathbb{E}\,|\Phi_u(x) - \Phi_u(y)|^{2\alpha\gamma}\,\mathrm{d}u\right)^{\frac{1}{2}} \left(\int_0^s \mathbb{E}\,|\eta_u(x)|^{2\gamma}\,\mathrm{d}u\right)^{\frac{1}{2}} \qquad (21) \\
&\quad + c_\gamma T^{\gamma-1}\|DK^\epsilon\|_{C^0}^\gamma M^\gamma \int_0^s \mathbb{E}\,|\eta_u(x) - \eta_u(y)|^\gamma\,\mathrm{d}u.
\end{aligned}$$

In a similar way, we can apply the Burkholder-Davis-Gundy inequality to (20), then apply the same reasoning as before to obtain

$$\begin{aligned}
c_\gamma \mathbb{E}\left|\int_0^s \sum_k (D\sigma_k(\Phi_u(x))\eta_u(x) - D\sigma_k(\Phi_s(y))\eta_u(y))\,\mathrm{d}W_u^k\right|^\gamma \\
&\le c_\gamma T^{\frac{\gamma}{2}-1}\|D\sigma\|_{C^0}^\gamma \int_0^s \mathbb{E}\,|\eta_u(x) - \eta_u(y)|^\gamma\,\mathrm{d}u \\
&\quad + c_\gamma T^{\frac{\gamma}{2}-1}\|D\sigma\|_{C^\alpha}^\gamma \left(\int_0^s \mathbb{E}\,|\Phi_u(x) - \Phi_u(y)|^{2\alpha\gamma}\,\mathrm{d}u\right)^{\frac{1}{2}} \left(\int_0^s \mathbb{E}\,|\eta_u(x)|^{2\gamma}\,\mathrm{d}u\right)^{\frac{1}{2}}. \qquad (22)
\end{aligned}$$

In a standard way we estimate the difference (using $a^{2\alpha\gamma} \le 1 + a^{(2\alpha\gamma)\vee 1}$:

$$\begin{aligned}
\mathbb{E}\,|\Phi_u(x) - \Phi_u(y)|^{2\alpha\gamma} &\le \int_0^1 (1 + \mathbb{E}\,|\eta_u(\xi x + (1-\xi)y)|^{(2\alpha\gamma)\vee 1})\mathrm{d}\xi |x-y|^{2\alpha\gamma} \\
&\le \left(1 + \sup_z \mathbb{E}\,|\eta_u(z)|^{(2\alpha\gamma)\vee 1}\right)|x-y|^{2\alpha\gamma}. \qquad (23)
\end{aligned}$$

We put together now estimates (21), (22), (23) and (17) to get (using $a^\gamma \le 1 + a^{2\gamma}$ and $a^{(2\alpha\gamma)\vee 1} \le 1 + a^{2\gamma}$ for every $a \ge 0$, $0 < \alpha \le 1$)

$$\begin{aligned}
\mathbb{E}\,|\eta_s(x) - \eta_s(x)|^\gamma \le & C(1 + M^\gamma \|DK^\epsilon\|_{C^\alpha}^\gamma) \exp\left(C(1 + M^{2\gamma}\|DK^\epsilon\|_{C^0}^{2\gamma})\right)|x-y|^{\alpha\gamma} \\
& + C(1 + M^\gamma \|DK^\epsilon\|_{C^0}^\gamma) \int_0^s \mathbb{E}|\eta_u(x) - \eta_u(y)|^\gamma \mathrm{d}u. \qquad (24)
\end{aligned}$$

Here and in the rest of the proof $C = C(\gamma, T, \|D\sigma\|_{C^\alpha}, \|D\sigma\|_{C^0})$. By Gronwall's Lemma we obtain

$$\mathbb{E}\,[|\eta_s(x) - \eta_s(y)|^\gamma] \le C(1 + M^\gamma \|DK^\epsilon\|_{C^\alpha}^\gamma) \exp\left(C(1 + M^{2\gamma}\|DK^\epsilon\|_{C^0}^{2\gamma})\right)|x-y|^{\alpha\gamma}. \qquad (25)$$

For the second term in (18), we have

$$\mathbb{E}\,|\eta_t(x) - \eta_s(x)|^\gamma \le c_\gamma \mathbb{E}\left|\int_s^t Db(\Phi_u(x))\eta_u(x)\mathrm{d}u\right|^\gamma + c_\gamma \mathbb{E}\left|\int_s^t \sum_k D\sigma_k(\Phi_u(x))\eta_u(x)\mathrm{d}W_u^k\right|^\gamma. \qquad (26)$$

