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Global existence of weak solutions to a nonlocal Cahn–Hilliard–Navier–Stokes system

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ABSTRACT

A well-known diffuse interface model consists of the Navier–Stokes equations nonlinearly coupled with a convective Cahn–Hilliard type equation. This system describes the evolution of an incompressible isothermal mixture of binary fluids and it has been investigated by many authors. Here we consider a variant of this model where the standard Cahn–Hilliard equation is replaced by its nonlocal version. More precisely, the gradient term in the free energy functional is replaced by a spatial convolution operator acting on the order parameter φ, while the potential F may have any polynomial growth. Therefore the coupling with the Navier–Stokes equations is difficult to handle even in two spatial dimensions because of the lack of regularity of φ. We establish the global existence of a weak solution. In the two-dimensional case we also prove that such a solution satisfies the energy identity and a dissipative estimate, provided that F fulfills a suitable coercivity condition.

1. Introduction

A well-known model which describes the evolution of an incompressible isothermal mixture of two immiscible fluids is the so-called model H (see [29,26], cf. also [17,35,37] and references therein). This is a diffuse interface model (cf. [4]) in which the sharp interface separating the two fluids (e.g., oil and water) is replaced by a diffuse one by introducing an order parameter φ. The dynamics of φ, which represents the (relative) concentration of one of the fluids (or the difference of the two concentrations), is governed by a Cahn–Hilliard type equation with a transport term. This parameter influences the (average) fluid velocity u through a capillarity force (called Korteweg force) proportional to $\mu\nabla\varphi$, where μ is the chemical potential (see, e.g., [30, Appendix] and references therein). Note that this force is concentrated close to the diffuse interface.

In a simplified setting where the density ϱ and the mobility m of the mixture are supposed to be constant, the model reduces to

$$\varphi_t + u\cdot\nabla\varphi = m\Delta\mu, \tag{1.1}$$

$$\varrho u_t - \operatorname{div}\big(\nu(\varphi)2Du\big) + (u\cdot\nabla)u + \nabla\pi = \kappa\mu\nabla\varphi + h, \tag{1.2}$$

$$\operatorname{div}(u) = 0 \tag{1.3}$$

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in $\Omega \times (0, T)$, where Ω is a domain in $\mathbb{R}^d$, $d = 2, 3$, and $T > 0$ is a given final time. Here π is the pressure, ν denotes the viscosity, $2Du := \nabla u + (\nabla u)^{tr}$, κ is a given positive (capillarity) constant and h represents volume forces applied to the binary mixture fluid. The chemical potential μ is the first variation of the free energy functional (see [14])

$$E(\varphi) = \int_\Omega \left(\frac{\xi}{2} \left| \nabla \varphi(x) \right|^2 + \eta F\big(\varphi(x)\big) \right) dx. \tag{1.4}$$

Here F represents the (density of) potential energy. This function is usually a double-well potential whose wells are located in the pure phases, while ξ and η are given positive constants. The potential can be defined either on the whole real line (smooth potential) or on a bounded interval (singular potential). The latter case (in a logarithmic form) is the most appropriate choice from the modeling viewpoint (cf. [14]), while the former can be considered as an approximation.

In the context of statistical mechanics, the square gradient term in (1.4) arises from attractive long-ranged interactions between the molecules of the fluid and ξ can be related to the pair correlation function (see, e.g., [4] and references therein). We also recall that κ and ξ are of the same order as the interface thickness $\varepsilon > 0$, while η is proportional to ε^{-1}. On account of (1.4), the chemical potential takes the following form

$$\mu = -\xi \Delta \varphi + \eta F'(\varphi). \tag{1.5}$$

Systems like (1.1)–(1.5), also known as Cahn–Hilliard–Navier–Stokes systems, have been studied from the mathematical viewpoint by several authors (see, for instance, [1–3,11–13,22,23,40,42], cf. also [6,18,31,19,33,39] for numerical issues).

A different form of the free energy has been proposed in [24,25] and rigorously justified as a macroscopic limit of microscopic phase segregation models with particle conserving dynamics (see also [15]). In this case the gradient term is replaced by a nonlocal spatial operator, namely,

$$\mathcal{E}(\varphi) = \frac{1}{4} \int_\Omega \int_\Omega J(x-y)\big(\varphi(x) - \varphi(y)\big)^2 \, dx\, dy + \eta \int_\Omega F\big(\varphi(x)\big)\, dx, \tag{1.6}$$

where $J : \mathbb{R}^d \to \mathbb{R}$ is a smooth function such that $J(x) = J(-x)$. Taking the first variation of $\mathcal{E}$ we can define the chemical potential associated with the nonlocal model

$$\mu = a\varphi - J * \varphi + \eta F'(\varphi) \tag{1.7}$$

where

$$(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)\, dy, \qquad a(x) := \int_\Omega J(x-y)\, dy, \quad x \in \Omega. \tag{1.8}$$

The corresponding nonlocal Cahn–Hilliard equation $\varphi_t = m\Delta\mu$ can be derived from idealized microscopic models through suitable limits like the diffusion equation and the Boltzmann equation. Moreover, the evolution in the sharp interface limits are the same as those derived from the classical Cahn–Hilliard equation in the corresponding limits (see [25]). However, from the mathematical viewpoint, the nonlocal Cahn–Hilliard equation, due to its integrodifferential nature, is rather difficult to handle (see, e.g., [8,9,16,20,21,27,34]). Here we consider system (1.1)–(1.3) with (1.7). More precisely, taking for simplicity $\varrho = m = \kappa = 1$, we want to study the following initial and boundary value problem

$$\varphi_t + u \cdot \nabla \varphi = \Delta \mu, \tag{1.9}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{1.10}$$

$$u_t - \operatorname{div}\big(\nu(\varphi) 2Du\big) + (u \cdot \nabla)u + \nabla \pi = \mu \nabla \varphi + h, \tag{1.11}$$

$$\operatorname{div}(u) = 0, \tag{1.12}$$

$$\frac{\partial \mu}{\partial n} = 0, \qquad u = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.13}$$

$$u(0) = u_0, \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{1.14}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with sufficiently smooth boundary and unit outward normal n. The no-flux boundary condition for μ is the usual one for Cahn–Hilliard type equations (cf., e.g., [8]) and implies the conservation of mass (see Remark 5 below). The no-slip boundary condition for u is also standard especially when one wants to investigate a new model involving Navier–Stokes equations (periodic boundary conditions can also be considered).

In this contribution we prove the existence of a global weak solution for smooth potentials F of arbitrary polynomial growth. Moreover, if F satisfies a suitable coercivity condition then we can slightly improve the smoothness properties of the solution. In particular, we show the validity of an energy identity if $d = 2$. These results are a first step towards the mathematical analysis of problem (1.9)–(1.14). However, further issues (such as, e.g., uniqueness in two dimensions) do not seem so straightforward to prove. The main difficulty arises from the presence of the nonlocal term which implies that φ is not as regular as for the standard (local) Cahn–Hilliard–Navier–Stokes system (cf. Remark 8 below). For this reason, we have not been able even to establish uniqueness of weak solutions in two dimensions. We conclude by mentioning that there are some related works on models for liquid-vapor phase transitions in which nonlocal energy functionals are considered, i.e., the so-called nonlocal Navier–Stokes–Korteweg systems (see, for instance, [28,38]).

2. Notation and functional setup

Let us set $V_s := D(B^{s/2})$ for every $s \in \mathbb{R}$, where $B = -\Delta + I$ with homogeneous Neumann boundary conditions. Hence we have

$$V_2 = D(B) = \left\{ v \in H^2(\Omega)\colon \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

We also define $H := V_0 = L^2(\Omega)$ and $V := V_1 = H^1(\Omega)$. Then we introduce the classical Hilbert spaces for the Navier–Stokes equations (see, e.g., [41])

$$G_{\text{div}} := \overline{\left\{ u \in C_0^\infty(\Omega)^d\colon \operatorname{div}(u) = 0 \right\}}^{L^2(\Omega)^d},$$

and

$$V_{\text{div}} := \left\{ u \in H_0^1(\Omega)^d\colon \operatorname{div}(u) = 0 \right\}.$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the scalar product, respectively, on both H and G_{div}. We recall that V_{div} is endowed with the scalar product

$$(u, v)_{V_{\text{div}}} = (\nabla u, \nabla v), \quad \forall u, v \in V_{\text{div}}. \tag{2.1}$$

Let us introduce the Stokes operator $A : D(A) \cap G_{\text{div}} \to G_{\text{div}}$. Recall that, in the case of no-slip boundary condition (1.13)

$$A = -P\Delta, \qquad D(A) = H^2(\Omega)^d \cap V_{\text{div}},$$

where $P : L^2(\Omega)^d \to G_{\text{div}}$ is the Leray projector. Notice that we have

$$(Au, v) = (u, v)_{V_{\text{div}}} = (\nabla u, \nabla v), \quad \forall u \in D(A),\ \forall v \in V_{\text{div}}.$$

We also recall that $A^{-1} : G_{\text{div}} \to G_{\text{div}}$ is a self-adjoint compact operator in G_{div} and by the classical spectral theorems there exists a sequence λ_j with

