SEMI-GLOBAL INVARIANTS OF PIECEWISE SMOOTH LAGRANGIAN FIBRATIONS

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Abstract. We study certain types of piecewise smooth Lagrangian fibrations of smooth symplectic manifolds, which we call *stitched Lagrangian fibrations*. We extend the classical theory of action-angle coordinates to these fibrations by encoding the information on the non-smoothness into certain invariants consisting, roughly, of a sequence of closed 1-forms on a torus. The main motivation for this work is given by the piecewise smooth Lagrangian fibrations previously constructed by the authors [3], which topologically coincide with the local models used by Gross in Topological Mirror Symmetry [5].

1. Introduction

Lagrangian fibrations arise naturally from integrable systems. It is a standard fact of Hamiltonian mechanics that such fibrations are locally given by maps of the type:

$$f = (f_1, \dots, f_n),$$

where the function components of f are Poisson commuting functions on a symplectic manifold and such that the differentials $df_1, \dots, df_n$ are pointwise linearly independent almost everywhere. It is customary to assume f to be C^∞ differentiable (smooth). Under this regularity assumption, a classical theorem of Arnold-Liouville says that a smooth proper Lagrangian submersion with connected fibres has locally the structure of a trivial Lagrangian T^n-bundle. In particular, all proper Lagrangian submersions are locally modelled on $U \times T^n$, where $U \subseteq \mathbb{R}^n$ is a contractible open set and $U \times T^n$ has the standard symplectic form induced from $\mathbb{R}^{2n}$. Standard coordinates with values in $U \times T^n$ are known as action-angle coordinates. Since these are defined on a fibred neighbourhood, action-angle coordinates are *semi-global* canonical coordinates. Thus proper Lagrangian submersions have no semi-global symplectic invariants.

In this article we investigate the semi-global symplectic topology of proper Lagrangian fibrations given by piecewise smooth maps. In [3]§6 we introduced the notion of *stitched Lagrangian fibration*. These are continuous proper S^1 invariant fibrations of smooth symplectic manifolds X which fail to be smooth only along the zero level set $Z = \mu^{-1}(0)$ of the moment map of the S^1 action and whose fibres are all smooth Lagrangian n-tori. Essentially, these fibrations consist of two honest smooth pieces $X^+ = \{\mu \geq 0\}$ and $X^- = \{\mu \leq 0\}$, stitched[1] together along Z, which we call the *seam*. These fibrations, roughly speaking, can be expressed locally as:

$$f = (\mu, f_2^\pm, \dots, f_n^\pm),$$

where f_j^+ and f_j^- are smooth functions defined on X^+ and X^-, respectively, whose differentials do not necessarily coincide along Z. Fibrations of this type are implicit in the examples proposed earlier by the authors [3]§5 and may also be implicit in those in [14]. In this paper, we develop a theory of action-angle coordinates for this class of piecewise smooth fibrations. Contrary to what happens in the smooth case, we found that these fibrations do give rise to semi-global symplectic invariants.

To the authors' knowledge, the kind of non-smoothness we investigate here does not seem to be of relevance to Hamiltonian mechanics. Nevertheless it is an important issue in symplectic topology and mirror symmetry. Over the past ten years, Lagrangian torus fibrations, in particular those which are *special Lagrangian*, have been discovered to play a fundamental

[1] We have chosen to use 'stitching' rather than 'gluing' since the resulting map is in general non smooth; the term 'gluing' usually has a smoothness meaning attached to it.

role in mirror symmetry [16]. One should expect mirror pairs of Calabi-Yau manifolds to be fibred by Lagrangian tori and the mirror relation to be expressed in terms of a Legendre transform between the corresponding affine bases [10], [6], [12]. This approach to mirror symmetry has some intricacies. For instance, there are examples of (non proper) special Lagrangian fibrations which are not given by smooth maps. Actually, one should expect a generic special Lagrangian fibration to be piecewise smooth [11]. Non-smoothness may also arise even in the purely Lagrangian case. In fact, there are examples of Lagrangian torus fibrations of Calabi-Yau manifolds which are piecewise smooth [14]. This lack of regularity has two important consequences. In first place, the discriminant locus of the fibration –i.e. the set of points in the base corresponding to singular fibres– may have codimension less than 2. Secondly, the base of the fibration may no longer carry the structure of an integral affine manifold away from the discriminant locus. In fact, the affine structure may break off not only along the discriminant –as it normally occurs in the smooth case– but also along a larger set containing the discriminant. Under these circumstances, it may become problematic to interpret the SYZ duality as a Legendre transform between affine manifolds. One should therefore understand the symplectic topology of piecewise smooth Lagrangian fibrations.

