Equilibrium model with default and insider's dynamic information[*]

Luciano Campi[†]   Umut Çetin[‡]   Albina Danilova[§]

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Abstract

We consider an equilibrium model à la Kyle-Back for a defaultable claim issued by a given firm. In such a market the insider observes *continuously in time* the value of firm, which is unobservable by the market makers. Using the construction of a dynamic Bessel bridge of dimension 3 in [5], we provide the equilibrium price and the optimal insider's strategy. As in [3], the information released by the insider while trading optimally makes the default time predictable in market's view at the equilibrium. We conclude the paper by comparing the insider's expected profits in the static and dynamic private information case. We also compute explicitly the value of insider's information in the special cases of a defaultable stock and a bond.

Key-words: Default time, defaultable claim, equilibrium, dynamic information, insider trading, dynamic Bessel bridge.
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JEL classification: D82, G14.

1   Introduction

Consider a defaultable claim issued by a company with no recovery and the payoff $f(1 + \beta_1)$ in case of no-default. β denotes the fundamental value process and is assumed to be a standard Brownian motion with $\beta_0 = 0$. The default time, T_0, is given by

$$T_0 := \inf\{t > 0 : 1 + \beta_t = 0\}.$$

Campi and Çetin [3] study an equilibrium model à la Kyle-Back in the case when T_0 is known to the insider at time 0. The main result therein is that a risk-neutral insider who is thus maximizing her expected profit reveals part of her private information making the default time predictable in market's view, while that very default time was, by assumption, totally inaccessible before the trading started. Moreover, it is shown in [3] that the equilibrium demand is a 3-dimensional

[†]CEREMADE, University Paris-Dauphine, campi@ceremade.dauphine.fr.
[‡]Corresponding author. Department of Statistics, London School of Economics, u.cetin@lse.ac.uk.
[§]Department of Mathematics, London School of Economics, a.danilova@lse.ac.uk.

Bessel bridge for the insider and a Brownian motion for the market (so-called "Inconspicuous trade theorem"), meaning that the insider hides herself (behind the noise traders) while optimally trading.

The model assumption of an insider knowing the default time from the beginning of the trading period is very strong. The goal of this paper is to generalize the results in [3] to the more realistic case of dynamic insider's information, where the insider observes the fundamental value of the company β *continuously on time*.

However, in this case an equilibrium would not exist (for a discussion of this in a related context, see the discussion after Remark 2.2, and Remark 5.1 in [4] as well as the discussion following Theorem 2.1 in [5]). Thus, to relax the assumption of static information while ensuring the existence of a solution, we allow the insider to look into the future, i.e. we assume that she observes $Z = (1 + \beta_{V(t)})_{t \in [0,1]}$ where V is any continuous and increasing function with $V(0) = 0$, $V(t) > t$ for $t \in (0,1)$ and $V(1) = 1$. Note that the assumption that the insider observes $(Z_t)_{t \in [0,1]}$ rather than β itself is a standard assumption[1] in dynamic information models, see, e.g., Back and Pedersen [2] and Wu [13]. Note that one can find another Brownian motion, B^Z, such that

$$Z_t = 1 + \int_0^t \sqrt{V'(s)} dB_s^Z \quad t \geq 0,$$

where V' is the left derivative of V. Also observe that $Z_1 = 1 + \beta_1$ and $T_0 = V(\inf\{t > 0 : Z_t = 0\})$. A precise description of the market model, based on the latter observation, will be the content of the next section.

The main results of the present paper can be summarized as follows. We obtain even in this case the existence of an equilibrium where the insider maximizes her profit while the market makers set a unique rational pricing rule based on the observation of the total demand. This is the content of the main Theorem 3.2. In particular, we compute explicitly both the equilibrium pricing rule and the optimal insider's strategy. The corresponding equilibrium total demand X^* is a semimartingale behaving like a bridge hitting the default barrier 0 for the first time at $V(\tau)$ on the default event $[\tau \leq 1]$ while reaching the fundamental value Z_1 at maturity in case of no-default, i.e. on $[\tau > 1]$. Moreover we obtain, as in the static information case in [3], that the equilibrium total demand X^* is a Brownian motion for the market makers, that is the insider hides her actions behind noise trading so that the "Inconspicuous trade theorem" holds true.

Using the characterization of the equilibrium we have just described, we can compare the static and dynamic information case. It turns out that an agent willing to pay a price (the value of information) for getting some private information is indifferent between knowing τ and Z_1 from the beginning or only at the end through a continuous-in-time monitoring of the fundamental value Z. This is due to the fact that from her viewpoint the expected profits in both cases are the same (Theorem 4.1 and the discussion below it). Finally, we computed explicitly the value of private information in the two important cases of a defaultable stock and a corporate bond, finding that the longer the defaultable claim is traded, the higher such a value is.

Since this paper deals with a Kyle-Back equilibrium model with default and gradually revealing information, it could also be viewed as a generalization of Back and Pedersen [2] and Campi, Çetin and Danilova [4] to a financial market with default.

The structure of the paper is the following: Section 2 sets the model, while Section 3 gives the existence and the characterization of the equilibrium (Theorem 3.2) via an explicit computation of

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[1]Observe that the signal process of the insider in Back and Pedersen [2] and Wu [13] can be rewritten as a time-changed Brownian motion where the time-change is given by $V(t)$ with $V(t) > t$ for every $t \in (0,1)$.

the optimal insider's expected profit (Proposition 3.1). In the proofs of that section a crucial role is played by the construction of a 3-dimensional dynamic Bessel bridge in [5]. Finally, Section 4 provides a comparison between the static and the dynamic information case.

2 Description of the market model

To formulate the model of the market precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and $\mathbb{P}$-saturatedness. Assume that on this probability space there exist two independent standard Brownian motions, B and B^Z.

Consider now a defaultable claim issued by a firm with no recovery and payoff $f(Z_1)$ in case of no-default, where Z denotes the fundamental value and follows

$$Z_t = 1 + \int_0^t \sigma(s) dB_s^Z \quad t \geq 0, \tag{2.1}$$

where $\sigma : [0, 1] \mapsto \mathbb{R}_+$ is a measurable deterministic function and $V(t) := \int_0^t \sigma^2(s) \, ds$.

Assumption 2.1 V and f satisfy the following:

1. $V(t) > t$ for all $t \in (0, 1)$ and $V(1) = 1$;

2. there exists some $\varepsilon > 0$ such that $\int_0^\varepsilon \frac{1}{(V(t)-t)^2} dt < \infty$.

3. $f : [0, \infty) \mapsto \mathbb{R}_+$ is a nondecreasing function which is not identically 0 and such that

$$|f(z)| \leq k_1 \exp(k_2 z), \qquad \forall z \in [0, \infty),$$

for some constants k_1 and k_2.

The firm's default time is given by the random time $V(\tau)$ where

$$\tau := \inf\{t > 0 : Z_t = 0\}. \tag{2.2}$$

and the price of the defaultable claim is determined in equilibrium. We generalize, by incorporating dynamic information, the equilibrium framework of Campi and Çetin [3], which in turn is an extension of that of Back [1] to a market with a defaultable bond. We refer the reader to Back [1] for motivation and details that are not explained in what follows.

Remark 1 The modeling of the default can also be interpreted in terms of economic, τ, and recorded, $V(\tau)$, default. It is documented that these two notions of default do not necessarily coincide and the latter is typically later than the former (see Guo et al. [8]).

The microstructure of the market, and the interaction of market participants, is modeled as follows. There are three types of agents: noisy/liquidity traders, an informed trader (insider), and competitive market makers, all of whom are risk neutral. The agents differ in their information sets, and objectives, as follows.

