Generation type inequalities for closed linear operators related to domains with conical points

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Abstract. Let $\mathcal{A}(x;D_x)$ be a second-order linear differential operator in divergence form. We prove that the operator $\lambda I-\mathcal{A}(x;D_x)$, where $\lambda\in\mathbf{C}$ and I stands for the identity operator, is closed and injective when $\mathrm{Re}\lambda$ is large enough and the domain of $\mathcal{A}(x;D_x)$ consists of a special class of weighted Sobolev function spaces related to conical open bounded sets of $\mathbf{R}^n$, $n\geq 1$.

Key words and phrases. Resolvent estimates. Weighted Sobolev function spaces. Conical bounded domains of $\mathbf{R}^n$.

1 Introduction and plan of the paper

In this paper we present a new approach for proving an estimate of generation type for the norm of the resolvent $[\lambda I-\mathcal{A}(x;D_x)]^{-1}$ of the operator $\lambda I-\mathcal{A}(x;D_x)$, where $\lambda\in\mathbf{C}$, I stands for the identity operator and $\mathcal{A}(x;D_x)$ denotes the second-order linear differential operator in divergence form

$$\mathcal{A}(x;D_x)=\sum_{j=1}^{n}D_{x_j}\big(\sum_{k=1}^{n}a_{j,k}(x)D_{x_k}\big). \tag{1.1}$$

We stress that in our paper the domain of $\mathcal{A}(x;D_x)$ will consist of an appropriate class of weighted Sobolev spaces whose elements will be functions taking their values in conical open bounded sets of $\mathbf{R}^n$, $n\geq 1$.
With the language of the modern semigroup theory a generation type estimate means that, denoted with $\mathcal{L}(X)$ the Banach space of the linear bounded operators from X to X, X being a Banach space, and endowed $\mathcal{L}(X)$ with the usual uniform operatorial norm, then $\|[\lambda I-\mathcal{A}(x;D_x)]^{-1}\|_{\mathcal{L}(X)}$ is bounded from above by some constant times $|\lambda|^{-1}$, at least for large enough $\mathrm{Re}\lambda$.
Even if in order to prove our main result we adopt an idea that goes back to [1] and [5], i.e. the procedure of increasing the dimension from n to $n+1$, in our proof there are so many different elements with respect to the proof of the estimates in the quoted papers that we may consider our results totally independent of those.
The novelties arise fundamentally from the fact that we consider bounded domain G of $\mathbf{R}^n$

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having a singular point O situated in a part of the boundary ∂G with a conical structure. This forces us to consider weighted Sobolev function spaces for which, unfortunately, the classical *a priori* estimates of [2] are not available. The role of that estimates will be played here by some estimates of the same type proven in [10], but these estimates, when applied in dimension $n+1$, require the conical structure of G to be preserved when we increase the dimension. Therefore, denoted by x_0 the added variable, unlike standard procedures making use of $\Gamma = (-\infty, +\infty) \times G$, we will consider, as a new domain in dimension $n+1$, a domain $\widetilde{G}$ which can be regarded as a rotation of G around its symmetry axis.
Of course, when $n = 1$, G simply coincides with a bounded open interval of $\mathbf{R}$ and rotations have no meaning. However, in this situation too, it will always be possible to consider two-dimensional conical domains $\widetilde{G}$ having G as their symmetry axis.
The main difficulties arising by the use of $\widetilde{G}$ instead of that of Γ consist in the following:

(i) the proof that the property of the boundary conditions to cover $\mathcal{A}(x; D_x)$ on $\partial G \backslash \{O\}$ in the sense of [3] continues to hold when we increase the dimension. This is not a straightforward fact and forces us to implement a new set of boundary operators coinciding with the original ones on ∂G;

(ii) the necessity of considering cut-off functions depending on both variables x_0, x, where $x \in G$, instead of cut-off functions as those considered in [1] and [5] and depending only on the added variable x_0. As a consequence, our computations will be heavier and longer than those in the quoted papers (cf. also [11]).

Observe that we will consider bounded domains of conical type. Since a lot of papers have been devoted in the past to the investigation of boundary value problems in such domains, we prefer here not to mention any of them, but only to refer the interested reader to [8], where some examples of admissible domains and an exhaustive list of references for this kind of problems are given.
We would like to emphasize that generation type estimates are one of the main tools needed to prove that a linear operator generates an analytic semigroup of linear bounded operators. Hence, if the generation in our functional setting could be guaranteed, by showing the surjectivity of $\lambda I - \mathcal{A}(x; D_x)$ too, the range of applications of our result would be extremely large. Indeed, nowadays semigroup theory is one of the most used tool in both direct and inverse problems related parabolic differential equations. However, while for regular domains and classical Sobolev spaces many generation results are available, the same, to the author's knowledge, is not true for conical domains and weighted Sobolev spaces.
The plan of this paper is the following. In Section 2, using notations of [10], we introduce the class of domains and of weighted Sobolev function spaces we will deal with. Moreover, we introduce also the correspondent spaces of traces for the boundary values.
Section 3 is devoted to recall the *a priori* estimates of [10] for boundary value problems in the functional setting of Section 2. To this purpose we need to introduce some further technical definitions and a rather heavy notation which, however, having to deal with scalar and not matrix differential operators, turns out to be quite simple in our case.
In the first part of Section 4 we list all the basic assumptions on the domain G, on the operator $\mathcal{A}(x; D_x)$ and on the boundary operator $\mathcal{B}(x; D_x)$ associated with $\mathcal{A}(x; D_x)$. Under these assumptions, in the second part of Section 4 we will introduce the concept

of regular boundary value problem and, for such a problem, we will state our main result (Theorem 4.6). We conclude the section by showing some easy corollaries to our estimate and related to the analytic semigroups theory.
Section 5 contains the proof of the preliminary Lemma 4.2. Essentially, it states that the property of $\mathcal{B}(x; D_x)$ to cover $\mathcal{A}(x; D_x)$ on $\partial G \backslash \{O\}$ in the sense of [3] continues to hold when we increase the dimension, provided we replace the triplet $\{\mathcal{A}(x; D_x), \mathcal{B}(x; D_x), G\}$ with the triplet $\{\mathcal{A}(x; D_x) + e^{i\psi} D_{x_0}^2, \mathcal{B}(x; D_x) + x_0 D_{x_0}, \widetilde{G}\}$, $\psi \in [-\pi/2, \pi/2]$.
In Section 6 we introduce the class of our admissible cut-off functions. For the reasons we said before, they have a structure more complicated (cf. (6.2)) than those used in [1] and [5] and hence, for clarity's sake, we report all the necessary computations we need in order to perform the technicalities of Section 7.
Finally, in Section 7 we prove our main result. The proof will be derived simply by taking advantage of the assumptions on G and by combining Theorem 3.2 with Lemma 4.2 and with the further preliminary estimates of Lemma 7.1 and Lemma 7.2.

2 The spaces $V^l_{p,\beta}(G)$, $W^l_{p,\beta}(G)$, $V^{l-p^{-1}}_{p,\beta}(\partial G)$, $W^{l-p^{-1}}_{p,\beta}(\partial G)$

Let $B(0, 1)$ be the unit open ball of $\mathbf{R}^n$, $n \geq 1$, and denote by K an open cone of $\mathbf{R}^n$ having its vertex at the origin and cutting out on the unit sphere $\partial B(0, 1)$ a domain Ω. From now on, with G we will denote an open subset of $\mathbf{R}^n$ having compact closure $\overline{G}$ and boundary ∂G on which there is a point O such that:

(i) $\partial G \backslash \{O\}$ is a smooth, $(n-1)$–dimensional submanifold of $\mathbf{R}^n$;

(ii) near O the domain G coincides with $K \cap B(0, 1)$.

Using a multi-index notation, for $1 < p < +\infty$, $\beta \in \mathbf{R}$, $l = 0, 1, \ldots$, we define the weighted spaces $V^l_{p,\beta}(G)$ and $W^l_{p,\beta}(G)$ as the spaces of functions u in G endowed, respectively, with the following norm $\|\cdot\|_{V^l_{p,\beta}(G)}$ and $\|\cdot\|_{W^l_{p,\beta}(G)}$, where $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$:

$$\|u\|_{V^l_{p,\beta}(G)} = \Big(\sum_{\alpha=0}^{l} \int_G |x|^{p(\beta - l + |\alpha|)} |D^\alpha u(x)|^p \, \mathrm{d}x \Big)^{1/p} < +\infty \tag{2.1}$$

$$\|u\|_{W^l_{p,\beta}(G)} = \Big(\sum_{\alpha=0}^{l} \int_G |x|^{p\beta} |D^\alpha u(x)|^p \, \mathrm{d}x \Big)^{1/p} < +\infty \tag{2.2}$$

Since (2.1) and (2.2) coincide if $l = 0$ we set $L_{p,\beta}(G)$ to be the weighted L_p space of functions in G endowed with norm

$$\|u\|_{L_{p,\beta}(G)} = \Big(\int_G |x|^{p\beta} |u(x)|^p \, \mathrm{d}x \Big)^{1/p}.$$

As shown in [10], the space $C_0^\infty(\overline{G} \backslash \{O\})$ of the infinitely differentiable functions having compact support on $\overline{G} \backslash \{O\}$ is dense in $V^l_{p,\beta}(G)$ and the following theorem holds true.

Theorem 2.1. *1) If $\beta < -np^{-1}$ or $\beta > l - np^{-1}$ then the spaces $V^l_{p,\beta}(G)$ and $W^l_{p,\beta}(G)$ coincide and the norm (2.1), (2.2) are equivalent.*
2) If for some number $\nu = 0, 1, \ldots, l-1$ the inequalities $\nu - np^{-1} < \beta < \nu + 1 - np^{-1}$ are

satisfied, then the space $W^l_{p,\beta}(G)$ *is the direct sum of* $V^l_{p,\beta}(G)$ *and* $\Pi_{l-\nu-1}$*, where* $\Pi_{l-\nu-1}$ *is the space of polynomials in* x *of degree at most equal to* $l-\nu-1$ ($\Pi_q = \{0\}$ *if* $q<0$).

Proof. See the proof of Theorem 2.1 in [10]. □

In order to consider boundary value problems we need to define also the spaces $V^{l-p^{-1}}_{p,\beta}(\partial G)$ and $W^{l-p^{-1}}_{p,\beta}(\partial G)$, i.e. the spaces of traces on ∂G of functions in $V^l_{p,\beta}(G)$ and $W^l_{p,\beta}(G)$, respectively. It turns out that $V^{l-p^{-1}}_{p,\beta}(\partial G)$ is the quotient space $V^l_{p,\beta}(G)\backslash\widetilde{V}^l_{p,\beta}(G)$, where $\widetilde{V}^l_{p,\beta}(G)$ is the completion with respect to the $V^l_{p,\beta}(G)$–norm of the set of smooth functions in $V^l_{p,\beta}(G)$ equal to zero on ∂G. $V^{l-p^{-1}}_{p,\beta}(\partial G)$ is endowed with the norm:

$$\|u\|_{V^{l-p^{-1}}_{p,\beta}(\partial G)} \;=\; \inf\left\{\|v\|_{V^l_{p,\beta}(G)} : v-u\in \widetilde{V}^l_{p,\beta}(G)\right\}. \tag{2.3}$$

Replacing V with W in the above definitions we obtain the description of $W^{l-p^{-1}}_{p,\beta}(\partial G)$.
From the fact that $C^\infty_0(\overline{G}\backslash\{O\})$ is dense in $V^l_{p,\beta}(G)$ it easily follows that $C^\infty_0(\partial\overline{G}\backslash\{O\})$ is dense in $V^{l-p^{-1}}_{p,\beta}(\partial G)$. Moreover, Theorem 2.1 ensures that if $\beta<-np^{-1}$ or $\beta>l-np^{-1}$ then $V^{l-p^{-1}}_{p,\beta}(\partial G)$ and $W^{l-p^{-1}}_{p,\beta}(\partial G)$ coincide whereas (cf. [10, Theorem 3.1]) if for some number ν the inequalities $\nu-np^{-1}<\beta<\nu+1-np^{-1}$ are satisfied then $W^{l-p^{-1}}_{p,\beta}(\partial G)$ is the direct sum of $V^{l-p^{-1}}_{p,\beta}(\partial G)$ and the space $Y_{l-\nu-1}$ of polynomials of degree at most $l-\nu-1$ which are not identically zero on $\partial\Omega\times\mathbf{R}_+$.

3 Admissible operators and boundary value problems in $W^l_{p,\beta}(G)$

Let D_x denotes the n-uple $(D_{x_1},\dots,D_{x_n})$ and let $\mathfrak{C}(\mu,s)$ to be the class of differential operators $\mathcal{M}(x;D_x)$ of order μ with coefficients in $C^s(\overline{G}\backslash\{O\};\mathbf{C})$ and admitting, near O, the following representation in local spherical co-ordinates (r,ω):

$$\mathcal{M}(x,D_x) \;=\; r^{-\mu}\sum_{k+|\gamma|\le\mu} p_{k,\gamma}(r,\omega)(rD_r)^k D^\gamma_\omega \;\equiv\; r^{-\mu}M(r,\omega;rD_r,D_\omega)\,, \tag{3.1}$$

where the functions $p_{h,\alpha}(r,\omega)$, $h+|\alpha|\le\mu$, satisfy the condition

$$(rD_r)^q D^\gamma_\omega\, p_{h,\alpha}\in C([0,\delta]\times\overline{\Omega};\mathbf{C})\,,\qquad q+|\gamma|\le s,\;\; \delta=\text{const}>0. \tag{3.2}$$

Recall that, for $h>0$, we have

$$D^h_r \;=\sum_{|\alpha|=h} a_{h,\alpha}(\omega)D^\alpha_x\,,\qquad D^h_\omega \;=\sum_{0<|\alpha|\le h} r^{|\alpha|}b_{h,\alpha}(\omega)D^\alpha_x\,, \tag{3.3}$$

where $a_{h,\alpha}$ and $b_{h,\alpha}$, $0<|\alpha|\le h$, are smooth functions on $\partial B(0,1)$.
From (2.1) it is easy to prove that any $\mathcal{M}\in\mathfrak{C}(\mu,s)$ realizes a continuous mapping $V^l_{p,\beta}(G)\to V^{l-\mu}_{p,\beta}(G)$ for $s\ge l-\mu$ and, by Theorem 2.1, if $\beta<-np^{-1}$ or $\beta>l-np^{-1}$ the same property holds true with $V^l_{p,\beta}(G)$ and $V^{l-\mu}_{p,\beta}(G)$ replaced by $W^l_{p,\beta}(G)$ and $W^{l-\mu}_{p,\beta}(G)$, respectively. In the case there exists $\nu=0,\dots,l-1$ such that $\nu-np^{-1}<\beta<\nu+1-np^{-1}$ the map $\mathcal{M}: W^l_{p,\beta}(G)\to W^{l-\mu}_{p,\beta}(G)$ is still continuous if $\mathcal{M}(\Pi_{l-\nu-1})\subset W^{l-\mu}_{p,\beta}(G)$.