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_j \to \infty,$$

and a family of $w_j \in D(A)$ which is orthonormal in G_{div} and such that

$$Aw_j = \lambda_j w_j.$$

We also need to define the map $\mathcal{A} : V_{\text{div}} \times H \to V'_{\text{div}}$ in the following way. For every $u \in V_{\text{div}}$ and every $\varphi \in H$ we set

$$\langle \mathcal{A}(u,\varphi), v \rangle := \big(\nu(\varphi) 2Du, Dv\big), \quad \forall v \in V_{\text{div}},$$

where ν is a continuous function satisfying $\nu_1 \leqslant \nu(s) \leqslant \nu_2$, for all $s \in \mathbb{R}$, with $\nu_1, \nu_2 > 0$. Notice that if $\nu = 1$ we have

$$\langle \mathcal{A}(u,\varphi), v \rangle = (2Du, Dv) = (\nabla u, \nabla v), \quad \forall u, v \in V_{\text{div}},$$

and hence in this case we have $\mathcal{A}(u,\varphi) = Au$ for every $u \in D(A)$. Finally, for u, v, $w \in V_{\text{div}}$ we define the trilinear V_{div}-continuous form

$$b(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w,$$

and the bilinear operator $\mathcal{B}$ from $V_{\text{div}} \times V_{\text{div}}$ into V'_{div} defined by

$$\langle \mathcal{B}(u, v), w \rangle = b(u, v, w), \quad \forall u, v, w \in V_{\text{div}}.$$

We recall that we have

$$b(u, w, v) = -b(u, v, w), \quad \forall u, v, w \in V_{\text{div}}, \tag{2.2}$$

and that, for every u, v and $w \in V_{\text{div}}$, the following estimates hold

$$\big|b(u, v, w)\big| \leqslant c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|\nabla w\|, \quad \text{for } d = 3, \tag{2.3}$$

$$\big|b(u, v, w)\big| \leqslant c\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2}, \quad \text{for } d = 2. \tag{2.4}$$

The following simple lemma, which will be useful to deduce the energy inequality for the weak solutions, can be easily proved using the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$.

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\{f_n\} \subset L^\infty(\Omega)$ be a sequence such that $\|f_n\|_\infty \leqslant c$ and $f_n \to f$ strongly in $L^2(\Omega)$. Let $\{g_n\} \subset L^2(\Omega)$ be another sequence such that $g_n \rightharpoonup g$ weakly in $L^2(\Omega)$. Then $f_n g_n \rightharpoonup fg$ weakly in $L^2(\Omega)$.*

In this paper c will stand for a nonnegative constant depending on J, F, Ω, ν_1, ν_2 and T, at most. The value of c may vary even within the same line. We shall denote by N, M or L generic nonnegative constants that depend on the initial data u_0, φ_0 and on h and whose values will be explicitly pointed out if needed.

3. Main result

In this section we first define the notion of weak solution to problem (1.9)–(1.14) which will be called Problem P. Then we state the main result of this paper and a related corollary.

Our assumptions on the kernel J, the viscosity ν, the potential F and the forcing term h are the following (cf. also (1.8))

(H1) $$J \in W^{1,1}(\mathbb{R}^d), \qquad J(x) = J(-x), \qquad a(x) := \int_\Omega J(x-y)\,dy \geqslant 0, \quad \text{a.e. } x \in \Omega.$$

(H2) The function ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_1, \nu_2 > 0$ such that

$$\nu_1 \leqslant \nu(s) \leqslant \nu_2, \quad \forall s \in \mathbb{R}.$$

(H3) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geqslant c_0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

(H4) There exist $c_1 > \frac{1}{2}\|J\|_{L^1(\mathbb{R}^d)}$ and $c_2 \in \mathbb{R}$ such that

$$F(s) \geqslant c_1 s^2 - c_2, \quad \forall s \in \mathbb{R}.$$

(H5) There exist $c_3 > 0$, $c_4 \geqslant 0$ and $p \in (1,2]$ such that

$$\left|F'(s)\right|^p \leqslant c_3\left|F(s)\right| + c_4, \quad \forall s \in \mathbb{R}.$$

(H6) $h \in L^2(0,T;V'_{\mathrm{div}})$ for all $T > 0$.

Remark 1. The requirements of assumption (H1) are standard for the nonlocal Cahn–Hilliard equation (see, e.g., [8] for slightly stronger hypotheses).

Remark 2. Assumption (H3) implies that the potential F is a quadratic perturbation of a (strictly) convex function. Indeed, if we set $a^* := \|a\|_\infty$, then F can be represented as

$$F(s) = G(s) - \frac{a^*}{2}s^2, \tag{3.1}$$

with $G \in C^{2,1}(\mathbb{R})$ strictly convex, since $G'' \geqslant c_0$ in Ω.

Remark 3. Assumption (H5) is fulfilled by a potential of arbitrary polynomial growth. In particular, (H3)–(H5) are satisfied for the case of the physically relevant double-well potential, i.e.

$$F(s) = \left(s^2 - 1\right)^2.$$

In this case we take $p = 4/3$ in (H5), while assumption (H3) is satisfied if and only if we have $a \geqslant c_0 + \theta$, where $\theta = -\min_{s\in\mathbb{R}} F''(s)$.

By weak solution we mean

Definition 1. Let $u_0 \in G_{\mathrm{div}}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. Then $[u, \varphi]$ is a weak solution to Problem P on $[0,T]$ corresponding to u_0 and φ_0 if

- u, φ and μ satisfy

$$u \in L^\infty(0,T;G_{\mathrm{div}}) \cap L^2(0,T;V_{\mathrm{div}}), \tag{3.2}$$

$$u_t \in L^{4/3}\big(0,T;V'_{\mathrm{div}}\big), \quad \text{if } d = 3, \tag{3.3}$$

$$u_t \in L^{2-\gamma}\big(0,T;V'_{\mathrm{div}}\big), \quad \forall\gamma \in (0,1), \text{ if } d = 2, \tag{3.4}$$

$$\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.5}$$

$$\varphi_t \in L^{4/3}\big(0,T;V'\big), \quad \text{if } d = 3, \tag{3.6}$$

$$\varphi_t \in L^{2-\delta}\big(0,T;V'\big), \quad \forall \delta \in (0,1), \text{ if } d = 2, \tag{3.7}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0,T;V); \tag{3.8}$$

- setting

$$\rho(x,\varphi) := a(x)\varphi + F'(\varphi), \tag{3.9}$$

then, for every $\psi \in V$, every $v \in V_{\mathrm{div}}$ and for almost any $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + (\nabla \rho, \nabla \psi) = \int_\Omega (u \cdot \nabla \psi)\varphi + \int_\Omega (\nabla J * \varphi) \cdot \nabla \psi, \tag{3.10}$$

$$\langle u_t, v \rangle + \big(\nu(\varphi) 2Du, Dv\big) + b(u,u,v) = -\int_\Omega (v \cdot \nabla \mu)\varphi + \langle h, v \rangle; \tag{3.11}$$

- the following initial conditions hold

$$u(0) = u_0, \qquad \varphi(0) = \varphi_0. \tag{3.12}$$

Remark 4. Since $\rho = \mu + J * \varphi$, from Definition 1 we have that $\rho \in L^2(0,T;V)$.

Remark 5. It is immediate to see that the total mass is conserved. Indeed, choosing $\psi = 1$ in (3.10), we have $\langle \varphi_t, 1 \rangle = 0$ whence $(\varphi(t), 1) = (\varphi_0, 1)$ for all $t \geqslant 0$.

Remark 6. The initial conditions (3.12) are meant in the weak sense, i.e., for every $v \in V_{\mathrm{div}}$ we have $(u(t), v) \to (u_0, v)$ as $t \to 0$, and for every $\chi \in V$ we have $(\varphi(t), \chi) \to (\varphi_0, \chi)$ as $t \to 0$. It can be proved that $u \in C_w([0,T]; G_{\mathrm{div}})$ and $\varphi \in C_w([0,T]; H)$.

Theorem 1. *Let $u_0 \in G_{\mathrm{div}}$, $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that* (H1)–(H6) *are satisfied. Then, for every $T > 0$ there exists a weak solution $[u,\varphi]$ to Problem* P *on $[0,T]$ corresponding to u_0, φ_0 with φ_t satisfying*

$$\varphi_t \in L^\infty\big(0,T;V_s'\big), \quad \textit{if } < p < \frac{d}{d-1},\ s = \frac{(4-d)p + 2d}{2p},$$

$$\varphi_t \in L^\infty\big(0,T;V_s'\big) \cap L^r\big(0,T;V'_{\frac{d+2}{2}}\big), \quad \textit{if } p = \frac{d}{d-1},\ s > \frac{d+2}{2},\ r \geqslant 2,$$

$$\varphi_t \in L^{2p/(2p-3)}\big(0,T;V_s'\big), \quad \textit{if } d = 3,\ 3/2 < p \leqslant 2,\ s = \frac{p+6}{2p}.$$

Furthermore, setting

$$\mathcal{E}\big(u(t),\varphi(t)\big) = \frac{1}{2}\big\|u(t)\big\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x,t) - \varphi(y,t)\big)^2\,dx\,dy + \int_\Omega F\big(\varphi(t)\big),$$

the following energy inequality holds for almost any $t > 0$

$$\mathcal{E}\big(u(t),\varphi(t)\big) + \int_0^t \big(\big\|\sqrt{\nu(\varphi)}Du\big\|^2 + \big\|\nabla\mu(\tau)\big\|^2\big)\,d\tau \leqslant \mathcal{E}(u_0,\varphi_0) + \int_0^t \big\langle h(\tau), u(\tau)\big\rangle\,d\tau. \tag{3.13}$$

On account of the typical examples of double-well smooth potentials (cf. Remark 3), the following additional assumption sounds reasonable (see, e.g., [8, (A2)])