The two terms in (26) can be estimated using the boundedness of the coefficients, Burkholder-Davis-Gundy inequality for the second one, Hölder inequality and (17) to obtain

$$\begin{aligned}
\mathbb{E}\,|\eta_t(x) - \eta_s(x)|^\gamma &\le C\left(1 + M^\gamma \|DK^\epsilon\|_{C^0}^\gamma\right)|t-s|^{\frac{\gamma}{2}} \sup_{u\in[0,T]} \mathbb{E}|\eta_u(x)|^\gamma \\
&\le C\left(1 + M^\gamma \|DK^\epsilon\|_{C^0}^\gamma\right) \exp\left(C(1 + M^\gamma \|DK^\epsilon\|_{C^0}^\gamma)\right)|t-s|^{\frac{\gamma}{2}}. \qquad (27)
\end{aligned}$$

Finally, we use (25) and (27) and the inequality $\|DK^\epsilon\|_{C^\alpha} \le \|DK^\epsilon\|_{C^1}$ to obtain

$$\mathbb{E}\,[|\eta_t(x) - \eta_s(y)|^\gamma] \le \Lambda(\gamma)\left(|x-y|^{\alpha\gamma} + |t-s|^{\frac{\gamma}{2}}\right), \qquad (28)$$

where $\Lambda(\gamma) = C\left(1 + M^{\gamma}\|DK^{\epsilon}\|_{C^1}^{\gamma}\right)\exp\left(C(1 + M^{2\gamma}\|DK^{\epsilon}\|_{C^0}^{2\gamma})\right)$. In order to conclude, we recall the following inequality, a consequence of the Sobolev Embedding Theorem, valid for $\alpha' > 0$, $\beta := \alpha' - 3/p > 0$:

$$\mathbb{E}\left[\sup_{t,s\in[0,T]}\sup_{x,y\in\mathbb{T}^2}\frac{|D\Phi_t(x) - D\Phi_s(y)|^p}{(|t-s|^2 + |x-y|^2)^{\frac{\beta p}{2}}}\right] \le \iint_{[0,T]\times\mathbb{T}^2}\frac{\mathbb{E}|D\Phi_t(x) - D\Phi_s(y)|^p}{(|t-s|^2+|x-y|^2)^{\frac{3}{2}+\frac{\alpha' p}{2}}}\mathrm{d}t\mathrm{d}s\mathrm{d}x\mathrm{d}y.$$

Now we use (28) with $\gamma = p$ to obtain

$$\mathbb{E}\left[\sup_{t,s\in[0,T]}\sup_{x,y\in\mathbb{T}^2}\frac{|D\Phi_t(x) - D\Phi_s(y)|^p}{(|t-s|^2 + |x-y|^2)^{\frac{\beta p}{2}}}\right] \le \Lambda(p)\iint_{[0,T]\times\mathbb{T}^2}\frac{|t-s|^{\frac{p}{2}} + |x-y|^{\alpha p}}{(|t-s|^2+|x-y|^2)^{\frac{3}{2}+\frac{\alpha' p}{2}}}\mathrm{d}t\mathrm{d}s\mathrm{d}x\mathrm{d}y. \quad (29)$$

The condition $p > 3$ guarantees that we can find α and α' in $(0,1)$ with $\alpha' - 3/p > 0$ and $\alpha p - \alpha' p - 3 > -3$ so that the integral in the right hand side of (29) is finite. The proof is complete. □

Lemma 15. *Let $T > 0$ and $p > 2$, let c be given as in Lemma 11. Given $\mu_0, \nu_0 \in \mathcal{M}_M(\mathbb{T}^2)$ there exists a positive constant $\Gamma = \Gamma(p, T, \sigma, M, \epsilon) := \Lambda(p)^{\frac{1}{p}} \vee \Lambda(4)^{\frac{1}{4}}$, such that*

$$d_p^c(\mu,\nu) \le 2\Gamma W_1(\mu_0,\nu_0),$$

where $\mu, \nu \in V$ are the fixed points of operators $\Psi^{\mu_0}, \Psi^{\nu_0}$ respectively.

Proof. We use a triangular inequality to get

$$d_p^c(\mu,\nu) = d_p^c(\Phi^{\mu}_{\#}\mu_0, \Phi^{\nu}_{\#}\nu_0) \le d_p^c(\Phi^{\mu}_{\#}\mu_0, \Phi^{\mu}_{\#}\nu_0) + d_p^c(\Phi^{\mu}_{\#}\nu_0, \Phi^{\nu}_{\#}\nu_0). \quad (30)$$