- *Noisy/liquidity traders* trade for liquidity reasons, and their total demand at time t is given by a standard $(\mathcal{F}_t)$-Brownian motion B independent of B^Z.

- *The market makers* set the price of the defaultable claim using their information set. Since the market makers are risk neutral and identical in their information sets, they set the same price. Their information consists of two parts. The first component is the total order of the noise traders and the insider, which is denoted with $\tilde{X}$ and has the decomposition

$$\tilde{X} = \tilde{X}^\theta = B^{V(\tau)} + \theta^{V(\tau)}, \qquad (2.3)$$

where θ is the position of the insider in the defaultable claim so that the total demand right before the insider starts trading at time 0 equals 0. Note that we stop the market at time $V(\tau)$ so that there is no trading in the defaultable claim once the default has occurred. Let $X = 1 + \tilde{X}$ and observe that X and $\tilde{X}$ generate the same filtration. We will denote the minimal right continuous and complete filtration generated by X with $\mathcal{F}^X$, where we suppress the dependency on θ in the notation.

The second part of the market makers' information comes from the observation of the recorded default event, i.e. the market makers also observe whether the recorded default has happened or not. In mathematical terminology, this makes $V(\tau)$ a stopping time in his filtration. Therefore, the market makers' information is modeled by the filtration $\mathcal{F}^M = (\mathcal{F}_t^M)_{0 \leq t \leq 1}$ where $\mathcal{F}_t^M := \mathcal{F}_t^X \vee \sigma(V(\tau) \wedge t)$.

Similar to Back [1] and Campi and Çetin [3], we assume that the market makers set their price as a function of the total order process at time t, i.e. we assume that

$$S_t = D_t H(t, X_t), \qquad \forall t \in [0, 1) \qquad (2.4)$$

where D is the non-default process, i.e. $D_t = \mathbf{1}_{[V(\tau) > t]}$. Moreover, a pricing rule H has to be admissible in the sense of Definition 2.1. In particular, $H \in C^{1,2}$ and, therefore, S is a semimartingale on $[0, 1)$.

- *The informed investor* observes the price process $S_t = D_t H(t, X_t)$ where X is given by (2.3), and the fundamental value Z_t, i.e. her filtration, $(\mathcal{F}_t^I)_{t \in [0,1]}$, is given by $(\mathcal{F}_t^{Z,S})$. She is risk-neutral, thus, her objective is to maximize the expected final wealth, i.e.

$$\sup_{\theta \in \mathcal{A}(H)} \mathbb{E}\left[W_1^\theta\right] = \sup_{\theta \in \mathcal{A}(H)} \mathbb{E}\left[(f(Z_1)\mathbf{1}_{[V(\tau) > 1]} - S_{1 \wedge V(\tau)-})\theta_{1 \wedge V(\tau)} + \int_0^{V(\tau) \wedge 1-} \theta_s dS_s\right] \qquad (2.5)$$

where $\mathcal{A}(H)$ is the set of admissible trading strategies for the given pricing rule H, which will be defined in Definition 2.2. In particular, θ is an absolutely continuous process. That is, the insider maximizes the expected value of her final wealth W_1^θ, where the first term on the right hand side of equation (2.5) is the contribution to the final wealth due to a potential differential between price and fundamental value at time $V(\tau) \wedge 1$, when the market terminates, and the second term is the contribution to final wealth coming from the trading activity.

Note also that the above market structure implies that the insider's optimal trading strategy takes into account the *feedback effect* i.e. that prices react to her trading strategy according to (2.4). Our goal is to find the rational expectations equilibrium of this market, i.e. a pair consisting of a pricing rule and an *admissible* trading strategy such that: *a)* given the pricing rule the trading

strategy is optimal, *b)* given the trading strategy, the pricing rule is *rational*[2] in the following sense:

$$D_t H(t, X_t) = S_t = \mathbb{E}\left[f(Z_1)\mathbf{1}_{[V(\tau)>t]} | \mathcal{F}_t^M \right], \quad t \in [0, 1]. \tag{2.6}$$

This in particular implies $S_{1 \wedge V(\tau)} = f(Z_1)\mathbf{1}_{[V(\tau)>1]}$ because true prices are revealed to the market at $1 \wedge V(\tau)$. To formalize this definition of equilibrium, we first need to define the sets of admissible pricing rules and trading strategies.

Definition 2.1 *A measurable function $H : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ is a pricing rule if*

1. *$H \in C^{1,2}([0, 1) \times \mathbb{R})$;*

2. *$\mathbb{E}[D_1 H(1, B_1)] < \infty$ and $\mathbb{E}[\int_0^1 D_t H(t, B_t)^2 dt] < \infty$;*

3. *$x \mapsto H(t, x)$ is strictly increasing for every $t \in [0, 1)$.*

Moreover, let θ be a trading strategy of the insider. Given θ, a pricing rule H is said to be rational if it satisfies equation (2.6).

Remark 2 The strict monotonicity of H in the space variable implies $x \mapsto H(t, x)$ is invertible for $t \in [0, 1)$, thus, the filtration of the insider is generated by X and Z. This in turn implies that $\mathcal{F}_t^{S,Z} = \mathcal{F}_t^{B,Z}$, for all $t \in [0, 1)$. Using the continuity of the processes involved we get $(\mathcal{F}_t^{S,Z})_{t \in [0,1]} = (\mathcal{F}_t^{B,Z})_{t \in [0,1]}$, i.e. the insider has full information about the market.

It is standard (see, e.g., [2], [6] or [13]) in the insider trading literature to limit the set of admissible strategies to absolutely continuous ones motivated by the result in Back [1], and we do so. The formal definition of the set of admissible trading strategies is summarized in the following definition.

Definition 2.2 *An $\mathcal{F}^{B,Z}$-adapted θ is said to be an admissible trading strategy for a given pricing rule H if*

1. *it is absolutely continuous with respect to the Lebesgue measure, i.e. $\theta_t = \int_0^t \alpha_s ds$;*

2. *(X, Z) is a Markov process adapted to $(\mathcal{F}_t)$;*

3. *and no doubling strategies are allowed i.e.*

$$\mathbb{E}\left[\int_0^1 D_t H^2(t, X_t) \, dt \right] < \infty. \tag{2.7}$$

The set of admissible trading strategies for a given H is denoted with $\mathcal{A}(H)$.

Given these definitions of admissible pricing rules and trading strategies, it is now possible to formally define the market equilibrium as follows.

Definition 2.3 *A couple (H^*, θ^*) is said to form an equilibrium if H^* is an admissible pricing rule, $\theta^* \in \mathcal{A}(H^*)$, and the following conditions are satisfied:*

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[2]This rational pricing rule corresponds to perfect competition among the risk neutral market makers.

1. Market efficiency condition: *given θ^*, H^* is a rational pricing rule.*

2. Insider optimality condition: *given H^*, θ^* solves the insider optimization problem:*

$$\mathbb{E}[W_1^{\theta^*}] = \sup_{\theta \in \mathcal{A}(H^*)} \mathbb{E}[W_1^{\theta}].$$

3 Equilibrium with dynamic insider's information

In order to determine the conditions for equilibrium we start with the optimality conditions for the insider. The next proposition characterizes the optimal insider's strategy for a given class of pricing rules set by the market makers. This class contains the solutions to the heat equation, which are fully determined by the terminal value $h(x)$. Therefore, choice of the pricing rule $H(t, x)$ is equivalent to the choice of $h(x)$ by the market makers. The following proposition is valid for a general $h(x)$ and we will establish in Theorem 3.2 the existence of a particular h ensuring the equilibrium.