(H7) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exist $c_5 > 0$, $c_6 > 0$ and $q > 0$ such that

$$F''(s) + a(x) \geqslant c_5|s|^{2q} - c_6, \quad \forall s \in \mathbb{R},\ \text{a.e. } x \in \Omega.$$

This requirement can replace (H4) in the proof of Theorem 1 (see (3.14) below). Indeed, (H7) implies the existence of $c_7 > 0$ and $c_8 > 0$ such that

$$F(s) \geqslant c_7|s|^{2+2q} - c_8, \quad \forall s \in \mathbb{R}. \tag{3.14}$$

Moreover, (H7) leads to establish further regularity properties for φ, φ_t, u_t. This is stated in the following:

Corollary 1. *Suppose that the assumptions of Theorem* 1 *hold with* (H4) *replaced by* (H7)*. Then, for every $T > 0$ there exists a weak solution $[u, \varphi]$ to Problem* P *on $[0, T]$ corresponding to $[u_0, \varphi_0]$ such that*

$$\varphi \in L^\infty\big(0, T; L^{2+2q}(\Omega)\big), \tag{3.15}$$

$$\varphi_t \in L^2\big(0, T; V'\big), \quad \text{if } d = 2 \text{ or } d = 3 \text{ and } q \geqslant 1/2, \tag{3.16}$$

$$u_t \in L^2\big(0, T; V'_{\mathrm{div}}\big), \quad \text{if } d = 2, \tag{3.17}$$

and

$$\varphi_t \in L^\infty\big(0, T; V'_s\big), \quad \text{if } \begin{cases} d = 2, 3, \quad 1 < p \leqslant \dfrac{d}{d-1}, \\ d = 3, \quad 3/2 < p \leqslant 2, \quad q \geqslant \dfrac{2(2p-3)}{6-p}, \end{cases} \tag{3.18}$$

$$\varphi_t \in L^\sigma\big(0, T; V'_s\big), \quad \text{if } d = 3,\ 3/2 < p \leqslant 2,\ 0 < q < \frac{2(2p-3)}{6-p}, \tag{3.19}$$

where $s = ((4-d)p + 2d)/2p$ and in (3.19) *the exponent σ is given by*

$$\sigma = \frac{2p(1-\frac{q}{2})}{(2p-3) - q(3-\frac{p}{2})}.$$

In two dimensions, as further consequences of (H7), we can prove the energy identity and a dissipative estimate, provided that $h \in L^2_{tb}(0, \infty; V'_{\mathrm{div}})$, that is

$$\|h\|^2_{L^2_{tb}(0,\infty;V'_{\mathrm{div}})} := \sup_{t\geqslant 0} \int_t^{t+1} \big\|h(\tau)\big\|^2_{V'_{\mathrm{div}}}\, d\tau < \infty.$$

Indeed, we have:

Corollary 2. *Let $d = 2$ and suppose that the assumptions of Theorem* 1 *with* (H4) *replaced by* (H7) *hold. Then the weak solution $[u, \varphi]$ to Problem* P *corresponding to $[u_0, \varphi_0]$ satisfies*

$$\frac{d}{dt}\mathcal{E}(u, \varphi) + 2\big\|\sqrt{\nu(\varphi)}Du\big\|^2 + \|\nabla\mu\|^2 = \langle h, u\rangle. \tag{3.20}$$

Therefore, (3.13) *with the equal sign holds for every $t \geqslant 0$. Furthermore, if in addition $h \in L^2_{tb}(0, \infty; V'_{\mathrm{div}})$, then the following dissipative estimate is satisfied*

$$\mathcal{E}\big(u(t), \varphi(t)\big) \leqslant \mathcal{E}(u_0, \varphi_0)e^{-kt} + F(m_0)|\Omega| + K, \quad \forall t \geqslant 0, \tag{3.21}$$

where $m_0 = (\varphi_0, 1)$ and k, K are two positive constants which are independent of the initial data. Moreover, K depends on Ω, ν_1, J, F, $\|h\|_{L^2_{tb}(0,\infty;V'_{\mathrm{div}})}$ only.

Remark 7. It follows from Corollary 1 that, in two dimensions, $u \in C([0, T]; G_{\mathrm{div}})$ and $\varphi \in C([0, T]; H)$. This fact along with the validity of an energy identity suggests that the generalized semiflow approach devised in [5] (see also [36]) might be applied to our system. If so, one should be able to establish the existence of a global attractor. This is one of the issues which will be investigated in a forthcoming paper.

4. Proof of Theorem 1

The proof will be carried out by means of a Faedo–Galerkin approximation scheme. We will assume first that $\varphi_0 \in D(B)$. The existence under the stated assumption on φ_0 will be recovered by a density argument by exploiting the form of the potential F as a quadratic perturbation of a convex function (see Remark 2).

We introduce the family $\{w_j\}_{j\geqslant 1}$ of the eigenfunctions of the Stokes operator A as a Galerkin base in V_{div} and the family $\{\psi_j\}_{j\geqslant 1}$ of the eigenfunctions of the Neumann operator

$$B = -\Delta + I$$

as a Galerkin base in V. We define the n-dimensional subspaces $\mathcal{W}_n := \langle w_1, \dots, w_n\rangle$ and $\Psi_n := \langle \psi_1, \dots, \psi_n\rangle$ and consider the orthogonal projectors on these subspaces in G_{div} and H, respectively, i.e., $\widetilde{P}_n := P_{\mathcal{W}_n}$ and $P_n := P_{\Psi_n}$. We then look for three functions of the form

$$u_n(t) = \sum_{k=1}^{n} a_k^{(n)}(t) w_k, \qquad \varphi_n(t) = \sum_{k=1}^{n} b_k^{(n)}(t)\psi_k, \qquad \mu_n(t) = \sum_{k=1}^{n} c_k^{(n)}(t)\psi_k$$

that solve the following approximating problem

$$\big(\varphi_n', \psi\big) + \big(\nabla\rho(\cdot,\varphi_n), \nabla\psi\big) = \int_\Omega (u_n\cdot\nabla\psi)\varphi_n + \int_\Omega (\nabla J * \varphi_n)\cdot\nabla\psi, \tag{4.1}$$

$$\big(u_n', w\big) + \big(\nu(\varphi_n)2Du_n, Dw\big) + b(u_n, u_n, w) = -\int_\Omega (w\cdot\nabla\mu_n)\varphi_n + (h_n, w), \tag{4.2}$$

$$\rho(\cdot,\varphi_n) := a(\cdot)\varphi_n + F'(\varphi_n), \tag{4.3}$$

$$\mu_n = P_n\big(\rho(\cdot,\varphi_n) - J*\varphi_n\big), \tag{4.4}$$

$$\varphi_n(0) = \varphi_{0n}, \qquad u_n(0) = u_{0n}, \tag{4.5}$$

for every $\psi \in \Psi_n$ and every $w \in \mathcal{W}_n$, where $\varphi_{0n} = P_n\varphi_0$ and $u_{0n} = \widetilde{P}_n u_0$ (primes denote derivatives with respect to time). In (4.2) $h_n \in C^0([0,T]; G_{\mathrm{div}})$ and, on account of (H6), we choose the sequence of h_n in such a way that $h_n \to h$ in $L^2(0,T; V_{\mathrm{div}}')$. It is easy to see that this approximating problem is equivalent to solving a Cauchy problem for a system of ordinary differential equations in the $2n$ unknowns $a_i^{(n)}$, $b_i^{(n)}$. Since $F' \in C^{1,1}_{loc}(\mathbb{R})$, the Cauchy–Lipschitz theorem ensures that there exists $T_n^* \in (0,+\infty]$ such that this system has a unique maximal solution $\mathbf{a}^{(n)} := (a_1^{(n)}, \dots, a_n^{(n)})$, $\mathbf{b}^{(n)} := (b_1^{(n)}, \dots, b_n^{(n)})$ on $[0, T_n^*)$ and $\mathbf{a}^{(n)}, \mathbf{b}^{(n)} \in C^1([0,T_n^*); \mathbb{R}^n)$.

We now derive some a priori estimates in order to show that $T_n^* = +\infty$ for every $n \geqslant 1$ and that the sequences of φ_n, u_n and μ_n are bounded in suitable functional spaces. By using μ_n as a test function in (4.1), u_n as a test function in (4.2) and recalling that $b(u_n, u_n, u_n) = 0$ (see (2.2)), we obtain

$$\big(\varphi_n', \mu_n\big) + \big(\nabla\rho(\cdot,\varphi_n), \nabla\mu_n\big) = \int_\Omega (u_n\cdot\nabla\mu_n)\varphi_n + \int_\Omega (\nabla J*\varphi_n)\cdot\nabla\mu_n,$$

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + 2\big\|\sqrt{\nu(\varphi_n)}Du_n\big\|^2 = -\int_\Omega (u_n\cdot\nabla\mu_n)\varphi_n + (h_n, u_n).$$

We now have

$$\begin{aligned}\big(\varphi_n', \mu_n\big) &= \big(\varphi_n', a\varphi_n + F'(\varphi_n) - J*\varphi_n\big)\\ &= \frac{d}{dt}\bigg(\frac{1}{2}\|\sqrt{a}\varphi_n\|^2 + \int_\Omega F(\varphi_n) - \frac{1}{2}(\varphi_n, J*\varphi_n)\bigg)\\ &= \frac{d}{dt}\bigg(\frac{1}{4}\int_\Omega\int_\Omega J(x-y)\big(\varphi_n(x)-\varphi_n(y)\big)^2\,dx\,dy + \int_\Omega F(\varphi_n)\bigg).\end{aligned} \tag{4.6}$$

