Rend. Sem. Mat. Univ. Pol. Torino
Vol. 68, 3 (2010), 235 – 250
Workshop on Hodge Theory and Algebraic Geometry

R. Castaño-Bernard and D. Matessi

THE FIXED POINT SET OF ANTI-SYMPLECTIC INVOLUTIONS OF LAGRANGIAN FIBRATIONS

Abstract. We discuss some results and ideas on the topology of Lagrangian submanifolds obtained as the fixed point locus of certain anti-symplectic involutions preserving the fibres of a Lagrangian fibration $f : X \to B$. Here X is a symplectic manifold diffeomorphic to a Calabi–Yau manifold.

1. Introduction

The fixed point locus of an anti-symplectic involution (i.e. a map $\iota : X \to X$ such that $\iota^2 = \mathrm{Id}_X$ and $\iota^*\omega = -\omega$) is an interesting type of Lagrangian submanifold of a symplectic manifold (X, ω). An easy, classical, construction of such an involution is given by complex conjugation when X is a smooth complex subvariety of $\mathbb{P}^n$ cut out by polynomials with real coefficients. In this case the fixed point locus is just the intersection with $\mathbb{RP}^n$. Understanding the topology of such varieties is generally a difficult problem. One reason why the fixed point set of an anti-symplectic involution is interesting is that its Floer homology is particularly well behaved ([5, 16]). In [3], together with Jake P. Solomon, we constructed a class of anti-symplectic involutions by requiring that they preserve the fibres of the Lagrangian fibrations $f : X \to B$ constructed in [2]. In this case X is diffeomorphic to a Calabi–Yau manifold (of complex dimension 2 or 3), e.g. a $K3$ surface or a quintic hypersurface in $\mathbb{P}^4$.

In this note we review the constructions in [2] and [3] and we report, in an informal way and with almost no proofs, on some work in progress on the topology of the fixed point locus of these anti-symplectic involutions. Proofs and details, together with other results, will appear in [1]. Many of the results and ideas mentioned here on Lagrangian fibrations are based on, or inspired by, the work of M. Gross [6, 7, 8, 10] and M. Gross–B. Siebert [12]. In particular, it follows from results in these articles and the construction in [2], that in most cases also the mirror Calabi–Yau $\check{X}$ comes with a “dual” Lagrangian fibration and anti-symplectic involution. In our fibrations the general fibre of f is a smooth Lagrangian torus, while fibres over points in a set $\Delta \subset B$ are singular. In the 2-dimensional case the fibrations we consider are topologically identical to stable elliptic fibrations, i.e. $B = S^2$ and there are 24 singular fibres of Kodaira type I_1, i.e. once pinched tori. In the 3-dimensional case the base is homeomorphic to S^3 and the discriminant locus is a 3-valent graph (with the possibility that some connected components are just circles with no vertices). The singular fibres are also “stable” in some sense but their topology is more complicated.

The fibrations we consider also have a Lagrangian section, therefore the smooth fibres have naturally a group structure isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$. The anti-symplectic involution fixes such a section and restricted to smooth fibres is just $\alpha \mapsto -\alpha$, and therefore the fixed point set in each smooth fibre is just 2^n points. So, if we call Σ the fixed point locus, then Σ is a Lagrangian submanifold of X and f restricted to Σ is a branched covering of B, of degree 2^n, branching over Δ. In the 2-dimensional case it is not difficult to show that Σ has two connected components, one being the fixed Lagrangian section, the other being a genus 10 curve. In this case Σ has the largest possible total cohomology group for the fixed point locus of an involution, and is therefore maximal. The 3-dimensional case is more complicated. We discuss a long exact sequence linking the $\mathbb{Z}/2\mathbb{Z}$ cohomology of Σ to the cohomology of X. It is inspired by a Leray spectral sequence studied by Gross (op. cit.), computing the cohomology of X in terms of the fibration. As a corollary we obtain that if X and its mirror $\check{X}$ are simply connected, then Σ has just two connected components. Computing the $\mathbb{Z}/2\mathbb{Z}$-cohomology of Σ is reduced to computing a map β appearing in the long exact sequence (cf. Section 3). When $\beta = 0$, Σ has the largest possible total cohomology. We describe an explicit example coming from the so-called Schoen's Calabi–Yau, where Σ can be described in a sufficiently simple way so to apply standard techniques for the computation of cohomology. The result is that for Schoen's 3-fold, β is not zero.

A few questions remain open. Can we compute β explicitly in more complicated known examples such as the quintic or complete intersections in toric manifolds? What is the relationship between Σ and the corresponding fixed point locus $\check{\Sigma}$ inside the mirror manifold $\check{X}$? What is the relation between the involutions we study and the more classical ones constructed algebraically, for instance by conjugation in $\mathbb{P}^n$? Is Σ in our case somehow special among other possible constructions, i.e. is it maximal in some other sense? These and other questions will be addressed in [1] and further work.

Acknowledgments. This project was partially supported by NSF award DMS-0854989: *FRG: Mirror Symmetry & Tropical Geometry*. Matessi was partially supported by MIUR (*Geometria Differenziale e Analisi Globale*, PRIN07).

2. Lagrangian fibrations and involutions

2.1. Affine manifolds with singularities

Let (X,ω) be a smooth symplectic $2n$-dimensional manifold, B a smooth n-dimensional manifold and $\Delta \subset B$ a closed subset with $B_0 = B - \Delta$ dense in B. A *Lagrangian fibration* on X is a smooth map $f : X \to B$ such that the fibres of f over B_0 are Lagrangian submanifolds (i.e. dim $f^{-1}(b) = n$ and $\omega|_{f^{-1}(b)} = 0$), and f restricted to B_0 is a submersion. The fibres over Δ are called singular. If the top dimensional stratum of a singular fibre is n-dimensional, we require the smooth part of it, $f^{-1}(b) - \{\mathrm{Crit}(f) \cap f^{-1}(b)\}$, to be a Lagrangian submanifold as well. When the fibres are compact and connected, then the Arnold-Liouville theorem implies that the fibres over B_0 are all n-tori. Moreover, we can cover the subset B_0 with an atlas, $\{(U_j,\phi_j)\}_{j\in J}$ such that the transition maps are affine transformations whose linear part has integral coefficients, i.e. $\phi_j \circ \phi_k^{-1} \in \mathbb{R}^n \rtimes \mathrm{Sl}_n(\mathbb{Z})$.