Proposition 3.1 *Assume that a pricing rule H is a classical solution to*

$$H_t(t, x) + \frac{1}{2} H_{xx}(t, x) = 0, \qquad H(1, x) = h(x), \tag{3.8}$$

where h is nondecreasing right continuous function on $\mathbb{R}$ with at most exponential growth, the range of h contains that of f and $0 \in (\inf_x h(x), \sup_x h(x))$.

If $\theta^* \in \mathcal{A}(H)$ satisfies $\lim_{t \uparrow 1} H(V(\tau) \wedge t, X^*_{V(\tau) \wedge t}) = f(Z_1)\mathbf{1}_{[\tau > 1]}$ a.s., where $X^* = B + \theta^*$, then θ^* is an optimal strategy, i.e., for all $\theta \in \mathcal{A}(H)$,

$$\mathbb{E}[W_1^{\theta}] \leq \mathbb{E}[W_1^{\theta^*}] = \mathbb{E}\left[\int_{\xi(0, a^*)}^{1} (H(0, u) - a^*) du + \frac{1}{2} \int_0^{1 \wedge V(\tau)} H_x(s, \xi(s, a^*)) ds\right], \tag{3.9}$$

where $a^* = \mathbf{1}_{[\tau > 1]} f(Z_1)$ and $\xi(t, a)$ is the unique solution of $H(t, \xi(t, a)) = a$ for all a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$ and $t < 1$.

PROOF. We will adapt Wu's proof of his Lemma 4.2 in [13]. This proof can be splitted into two main steps.

Step 1. Fix an a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$. Suppose that a is the maximum of h, then $\xi(t, a) = \infty$ for all $t \in [0, 1)$ since H is strictly increasing with supremum being equal to a. Similarly if a is the minimum of h, then $\xi(t, a) = -\infty$ for all $t \in [0, 1)$. If a is neither the minimum nor the maximum, $\xi(t, a)$ exists since $H(t, x)$ is strictly increasing in x for each $t \in [0, 1)$. Moreover, for all $a \in (\inf_x h(x), \sup_x h(x))$, the mapping $t \mapsto \xi(t, a)$ is uniformly bounded on the interval $[0, 1)$ due to Lemma A.1 and the continuity of H.

Next, consider the function

$$\Phi^a(t, x) := \int_{-\infty}^{\infty} \int_{X_{min}^a}^{y+x} (h(u) - a) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{y^2}{2(1-t)}\right) du \, dy, \tag{3.10}$$

where

$$X_{min}^a := \inf\{x : h(x) \geq a\}.$$

Direct calculations show that

$$\Phi_t^a + \frac{1}{2}\Phi_{xx}^a = 0,$$

with the boundary condition

$$\Phi^a(1, x) = \int_{X_{min}^a}^x (h(u) - a) du.$$

Therefore, Φ^a is jointly continuous and nonnegative on $[0,1] \times \mathbb{R}$ and $C^{1,2}([0,1) \times \mathbb{R})$. Moreover, for $t < 1$,

$$\Phi_x^a(t, x) = H(t, x) - a,$$

so that the minimum of Φ^a is achieved at $\xi(t, a)$.

Suppose that $|\xi(t, a)| = \infty$ so that a is either the minimum or the maximum of h and that $\Phi^a(1, x)$ is positive and decreasing as $x \to \xi(t, a)$. Then,

$$
\begin{aligned}
\Phi^a(t, \xi(t, a)) \quad &:= \quad \lim_{x \to \xi(t,a)} \Phi^a(t, x) = \lim_{x \to \xi(t,a)} \int_{-\infty}^{\infty} \Phi^a(1, x + y) \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) dy \\
&= \quad \int_{-\infty}^{\infty} \lim_{x \to \xi(t,a)} \Phi^a(1, x + y) \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) dy = 0. \qquad (3.11)
\end{aligned}
$$

Moreover, when $\xi(t, a)$ is finite,

$$
\begin{aligned}
0 \quad &\leq \quad \lim_{t \uparrow 1} \Phi^a(t, \xi(t, a)) \leq \lim_{t \uparrow 1} \Phi^a(t, X_{min}^a) \\
&= \quad \lim_{t \uparrow 1} \int_{-\infty}^{\infty} \int_{X_{min}^a}^{y + X_{min}^a} (h(u) - a) \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) du \, dy = 0. \qquad (3.12)
\end{aligned}
$$

Next observe that

$$
\begin{aligned}
\Phi^a(t, x) - \Phi^a(t, \xi(t, a)) \quad &= \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) \int_{y + \xi(t,a)}^{y + x} (h(u) - a) du \, dy \\
&= \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) \int_{\xi(t,a)}^{x} (h(u + y) - a) du \, dy \\
&= \quad \int_{\xi(t,a)}^{x} (H(t, u) - a) du. \qquad (3.13)
\end{aligned}
$$

Step 2. Consider for all $\nu < 1$

$$\Psi^{a,\nu}(t, x) := \int_{\xi(t,a)}^{x} (H(t, u) - a) du + \frac{1}{2}\int_t^{\nu} H_x(s, \xi(s, a)) ds, \qquad t \leq \nu.$$

Notice that the both integrals in the RHS are well-defined: the first one is well-defined for all values of $\xi(t, a)$ thanks to Step 1, and the second one is well-defined due to the fact that $t \mapsto H_x(t, \xi(t, a))$ is uniformly bounded on $[0, \nu]$. Direct differentiation with respect to x gives that

$$\Psi_x^{a,\nu}(t, x) = H(t, x) - a. \qquad (3.14)$$

Differentiating above with respect to x gives

$$\Psi_{xx}^{a,\nu}(t, x) = H_x(t, x). \qquad (3.15)$$

7

Direct differentiation of $\Psi^{a,\nu}(t,x)$ with respect to t gives

$$\Psi_t^{a,\nu}(t,x) = \int_{\xi(t,a)}^x H_t(t,u)du - \frac{1}{2}H_x(t,\xi(t,a)) = -\frac{1}{2}H_x(t,x)$$

where in order to obtain the last equality we used (3.8). Combining this and (3.15) gives

$$\Psi_t^{a,\nu} + \frac{1}{2}\Psi_{xx}^{a,\nu} = 0.$$

Therefore from (3.14) and Itô's formula it follows that

$$\Psi^{a,\nu}(\nu, X_\nu) - \Psi^{a,\nu}(0,1) = \int_0^\nu (H(u, X_u) - a)dX_u,$$

and in particular, when $a := \mathbf{1}_{[\tau>1]}f(Z_1)$, $\nu(t) = t \wedge V(\tau)$,

$$\lim_{t\uparrow 1}\left(\Psi^{a,\nu(t)}(t \wedge V(\tau), X_{t\wedge V(\tau)}) - \Psi^{a,\nu(t)}(0,1)\right) = \int_0^{1\wedge V(\tau)}(H(t,X_t) - \mathbf{1}_{[\tau>1]}f(Z_1))dX_t. \quad (3.16)$$

By the admissibility properties of θ, in particular $d\theta_t = \alpha_t dt$, the insider's optimization problem becomes

$$\begin{aligned}
\sup_{\theta\in\mathcal{A}(H)}\mathbb{E}[W_1^\theta] &= \sup_{\theta\in\mathcal{A}(H)}\mathbb{E}\left[(f(Z_1)\mathbf{1}_{[\tau>1]} - S_{1\wedge V(\tau)-})\theta_{1\wedge V(\tau)} + \int_0^{V(\tau)\wedge 1-}\theta_s dS_s\right] \\
&= \sup_{\theta\in\mathcal{A}(H)}\mathbb{E}\left[\int_0^{V(\tau)\wedge 1}(f(Z_1)\mathbf{1}_{[\tau>1]} - H(t,X_t))\alpha_t dt\right]
\end{aligned}$$

where we used integration-by-parts formula and the fact that $\theta_0 = 0$ to obtain the second equality. Due to above and (3.16), we have

$$\mathbb{E}[W_1^\theta] = -\mathbb{E}\left[\lim_{t\uparrow 1}\left(\Psi^{a,\nu(t)}(t \wedge V(\tau), X_{t\wedge V(\tau)}) - \Psi^{a,\nu(t)}(0,1)\right)\right]. \quad (3.17)$$

Notice that all the Brownian integrals vanish due to (2.7) in Definition 2.2, and

$$\mathbb{E}\left[\left(\int_0^{1\wedge\tau}f(Z_1)dB_t\right)^2\right] \leq \mathbb{E}\left[f(Z_1)^2\right]\mathbb{E}[B_1^2] < \infty,$$

since Z and B are independent.