Here we have used the fact that $(\phi, J*\psi) = (\psi, J*\phi)$ since $J(x) = J(-x)$. Furthermore, observe that

$$\big(\nabla\rho(\cdot,\varphi_n), \nabla\mu_n\big) = \big(-\rho(\cdot,\varphi_n), \Delta\mu_n\big) = (-\rho_n, \Delta\mu_n) = (\nabla\rho_n, \nabla\mu_n),$$

where $\rho_n := P_n\rho(\cdot,\varphi_n) = \mu_n + P_n(J*\varphi_n)$. Summing the first two identities and taking the previous relations into account we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2+\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi_n(x)-\varphi_n(y)\big)^2\,dx\,dy+2\int_\Omega F(\varphi_n)\right)+2\big\|\sqrt{\nu(\varphi_n)}Du_n\big\|^2+\|\nabla\mu_n\|^2$$
$$+\big(\nabla\big(P_n(J*\varphi_n)\big),\nabla\mu_n\big)$$
$$=\int_\Omega(\nabla J*\varphi_n)\cdot\nabla\mu_n+(h_n,u_n). \tag{4.7}$$

Now, it is easy to see that

$$\big\|\nabla\big(P_n(J*\varphi_n)\big)\big\|\leqslant\big\|B^{1/2}P_n(J*\varphi_n)\big\|\leqslant\|\nabla J*\varphi_n\|+\|J*\varphi_n\|\leqslant\|J\|_{W^{1,1}}\|\varphi_n\|, \tag{4.8}$$

and that, by means of (H4), we have

$$\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi_n(x)-\varphi_n(y)\big)^2\,dx\,dy+2\int_\Omega F(\varphi_n)=\|\sqrt{a}\varphi_n\|^2+2\int_\Omega F(\varphi_n)-(\varphi_n,J*\varphi_n)$$
$$\geqslant\int_\Omega\big(a+2c_1-\|J\|_{L^1}\big)\varphi_n^2-2c_2|\Omega|$$
$$\geqslant\alpha\|\varphi_n\|^2-c, \tag{4.9}$$

where $\alpha=2c_1-\|J\|_{L^1}>0$. Hence, integrating (4.7) with respect to time between 0 and $t\in(0,T_n^*)$ and using (4.8), (4.9), we are led to the following integral inequality

$$\|u_n\|^2+\alpha\|\varphi_n\|^2+\int_0^t\big(\nu_1\|\nabla u_n\|^2+\|\nabla\mu_n\|^2\big)\,d\tau$$
$$\leqslant c\|J\|_{W^{1,1}}^2\int_0^t\|\varphi_n\|^2\,d\tau+\|u_{0n}\|^2+\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi_{0n}(x)-\varphi_{0n}(y)\big)^2\,dx\,dy$$
$$+2\int_\Omega F(\varphi_{0n})+\frac{1}{2\nu}\int_0^t\|h_n\|_{V'_{\mathrm{div}}}^2\,d\tau+c$$
$$\leqslant M+\frac{1}{2\nu}\int_0^t\|h\|_{V'_{\mathrm{div}}}^2\,d\tau+c\int_0^t\|\varphi_n\|^2\,d\tau,\quad\forall t\in\big[0,T_n^*\big), \tag{4.10}$$

where c only depends on $\|J\|_{W^{1,1}}$ and on $|\Omega|$, while M is given by

$$M=c\left(1+\|u_0\|^2+\|\varphi_0\|^2+\int_\Omega F(\varphi_0)\right). \tag{4.11}$$

Here we have used the fact that, since φ_0 is supposed to belong to $D(B)$, then we have $\varphi_{0n}\to\varphi_0$ in $H^2(\Omega)$ and hence also in $L^\infty(\Omega)$ (for $d=2,3$). Since we have $\|u_n(t)\|=|\mathbf{a}^{(n)}(t)|$ and $\|\varphi_n(t)\|=|\mathbf{b}^{(n)}(t)|$, by means of Gronwall lemma we deduce that $T_n^*=+\infty$, for every $n\geqslant1$, i.e., problem (4.1)–(4.5) has a unique global in time solution, and that (4.10) is satisfied for every $t\geqslant0$. Furthermore, we obtain the following estimates holding for any given $0<T<+\infty$,

$$\|u_n\|_{L^\infty(0,T;G_{\mathrm{div}})\cap L^2(0,T;V_{\mathrm{div}})}\leqslant N, \tag{4.12}$$
$$\|\varphi_n\|_{L^\infty(0,T;H)}\leqslant N, \tag{4.13}$$
$$\|\nabla\mu_n\|_{L^2(0,T;H)}\leqslant N, \tag{4.14}$$

where

$$N=cM^{1/2}+c\|h\|_{L^2(0,T;V'_{\mathrm{div}})}.$$

Here c also depends on T and on ν_1. From (4.4), (4.14) and recalling (1.8) we now deduce an estimate for φ_n in $L^2(0,T;V)$. We have

$$\begin{aligned}(\mu_n, -\Delta\varphi_n) &= (\nabla\mu_n, \nabla\varphi_n) = \big(-\Delta\varphi_n, a\varphi_n + F'(\varphi_n) - J*\varphi_n\big)\\ &= \big(\nabla\varphi_n, a\nabla\varphi_n + \varphi_n\nabla a + F''(\varphi_n)\nabla\varphi_n - \nabla J*\varphi_n\big)\\ &\geqslant c_0\|\nabla\varphi_n\|^2 - 2\|\nabla J\|_{L^1}\|\nabla\varphi_n\|\|\varphi_n\|\\ &\geqslant \frac{c_0}{2}\|\nabla\varphi_n\|^2 - k\|\varphi_n\|^2, \end{aligned} \tag{4.15}$$

where $k = (2/c_0)\|\nabla J\|_{L^1}^2$ and where we have used (H3). Since

$$(\nabla\mu_n, \nabla\varphi_n) \leqslant \frac{c_0}{4}\|\nabla\varphi_n\|^2 + \frac{1}{c_0}\|\nabla\mu_n\|^2,$$

we get

$$\|\nabla\mu_n\|^2 \geqslant \frac{c_0^2}{4}\|\nabla\varphi_n\|^2 - c\|\varphi_n\|^2, \tag{4.16}$$

and (4.13), (4.14), (4.16) yield

$$\|\varphi_n\|_{L^2(0,T;V)} \leqslant N. \tag{4.17}$$

The next step is to deduce an estimate for the sequence of μ_n in $L^2(0,T;V)$. To this aim we first observe that (H5) implies that $|F'(s)| \leqslant c|F(s)| + c$ for every $s \in \mathbb{R}$ and therefore we obtain

$$\left|\int_\Omega \mu_n\right| = \big|(\mu_n, 1)\big| = \big|\big(F'(\varphi_n), 1\big)\big| \leqslant \int_\Omega \big|F'(\varphi_n)\big| \leqslant c\int_\Omega \big|F(\varphi_n)\big| + c \leqslant N. \tag{4.18}$$

Here we have used the identity $(P_n(-J*\varphi_n + a\varphi_n), 1) = 0$, the bound $\|F(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leqslant N$ (implied by (4.7) integrated in time between 0 and $t \in (0,T]$) and (4.9). We have also used the estimates (4.12)–(4.14). Hence, by means of the Poincaré–Wirtinger inequality, from (4.14) and (4.18) we get

$$\|\mu_n\|_{L^2(0,T;V)} \leqslant N. \tag{4.19}$$

We also need an estimate for the sequence $\{\rho(\cdot,\varphi_n)\}$. From (H5) we immediately get

$$\big\|\rho(\cdot,\varphi_n)\big\|_{L^p} \leqslant \big(c\|a\|_{L^\infty}\|\varphi_n\| + \big\|F'(\varphi_n)\big\|_{L^p}\big) \leqslant c\left(\left(\int_\Omega \big|F(\varphi_n)\big|\right)^{1/p} + 1\right) \leqslant N, \tag{4.20}$$

and hence we have

$$\big\|\rho(\cdot,\varphi_n)\big\|_{L^\infty(0,T;L^p(\Omega))} \leqslant N. \tag{4.21}$$

The final estimates we need are for the sequences of time derivatives u_n' and φ_n'. Let us start from the sequence of u_n'. Eq. (4.2) can be written as

$$u_n' + \widetilde{P}_n\mathcal{A}(u_n,\varphi_n) + \widetilde{P}_n\mathcal{B}(u_n,u_n) = -\widetilde{P}_n(\varphi_n\nabla\mu_n) + \widetilde{P}_n h_n. \tag{4.22}$$

We now have, for $d = 3$, by using Sobolev embeddings, interpolation between L^p spaces and (4.13)

$$\big\|\widetilde{P}_n(\varphi_n\nabla\mu_n)\big\|_{V'_{\mathrm{div}}} \leqslant c\|\varphi_n\|_{L^3}\|\nabla\mu_n\| \leqslant c\|\varphi_n\|^{1/2}\|\varphi_n\|_{L^6}^{1/2}\|\nabla\mu_n\| \leqslant N^{1/2}\|\varphi_n\|_V^{1/2}\|\nabla\mu_n\|. \tag{4.23}$$

Therefore, thanks to (4.14) and (4.17), we get

$$\big\|\widetilde{P}_n(\varphi_n\nabla\mu_n)\big\|_{L^{4/3}(0,T;V'_{\mathrm{div}})} \leqslant N^2. \tag{4.24}$$