Remark 3.1. For reasons that will be clearer in Section 4, denote by (x_0,x) the points of $\mathbf{R}^{n+1}$ and by $\widetilde{G}$ an $(n+1)$-dimensional domain which, close to the origin, coincides with the cone $\{(x_0,x)\in\mathbf{R}^{n+1}: |(x_0,x')|\le C_1x_n,\ x'=(x_1,\dots,x_{n-1}),\ x_n>0\}$, $C_1>0$. We will show here that if $a_{j,k}\in C^1(\overline{G};\mathbf{C})$, $j,k=1,\dots,n$, and $G=\{(x_0,x)\in\widetilde{G}: x_0=0\}$, then the operator $\mathcal{A}_\psi(x;D_x,D_{x_0})=\mathcal{A}(x;D_x)+e^{i\psi}D_{x_0}^2$, where $\psi\in[-\pi/2,\pi/2]$ and $\mathcal{A}(x;D_x)$ is defined by (1.1), belongs to the class $\mathfrak{C}(2,0)$.
We introduce in the space $\mathbf{R}^{n+1}$ the $(n+1)$-dimensional spherical co-ordinates, related to the Cartesian ones by the well-known relationships:

$$\begin{aligned}&(x_0,x_1,\dots,x_{n-1},x_n)\\ =\ &\big(r\cos\theta_0, r\sin\theta_0\cos\theta_1,\dots, r\textstyle\prod_{h=0}^{n-2}\sin\theta_h\cos\theta_{n-1}, r\prod_{h=0}^{n-1}\sin\theta_h\big)\,,\end{aligned}\tag{3.4}$$

where $r=|(x_0,x)|$, $\theta_h\in[0,\pi]$, $h=0,\dots,n-2$, $\theta_{n-1}\in[0,2\pi)$ and where $\prod_{h=l_1}^{l_2}\sin\theta_h$ has to be intending equal to one if $l_2<l_1$.
Denoting by ω the $(n-1)$-uple $(\theta_0,\dots,\theta_{n-1})$, with the help of (3.4) it is not too difficult to show that the gradient (D_{x_0},D_x) can be expressed in terms of (D_r,D_ω), $D_\omega=(D_{\theta_0},\dots,D_{\theta_{n-1}})$, by the following formulae:

$$D_{x_j}=\tau_{j,r}(\omega)D_r+r^{-1}\big(\textstyle\sum_{k=0}^{n-2}\tau_{j,\theta_k}(\omega)D_{\theta_k}+\tau_{j,\theta_{n-1}}(\omega)D_{\theta_{n-1}}\big)\,,\qquad j=0,\dots,n\,,\tag{3.5}$$

where, $\delta_{i,l}$ standing for the Kronecher symbol, for any $j\in\{0,\dots,n-1\}$ and any $k\in\{0,\dots,n-2\}$ we have

$$\begin{cases}\tau_{j,r}(\omega)=\prod_{h=0}^{j-1}\sin\theta_h\cos\theta_j\,,\qquad \tau_{n,r}(\omega)=\prod_{h=0}^{n-1}\sin\theta_h\,,\\ \tau_{j,\theta_k}(\omega)=\begin{cases}\big(\prod_{h=0}^{k}\sin\theta_h\big)^{-1}\big[\prod_{h=k}^{j-1}\sin\theta_h\cos\theta_k\cos\theta_j-\delta_{k,j}\big]\,, & \text{if } k\le j\,,\\ 0\,, \quad \text{if } k>j\,,\end{cases}\\ \tau_{n,\theta_k}(\omega)=\big(\prod_{h=0}^{k-1}\sin\theta_h\big)^{-1}\prod_{h=k+1}^{n-1}\sin\theta_h\cos\theta_k\,,\\ \tau_{j,\theta_{n-1}}(\omega)=-\big(\prod_{h=0}^{n-2}\sin\theta_h\big)^{-1}\delta_{(n-1),j}\sin\theta_j\,,\qquad \tau_{n,\theta_{n-1}}(\omega)=\big(\prod_{h=0}^{n-2}\sin\theta_h\big)^{-1}\cos\theta_{n-1}\,.\end{cases}\tag{3.6}$$

Hence, if we set $\widetilde{a}_{j,k}(r,\omega)=a_{j,k}(r\sin\theta_0\cos\theta_1,\dots,r\prod_{h=0}^{n-2}\sin\theta_h\cos\theta_{n-1},r\prod_{h=0}^{n-1}\sin\theta_h)$, $j,k=1,\dots,n$, we obtain

$$\textstyle\sum_{k=1}^{n}a_{j,k}(x)D_{x_k}=f_{j,r}(r,\omega)D_r+r^{-1}\sum_{h=0}^{n-1}f_{j,\theta_h}(r,\omega)D_{\theta_h},\qquad j=1,\dots,n\,,\tag{3.7}$$

functions $f_{j,r}$, f_{j,θ_l}, $j=1,\dots,n$, $l=0,\dots,n-1$, being defined by

$$f_{j,r}(r,\omega):=\textstyle\sum_{k=1}^{n}\widetilde{a}_{j,k}(r,\omega)\tau_{k,r}(\omega)\,,\qquad f_{j,\theta_l}(r,\omega):=\sum_{k=1}^{n}\widetilde{a}_{j,k}(r,\omega)\tau_{k,\theta_l}(\omega)\,.\tag{3.8}$$

Using again (3.5) and applying it to relations (3.7), performing easy computations we get

$$\mathcal{A}(x;D_x)=\mathcal{Q}_r(r,\omega;D_r,D_\omega)+\textstyle\sum_{l=0}^{n-1}\mathcal{Q}_{\theta_l}(r,\omega;D_r,D_\omega),\tag{3.9}$$

where $\mathcal{Q}_r(r,\omega;D_r,D_\omega)$, $\mathcal{Q}_{\theta_l}(r,\omega;D_r,D_\omega)$, $l=0,\dots,n-1$, stand for the second-order linear differential operator

$$\begin{aligned}&\mathcal{Q}_r(r,\omega;D_r,D_\omega)=D_r\big(k_r(r,\omega)D_r+r^{-1}\textstyle\sum_{h=0}^{n-1}k_{\theta_h}(r,\omega)D_{\theta_h}\big),\\ &\mathcal{Q}_{\theta_l}(r,\omega;D_r,D_\omega)=r^{-1}\textstyle\sum_{j=1}^{n}\tau_{j,\theta_l}(\omega)D_{\theta_l}\big(f_{j,r}(r,\omega)D_r+r^{-1}\sum_{h=0}^{n-1}f_{j,\theta_h}(r,\omega)D_{\theta_h}\big),\end{aligned}\tag{3.10}$$

functions k_r, k_{θ_l}, $l = 0, \ldots, n-1$, appearing in (3.10) being defined by

$$k_r(r,\omega) := \textstyle\sum_{j=1}^n \tau_{j,r}(\omega) f_{j,r}(r,\omega)\,, \qquad k_{\theta_l}(r,\omega) := \textstyle\sum_{j=1}^n \tau_{j,r}(\omega) f_{j,\theta_l}(r,\omega)\,. \tag{3.11}$$

Moreover, since (3.5), (3.6) imply $D_{x_0} = \cos\theta_0 D_r - r^{-1}\sin\theta_0 D_{\theta_0}$, taking advantage from

$$(rD_r)^2 = r^2 D_r^2 + rD_r \tag{3.12}$$

we obtain

$$D_{x_0}^2 = r^{-2}\{\cos^2\theta_0 (rD_r)^2 + \sin(2\theta_0)[I - (rD_r)]D_{\theta_0} + \sin^2\theta_0 D_{\theta_0}^2 - \cos(2\theta_0)(rD_r)\}. \tag{3.13}$$

Therefore, assuming $a_{i,j} \in C^1(\overline{G};\mathbf{C})$, $i,j = 1,\ldots,n$, taking into account the following formulae (cf. (3.3) with $h = 1$)

$$\begin{cases} D_r \;=\; \sum_{k=0}^{n-1}(\prod_{h=0}^{k-1}\sin\theta_h\cos\theta_k)D_{x_k} + \prod_{h=0}^{n-1}\sin\theta_h D_{x_n}\,, \\ D_{\theta_l} \;=\; \sum_{j=l}^{n}(D_{\theta_k}x_j)D_{x_j}\,, \qquad l = 0,\ldots,n-1\,, \end{cases} \tag{3.14}$$

and recalling (3.8) and (3.11), if we differentiate, respectively with respect to r and θ_l, $l = 0,\ldots,n-1$, each term in the brackets of (3.10) and we rearrange the term using (3.12), from (3.9), (3.13) we can easily see that $\mathcal{A}_\psi(x;D_x,D_{x_0}) = \mathcal{A}(x;D_x) + e^{i\psi}D_{x_0}^2$, admits representation (3.1) with $\mu = 2$. In addition, since the points $(0,\ldots,0,x_j,0,\ldots,0)$, $j = 0,\ldots,n-1$, with $x_j \neq 0$ do not belong to $\widetilde{G}$, we have $\sin\theta_j \neq 0$ for any $j \in \{0,\ldots,n-1\}$ and hence (cf. (3.6)) condition (3.2) is satisfied, too, with $s = 0$.

Coming back to our purposes, we consider the boundary value problem:

$$\mathcal{L}(x;D_x)u \;=\; \mathcal{F} \quad \text{in } G\,; \qquad \mathcal{B}(x;D_x)u \;=\; \mathcal{G} \quad \text{on } \partial G\backslash\{O\}\,, \tag{3.15}$$

$\mathcal{L}$ and $\mathcal{B}$ being matrix differential operators in G of dimension $k\times k$ and $m\times k$, respectively, with elements $\mathcal{L}_{h,j}(x;D_x)$ and $\mathcal{B}_{q,j}(x;D_x)$, $h,j = 1,\ldots,k$, $q = 1,\ldots,m$. The orders of operators $\mathcal{L}_{h,j}$ and $\mathcal{B}_{q,j}$ are equal to $(s_h + t_j)$ and $(\sigma_q + t_j)$, respectively, where $\{s_h\}_{h=1}^k$, $\{t_j\}_{j=1}^k$ and $\{\sigma_q\}_{q=1}^m$ are collections of integers with $\max_{h=1,\ldots,k} s_h = 0$, $t_j > 0$, $j = 1,\ldots,k$, and $\sum_{j=1}^k (s_j + t_j) = 2m$. Clearly, $\mathcal{L}_{h,j} \equiv 0$ and $\mathcal{B}_{q,j} \equiv 0$ if $s_h + t_j < 0$ and $\sigma_q + t_j < 0$. Moreover, taken $l \geq \max\{0, \max_{q=1,\ldots,m}\sigma_q\}$, we assume $\mathcal{L}_{h,j}(x;D_x) \in \mathfrak{C}(s_h + t_j, l - s_h)$ and $\mathcal{B}_{q,j}(x;D_x) \in \mathfrak{C}(\sigma_q + t_j, l - \sigma_q)$ in a neighborhood of O (cf. [9] p.76).
We require $\mathcal{L}$ to be *uniformly elliptic in* $\overline{G}\backslash\{O\}$ in the sense of [3] and we impose that the boundary conditions $\mathcal{B}$ cover $\mathcal{L}$ on $\partial G\backslash\{O\}$ (cf. [3] or [7]).
Problem (3.15) generates a model problem in the cone $(0,+\infty)\times\Omega$. With the pair $\{\mathcal{L},\mathcal{B}\}$ we associate the operator $\mathcal{U}(0,\omega,z,D_\omega) = \{L(0,\omega,z,D_\omega), B(0,\omega,z,D_\omega)\}$ where $\omega \in \Omega$, $z \in \mathbf{C}$ and the matrix differential operators $L(0,\omega,z,D_\omega)$ and $B(0,\omega,z,D_\omega)$ are defined, respectively, by

$$L(0,\omega,z,D_\omega) \;=\; \big(L_{h,j}(0,\omega;z - it_j, D_\omega)\big)_{h=1\ldots,k}^{j=1\ldots,k}\,, \tag{3.16}$$

$$B(0,\omega,z,D_\omega) \;=\; \big(B_{q,j}(0,\omega;z - it_j, D_\omega)\big)_{q=1,\ldots,m}^{j=1,\ldots,k}\,, \tag{3.17}$$

the operators $L_{h,j}$ and $B_{q,j}$ being determined from $\mathcal{L}_{h,j}$ and $\mathcal{B}_{q,j}$ by means of (3.1) replacing $p_{k,\gamma}(r,\omega)$ with $p_{k,\gamma}(0,\omega)$. As shown in the Appendix the ellipticity of system (3.15) implies that $\mathcal{U}(0,\omega,z,D_\omega)$ is elliptic with parameter in the sense of [6].

