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# Insider trading in an equilibrium model with default: a passage from reduced-form to structural modelling

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**Abstract** We study an equilibrium model for the pricing of a defaultable zero coupon bond issued by a firm in the framework of Back [2]. The market consists of a risk-neutral informed agent, noise traders and a market maker who sets the price using the total order. When the insider does not trade, the default time possesses a default intensity in market's view as in reduced-form credit risk models. However, we show that, in the equilibrium, the modelling becomes structural in the sense that the default time becomes the first time that some continuous observation process falls below a certain barrier. Interestingly, the firm value is still not observable. We also establish the no expected trade theorem that the insider's trades are inconspicuous.

**Keywords** Default · structural models · reduced-form models · equilibrium · insider trading · Bessel bridge

**Mathematics Subject Classification (2000)** 93E11 · 93E20
**JEL Classification** D82 · G12

## 1 Introduction

In the valuation of credit derivatives the key issue is the calculation of the probabilities associated to the default event for which the product is written. There are essentially two approaches in the literature to model the default probabilities: *structural approach* and *reduced-form approach*. The structural approach dates back to Black and Scholes [4] and Merton [19] while the reduced-form models originated with Jarrow and Turnbull [14]. As argued by Jarrow and Protter [13] the difference between these two approaches lies on the amount of information available to the modeler. Structural models assume the modeler has the same information as the manager of the firm and, thus, has the

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continuous knowledge of the firm's assets and liabilities. In these models the default time is modelled to be the first time that the firm's value falls below a certain barrier. Consequently, this approach might come up with a model, e.g. if the firm value is assumed to be continuous, where the default time is predictable to the modeler. This feature of the model makes the yield spreads on the defaultable bonds approach to zero very quickly as one gets close to maturity. However, such behavior in the yield spreads is not common in practice (see, e.g. [11]).

In contrast the reduced-form models in general take the default time as exogenous and model the information available to the modeler. The modeler does not have the full information that the manager of the firm possesses but only a subset generated by the default process and several other related state variables. This approach was originated by Jarrow and Turnbull [14] and there has been a considerable literature on these models since then. In this respect one can mention the works of Jarrow and Turnbull [15], Artzner and Delbaen [1], Duffie, et al. [10], Lando ([17], [18]) and Duffie and Singleton [9] to name a few. The common characteristic of these models is that there exists a default arrival intensity as a function of state variables and the default process. This aspect of reduced form modelling excludes such abnormal behavior of yield spreads as observed in the structural models. The existence of the default intensity implies that, in mathematical terms, the default time is a *totally inaccessible* stopping time[1]. Consequently, the default time cannot be anticipated by the market and comes as a total surprise.

Although they seem conceptually different, one may pass from a structural model to a reduced form model. One can do this by assuming that the market's information set is that of the manager plus some noise as in Duffie and Lando [7], or restricting the information set of the market by assuming that the market only knows whether the firm is in financial distress or not as in Çetin, et al. [5]. The common feature of the both models is that they start with a structural model to define the default time, however, they are still able to come up with default intensities although the firm value is assumed to be continuous. We will not elaborate further on the differences of these two model but refer the reader to the recent works by Bielecki and Rutkowski [3], Duffie and Singleton [8] and Jarrow and Protter [13].

In the model of Duffie and Lando [7] the market has imperfect information about the default event due to the noisy and infrequent accounting data. Duffie and Lando assume that managers/owners, who have the perfect information about firm value, is precluded from trading in order to avoid complex equilibrium problems involving asymmetric information. In this paper we address such asymmetric information problem in a market for a defaultable bond. We suppose that the default time is exogenously given to the market. However, there is one trader, that we call *insider*, who has an extra information about the default time. The other two market participants are the noise traders who trade for liquidity reasons and the market maker who clears the market given the total demand. The insider is assumed to know the default time. The default event is obviously predictable to the insider as is the case in the structural models with continuous firm value, but it is totally inaccessible to the market maker when the insider doesn't trade, and it has an intensity, as in reduced form credit risk models. The market maker chooses a pricing rule and the insider chooses a trading strategy where the cumulative

[1] Default intensity could exist in a structural model if the firm value is allowed to jump, see, e.g., Huang and Huang [12]; hence, the essential difference between the structural and the reduced form models is the level of the information available to the market.

demand of the noise traders is modelled by a Brownian motion. The equilibrium is defined similar to that in Back [2]. We show that in the equilibrium the insider's trades cannot be seen in the market and the default time becomes the first time that the continuous total order process falls below a certain barrier. Consequently, the model becomes structural although the firm value is still not observable. We also show that the equilibrium total order process is a Brownian motion in its own filtration, hence the insider's trades are inconspicuous, but a 3-dimensional Bessel bridge of length $\tau$ in the insider's view, where $\tau$ is the default time (see Theorem 3.6 for the precise statement). The outline of the paper is as follows. Section 2 introduces the model. Section 3 solves the equilibrium pricing rule and the demand for the defaultable bond while Section 4 concludes.