This motivates the definition of an *integral affine manifold with singularities*: a topological manifold B with a closed subset $\Delta \subset B$, with $B_0 = B - \Delta$ dense in B, such that on B_0 there exists an atlas $\mathcal{A}$ whose change of coordinates are in $\mathbb{R}^n \rtimes \mathrm{Sl}_n(\mathbb{Z})$. If $(y_1, \dots, y_n)$ are affine coordinates, then the $\mathbb{Z}$-linear combinations of the 1-forms $dy_1, \dots dy_n$ span a maximal lattice $\Lambda \subset T^*B_0$ which is well defined independently of the chosen affine coordinates (this follows from the fact that the linear part of the change of coordinates is in $\mathrm{Sl}_n(\mathbb{Z})$). We can use this to form the n-torus bundle $X_0 = T^*B_0/\Lambda$. The standard symplectic form on T^*B_0 descends to X_0 so that the standard projection $f_0 : X_0 \to B_0$ is a Lagrangian submersion.

In general, if we start with a given integral affine manifold with singularities $(B, \Delta, \mathcal{A})$, we may ask whether we can find a symplectic manifold X and extend the bundle $f_0 : X_0 \to B_0$ to a Lagrangian fibration $f : X \to B$ by inserting singular Lagrangian fibres over the set Δ. More precisely we want the following commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{j} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_0 & \xrightarrow{i} & B \end{array} \tag{1}$$

where j is a symplectomorphism and i is the inclusion. This is the starting point for the construction of the Lagrangian fibrations in [2]. If we ask the question at the purely topological level (i.e. without requiring a symplectic form on X and the Lagrangian condition on f) then, for the cases we consider here, the answer was provided by Gross in [8]. In particular Gross finds a topological torus fibration on the quintic threefold in $\mathbb{P}^4$.

Let us now give some examples of affine manifolds with singularities.

EXAMPLE 1 (Focus-focus). We start with a 2-dimensional example. We define an affine structure with singularities on $B = \mathbb{R}^2$. Let $\Delta = \{0\}$ and let (x_1, x_2) be the standard coordinates on B. As the covering $\{U_i\}$ of $B_0 = \mathbb{R}^2 - \Delta$ we take the following two sets

$$U_1 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \geq 0\},$$
$$U_2 = \mathbb{R}^2 - \{x_2 = 0 \text{ and } x_1 \leq 0\}.$$

Denote by H^+ the set $\{x_2 > 0\}$ and by H^- the set $\{x_2 < 0\}$. Let T be the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{2}$$

The coordinate maps ϕ_1 and ϕ_2 on U_1 and U_2 are defined as follows

$$\phi_1 = \mathrm{Id},$$

$$\phi_2 = \begin{cases} \mathrm{Id} & \text{on } \bar{H}^+ \cap U_2 \\ (T^{-1})^t & \text{on } H^-. \end{cases}$$

The atlas $\mathcal{A} = \{U_i, \phi_i\}_{i=1,2}$ is clearly an affine structure on B_0. We can easily check that if we consider the 2-torus bundle $X_0 = T^*B_0/\Lambda$, then the monodromy of the H_1 homology of the fibres at a point $b \in B_0$, associates the matrix T to the anti-clockwise oriented generator of $\pi_1(B_0)$.

This is a compact example:

EXAMPLE 2. In $\mathbb{R}^3$ consider the 3-dimensional simplex Ξ spanned by the points $P_0 = (-1,-1,-1)$, $P_1 = (3,-1,-1)$, $P_2 = (-1,3,-1)$, $P_3 = (-1,-1,3)$. Let $B = \partial\Xi$. We explain how to construct an affine structure with singularities on B. Each edge ℓ_j of Ξ has 5 integral points (i.e. belonging to $\mathbb{Z}^3$), which divide ℓ_j into 4 segments. For each $j = 1,\dots,6$ denote by Δ_k^j, $k = 1,\dots,4$ the four barycenters of these four segments. We let

$$\Delta = \{\Delta_k^j; j = 1\dots 6 \text{ and } k = 1,\dots,4\}.$$

A covering of $B_0 = B - \Delta$ can be defined as follows. The first four open sets consist of the four open faces Σ_i, $i = 1\dots,4$ with the affine coordinate maps ϕ_i induced by their affine embeddings in $\mathbb{R}^3$. Denote by I the set of integral points of B which lie on an edge. For every $Q \in I$ we can choose a small open set U_Q in B_0 such that $\{\Sigma_i\}_{i=1,\dots,4} \cup \{U_Q\}_{Q\in I}$ is a covering of B_0. Let R_Q denote the 1-dimensional subspace of $\mathbb{R}^3$ generated by $Q \in I$. One can verify that if U_Q is small enough, the projection $\phi_Q : U_Q \to \mathbb{R}^3/R_Q$ is a homeomorphism. A computation shows that the atlas $\mathcal{A} = \{\Sigma_i, \phi_i\}_{i=1,\dots,4} \cup \{U_Q, \phi_Q\}_{Q\in I}$ defines an integral affine structure on B_0.

In the latter example it can be easily checked that a neighbourhood of the singular points in Δ is affine isomorphic to a neighbourhood of $0 \in \mathbb{R}^2$ in Example 1. In dimension 2, an affine manifold with singularities $(B, \Delta, \mathcal{A})$ is called *simple* if Δ consists of isolated points and each point has a neighbourhood affine isomorphic to a neighbourhood of 0 in Example 1.

We now present some 3-dimensional examples.

EXAMPLE 3 (The edge). Let $I \subseteq \mathbb{R}$ be an open interval. Consider $B = \mathbb{R}^2 \times I$ and $\Delta = \{0\} \times I$. On $B_0 = (\mathbb{R}^2 - \{0\}) \times I$ we take the product affine structure between the affine structure on $\mathbb{R}^2 - \{0\}$ described in Example 1 and the standard affine structure on I.