Denoting by $\vec{t}$, $\vec{s}$ and $\vec{\sigma}$ the vectors $(t_1, \dots, t_k)$, $(s_1, \dots, s_k)$ and $(\sigma_1, \dots, \sigma_m)$, respectively, we introduce the spaces of vector-valued functions

$$V^{l+\vec{t}}_{p,\beta}(G) = \prod_{j=1}^{k} V^{l+t_j}_{p,\beta}(G)\,, \qquad V^{l-\vec{s}}_{p,\beta}(G) = \prod_{j=1}^{k} V^{l-s_j}_{p,\beta}(G)\,, \\ V^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G) = \prod_{q=1}^{m} V^{l-\sigma_q-p^{-1}}_{p,\beta}(\partial G)\,, \tag{3.18}$$

and the correspondent spaces $W^{l+\vec{t}}_{p,\beta}(G)$, $W^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G)$ obtained by replacing V with W in (3.18). By the assumptions it is obvious that the map

$$\{\mathcal{L}, \mathcal{B}\} : V^{l+\vec{t}}_{p,\beta}(G) \to V^{l-\vec{s}}_{p,\beta}(G) \times V^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G)\,, \tag{3.19}$$

is continuous and from Theorem 2.1 we deduce that if $\beta < -np^{-1}$ or $\beta > l + t_{\max} - np^{-1}$, $t_{\max} = \max_{j=1,\dots,k} t_j$, the same regularity holds true by replacing V with W. In addition, Theorem 4.2 in [10] shows that if there exists some $\nu = 0, 1, \dots, l + t_{\max} - 1$ such that $\nu - np^{-1} < \beta < \nu + 1 - np^{-1}$ then the map $\{\mathcal{L}, \mathcal{B}\} : W^{l+\vec{t}}_{p,\beta}(G) \to W^{l-\vec{s}}_{p,\beta}(G) \times W^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G)$ still remains continuous provided that $\{\mathcal{L}, \mathcal{B}\}(\Pi_{l+\vec{t}-\nu-1}) \subset W^{l-\vec{s}}_{p,\beta}(G) \times W^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G)$, where $\Pi_{l+\vec{t}-\nu-1} = \prod_{j=1}^{k} \Pi_{l+t_j-\nu-1}$. We can now state the following result corresponding to Theorem 4.3 in [10] and to which we refer the reader for the proof.

Theorem 3.2. *If $\beta \notin [-np^{-1}, l + t_{\max} - np^{-1}]$ or if there exists $\nu = 0, 1, \dots, l + t_{\max} - 1$ such that $\nu - np^{-1} < \beta < \nu + 1 - np^{-1}$ then for $p \in (1, +\infty)$ the operator (3.19) is Fredholm if and only if the line $\mathrm{Im} z = \beta - l + np^{-1}$ contains no poles of the operator $\mathcal{U}(0, \omega, z, D_\omega)^{-1}$ which are the eigenvalues of $\mathcal{U}(0, \omega, z, D_\omega)$. Under this condition, for any vector-valued function $w \in W^{l+\vec{t}}_{p,\beta}(G)$ the following estimate holds*

$$\|w\|_{W^{l+\vec{t}}_{p,\beta}(G)} \le c_1 \big\{ \|\mathcal{L}w\|_{W^{l-\vec{s}}_{p,\beta}(G)} + \|\mathcal{B}w\|_{W^{l-\vec{\sigma}-p^{-1}}_{p,\beta}(\partial G)} + \|w\|_{W^{l-1+\vec{t}}_{p,\beta}(G)} \big\}. \tag{3.20}$$

In the next, Theorem 3.2 will be applied to the case in which $\mathcal{L}$ and $\mathcal{B}$ are single and not matrix differential operators. Therefore, from now on the parameters appearing from formula (3.15) onward will be assumed to be the following:

$$k = m = 1\,, \quad s_1 = 0\,, \quad t_1 = 2 \quad \sigma_1 = -1\,, \quad l = 0\,. \tag{3.21}$$

4 Basic assumptions and main result

With all the necessary background introduced in the previous sections, here we will be finally able to state our *a priori* estimate for a solution $u \in W^2_{p,-1}(G)$ to the boundary value problem

$$\begin{cases} \lambda u(x) - \mathcal{A}(x; D_x)u(x) = f(x)\,, & x \in G\,, \\ \mathcal{B}(x; D_x)u(x) = g(x)\,, & x \in \partial G\,, \end{cases} \tag{4.1}$$

where $f \in L_{p,-1}(G)$, $g \in W^{1-p^{-1}}_{p,-1}(\partial G)$ and $\lambda \in \mathbf{C}$.
However, to state the main result, some basic assumptions on the domain G and on the differential operators $\mathcal{A}(x; D_x)$ and $\mathcal{B}(x; D_x)$ are needed. We are going to list them.
Let C_j, $j = 1, 2, 3$, be three positive constants such that, denoted with ϕ the angle $\arctan[(C_1)^{-1}] \in (0, \pi/2)$, they satisfy $C_2 > \sin\phi$ and $C_3 > C_1$ and let $\eta : [0, C_2] \to \mathbf{R}$ be a function of class C^2 satisfying the following properties:

i) $\eta(y) = C_1 y$, if $y \in [0, \sin\phi]$;

ii) $0 < \eta(y) < C_3 y$, if $y \in (\sin\phi, C_2)$;

iii) $\eta(C_2) = 0$, $\eta'(C_2) = -\infty$.

Having such a η, for the rest of the paper with G we will denote the domain

$$G = \big\{x \in \mathbf{R}^n : \ |x'| < \eta(x_n),\ x' = (x_1, \ldots, x_{n-1}),\ 0 < x_n < C_2\big\}. \tag{4.2}$$

When $n \geq 2$, due to i), $G \cap B(0,1)$ coincides with the cone $\{x \in \mathbf{R}^n : |x'| < C_1 x_n,\ x_n > 0\}$, whereas, when $n = 1$, we have $x' = 0$ and G simply coincides with the interval $(0, C_2)$.

Remark 4.1. To clarify the meaning of the constant C_1 and of the choice of the interval $[0, \sin\phi]$ in the assumption i) on η, observe first what happens when $n = 2$. In this case $G \cap B(0,1)$ coincides with the cone $\{(x_1, x_2) \in \mathbf{R}^2 : |x_1| < C_1 x_2, x_2 > 0\}$ and therefore, using polar co-ordinates in $\mathbf{R}^2$ (set $\theta_0 = \pi/2$ and $n = 2$ in formulae (3.4)), we deduce that for any $x \in G \cap B(0,1)$ the angle θ_1 belongs to $(\phi, \pi - \phi)$. Hence if $x \in \partial G \cap \overline{B(0,1)}$ then $x_2 \in [0, \sin\phi]$. Generalizing to the case $n > 2$, by setting $\theta_0 = \pi/2$ in formulae (3.4) we deduce that for any $x \in G \cap B(0,1)$ all the angles θ_i, $i = 1, \ldots, n-1$, belong to the same interval $(\phi, \pi - \phi)$.

Now, $\mathcal{A}(x; D_x)$ being defined by (1.1), we assume

$$a_{i,j} \in C^1(\overline{G}; \mathbf{C}), \quad a_{i,j} = a_{j,i}, \quad i, j = 1, \ldots, n, \tag{4.3}$$

$$\mathrm{Re} \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq C_0|\xi|^2, \quad \forall\,(x, \xi) \in (\overline{G}\backslash\{O\}) \times \mathbf{R}^n \text{ and some } C_0 > 0. \tag{4.4}$$

As it is well-know, if $n \geq 2$ then assumption (4.4) implies the following:

$$\begin{cases} \textit{for any } x \in \overline{G} \textit{ and any linearly independent vectors } \xi, \zeta \in \mathbf{R}^n \\ \textit{the polynomial } \tau \to \mathcal{A}(x; \xi + \tau\zeta) = \sum_{i,j=1}^{n} a_{i,j}(x)(\xi_i + \tau\zeta_i)(\xi_j + \tau\zeta_j) \\ \textit{has a unique root with positive imaginary part.} \end{cases} \tag{4.5}$$

With $\mathcal{A}(x; D_x)$ we associate the boundary operator

$$\mathcal{B}(x; D_x) = \sum_{i=1}^{n} b_i(x) D_{x_i} + b_0(x) I, \quad x \in \partial G, \tag{4.6}$$

where, $\mathcal{V}$ being an open neighborhood of $\overline{G}$, we have

$$b_j \in C^1(\mathcal{V}; \mathbf{R}), \qquad j = 0, 1, \ldots, n. \tag{4.7}$$

Moreover, if $n \geq 2$ we assume that the b_j's satisfy also the following two requirements:

$$\begin{cases} |\sum_{i=1}^{n-1} b_i(x'\cos\gamma, x_n) v_i(x)\cos\gamma + b_n(x'\cos\gamma, x_n) v_n(x) + |v'(x)|^2 \sin^2\gamma| \geq m > 0, \\ \textit{for any } \gamma \in [0, 2\pi] \textit{ and any } x \in \partial G\backslash\{O\}, \\ v(x) = (v'(x), v_n(x)) \textit{ being the outer normal to } \partial G\backslash\{O\} \textit{ at } x, \end{cases} \tag{4.8}$$

$$\begin{cases} for\ any\ x \in \partial G\backslash\{O\}\ and\ for\ any\ \xi,\zeta \in \mathbf{R}^n, repectively\ tangent\ and\ normal \\ to\ \partial G\backslash\{O\}\ at\ x, the\ polynomial\ \mathcal{B}(x;\xi+\tau\zeta) = \sum_{i=1}^n b_i(x)(\xi_i+\tau\zeta_i)\ is\ not \\ divisible\ by\ (\tau-\tau^+(x,\xi,\zeta))\ without\ remainder,\ \tau^+(x,\xi,\zeta)\ being\ the\ unique \\ root\ with\ positive\ imaginary\ part\ of\ the\ polynomial\ \mathcal{A}(x;\xi+\tau\zeta)\ in\ (4.5)\,. \end{cases} \tag{4.9}$$

Observe that (4.8) is well defined by virtue of (4.7) since if $x \in \partial G\backslash\{O\}$ then $(x'\cos\gamma, x_n)$, $\gamma \in [0, 2\pi]$, belongs to G. Assumption (4.8) can be considered as an improvement of the standard assumption for the coefficients of $\mathcal{B}(x; D_x)$, corresponding to $\gamma = 0$ in (4.8). In Section 5 we will exhibit a concrete class of functions b_j, $j = 1, \ldots, n$, satisfying (4.8). Instead, in the case $n = 1$ we assume

$$b_1(x_1) \neq [\eta'(x_1)]^{-1}\eta(x_1), \qquad \text{for every}\, x_1 \in (0, C_2] \text{ such that } \eta'(x_1) \neq 0\,, \tag{4.10}$$

with the convention that when $x_1 = C_2$ then (4.10) should be understood as $b_1(C_2) \neq 0$. In accordance with Definition 1.5 on p. 113 in [7], assumption (4.9) means that $\mathcal{B}(x; D_x)$ covers $\mathcal{A}(x; D_x)$ on $\partial G\backslash\{O\}$. We will need the following preliminary result.

Lemma 4.2. *Let G be the domain defined by (4.2) and let $\mathcal{A}(x; D_x)$ and $\mathcal{B}(x; D_x)$ be the differential operators defined respectively by (1.1) and (4.6) and assume that the coefficients of $\mathcal{A}(x; D_x)$ satisfy (4.3), (4.4) whereas the coefficients of $\mathcal{B}(x; D_x)$ satisfy (4.7)–(4.10). Denote with (x_0, x), $x = (x', x_n)$, the points of $\mathbf{R}^{n+1}$ and with $\widetilde{G}$ the domain*

$$\widetilde{G} = \big\{(x_0, x) \in \mathbf{R}^{n+1} : |(x_0, x')| < \eta(x_n)\,,\ 0 < x_n < C_2\big\}. \tag{4.11}$$

Then $\mathcal{B}((x_0, x); D_x, D_{x_0}) = \mathcal{B}(x; D_x) + x_0 D_{x_0}$ covers $\mathcal{A}_\psi(x; D_x, D_{x_0}) = \mathcal{A}(x; D_x) + e^{i\psi} D_{x_0}^2$, $\psi \in [-\pi/2, \pi/2]$, on $\partial\widetilde{G}\backslash\{O\}$.

The proof Lemma 4.2 will be given in Section 5. Here we make only two easy remarks.

Remark 4.3. Observe that if $n = 2$ then $\widetilde{G}$ is a 3–dimensional domain generated by a rotation of G around the x_2–axis. In this sense, when $n \geq 2$, the $(n + 1)$–dimensional domain $\widetilde{G}$ can always be viewed as a rotation of the n–dimensional domain G around the x_n–axis. In addition, in the critical case $n = 1$, definition (4.11) ensures that $\widetilde{G}$ is 2–dimensional domain, symmetric with respect to the x_1–axis and coinciding with a cone near the origin. This will be important in the following, since we will use Theorem 3.2 in dimension $n + 1$ and therefore we will need to consider $(n + 1)$–dimensional domain having the properties for which that theorem is true.

Remark 4.4. Under our assumptions on $\mathcal{A}(x; D_x)$ the operator $\mathcal{A}_\psi(x; D_x, D_{x_0})$, $\psi \in [-\pi/2, \pi/2]$, does not necessarily satisfy (4.4), but only the weaker ellipticity condition

$$\Big|\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j + e^{i\psi}\xi_0^2\Big| \geq 2^{-1/2}\min\{C_0, 1\}|\xi|^2 \qquad \forall\,(x, \xi) \in (\overline{G}\backslash\{O\}) \times \mathbf{R}^{n+1}. \tag{4.12}$$

However, if $n \geq 2$ then $n + 1 \geq 3$ and it is well-known that in this case (4.12) implies (4.5). If $n = 1$, by computing explicitly the roots of the polynomial $\mathcal{A}_\psi(x_1; \xi + \tau\zeta)$ for the operator $\mathcal{A}_\psi(x_1; D_{x_1}, D_{x_0})$, it can be checked that (4.5) is satisfied, too. Indeed, by virtue of (4.4), when $n = 1$ we may for simplicity assume $a_{1,1}$ to be real and positive and hence, given two linearly independent vectors $\xi = (\xi_0, \xi_1), \zeta = (\zeta_0, \zeta_1) \in \mathbf{R}^2$, it follows that the polynomial $\mathcal{A}_\psi(x_1; \xi + \tau\zeta) = a_{1,1}(x_1)(\xi_1 + \tau\zeta_1)^2 + e^{i\psi}(\xi_0 + \tau\zeta_0)^2$ has the roots:

$$\begin{aligned}\tau_\pm(x_1, \xi, \zeta) = {} & [\zeta_0^4 + (a_{1,1}(x_1))^2\zeta_1^4 + 2a_{1,1}(x_1)\zeta_0^2\zeta_1^2\cos\psi]^{-1} \\ & \times\Big\{ -\xi_0\zeta_0^3 - (a_{1,1}(x_1))^2\xi_1\zeta_1^3 - a_{1,1}(x_1)\zeta_0\zeta_1 e^{-i\psi}[\xi_0\zeta_1 e^{2i\psi} + \xi_1\zeta_0] \\ & \pm i\,[a_{1,1}(x_1)\xi_0\zeta_1 - \xi_1\zeta_0]e^{-i\psi/2}[a_{1,1}(x_1)\zeta_1^2 e^{i\psi} + \zeta_0^2]\Big\}. \end{aligned} \tag{4.13}$$