## 2 The model

A company issues a bond that pays 1 unit of a currency at time 1 unless it defaults before that time. We suppose that the defaultable bond's recovery rate is 0 for simplicity. The company's default time is modelled by a random time $\tau$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The equilibrium framework of our model follows closely that of Back [2]. We refer the reader to Back [2] for motivation and details that are not explained in what follows. We suppose the default time is given by

$$\tau := \inf\{t > 0 : Z_t = -1\}, \tag{2.1}$$

where $Z$ is a standard Brownian motion with $Z_0 = 0$. It is well-known that

$$\mathbb{P}[\tau > 1 | Z_t] = \mathbf{1}_{[\tau > t]} F(t, Z_t),$$

where

$$F(t, y) := \int_{1-t}^{\infty} \frac{y+1}{\sqrt{2\pi x^3}} e^{-\frac{(y+1)^2}{2x}} dx. \tag{2.2}$$

We may view $Z$ as the value of the firm under a risk-neutral measure and $-1$ can be considered as default barrier. More general forms for the firm's value process can be chosen. However, we retain this Brownian assumption for the firm's value process for transparency of our results.

Three types of agents act in this market:

1. *The noise traders:* as in Back's model [2], they can only observe their own cumulative demands and whether the default has happened or not. Their cumulative demand is modelled by a standard Brownian motion $B$, with $B_0 = 0$, whose completed natural filtration is denoted by $\mathcal{F}^B = (\mathcal{F}^B_t)_{t \geq 0}$ and independent of $Z$.
2. *The informed trader:* apart from observing continuously in time the defaultable bond prices, the insider knows the default time, $\tau$. We denote $\mathcal{F}^I$ her filtration and assume that she is *risk-neutral*, so that her objective is to maximize her expected profit.
3. *The market maker:* the market maker observes the total order of the noise traders and the insider and sets the price of the risky asset to clear the market.

We further suppose $Z_0$ is known to the market maker and the insider at time 0.

*Insider's objective.* As in Back's model [2], insider's trading strategies, denoted with $\theta$, will be assumed to be absolutely continuous for optimality reasons so that $d\theta_t = \alpha_t dt$, where $\alpha$ is an $\mathcal{F}^I$-adapted process such that $\int_0^1 |\alpha_t| dt < \infty$.

Being risk-neutral, the insider has the objective to maximize her expected wealth at time 1. We suppose the default-free interest rate equals 0. Note that the value of the zero-coupon bond to the insider equals $\mathbf{1}_{[\tau>1]}$ all the time. Using this insight and following the arguments leading to the wealth process of the insider in Back [2], we find that

$$W_1^\theta = \int_0^1 (\mathbf{1}_{[\tau>1]} - S_t)\alpha_t dt, \tag{2.3}$$

where $S_t$ denotes the market price of the defaultable bond at time $t$, which is assigned by the market-maker. We will give the precise definition of admissible strategies for the insider after explaining what a *pricing rule* is in this framework.

*Market maker's objective.* The market maker sets the price of the defaultable bond using his information set, which consists of two parts. The first component is the total order of the noise traders and the insider, which is denoted with $Y$ and has the decomposition

$$Y = Y^\theta = B^\tau + \theta^\tau,$$

where $\theta$ is the position of the insider in the defaultable bond so that the total demand right before the insider starts trading at time 0 equals 0. Note that we stop the market at time $\tau$ so that there is no trading in the defaultable bond once the default has occurred. We denote the minimal right continuous and complete filtration generated by $Y^\theta$ with $\mathcal{F}^Y$, where we suppress the dependency on $\theta$ in the notation. The second part of the market maker's information comes from the observation of the default event, i.e. the market maker also observes whether the default has happened or not. In mathematical terminology, this makes $\tau$ a stopping time in his filtration. Therefore, the market maker's information is modelled by the filtration $\mathcal{F}^M = (\mathcal{F}_t^M)_{0\le t\le 1}$ where $\mathcal{F}_t^M := \mathcal{F}_t^Y \vee \sigma(\tau \wedge t)$.