It follows from Lemma 11 that the second term in the right hand side is less than or equal to $\frac{1}{2}d_p^c(\mu,\nu)$. We look at the first term, which is, by definition,

$$d_p^c(\Phi^{\mu}_{\#}\mu_0, \Phi^{\mu}_{\#}\nu_0)^p = \mathbb{E}\sup_{t\in[0,T]} e^{-ct}\left|\sup_{\varphi\in Lip_1(\mathbb{T}^2)}\left(\int \varphi\circ\Phi^{\mu}(x)\,(d\mu_0 - d\nu_0)(x)\right)\right|^p.$$

It follows from Remark 10 that the flow Φ^{μ} is a Lipschitz function on $\mathbb{T}^2$. Hence, for any Lipschitz function φ, also the function $\varphi\circ\Phi^{\mu}$ is Lipschitz. Hence we have

$$d_p(\Phi^{\mu}_{\#}\mu_0, \Phi^{\mu}_{\#}\nu_0) \le \mathbb{E}[\sup_{t\in[0,T]} e^{-ct}|\mathrm{Lip}(\Phi^{\mu})|^p]^{\frac{1}{p}} W_1(\mu_0,\nu_0).$$

We recall that

$$e^{-ct}|\mathrm{Lip}(\Phi^{\mu}_t)| \le \sup_{x\in\mathbb{T}^2}|D\Phi^{\mu}_t(x)|,$$

where we used that $e^{-ct} < 1$. To estimate this last term we use Lemma 14: for any $p > 3$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in\mathbb{T}^2}|D\Phi_t(x)|^p\right] \le \Lambda(p).$$

If $2 < p \le 3$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in\mathbb{T}^2}|D\Phi_t(x)|^p\right] \le \Lambda(4)^{\frac{p}{4}}.$$

Using this, we find that

$$d_p^c(\Phi^{\mu}_{\#}\mu_0, \Phi^{\mu}_{\#}\nu_0) \le \mathbb{E}\left[\sup_{t\in[0,T]} e^{-ct}\mathrm{Lip}(\Phi^{\mu})^p\right]^{\frac{1}{p}} W_1(\mu_0,\nu_0) \le \Gamma W_1(\mu_0,\nu_0).$$

Putting together the estimates on the two terms of (30) we conclude the proof. □

Lemma 15 states the continuity of the fixed point of Ψ^{μ_0} with respect to the initial condition μ_0. We use this to study the mean-field convergence of the particle system (3). We recall that the particle system reads, for $N \in \mathbb{N}$ and $1 \le i \le N$,

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^N \xi^{j,N} K^\epsilon(X_t^{i,N} - X_t^{j,N})dt + \sum_{k=1}^\infty \sigma_k(X_t^{i,N}) \circ dW_t^k, \qquad X_t^{i,N}|_{t=0} = x^{i,N}. \tag{31}$$

Here $(x^{i,N})_{1\le i\le N} \subseteq \mathbb{T}^2$ and $(\xi^{i,N})_{1\le i\le N} \subseteq \mathbb{R}$ are given.

Corollary 16. *Let $T, M > 0$ and $p > 2$. Let μ be a solution to equation (6) and $S_t^{N,\epsilon} = \frac{1}{N}\sum_{i=1}^N \xi^{i,N}\delta_{X_t^{i,N}}$ the empirical measure associated to the system of particles (31) with $\|\mu_0\| \vee \|S_0^{N,\epsilon}\| \le M$. Then,*

$$d_p(S^N, \mu) \le e^{cT} d_p^c(S^N, \mu) \le 2\Gamma e^{cT} W_1(S_0^N, \mu_0),$$

where c is given in (9) and Γ is defined in Lemma 15.

Proof. We show now that for every $N \in \mathbb{N}$ the empirical measure $S^{\epsilon,N}$ associated to the system of interacting particles (31) driven by K^ϵ is indeed a fixed point for the operator $\Psi^{S_0^N}$. We must prove

$$S_t^{\epsilon,N} = (\Phi_t^{S^{\epsilon,N}})_\sharp S_0^N, \qquad t \in [0,T].$$

We evaluate the right hand side in a test function $\varphi \in C(\mathbb{T}^2; \mathbb{R}^2)$,

$$(\Phi_\sharp^{S_t^{\epsilon,N}} S_0^N)(\varphi) = \sum_{i=1}^N \varphi(\Phi^{S_t^{\epsilon,N}}(x^{i,N})). \tag{32}$$

Since, by definition, $\Phi^{S_T^{\epsilon,N}}$ is the flow associated with the equation (8) with drift depending on the empirical measure, it is immediate to see that $\Phi^{S_t^{\epsilon,N}}(x^{i,N}) = X_t^{i,N}$. Thus, the right hand side of (32) is exactly $S_t^{\epsilon,N}(\varphi)$.