Now, triplet $\{\mathcal{A}_\psi(x; D_x, D_{x_0}), \mathcal{B}((x_0, x); D_x, D_{x_0}); \widetilde{G}\}$ being defined as in Lemma 4.2, with problem (4.1) we associate the following boundary value problem, where $\mathcal{F} \in L_{p,-1}(\widetilde{G})$ and $\mathcal{G} \in W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})$

$$\begin{cases} \mathcal{A}_\psi(x; D_x, D_{x_0})v(x_0, x) = \mathcal{F}(x_0, x)\,, & (x_0, x) \in \widetilde{G}\,, \\ \mathcal{B}((x_0, x); D_x, D_{x_0})v(x_0, x) = \mathcal{G}(x_0, x)\,, & (x_0, x) \in \partial\widetilde{G}\,, \end{cases} \tag{4.14}$$

As shown in Remark 3.1 when the coefficients $a_{i,j}$, $i, j = 1, \ldots, n$, of $\mathcal{A}(x; D_x)$ satisfy (4.3) then $\mathcal{A}_\psi(x; D_x, D_{x_0})$ belongs to the class $\mathfrak{C}(2, 0)$. Moreover, when the coefficients b_h, $h = 0, 1, \ldots, n$, of $\mathcal{B}(x; D_x)$ satisfy (4.7) then, using formulae (3.4), (3.5) and (3.14), it is easy to see that $\mathcal{B}((x_0, x); D_x, D_{x_0})$ belongs to $\mathfrak{C}(1.1)$. In addition formula (4.12) shows that $\mathcal{A}_\psi(x; D_x, D_{x_0})$ is uniformly elliptic in $\widetilde{G}\backslash\{O\}$, whereas Lemma 4.2 establishes that the boundary conditions $\mathcal{B}((x_0, x); D_x, D_{x_0})$ covers $\mathcal{A}_\psi(x; D_x, D_{x_0})$ on $\partial\widetilde{G}\backslash\{O\}$.
Hence, denoted with Ω the intersection $\widetilde{G} \cap \partial B(0, 1)$, problem (4.14) generates a model problem in the cone $(0, +\infty) \times \Omega$. Indeed, θ_i, $i = 0, \ldots, n-1$, being defined by (3.4), with the pair $\{\mathcal{A}_\psi(x; D_x, D_{x_0}), \mathcal{B}((x_0, x); D_x, D_{x_0})\}$ we associate the operator

$$\mathcal{U}(0, \omega, z, D_\omega) = \{A_\psi(0, \omega, z - 2i, D_\omega), B(0, \omega, z - 2i, D_\omega)\}\,, \tag{4.15}$$

where $\omega = (\theta_0, \ldots, \theta_{n-1})$, $D_\omega = (\theta_0, \ldots, \theta_{n-1})$, and the operators A_ψ and B are determined from $\mathcal{A}_\psi(x; D_x, D_{x_0})$ and $\mathcal{B}((x_0, x); D_x, D_{x_0})$ by means of (3.1) replacing the coefficients $p_{h,\alpha}(r, \omega)$ with $p_{h,\alpha}(0, \omega)$. Since problem (4.14) is uniformly elliptic in $\widetilde{G}\backslash\{O\}$ and the boundary condition covers $\mathcal{A}_\psi(x; D_x, D_{x_0})$ on $\partial\widetilde{G}\backslash\{O\}$ then the operator $\mathcal{U}(0, \omega, z, D_\omega)$ is elliptic with complex parameter in the sense of [6]. Therefore, with the choice of the parameter as in (3.21) and G replaced by $\widetilde{G}$, all the assumptions on problem (3.15) which are necessary in order to state Theorem 3.2 are satisfied even for problem (4.14).
We now give the following definition, arising from the necessity to use Theorem 3.2 in dimension $n + 1$, with $\beta = -1$ and $l = 0$.

Definition 4.5. *The boundary value problem (4.14) will be said* regular *if the line* $\operatorname{Im} z = -1 + (n+1)p^{-1}$ *contains no eigenvalue of the operator* $\mathcal{U}(0, \omega, z, D_\omega)$ *defined by (4.15). In this case the triplet* $\{\mathcal{A}(x; D_x), \mathcal{B}(x; D_x); G\}$ *will be said the restriction to the* x *variable of the regular boundary value problem (4.14) in the domain* $\widetilde{G}$ *related to* G *by (4.11).*

Definition 4.5 can be considered as the equivalent, in the setting of weighted Sobolev function spaces, of Definition 6.2 in [5]. In this sense our results are in accordance with those proven in [5] for the subclass of problems consisting in the restriction to the x variable of regular elliptic problems in one more variable. Two simple examples for the Definition 4.5 are those given on p. 45 in [8] and p. 86 in [9] and related to the homogeneous Dirichlet boundary conditions.

We can finally state our main result.

Theorem 4.6. *Let $p > n$, $p \neq n+1$, and let the triplet $\{\mathcal{A}(x; D_x), \mathcal{B}(x; D_x); G\}$ be the restriction to the x variable of the regular boundary value problem (4.14) in the sense of Definition 4.5. Then, $g_0 \in W^1_{p,-1}(G)$ being any extension to G of $\mathcal{B}(x; D_x)u$, there exists $\omega > 0$ such that if $\mathrm{Re}\lambda \geq \omega$ for every $u \in W^2_{p,-1}(G)$ the following estimate hold*

$$
\begin{aligned}
&|\lambda|\|u\|_{L_{p,-1}(G)} + |\lambda|^{1/2}\|Du\|_{L_{p,-1}(G)} + \|D^2u\|_{L_{p,-1}(G)} \\
&\leq M\big\{\|\big(\lambda I - \mathcal{A}(x; D_x)\big)u\|_{L_{p,-1}(G)} + (1+|\lambda|^{1/2})\|g_0\|_{L_{p,-1}(G)} + \|Dg_0\|_{L_{p,-1}(G)}\big\}. \quad (4.16)
\end{aligned}
$$

The positive constant M in (4.16) depends only on p, n, the $C^1(G)$-norm of the coefficients of $\mathcal{A}(x; D_x)$ and of $\mathcal{B}(x; D_x)$ and the constants C_j, $j = 2, 3$, intervening in the properties i)–iii) for the function η which describes the boundary ∂G of G.

As announced in the Introduction, the proof of Theorem 4.6 will be given in Section 7. Here, instead, we want to show some easy consequence of estimate (4.16). We set

$$
\begin{cases}
\mathcal{D}(A) = \{u \in W^2_{p,-1}(G) : \ \mathcal{B}(x; D_x)u = 0 \ \text{ in } \partial G\}, \\
Au = \mathcal{A}(x; D_x)u, \qquad \forall\, u \in \mathcal{D}(A).
\end{cases} \quad (4.17)
$$

A is said the realization of $\mathcal{A}(x; D_x)$ in $L_{p,-1}(G)$ with homogeneous boundary condition.

Corollary 4.7. *Let assumptions of Theorem 4.6 be fulfilled and let the pair $(A, \mathcal{D}(A))$ be defined by (4.17). There exists $\omega > 0$ such that if $\mathrm{Re}\lambda \geq \omega$ then the operator $\lambda I - A$ is closed and injective in $L_{p,-1}(G)$. As a consequence, A is closed in $L_{p,-1}(G)$.*

Proof. By taking as g_0 the null function, the injectivity of $\lambda I - A$, $\mathrm{Re}\lambda \geq \omega$, trivially follows from (4.16). Now, let $\{u_n\}_{n\in\mathbf{N}} \subset \mathcal{D}(A)$ such that $u_n \to u$ in $L_{p,-1}(G)$ and $(\lambda I - A)u_n \to v$ in $L_{p,-1}(G)$. If we set $g_0 = 0$, from (4.16) it clearly follows that $\{u_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $W^2_{p,-1}(G)$ and hence $u \in W^2_{p,-1}(G)$. Moreover, due to assumptions (4.3), (4.7) on the coefficients of $\mathcal{A}(x; D_x)$ and $\mathcal{B}(x; D_x)$, respectively, it is easy to deduce $(\lambda I - A)u_n \to (\lambda I - A)u$ in $L_{p,-1}(G)$ and $0 = \mathcal{B}(x; D_x)u_n \to \mathcal{B}(x; D_x)u$ in $W^{1-p^{-1}}_{p,-1}(\partial G)$. Therefore $u \in \mathcal{D}(A)$ and $(\lambda I - A)u = v$, i.e. $\lambda I - A$ is closed in $L_{p,-1}(G)$.
The last assertion follows from $A = \lambda I - (\lambda I - A)$. □

As a further corollary of Theorem 4.6 we show that, if a solution $u \in W^2_{p,-1}(G)$ of problem (4.1) exists, then the operator A defined via (4.17) is sectorial, in accordance with the Definition 2.0.1 in [11] which we report for reader's convenience.

Definition 4.8. *Let X be a complex Banach space, with norm $\|\cdot\|$. A linear operator $B : \mathcal{D}(B) \subset X \to X$ is said to be sectorial if there are constants $\omega_1 \in \mathbf{R}$, $\vartheta \in (\pi/2, \pi)$, $C > 0$ such that, denoted with $\rho(B)$ the resolvent set of B, the following hold:*
(i) $\rho(B) \supset S_{\vartheta,\omega_1} = \{z \in \mathbf{C} : \ z \neq \omega_1, |\arg(z - \omega_1)| < \vartheta\}$,
(ii) $\|(zI - B)^{-1}\|_{\mathcal{L}(X)} \leq C|z - \omega_1|^{-1}, \quad \forall\, z \in S_{\vartheta,\omega_1}$.

We recall also the following sufficient condition for an operator to be sectorial and for the proof of which we refer to [11, Proposition 2.1.11].

Proposition 4.9. *Let $\omega_1 \in \mathbf{R}$ and let $B : \mathcal{D}(B) \subset X \to X$ be a linear operator such that $\rho(B) \supset S_{\omega_1} = \{z \in \mathbf{C} : \ \mathrm{Re} z \geq \omega_1\}$ and $\|(zI - B)^{-1}\|_{\mathcal{L}(X)} \leq C|z|^{-1}$ for any $z \in S_{\omega_1}$ and some $C > 0$. Then B is sectorial.*

Consequently, we have the following corollary.

Corollary 4.10. *Let assumptions of Theorem 4.6 be fulfilled and let us suppose that for any pair $(f, g) \in L_{p,-1}(G) \times W^{1-p^{-1}}_{p,-1}(\partial G)$ there exists a solution $u \in W^2_{p,-1}(G)$ to problem (4.1). Then the operator A defined by (4.17) is sectorial.*

Proof. If for any pair $(f, g) \in L_{p,-1}(G) \times W^{1-p^{-1}}_{p,-1}(\partial G)$ a solution $u \in W^2_{p,-1}(G)$ of problem (4.1) exists, then, when $\mathrm{Re}\lambda \geq \omega$, the solution is unique by virtue of estimate (4.16). Moreover, from (4.16) with g_0 equal to zero, we deduce $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(L_{p,-1}(G))} \leq M|\lambda|^{-1}$ for any $\lambda \in \mathbf{C}$ such that $\mathrm{Re}\lambda \geq \omega$. Hence, the assertion follows from Proposition 4.9. □

Since it is well-known that sectorial operators generate analytic semigroups, we have the following further corollary.

Corollary 4.11. *Under the hypotheses of Corollary 4.10 the realization A of $\mathcal{A}(x; D_x)$ in $L_{p,-1}(G)$ with homogeneous boundary condition generates an analytic semigroup of linear bounded operators $\{\mathcal{T}(t)\}_{t\geq 0} \subset \mathcal{L}(L_{p,-1}(G))$.*

5 Proof of Lemma 4.2

First, accordingly to Remark 4.3, we observe that if $n \geq 2$ and $\widetilde{G}$ is related to G by (4.11) then a very special characterization of the points in $\partial\widetilde{G}\backslash\{O\}$ in terms of those in $\partial G\backslash\{O\}$ can be given. Indeed, when $n \geq 2$ and $(\widetilde{x}_0, \widetilde{x}) \in \partial\widetilde{G}\backslash\{O\}$ (i.e. $|(\widetilde{x}_0, \widetilde{x}')| = \eta(\widetilde{x}_n)$) we set $\alpha = |(\widetilde{x}_0, \widetilde{x}')|^{-1}|\widetilde{x}'|$, $\beta = |(\widetilde{x}_0, \widetilde{x}')|^{-1}\widetilde{x}_0$. Since $\alpha^2 + \beta^2 = 1$ there exists $\varphi \in [0, 2\pi]$ such that $\alpha = |\cos\varphi|$ and $\beta = \sin\varphi$. Let us set $x_i = \widetilde{x}_i/\cos\varphi$, $i = 1, \ldots, n-1$, $x_n = \widetilde{x}_n$. If $\cos\varphi = 0$, i.e. when $(\widetilde{x}_0, \widetilde{x})$ is of the form $(\widetilde{x}_0, 0, \ldots, 0, \widetilde{x}_n)$, we set $x_i = 0$, $i = 1, \ldots, n-1$, $x_n = C_2$. In this way we have defined a point $x \in \partial G\backslash\{O\}$. In fact, $|x'| = |\widetilde{x}'|/|\cos\varphi| = |\widetilde{x}'|/\alpha = |(\widetilde{x}_0, \widetilde{x}')|$ if $\alpha \neq 0$ and $x = (0, \ldots, 0, C_2)$ if $\alpha = 0$. Summing up, if $n \geq 2$, given $(\widetilde{x}_0, \widetilde{x}) \in \partial\widetilde{G}\backslash\{O\}$ there exists an angle $\varphi \in [0, 2\pi]$ such that

$$(\widetilde{x}_0, \widetilde{x}', \widetilde{x}_n) = (|x'|\sin\varphi, x'\cos\varphi, x_n), \quad x = (x', x_n) \in \partial G\backslash\{O\}, \tag{5.1}$$

where boundary points of the form $(\widetilde{x}_0, 0, \ldots, 0, \widetilde{x}_n)$ correspond to the choice $\varphi = \pi/2$ if $\widetilde{x}_0 > 0$ and $\varphi = 3\pi/2$ if $\widetilde{x}_0 < 0$.