Let $D_t = \mathbf{1}_{[\tau>t]}$ denote the indicator function of no-default by time $t$. The modelling idea, as borrowed from Back [2], is that the market maker assigns the price looking at the current total order and whether the default has happened. Thus, $S_t$ denoting the market price of the bond at time $t$, we expect

$$S_t = D_t H(t, Y_t),$$

where $H : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ is the pricing rule of the market maker. This justifies the following definition:

**Definition 2.1** A measurable function $H : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$ is a *pricing rule* if

1. $H \in C^{1,2}([0,1) \times \mathbb{R})$ and $H(1,\cdot) \equiv 1$[2];
2. $\mathbb{E}[D_1 H(1, B_1)] < \infty$ and $\mathbb{E}[\int_0^1 D_t H(t, B_t)^2 dt] < \infty$;
3. $y \mapsto H(t, y)$ is strictly increasing for every $t \in [0, 1)$.

Moreover, let $\theta$ be a trading strategy of the insider. Given $\theta$, a pricing rule $H$ is said to be *rational* if it satisfies

$$D_t H(t, Y_t) = \mathbb{E}[\mathbf{1}_{[\tau>1]} | \mathcal{F}_t^M], \quad t \in [0, 1]. \tag{2.4}$$

[2] Note that these two conditions ensure that $(H(t, Y_t))_{t\in[0,1]}$ is a semimartingale.

*Remark 2.2* Since the insider observes the price and the pricing rule is monotone over [0,1) and $B$ is continuous, the insider's filtration, $\mathcal{F}^I$, is generated by $B$ and $\tau$. Since $B$ and $\tau$ are independent, $B$ is also an $\mathcal{F}^I$-Brownian motion. Thus, the total order $Y$ is the sum of a $\mathcal{F}^I$-Brownian motion and an absolutely continuous $\mathcal{F}^I$-adapted process. Now, a classical result in optimal filtering implies all $\mathcal{F}^Y$-martingales are continuous (see e.g. Corollary 8.10 in Sect. VI.8 of Rogers and Williams [20]).

We are now able to give the definitions of admissible trading strategies for the insider, and equilibrium in this setting:

**Definition 2.3** An insider's admissible trading strategy is an $\mathcal{F}^I$-adapted process $\theta$ such that

1. $d\theta_t = \alpha_t dt$ for $\alpha$ an $\mathcal{F}^I$-adapted process with $\mathbb{E}\int_0^1 |\alpha_t| dt < \infty$,
2. the corresponding total order $Y^\theta$ is (in its own filtration) a Markov process given $\tau$,
3. for every pricing rule $H$, one has

$$\mathbb{E}\left[\int_0^1 D_t H(Y_t^\theta, t)^2 dt\right] < \infty.$$

The set of all admissible strategies is denoted with $\mathcal{A}$.

The condition 3 above is set to rule out doubling strategies. For a more detailed discussion of this condition, we refer the reader to Back [2].

**Definition 2.4** A pair $(H^*, \theta^*)$ is said to form an equilibrium if $H^*$ is a pricing rule, $\theta^* \in \mathcal{A}$, and the following conditions are satisfied:

1. *Market efficiency condition:* given $\theta^*$, $H^*$ is a rational pricing rule.
2. *Insider optimality condition:* given $H^*$, $\theta^*$ solves the insider optimization problem:

$$W_1^{\theta^*} = \sup_{\theta \in \mathcal{A}} \mathbb{E}[W_1^\theta | \mathcal{F}_0^I].$$

## 3 Equilibrium

In this section, we will look for the existence of an equilibrium as defined in Definition 2.4. We'll first address the optimality condition for the insider. Throughout this section, we will use the standard notation $[\![0,\tau]\!] := \{(\omega, t) \in [0,1] \times \Omega : \tau(\omega) \geq t\}$. This is clearly a measurable subset of $(\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]), d\mathbb{P}dt)$.