For the case $d = 2$, by means of Gagliardo–Nirenberg interpolation inequality in dimension 2 we have, for every $0 < \gamma < 1$,

$$\begin{aligned}\big\|\widetilde{P}_n(\varphi_n\nabla\mu_n)\big\|_{V'_{\mathrm{div}}} &\leqslant c\|\varphi_n\|_{L^{2+\gamma/(1-\gamma)}}\|\nabla\mu_n\|\\ &\leqslant c\|\varphi_n\|^{2(1-\gamma)/(2-\gamma)}\|\varphi_n\|_V^{\gamma/(2-\gamma)}\|\nabla\mu_n\|\\ &\leqslant N^{2(1-\gamma)/(2-\gamma)}\|\varphi_n\|_V^{\gamma/(2-\gamma)}\|\nabla\mu_n\|, \end{aligned} \tag{4.25}$$

so that (4.14) and (4.17) yield

$$\big\|\widetilde{P}_n(\varphi_n\nabla\mu_n)\big\|_{L^{2-\gamma}(0,T;V'_{\mathrm{div}})} \leqslant N^2. \tag{4.26}$$

Moreover, it is easy to check that

$$\big\|\widetilde{P}_n\mathcal{A}(u_n,\varphi_n)\big\|_{V'_{\mathrm{div}}}\leqslant \nu_2\|u_n\|_{V_{\mathrm{div}}},$$

while the treatment of the term $\widetilde{P}_n\mathcal{B}(u_n,u_n)$ is classical and, by means of (2.3) and (2.4) we have

$$\big\|\widetilde{P}_n\mathcal{B}(u_n,u_n)\big\|_{V'_{\mathrm{div}}}\leqslant c\|u_n\|^{1/2}\|u_n\|^{3/2}_{V_{\mathrm{div}}},\quad \text{for } d=3, \tag{4.27}$$

$$\big\|\widetilde{P}_n\mathcal{B}(u_n,u_n)\big\|_{V'_{\mathrm{div}}}\leqslant c\|u_n\|\|u_n\|_{V_{\mathrm{div}}},\quad \text{for } d=2. \tag{4.28}$$

Hence, by using (4.24), (4.26) and (4.27), (4.28), and recalling that $\widetilde{P}_n\in\mathcal{L}(V'_{\mathrm{div}},V'_{\mathrm{div}})$, which implies that

$$\|\widetilde{P}_n h_n\|_{L^2(0,T;V'_{\mathrm{div}})}\leqslant c\big(1+\|h\|_{L^2(0,T;V'_{\mathrm{div}})}\big),$$

from (4.22) we obtain

$$\big\|u'_n\big\|_{L^{4/3}(0,T;V'_{\mathrm{div}})}\leqslant L,\quad \text{for } d=3, \tag{4.29}$$

$$\big\|u'_n\big\|_{L^{2-\gamma}(0,T;V'_{\mathrm{div}})}\leqslant L,\quad \forall\gamma\in(0,1),\ \text{for } d=2, \tag{4.30}$$

where $L=N^2+N$.

In order to derive an estimate for the sequence of φ'_n, we aim to take the test function $\psi\in V_s$ in (4.1), where $s\geqslant 2$ is such that $\Delta\psi\in H^{s-2}(\Omega)\hookrightarrow L^{p'}(\Omega)$ (p' is the conjugate index to p). Since $H^{s-2}\hookrightarrow L^{p^*}$, where $p^*=2d/(d+4-2s)$, we see that it is enough to take

$$s\geqslant\frac{(4-d)p+2d}{2p}. \tag{4.31}$$

Let us now decompose ψ as

$$\psi=\psi_I+\psi_{II},$$

where $\psi_I=P_n\psi=\sum_{k=1}^{n}(\psi,\psi_k)\psi_k\in\Psi_n$ and $\psi_{II}=(I-P_n)\psi=\sum_{k=n+1}^{\infty}(\psi,\psi_k)\psi_k\in\Psi_n^{\perp}$ (recall that ψ_I and ψ_{II} are orthogonal in all the Hilbert spaces V_r, for every $0\leqslant r\leqslant s$), and notice that we have, due to (4.21)

$$\big|\big(\nabla\rho(\cdot,\varphi_n),\nabla\psi_I\big)\big|=\big|\big(\rho(\cdot,\varphi_n),\Delta\psi_I\big)\big|\leqslant N\|\Delta\psi_I\|_{L^{p'}}\leqslant N\|\psi_I\|_{V_s}\leqslant N\|\psi\|_{V_s}. \tag{4.32}$$

Furthermore, it is easy to see that

$$\left|\int_\Omega(\nabla J*\varphi_n)\cdot\nabla\psi_I\right|\leqslant c\|\nabla J\|_{L^1}\|\varphi_n\|\|\psi\|_{V_s}\leqslant N\|\psi\|_{V_s}. \tag{4.33}$$

As far as the first term in the right hand side of (4.1) (written with $\psi=\psi_I$) is concerned we notice that $\nabla\psi_I\in H^{s-1}(\Omega)$. Therefore, when $1<p<d/(d-1)$ and $s=((4-d)p+2d)/2p$ or $p=d/(d-1)$ and $s>((4-d)p+2d)/2p=(d+2)/2$, due to the embedding $H^{s-1}\hookrightarrow L^\infty$, we have

$$\left|\int_\Omega(u_n\cdot\nabla\psi_I)\varphi_n\right|\leqslant c\|u_n\|\|\varphi_n\|\|\psi\|_{V_s}\leqslant N^2\|\psi\|_{V_s}. \tag{4.34}$$

When $p=d/(d-1)$ and $s=((4-d)p+2d)/2p=(d+2)/2$, due to the embedding $H^{s-1}\hookrightarrow L^q$ for every $1\leqslant q<+\infty$ and interpolation in L^p spaces, we have, for every $r\geqslant 2$, that

$$\left|\int_\Omega(u_n\cdot\nabla\psi_I)\varphi_n\right|\leqslant c\|u_n\|\|\psi\|_{V_s}\|\varphi_n\|_{L^{2r/(r-1)}}\leqslant c\|u_n\|\|\psi\|_{V_s}\|\varphi_n\|^{(r-2)/r}\|\varphi_n\|^{2/r}_{L^4}\leqslant N^{2/r'}\|\psi\|_{V_s}\|\varphi_n\|^{2/r}_{V}. \tag{4.35}$$

Finally, in the case $d=3$, when $3/2<p\leqslant 2$ and $s=((4-d)p+2d)/2p=(p+6)/2p$, due to the embedding $H^{s-1}\hookrightarrow L^{3p/(2p-3)}$, we obtain

$$\begin{aligned}\left|\int_\Omega(u_n\cdot\nabla\psi_I)\varphi_n\right|&\leqslant c\|u_n\|\|\psi\|_{V_s}\|\varphi_n\|_{L^{6p/(6-p)}}\leqslant c\|u_n\|\|\psi\|_{V_s}\|\varphi_n\|^{(3-p)/p}\|\varphi_n\|^{(2p-3)/p}_{L^6}\\&\leqslant N^{3/p}\|\psi\|_{V_s}\|\varphi_n\|^{(2p-3)/p}_{V}.\end{aligned} \tag{4.36}$$

Collecting (4.32)–(4.36), from (4.1) (written with $\psi = \psi_I$) we then get

$$\|\varphi_n'\|_{L^\infty(0,T;V_s')} \leqslant L \quad \text{if } 1 < p < d',\ s = \frac{(4-d)p + 2d}{2p}, \tag{4.37}$$

$$\|\varphi_n'\|_{L^\infty(0,T;V_s') \cap L^r(0,T;V'_{\frac{d+2}{2}})} \leqslant L \quad \text{if } p = d',\ s > \frac{d+2}{2},\ r \geqslant 2, \tag{4.38}$$

where $d' = d/(d-1)$, while in the case $d = 3$, if $3/2 < p \leqslant 2$ we find

$$\|\varphi_n'\|_{L^{2p/(2p-3)}(0,T;V_s')} \leqslant L, \quad s = \frac{p+6}{2p}, \tag{4.39}$$

where $L = N + N^2$ in all cases.