EXAMPLE 4 (Positive vertex). Let $B = \mathbb{R} \times \mathbb{R}^2$ and let (x_1, x_2, x_3) be coordinates in B. Identify $\mathbb{R}^2$ with $\{0\} \times \mathbb{R}^2$. Inside $\mathbb{R}^2$ consider the cone over three points:

$$\Delta = \{x_2 = 0, x_3 \leq 0\} \cup \{x_3 = 0, x_2 \leq 0\} \cup \{x_2 = x_3, x_3 \geq 0\}.$$

Now define closed sets in B

$$\begin{aligned} R &= \mathbb{R} \times \Delta, \\ R^+ &= \mathbb{R}_{\geq 0} \times \Delta, \\ R^- &= \mathbb{R}_{\leq 0} \times \Delta, \end{aligned}$$

and consider the following cover $\{U_i\}$ of $\mathbb{R}^3 - \Delta$:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - R^+, \\ U_2 &= \mathbb{R}^3 - R^-. \end{aligned}$$

It is clear that $U_1 \cap U_2$ has the following three connected components

$$\begin{aligned} V_1 &= \{x_2 < 0,\ x_3 < 0\}, \\ V_2 &= \{x_2 > 0,\ x_2 > x_3\}, \\ V_3 &= \{x_3 > 0,\ x_3 > x_2\}. \end{aligned}$$

Take two matrices

$$T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Now on U_1, U_2 we define coordinate maps ϕ_1, ϕ_2 as follows

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} \mathrm{Id} & \text{on } \overline{V_1} \cap U_2 \\ T_1^{-1} & \text{on } \overline{V_2} \cap U_2 \\ T_2 & \text{on } \overline{V_3} \cap U_2. \end{cases} \end{aligned}$$

Again we see that $\mathcal{A} = \{U_i, \phi_i\}_{i=1,2}$ gives an affine structure on $B_0 = \mathbb{R}^3 - \Delta$. One can compute that, if we form the 3-torus bundle X_0, given a point $b \in B_0$ and two closed paths generating $\pi_1(B_0)$, then the H_1 monodromy of the fibre of X_0 associates to these two paths the matrices $(T_j^{-1})^t$ for $j = 1, 2$.

EXAMPLE 5 (Negative vertex). Let B and Δ be as in Example 4. Then, $\mathbb{R}^2 - \Delta$ has three connected components, which we denote C_1, C_2 and C_3. Let $\overline{C_j} = C_j \cup \partial C_j$. Consider the following three open subsets of B_0:

$$\begin{aligned} U_1 &= \mathbb{R}^3 - (\overline{C_2} \cup \overline{C_3}), \\ U_2 &= \mathbb{R}^3 - (\overline{C_1} \cup \overline{C_3}), \\ U_3 &= \mathbb{R}^3 - (\overline{C_1} \cup \overline{C_2}). \end{aligned}$$

Let

$$\begin{aligned} V^+ &= \{x_1 > 0\}, \\ V^- &= \{x_1 < 0\}. \end{aligned}$$

Clearly $U_i \cap U_j = V^+ \cup V^-$ when $i \neq j$. If T_1 and T_2 are as in (3), define the following coordinate charts on U_1, U_2, U_3 respectively:

$$\begin{aligned} \phi_1 &= \mathrm{Id}, \\ \phi_2 &= \begin{cases} (T_1^{-1})^t & \text{on } \overline{V}^+ \cap U_2 \\ \mathrm{Id} & \text{on } \overline{V}^- \cap U_2, \end{cases} \\ \phi_3 &= \begin{cases} \mathrm{Id} & \text{on } \overline{V}^+ \cap U_3 \\ (T_2^{-1})^t & \text{on } \overline{V}^- \cap U_3. \end{cases} \end{aligned}$$

We can check that the affine structure defined by these charts is such that, on the 3-torus bundle X_0, given a point $b \in B_0$, then the H_1 monodromy of the fibre of X_0 associates to two generators of $\pi_1(B_0)$ the matrices $T_j,\ j = 1,2$. In particular, monodromy is given by the inverse transpose matrices of the monodromy in the previous example.

These three examples are the building blocks of so-called 3-dimensional simple affine structures with singularities. A 3-dimensional compact example is the following:

EXAMPLE 6. This three dimensional example is taken from [11, §19.3]. Let Ξ be the 4-simplex in $\mathbb{R}^4$ spanned by

$$P_0 = (-1,-1,-1,-1),\ P_1 = (4,-1,-1,-1),\ P_2 = (-1,4,-1,-1),$$
$$P_3 = (-1,-1,4,-1),\ \ P_4 = (-1,-1,-1,4).$$

Let $B = \partial\Xi$. Denote by Σ_j the open 3-face of B opposite to the point P_j and by F_{ij} the closed 2-face separating Σ_i and Σ_j. Each F_{ij} contains 21 integral points (including those on its boundary). These form the vertices of a triangulation of F_{ij} as in Figure 1. By joining the barycenter of each triangle with the barycenters of its sides we form a trivalent graph as in Figure 1. Define the set Δ to be the union of all such graphs in each 2-face. Denote by I the set of integral points of B. Just as in the previous example, we can form a covering of $B_0 = B - \Delta$ by taking the open 3-faces Σ_j and small open neighborhoods U_Q inside B_0 of $Q \in I$. A coordinate chart ϕ_i on Σ_i can be obtained from its affine embedding in $\mathbb{R}^4$. If we denote again by R_Q the linear space spanned by $Q \in I$, as a chart on U_Q we take the projection $\phi_Q : U_Q \to \mathbb{R}^4/R_Q$.

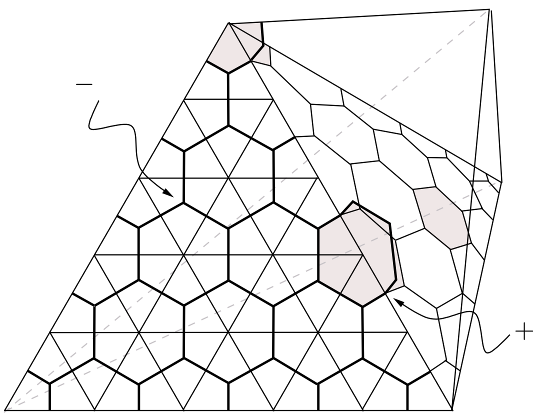

Figure 1: Affine S^3 with singularities.

In the above example one can check that points in the interior of the edges of the graph Δ have neighbourhoods which are affine isomorphic to neighbourhoods of points on the line of singularities in Example 3. The vertices of Δ which are in the interior of 2-faces have neighbourhoods affine isomorphic to the vertex in Example 5 (these are called negative vertices), while vertices of Δ on 1-faces have neighbourhoods affine isomorphic to a neighbourhood of the vertex in Example 4 (positive vertices).