### 3.1 Optimality conditions for the insider's problem

Let us fix a pricing rule $H$. Using (2.3) and noting that $\lim_{s \uparrow t} H(s, Y_s) = H(t, Y_t)$ for all $t \in (0,1)$, the insider's problem becomes

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{1-} \left\{\mathbf{1}_{[\tau > 1]} - D_{t-} H(t, Y_t)\right\} \alpha_t dt \middle| \mathcal{F}_0^I\right] \tag{3.1}$$

For a given strategy $\alpha$, let us write for $t<1$

$$J(t,Y_t,\mathbf{1}_{[\tau>1]}) = \text{ess sup}_{\alpha^{(t)}\in\mathcal{A}(t,\alpha)}\mathbb{E}\left[\int_{t\wedge\tau}^{1\wedge\tau}\mathbf{1}_{[u<1]}\left\{\mathbf{1}_{[\tau>1]}-H(u,Y_u)\right\}\alpha_u^{(t)}du\middle|\mathcal{F}_t^I\right],$$

where $\mathcal{A}(t,\alpha)=\{\alpha^{(t)}\in\mathcal{A}:\alpha_u^{(t)}=\alpha_u, 0\le u\le t\}$, $t\ge 0$, and define

$$J(1,Y_1,\mathbf{1}_{[\tau>1]}) := \lim_{t\uparrow 1} J(t,Y_t,\mathbf{1}_{[\tau>1]}).$$

Note that we are in a similar situation as in Back [2] (see also [6]). Using the arguments therein, the solution to (3.1) exists if the following system has a solution:

$$\frac{\partial}{\partial y}J(t\wedge\tau,y,\mathbf{1}_{[\tau>1]})+\mathbf{1}_{[\tau>1]}-H(t\wedge\tau,y)=0, \text{ for } t<1, \tag{3.2}$$

$$\frac{\partial}{\partial t}J(t\wedge\tau,y,\mathbf{1}_{[\tau>1]})+\frac{1}{2}\frac{\partial^2}{\partial y^2}J(t\wedge\tau,y,\mathbf{1}_{[\tau>1]})=0, \text{ for } t<1. \tag{3.3}$$

This implies that for any total order process $Y$, associated to an admissible strategy of the insider, $H$ satisfies

$$\frac{\partial}{\partial t}H(t,Y_t)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H(t,Y_t)=0, \quad t\in[\![0,\tau]\!]\cap[0,1). \tag{3.4}$$

Although the dependency of $J$ on $\mathbf{1}_{[\tau>1]}$ is obvious, we'll suppress this dependency in the notation as long as no confusion arises.

**Lemma 3.1** *(Theorem 2, Back [2]) If a pricing rule $H$ verifies equation (3.4), then there exist a function $J$ satisfying the system given by (3.2) and (3.3). Moreover, one has*

$$J(\tau\wedge 1, Y_{\tau\wedge 1})\ge 0, \tag{3.5}$$

*for all admissible $Y$ and the equality holds if and only if $Y$ satisfies $\lim_{t\uparrow 1}H(\tau\wedge t, Y_{\tau\wedge t})=\mathbf{1}_{[\tau>1]}$.*

*Proof* Keeping in mind that $\tau\wedge 1\in\mathcal{F}_0^I$, we refer to Back [2] (Theorem 2, p. 396) or Cho [6] (Lemmas 4, 5, p. 56).

In the sequel, a rational pricing rule $H$ will be given. We will investigate what properties an admissible insider strategy should satisfy in order to be optimal.

**Proposition 3.2** *Let $H$ be a rational pricing rule and assume $\lim_{t\uparrow 1}H(\tau\wedge t,Y_{\tau\wedge t})=\mathbf{1}_{[\tau>1]}$. The following are equivalent:*

1. *$H$ satisfies (3.4).*
2. Inconspicuous insider trade theorem holds, *i.e.* $\mathbb{E}[\alpha_t|\mathcal{F}_t^M]=0$ *$d\mathbb{P}dt$-a.e. on $[\![0,\tau]\!]$.*
3. *$Y$ stopped at $\tau$ is a Brownian motion for the market maker, more precisely there exists an $\mathcal{F}^M$-Brownian motion $B^M$ such that $Y_{t\wedge\tau}=B^M_{t\wedge\tau}$ for every $t\in[0,1]$.*

*Proof* First, we prove that 1 and 2 are equivalent. Itô's formula applied to $D_tH(t,Y_t)$ gives, for $t<1$,

$$dD_tH(t,Y_t) = D_{t-}\left\{\frac{\partial}{\partial t}H(t,Y_t)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H(t,Y_t)\right\}dt$$
$$+D_{t-}\frac{\partial}{\partial y}H(t,Y_t)dY_t + H(t,Y_t)dD_t.$$