From the estimates (4.12)–(4.14), (4.17), (4.19), (4.21), (4.29), (4.30), (4.37)–(4.39) and on account of the compact embeddings

$$L^2(0,T;V) \cap H^1(0,T;V_s') \hookrightarrow\hookrightarrow L^2(0,T;H),$$
$$L^2(0,T;V_{\mathrm{div}}) \cap W^{1,q}(0,T;V_{\mathrm{div}}') \hookrightarrow\hookrightarrow L^2(0,T;G_{\mathrm{div}}), \quad \forall q > 1$$

we deduce that there exist

$$u \in L^\infty(0,T;G_{\mathrm{div}}) \cap L^2(0,T;V_{\mathrm{div}}), \tag{4.40}$$
$$\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{4.41}$$
$$\mu \in L^2(0,T;V), \tag{4.42}$$
$$\rho \in L^\infty(0,T;L^p(\Omega)) \tag{4.43}$$

with

$$u_t \in L^{4/3}(0,T;V_{\mathrm{div}}'), \quad \text{if } d = 3,$$
$$u_t \in L^{2-\gamma}(0,T;V_{\mathrm{div}}'), \quad \forall\gamma \in (0,1),\ \text{if } d = 2, \tag{4.44}$$

and

$$\varphi_t \in L^\infty(0,T;V_s'), \quad \text{if } 1 < p < d',\ s = \frac{(4-d)p + 2d}{2p},$$
$$\varphi_t \in L^\infty(0,T;V_s') \cap L^r(0,T;V'_{\frac{d+2}{2}}), \quad \text{if } p = d',\ s > \frac{d+2}{2},\ r \geqslant 2,$$
$$\varphi_t \in L^{2p/(2p-3)}(0,T;V_s'), \quad \text{if } d = 3,\ 3/2 < p \leqslant 2,\ s = \frac{p+6}{2p}, \tag{4.45}$$

such that, for a not relabeled subsequence, we deduce

$$u_n \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty(0,T;G_{\mathrm{div}}), \tag{4.46}$$
$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T;V_{\mathrm{div}}), \tag{4.47}$$
$$u_n \to u \quad \text{strongly in } L^2(0,T;G_{\mathrm{div}}),\ \text{a.e. in } \Omega \times (0,T), \tag{4.48}$$
$$u_n' \rightharpoonup u_t \quad \text{weakly in } L^{4/3}(0,T;V_{\mathrm{div}}'),\ d = 3, \tag{4.49}$$
$$u_n' \rightharpoonup u_t \quad \text{weakly in } L^{2-\gamma}(0,T;V_{\mathrm{div}}'),\ \forall\gamma \in (0,1),\ d = 2, \tag{4.50}$$
$$\varphi_n \rightharpoonup \varphi \quad \text{weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.51}$$
$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in } L^2(0,T;V), \tag{4.52}$$
$$\varphi_n \to \varphi \quad \text{strongly in } L^2(0,T;H),\ \text{a.e. in } \Omega \times (0,T), \tag{4.53}$$
$$\mu_n \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;V), \tag{4.54}$$
$$\rho(\cdot,\varphi_n) \rightharpoonup \rho \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^p(\Omega)), \tag{4.55}$$

and

$$\varphi_n' \rightharpoonup \varphi_t \quad \text{weakly}^* \text{ in } L^\infty(0,T;V_s'), \tag{4.56}$$

if $1 < p < d'$, with $s = ((4-d)p + 2d)/2p$,

$$\varphi_n' \rightharpoonup \varphi_t \quad \text{weakly* in } L^\infty\big(0,T;V_s'\big), \ \text{weakly in } L^r\big(0,T;V'_{\frac{d+2}{2}}\big), \tag{4.57}$$

if $p = d'$, with $s > (d+2)/2$ and $r \geqslant 2$,

$$\varphi_n' \rightharpoonup \varphi_t \quad \text{weakly in } L^{2p/(2p-3)}\big(0,T;V_s'\big), \tag{4.58}$$

if $d = 3$ and $3/2 < p \leqslant 2$, with $s = (p+6)/2p$.

We can now pass to the limit in (4.1)–(4.5) in order to prove that the functions u and φ yield a weak solution to Problem P in the sense of Definition 1, i.e., u, φ, μ and ρ satisfy (1.7), (3.9) and (3.10), (3.11), (3.12). First of all, from the pointwise convergence (4.53) we have $\rho(\cdot,\varphi_n) \to a\varphi + F'(\varphi)$ almost everywhere in $\Omega \times (0,T)$ and therefore from (4.55) we have $\rho = a\varphi + F'(\varphi)$, i.e. (3.9). Moreover, since $\mu_k = P_k(\rho(\cdot,\varphi_k) - J * \varphi_k)$, we have, for every $v \in \Psi_n$ and every $k \geqslant n$ (n is fixed)

$$\int_0^T \big(\mu_k(t), v\big)\chi(t)\,dt = \int_0^T \big(\rho(\cdot,\varphi_k) - J*\varphi_k, v\big)\chi(t)\,dt, \quad \forall \chi \in C_0^\infty(0,T).$$

By passing to the limit as $k \to \infty$ in this identity and using the convergences (4.54), (4.53) (which implies $J * \varphi_k \to J * \varphi$ strongly in $L^2(0,T;V)$) and (4.55), on account of the density of $\{\Psi_n\}_{n\geqslant 1}$ in H we get $\mu = \rho - J*\varphi = a\varphi + F'(\varphi) - J*\varphi$, i.e. (1.7). In particular we obtain $\rho \in L^2(0,T;V)$.

The argument used to recover (3.10) and (3.11) by passing to the limit in (4.1) and (4.2) of the approximate problem and by exploiting the above convergences is standard and we only limit ourselves to give a sketch of it. We multiply (4.1) by χ and (4.2) by ω, where χ, $\omega \in C_0^\infty(0,T)$ and integrate in time between 0 and T. Due to the above convergences we can pass to the limit in these equations. In particular the term $(\nabla\rho(\cdot,\varphi_n), \nabla\psi)$ can be rewritten as $(\rho(\cdot,\varphi_n), -\Delta\psi)$ and (4.55) is used. We also recall that in the nonlinear term $b(u_n,u_n,w)\omega$ we exploit the strong convergence (4.48) to pass to the limit. Moreover, observe that, due to (H2), to the a.e. convergence (4.53) and to Lebesgue's theorem we have $\nu(\varphi_n) \to \nu(\varphi)$ strongly in $L^2(0,T;H)$. The limit equations thus obtained hold for every $\psi \in \Psi_n$, every $w \in \mathcal{W}_n$ (where n is fixed) and every χ, $\omega \in C_0^\infty(0,T)$. The density of $\{\Psi_n\}_{n\geqslant 1}$ and $\{\mathcal{W}_n\}_{n\geqslant 1}$ in V_s and V_{div}, respectively, allows us to conclude that u, φ, μ and ρ satisfy (3.10) for every $\psi \in V_s$ and (3.11) for every $v \in V_{\mathrm{div}}$. Furthermore, observe that (3.10) can be written in the form

$$\langle \varphi_t, \psi\rangle = -(\nabla\mu, \nabla\psi) + (u, \varphi\nabla\psi), \tag{4.59}$$

and consider the contribution of the transport term in (4.59). In the case $d = 3$, by arguing as in (4.23) we have (cf. (2.1))

$$\big|(u,\varphi\nabla\psi)\big| \leqslant N^{1/2}\|\nabla u\|\|\varphi\|_V^{1/2}\|\nabla\psi\|, \tag{4.60}$$

while, in the case $d = 2$, by arguing as in (4.25) we have

$$\big|(u,\varphi\nabla\psi)\big| \leqslant N^{2(1-\delta)/(2-\delta)}\|\nabla u\|\|\varphi\|_V^{\delta/(2-\delta)}\|\nabla\psi\|, \tag{4.61}$$

for every $\delta \in (0,1)$. From (4.60) and (4.61) we deduce that $\varphi_t(t)$ can be continuously extended to V for almost any $t > 0$ and from these equations and (4.59) we also infer that

$$\varphi_t \in L^{4/3}\big(0,T;V'\big), \quad \text{if } d=3; \qquad \varphi_t \in L^{2-\delta}\big(0,T;V'\big), \quad \forall\delta \in (0,1), \ \text{if } d=2.$$

We hence get (3.6), (3.7) and furthermore, (3.10) and (4.59) hold also for every $\psi \in V$.

Finally, in order to get (3.12), it is enough to integrate (4.1), (4.2) between 0 and t and pass to the limit for $n \to \infty$ by using the weak convergences above. By integrating between t_0 and t we prove the weak continuity of u and φ in G_{div} and H, respectively.

We now prove that the energy inequality (3.13) holds for the weak solution $[u,\varphi]$ corresponding to the initial data $u_0 \in G_{\mathrm{div}}$ and $\varphi_0 \in D(B)$. To this aim let us first observe that, for almost any $t \in (0,T)$ and for a not relabeled subsequence we have

$$u_n(t) \to u(t) \quad \text{strongly in } G_{\mathrm{div}}, \tag{4.62}$$

$$\varphi_n(t) \to \varphi(t) \quad \text{strongly in } H \text{ and a.e. in } \Omega, \tag{4.63}$$

and that, by means of (H4) and of Fatou's lemma we have

$$\int_\Omega F\big(\varphi(t)\big) \leqslant \liminf_{n\to\infty} \int_\Omega F\big(\varphi_n(t)\big). \tag{4.64}$$

In addition, it is easy to see that

$$P_n(J*\varphi_n)\to J*\varphi \quad \text{in } L^2(0,T;V), \tag{4.65}$$

as a consequence of the convergence $J*\varphi_n \to J*\varphi$ strongly in $L^2(0,T;V)$ and of the fact that $P_n \in \mathcal{L}(V,V)$. Also, we have that

$$\sqrt{\nu(\varphi_n)}Du_n \to \sqrt{\nu(\varphi)}Du, \quad \text{weakly in } L^2(0,T;H). \tag{4.66}$$

This convergence in ensured by applying Lemma 1 on account of the uniform bound $\|\sqrt{\nu(\varphi_n)}\|_\infty \leqslant \sqrt{\nu_2}$, the strong convergence $\sqrt{\nu(\varphi_n)} \to \sqrt{\nu(\varphi)}$ in $L^2(0,T;H)$ and the weak convergence (4.47). We now integrate (4.7) between 0 and t and we pass to the limit using (4.47), (4.54) and (4.62)–(4.66). Taking advantage of the weak lower semicontinuity of the $L^2(0,T;H)$-norm, we therefore get (3.13).