The conclusion follows from the fact that $\lim_{t\uparrow 1}\Psi^{a,\nu(t)}(t \wedge V(\tau), X_{t\wedge V(\tau)})$ is nonnegative and equals 0 if $\lim_{t\uparrow 1}H(V(\tau)\wedge t, X_{V(\tau)\wedge t}^*) = f(Z_1)\mathbf{1}_{[\tau>1]}$. Indeed, observe that

$$\lim_{t\uparrow 1}\Psi^{a,\nu(t)}(t \wedge V(\tau), X_{t\wedge V(\tau)}) = \lim_{t\uparrow 1}\int_{\xi(t\wedge V(\tau),a)}^{X_{t\wedge V(\tau)}}(H(t \wedge V(\tau), u) - a)du.$$

On the set $[V(\tau) \geq 1]$ we have

$$
\begin{aligned}
\lim_{t \uparrow 1} \Psi^{a,\nu(t)}(t \wedge V(\tau), X_{t \wedge V(\tau)}) &= \lim_{t \uparrow 1} \int_{\xi(t,a)}^{X_t} (H(t,u) - a)du \\
&= \lim_{t \uparrow 1} \{\Phi^a(t, X_t) - \Phi^a(t, \xi(t,a))\} = \Phi^a(1, X_1),
\end{aligned}
$$

which is nonnegative and is 0 if $\lim_{t \uparrow 1} H(t, X_t^*) = f(Z_1)$ in view of Lemma A.1. Observe that we used (3.13) for the second equality above while (3.11) and (3.12) for the third one.

On the set $[V(\tau) < 1]$,

$$
\lim_{t \uparrow 1} \int_{\xi(t \wedge V(\tau),a)}^{X_{t \wedge V(\tau)}} (H(t \wedge V(\tau), u) - a)du = \int_{\xi(V(\tau),a)}^{X_{V(\tau)}} (H(V(\tau), u) - a)du,
$$

which is nonnegative and equals 0 if $X_{V(\tau)} = \xi(V(\tau), a)$ due to the invertibility of H. Therefore, an insider trading strategy which gives $\lim_{t \uparrow 1} H(V(\tau) \wedge t, X_{V(\tau) \wedge t}^*) = f(Z_1)\mathbf{1}_{[\tau > 1]}$ is optimal. ∎

Remark 3 The same results as in Proposition 3.1 above apply when the initial insider's information $\mathcal{F}_0^I$ is not trivial provided one replaces expectations with conditional expectations given $\mathcal{F}_0^I$ in the statement as well as in its proof.

Combining Proposition 3.1 and the dynamic Bessel bridge construction performed in [5], we can finally state and prove the main result of this paper. Before that, we need some preliminary notation.

Recall that $\tau = \inf\{t > 0 : Z_t = 0\}$. Since Z is a time-changed Brownian motion, one can characterize the distribution of τ using the well-known distributions of first hitting times of a standard Brownian motion. To this end let

$$
\mathbb{P}[T_a > t] = \int_t^\infty \ell(u, a)\, du, \tag{3.18}
$$

for $a > 0$ where

$$
T_a := \inf\{t > 0 : B_t = a\}, \quad \text{and} \quad \ell(t, a) := \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).
$$

Another formulation for the distribution of T_a can be stated in terms of the transition density of a *Brownian motion killed at 0*. Recall that this transition density is given by

$$
q(t, x, y) := \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(x - y)^2}{2t}\right) - \exp\left(-\frac{(x + y)^2}{2t}\right)\right),
$$

for $x > 0$ and $y > 0$ (see Exercise (1.15), Chapter III in [12]). Then one has the identity

$$
\mathbb{P}[T_a > t] = \int_0^\infty q(t, a, y)\, dy. \tag{3.19}
$$

Moreover, if f is a function satisfying the conditions in Assumption 2.1, then one can define

$$
P(t, z) := \int_0^\infty f(y)q(1 - V(t), z, y)dy \tag{3.20}
$$

9

so that $\mathbf{1}_{[\tau>t]}P(t, Z_t)$ is the value of the defaultable claim for the insider at time t and the process $(\mathbf{1}_{[\tau>t]}P(t, Z_t))_{t\in[0,1]}$ is a martingale for the insider's filtration.

For reader's convenience, we recall here the main result of the paper [5]. This is the key ingredient to solve our equilibrium model.

Theorem 3.1 ([5]) *Suppose that $V(t) > t$ for all $t > 0$ and satisfies Assumption 2.1.2. Then, there exists a unique strong solution to*

$$X_t = 1 + B_t + \int_0^{\tau\wedge t} \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)}\, ds + \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)}\, ds.$$

Moreover,

i) Let $\mathcal{F}_t^X = \mathcal{N}\bigvee\sigma(X_s; s \leq t)$, where $\mathcal{N}$ is the set of $\mathbb{P}$-null sets. Then, X is a standard Brownian motion with respect to $\mathbb{F}^X = (\mathcal{F}_t^X)$;

ii) $V(\tau) = \inf\{t > 0 : X_t = 0\}$.

Notice that we cannot apply directly the previous result to get the solution to our equilibrium problem, since we impose $V(1) = 1$ for the time-change $V(t)$. Nonetheless, Theorem 3.1 will be very useful to prove our main result, which is stated in the following theorem.

Theorem 3.2 *Under Assumption 2.1 there exists an equilibrium (H^*, θ^*), where*

(i) $H^*(t, x) = P(V^{-1}(t), x)$ where $P(t, x)$ is given by (3.20) for $(t, x) \in [0, 1] \times \mathbb{R}$.

(ii) $\theta_t^* = \int_0^t \alpha_s^* ds$ where

$$\alpha_s^* = \frac{q_x(V(s) - s, X_s^*, Z_s)}{q(V(s) - s, X_s^*, Z_s)}\mathbf{1}_{[s\leq\tau\wedge 1]} + \frac{\ell_a(V(\tau) - s, X_s^*)}{\ell(V(\tau) - s, X_s^*)}\mathbf{1}_{[\tau\wedge 1 < s \leq V(\tau)\wedge 1]} \tag{3.21}$$

where the process X^* is the unique strong solution under insider's filtration $\mathcal{F}^{X,Z}$ of the following SDE:

$$X_t = 1 + B_{V(\tau)\wedge t} + \int_0^{\tau\wedge t} \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)}\, ds + \int_{\tau\wedge t}^{V(\tau)\wedge t} \frac{\ell_a(V(\tau) - s, X_s)}{\ell(V(\tau) - s, X_s)}\, ds. \tag{3.22}$$

Moreover, $V(\tau) = \inf\{t \in [0, 1] : X_t^* = 0\}$, where $\inf\emptyset = 1$ by convention, and one has $\lim_{t\uparrow 1} X_t^* = Z_1$ on the set of non-default $[\tau > 1]$. As a consequence, $V(\tau)$ is a predictable stopping time in the market filtration $\mathcal{F}^M$.