Now, since both μ and the empirical measure $S^{\epsilon,N}$ are solutions in $V_M^{p,T}$ to the limit equation (6), given as a fixed point of the map Ψ, we conclude using Lemma 15. □

4 Convergence of regularized Euler equations

In this section we show the convergence of the regularized Euler equation to the (true) Euler equation.

For a given initial condition $\xi_0 \in L^\infty(\mathbb{T}^2)$, we consider the flow associated with the approximated kernel K^ϵ, namely

$$d\Phi^\epsilon(x) = \int_{\mathbb{T}^2} K^\epsilon(\Phi^\epsilon(x) - \Phi^\epsilon(y))\xi_0(y)\,\mathrm{d}y dt + \sum_{k=1}^\infty \sigma_k(\Phi^\epsilon(x))dW^k.$$

The existence and uniqueness of Φ^ϵ follows from the previous section. We also consider the flow Φ associated with the true Euler equation, namely

$$d\Phi(x) = \int_{\mathbb{T}^2} K(\Phi(x) - \Phi(y))\xi_0(y)\,\mathrm{d}y dt + \sum_{k=1}^\infty \sigma_k(\Phi(x))dW^k.$$

In [9] the authors show existence and uniqueness for Φ and proves that the measure $\xi_t := (\Phi_t)_\sharp\xi_0$ is a solution to the stochastic Euler vorticity equation (1). The following result shows the convergence of Φ^ϵ to Φ. The result is adapted from [9]; the main improvement here is to bring the supremum over time inside the expectation and take the L^p norm in ω.

Theorem 17. *For every $p \ge 1$ finite, the family $(\Phi^\epsilon)_\epsilon$ converges to Φ in $L^p_\omega(C_t(L^1_x))$ (as $\epsilon \to 0$). Moreover it holds, for some positive constants C (depending on p, T, $\|\xi_0\|_{L^\infty_x}$ and $\sum_k \|\sigma_k\|^2_{W^{1,\infty}_x}$) and c (depending on p and T)*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \int_{\mathbb{T}^2} |\Phi_t^\epsilon(x) - \Phi_t(x)|\,\mathrm{d}x\right)^p \le C\|K^\epsilon - K\|_{L^1_x}^{p\exp(-c\|\xi_0\|_{L^\infty_x} t)}.$$

Proof. In the proof, unless otherwise stated, $C, C', c, \dots$ denote constants that can depend on p and T. Call $Z^\epsilon_t(x) = \Phi^\epsilon_t(x) - \Phi_t(x)$. We expect, from the deterministic theory (see e.g. [29]) and the stochastic counterpart (see [9]), that $\mathbb{E}\sup_{s\in[0,t]} \|Z^\epsilon_s\|^p_{L^1_x}$ satisfies a differential inequality involving a log-Lipschitz drift, and therefore we expect to get an estimate of the form $\mathbb{E}\sup_{s\in[0,t]} \|Z^\epsilon_s\|^p_{L^1_x} \le \|K_\epsilon - K\|^{pe^{-\lambda t}}$. The problem, with respect to [9], comes from the supremum over time inside the expectation, which does not allow easily a comparison principle. For this reason, we choose not to control directly the $L^p_\omega(C_t(L^1_x))$ norm of Z, but rather

$$\mathbb{E}\sup_{s\in[0,t]} \|Z^\epsilon_s\|^{p(t)}_{L^1_x},$$

where $p(t) = pe^{\lambda t}$ for some $\lambda \ge 0$ to be determined later and for $p \ge 2$. A bound on this quantity will imply the final estimate.

As first step we compute the SDE for $\|Z^\epsilon_t\|_{L^1_x}$. We would like to apply Itô formula to $f(z) = |z|$, since this function is not C^2 we use the approximate function $f_\delta(z) = (|z|^2 + \delta)^{1/2}$, $\delta > 0$; we recall that $|\nabla f_\delta(z)| \le 1$ and $|D^2 f_\delta(z)| \le |z|^{-1}$. Applying Itô formula to $f_\delta(Z^\epsilon)$ we get

$$\begin{aligned} df_\delta(Z^\epsilon) =& \nabla f_\delta(Z^\epsilon)(u^\epsilon(\Phi^\epsilon) - u(\Phi))dt + \sum_k \nabla f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))dW^k \\ &+ \frac{1}{2}\sum_k (\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi)) \cdot D^2 f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))dt. \end{aligned}$$