We say that a 3-dimensional affine manifold with singularities is *simple* if Δ is a 3-valent graph, with vertices labelled as positive or negative. The affine structure near points on the edges of Δ is locally affine isomorphic to Example 3, near positive (resp. negative) vertices it is locally affine isomorphic to Example 4 (resp. Example 5).

2.2. Glueing singular fibres

Given the symplectic manifold $X_0 = T^*B_0/\Lambda$, how do we glue singular fibres to X_0? The 2-dimensional case can be achieved as follows. First consider the following example of Lagrangian fibration:

EXAMPLE 7. Let $X = \mathbb{C}^2 - \{z_1 z_2 + 1 = 0\}$ and let ω be the restriction to X of the standard symplectic form on $\mathbb{C}^2$. One can easily check that the following map $f : X \to \mathbb{R}^2$ is a Lagrangian fibration:

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log|z_1 z_2 + 1| \right). \tag{4}$$

The only singular fibre is $f^{-1}(0)$, which has the topology of a I_1 fibre (a pinched torus).

This is an example of a fibration of “focus-focus” type. One can explicitly compute the affine coordinates on the base, away from the singular point $(0,0) \in \mathbb{R}^2$. It can be shown that this affine structure is isomorphic (in a neighbourhood of $(0,0)$) to the one given in Example 1. This implies that given a 2-dimensional, simple, affine manifold with singularities and a point $p \in \Delta$, we can glue, via a fibre-preserving symplectomorphism, a neighbourhood of the singular fibre in the above example to $(f_0)^{-1}(U - p) \subset X_0$ for a suitable neighbourhood U. For the details of this construction consult [2]. If we do this at all 24 points in Example 2, in the end we obtain a symplectic manifold diffeomorphic to a K3 surface and a Lagrangian fibration $f : X \to S^2$ with 24 singular fibres and a Lagrangian section of f.

A similar, but rather more complicated, construction can be carried out in the case of a 3-dimensional, simple affine manifold with singularities. Thus obtaining a 6-dimensional (compact) symplectic manifold X with a Lagrangian fibration $f : X \to B$, together with a Lagrangian section. This is the main result of [2]. The idea is to find suitable models of Lagrangian fibrations with singular fibres which can be glued over Δ. When compactified in this way, Example 6 gives a manifold diffeomorphic to a quintic in $\mathbb{P}^4$. We should warn the reader that in the final result of [2] the map f is not smooth but just piecewise smooth, it fails to be smooth only along the preimage of small 2-dimensional discs containing negative vertices. Also, the discriminant locus Δ has to be enlarged slightly, so that near a negative vertex it is a codimension 1 thickening of the graph. The total space X obtained is nevertheless smooth. When the integral affine base is as the ones considered by Gross and Siebert, X turns out to be diffeomorphic to a Calabi–Yau.

2.3. Anti-symplectic involutions

An *anti-symplectic involution* on a symplectic manifold (X,ω) is a map $\iota : X \to X$ such that $\iota^*\omega = -\omega$ and $\iota^2 = \mathrm{Id}_X$. The fixed point set of an anti-symplectic involution is always a Lagrangian submanifold. In [3], together with Jake P. Solomon, we showed that, given a Lagrangian fibration $f : X \to B$ with a Lagrangian section constructed as above, one can also find an anti-symplectic involution $\iota : X \to X$ which preserves the fibres and fixes the section. The idea for the construction is as follows. Consider the fibre-preserving anti-symplectic involution ι_0 on T^*B_0/Λ induced by $(p,\alpha) \mapsto (p,-\alpha)$ for every $p \in B_0$ and $\alpha \in T_p^*B_0$. We show that ι_0 extends to a smooth fibre-preserving anti-symplectic involution ι on X. This is done by first studying anti-symplectic involutions on local models of singular fibres and then refining the gluing by also matching the involutions.

In this note, we would like to discuss the topology of the fixed point set of this type of involutions. The fixed point set Σ is the closure in X of the image of the set $\frac{1}{2}\Lambda$ inside T^*B_0/Λ. The map $f|_\Sigma : \Sigma \to B$ is a branched covering of B, branching over Δ. In dimension 2 this branched covering is of degree 4 and of degree 8 in dimension 3. Let us look more closely at some examples.

EXAMPLE 8. In the "focus-focus" case, i.e. Examples 1 and 7, fix a point $b \in \mathbb{R}^2 - \{(0,0)\}$. Then we can find a basis of Λ_b with respect to which, monodromy is the matrix (2). With respect to this basis, we can identify Λ_b with $\mathbb{Z}^2$. Then $\Sigma \cap T_b^*B_0/\Lambda_b$ consists of points $s_0 = (0,0), s_1 = (1/2,0), s_2 = (0,1/2)$ and $s_3 = (1/2,1/2)$. As we go around the singular point, monodromy maps s_0 and s_2 to themselves and s_1 to s_3. Therefore, if U is a neighbourhood of $(0,0)$, $f^{-1}(U) \cap \Sigma$ has 3 connected components. There are two which map 1 to 1 to U, these are the ones containing s_0 and s_2 respectively. Then there is one mapping 2 to 1, which contains both s_1 and s_3. The map f restricted to this latter component is a 2 to 1 branched covering, with branched point inside the singular fibre.

EXAMPLE 9. In the case of Example 2, where the compactified X is diffeomorphic to a K3 surface, the fixed point set Σ is a compact Lagrangian surface. It is not difficult to check that Σ has 2 connected components, one of them is the zero section and the other one is a degree 3 branched covering over S^2 with 24 branched points of ramification index 2. The Riemann–Hurwitz formula tells us that this connected component is a genus 10 surface.

Observe that in the previous example $b^1(\Sigma) = 20 = h^{1,1}(X)$. We will discuss in the following sections the reason why this equality is not a coincidence. Moreover it is known that the fixed point set Σ of an involution $\iota : X \to X$ on a compact manifold satisfies the following inequality in cohomology (see [4]):

$$\dim\left(\sum_{*} H^{*}(\Sigma,\mathbb{Z}/2\mathbb{Z})\right) \leq \dim\left(\sum_{*} H^{*}(X,\mathbb{Z}/2\mathbb{Z})\right) \tag{5}$$

In the case of our involution, Σ satisfies the equality, so it is in some sense maximal.