Furthermore, the expected profit of the insider is

$$\mathbb{E}\left[\int_{\xi(0,a^*)}^1 (H^*(0, u) - a^*)du + \frac{1}{2}\int_0^{1\wedge V(\tau)} H_x^*(s, \xi(s, a^*))ds\right], \tag{3.23}$$

where $a^* = \mathbf{1}_{[\tau>1]}f(Z_1)$ and $\xi(t, a)$ is the unique solution of $H(t, \xi(t, a)) = a$ for all $a \geq 0$.

PROOF.   Observe that

$$H^*(t, x) = \int_{-\infty}^{\infty} \tilde{f}(x + y) \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) dy,$$

where

$$\tilde{f}(y) = \begin{cases} f(y), & y \geq 0; \\ -f(-y), & y < 0. \end{cases}$$

Thus, clearly, $H^* \in C^{1,2}([0, 1) \times \mathbb{R})$, $\mathbb{E}[D_1 H^*(1, B_1)] < \infty$, and $\mathbb{E}[\int_0^1 D_t H^*(t, B_t)^2 dt] < \infty$. To show that it is a pricing rule in the sense of Definition 2.1, it is enough to show that it is strictly increasing for any $t \in [0, 1)$. We have, due Assumption 2.1.3, for $x > z$ and $t < 1$,

$$H(t, x) - H(t, z) = \int_{-\infty}^{\infty} (\tilde{f}(x + y) - \tilde{f}(z + y)) \frac{1}{\sqrt{2\pi(1 - t)}} \exp\left(-\frac{y^2}{2(1 - t)}\right) dy \geq 0.$$

Note that the above holds in equality if and only if $\tilde{f}(x + y) = \tilde{f}(z + y)$ for almost all y, which is not possible due to the construction of $\tilde{f}$ and the assumption on f. Therefore H^* is a pricing rule in the sense of Definition 2.1.

Moreover, direct calculations show that H^* as defined above satisfies (3.8). Since the process $(\mathbf{1}_{[\tau > t]} P(t, Z_t))_{t \in [0,1]}$ is a martingale for the insider's filtration, $(D_t H^*(t, X_t^*))_{t \in [0,1]}$ will be an $\mathcal{F}^M$-martingale as soon as we show that $(X_t^*)_{t \in [0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration, where $V(\tau)$ is the first time that it hits 0. This implies that $\mathcal{F}^{X^*} = \mathcal{F}^M$ and that H^* is a rational pricing rule in the sense of Definition 2.1 and that the proposed optimal strategy is admissible for the insider.

To do so, we prove that there exists a unique strong solution of (3.22) on $[0, 1)$, X^*, satisfying the following properties:

1) $\lim_{t\uparrow1} X_{t\wedge V(\tau)}^* = 0$ a.s. on $[\tau < 1]$,

2) $\lim_{t\uparrow1} X_t^* = Z_1$ a.s. on the set $[\tau > 1]$,

3) $(X_t^*)_{t \in [0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration.

This will establish (H^*, θ^*) as an equilibrium in view of Proposition 3.1, where $h(x)$ is chosen to be equal to $\tilde{f}(x)$, since, due to 1) and 2), we have $\lim_{t\uparrow1} H^*(t \wedge V(\tau), X_{t\wedge V(\tau)}) = \mathbf{1}_{[\tau>1]} f(Z_1)$. Moreover, the expected profit of the insider is given by (3.23) due to (3.9).

Due to Theorem 3.1 there exists unique strong solution, X^*, to (3.22) on $[0, 1)$. Moreover, $V(\tau) = \inf\{t > 0 : X_t^* = 0\}$ on the set $[\tau < 1]$, so that property 1) above is satisfied.

On the non-default set $[\tau > 1]$, which is the same as $[V(\tau) > 1]$, the SDE (3.22) becomes

$$X_t = 1 + B_t + \int_0^t \frac{q_x(V(s) - s, X_s, Z_s)}{q(V(s) - s, X_s, Z_s)} ds, \quad t \in [0, 1).$$

The function $\frac{q_x(t,x,z)}{q(t,x,z)}$ appearing in the drift above can be decomposed as follows

$$\frac{q_x(t, x, z)}{q(t, x, z)} = \frac{z - x}{t} + b(t, x, z) := \frac{z - x}{t} + \frac{\exp\left(\frac{-2xz}{t}\right)}{1 - \exp\left(\frac{-2xz}{t}\right)} \frac{2z}{t}.$$

We want to prove that $\lim_{t\uparrow 1} X_t^* = Z_1$ a.s. on the set $[\tau > 1]$. Consider the process

$$R_t := X_t^* - \lambda(t) \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)},$$

where

$$\lambda(t) = \exp\left(-\int_0^t \frac{ds}{V(s) - s}\right).$$

Direct calculations give that, on $[\tau > 1]$, $dR_t = \frac{Z_t - R_t}{V(t) - t} dt + dB_t$, thus we can apply the results in Back and Pedersen [2] (see also [4], Proposition 3.2) to conclude that R_t goes to Z_1 as $t \uparrow 1$ a.s. on the set of non-default $[\tau > 1]$. To deduce from it that $X_t \to Z_1$ a.s. on $[\tau > 1]$ when $t \uparrow 1$, we have to show

$$\lim_{t\uparrow 1} \lambda(t) \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)} = 0,$$

a.s. on $[\tau > 1]$.

Observe that $b(V(t) - t, X_t^*, Z_t) = g\left(\frac{2X_t^* Z_t}{V(t) - t}\right) \frac{1}{X_t^*}$, with $g(u) = \frac{e^{-u}}{1 - e^{-u}} u$, and $g(u) \in [0, 1]$ for all $u \in [0, +\infty]$. Since on the set $[\tau > 1]$ we have $\inf_{t\in[0,1]} Z_t > 0$ and $\inf_{t\in[0,1]} X_t^* \geq 0$ (as, due to Theorem 3.1, $\inf_{t\in[0,1)} X_t^* > 0$), we obtain $\inf_{t\in[0,1]} b(V(t) - t, X_t^*, Z_t) \geq 0$ on $[\tau > 1]$, and therefore the following two cases are possible:

- Case 1: $\lim_{t\uparrow 1} \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)} < \infty$. Then, since

$$0 \leq \lim_{t\uparrow 1} \lambda(t) \leq \lim_{t\uparrow 1} \exp\left(-\int_0^t \frac{ds}{1 - s}\right) = 0,$$

  we are done.

- Case 2: $\lim_{t\uparrow 1} \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)} = \infty$. Since both $\lambda(t)$ and $\int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)}$ are differentiable for fixed ω in $[\tau > 1]$, we can use de l'Hôpital's rule to get:

$$\lim_{t\uparrow 1} \lambda(t) \int_0^t b(V(s) - s, X_s^*, Z_s) \frac{ds}{\lambda(s)} ds = \lim_{t\uparrow 1}(V(t) - t)b(V(t) - t, X_t^*, Z_t) = 0 \qquad (3.24)$$

  a.s. on the set of non-default $[\tau > 1]$ provided $\limsup_{t\uparrow 1} b(V(t) - t, X_t^*, Z_t) < \infty$ a.s..