In the following, we use the notation $H_t = \int_{\mathbb{T}^2} |Z^\epsilon_t|\,\mathrm{d}x$ and $H(\delta)_t = \int_{\mathbb{T}^2} f_\delta(Z^\epsilon_t)\,\mathrm{d}x$. In order to estimate $H(\delta)$, we integrate in space and exchange integrals in space and in time using Fubini theorem and stochastic Fubini theorem: it holds

$$\begin{aligned} dH(\delta) =& \int_{\mathbb{T}^2} \nabla f^\delta(Z^\epsilon)(u^\epsilon(\Phi^\epsilon) - u(\Phi))\,\mathrm{d}x\,\mathrm{d}t + \sum_k \int_{\mathbb{T}^2} \nabla f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))\,\mathrm{d}x\,\mathrm{d}W^k \\ &+ \frac{1}{2}\sum_k \int_{\mathbb{T}^2} (\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi)) \cdot D^2 f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

To control $\|Z^\epsilon_s\|^{p(t)}_{L^1_x}$, we apply again Itô formula to $H(\delta)^{p(t)/2} = \exp[p(t)\log H(\delta)/2]$ (the $p(t)/2$-power can be regarded as regular since $H(\delta) \ge \delta^{1/2}|\mathbb{T}^2| > 0$): we get

$$\begin{aligned} H(\delta)^{p(t)/2}_t - H(\delta)^{p/2}_0 =& \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2} \nabla f^\delta(Z^\epsilon)(u^\epsilon(\Phi^\epsilon) - u(\Phi))\,\mathrm{d}x\,\mathrm{d}r \\ &+ \sum_k \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2} \nabla f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))\,\mathrm{d}x\,\mathrm{d}W^k \\ &+ \frac{1}{2}\sum_k \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2} (\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi)) \cdot D^2 f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))\,\mathrm{d}x\,\mathrm{d}r \\ &+ \frac{1}{2}\sum_k \int_0^t \frac{p(r)(p(r)-2)}{4} H(\delta)^{p(r)/2-2} \left(\int_{\mathbb{T}^2} \nabla f_\delta(Z^\epsilon)(\sigma_k(\Phi^\epsilon) - \sigma_k(\Phi))\,\mathrm{d}x\right)^2 \mathrm{d}r \\ &+ \int_0^t \frac{p'(r)}{2} \log(H(\delta)) H(\delta)^{p(r)/2}\,\mathrm{d}r \\ =:& A_u + A_{stoch} + A_{second-order-1} + A_{second-order-2} + A_{log}. \end{aligned}$$

We aim at controlling the (square of) the $L^2_\omega(C_t)$ norm in the equation above, but before doing this, we want to get rid of the log-Lipschitz dependency coming from $u^\epsilon(\Phi^\epsilon) - u(\Phi)$, which would otherwise cause problems: for this we will use the term A_{log}, which comes from p'. We start splitting A_u as follows:

$$\begin{aligned}
A_u \leq& \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2} |u^\epsilon(\Phi^\epsilon) - u(\Phi)| \,\mathrm{d}x \,\mathrm{d}r \\
\leq& \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K^\epsilon(\Phi^\epsilon(x)-\Phi^\epsilon(y)) - K(\Phi(x)-\Phi(y))||\xi_0(x)| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}r \\
\leq& \|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K^\epsilon(\Phi^\epsilon(x)-\Phi^\epsilon(y)) - K(\Phi^\epsilon(x)-\Phi^\epsilon(y))| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}r \\
&+ \|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(\Phi^\epsilon(x)-\Phi^\epsilon(y)) - K(\Phi(x)-\Phi^\epsilon(y))| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}r \\
&+ \|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(\Phi(x)-\Phi^\epsilon(y)) - K(\Phi(x)-\Phi(y))| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}r \\
=:& A_{u1} + A_{u2} + A_{u3}.
\end{aligned}$$

Leaving the term A_{u1} for later, we estimate A_{u2} and A_{u3}. For the term A_{u2}, we make the change of variable $y' = \Phi^\epsilon(y)$ and we use the measure preserving property of Φ^ϵ, the log-Lipschitz property associated with K and Jensen inequality for the concave function γ defined in (5). We get:

$$\begin{aligned}
A_{u2} =& \|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} |K(\Phi^\epsilon(x)-\Phi^\epsilon(y)) - K(\Phi(x)-\Phi^\epsilon(y))| \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}r \\
\leq& C'\|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \int_{\mathbb{T}^2} \gamma(|Z^\epsilon(x)|) \,\mathrm{d}x \,\mathrm{d}r \\
\leq& C'\|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \gamma(H(\delta)) \,\mathrm{d}r.
\end{aligned}$$

For the term A_{u3} we proceed similarly with the change of variable $x' = \Phi(x)$, getting

$$A_{u3} \leq C'\|\xi_0\|_{L^\infty_x} \int_0^t \frac{p(r)}{2} H(\delta)^{p(r)/2-1} \gamma(H(\delta)) \,\mathrm{d}r.$$