2.4. Mirror symmetry

For a more thorough explanation of the relevance of affine manifolds with singularities and Lagrangian torus fibrations in the context of mirror symmetry the reader may consult [9] and the references therein. We just mention a few facts that are related to the topic of this note. Given an affine manifold with singularities B, besides $X_0 = T^*B_0/\Lambda$, we can also construct $\check{X}_0 = TB_0/\Lambda^*$, where Λ^* is the dual lattice. If B is simple, then also $\check{X}_0$ can be topologically compactified. This follows from the Gross's topological compactification, in fact in this case the role of vertices is inverted: where we had negative vertices, we glue fibres of positive type and viceversa. We thus obtain a (compact) smooth manifold $\check{X}$ together with a torus fibration $\check{f}: \check{X} \to B$. It was shown by Gross (see next subsection) that the manifolds X and $\check{X}$ satisfy the topological conditions required for them to be mirror manifolds. In the case of Example 6, Gross also shows that $\check{X}$ is diffeomorphic to the mirror of the quintic.

The existence of a good symplectic structure on $\check{X}$ is not immediately apparent from this description, since the tangent bundle does not carry a natural symplectic structure. To solve this problem one needs the extra data on B of a (multivalued) strictly convex function ϕ, which can be used to define a symplectic form on TB_0. Equivalently, via the Legendre transform applied to ϕ, one defines a new affine structure on B_0, giving a new lattice in T^*B_0, which we denote by $\check{\Lambda}$. It can be checked that $T^*B_0/\check{\Lambda}$ and $\check{X}_0$ are isomorphic torus bundles over B_0, therefore also $\check{X}_0$ has a symplectic structure (inherited from T^*B_0). Thus, also $\check{X}_0$ can be symplectically compactified to give a Lagrangian fibration $\check{f}: \check{X} \to B$, with a fibre-preserving anti-symplectic involution. Using ideas from toric geometry, Gross and Siebert also introduce the discrete Legendre transform, which is a combinatorial version of the standard Legendre transform. In [10], Gross shows that this construction can be applied to all the examples of Batirev–Borisov's pairs of mirror Calabi–Yau's.

In [3] we also discuss the relevance of our construction of anti-symplectic involutions in the context of the Homological Mirror Symmetry conjecture. This conjecture states that given mirror manifolds X and $\check{X}$, there should be an equivalence of categories between the derived category of coherent sheaves on $\check{X}$ and the derived Fukaya category on X. The objects in this latter category are Lagrangian submanifolds of X, with some other data attached. Since the two categories are conjectured to be equivalent, an autoequivalence on one category should correspond to one on the other. The category of coherent sheaves has a natural autoequivalence which consists in mapping a sheaf to its dual. In [3] we discussed some evidence of a conjecture claiming that the autoequivalence on the Fukaya category, corresponding to dualization, should be given by the anti-symplectic involution ι that we constructed, where a Lagrangian submanifold is mapped to its image under ι.

2.5. Topology of Lagrangian 3-torus fibrations

We describe here some of Gross's results on the Leray spectral sequence applied to the torus fibrations $f : X \to B$ of the type discussed in the previous section. We assume that $(B, \Delta, \mathcal{A})$ is a compact simply-connected, 3-dimensional, simple integral affine manifold with singularities. The arguments are entirely topological, without any reference to the fact that X is symplectic and the fibres are Lagrangian. Given an abelian group G, we denote by $R^k f_* G$ the sheaf associated to the presheaf on B given by $U \mapsto H^k(f^{-1}(U), G)$. The Leray spectral sequence associated to f has as E_2 terms the groups $H^j(B, R^k f_* G)$. We recall Gross's definition:

DEFINITION 1. *Let $i : B_0 \to B$ be the inclusion. The fibration $f : X \to B$ is G-simple if*

$$i_* R^k f_{0*} G = R^k f_* G$$

We will assume in the following that X is simply connected. For some of the arguments, this condition can be relaxed, e.g. it could be replaced with $H^1(X, \mathbb{R}) = 0$. In [8] Gross showed that the fibrations considered here are always G-simple, when G is $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. Moreover $\mathbb{Z}$ simplicity implies $\mathbb{Q}$ simplicity. Notice also that, since affine coordinates on B_0 have linear part in $\mathrm{Sl}(n, \mathbb{Z})$, the fibres are canonically oriented, so that

$$R^3 f_* G = G.$$

Moreover, f has a smooth section (extending the zero section on $T^* B_0$). We also consider the mirror dual fibration $\check{f} : \check{X} \to B$, which is also a G-simple fibration with a section and we will assume that also $\check{X}$ is simply connected.

Now, let $G = \mathbb{Q}$ (but the following also holds for $G = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$). By Poincaré duality applied to the fibres, we have that

$$(R^j f_{0*} \mathbb{Q})^\vee = R^{n-j} f_{0*} \mathbb{Q}.$$

Moreover, from the definition of dual torus fibration, we also have:

$$(R^j f_{0*} \mathbb{Q})^\vee = R^j \check{f}_{0*} \mathbb{Q}.$$

By applying i_* to the above and using $\mathbb{Q}$-simplicity we obtain

$$R^j \check{f}_* \mathbb{Q} = R^{n-j} f_* \mathbb{Q}. \tag{6}$$

The E_2 page for the Leray spectral sequence for f with $G = \mathbb{Q}$, looks like the following

$$\begin{array}{cccc} \mathbb{Q} & 0 & 0 & \mathbb{Q} \\ 0 & H^1(B, R^2 f_* \mathbb{Q}) & H^2(B, R^2 f_* \mathbb{Q}) & 0 \\ 0 & H^1(B, R^1 f_* \mathbb{Q}) & H^2(B, R^1 f_* \mathbb{Q}) & 0 \\ \mathbb{Q} & 0 & 0 & \mathbb{Q} \end{array}$$

For the proof of this, one can argue as follows. Since the fibres are connected, we have $R^0 f_*\mathbb{Q} = \mathbb{Q}$. Moreover $R^3 f_*\mathbb{Q} = \mathbb{Q}$, as we already mentioned. So, together with the fact that B is simply connected, we obtain the zeroes in the top and bottom row. The zeroes in the first and last column come from the fact that $H^1(X,\mathbb{Q}) = H^5(X,\mathbb{Q}) = H^1(\check{X},\mathbb{Q}) = H^5(\check{X},\mathbb{Q}) = 0$ together with (6). The E_2 term for $\check{f}$ is obtained by exchanging the first and second row of the E_2 term of f.