Proof of Lemma 4.2. We consider the following two distinct cases: *i)* $n \geq 2$, *ii)* $n = 1$.
i) $n \geq 2$. Let $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}$ to be the function defined by

$$\Phi(y_0, y) = 2^{-1}\{|(y_0, y')|^2 - [\eta(y_n)]^2\}, \quad y_n > 0. \tag{5.2}$$

It follows $\partial\widetilde{G}\backslash\{O\} = \{(y_0, y) \in \mathbf{R}^{n+1} : \Phi(y_0, y) = 0, \, y_n > 0\}$ and hence, since the only point of $\partial\widetilde{G}\backslash\{O\}$ with $x_n = C_2$ is the point $(0, \ldots, 0, C_2)$ with normal $(0, \ldots, 0, 1)$, the normal $\widetilde{\zeta}$ to $\partial\widetilde{G}\backslash\{O\}$ at $(\widetilde{x}_0, \widetilde{x})$ is given by

$$\widetilde{\zeta} = \begin{cases} (\widetilde{x}_0, \widetilde{x}', -\eta(\widetilde{x}_n)\eta'(\widetilde{x}_n)), & \text{if } \ x_n \in (0, C_2), \\ (0, \ldots, 0, 1), & \text{if } \ x_n = C_2. \end{cases} \tag{5.3}$$

From (5.2) it follows also that the normal $v(x)$ to $\partial G\backslash\{O\}$ at x is the vector

$$v(x)=\begin{cases}(x',-\eta(x_n)\eta'(x_n))\,, & \text{if } x_n\in(0,C_2)\,,\\ (0,\dots,0,1)\,, & \text{if } x_n=C_2\,,\end{cases}\tag{5.4}$$

and, since $v'(x)=x'$, we see that (5.3) can be rewritten in the more compact way

$$\widetilde{\zeta}=(|v'(x)|\sin\varphi, v'(x)\cos\varphi, v_n(x))\,,\tag{5.5}$$

where $v(x)$ defined by (5.4) is the normal at the point x such that (5.1) holds.
It remains to characterize the tangent vectors $\widetilde{\xi}$ to $\partial\widetilde{G}\backslash\{O\}$. Taking advantage from (5.4), (5.5) it is not too difficult to show that any vector $\widetilde{\xi}$ tangent to $\partial\widetilde{G}\backslash\{O\}$ at $(\widetilde{x}_0,\widetilde{x})$ has one of the following three representations

$$\begin{cases}\big(a\cos\varphi+c|v'(x)|\sin\varphi, by'+cv'(x)\cos\varphi-a\,\dfrac{v'(x)}{|v'(x)|}\sin\varphi, c\,\dfrac{\eta(x_n)}{\eta'(x_n)}\big), & \text{if } \eta'(x_n)\neq 0,\\ \big(a\cos\varphi, by'-a\,\dfrac{v'(x)}{|v'(x)|}\sin\varphi, c\big), & \text{if } \eta'(x_n)=0 \text{ and } \varphi\notin\{\pi/2,3\pi/2\}\\ (0,y),\ y\in\mathbf{R}^n,\ |y|\neq 0, & \text{if } \eta'(x_n)=0 \text{ and } \varphi\in\{\pi/2,3\pi/2\}\end{cases}\tag{5.6}$$

where $y'\in\mathbf{R}^{n-1}$, $|y'|\neq 0$, satisfies $y'\cdot x'=0$ if $x_n\neq C_2$ or $\varphi\notin\{\pi/2,3\pi/2\}$, a,b,c are real numbers not all equal to zero and $x\in\partial G\backslash\{O\}$ is the point in (5.1).
Now, let $(\widetilde{x}_0,\widetilde{x})\in\partial\widetilde{G}\backslash\{O\}$ with $\widetilde{x}_0\neq 0$ and assume condition (4.9) is violated for the operator $\mathcal{B}((\widetilde{x}_0,\widetilde{x});D_x,D_{x_0})$. Hence, denoted with $\widetilde{\tau}^+((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})$ the unique root with positive imaginary part of the polynomial $\mathcal{A}_\psi(\widetilde{x},\widetilde{\xi}+\tau\widetilde{\zeta})$ we have, for any $\tau\in\mathbf{C}$

$$\sum_{i=1}^n b_i(\widetilde{x})(\widetilde{\xi}_i+\tau\widetilde{\zeta}_i)+\widetilde{x}_0(\widetilde{\xi}_0+\tau\widetilde{\zeta}_0)=\chi((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})[\tau-\widetilde{\tau}^+((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})]\,.\tag{5.7}$$

From (5.6) we deduce that there are three different situations to take into examination.
1) *Case* $\eta'(x_n)\neq 0$. In this case from (5.1), (5.5)–(5.7) and assumption (4.8) we easily deduce

$$\begin{aligned}&\sum_{i=1}^{n-1} b_i(x'\cos\varphi,x_n)v_i(x)\cos\varphi+b_n(x'\cos\varphi,x_n)v_n(x)+|v'(x)|^2\sin^2\gamma\\ =\ &\chi((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})\neq 0\,,\end{aligned}\tag{5.8}$$

$$\begin{aligned}&\Big\{\sum_{i=1}^{n-1} b_i(x'\cos\varphi,x_n)[by_i'+cv_i(x)\cos\varphi-a|v'(x)|^{-1}v_i(x)\sin\varphi]\\ &\quad+c\,b_n(x'\cos\varphi,x_n)[\eta'(x_n)]^{-1}\eta(x_n)+|v'(x)|\big(a\cos\varphi+c|v'(x)|\sin\varphi\big)\sin\varphi\Big\}\\ =\ &-\chi((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})\,\widetilde{\tau}^+((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})\end{aligned}\tag{5.9}$$

From (5.8), (5.9) and the fact that b_j, $j=1,\dots,n$, assume only real values (cf. (4.7)) we get a contradiction since $\operatorname{Im}\tau^+((\widetilde{x}_0,\widetilde{x}),\widetilde{\xi},\widetilde{\zeta})>0$.
2) *Case* $\eta'(x_n)=0$, $\varphi\notin\{\pi/2,3\pi/2\}$. From (5.4) we see that in this case we have $v_n(x)=0$

and hence from (5.8) the contradiction follows as in the case before using assumption (4.8) and changing the left-hand side of (5.9) in accordance with (5.6).
3) *Case* $\eta'(x_n) = 0$, $\varphi \in \{\pi/2, 3\pi/2\}$. From (5.1), (5.3) and (5.6) we obtain that $\widetilde{\zeta}$ and $\widetilde{\xi}$ are respectively of the form $(\widetilde{\zeta}_0, 0, \dots, 0)$ and $(0, y)$ with $\widetilde{\zeta}_0 \neq 0$ and $y \in \mathbf{R}^n$, $|y| \neq 0$. However, due to the assumptions on η it follows $x_n \in (0, C_2)$ and hence, since $(\widetilde{x}_0, 0, \dots, 0, x_n) \in \partial\widetilde{G}\backslash\{O\}$, we have $\widetilde{x}_0 = \eta(x_n) \neq 0$. Therefore, from (5.7) we get

$$\widetilde{x}_0\widetilde{\zeta}_0 = \chi((\widetilde{x}_0, 0, \dots, 0, x_n), \widetilde{\xi}, \widetilde{\zeta}) \neq 0\,, \tag{5.10}$$

$$\sum_{i=1}^{n} b_i(0, \dots, 0, x_n)y_i = -\chi((\widetilde{x}_0, 0, \dots, 0, x_n), \widetilde{\xi}, \widetilde{\zeta})\, \widetilde{\tau}^+((\widetilde{x}_0, 0, \dots, 0, x_n), \widetilde{\xi}, \widetilde{\zeta})\,, \tag{5.11}$$

and again the contradiction follows from the fact that the b_i's assume only real values whereas $\operatorname{Im}\tau^+((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta}) > 0$.
Contradictions we get in 1)–3) mean that the assumption that condition (4.9) was violated for $\mathcal{B}((\widetilde{x}_0, \widetilde{x}); D_x, D_{x_0})$ was wrong and so, if $n \geq 2$, the proof is complete.
ii) $n = 1$. In this case, since $x' = 0$, no relationship of type (5.1) is possible and we can not reason as before. However, since the points of $\partial\widetilde{G}\backslash\{O\}$ are the points $(\widetilde{x}_0, \widetilde{x}_1) = (\pm\eta(x_1), x_1)$, from (5.2) with $n = 1$ we deduce that the normal $\widetilde{\zeta}$ and the tangent $\widetilde{\xi}$ to $\partial\widetilde{G}\backslash\{O\}$ at $(\pm\eta(x_1), x_1)$ have the following form

$$\widetilde{\zeta} = \begin{cases} (\pm\eta(x_1), -\eta(x_1)\eta'(x_1))\,, & \text{if } \ x_1 \in (0, C_2)\,, \\ (0, 1)\,, & \text{if } \ x_1 = C_2\,, \end{cases} \tag{5.12}$$

$$\widetilde{\xi} = \begin{cases} (\pm\eta(x_1), [\eta'(x_1)]^{-1}\eta(x_1))\,, & \text{if } \ \eta'(x_1) \neq 0\,, \text{ and } \ x_1 \in (0, C_2)\,, \\ (0, 1)\,, & \text{if } \ \eta'(x_1) = 0\,, \\ (1, 0)\,, & \text{if } \ x_1 = C_2\,. \end{cases} \tag{5.13}$$

Now, assume that (4.9) does not hold for the operator $\mathcal{B}((\widetilde{x}_0, \widetilde{x}); D_x, D_{x_0})$. Hence, denoted with $\widetilde{\tau}^+((\widetilde{x}_0, \widetilde{x}_1), \widetilde{\xi}, \widetilde{\zeta})$ the unique root with positive imaginary part of the polynomial $\mathcal{A}_\psi(\widetilde{x}, \widetilde{\xi} + \tau\widetilde{\zeta})$ (cf. (4.13)) we have that (5.7) reduces to

$$b_1(x_1)(\widetilde{\xi}_1 + \tau\widetilde{\zeta}_1) + \widetilde{x}_0(\widetilde{\xi}_0 + \tau\widetilde{\zeta}_0) = \chi((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta})[\tau - \widetilde{\tau}^+((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta})]\,. \tag{5.14}$$

From (5.12), (5.13) we deduce that only two situations have to be examined.
1) *Case* $x_1 \in (0, C_2)$. If $\eta'(x_1) \neq 0$, from (5.12)–(5.14) and assumption (4.10) we find

$$-b_1(x_1)\eta(x_1)\eta'(x_1) + [\eta(x_1)]^2 = \chi((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta}) \neq 0 \tag{5.15}$$

$$b_1(x_1)[\eta'(x_1)]^{-1}\eta(x_1) + [\eta(x_1)]^2 = -\chi((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta})\widetilde{\tau}^+((\widetilde{x}_0, \widetilde{x}), \widetilde{\xi}, \widetilde{\zeta}) \tag{5.16}$$

which is a contradiction since on the left-hand side of (5.16) we have a real value whereas on the right-hand side we have a complex number with positive imaginary part. It is easy to observe that if $\eta'(x_1) = 0$ we still get a contradiction. Indeed, due to the fact that $x_1 \in (0, C_2)$, on the left-hand side of (5.15) we have $[\eta(x_1)]^2 \neq 0$ whereas (cf. (5.13)) on the left-hand side of (5.16) we have only the real value $b_1(x_1)$.
2) *Case* $x_1 = C_2$. Since $\eta(C_2) = 0$, from (5.12)–(5.14) we find

$$b_1(C_2)\tau = \chi((0, C_2), \widetilde{\xi}, \widetilde{\zeta})[\tau - \widetilde{\tau}^+((0, C_2), \widetilde{\xi}, \widetilde{\zeta})]\,,$$

which is a contradiction due to assumption (4.10). Hence, also in the case $n = 1$ we are done and the proof of Lemma 4.2 is complete.

Remark 5.1. With the help of (5.4) we present here a class of coefficients b_j, $j = 1, \dots, n$, which satisfy assumption (4.8). To this purpose, for any $x \in \overline{G}$ let set

$$b_j(x) = x_j\,, \quad j = 1, \dots, n-1, \qquad b_n(x) \neq [\eta'(x_n)]^{-1}\eta(x_n)\,, \quad \text{if } \eta'(x_n) \neq 0. \tag{5.17}$$

Since from (5.4) it follows $v'(x) = x'$ for any $x \in \partial G \backslash \{O\}$, with the coefficients defined by (5.17) and using $|x'| = \eta(x_n)$ we see that (4.8) is equivalent to require

$$\begin{aligned}&[\eta(x_n)]^2 - b_n(x'\cos\gamma, x_n)\eta(x_n)\eta'(x_n) \neq 0 \qquad \text{if } x_n \in (0, C_2)\,,\\ &-b_n(0, \dots, 0, C_2) \neq 0 \qquad \text{if } x_n \in (0, C_2)\,.\end{aligned}$$

Therefore, with the convention that the assumption on b_n in (5.17) should be intended as $b_n(x) \neq 0$ if $x_n = C_2$ (i.e. when $\eta'(x_n) = -\infty$), the previous two inequality are both satisfied even in the case $\eta'(x_n) = 0$ since in this case we have $x_n \in (0, C_2)$ and hence $\eta(x_n) \neq 0$. Observe also that in the case $n = 1$ then (5.17) corresponds to (4.10).

Remark 5.2. From (5.10) and (5.11) we see that in the case $n > 1$ Lemma 4.2 fails if instead of $\mathcal{B}((x_0, x); D_x, D_{x_0})$ we consider only the operator $\mathcal{B}(x; D_x)$. In the case $n = 2$, using complex valued coefficients b_j, $j = 1, 2$, there could be still the possibility to conclude the proof considering only $\mathcal{B}(x; D_x)$, but surely no if $n \geq 3$.