Hence we can bound $A_{u2} + A_{u3} + A_{log}$ with

$$A_{u2} + A_{u3} + A_{log} \leq \frac{1}{2} \int_0^t 2C'\|\xi_0\|_{L^\infty_x} p(r) H(\delta)^{p(r)/2-1} \gamma(H(\delta)) + p'(r) \log(H(\delta)) H(\delta)^{p(r)/2}] \,\mathrm{d}r.$$

Now we choose $\lambda = 2C'\|\xi_0\|_{L^\infty_x}$, which gives (recall the definition of γ in (5))

$$p'(r)\log(H(\delta))H(\delta) + 2C'\|\xi_0\|_{L^\infty_x} p(r)\gamma(H(\delta)) = p(r)(\log(H(\delta))H(\delta) + \gamma(H(\delta))).$$

We use the following inequality for γ, valid for r in a bounded interval $[0, R]$:

$$\gamma(r) + r\log r \leq Cr,$$

for some C depending only on R. Hence we get

$$A_{u2} + A_{u3} + A_{log} \leq C\|\xi_0\|_{L^\infty_x} \int_0^t p(r) H(\delta)^{p(r)/2} \,\mathrm{d}r,$$

and so

$$H(\delta)_t^{p(t)/2} - H(\delta)_0^{p/2} \leq A_{u1} + A_{stoch} + A_{second-order-1} + A_{second-order-2} + C\|\xi_0\|_{L^\infty_x} \int_0^t p(r) H(\delta)^{p(r)/2} \,\mathrm{d}r.$$

At this point we control the square of the $L^2_\omega(C_t)$ norm of each addend of the right hand side. For the term A_{u1}, we make the change of variable $x' = \Phi^\epsilon(x)$, $y' = \Phi^\epsilon(y)$ and use the measure preserving property of Φ^ϵ, getting, via Young inequality (with exponents $p(r)/2$ and its conjugate),

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}A_{u1}^2 \leq&\mathbb{E}\|\xi_0\|_{L^\infty_x}^2\int_0^t\frac{p(r)^2}{4}H(\delta)^{p(r)-2}\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}|K^\epsilon(\Phi^\epsilon(x)-\Phi^\epsilon(y))-K(\Phi^\epsilon(x)-\Phi^\epsilon(y))|\,\mathrm{d}x\,\mathrm{d}y\right)^2\mathrm{d}r\\ \leq&C\mathbb{E}\|\xi_0\|_{L^\infty_x}^2\int_0^t p(r)^2(\|K^\epsilon-K\|_{L^1_x}^{p(r)}+H(\delta)^{p(r)})\,\mathrm{d}r\\ \leq&C\|\xi_0\|_{L^\infty_x}(e^{C\|\xi_0\|_{L^\infty_x}}-1)\|K^\epsilon-K\|_{L^1_x}^p+C\|\xi_0\|_{L^\infty_x}^2e^{C\|\xi_0\|_{L^\infty_x}}\int_0^t\mathbb{E}\sup_{s\in[0,r]}H(\delta)_s^{p(s)}\,\mathrm{d}r.\end{aligned}$$

For the term A_{stoch}, using Burkholder-Davis-Gundy inequality and the Lipschitz property of σ_k we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}A_{stoch}^2\leq&C\mathbb{E}\int_0^t p(r)^2H(\delta)^{p(r)-2}\int_{\mathbb{T}^2}\sum_k|\sigma_k(\Phi^\epsilon(x))-\sigma_k(\Phi(x))|^2\,\mathrm{d}x\,\mathrm{d}r\\ \leq&C\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2\int_0^t p(r)^2\mathbb{E}H(\delta)_r^{p(r)}\,\mathrm{d}r\\ \leq&Ce^{C\|\xi_0\|_{L^\infty_x}}\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2\int_0^t\mathbb{E}\sup_{s\in[0,r]}H(\delta)_s^{p(s)}\,\mathrm{d}r.\end{aligned}$$

For the term $A_{second-order-1}$, using again the Lipschitz property of σ_k we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}A_{second-order-1}^2\leq&C\mathbb{E}\int_0^t p(r)^2H(\delta)^{p(r)-2}\left(\int_{\mathbb{T}^2}\sum_k|\sigma_k(\Phi^\epsilon(x))-\sigma_k(\Phi(x))|^2|D^2f_\delta(Z^\epsilon)|\,\mathrm{d}x\right)^2\mathrm{d}r\\ \leq&C(\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2)^2\mathbb{E}\int_0^t p(r)^2H(\delta)^{p(r)-2}\left(\int_{\mathbb{T}^2}|Z^\epsilon||Z^\epsilon|^{-1}|Z^\epsilon|\,\mathrm{d}x\right)^2\mathrm{d}r\\ \leq&Ce^{C\|\xi_0\|_{L^\infty_x}}(\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2)^2\mathbb{E}\int_0^t\sup_{s\in[0,r]}H(\delta)_s^{p(s)}\,\mathrm{d}r.\end{aligned}$$