Gross proved that under these hypotheses the Leray spectral sequences of f and $\check{f}$ degenerate at the E_2 term, so that when X is a Calabi–Yau manifold

$$h^{1,1}(X) = \dim H^1(B, R^1 f_*\mathbb{Q}) = \dim H^1(B, R^2 \check{f}_*\mathbb{Q}) = h^{2,1}(\check{X}).$$

So the topology of these fibrations on X and $\check{X}$ guarantees that X and $\check{X}$ satisfy the basic topological requirement of mirror symmetry. These arguments also work if we replace $\mathbb{Q}$ with $\mathbb{Z}/2\mathbb{Z}$ (except, maybe, the equality with Hodge numbers, due to possible presence of 2-torsion):

$$\begin{array}{cccc}
\mathbb{Z}/2\mathbb{Z} & 0 & 0 & \mathbb{Z}/2\mathbb{Z} \\
0 & H^1(B,R^2 f_*\mathbb{Z}/2\mathbb{Z}) & H^2(B,R^2 f_*\mathbb{Z}/2\mathbb{Z}) & 0 \\
0 & H^1(B,R^1 f_*\mathbb{Z}/2\mathbb{Z}) & H^2(B,R^1 f_*\mathbb{Z}/2\mathbb{Z}) & 0 \\
\mathbb{Z}/2\mathbb{Z} & 0 & 0 & \mathbb{Z}/2\mathbb{Z}
\end{array}$$

From which we obtain that

$$H^2(X,\mathbb{Z}/2\mathbb{Z}) \cong H^1(B,R^1 f_*\mathbb{Z}/2\mathbb{Z})$$

$$H^3(X,\mathbb{Z}/2\mathbb{Z}) \cong H^1(B,R^2 f_*\mathbb{Z}/2\mathbb{Z}) \oplus H^2(B,R^1 f_*\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$H^4(X,\mathbb{Z}/2\mathbb{Z}) \cong H^2(B,R^2 f_*\mathbb{Z}/2\mathbb{Z})$$

3. A long exact sequence

We now wish to understand the $\mathbb{Z}_2$-cohomology of the fixed point locus of the anti-symplectic involutions constructed in the previous section. We sketch here the construction of a long exact sequence which links the cohomology of the ambient manifold X with the cohomology of Σ, details will appear in [1]. The assumptions on $f : X \to B$ are the same as those in the last subsection of the previous section (in particular X is 6-dimensional) and we let Σ be the fixed point locus of the anti-symplectic involution $\iota : X \to X$. Let

$$\sigma = f|_\Sigma$$

and denote by σ_0 the restriction of σ to $\sigma^{-1}(B_0)$. The idea is to consider the spectral sequence associated to the branched covering $\sigma : \Sigma \to B$ and compare it with the one associated to f. Observe that the E_2 term of the spectral sequence of σ consists of just one row of elements of the type $H^j(B, R^0\sigma_*\mathbb{Z}/2\mathbb{Z})$ and therefore the spectral sequence degenerates at E_2. It can also be shown that, since f is $\mathbb{Z}/2\mathbb{Z}$-simple then also $\sigma : \Sigma \to B$ is a $\mathbb{Z}/2\mathbb{Z}$-simple fibration.

We can now restrict our attention only to the sheaf $R^0\sigma_{0*}\mathbb{Z}/2\mathbb{Z}$. For every point $p \in B_0$, $\sigma^{-1}(p)$ consists of the 8 points in the image of $\frac{1}{2}\Lambda_p$ inside $T_p^* B_0/\Lambda_p$. Notice that $\sigma^{-1}(p)$ has a group structure isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Denote this group by $\mathcal{G}_p$ and let $\mathcal{G}$ be the sheaf over B_0 whose stalk is $\mathcal{G}_p$. Observe that

$$\mathcal{G} \cong R^2 f_{0*}\mathbb{Z}/2\mathbb{Z}, \tag{7}$$

in fact $\mathcal{G}$ is naturally isomorphic to $(R^1 f_{0*}\mathbb{Z}/2\mathbb{Z})^\vee$.

Now let us denote by $\mathcal{G}'$ the sheaf $R^0\sigma_{0*}\mathbb{Z}/2\mathbb{Z}$ and observe that $\mathcal{G}'_p$ is just the set of maps from $\mathcal{G}_p$ to $\mathbb{Z}/2\mathbb{Z}$. Clearly, constant maps are monodromy invariant, but since monodromy acts linearly on $\mathcal{G}_p$, also the map which is 1 at $0 \in \mathcal{G}_p$ and zero elsewhere is monodromy invariant. Let us denote by $\mathcal{C}$ the sheaf generated by the constant maps and this latter map. Since these maps are monodromy invariant, $\mathcal{C}$ is just the constant sheaf $(\mathbb{Z}/2\mathbb{Z})^2$. Also note that $\mathcal{G}^\vee$ is naturally a subsheaf of $\mathcal{G}'$. It can be shown that there is a short exact sequence of sheaves

$$0 \to \mathcal{G}^\vee \oplus \mathcal{C} \to \mathcal{G}' \to \mathcal{G} \to 0.$$

The map $\mathcal{G}' \to \mathcal{G}$ in the above sequence is defined as follows. Let $g \in \mathcal{G}_p$ and denote by $\delta_g \in \mathcal{G}'_p$ the map which is 1 at g and zero elsewhere. One can show that every class in the quotient of $\mathcal{G}'_p$ by $\mathcal{G}_p^\vee \oplus \mathcal{C}_p$ is represented by a δ_g for a unique g. So the map from $\mathcal{G}'_p$ to $\mathcal{G}_p$ maps every element in the class of δ_g to g. It can be shown that this map is linear and that it is a morphism of sheaves. Using (7) and $\mathbb{Z}/2\mathbb{Z}$-simplicity the above exact sequence becomes

$$0 \to (R^1 f_*\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to R^0\sigma_*\mathbb{Z}/2\mathbb{Z} \to R^2 f_*\mathbb{Z}/2\mathbb{Z} \to 0.$$