Therefore, letting $H_-(t,Y_t)=\lim_{s\uparrow t}H(s,Y_s)$ and $\frac{\partial}{\partial t}H_-(t,Y_t):=\lim_{s\uparrow t}\frac{\partial}{\partial t}H(s,Y_s)$, and defining $\frac{\partial^2}{\partial y^2}H_-(t,Y_t)$ and $\frac{\partial}{\partial y}H_-(t,Y_t)$ analogously, we have the following decomposition for $D_tH(t,Y_t)$ on the whole interval $[0,1]$:

$$\begin{aligned}
dD_tH(t,Y_t) &= D_{t-}\left\{\frac{\partial}{\partial t}H_-(t,Y_t)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H_-(t,Y_t)\right\}dt + D_{t-}\frac{\partial}{\partial y}H_-(t,Y_t)dY_t\\
&\quad+\mathbf{1}_{[t=1]}D_{1-}(1-H_-(1,Y_1))+H_-(t,Y_t)dD_t\\
&\quad+\{H(t,Y_t)-H_-(t,Y_t)\}(D_t-D_{t-})\\
&= D_{t-}\left\{\frac{\partial}{\partial t}H_-(t,Y_t)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H_-(t,Y_t)\right\}dt + D_{t-}\frac{\partial}{\partial y}H_-(t,Y_t)dY_t\\
&\quad+H_-(t,Y_t)dD_t+\mathbf{1}_{[t=1]}D_1(1-H_-(1,Y_1)).
\end{aligned}\tag{3.6}$$

By standard filtering arguments it is not difficult to see that

$$Y_t = B_t^M + \int_0^t \widehat{\alpha}_u du,\quad t\in[0,1],\tag{3.7}$$

for an $\mathcal{F}^M$-Brownian motion $B^M$ and $\widehat{\alpha}_t:=\mathbb{E}[\alpha_t|\mathcal{F}_t^M]$, $t\in[0,1]$. To see this, consider the process

$$N_t := \mathbb{E}\left[\int_0^t \alpha_s ds\,\middle|\,\mathcal{F}_t^M\right]-\int_0^t\widehat{\alpha}_s ds.$$

$N$ is clearly an $\mathcal{F}^M$-martingale. As a consequence, we have that

$$Y_t=\mathbb{E}[Y_t|\mathcal{F}_t^M]=\mathbb{E}[B_t|\mathcal{F}_t^M]+N_t+\int_0^t\widehat{\alpha}_s ds.$$

Now, we observe that $M_t:=\mathbb{E}[B_t|\mathcal{F}_t^M]$ is an $\mathcal{F}^M$-martingale which follows from that $B$ is an $\mathcal{F}^I$-Brownian motion and $\mathcal{F}^M\subset\mathcal{F}^I$. Moreover, $\langle Y\rangle_t=t$. Thus, the continuous martingale $M+N$ having $\langle M+N\rangle_t=\langle Y\rangle_t=t$ is an $\mathcal{F}^M$-Brownian motion and we get (3.7).

Using this result, we can rewrite the decomposition of $D_tH(t,Y_t)$ as follows

$$\begin{aligned}
dD_tH(t,Y_t) &= D_{t-}\left\{\frac{\partial}{\partial t}H_-(t,Y_t)+\widehat{\alpha}_t\frac{\partial}{\partial y}H_-(t,Y_t)+\frac{1}{2}\frac{\partial^2}{\partial y^2}H_-(t,Y_t)\right\}dt\\
&\quad+D_{t-}\frac{\partial}{\partial y}H_-(t,Y_t)dB_t^M\\
&\quad+H_-(t,Y_t)dD_t+\mathbf{1}_{[t=1]}D_1(1-H_-(1,Y_1)).
\end{aligned}$$

Under the assumption $\lim_{t\uparrow 1}H(\tau\wedge t,Y_{\tau\wedge t})=\mathbf{1}_{[\tau>1]}$, the last term in the above decomposition of $D_tH(t,Y_t)$ vanishes. Also note that $\int_0^t H_-(s,Y_s)dD_s=-H_-(\tau,Y_\tau)\mathbf{1}_{[\tau\le t]}$, which equals, for $t < 1$, $-H(\tau, Y_\tau)\mathbf{1}_{[\tau \le t]}$. Observe that for $u \in [t, 1)$, $H(\tau \wedge u, Y_{\tau\wedge u})\mathbf{1}_{[\tau \le t]} = H(\tau, Y_\tau)\mathbf{1}_{[\tau \le t]}$. Thus,

$$\mathbf{1}_{[\tau \le t]} \lim_{u \uparrow 1} H(\tau \wedge u, Y_{\tau\wedge u}) = \mathbf{1}_{[\tau \le t]} H(\tau, Y_\tau).$$