6 The cut-off function

The procedure we will perform in Section 7 to prove estimate (4.16) for a function $u \in W^2_{p,-1}(G)$, G being defined by (4.2), requires the implementing of a function v depending on $n + 1$ variables and having the following form

$$v(x_0, x) = \varkappa(x_0, x)e^{i\rho x_0}u(x)\,, \qquad (x_0, x) \in \widetilde{G}\,, \tag{6.1}$$

where $\rho > 0$, $\widetilde{G}$ is related to G by (4.11) and $\varkappa$ is an infinitely differentiable function having compact support on $\widetilde{G}$.
Functions of type (6.1), with the aim of proving estimates for the function u, are used, for instance, in [1], [5] and [11]. However, in that papers the domain $\widetilde{G}$ always consists in the infinite "cylinder" $\Gamma = (-\infty, +\infty) \times G$ and this, as remarked in the Introduction, allows the authors to use cut-off functions $\varkappa$ depending only on x_0.
In our case the situation is really different, since when $\widetilde{G}$ is related to G by (4.11) then (x_0, x) do not belongs to $\widetilde{G}$ if $|x_0| > \{[\eta(x_n)]^2 - |x'|^2\}^{1/2}$. By recalling definition of ϕ before the definition (4.2) of G, the right choice of function $\varkappa$ suitable to our purposes arise from Remark 4.1. Indeed, due to formulae (3.4), if we define $\widetilde{G}$ accordingly to (4.11) then, for any $(x_0, x) \in \widetilde{G}$, the angle θ_0 between the x_0 axis and the vector $|(x_0, x)|$ belongs to the interval $(\phi, \pi - \phi)$. This leads at once to consider the following cut-off function $\varkappa$:

$$\varkappa(x_0, x) := \mathcal{E}\Big(\arccos\frac{x_0}{|(x_0, x)|}\Big)\,, \qquad (x_0, x) \in \widetilde{G}\,, \tag{6.2}$$

where $\mathcal{E} \in C^\infty([0,\pi], \mathbf{R}_+)$, $\|\mathcal{E}\|_{C([0,\pi])} = 1$ and for some $\varepsilon \in (0, (\pi - 2\phi)/6)$ satisfies

$$\mathcal{E}(\varphi) \equiv 0\,, \qquad \forall\, \varphi \in [0, (\pi - 6\varepsilon)/2] \cup [(\pi + 6\varepsilon)/2, \pi]\,, \tag{6.3}$$

$$\mathcal{E}(\varphi) \equiv 1\,, \qquad \forall\, \varphi \in [(\pi - 2\varepsilon)/2, (\pi + 2\varepsilon)/2]\,. \tag{6.4}$$

In particular, by observing that our choice of ε guarantees $\mathcal{E} \equiv 1$ in an open interval containing $\pi/2$ and recalling $G = \{(x_0, x) \in \widetilde{G} : x_0 = 0\}$, we deduce that $\varkappa$ is equal to one on G whereas $\mathcal{E}^{(l)}(\pi/2) = 0$ for any $l \in \mathbf{N}\backslash\{0\}$. Moreover, for any $l \in \mathbf{N} \cup \{0\}$ and $i = 1, \dots, n$ we have

$$D_{x_i}\mathcal{E}^{(l)}\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big) = \frac{x_0 x_i}{|x||(x_0,x)|^2}\,\mathcal{E}^{(l+1)}\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)\,, \tag{6.5}$$

$$D_{x_0}\mathcal{E}^{(l)}\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big) = -\frac{|x|}{|(x_0,x)|^2}\,\mathcal{E}^{(l+1)}\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)\,. \tag{6.6}$$

Hence, when $\varkappa$ is defined by (6.2), from definition (6.1) we derive the following formulae for the first and the second derivatives of v, where $i, j = 1, \dots, n$:

$$D_{x_i}v(x_0,x) = \varkappa(x_0,x)e^{i\rho x_0}D_{x_i}u(x) + [D_{x_i}\varkappa(x_0,x)]e^{i\rho x_0}u(x)\,, \tag{6.7}$$

$$D_{x_0}v(x_0,x) = i\rho v(x_0,x) + [D_{x_0}\varkappa(x_0,x)]e^{i\rho x_0}u(x)\,, \tag{6.8}$$

$$\begin{aligned} D_{x_0}^2 v(x_0,x) = &-\rho^2 v(x_0,x) + \frac{|x|^2}{|(x_0,x)|^4}\,\mathcal{E}''\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)e^{i\rho x_0}u(x) \\ &-2\Big[\frac{x_0 - i\rho|(x_0,x)|^2}{|(x_0,x)|^2}\Big][D_{x_0}\varkappa(x_0,x)]e^{i\rho x_0}u(x)\,, \end{aligned} \tag{6.9}$$

$$\begin{aligned} D_{x_i}D_{x_j}v(x_0,x) = &\ \varkappa(x_0,x)e^{i\rho x_0}D_{x_i}D_{x_j}u(x) + 2[D_{x_i}\varkappa(x_0,x)]e^{i\rho x_0}D_{x_j}u(x) \\ &+\Big[\frac{\delta_{i,j}x_0}{|x||(x_0,x)|^2} - \frac{x_0x_ix_j(x_0^2 + 2|x|^2)}{|x|^3|(x_0,x)|^4}\Big]\mathcal{E}'\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)e^{i\rho x_0}u(x) \\ &+\frac{x_0^2x_ix_j}{|x|^2|(x_0,x)|^4}\,\mathcal{E}''\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)e^{i\rho x_0}u(x)\,, \end{aligned} \tag{6.10}$$

$$\begin{aligned} D_{x_0}D_{x_j}v(x_0,x) = &\ [i\rho\,\varkappa(x_0,x) + D_{x_0}\varkappa(x_0,x)]e^{i\rho x_0}D_{x_j}u(x) \\ &+\Big[\frac{i\rho x_0x_j}{|x||(x_0,x)|^2} + \frac{x_j(|x|^2 - x_0^2)}{|x||(x_0,x)|^4}\Big]\mathcal{E}'\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)e^{i\rho x_0}u(x) \\ &-\frac{x_0x_j|x|}{|x||(x_0,x)|^4}\,\mathcal{E}''\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)e^{i\rho x_0}u(x)\,. \end{aligned} \tag{6.11}$$

In addition, using (6.5), (6.6) with $l = 0$ we deduce, for any $j, k = 1, \dots, n$,

$$\begin{aligned} D_{x_j}D_{x_k}\varkappa(x_0,x) = &\ \frac{x_0^2x_jx_k}{|x|^2|(x_0,x)|^4}\,\mathcal{E}''\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big) \\ &-\frac{\delta_{j,k}x_0D_{x_0}\varkappa(x_0,x)}{|x|^2} - \frac{x_k(x_0^2 + 3|x|^2)D_{x_j}\varkappa(x_0,x)}{|x|^2|(x_0,x)|^2}\,. \end{aligned} \tag{6.12}$$

Now, let $\mathcal{A}_\psi(x; D_x, D_{x_0})$ and $\mathcal{B}((x_0, x); D_x, D_{x_0})$ be defined as in the statement of Lemma 4.2. Through easy but lengthy computations, from (6.7), (6.9), (6.12) we obtain

$$\begin{aligned}
&\mathcal{A}_\psi(x; D_x, D_{x_0})v(x_0, x)\\
&= \varkappa(x_0, x)e^{i\rho x_0}[\mathcal{A}(x; D_x) - \rho^2 e^{i\psi} I]u(x) + 2e^{i\rho x_0} \sum_{j,k=1}^{n} a_{j,k}(x)D_{x_k}u(x)D_{x_j}\varkappa(x_0, x)\\
&\quad + \mathcal{E}''\Big(\arccos \frac{x_0}{|(x_0, x)|}\Big)e^{i\rho x_0}u(x) \sum_{j,k=1}^{n} \frac{a_{j,k}(x)x_0^2 x_j x_k}{|x|^2|(x_0, x)|^4}\\
&\quad + e^{i(\psi+\rho x_0)}u(x)\Big\{\frac{|x|^2}{|(x_0, x)|^4}\,\mathcal{E}''\Big(\arccos \frac{x_0}{|(x_0, x)|}\Big) - 2\Big[\frac{x_0 - i\rho|(x_0, x)|^2}{|(x_0, x)|^2}\Big]D_{x_0}\varkappa(x_0, x)\Big\}\\
&\quad + e^{i\rho x_0}u(x)\Big\{\sum_{j,k=1}^{n} D_{x_k}a_{j,k}(x)D_{x_j}\varkappa(x_0, x) - \sum_{j=1}^{n} \frac{a_{j,j}(x)x_0 D_{x_0}\varkappa(x_0, x)}{|x|^2}\\
&\qquad - \sum_{j,k=1}^{n} \frac{a_{j,k}(x)x_k(x_0^2 + 3|x|^2)D_{x_j}\varkappa(x_0, x)}{|x|^2|(x_0, x)|^2}\Big\}\\
&=: \sum_{l=1}^{5} J_l(u, \mathcal{E}, (x_0, x))\,,
\end{aligned} \tag{6.13}$$

whereas, from (6.7) and (6.8), we get

$$\begin{aligned}
\mathcal{B}((x_0, x); D_x, D_{x_0})v(x_0, x) &= \varkappa(x_0, x)e^{i\rho x_0}\mathcal{B}(x; D_x)u(x) + i\rho x_0\varkappa(x_0, x)e^{i\rho x_0}u(x)\\
&\quad + e^{i\rho x_0}u(x)\Big[x_0 D_{x_0}\varkappa(x_0, x) + \sum_{i=1}^{n} b_i(x)D_{x_i}\varkappa(x_0, x)\Big]\\
&=: \sum_{l=6}^{8} J_l(u, \mathcal{E}, (x_0, x))\,.
\end{aligned} \tag{6.14}$$

In the next section, with the help of (6.13) and (6.14), we will upper bound the norms of $\mathcal{A}_\psi(x; D_x, D_{x_0})v$ and $\mathcal{B}((x_0, x); D_x, D_{x_0})v$, respectively in $L_{p,-1}(\widetilde{G})$ and $W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})$, in terms of the $W^k_{p,-1}(G)$-norms , $k = 0, 1, 2$, of u and of an any extension to G of its assigned boundary values. Just these estimates will be the argument of the forthcoming lemmata Lemma 7.1 and Lemma 7.2, which will be a fundamental step in the proof of our main result Theorem 4.6.

7 Proof of Theorem 4.6

As we said at the end of Section 6, Theorem 4.6 will be an easy consequence of two crucial lemmata, Lemma 7.1 and Lemma 7.2. We postpone such lemmata to the following considerations which strictly depends on the class (4.2) of domains G we restrict to work with. First, observe that $|(x_0, x)| \geq |x|$ implies

$$|(x_0, x)|^{-q} \leq |x|^{-q}\,, \qquad \forall\, q \geq 1\,. \tag{7.1}$$

Hence, if $w(x_0,x)$ is a function such that $|w(x_0,x)| \le |(x_0,x)|^{-k}|w_1(x_0,x)||w_2(x)|$, for some $k \in \mathbf{N} \cup \{0\}$ and some $w_1 \in C(\widetilde{G})$ vanishing for $x_0 \notin (-\delta_0,\delta_0)$, $0 < \delta_0 < \|\eta\|_{C([0,C_2])}$, then from (7.1) we easily find

$$\|w\|_{L_{p,-1}(\widetilde{G})} \le (2\delta_0)^{1/p}\|w_1\|_{C(\widetilde{G})}\|w_2\|_{L_{p,-(k+1)}(G)}\,. \tag{7.2}$$

Moreover, assumptions i), ii) on function η which describes the boundary ∂G of G imply

$$|(x_0,x)| \le \{[\eta(x_n)]^2 + x_n^2\}^{1/2} \le \{C_3^2+1\}^{1/2}x_n\,, \qquad \forall\,(x_0,x) \in \widetilde{G}\,. \tag{7.3}$$

Therefore, if we set $C_4 = \{C_3^2+1\}^{1/2}$ and we use $x_n \le |x|$, from (7.3) we deduce

$$C_4^{-q}|x|^{-q} \le |(x_0,x)|^{-q}\,, \qquad \forall\, q \ge 1\,, \qquad \forall\,(x_0,x) \in \widetilde{G}\,, \tag{7.4}$$

and hence, for any $w \in L_{p,-1}(\widetilde{G})$, we deduce also $\|w\|_{L_{p,-1}(\widetilde{G})} \ge C_4^{-1}\|w\|_{L_{p,-1}(G)}$.

Lemma 7.1. *Let $p > n$ and $u \in W^2_{p,-1}(G)$, where G is defined by (4.2), and let $\mathcal{A}(x;D_x)$ be the differential operator (1.1) with coefficients $a_{i,j}$, $i,j = 1,\ldots,n$, satisfying (4.3). Then, when v is defined by (6.1), (6.2) and $\mathcal{A}_\psi(x;D_x,D_{x_0})$ is defined as in the statement of Lemma 4.2, for any $\rho > 0$ and some $\delta_0(\varepsilon) > 0$ the following estimate holds:*

$$\begin{aligned}&\|\mathcal{A}_\psi(x;D_x,D_{x_0})v\|_{L_{p,-1}(\widetilde{G})}\\ &\le [2\delta_0(\varepsilon)]^{1/p}\big\{\|\big(\mathcal{A}(x;D_x) - \rho^2 e^{i\psi}I\big)u\|_{L_{p,-1}(G)} + M_1(1+\rho)\|u\|_{W^2_{p,-1}(G)}\big\}\,,\end{aligned} \tag{7.5}$$

The positive constant M_1 in (7.5) depends only on p, n, the $C^1(G)$-norm of the coefficients of $\mathcal{A}(x;D_x)$ and the constants C_j, $j = 2,3$, intervening in the properties i)–iii) for the function η which describes the boundary ∂G of G.