In order to complete the proof of the theorem we now assume that $u_0 \in G_{\mathrm{div}}$ and that $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$. For every $k \in \mathbb{N}$ let us define $\varphi_{0k} \in D(B)$ as

$$\varphi_{0k} := \left(I + \frac{1}{k}B\right)^{-1}\varphi_0.$$

Since B is a maximal monotone operator, we have $\varphi_{0k} \to \varphi_0$ in H. Let $[u_k,\varphi_k]$ be a weak solution corresponding to u_0 and φ_{0k}, satisfying (3.2)–(3.12) and constructed by the Faedo–Galerkin scheme as above. We know that $[u_k,\varphi_k]$ satisfies the energy inequality (3.13) for each k, on the right hand side of which we need to control the nonlinear term that, by virtue of (3.1), can be written as

$$\int_\Omega F(\varphi_{0k}) = \int_\Omega G(\varphi_{0k}) - \frac{a^*}{2}\|\varphi_{0k}\|^2. \tag{4.67}$$

To this aim we multiply the equation $\varphi_{0k} - \varphi_0 = -\frac{1}{k}B\varphi_{0k}$ by $g(\varphi_{0k})$ in $L^2(\Omega)$, where $g = G'$. We obtain

$$\int_\Omega g(\varphi_{0k})(\varphi_{0k}-\varphi_0) = -\frac{1}{k}\int_\Omega g(\varphi_{0k})B\varphi_{0k} = -\frac{1}{k}\int_\Omega g'(\varphi_{0k})|\nabla\varphi_{0k}|^2 - \frac{1}{k}\int_\Omega g(\varphi_{0k})\varphi_{0k} \leqslant 0, \tag{4.68}$$

since g is monotone nondecreasing and we can suppose that $g(0)=0$. Therefore, due to the convexity of G we can write

$$\int_\Omega G(\varphi_{0k}) \leqslant \int_\Omega G(\varphi_0) + \int_\Omega g(\varphi_{0k})(\varphi_{0k}-\varphi_0) \leqslant \int_\Omega G(\varphi_0). \tag{4.69}$$

Hence, on account of (4.67) and (4.69) we get the desired control and from (3.13), written for each weak solution $[u_k,\varphi_k]$, by means of (H4) and of Gronwall lemma, we deduce the estimates (4.12), (4.13) and (4.14) for u_k, φ_k and $\nabla\mu_k$, respectively. By taking the gradient of $\mu_k = a\varphi_k - J*\varphi_k + F'(\varphi_k)$, multiplying the resulting relation by $\nabla\varphi_k$ in $L^2(\Omega)$ and using (H3) we recover the control of the gradient of φ_k from the gradient of μ_k (see (4.16)) and therefore, for φ_k we get the estimate (4.17). Moreover, arguing as in the Faedo–Galerkin approximation scheme above we get (4.19) and (4.21) for μ_k and $\rho(\cdot,\varphi_k)$, and (4.29), (4.30) and (4.37)–(4.39) for the time derivatives u_k' and φ_k', respectively. By compactness we hence deduce the existence of four functions u, φ, μ and ρ satisfying (4.40)–(4.45) such that the convergences (4.46)–(4.55) hold. By passing to the limit in the variational formulation for $[u_k,\varphi_k]$ it is immediate to see that $[u,\varphi]$ is a solution corresponding to the initial data u_0 and φ_0. This completes the proof of the existence of a weak solution when $u_0 \in G_{\mathrm{div}}$ and $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$.

Finally, the energy inequality (3.13) for the solution $[u,\varphi]$ can be obtained by passing to the limit in the energy inequality (3.13) written for each approximating couple $[u_k,\varphi_k]$, using the weak/strong convergences (4.46)–(4.55) and Fatou's lemma, in a similar way as done above for the Faedo–Galerkin approximate solutions (see (4.62)–(4.64)). In particular, on account of (3.1), when we pass to the limit in the nonlinear term on the right hand side we have, by (4.67) and (4.69),

$$\limsup_{k\to\infty}\int_\Omega F(\varphi_{0k}) \leqslant \int_\Omega G(\varphi_0) - \frac{a^*}{2}\|\varphi_0\|^2 = \int_\Omega F(\varphi_0).$$

The proof of Theorem 1 is now complete.

Remark 8. If we compare estimates (4.29) and (4.30) for the time derivatives u_n' in the case $d=3$ and $d=2$, respectively, with the analogous estimates that hold in the case of the local Cahn–Hilliard–Navier–Stokes system (see, e.g., [11]), we see that in the case $d=3$ we obtain the same time regularity exponent 4/3 for both the local and nonlocal systems. However, in the local system we can estimate φ_n in $L^\infty(0,T;V)$ so that, in two dimensions we easily get the exponent 2. For the nonlocal system, this possibility seems out of reach since we can only estimate φ_n in $L^2(0,T;V)$. Also, for the same reason, the transport term in the Cahn–Hilliard equation is less regular so that the bound on φ_n' is weaker in comparison with the analog for the local system.

Remark 9. We point out that energy inequality (3.13) can be written in an alternative form, provided that a suitable condition holds. Indeed, suppose that

$$\int_\Omega \varphi_0 = 0, \tag{4.70}$$

and that c_0 (see (H3)) and J are such that

$$C_P < \frac{c_0}{2\|\nabla J\|_{L^1}}, \tag{4.71}$$

where C_P is the Poincaré–Wirtinger constant in the inequality

$$\|\varphi\| \leqslant C_P\|\nabla\varphi\|, \quad \forall\varphi \in V \quad \text{s.t.} \quad \int_\Omega \varphi = 0.$$

Then, we can get the following control of the gradient of φ by the gradient of μ

$$\|\nabla\mu\|^2 \geqslant \beta\|\nabla\varphi\|^2, \tag{4.72}$$

where $\beta = (c_0 - 2C_P\|\nabla J\|_{L^1})^2$ (compare (4.72) with (4.16)). Indeed, by taking the gradient of $\mu = a\varphi - J*\varphi + F'(\varphi)$, multiplying the resulting relation by $\nabla\varphi$ and using (H3) we have

$$\begin{aligned}\frac{\sqrt{\beta}}{2}\|\nabla\varphi\|^2 + \frac{1}{2\sqrt{\beta}}\|\nabla\mu\|^2 &\geqslant (\nabla\mu, \nabla\varphi)\\ &\geqslant c_0\|\nabla\varphi\|^2 - 2\|\nabla J\|_{L^1}\|\varphi\|\|\nabla\varphi\|\\ &\geqslant \big(c_0 - 2C_P\|\nabla J\|_{L^1}\big)\|\nabla\varphi\|^2\\ &= \sqrt{\beta}\|\nabla\varphi\|^2,\end{aligned} \tag{4.73}$$

whence (4.72). Therefore, as a consequence of (3.13), for the weak solution $[u,\varphi]$ of Theorem 1 the following energy inequality is satisfied as well

$$\begin{aligned}&\frac{1}{2}\bigg(\big\|u(t)\big\|^2 + \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x,t)-\varphi(y,t)\big)^2\,dx\,dy + 2\int_\Omega F\big(\varphi(t)\big)\bigg)\\ &\quad + \int_0^t \big(2\big\|\sqrt{\nu(\varphi)}Du(\tau)\big\|^2 + \beta\big\|\nabla\varphi(\tau)\big\|^2\big)\,d\tau\\ &\leqslant \frac{1}{2}\bigg(\|u_0\|^2 + \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi_0(x)-\varphi_0(y)\big)^2\,dx\,dy + 2\int_\Omega F(\varphi_0)\bigg) + \int_0^t \big\langle h(\tau), u(\tau)\big\rangle\,d\tau.\end{aligned} \tag{4.74}$$

We recall that C_P can be estimated for many important special classes of domains (cf., e.g., [32]). For example, if Ω is convex we can take $C_P = \operatorname{diam}(\Omega)/\pi$ and there exist convex domains for which this constant is optimal (see [10]).

5. Proofs of Corollaries 1 and 2

Proof of Corollary 1. Recalling (3.14) and repeating the proof of Theorem 1, in place of (4.9) we have

$$\begin{aligned}\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi_n(x)-\varphi_n(y)\big)^2\,dx\,dy + 2\int_\Omega F(\varphi_n) &= \|\sqrt{a}\varphi_n\|^2 + 2\int_\Omega F(\varphi_n) - (\varphi_n, J*\varphi_n)\\ &\geqslant \int_\Omega \big(\big(a - \|J\|_{L^1}\big)\varphi_n^2 + 2c_7|\varphi_n|^{2+2q}\big) - 2c_8|\Omega|\\ &\geqslant c\|\varphi_n\|^{2+2q}_{L^{2+2q}(\Omega)} - c,\end{aligned} \tag{5.1}$$

and this estimate, by integrating (4.7) as done above, allows to control the sequence of φ_n and yields (3.15). All the other estimates for φ_n, u_n, μ_n and $\rho(\cdot,\varphi_n)$ established in the proof of Theorem 1 still hold. The only estimates that can be improved are the ones for u_n' and φ_n'. Indeed, for $d = 2$, in place of (4.25) we can write

$$\big\|\widetilde{P}_n(\varphi_n\nabla\mu_n)\big\|_{V'_{\text{div}}} \leqslant c\|\varphi_n\|_{L^{2+2q}(\Omega)}\|\nabla\mu_n\| \leqslant N\|\nabla\mu_n\|,$$

and hence we can control the sequence of $\widetilde{P}_n(\varphi_n \nabla \mu_n)$ in $L^2(0,T;V'_{\mathrm{div}})$. This control, combined with the control for the other terms in (4.22), yields (3.17). Furthermore, as far as the sequence of φ_n' is concerned, we can improve estimates (4.34)–(4.36) by arguing as in the proof of Theorem 1 and by considering the following cases. Choosing $s = ((4-d)p + 2d)/2p$ (cf. (4.31) and (4.32)), when $1 < p \leqslant d' = d/(d-1)$, due to the embeddings $H^{s-1}(\Omega) \hookrightarrow L^\infty(\Omega)$ (if $1 < p < d'$) or $H^{s-1}(\Omega) \hookrightarrow L^r(\Omega)$ for every $r < \infty$ (if $p = d'$), we have

$$\left| \int_\Omega (u_n \cdot \nabla \psi_I)\varphi_n \right| \leqslant c\|u_n\| \|\varphi_n\|_{L^{2+2q}(\Omega)} \|\psi\|_{V_s} \leqslant N\|\psi\|_{V_s}. \tag{5.2}$$