However, by hypotheses, $\lim_{u\uparrow 1} H(\tau \wedge u, Y_{\tau\wedge u}) = \mathbf{1}_{[\tau > 1]}$,which in turn implies $H(\tau, Y_\tau)\mathbf{1}_{[\tau \le t]} = 0$ for $t < 1$. Taking limits we have $H_-(\tau, Y_\tau)\mathbf{1}_{[\tau \le 1]} = 0$, as well. Therefore,

$$\begin{aligned} dD_t H(t, Y_t) = D_{t-} &\left\{ \frac{\partial}{\partial t} H_-(t, Y_t) + \widehat{\alpha}_t \frac{\partial}{\partial y} H_-(t, Y_t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} H_-(t, Y_t) \right\} dt \\ &+ D_{t-} \frac{\partial}{\partial y} H_-(t, Y_t) dB_t^M. \end{aligned} \tag{3.8}$$

Since $H$ is rational, the process $D_t H(t, Y_t)$ must be an $\mathcal{F}^M$-martingale. Of course, this is the case if and only if the finite variation part in the above decomposition vanishes, i.e., for $t < 1$,

$$\frac{\partial H}{\partial t}(t, Y_t) + \widehat{\alpha}_t \frac{\partial H}{\partial y}(t, Y_t) + \frac{1}{2} \frac{\partial^2 H}{\partial y^2}(t, Y_t) = 0 \quad \text{on } [\tau \ge t].$$

This proves that 1 and 2 are equivalent since $H$ is strictly increasing in the second variable. It remains to show that 2 and 3 are equivalent. For this, it suffices to observe that, in view of (3.7), $\int_0^t \widehat{\alpha}_s ds = 0$ $d\mathbb{P}dt$-a.e. on $[\![0, \tau]\!]$ if and only if $Y_{t\wedge\tau} = B^M_{t\wedge\tau}$ for all $t \in [0, 1]$.

The next lemma gives a necessary and sufficient condition for an admissible strategy to be optimal given a rational pricing rule $H$.

**Lemma 3.3** *Given a rational pricing rule $H$, $\theta^* \in \mathcal{A}$ is an optimal insider strategy if and only if it satisfies the following two properties:*

1. *$\mathbb{E}[\alpha_t^* | \mathcal{F}_t^M] = 0$ $d\mathbb{P}dt$-a.e. on $[\![0, \tau]\!]$, where $\alpha_t^* = \frac{d\theta^*}{dt}$;*
2. *the corresponding optimal total order $Y^*$ satisfies $\lim_{t\uparrow 1} H(\tau \wedge t, Y^*_{\tau\wedge t}) = \mathbf{1}_{[\tau > 1]}$.*

*Proof* Let $H$ be a rational pricing rule. Suppose $\theta^*$ is a corresponding optimal strategy of the insider. HJB equations ((3.2) and (3.3)) require that $H$ satisfies equation (3.4), and by Lemma 3.1 there exists a function $J$ satisfying the system given by (3.2) and (3.3).

An application of Itô's formula for $t < 1$ gives, on the event $[t < \tau] \in \mathcal{F}_0^I$,

$$J(\tau \wedge 1, Y_{\tau\wedge 1}) = J(t, y) - \int_t^{\tau\wedge 1} (\mathbf{1}_{[\tau > 1]} - S_u)\alpha_u du - \int_t^{\tau \wedge 1} (\mathbf{1}_{[\tau > 1]} - S_u) dB_u.$$

Now, condition 3 of Definition 2.3 implies $\int_t^s (\mathbf{1}_{[\tau > 1]} - S_u) dB_u$, $t \le s \le 1$, is an $\mathcal{F}^I$-martingale so that

$$\mathbb{E}\left[ \int_t^{\tau\wedge 1} (\mathbf{1}_{[\tau > 1]} - S_u)\alpha_u du \middle| \mathcal{F}_t^I \right] = J(t, Y_t) - \mathbb{E}[J(\tau \wedge 1, Y_{\tau\wedge 1}) | \mathcal{F}_t^I].$$

The expected future wealth reaches its maximum when $\mathbb{E}[J(\tau \wedge 1, Y_{\tau\wedge 1})|\mathcal{F}_t^I]$ reaches its minimum. In view of Lemma 3.1, $\mathbb{E}[J(\tau \wedge 1, Y_{\tau\wedge 1})|\mathcal{F}_t^I]$ attains its minimum at

$Y^*_{\tau\wedge 1}$ if and only if $\lim_{t\uparrow 1} H(\tau\wedge t, Y^*_{\tau\wedge t}) = \mathbf{1}_{[\tau>1]}$. Thus, by Proposition 3.2, $\alpha^*$ is inconspicuous, i.e. $\mathbb{E}[\alpha^*_t|\mathcal{F}^M_t] = 0$ $d\mathbb{P}dt$-a.e. on $[\![0,\tau]\!]$.