The fact that the sequence remains short exact after applying i_* follows by directly computing the above maps on elements which are locally monodromy invariant near points $p \in \Delta$. With some abuse of notation, we continue to denote this sequence by

$$0 \to \mathcal{G}^\vee \oplus \mathcal{C} \to \mathcal{G}' \to \mathcal{G} \to 0.$$

Passing to the long exact sequence in sheaf cohomology, we obtain

THEOREM 1. *The sheaves $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}^\vee$ over B satisfy the following long exact sequence:*

$$\begin{array}{llll} 0 \to H^0(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^0(B, \mathcal{G}') \to & H^0(B, \mathcal{G}) \to \\ H^1(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^1(B, \mathcal{G}') \to & H^1(B, \mathcal{G}) \stackrel{\beta}{\to} \\ H^2(B, \mathcal{G}^\vee \oplus \mathcal{C}) & \to H^2(B, \mathcal{G}') \to & H^2(B, \mathcal{G}) \to 0 \end{array} \tag{8}$$

Observe that from the Leray spectral sequence for σ, we have that

$$H^j(B, \mathcal{G}') \cong H^j(\Sigma, \mathbb{Z}/2\mathbb{Z}).$$

and from the definitions of $\mathcal{G}$, $\mathcal{G}^\vee$ and $\mathcal{C}$, for $j = 1,2$, we have

$$H^j(B, \mathcal{G}^\vee \oplus \mathcal{C}) \cong H^j(B, \mathcal{G}^\vee) \cong H^j(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}),$$
$$H^j(B, \mathcal{G}) \cong H^j(B, R^2 f_* \mathbb{Z}/2\mathbb{Z}).$$

So we obtain

COROLLARY 1. *Σ has two connected components.*

Proof. Since $H^0(B, \mathcal{G}) = H^0(B, \mathcal{G}^\vee) = 0$ and $\mathcal{C} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the first row splits off from the rest and it tells us that $H^0(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence Σ has two connected components. □

One of the two components is the zero section, therefore diffeomorphic to S^3. Notice also that the second row tells us that $H^1(B, R^1 f_* \mathbb{Z}/2\mathbb{Z})$ injects into $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$.

COROLLARY 2. *With the above hypotheses*

$$\dim H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \geq \dim H^2(X, \mathbb{Z}/2\mathbb{Z})$$

Moreover if $\beta = 0$, for $j = 1,2$ we would have

$$H^j(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong H^j(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}) \oplus H^j(B, R^2 f_* \mathbb{Z}/2\mathbb{Z})$$

Observe that if $\beta = 0$ then Σ satisfies the equality in the inequality (5) and is therefore maximal. Observe also that in the 2-dimensional case of Example 9, a similar but smaller spectral sequence gives us:

$$H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \cong H^1(B, R^1 f_* \mathbb{Z}/2\mathbb{Z}) \oplus H^2(B, R^2 f_* \mathbb{Z}/2\mathbb{Z}).$$

Since the total space is a K3 surface and Σ is oriented, the above equality holds since $b_1(\Sigma) = 2h^{1,1}(X) = 20$, which is what we already noticed.

4. An example: Schoen's Calabi–Yau

4.1. The manifold and the fibration

At the time of writing this note, we were able to compute the cohomology of only one example of fixed point locus of an involution of the type described. It comes from a Lagrangian fibration of the so-called Schoen's Calabi–Yau. This manifold was studied in [15], and then described in terms of its associated affine manifold with singularities by Gross in [10]. Kovalev [13] described a 3-torus fibration, which inspired the construction we provide here. Consider $f_1 : Y_1 \to \mathbb{P}^1$ and $f_2 : Y_2 \to \mathbb{P}^1$ two rational elliptic surfaces with a section, such that there does not exist $x \in \mathbb{P}^1$ for which $f_1^{-1}(x)$ and $f_2^{-1}(x)$ are both singular. Then Schoen's Calabi–Yau is the fibred product $Y = Y_1 \times_{\mathbb{P}^1} Y_2$. It satisfies $\chi(X) = 0$, $h^{1,1}(X) = h^{1,2}(X) = 19$. It can also be written as a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ of hypersurfaces of tridegree $(1,3,0)$ and $(1,0,3)$.

A topological construction can be given as follows. Consider a 4-dimensional manifold with boundary $\overline{M}$ which fibres over the closed 2-disc D so that the general fibre is a 2-torus and such that there are 12 singular fibres of Kodaira type I_1 (pinched tori) over interior points of D. Assume also that the boundary of $\overline{M}$ is a trivial 2-torus bundle over $\partial D = S^1$, i.e. $\partial\overline{M} \cong T \times S^1$, where T is a 2-torus. To construct $\overline{M}$ we can proceed as follows. Take an elliptically or Lagrangian fibred $K3$, with 24 singular fibres of Kodaira type I_1. Then consider a simple closed curve γ on the base bounding a 2-disc D containing images of 12 singular fibres, and such that it does not pass through critical points. If, furthermore, we choose γ so that along it the H^1-monodromy of the fibres is trivial, then we can take $\overline{M}$ to be the union of the fibres over D. Another construction can be found in [14]. Now consider the 6-manifold with boundary $\overline{X} = \overline{M} \times T'$, where T' is a 2-torus. Clearly $\overline{X}$ fibres over $D \times S^1$ by taking the product of the given fibration of $\overline{M}$ with the standard S^1 fibration of T. The boundary of $\overline{X}$ is $S^1 \times T \times T'$, where $S^1 \times T$ is the boundary of $\overline{M}$. Consider coordinates on $\partial\overline{X}$ given by $(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2)$, where ϕ_1 is the (angle) coordinate on S^1, (ϕ_2, ϕ_3) and (θ_1, θ_2) are (angle) coordinates on T and T' respectively. Assume that the fibration restricted to $\partial\overline{X}$ is the projection onto the coordinates $(\phi_1, \theta_1) \in \partial D \times S^1$.

Now consider the homeomorphism $\Phi : \partial\overline{X} \to \partial\overline{X}$ given by

$$\Phi(\phi_1, \phi_2, \phi_3, \theta_1, \theta_2) = (\theta_1, \theta_2, -\phi_3, \phi_1, \phi_2).$$

We form the manifold Y by gluing two copies of $\overline{X}$ along their boundary using the homeomorphism Φ, i.e.

$$Y = \overline{X} \sqcup_\Phi \overline{X}.$$

It turns out that Y is diffeomorphic to Schoen's Calabi–Yau (see Gross [10] and Kovalev [13]). Notice that if we glue two copies of the base $D \times S^1$ of the fibration on $\overline{X}$ via the map $(\phi_1, \theta_1) \mapsto (\theta_1, \phi_1)$, then we obtain a 3-sphere S^3 and a 3-torus fibration of Y on S^3 induced from the fibration on $\overline{X}$. One can show, using the results of [2], that this fibration can be turned into a (smooth) Lagrangian fibration, by compactifying the affine manifold with singularities constructed by Gross [10, §4].