  Since $b(V(t) - t, X_t^*, Z_t) = g\left(\frac{2X_t^* Z_t}{V(t) - t}\right) \frac{1}{X_t^*}$ with g being a bounded function on $[0, +\infty]$, to show that $\limsup_{t\uparrow 1} b(V(t) - t, X_t^*, Z_t) < \infty$, it is sufficient to demonstrate that $\liminf_{t\uparrow 1} X_t^* > 0$ on $[\tau > 1]$.

  To prove it, consider two processes $\hat{X}$ and Y which follow

$$d\hat{X}_t = \left[\frac{Z_t - \hat{X}_t}{V(t) - t} + g\left(\frac{2\hat{X}_t Z_t}{V(t) - t}\right) \frac{1}{\hat{X}_t}\right] \mathbf{1}_{[\tau>t]} dt + dB_t, \quad t \in [0, 1),$$

$$dY_t = \frac{Z_t - Y_t}{V(t) - t} \mathbf{1}_{[\tau>t]} dt + dB_t, \quad t \in [0, 1].$$

The process $\hat{X}$ is well defined and is strictly positive for all $t \in [0,1)$ due to the Theorem 3.1. Moreover, for all $t \in [0,1)$ we have $Y_t \mathbf{1}_{[\tau>1]} = R_t \mathbf{1}_{[\tau>1]}$ and $\hat{X}_t \mathbf{1}_{[\tau>1]} = X_t^* \mathbf{1}_{[\tau>1]}$ and therefore it is sufficient to show that $\liminf_{t \uparrow 1} \hat{X}_t > 0$ on $[\tau > 1]$.

Observe that

$$d(Y_t - \hat{X}_t) = \left[\frac{\hat{X}_t - Y_t}{V(t) - t} - g\left(\frac{2\hat{X}_t Z_t}{V(t) - t} \right) \frac{1}{\hat{X}_t} \right] \mathbf{1}_{[\tau>t]} dt,$$

and g and $\hat{X}$ are strictly positive, so that by Tanaka's formula (see Theorem 1.2 in Chap. VI of [12])

$$
\begin{aligned}
(Y_t - \hat{X}_t)^+ &= \int_0^t \mathbf{1}_{[Y_s > \hat{X}_s]} \left[\frac{\hat{X}_s - Y_s}{V(s) - s} - g\left(\frac{2\hat{X}_s Z_s}{V(s) - s} \right) \frac{1}{\hat{X}_s} \right] \mathbf{1}_{[\tau>s]} ds \\
&\leq \int_0^t \mathbf{1}_{[Y_s > \hat{X}_s]} \left[\frac{\hat{X}_s - Y_s}{V(s) - s} \right] \mathbf{1}_{[\tau>s]} ds \leq 0
\end{aligned}
$$

since the local time of $Y - \hat{X}$ at 0 is identically 0 (see Corollary 1.9 in Chap. VI of [12]). Thus, on the set $[\tau > 1]$ we have

$$\liminf_{t \uparrow 1} X_t^* = \liminf_{t \uparrow 1} \hat{X}_t \geq Y_1 = R_1 = Z_1 > 0$$

as required.

Recall that $\mathbb{P}[\tau = 1] = 0$, therefore, one has $V(\tau) = \inf\{t \in [0,1] : X_t^* = 0\}$, where $\inf \emptyset = 1$ by convention. This makes $V(\tau)$ a stopping time with respect to $\mathcal{F}^{X^*}$ and yields that $\mathcal{F}^{X^*} = \mathcal{F}^M$. To complete the proof we need to show that X^* is a Brownian motion in its own filtration, stopped at $V(\tau)$. Notice first that the construction of the dynamic Bessel bridge in [5] implies in particular that X^* is a Brownian motion in its own filtration over each interval $[0,T]$ for every $T < 1$, i.e. $(X_{t \wedge V(\tau)}^*)_{t \in [0,1)}$ is a Brownian motion. As we have seen, $\lim_{t \uparrow 1} X_{t \wedge V(\tau)}^* = Z_{1 \wedge \tau}$; thus, it follows from Fatou's lemma that $(X_{t \wedge V(\tau)}^*)_{t \in [0,1]}$ is a supermartingale. In order to obtain the martingale property over the whole interval $[0,1]$, it suffices to show that $\mathbb{E}[X_{1 \wedge V(\tau)}^*] = 1$. However, since $X_{1 \wedge V(\tau)}^* = Z_{1 \wedge \tau}$, this follows from the fact that Z is a martingale. In view of Lévy's characterization, we conclude that X^* is a Brownian motion in its own filtration, stopped at $V(\tau)$. ∎

In the next section we will compare properties of the equilibriums with dynamic versus static private information. However, we will analyze the effect of default on the market parameters such as volatility and market depth in the next remark.

Remark 4 In order to understand the effect of default risk on the market parameters, let us choose an absolutely continuous pay-off $f(x)$ which is null on $(-\infty, 0]$. We have seen in the theorem above that the equilibrium price of this defaultable claim is given by

$$H^*(t,x) = \int_0^\infty f(y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)} \right) dy - \int_{-\infty}^0 f(-y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)} \right) dy$$

while equation (9) in [2] gives the price of this claim in the absence of default as

$$\widehat{H}(t,x) = \int_0^\infty f(y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)} \right) dy.$$

It is clear from above that $H^*(t,x)$ is smaller than $\widehat{H}(t,x)$. Kyle's λ which measures the market depth is given by $1/H_x$ where H is the equilibrium price in the corresponding market. Direct calculations show that

$$\widehat{H}_x - H_x^* = -\int_{-\infty}^{0} f'(-y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy < 0.$$

Thus, the market depth decreases with default risk. This is quite intuitive, since the presence of default risk increases market makers' informational disadvantage thereby causing them to decrease the market depth to compensate for the additional risk and informational disadvantage. Similarly default risk also increases the volatility of the log-returns of prices. Indeed, the above comparisons lead to $H_x^*/H^* > \widehat{H}_x/\widehat{H}$.

4 Comparison of dynamic and static private information

In this section we compare the expected profits of the insider in the cases of dynamic and static private information. By static private information we mean that the insider knows τ and Z_1 in advance. In order to do the comparison, we first need to obtain the equilibrium and the associated expected profit when the private information is static. The concepts of equilibrium, admissibility and the market microstructure are analogous to the definitions in Section 2.

Recall that the proof of Proposition 3.1 did not depend on the type of the private information, therefore, the optimality conditions for the insider with a static information are still described by it after replacing expectations with conditional expectations (see Remark 3).

Theorem 4.1 *Suppose that the insider observes τ and Z_1 at time 0. Then, under Assumption 2.1 there exists an equilibrium (H^*, θ^*), where*

(i) $H^*(t,x) = P(V^{-1}(t), x)$ where $P(t,x)$ is given by (3.20) for $(t,x) \in [0,1] \times \mathbb{R}_+$.

(ii) $\theta_t^* = \int_0^t \alpha_s^* ds$ for $t \in [0, 1 \wedge V(\tau)]$, where

$$\alpha_s^* = \frac{q_x(1-s, X_s^*, Z_1)}{q(1-s, X_s^*, Z_1)} \mathbf{1}_{[\tau>1]} + \frac{\ell_a(V(\tau) - s, X_s^*)}{\ell(V(\tau) - s, X_s^*)} \mathbf{1}_{[\tau \leq 1]} \tag{4.25}$$

where the process X^* is the unique strong solution under insider's filtration $\mathcal{F}^{X,Z_1,\tau}$ of the following SDE:

$$X_t^* = 1 + B_{V(\tau) \wedge t} + \int_0^{V(\tau) \wedge t} \left\{ \frac{q_x(1-s, X_s^*, Z_1)}{q(1-s, X_s^*, Z_1)} \mathbf{1}_{[\tau>1]} + \frac{\ell_a(V(\tau) - s, X_s^*)}{\ell(V(\tau) - s, X_s^*)} \mathbf{1}_{[\tau \leq 1]} \right\} ds. \tag{4.26}$$

Moreover, $V(\tau) = \inf\{t \in [0,1] : X_t^* = 0\}$, where $\inf \emptyset = 1$ by convention, and one has $\lim_{t \uparrow 1} X_t^* = Z_1$ on the set of non-default $[\tau > 1]$. As a consequence, $V(\tau)$ is a predictable stopping time in the market filtration $\mathcal{F}^M$.