The same estimate also holds for the case $d = 3$ when $3/2 < p \leqslant 2$ and $q \geqslant 2(2p-3)/(6-p)$, where here we use the embedding $H^{s-1}(\Omega) \hookrightarrow L^{3p/(2p-3)}(\Omega)$ and the fact that $6p/(6-p) \leqslant 2 + 2q$. Finally, when $d = 3$, $3/2 < p \leqslant 2$ and $0 < q < 2(2p-3)/(6-p)$ we have

$$\begin{aligned} \left| \int_\Omega (u_n \cdot \nabla \psi_I)\varphi_n \right| &\leqslant c\|u_n\| \|\varphi_n\|_{L^{6p/(6-p)}(\Omega)} \|\psi\|_{V_s} \\ &\leqslant c\|u_n\| \|\varphi_n\|_{L^{2+2q}(\Omega)}^{2(3-p)(1+q)/p(2-q)} \|\varphi_n\|_{L^6(\Omega)}^{(4p-6q+pq-6)/p(2-q)} \|\psi\|_{V_s} \\ &\leqslant N\|\varphi_n\|_V^{(4p-6q+pq-6)/p(2-q)} \|\psi\|_{V_s}. \end{aligned} \tag{5.3}$$

Hence, on account of (5.2) and (5.3), from (4.1) (written with $\psi = \psi_I$) we deduce (3.18) and (3.19). The improved regularity (3.16) for φ_t can be obtained by estimating the term $(u, \varphi\nabla\psi)$ in (4.59) for the case $d = 2$ as

$$\left|(u, \varphi\nabla\psi)\right| \leqslant c\|\nabla u\| \|\varphi\|_{L^{2+2q}(\Omega)} \|\nabla\psi\| \leqslant N\|\nabla u\| \|\nabla\psi\|,$$

and for the case $d = 3$ and $q \geqslant 1/2$ as

$$\left|(u, \varphi\nabla\psi)\right| \leqslant c\|u\|_{L^{2(1+1/q)}(\Omega)} \|\varphi\|_{L^{2+2q}(\Omega)} \|\nabla\psi\| \leqslant N\|\nabla u\| \|\nabla\psi\|. \qquad \square$$

Proof of Corollary 2. For $d = 2$ the regularity properties (3.16) and (3.17) allow us to deduce the energy identity for the weak solution. Indeed, in this case we can take and $v = u(\tau)$ in (3.11) and $\psi = \mu(\tau)$ in (4.59), sum the resulting equations and then integrate with respect to τ between 0 and t. When we consider the duality product $\langle \varphi_t, \mu \rangle$, we are led to the duality $\langle \varphi_t, F'(\varphi) \rangle$ which can be rewritten by taking into account that $F'(\varphi) = g(\varphi) - a^*\varphi$, with $g \in C^1(\mathbb{R})$ monotone increasing. Now, introducing the functional $\mathcal{G} : H \to \mathbb{R} \cup \{+\infty\}$ defined as $\mathcal{G}(\varphi) = \int_\Omega G(\varphi)$ if $G(\varphi) \in L^1(\Omega)$ and $\mathcal{G}(\varphi) = +\infty$ otherwise, we have (see [7, Proposition 2.8, Chapter II]) that $\mathcal{G}$ is convex, lower semicontinuous on H and $\xi \in \partial\mathcal{G}(\varphi)$ if and only if $\xi = G'(\varphi) = g(\varphi)$ almost everywhere in Ω. In view of (3.16) and of the fact that $g(\varphi) \in L^2(0,T;V)$, we can use [16, Proposition 4.2] and get, for almost any $t \in (0,T)$,

$$\langle \varphi_t, F'(\varphi) \rangle = \langle \varphi_t, g(\varphi) \rangle - a^*\langle \varphi_t, \varphi \rangle = \frac{d}{dt}\left(\mathcal{G}(\varphi) - \frac{a^*}{2}\|\varphi\|^2\right) = \frac{d}{dt}\int_\Omega F(\varphi).$$

Therefore, on account of this identity, from (3.11) and (4.59) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|\sqrt{a}\varphi\|^2 - (\varphi, J * \varphi) + 2\int_\Omega F(\varphi)\right) + 2\left\|\sqrt{\nu(\varphi)}Du\right\|^2 + \|\nabla\mu\|^2 \\ &= \frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x) - \varphi(y)\big)^2\,dx\,dy + 2\int_\Omega F(\varphi)\right) + 2\left\|\sqrt{\nu(\varphi)}Du\right\|^2 + \|\nabla\mu\|^2 \\ &= \langle h, u \rangle. \end{aligned} \tag{5.4}$$

Hence we get (3.20). Furthermore, by integrating between 0 and t we recover the energy identity in integral form, i.e, (3.13) holds with the equal sign for every $t \geqslant 0$.

In order to obtain (3.21), let us multiply equation $\mu = a\varphi - J * \varphi + F'(\varphi)$ by φ in $L^2(\Omega)$. We obtain

$$(\mu, \varphi) = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x) - \varphi(y)\big)^2\,dx\,dy + \big(F'(\varphi), \varphi\big). \tag{5.5}$$

Now, observe that, due to (3.18) and to the convexity of G we have

$$F(0) \geqslant F(s) + \frac{a^*}{2}s^2 - \big(F'(s) + a^*s\big)s$$

and hence

$$F'(s)s \geqslant F(s) - \frac{a^*}{2}s^2 - F(0).$$

Therefore, from (5.5) we get

$$(\mu,\varphi) \geqslant \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x)-\varphi(y)\big)^2\,dx\,dy + \int_\Omega F\big(\varphi(t)\big) - \frac{a^*}{2}\|\varphi\|^2 - c. \tag{5.6}$$

Set $\overline{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu$ and suppose $(\varphi_0, 1) = 0$ first. Then we have

$$(\mu,\varphi) = (\mu - \overline{\mu}, \varphi) \leqslant C_p\|\nabla\mu\|\|\varphi\|,$$

and then, by means of (H7), from (5.6) we have

$$\frac{1}{8}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x)-\varphi(y)\big)^2\,dx\,dy + \frac{1}{2}\int_\Omega F(\varphi) + \frac{c_7}{2}\int_\Omega |\varphi|^{2+2q} - c_9 - \frac{a^*}{2}\|\varphi\|^2 - c$$
$$\leqslant \frac{3}{2}\|J\|_{L^1}\|\varphi\|^2 + \|\nabla\mu\|^2 + \frac{C_P^2}{2}\|\varphi\|^2.$$

Therefore, we deduce

$$\frac{1}{8}\int_\Omega\int_\Omega J(x-y)\big(\varphi(x)-\varphi(y)\big)^2\,dx\,dy + \frac{1}{2}\int_\Omega F(\varphi) \leqslant \|\nabla\mu\|^2 + c_{10}$$

and hence

$$\frac{1}{2}\mathcal{E}(u,\varphi) \leqslant c_{11}\left(\frac{\nu_1}{2}\|\nabla u\|^2 + \|\nabla\mu\|^2\right) + c_{10}, \tag{5.7}$$

where $c_{11} = \max(1, 1/2\lambda_1\nu_1)$, λ_1 being the lowest eigenvalue of the Stokes operator A. We point out that all constants only depend on the parameters of the problem and are independent of the initial data. Now, by virtue of (3.20) and (5.7) we have

$$\frac{d}{dt}\mathcal{E}(u,\varphi) + k\mathcal{E}(u,\varphi) \leqslant l + \frac{1}{2\nu_1}\|h\|^2_{V'_{\mathrm{div}}}, \tag{5.8}$$

where $k = 1/2c_{11}$ and $l = c_{10}/c_{11}$. By means of Gronwall lemma we hence deduce

$$\mathcal{E}\big(u(t),\varphi(t)\big) \leqslant \mathcal{E}(u_0,\varphi_0)e^{-kt} + K, \tag{5.9}$$

with

$$K = \frac{l}{k} + \frac{1}{2\nu_1(1-e^{-k})}\|h\|^2_{L^2_{tb}(0,\infty;V'_{\mathrm{div}})}.$$

If $m_0 := (\varphi_0, 1) \neq 0$, observe that if $[u,\varphi]$ is a weak solution with data $[u_0,\varphi_0]$ for the problem with potential F, then $[u,\widetilde{\varphi}]$, where $\widetilde{\varphi} = \varphi - m_0$ is a weak solution with data $[u_0, \varphi_0 - m_0]$ for the same problem with potential $\widetilde{F}$ given by

$$\widetilde{F}(s) := F(s + m_0) - F(m_0).$$

By relying on (5.9) satisfied by the solution $[u,\widetilde{\varphi}]$, we easily get (3.21). □

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