4.2. The involution and the fixed point locus

We now describe the involution on Y. On $\overline{M}$ we can construct a fibrepreserving involution, simply by considering the involution on the K3 as described in Example 9 and restricting it to $\overline{M}$. On T' we take the involution which preserves the fibres of the fibration on S^1, i.e. in coordinates (θ_1, θ_2), the involution is $(\theta_1, \theta_2) \mapsto (\theta_1, -\theta_2)$. Then on $\overline{X} = \overline{M} \times T'$ we take the product involution. It clearly descends to an involution on Y.

Let us now describe the fixed point locus of this involution. The fixed point locus of the involution on $\overline{M}$ is the disjoint union of a 2-disc, which we denote by S_0 (corresponding to the zero section on the $K3$) and a genus 4 surface S_1 with 3 open discs removed. The boundary of the 2-disc and the three copies of S^1 forming the boundary of S_1 are mapped by the fibration to the boundary of the base ∂D. In fact, we may assume that, with respect to the coordinates (ϕ_1, ϕ_2, ϕ_3) of $\partial \bar{M} = S^1 \times T$ the involution is $(\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, -\phi_3)$ so the four circles are given by $(\phi_1, 0, 0)$ (which is the boundary of S_0), $(\phi_1, 1/2, 0)$, $(\phi_1, 0, 1/2)$, $(\phi_1, 1/2, 1/2)$. The latter three form the boundary of S_1. Now on T', the fixed locus of the involution is given by a pair of circles, corresponding to $(\theta_1, 0)$ and $(\theta_1, 1/2)$. Therefore the fixed point locus of the involution on $\bar{M} \times T'$ is given by two copies of $S_0 \times S^1$ and two copies of $S_1 \times S^1$, i.e.

$$\bar{\Sigma} = (S_0 \times S^1) \sqcup (S_0 \times S^1) \sqcup (S_1 \times S^1) \sqcup (S_1 \times S^1).$$

Then, the fixed point locus of the involution on Y is obtained by gluing together two copies of $\bar{\Sigma}$ via the the homeomorphism Φ restricted to the boundary components of $\bar{\Sigma}$. The result is a 3-manifold Σ with two connected components, one of which is a 3-sphere. According to our computations (based on the Mayer–Vietoris Theorem) $H^1(\Sigma, \mathbb{Z})$ is $\mathbb{Z}^{34}$. Since $H^2(Y, \mathbb{Z}) = \mathbb{Z}^{19}$ and $H^3(Y, \mathbb{Z}) = \mathbb{Z}^{40}$, in the long exact sequence of Theorem 1 applied to Y and Σ, we have

$$H^1(B, \mathcal{G}) = (\mathbb{Z}/2\mathbb{Z})^{19},$$

and

$$\dim \ker \beta = 15.$$

So this is an example where β is not zero.

References

[1] CASTANO-BERNARD R. AND MATESSI D. The fixed point set of anti-symplectic involutions of Lagrangian fibrations. In progress.

[2] CASTANO-BERNARD R. AND MATESSI D. Lagrangian 3-torus fibrations. *J. Differential Geom. 81* (2009), 483–573.

[3] CASTANO-BERNARD R., MATESSI D. AND SOLOMON J. Symmetries of Lagrangian fibrations. *Adv. in Math. 225*, 3 (2010), 1341–1386.

[4] FLOYD E. E. On periodic maps and the Euler characteristics of associated spaces. *Trans. Amer. Math Soc. 72* (1952), 138–147.

[5] FUKAYA K., OH Y.-G., OHTA H. AND ONO K. Anti-symplectic involution and Floer cohomology. `arXiv:0912.2646v2`.

[6] GROSS M. Special Lagrangian fibrations I: Topology. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)* (River Edge NJ, 1998), World Sci. Publishing, pp. 156–193.

[7] GROSS M. Special Lagrangian fibrations II. Geometry. *Surv. Differ. Geom. 5* (1999), 341–403.

[8] GROSS M. Topological mirror symmetry. *Invent. Math. 144* (2001), 75–137.

[9] GROSS M. The Strominger-Yau-Zaslow conjecture: From torus fibrations to degenerations. In *Algebraic Geometry, Seattle 2005* (2005), vol. 80 of *Proceedings of Symposia in Pure Mathematics*, Amer. Math. Soc., pp. 149–192.

[10] GROSS M. Toric degenerations and Batirev-Borisov duality. *Math. Ann. 333* (2005), 645–688.

[11] GROSS M., HUYBRECHTS D. AND JOYCE D. *Calabi-Yau Manifolds and Related geometries*. Springer-Verlag, 2003. Lecture notes at a summer school in Nordfjordeid, Norway, June 2001.

[12] GROSS M. AND SIEBERT B. Mirror symmetry via logarithmic degeneration data I. *J. Differential Geom. 72* (2006), 169–338.

[13] KOVALEV A. A special Lagrangian fibration of smooth compact Calabi-Yau 3-fold. IPAM lecture, April 2003.

[14] LEUNG N. AND SYMINGTON M. Almost toric symplectic four-manifolds. *J. Symplectic Geom. 8* (2010), 143–187.

[15] SCHOEN C. On fiber products of rational elliptic surfaces with section. *Math. Z. 197* (1988), 177–197.

[16] SOLOMON J. Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions. `arXiv:math.SG/0606429`.

AMS Subject Classification: 53D37, 14J32

Ricardo CASTAÑO–BERNARD,
Mathematics Departmnent, Kansas State University,
138 Cardwell Hall, Manhattan, KS 66502, USA
e-mail: `rcastano@math.ksu.edu`

Diego MATESSI,
Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale,
Via Bellini 25/G, 15100 Alessandria, ITALIA
e-mail: `matessi@unipmn.it`

Lavoro pervenuto in redazione il 30.07.2010, e in forma definitiva, il 16.10.2010