Proof. Since from formula (6.13) it follows

$$\|\mathcal{A}_\psi(x;D_x,D_{x_0})v\|_{L_{p,-1}(\widetilde{G})} \le \sum_{l=1}^{5}\|J_l(u,\mathcal{E},(x_0,x))\|_{L_{p,-1}(\widetilde{G})}\,. \tag{7.6}$$

we need only to estimate from above each norm $\|J_l(u,\mathcal{E},(x_0,x))\|_{L_{p,-1}(\widetilde{G})}$, $l = 1,\ldots,5$, and then to rearrange the term. First, from (7.2) and $\|\mathcal{E}\|_{C([0,\pi])} = 1$ we immediately get

$$\|J_1(u,\mathcal{E},(x_0,x))\|_{L_{p,-1}(\widetilde{G})} \le [2\delta_0(\varepsilon)]^{1/p}\|\big(\mathcal{A}(x;D_x) - \rho^2 e^{i\psi}I\big)u\|_{L_{p,-1}(G)}\,. \tag{7.7}$$

where using (6.3), (6.4) and (7.3) we have set

$$\delta_0(\varepsilon) = C_4C_2\cos[(\pi-6\varepsilon)/2], \qquad \varepsilon \in (0,(\pi-2\phi)/6)\,. \tag{7.8}$$

Now, observe that for any $(x_0,x) \in \widetilde{G}$ and $l \in \mathbf{N} \cup \{0\}$ from (6.5), (6.6) we derive

$$\big|D_{x_j}\mathcal{E}^{(l)}\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)\big| \le \frac{1}{|(x_0,x)|}\,\|\mathcal{E}^{(l+1)}\|_{C([0,\pi])}\,, \qquad j = 0,\ldots,n\,, \tag{7.9}$$

whereas, since $u \in W^2_{p,-1}(G)$ and $-1 < -np^{-1}$, from Theorem 2.1 it follows

$$\big\{\|u\|^p_{L_{p,-3}(G)} + \sum_{k=1}^n \|D_{x_k}u\|^p_{L_{p,-2}(G)}\big\}^{1/p} \le \|u\|_{V^2_{p,-1}(G)} \le c\|u\|_{W^2_{p,-1}(G)} \,. \tag{7.10}$$

Hence, recalling the definition of $J_2(u, \mathcal{E}, (x_0, x))$ in (6.13) and that of $\varkappa$ in (6.2), from (7.2), (7.9) and (7.10) we deduce

$$\begin{aligned}
&\|J_2(u, \mathcal{E}, (x_0, x))\|_{L_{p,-1}(\widetilde{G})}\\
&\le 2n[2\delta_0(\varepsilon)]^{1/p}\|\mathcal{E}'\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C(G)}\Big(\sum_{k=1}^n \|D_{x_k}u\|^p_{L_{p,-2}(G)}\Big)^{1/p}\\
&\le 2cn[2\delta_0(\varepsilon)]^{1/p}\|\mathcal{E}'\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C(G)}\|u\|_{W^2_{p,-1}(G)} \,,
\end{aligned} \tag{7.11}$$

and similarly, but taking advantage from

$$\Big|\frac{x_0^2 x_j x_k}{|x|^2|(x_0, x)|^4}\Big| \le \frac{1}{|(x_0, x)|^2} \,, \qquad j, k = 1, \dots, n \,, \tag{7.12}$$

we obtain

$$\begin{aligned}
\|J_3(u, \mathcal{E}, (x_0, x))\|_{L_{p,-1}(\widetilde{G})} &\le n^2[2\delta_0(\varepsilon)]^{1/p}\|\mathcal{E}''\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C(G)}\|u\|_{L_{p,-3}(G)}\\
&\le cn^2[2\delta_0(\varepsilon)]^{1/p}\|\mathcal{E}''\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C(G)}\|u\|_{W^2_{p,-1}(G)}.
\end{aligned} \tag{7.13}$$

Finally, using (7.3) and (7.9), it is easy to prove that the factors on the braces in the definition of $J_4(u, \mathcal{E}, (x_0, x))$ and $J_5(u, \mathcal{E}, (x_0, x))$ have their absolute values which are bounded from above respectively by $|(x_0, x)|^{-2}(3 + 2\rho C_4C_2)\|\mathcal{E}\|_{C^2([0,\pi])}$ and $[|x||(x_0, x)|]^{-1}n^2[4 + C_4(1 + C_2)]\|\mathcal{E}'\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C^1(G)}$.
Therefore, if we set $M_0 = \{\max[2^{p-1}3^p, 2^{2p-1}C_4^pC_2^p]\}^{1/p}$, from (7.2) and (7.10) we find

$$\begin{aligned}
\|J_4(u, \mathcal{E}, (x_0, x))\|_{L_{p,-1}(\widetilde{G})} &\le [2\delta_0(\varepsilon)]^{1/p}M_0(1 + \rho)\|\mathcal{E}\|_{C^2([0,\pi])}\|u\|_{L_{p,-3}(G)}\\
&\le c[2\delta_0(\varepsilon)]^{1/p}M_0(1 + \rho)\|\mathcal{E}\|_{C^2([0,\pi])}\|u\|_{W^2_{p,-1}(G)} \,,
\end{aligned} \tag{7.14}$$

$$\begin{aligned}
&\|J_5(u, \mathcal{E}, (x_0, x))\|_{L_{p,-1}(\widetilde{G})}\\
&\le n^2[2\delta_0(\varepsilon)]^{1/p}[4 + C_4(1 + C_2)]\|\mathcal{E}'\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C^1(G)}\|u\|_{L_{p,-3}(G)}\\
&\le cn^2[2\delta_0(\varepsilon)]^{1/p}[4 + C_4(1 + C_2)]\|\mathcal{E}'\|_{C([0,\pi])} \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C^1(G)}\|u\|_{W^2_{p,-1}(G)} \,.
\end{aligned} \tag{7.15}$$

By replacing (7.7), (7.11), (7.13)–(7.15) in (7.6) and rearranging the term we obtain (7.5) with $M_1 = c\{M_0 + n[2 + 5n + nC_4(1 + C_2)] \max_{j,k=1,\dots,n} \|a_{j,k}\|_{C^1(G)}\}\|\mathcal{E}\|_{C^2([0,\pi])}$. □

Lemma 7.2. *Let $p > n$ and $u \in W^2_{p,-1}(G)$ where G is defined by (4.2) and let $\mathcal{B}(x; D_x)$ be the differential operator (4.6) with coefficients b_j, $j = 0, \dots, n$, satisfying (4.7). Then, $g_0 \in W^1_{p,-1}(G)$ being any extension to G of $\mathcal{B}(x; D_x)u$, when v is defined by (6.1), (6.2)* *and $\mathcal{B}((x_0,x);D_x,D_{x_0})$ is defined as in the statement of Lemma 4.2, for any $\rho>0$ and some $\delta_0(\varepsilon)>0$ the following estimate holds:*

$$\begin{aligned}&\|\mathcal{B}((x_0,x);D_x,D_{x_0})v\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\\ &\le 2[\delta_0(\varepsilon)]^{1/p}M_2\big\{\|g_0\|_{W^1_{p,-1}(G)}+\rho\|g_0\|_{L_{p,-1}(G)}+\|u\|_{W^2_{p,-1}(G)}\\ &\qquad+(1+2\rho)\|u\|_{W^1_{p,-1}(G)}+(\rho+\rho^2)\|u\|_{L_{p,-1}(G)}\big\}\,, \end{aligned}\tag{7.16}$$

The constant $M_2>1$ in (7.16) depends only on p, n, the $C^1(G)$-norm of the coefficients of $\mathcal{B}(x;D_x)$ and the constants C_j, $j=2,3$, intervening in the properties i)–iii) for the function η which describes the boundary ∂G of G.

Proof. First, from (6.14) we get

$$\|\mathcal{B}((x_0,x);D_x,D_{x_0})v\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\le\sum_{l=6}^{8}\|J_l(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\,,\tag{7.17}$$

and observe that, due to the definition (4.11) of $\widetilde{G}$, if (x_0,x) belong to $\partial\widetilde{G}$ then it is of the form $(0,x)$ with $x\in\partial G$ or (x_0,x) with $x_0\neq 0$ and $x\in G$. Therefore (cf. also (4.7)), the term $J_l(u,\mathcal{E},(x_0,x))$, $l=6,7,8$, in (7.17) are well defined for any $(x_0,x)\in\widetilde{G}\cup\partial\widetilde{G}$. Hence, recalling the definition (2.3) of the norm in the spaces of traces and using (6.7), (6.8) with u replaced by g_0 and the inequality $|a+b|^q\le 2^{q-1}(|a|^q+|b|^q)$, $a,b\in\mathbf{C}$, $q\ge 1$, from (6.5), (6.6), (7.9) and (7.2) we obtain

$$\begin{aligned}&\|J_6(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\le\|J_6(u,\mathcal{E},(x_0,x))\|_{W^1_{p,-1}(\widetilde{G})}\\ &\le 2[\delta_0(\varepsilon)]^{1/p}\big\{\|g_0\|_{W^1_{p,-1}(G)}+(n+1)^{1/p}\|\mathcal{E}'\|_{C([0,\pi])}\|g_0\|_{L_{p,-2}(G)}+\rho\|g_0\|_{L_{p,-1}(G)}\big\},\end{aligned}\tag{7.18}$$

$\delta_0(\varepsilon)$ being defined by (7.8). Now, for any $w\in W^1_{p,-1}(G)$ with $p>n$ Theorem 2.1 imply

$$\Big\{\|w\|^p_{L_{p,-2}(G)}+\sum_{k=1}^{n}\|D_{x_k}w\|^p_{L_{p,-1}(G)}\Big\}^{1/p}\le\|w\|_{V^1_{p,-1}(G)}\le c\|w\|_{W^1_{p,-1}(G)}\,,\tag{7.19}$$

and consequently, if we set $M_3=[1+c(n+1)^{1/p}\|\mathcal{E}'\|_{C([0,\pi])}]$, from (7.18) we get

$$\|J_6(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\le 2[\delta_0(\varepsilon)]^{1/p}\big\{M_3\|g_0\|_{W^1_{p,-1}(G)}+\rho\|g_0\|_{L_{p,-1}(G)}\big\}\,.\tag{7.20}$$

Similarly, from (6.5)–(6.8), (7.9), (7.2) and (7.19) we obtain

$$\begin{aligned}&\|J_7(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}\le\|J_7(u,\mathcal{E},(x_0,x))\|_{W^1_{p,-1}(\widetilde{G})}\\ &\le 2M_4[\delta_0(\varepsilon)]^{1/p}\big\{M_3\rho\|u\|_{W^1_{p,-1}(G)}+2^{1-1/p}(\rho+\rho^2)\|u\|_{L_{p,-1}(G)}\big\}\,,\end{aligned}\tag{7.21}$$

where $M_4=\max[1,\delta_0(\varepsilon)]$.
Before to estimate the term $J_8(u,\mathcal{E},(x_0,x))$ in (7.17) observe that from (6.5) and (6.6) it follows

$$\begin{aligned}&x_0D_{x_0}\varkappa(x_0,x)+\sum_{i=1}^{n}b_i(x)D_{x_i}\varkappa(x_0,x)\\ &=-\Big[\frac{x_0|x|}{|(x_0,x)|^2}-\sum_{i=1}^{n}\frac{b_i(x)x_0x_i}{|x||(x_0,x)|^2}\Big]\mathcal{E}'\Big(\arccos\frac{x_0}{|(x_0,x)|}\Big)\,,\end{aligned}\tag{7.22}$$

so that, using (7.3), we easily get

$$\Big| x_0 D_{x_0} \varkappa(x_0,x) + \sum_{i=1}^{n} b_i(x) D_{x_i} \varkappa(x_0,x) \Big| \le \frac{M_5 \|\mathcal{E}'\|_{C([0,\pi])}}{|(x_0,x)|}\,, \tag{7.23}$$

where $M_5 = [n \max_{i=1,\dots,n} \|b_i\|_{C(G)} + C_4 C_2]$. In addition, for any $k = 1,\dots,n$ we have

$$D_{x_0}\Big[\frac{x_0|x|}{|(x_0,x)|^2} - \sum_{i=1}^{n} \frac{b_i(x) x_0 x_i}{|x||(x_0,x)|^2}\Big] = \frac{(|x|^2 - x_0^2)}{|(x_0,x)|^4}\Big[|x| - \sum_{i=1}^{n} \frac{b_i(x) x_i}{|x|}\Big],$$

$$D_{x_k}\Big[\frac{x_0|x|}{|(x_0,x)|^2} - \sum_{i=1}^{n} \frac{b_i(x) x_0 x_i}{|x||(x_0,x)|^2}\Big]$$
$$= \frac{x_0 x_k (x_0^2 - |x|^2)}{|x||(x_0,x)|^4} + \sum_{i=1}^{n} \frac{x_0[x_i D_{x_k} b_i(x) + \delta_{i,k} b_i(x)]}{|x||(x_0,x)|^2} - \sum_{i=1}^{n} \frac{b_i(x) x_0 x_i x_k [x_0^2 + 3|x|^2]}{|x|^3 |(x_0,x)|^4}\,,$$

and hence, applying the Leibniz's formula to the right-hand side of (7.22) and using (7.1), (7.3), (7.9) and (7.12), it is easy to obtain

$$\Big| D_{x_0}\Big[x_0 D_{x_0} \varkappa(x_0,x) + \sum_{i=1}^{n} b_i(x) D_{x_i} \varkappa(x_0,x)\Big]\Big| \le \frac{M_5}{|(x_0,x)|^2} \sum_{k=1}^{2} \|\mathcal{E}^{(k)}\|_{C([0,\pi])}\,, \tag{7.24}$$

$$\Big| D_{x_k}\Big[x_0 D_{x_0} \varkappa(x_0,x) + \sum_{i=1}^{n} b_i(x) D_{x_i} \varkappa(x_0,x)\Big]\Big| \le \frac{M_6}{|x||(x_0,x)|} \sum_{k=1}^{2} \|\mathcal{E}^{(k)}\|_{C([0,\pi])}\,, \tag{7.25}$$

where $M_6 = [n(5+C_4C_2) \max_{i=1,\dots,n} \|b_i\|_{C^1(G)} + C_4 C_2]$ and in (7.25) we have used $M_5 < M_6$. Therefore, combining (7.2) and (7.23)–(7.25) we deduce

$$\|J_8(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})} \le \|J_8(u,\mathcal{E},(x_0,x))\|_{W^{1}_{p,-1}(\widetilde{G})}$$
$$\le 2M_6[\delta_0(\varepsilon)]^{1/p} \|\mathcal{E}\|_{C^2([0,\pi])} \big\{ 2^{1-1/p}(1+\rho)\|u\|_{L_{p,-2}(G)} + 2^{1-1/p}(1+2n)^{1/p}\|u\|_{L_{p,-3}(G)}$$
$$+ \big[\|u\|^p_{L_{p,-3}(G)} + \sum_{k=1}^{n} \|D_{x_k} u\|^p_{L_{p,-2}(G)}\big]^{1/p} \big\}\,,$$

and hence, using (7.10) and (7.19),

$$\|J_8(u,\mathcal{E},(x_0,x))\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})}$$
$$\le 2M_7[\delta_0(\varepsilon)]^{1/p} \big\{ M_8 \|u\|_{W^2_{p,-1}(G)} + 2^{1-1/p}(1+\rho)\|u\|_{W^1_{p,-1}(G)} \big\}\,, \tag{7.26}$$

where $M_7 = cM_6 \|\mathcal{E}\|_{C^2([0,\pi])}$ and $M_8 = [1 + 2^{1-1/p}(1+2n)^{1/p}]$. Rearranging (7.20), (7.21) and (7.26) from (7.17) we derive (7.16) with the constant $M_2 = \max[M_4, M_7] \times \max[M_3, M_8]$. □

We can now prove the main result of the paper. To simplify notations, from now on for any $u \in W^l_{p,-1}(G)$, $l \ge 0$, we will set $\|D^l u\|_{L_{p,-1}(G)} = \sum_{|\alpha|=l} \|D^\alpha u\|_{L_{p,-1}(G)}$.