Similarly for the term $A_{second-order-2}$ we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}A_{second-order-2}^2\leq&C\mathbb{E}\int_0^t p(r)^2(p(r)-2)^2H(\delta)^{p(r)-4}\left(\sum_k\left(\int_{\mathbb{T}^2}|\sigma_k(\Phi^\epsilon(x))-\sigma_k(\Phi(x))|\,\mathrm{d}x\right)^2\right)^2\mathrm{d}r\\ \leq&C(\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2)^2\mathbb{E}\int_0^t p(r)^2(p(r)-2)^2H(\delta)^{p(r)-4}\left(\int_{\mathbb{T}^2}|Z^\epsilon|\,\mathrm{d}x\right)^4\mathrm{d}r\\ \leq&Ce^{C\|\xi_0\|_{L^\infty_x}}(\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2)^2\mathbb{E}\int_0^t\sup_{s\in[0,r]}H(\delta)_s^{p(s)}\,\mathrm{d}r.\end{aligned}$$

Putting all together, we obtain, for some $C=C(\sum_k\|\sigma_k\|_{W^{1,\infty}_x}^2)$ (possibly depending also on p and T),

$$\begin{aligned}\mathbb{E}[\sup_{s\in[0,t]}H(\delta)_s^{p(s)}]\leq&\mathbb{E}[H(\delta)_0^p]+C\|\xi_0\|_{L^\infty_x}(e^{C\|\xi_0\|_{L^\infty_x}}-1)\|K^\epsilon-K\|_{L^1_x}^p\\ &+C(1+\|\xi_0\|_{L^\infty_x}^2)e^{C\|\xi_0\|_{L^\infty_x}}\int_0^t\mathbb{E}\sup_{s\in[0,r]}H(\delta)_s^{p(s)}\,\mathrm{d}r.\end{aligned}$$

Applying first Young inequality and then letting $\delta\to 0$ (f_δ converges to f uniformly), we obtain

$$\mathbb{E}[\sup_{s\in[0,t]}H_s^{p(s)}]\leq C\|\xi_0\|_{L^\infty_x}e^{C\|\xi_0\|_{L^\infty_x}+C\|\xi_0\|_{L^\infty_x}^2+\exp[C\|\xi_0\|_{L^\infty_x}]}\|K^\epsilon-K\|_{L^1_x}^p.$$

Since H is uniformly bounded (as Φ and Φ^ϵ take values on $\mathbb{T}^2$) and $p(t)$ is increasing in t, then $H^{p(s)} \geq cH^{p(t)}$ for any $s \leq t$, for some $c = c(\|\xi_0\|_{L^\infty}) > 0$ (depending also on T and p). Therefore we conclude

$$\mathbb{E}[\sup_{s\in[0,t]} H_s^p] \leq \mathbb{E}[\sup_{s\in[0,t]} H_s^{p(t)}]^{p/p(t)} \leq C\mathbb{E}[\sup_{s\in[0,t]} H_s^{p(s)}]^{p/p(t)} \leq C\|K^\epsilon - K\|_{L_x^1}^{p^2/p(t)} = C\|K^\epsilon - K\|_{L_x^1}^{pe^{-\lambda t}}.$$

□

5 Vortex approximation for Euler equations

Theorem 18. *Let $M \in \mathbb{R}_+$ and $\xi_0 \in L^\infty(\mathbb{T}^2)$ such that $\|\xi_0\|_{L^\infty} \leq M$. Let $(x^{i,N})_{1\leq i\leq N} \subset \mathbb{T}^2$ and $(\xi^{i,N})_{1\leq i\leq N} \subset \mathbb{R}$ such that $S_0^N := \frac{1}{N}\sum_{i=1}^N \xi^{i,N}\delta_{x^{i,N}}$ is in $\mathcal{M}_M$. Assume that*

$$W_1(S_0^N, \xi_0) =: \zeta_N \to 0, \quad \text{as } N \to \infty. \tag{33}$$

Call $S_t^{N,\epsilon}$ the empirical measure associated to the system (31) *with initial condition $(x^{i,N})_{1\leq i\leq N}$ and ξ_t the solution to the Euler equation* (1) *starting form ξ_0. Then, taking the approximation $\epsilon(N) = o\left((-\log(\zeta_N))^{-(4(2+\delta))^{-1}}\right)$, the following convergence holds true, on every time interval $[0,T]$,*

$$d_1(S^{N,\epsilon(N)}, \xi) \to 0, \quad \text{as } N \to \infty.$$

Proof. Let ξ^ϵ be a solution to equation (6). We split

$$d_1(S^{N,\epsilon}, \xi) \leq d_1(S^{N,\epsilon}, \xi^\epsilon) + d_1(\xi^\epsilon, \xi). \tag{34}$$

We will obtain the estimate of the two terms on the right-hand side as a consequence of Corollary 16 and Theorem 17 respectively.