Furthermore, the expected profit of the insider is

$$\int_{\xi(0,a^*)}^{1} (H^*(0,u) - a^*) du + \frac{1}{2} \int_0^{1 \wedge V(\tau)} H_x^*(s, \xi(s, a^*)) ds, \tag{4.27}$$

where $a^* = \mathbf{1}_{[\tau>1]} f(Z_1)$ and $\xi(t,a)$ is the unique solution of $H(t, \xi(t,a)) = a$ for all $a \geq 0$.

PROOF.   Since the optimality conditions for the insider are still described by Proposition 3.1, the proof will follow the same lines as the proof of Theorem 3.2 once we show that there exist unique strong solution of (4.26) on $[0, 1)$, X^*, satisfying the following properties:

1) $\lim_{t \uparrow 1} X^*_{t \wedge V(\tau)} = 0$ a.s. on $[\tau < 1]$,

2) $\lim_{t \uparrow 1} X^*_t = Z_1$ a.s. on the set $[\tau > 1]$,

3) $(X^*_t)_{t \in [0,1]}$ is a Brownian motion stopped at $V(\tau)$ in its own filtration.

To see this consider a Brownian motion β, in a possibly different probability space, with $\beta_0 = 1$ and $T_0 = \inf\{t > 0 : \beta_t = 0\}$. Let $(\mathcal{G}_t)_{t \geq 0}$ be the minimal filtration satisfying usual conditions and to which β is adapted and $\mathcal{G}_0 \supset \sigma(\beta_1, T_0)$. Direct calculations show that

$$\mathbb{P}[T_0 \in du, T_0 > 1, \beta_1 \in dy | \mathcal{F}^\beta_t] \;=\; \mathbf{1}_{[1 \wedge T_0 > t]} \ell(u-1, y) q(1-t, \beta_t, y) \, dy \, du + \mathbf{1}_{[T_0 > t \geq 1]} \ell(u-t, \beta_t) \, du$$

$$\mathbb{P}[T_0 \in du, T_0 \leq 1, \beta_1 \in dy | \mathcal{F}^\beta_t] \;=\; \mathbf{1}_{[T_0 > t]} \frac{1}{\sqrt{2\pi(1-u)}} \exp\left(-\frac{y^2}{2(1-u)}\right) \ell(u-t, \beta_t) \, dy \, du$$

$$+ \mathbf{1}_{[T_0 \leq t]} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(y-\beta_t)^2}{2(1-t)}\right) dy$$

Thus, it follows from Theorem 1.6 in [11] that

$$\beta_t \;=\; 1 + \tilde{\beta}_t + \int_0^{t \wedge 1 \wedge T_0} \left\{ \frac{q_x(1-s, \beta_s, \beta_1)}{q(1-s, \beta_s, \beta_1)} \mathbf{1}_{[T_0 > 1]} + \frac{\ell_a(T_0 - s, \beta_s)}{\ell(T_0 - s, \beta_s)} \mathbf{1}_{[T_0 \leq 1]} \right\} ds$$

$$+ \int_{t \wedge 1 \wedge T_0}^{t \wedge T_0} \frac{\ell_a(T_0 - s, \beta_s)}{\ell(T_0 - s, \beta_s)} \, ds + \int_{t \wedge 1 \wedge T_0}^{t \wedge 1} \frac{\beta_1 - \beta_s}{1 - s} \, ds,$$

where $\tilde{\beta}$ is a $\mathcal{G}$-Brownian motion independent of β_1 and T_0. Observe that $Z_{V^{-1}(t)}$ is a standard Brownian motion starting at 1 with $V(\tau)$ as its first hitting time of 0. Moreover, the SDE satisfied by β until $T_0 \wedge 1$ is the same as (4.26) until time 1 since $(\beta_1, T_0, \tilde{\beta})$ has the same law as $(Z_1, V(\tau), B)$ due to $V(1) = 1$. Therefore, the law of $(X^*_{t \wedge V(\tau) \wedge 1})_{t \geq 0}$ is the same as that of $(\beta_{t \wedge T_0 \wedge 1})_{t \geq 0}$ since the solution of the SDE for β has strong uniqueness. In particular, properties 1), 2) and 3) above are satisfied.   ∎

Now, we are in a position to compare the value of static and dynamic information. This comparison is relevant to an uninformed and risk-neutral investor at time 0 who is about to decide whether to purchase a particular private information at a given price. Obviously, as the investor is uninformed her information prior to making this decision is trivial. Thus, the decision will be based on the comparison of the expected profits resulting from the purchased information, and the expectation will be taken with respect to the trivial σ-algebra. Comparison of (3.23) and (4.27) leads to the immediate conclusion that this risk-neutral investor is indifferent between purchasing static or dynamic information, whose value is given by (3.23).

This indifference might appear counterintuitive at first. However, it is clear that a necessary condition for the optimality is that the insider drives the market price to the fundamental value of the asset at the termination of the market since otherwise she wouldn't have used all her informational advantage. On the other hand, Proposition 3.1 demonstrates that this is also sufficient.

Consequently, the only thing she strives to achieve is to make sure that the price converges to the fundamental value. This observation together with her risk-neutrality will lead her to value both types of information same since the variance of the signals does not affect her valuation.

The same phenomenon is also responsible for the fact that the price of information does not depend on V, which also manifests itself in expression (3.23) since the distribution of $V(\tau)$ is the same as that of the first hitting time of 0 by a Brownian starting at 1. In fact, it is easy to observe that the static information is the limiting case of dynamic ones characterized by an increasing sequence of functions V^n with $\lim_{n\to\infty} V^n(t) = 1$ for all $t \in (0, 1]$.

The value of information, (3.23), can be computed more explicitly as the following proposition shows.

Proposition 4.1 *Suppose f is invertible with $f(0) = 0$ and satisfies Assumption 2.1. Then, (3.23) becomes*

$$
\begin{aligned}
\mathbb{E}[W_1^{\theta^*}] &= \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}\left[F\left(|\beta_1|\sqrt{1 - V(\tau)}\right)\mathbf{1}_{[\tau<1]}\right] - F(1)\mathbb{P}[\tau > 1] \\
&\quad + \mathbb{E}\left[\{(Z_1 - 1)f(Z_1) - (F(Z_1) - F(1))\}\mathbf{1}_{[\tau>1]}\right],
\end{aligned}
\tag{4.28}
$$

where β is a standard Brownian motion independent of B^Z with $\beta_0 = 0$, and $F(z) := \int_0^z f(y)\,dy$.

PROOF. For any $a \geq 0$ let

$$
g(a) := \int_{\xi(0,a)}^1 (H^*(0, u) - a)du + \frac{1}{2}\int_0^{1 \wedge V(\tau)} H_x^*(s, \xi(s, a))ds.
$$

Since $\xi(s, 0) = 0$ for any $s \geq 0$,

$$
g(0) = \int_0^1 H^*(0, u)du + \frac{1}{2}\int_0^{1 \wedge V(\tau)} H_x^*(s, 0)ds.
$$

We will first compute the first term in the equation above.

$$
\begin{aligned}
\int_0^1 H^*(0, u)du &= \int_0^1 \int_0^\infty f(y)q(1, u, y)\,dy\,du = -\int_0^\infty F(y)\int_0^1 q_y(1, u, y)\,du\,dy \\
&= \int_0^\infty F(y)\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y+1)^2}{2}}\right)dy - 2\int_0^\infty F(y)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy \\
&= \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(|\beta_1|)].
\end{aligned}
$$

In particular, this implies $\mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(c|\beta_1|)] > 0$ for any constant $0 \leq c \leq 1$ since F is increasing.