Proof of Theorem 4.6. For every $u \in W^2_{p,-1}(G)$ and $\rho > 0$ we define the function v accordingly to (6.1) (6.2) and we observe that $p > n$, $p \neq n+1$, imply

$$\begin{cases} -1 < -(n+1)p^{-1}\,, & \text{if} \quad p > n+1\,, \\ -(n+1)p^{-1} < -1 < 1-(n+1)p^{-1}\,, & \text{if} \quad n < p < n+1\,. \end{cases}$$

Therefore, recalling (3.21), the assumptions of Theorem 3.2 for $\beta = -1$ are both satisfied, the second one with $\nu = 0$. Moreover, since we have assumed problem (4.14) to be regular in the sense of Definition 4.5, we can apply to the function v the estimate (3.20) with the choice of the parameter l, $\vec{t}$, $\vec{s}$ and $\vec{\sigma}$ as in (3.21):

$$\|v\|_{W^2_{p,-1}(\widetilde{G})} \leq c_1\big\{\|\mathcal{A}_\psi(x;D_x,D_{x_0})v\|_{L_{p,-1}(\widetilde{G})} + \|\mathcal{B}((x_0,x);D_x,D_{x_0})v\|_{W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})} + \|v\|_{W^1_{p,-1}(\widetilde{G})}\big\}. \tag{7.27}$$

Since the norms $L_{p,-1}(\widetilde{G})$ and $W^{1-p^{-1}}_{p,-1}(\partial\widetilde{G})$ of $\mathcal{A}_\psi(x;D_x,D_{x_0})v$ and $\mathcal{B}((x_0,x);D_x,D_{x_0})v$ have been estimated in Lemma 7.1 and Lemma 7.2, respectively, it remains only to analyze the term $\|v\|_{W^1_{p,-1}(\widetilde{G})}$ in (7.27). But, due to definition (6.1), as in (7.18) and (7.20) with g_0 replaced by u, we obtain

$$\|v\|_{W^1_{p,-1}(\widetilde{G})} \leq 2[\delta_0(\varepsilon)]^{1/p}\big\{M_3\|u\|_{W^1_{p,-1}(G)} + \rho\|u\|_{L_{p,-1}(G)}\big\}\,. \tag{7.28}$$

Since for any $w \in W^l_{p,-1}(G)$, $l \geq 0$, we have $\|w\|_{W^l_{p,-1}(G)} \leq \sum_{|\alpha|=0}^{l}\|D^\alpha w\|_{L_{p,-1}(G)}$, if we set $M_9 = M_2\max[1,M_1]$, M_2 being defined at the end of the proof of Lemma 7.2, by combining (7.5), (7.16) and (7.28) from (7.27) we obtain

$$\begin{aligned} \|v\|_{W^2_{p,-1}(\widetilde{G})} \leq 2c_1M_9[\delta_0(\varepsilon)]^{1/p}\Big\{&\|(\mathcal{A}(x;D_x)-\rho^2e^{i\psi}I)u\|_{L_{p,-1}(G)} + (4+5\rho+\rho^2)\|u\|_{L_{p,-1}(G)} \\ &+(4+3\rho)\|Du\|_{L_{p,-1}(G)} + (2+\rho)\|D^2u\|_{L_{p,-1}(G)} \\ &+(1+\rho)\|g_0\|_{L_{p,-1}(G)} + \|Dg_0\|_{L_{p,-1}(G)}\Big\}\,, \end{aligned} \tag{7.29}$$

On the other hand, using $G = \{(x_0,x) \in \widetilde{G} : x_0 = 0\}$, (6.4) and $\mathcal{E}^{(k)}(\pi/2) = 0$, $k = 1,2$, from (6.1) and (6.7)–(6.11) for any $(x_0,x) \in \widetilde{G}$ we deduce the following inequalities

$$\begin{cases} |v(x_0,x)| \geq |v(0,x)| = |u(x)|\,, \\ |D_{x_i}v(x_0,x)| \geq |D_{x_i}v(0,x)| = |D_{x_i}u(x)|\,, \quad i = 1,\ldots,n\,, \\ |D_{x_0}v(x_0,x)| \geq |D_{x_0}v(0,x)| = \rho|u(x)|\,, \\ |D^2_{x_0}v(x_0,x)| \geq |D^2_{x_0}v(0,x)| = \rho^2|u(x)|\,, \\ |D_{x_i}D_{x_j}v(x_0,x)| \geq |D_{x_i}D_{x_j}v(0,x)| = |D_{x_i}D_{x_j}u(x)|\,, \quad i,j = 1,\ldots,n, \\ |D_{x_0}D_{x_j}v(x_0,x)| \geq |D_{x_0}D_{x_j}v(0,x)| = \rho|D_{x_j}u(x)|\,, \quad j = 1,\ldots,n. \end{cases}$$

Hence, using (7.4) we obtain

$$C_4^p \|v\|^p_{W^2_{p,-1}(\widetilde{G})} = \sum_{0\le|\alpha|\le 2} \int_{\widetilde{G}} C_4^p |(x_0,x)|^{-p} \big|D^\alpha v(x_0,x)\big|^p \,\mathrm{d}x_0\,\mathrm{d}x$$
$$\ge \int_{\widetilde{G}} |x|^{-p}\Big[|v(x_0,x)|^p + \sum_{i=0}^n |D_{x_i} v(x_0,x)|^p + \sum_{i,j=0}^n |D_{x_i} D_{x_i} v(x_0,x)|^p\Big] \,\mathrm{d}x_0\,\mathrm{d}x$$
$$\ge \int_G |x|^{-p}\Big[(1+\rho^p+\rho^{2p})|u(x)|^p + (1+\rho^p)\sum_{i=1}^n |D_{x_i} u(x)|^p + \sum_{i,j=1}^n |D_{x_i} D_{x_i} u(x)|^p\Big] \,\mathrm{d}x$$
$$\ge \rho^{2p}\|u\|^p_{L_{p,-1}(G)} + \rho^p\|Du\|^p_{L_{p,-1}(G)} + \|D^2 u\|^p_{L_{p,-1}(G)}\,.$$

Taking into account (7.29), it follows

$$\rho^2\|u\|_{L_{p,-1}(G)} + \rho\|Du\|_{L_{p,-1}(G)} + \|D^2u\|_{L_{p,-1}(G)} \le 3C_4\,\|v\|_{W^2_{p,-1}(\widetilde{G})}$$
$$\le M_{10}(\varepsilon)\Big\{\|\big(\mathcal{A}(x;D_x) - \rho^2 e^{i\psi} I\big)u\|_{L_{p,-1}(G)} + (4+5\rho+\rho^2)\|u\|_{L_{p,-1}(G)}$$
$$+ (4+3\rho)\|Du\|_{L_{p,-1}(G)} + (2+\rho)\|D^2u\|_{L_{p,-1}(G)}$$
$$+ (1+\rho)\|g_0\|_{L_{p,-1}(G)} + \|Dg_0\|_{L_{p,-1}(G)}\Big\} \tag{7.30}$$

where we have set $M_{10}(\varepsilon) = 6c_1C_4M_9[\delta_0(\varepsilon)]^{1/p}$. Now, from (7.8) we deduce that $M_{10}(\varepsilon)$ goes to zero as $\varepsilon \to 0^+$. Therefore, if we take $\lambda = \rho^2 e^{i\psi}$ and we assume ε sufficiently small, we can take ρ so large so that the following inequalities are satisfied

$$\begin{cases} M_{10}(\varepsilon)(4+5\rho+\rho^2) \le \rho^2/2\,, \\ M_{10}(\varepsilon)(4+3\rho) \le \rho/2\,, \\ M_{10}(\varepsilon)(2+\rho) \le 1/2\,. \end{cases} \tag{7.31}$$

From (7.30) and (7.31) our statement follows with $M = 2M_{10}(\varepsilon)$ in (4.16).

Remark 7.3. In the latter part of the proof of Theorem 4.6 we have assumed ε to be close to zero which, equivalently, means that the function $\mathcal{E}$ in (6.2) has its support in a small neighborhood of $\pi/2$ (cf. (6.3) and (6.4)). The sake for such a condition is due to estimate (7.21) where a factor ρ^2 appear in front of $\|u\|_{L_{p,-1}(G)}$. Since this factor takes origin from the definition of $J_7(u,\mathcal{E},(x_0,x))$ in (6.14), we can say that the restriction to considering small ε is a direct consequence of the necessity of introducing the boundary operator $\mathcal{B}((x_0,x);D_x,D_{x_0})$ in order to prove Lemma 4.2.

8 Appendix

We recall here the condition of ellipticity in the sense of Agranovich-Vishik for the operator $\mathcal{U}(0,\omega,z,D_\omega) = \{L(0,\omega,z,D_\omega), B(0,\omega,z,D_\omega)\}$, $\omega \in \Omega$, $z \in \mathbf{C}$, introduced in Section 3. Moreover, taking advantage from the discussions on pages 88–90 in [6], we sketch out how easily problems satisfying this condition can be constructed.

First of all, let G be a bounded domain of $\mathbf{R}^n$ whose boundary ∂G is an $(n-1)$–dimensional smooth surface locally admitting rectification by means of a C^∞ transformation of coordinates $x \to y$. As a result of such transformation ∂G becomes locally a hyperplane with equation $y_n = 0$ and G turns out to lie in the half-space $y_n > 0$.

Suppose now we are given the boundary value problem

$$\mathcal{L}(x; D_x, q)u(x, q) \;=\; f(x, q)\,, \qquad x \in G\,, \tag{8.1}$$

$$\mathcal{B}(x'; D_x, q)u(x', q) \;=\; g(x', q)\,, \qquad x' \in \partial G\,. \tag{8.2}$$

Here $\mathcal{L}$ and $\mathcal{B}$ are $k \times k$ and $m \times k$ matrix differential operators with sufficiently smooth complex coefficients polynomially depending on a parameter q which varies in the sector $Q = \{z \in \mathbf{C} : \theta_0 \le \arg z \le \theta_1\}$. In particular, for $\theta_0 = \theta_1$, Q can be a ray.
The assumptions on the orders of the differential operators being the same as those in Section 3, with $\mathcal{L}_0$ and $\mathcal{B}_0$ we denote here the principal parts consisting of the terms of higher order in $\mathcal{L}$ and $\mathcal{B}$, respectively. We impose two algebraic conditions on $\mathcal{L}$ and $\mathcal{B}$.

i) If $x \in \overline{G}$, $\xi \in \mathbf{R}^n$ and $q \in Q$, $|\xi| + |q| \neq 0$, then

$$\det \mathcal{L}_0(x; \xi, q) \neq 0\,.$$

Since the degree of the polynomial $\det \mathcal{L}_0(x; \xi, q)$ in ξ is $2r$, for $n \ge 2$ the equation $\lambda \to \det \mathcal{L}_0(x; \xi + \lambda\xi_0, q) = 0$, where $\xi_0 \neq 0$ and ξ is orthogonal to ξ_0, has exactly r roots with positive imaginary part.

ii) Let x' be any point on ∂G. We consider the problem in the half-line

$$\mathcal{L}_0((x', 0); \xi', -iD_y, q)v(y) \;= 0\,, \qquad y = x_n > 0\,, \tag{8.3}$$

$$\mathcal{B}_0((x', 0); \xi', -iD_y, q)v(y)|_{y=0} \;=\; h\,. \tag{8.4}$$

and we require that if $|\xi'| + |q| \neq 0$, $q \in Q$, for any vector $h \in \mathbf{C}^k$ this problem has one and only one solution in the class $\mathfrak{M}(\xi')$ of stable solutions of (8.3), i.e. solutions tending (exponentially) to zero together with all their derivatives as $y \to +\infty$.

For $q = 0$ conditions *i)* and *ii)* reduces, respectively, to the condition that system (8.1) is elliptic and to the condition of Shapiro-Lopatinskij for problem (8.1), (8.2).

Definition 8.1. With the problem (8.1), (8.2) we associate the operator

$$\mathcal{U}(x; D_x, q) = \{\mathcal{L}(x; D_x, q), \mathcal{B}(x'; D_x, q)\}\,.$$

If $\mathcal{L}$ and $\mathcal{B}$ satisfy the algebraic conditions *i)* and *ii)*, we say that $\mathcal{U}(x; D_x, q)$ is elliptic with parameter in the sense of Agranovich-Vishik.

Examples of problems satisfying *i)* and *ii)* can be constructed as follows.
Let $\mathcal{L}(x; D_x, D_{x_{n+1}})$ be an elliptic operator in the closure of the infinite cylinder $G_1 = G \times (-\infty, +\infty)$, connected on $\partial G_1 = \partial G \times (-\infty, +\infty)$ by the condition of Shapiro-Lopatinskij with the boundary operator $\mathcal{B}(x; D_x, D_{x_{n+1}})$. Then, the operators $\mathcal{L}(x; D_x, q)$ and $\mathcal{B}(x; D_x, q)$, obtained by replacing $D_{x_{n+1}}$ with q, satisfy conditions *i)* and *ii)* in each

section $x_{n+1} = \text{const}$ of G_1 if q belongs to $\{z \in \mathbf{C} : \arg z = 0\}$ or $\{z \in \mathbf{C} : \arg z = \pi\}$. Moreover, since it is well-known (cf. [3] or [4]) that the Shapiro-Lopatinskij condition is equivalent to require that $\mathcal{B}$ cover $\mathcal{L}$ on ∂G in the sense of [3] the former example shows that the uniform ellipticity of system (3.15) implies the ellipticity in the sense of Agranovich–Vishik for the operator $\mathcal{U}(0, \omega, z, D_\omega)$ defined through (3.16), (3.17).

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