For the second term in the right-hand side of (34), using the definition of d_1, we have

$$d_1(\xi^\epsilon, \xi) \leq \mathbb{E}\left[\sup_{t\in[0,T]} \int |\Phi_t^\epsilon(x) - \Phi_t(x)|\, |\xi_0(x)| dx\right].$$

Recall that the initial condition ξ_0 is deterministic and in $L^\infty(\mathbb{T}^2)$. Hence from Theorem 17 we obtain

$$d_1(\xi^\epsilon, \xi) \leq C(M)\|K^\epsilon - K\|_{L^1}, \tag{35}$$

which goes to 0 by Lemma 6. For the first term in the right-hand side of (34), we have the following estimate from Corollary 16,

$$d_1(S^{N,\epsilon}, \xi^\epsilon) \leq d_4(S^{N,\epsilon}, \xi^\epsilon) \leq 2e^{cT}\Gamma W_1(S_0^N, \xi_0).$$

From the definition of c and Γ and Lemma (iii) we have that for any $\delta > 0$,

$$\Gamma e^{cT} \sim C\|D^2K^\epsilon\|_{C^0} e^{C\|DK^\epsilon\|_{C^0}^p} e^{C\|DK^\epsilon\|_{C^0}^{p^2/(2p-4)}} \sim C\epsilon^{-(3+\delta)} e^{C\epsilon^{-4(2+\delta)}} \sim e^{C\epsilon^{-4(2+\delta)}}, \qquad \text{as } \epsilon \to 0,$$

with $C = C(T, \alpha, \sigma)$ (note that $p = 4$ minimizes $p^2/(2p-4)$). We conclude by taking $\epsilon(N) = o\left((-\log(\zeta_N))^{-(4(2+\delta))^{-1}}\right)$, so that

$$\epsilon(N) \to 0 \quad \text{and} \quad e^{\epsilon(N)^{-7(2+\delta)}} W_1(S_0^N, \xi_0) \to 0, \qquad \text{as } N \to \infty.$$

□

We show now that starting from a bounded initial vorticity it is always possible to construct a sequence of empirical measures that satisfies (33).

Lemma 19. *Let $\xi_0 \in L^\infty(\mathbb{T}^2)$ with $\|\xi_0\|_{L^\infty} \leq M < \infty$. There exist families $(\xi^i)_{i\in\mathbb{N}} \subset [-M, M]$ and $(x^i)_{i\in\mathbb{N}} \subset \mathbb{T}^2$ such that*

$$W_1\left(\frac{1}{N}\sum_{i=1}^N \xi^i \delta_{x^i}, \xi_0\right) \to 0, \qquad N \to \infty.$$

Proof. There exist two non negative functions $\xi_0^+, \xi_0^- \in L^\infty(\mathbb{T}^2)$ such that $\xi_0 = \xi_0^+ - \xi_0^-$, Lebesgue-almost surely. We define

$$\mu_0(dm, dx) := \frac{1}{M}\delta_M(dm)\xi_0^+(x)dx + \frac{1}{M}\delta_{-M}(dm)\xi_0^-(x)dx \in \mathcal{P}([-M, M] \times \mathbb{T}^2).$$

On an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider an independent and identically distributed sequence of random variables $(M^i, X^i)_{i\in\mathbb{N}}$ with law μ_0.

Let $\varphi \in BL_1(\mathbb{T}^2)$, then $(m, x) \mapsto \frac{m\varphi(x)}{M+1} \in BL_1([-M, M] \times \mathbb{T}^2)$. Hence, we have $\mathbb{P}$-a.s.,

$$W^1\left(\frac{1}{N}\sum_{i=1}^N M^i\delta_{X^i}, \xi_0\right) = \sup_{\varphi\in BL_1} \int_{[-M,M]\times\mathbb{T}^2} m\varphi(x) \left(\frac{1}{N}\sum_{i=1}^N \delta_{(M^i,X^i)} - \mu_0\right)(dm, dx).$$

By the law of large numbers, the right-hand side goes to zero almost surely. Thus, for any ω in a set of full measure we have that the lemma is satisfied for the families $(\xi^i, x^i)_{i\in\mathbb{N}} := (M^i(\omega), X^i(\omega))_{i\in\mathbb{N}}$. □

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