The second term is given by

$$
\begin{aligned}
\frac{1}{2}\int_0^{1 \wedge V(\tau)} H_x^*(s, 0)ds &= \int_0^{1 \wedge V(\tau)} \int_0^\infty f(y)\ell(1 - s, y)\,dy\,ds = \int_0^\infty f(y)\int_0^{1 \wedge V(\tau)} \ell(1 - s, y)\,ds\,dy \\
&= \int_0^\infty f(y)\mathbb{P}[T_y < 1]\,dy - \int_0^\infty f(y)\mathbb{P}[T_y < 1 - 1 \wedge V(\tau)|\tau]\,dy \\
&= \int_0^\infty f(y)\left\{\mathbb{P}[|\beta_1| > y] - \mathbb{P}[|\beta_{1-1\wedge V(\tau)}| > y|\tau]\right\}dy \\
&= \mathbb{E}[F(|\beta_1|)] - \mathbb{E}[F(\sqrt{1 - 1 \wedge V(\tau)}|\beta_1|)|\tau],
\end{aligned}
$$

where in the one to the last equality, we used the reflection principle and the last equality follows from the scaling property of Brownian motion. Thus,

$$g(0) = \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(\sqrt{1 - 1 \wedge V(\tau)}|\beta_1|)|\tau]. \tag{4.29}$$

Next, observe that

$$g'(a) = \xi(0, a) - 1 + \frac{1}{2} \int_0^{1 \wedge V(\tau)} H_{xx}^*(s, \xi(s, a))\xi_a(s, a)\, ds.$$

Differentiating the equality $H^*(s, \xi(s, a)) = a$ with respect to s and a yields

$$H_s^* + H_x^*\xi_s = 0, \text{ and } H_x^* = \frac{1}{\xi_a},$$

and using the fact that $H_t^* + \frac{1}{2}H_{xx}^* = 0$, we get $\frac{1}{2}H_{xx}^*\xi_a = \xi_s$. Therefore,

$$g'(a) = \xi(0, a) - 1 + \int_0^{1 \wedge V(\tau)} \xi_s(s, a)\, ds = \xi(1 \wedge V(\tau), a) - 1,$$

and thus

$$g(a) = \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(\sqrt{1 - 1 \wedge V(\tau)}|\beta_1|)|\tau] + \int_0^a (\xi(1 \wedge V(\tau), u) - 1)du.$$

Plugging $a = \mathbf{1}_{[\tau > 1]}f(Z_1)$ into above yields

$$
\begin{aligned}
g(a) &= \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(\sqrt{1 - 1 \wedge V(\tau)}|\beta_1|)|\tau] + \mathbf{1}_{[\tau > 1]} \int_0^{f(Z_1)} (f^{-1}(u) - 1)du \\
&= \mathbb{E}[F(|1 + \beta_1|)] - \mathbb{E}[F(\sqrt{1 - 1 \wedge V(\tau)}|\beta_1|)|\tau] + \mathbf{1}_{[\tau > 1]} \{(Z_1 - 1)f(Z_1) - F(Z_1)\}.
\end{aligned}
$$

Taking the expectation of above, it is easy to see that the conclusion holds. ∎

Below are some explicit examples where we can compute the value of information.

Example 1 In the case of defaultable stock, $f(x) = x$. Then, the value of information becomes

$$
\begin{aligned}
\mathbb{E}[W_1^{\theta^*}] &= \frac{1}{2} \left(\mathbb{E}\left[(1 + \beta_1)^2\right] - \mathbb{E}\left[\beta_1^2\right] \mathbb{E}\left[(1 - V(\tau))\mathbf{1}_{[\tau < 1]}\right] + \mathbb{E}\left[(Z_1^2 - 2Z_1)\mathbf{1}_{[\tau > 1]}\right] \right) \\
&= \mathbb{P}[V(\tau) \geq 1] + \mathbb{E}\left[V(\tau)\mathbf{1}_{[V(\tau) < 1]}\right] = \mathbb{E}[V(\tau) \wedge 1].
\end{aligned}
$$

According to the last equality above, the longer the defaultable stock is traded, the higher is the insider's expected profit. Such a result is to be expected since the insider can speculate on her private information only when the market operates.

Although Proposition 4.1 requires f is invertible, one can still calculate the value of the information even if f fails this condition.

Example 2 Consider the defaultable zero-coupon bond with payoff $f \equiv 1$. Then, observe that (4.29) is still valid as we did not use the conditions on f to obtain it. Moreover, in this case a^* takes values in $\{0, 1\}$. Thus, it remains to calculate $g(1)$. First, observe that H is bounded by 1 and strictly increasing, thus, $\xi(t, 1) = \infty$ and $H_x(t, \xi(t, 1)) = 0$. Therefore,

$$g(1) = \int_\infty^1 (H^*(0, u) - 1) du = \int_1^\infty \int_{-\infty}^\infty (1 - sgn(y + u)) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy du = \sqrt{\frac{2}{\pi e}} - 2\mathbb{P}[\beta_1 < -1].$$

Thus, the value of information is

$$\mathbb{E}[W_1^{\theta^*}] = \mathbb{E}\left[g(0) \mathbf{1}_{[\tau < 1]} + g(1) \mathbf{1}_{[\tau > 1]} \right] = \sqrt{\frac{2}{\pi e}} - \sqrt{\frac{2}{\pi}} \mathbb{E}\left[\sqrt{1 - V(\tau)} \mathbf{1}_{[V(\tau) < 1]} \right].$$

The last expectation on the RHS is an indicator on how far is, on average, the default time from the defaultable bond's maturity in the case of default before maturity. As in the previous example, the larger is that expectation, i.e. the larger is the average distance between market's default time and the maturity, the lower the value of the private information is.

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A   Appendix

Lemma A.1 *Suppose that h is a nondecreasing right-continuous function with at most an exponential growth. Let*

$$H(t,x) := \int_{-\infty}^{\infty} h(x+y) \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{y^2}{2(1-t)}\right) dy,$$

and $(\xi_n)_{n\geq 1}$ be a convergent sequence such that $\lim_{n\to\infty} H(t_n, \xi_n) = a$ for some a in the range of h or in the interval $(\inf_x h(x), \sup_x h(x))$, and some sequence $(t_n)_{n\geq 1} \subseteq [0,1)$ converging to 1. Then,

$$\lim_{n\to\infty} \xi_n \in [X_{min}^a, X_{max}^a],$$

where

$$X_{min}^a := \inf\{x : h(x) \geq a\} \qquad and \qquad X_{max}^a := \sup\{x : h(x) \leq a\}.$$

PROOF.   Suppose $\lim_{n\to\infty} \xi_n < X_{min}^a$. Then, there exists some ξ such that $\lim_{n\to\infty} \xi_n < \xi < X_{min}^a$. Since H is nondecreasing in x, one has

$$\lim_{n\to\infty} H(t_n, \xi_n) \leq \lim_{n\to\infty} H(t_n, \xi) = h(\xi) < a,$$

which is a contradiction. Similarly, we have that $\lim_{n\to\infty} \xi_n \leq X_{max}^a$.   ∎