To complete the proof it remains to show that if $\alpha^*$ satisfies 1 and 2 then it is optimal given $H$. Again by Proposition 3.2 one has that $H$ solves (3.4). By Lemma 3.1 $\alpha^*$ is optimal if and only if $\lim_{t\uparrow 1} H(\tau\wedge t, Y^*_{\tau\wedge t}) = \mathbf{1}_{[\tau>1]}$, $Y^*$ being the total order associated to $\alpha^*$. This equality is satisfied thanks to property 2.

## 3.2 The equilibrium and its interpretation

The following lemma is essential in the characterization of the equilibrium.

**Lemma 3.4** *The couple $(H^*, \theta^*)$ is an equilibrium if and only if the following two conditions hold:*

1. *$H^*$ solves (3.4),*
2. *$Y^*$ is an $\mathcal{F}^M$-Brownian motion stopped at $\tau$ such that $\lim_{t\uparrow 1} H(\tau\wedge t, Y_{\tau\wedge t}) = \mathbf{1}_{[\tau>1]}$.*

*Proof* We prove first that if $(H^*, \theta^*)$ satisfies 1 and 2 then it is an equilibrium, i.e. $H^*$ is rational and $\theta^*$ is an optimizer. Condition 2 implies that $\widehat{\alpha}^*_t = 0$ on $[t\le\tau]$ (use Proposition 3.2). Moreover, decomposition (3.8) together with Condition 1 gives that $D_tH(t, Y_t)$ is an $\mathcal{F}^M$-martingale, which implies that $H^*$ is rational.

Moreover, given $H^*$ satisfying the conditions in 1 and 2, $\theta^*$ with the associated total order process $Y^*$ satisfying the conditions in 2 is optimal. Indeed, the optimality of $\theta^*$ is a straightforward consequence of Lemma 3.3.

To finish, it remains to show that if $(H^*, \theta^*)$ is an equilibrium then it satisfies 1 and 2. This comes from a combination of Proposition 3.2 and Lemma 3.3. Indeed, $\theta^*$ optimal given $H^*$ implies that the inconspicuous trade theorem holds and that $\lim_{t\uparrow 1} H(\tau\wedge t, Y_{\tau\wedge t}) = \mathbf{1}_{[\tau>1]}$. so that, by Proposition 3.2, $H^*$ has to satisfy (3.4) and $Y^*$ is an $\mathcal{F}^M$-Brownian motion stopped at $\tau$.

Before we show the existence of an equilibrium, we present the following technical result.

**Lemma 3.5** *On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathcal{F}^I = \mathcal{F}^B \vee \sigma(\tau)$, there exists a unique strong solution to the SDE*

$$dY_t = dB_t + \left\{\frac{1}{1+Y_t} - \frac{1+Y_t}{\tau - t}\right\}\mathbf{1}_{[t\le\tau]}dt\,, \tag{3.9}$$

*with $Y_0 = 0$.*

*Proof* The result follows as soon as one observes that conditioned on $[\tau = \ell]$, $1+Y$ is a 3-dimensional Bessel bridge of length $\ell$ starting at 1 at time 0 and ending at 0 at time $\ell$, and that $\tau$ is independent of $B$. □

The following theorem is the main result of our paper. It gives the existence and a characterization of the equilibrium. Moreover, it shows that in the equilibrium the default time $\tau$ is a hitting time of the equilibrium total order process $Y^*$ so that the modelling changes to structural.

**Theorem 3.6** *There exists an equilibrium* $(H^*, \theta^*)$ *such that*

$$H^*(t, y) = F(t, y),$$

*for* $t < 1$ *and* $H^*(1, \cdot) \equiv 1$ *where* $F$ *is as defined in (2.2) and the equilibrium total order* $Y^*$ *solves*

$$dY_t = dB_t + \left\{ \frac{1}{1 + Y_t} - \frac{1 + Y_t}{\tau - t} \right\} \mathbf{1}_{[t \le \tau]} dt \,. \tag{3.10}$$

*Therefore, given* $\tau = \ell$, $1 + Y$ *is a* 3*-dimensional Bessel bridge of length* $\ell$ *starting at* 1 *at time* 0 *and ending at* 0 *at time* $\ell$.

*Moreover, one has* $\tau = \inf\{t > 0 : Y_t^* = -1\}$. *As a consequence,* $\tau$ *is a predictable stopping time under the market maker's filtration* $\mathcal{F}^M$.

*Proof* It is straightforward to show that $H^*$ is a pricing rule satisfying (3.4). In view of Lemma 3.4 it remains to show that the unique strong solution to the SDE in (3.10), which exists by Lemma 3.5, is an $\mathcal{F}^M$-Brownian motion stopped at $\tau$ with $\lim_{t \uparrow 1} H^*(\tau \wedge t, Y_{\tau \wedge t}) = \mathbf{1}_{[\tau > 1]}$. In order to do so we'll construct a Brownian motion that is a weak solution to (3.10) by enlargement of filtration techniques. Observe that this is the first time we use the assumption that the default time is given by the first hitting of $-1$ by the firm value, $Z$. Let $\mathcal{G}$ be the filtration generated by $\mathcal{F}^Z$ and $\tau$, where $\mathcal{F}^Z$ is the usual augmentation of $Z$. Then it readily follows from Jeulin [16] (Lemme 3.25, p. 52) that $Z$ has the following $\mathcal{G}$-decomposition:

$$dZ_t = d\beta_t + \left( \frac{1}{1 + Z_t} - \frac{1 + Z_t}{\tau - t} \right) \mathbf{1}_{[t \le \tau]} dt$$

where $\beta$ is a $\mathcal{G}$-Brownian motion and so independent of $\mathcal{G}_0 \supseteq \sigma(\tau)$. Then, due to the strong uniqueness of the solution as established in Lemma 3.5, $Y^*$ has the same law as $Z$; thus, $Y^*$ is a Brownian motion in its own filtration. The uniqueness of the solution implies also that

$$\tau = \inf\{t > 0 : Y_t^* = -1\}.$$

Thus $\tau$ is a stopping time with respect to $\mathcal{F}^{Y^*}$ and the filtrations $\mathcal{F}^{Y^*}$ and $\mathcal{F}^M$ coincide, so that $Y^*$ is an $\mathcal{F}^M$-Brownian motion, too. To finish the proof it remains to check $\lim_{t \uparrow 1} H^*(\tau \wedge t, Y^*_{\tau \wedge t}) = \mathbf{1}_{[\tau > 1]}$. We leave this simple task to the reader.

Now we turn to the interpretation of the equilibrium. It is easy to see that in the equilibrium described in Theorem 3.6

$$\frac{1}{1 + Y_t^*} = \mathbb{E}\left[ \frac{1 + Y_t^*}{\tau - t} \Big| \mathcal{F}_t^M \right] \tag{3.11}$$

since $\widehat{\alpha}^* = 0$. This is an easy consequence of decomposition (3.10). Thus, conditional on $\mathcal{F}_t^M$, $\frac{1}{1+Y_t^*}$ is the best approximation in market's view to the value $\frac{Y_t^*+1}{\tau - t}$. Recall that the default will happen when $Y^*$ hits $-1$. Therefore, $Y_t^* + 1$ can be viewed as a metric measuring the *distance to default*. Moreover, $\tau - t$ is the *time to default* if default has not happened by time $t$ so that one can define the ratio $\frac{Y_t^*+1}{\tau - t}$ as the true *speed of default*. Due to the information structure the true speed of default is known to the insider. However, the market maker has only an expectation of this speed given his information. In view of (3.11) the market's expectation for the speed of default conditional on $\mathcal{F}_t^M$ is given by $\frac{1}{Y_t^*+1}$.

Looking at (3.10), we see that the insider sells when the market's expectation of the speed of default is lower than the true speed and buys otherwise. This is quite intuitive. Indeed, when the market's expectation for the imminence of the default is low, the bond is relatively overpriced, so the insider sells to increase her profits.

## 4 Conclusions

We have analyzed the effects of asymmetric information in defaultable bond pricing. In an equilibrium setting à la Back [2] we have shown that the information asymmetries can change the nature of the modelling completely. We solved for the equilibrium pricing rule for the market maker, optimal strategy for the insider and equilibrium demand for the defaultable bond. It is shown that it is optimal for the insider to trade without being seen while driving the total demand to hit a certain level at the default time. The presence of a strategic insider turns the modelling into structural while the model is reduced-form in the absence of the insider.

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