Some of the piecewise smooth examples here actually resemble the singular behaviour expected to appear in generic special Lagrangian fibrations. What is more important for our purposes, however, is the fact that our Lagrangian models coincide topologically with the non-Lagrangian models used by Gross [5]; the discriminant locus in our case may jump to codimension 1 in some regions but the total spaces are the same. In some cases, the discriminant has the shape of a planar amoeba Δ (see Figure 1) and fails to be smooth over the hyperplane $\Gamma = \{\mu = 0\}$ containing Δ. Away from Δ these fibrations are stitched Lagrangian torus fibrations. In particular, the affine structure on the base breaks apart along $\Gamma \setminus \Delta$. In this paper we provide some useful techniques to understand how this degeneration of the affine structure occurs.

The material of this paper is organised as follows. In §2 we start reviewing the classical theory of action-angle coordinates for smooth fibrations. In §3 we recall the construction of piecewise smooth fibrations of [3], these are explicitly given examples, some of them with codimension 1 discriminant locus. Then we revise the definition of stitched fibration, introduced in [3]. We formalise the idea of action-angle coordinates for stitched fibrations, allowing us to define the first order invariant, ℓ_1, of a stitched fibration. This invariant measures the discrepancy along Z between the distributions spanned by the Hamiltonian vector fields $\eta_2^+, \dots, \eta_n^+$ and $\eta_2^-, \dots, \eta_n^-$ corresponding to $f_2^+, \dots, f_n^+$ and $f_2^-, \dots, f_n^-$, respectively. The seam Z is an S^1-bundle $p : Z \to \bar{Z} := Z/S^1$ such that:

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "f|_Z"'] & & \bar{Z} \arrow[dl, "\bar{f}"] \\ & \Gamma & \end{tikzcd}$$

where $\bar{f}$ is the reduced fibration over the wall $\Gamma = \{\mu = 0\}$, with tangent $(n-1)$-plane distribution:

$$\mathfrak{L} = \ker \bar{f}_* \subset T\bar{Z}.$$

Let $\mathscr{L}_{\bar{Z}}$ be the set of fibrewise closed sections of $\mathfrak{L}^*$, i.e. elements in $\mathscr{L}_{\bar{Z}}$ can be viewed as closed 1-forms on the fibres of $\bar{f}$. The first order invariant of f is defined as follows. There are smooth S^1-invariant functions $a_2, \dots, a_n$ on Z such that $a_j\eta_1 = \eta_j^+ - \eta_j^-$. In particular, this implies that η_j^+ and η_j^- are mapped under p_* to the same vector field $\bar{\eta}_j$ on $\bar{Z}$. The first order invariant ℓ_1 is defined to be the section of $\mathfrak{L}^*$ such that $\ell_1(\bar{\eta}_j) = a_j$. It turns out that $\ell_1 \in \mathscr{L}_{\bar{Z}}$. In §5 we investigate higher order invariants. These are sequences $\{\ell_k\}_{k\in\mathbb{N}}$, with each $\ell_k \in \mathscr{L}_{\bar{Z}}$. Given a stitched Lagrangian fibration f, we define $\mathrm{inv}(f)$ to consist of the data of $(\bar{Z}, \bar{f})$, suitably normalized, together with the sequence $\ell = \{\ell_k\}_{k\in\mathbb{N}}$. The main result of this paper is proved in §6 (cf. Theorem 6.11 and Theorem 6.12) where we give a classification of stitched fibrations up to fibre-preserving symplectomorphims. Roughly, this can be stated as follows:

Theorem. There are stitched Lagrangian fibrations f having any specified set of data $\mathrm{inv}(f)$. Moreover, given stitched Lagrangian fibrations f and f' with invariants $\mathrm{inv}(f)$ and $\mathrm{inv}(f')$, respectively, there is a smooth symplectomorphism Φ, defined on a neighbourhood of Z, and a smooth diffeomorphism ϕ preserving $\Gamma \subset B$ and a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X' \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ B & \xrightarrow{\phi} & B' \end{array}$$

if and only if $\mathrm{inv}(f) = \mathrm{inv}(f')$.

This result extends Arnold-Liouville's theorem to this piecewise smooth setting. In §7 we study stitched fibrations over non simply connected bases. We show that one can read the monodromy of a stitched fibration as a jump of the cohomology class $[\ell_1(b)]$ as $b \in \Gamma$ traverses a component of the discriminant locus.

In the last section, we propose the following:

Conjecture. Let $Y \subseteq (\mathbb{C}^*)^{n-1}$ be a smooth algebraic hypersurface, $\mathrm{Log} : (\mathbb{C}^*)^{n-1} \to \mathbb{R}^{n-1}$ be the map defined by:

$$\mathrm{Log}(z_2, \ldots, z_n) = (\log|z_2|, \ldots, \log|z_n|).$$

Then there is a piecewise smooth Lagrangian n-torus fibration with discriminant locus being the amoeba $\Delta = \mathrm{Log}(Y)$ inside $\{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Away from Δ these fibrations are stitched Lagrangian fibrations.

To support this conjecture we propose a construction.

The results of this article allow us to have good control on the regularity of a large class of proper Lagrangian fibrations. Using simple techniques, one may deform the invariants of a given stitched fibration and produce proper Lagrangian fibrations with S^1 symmetry which are smooth on prescribed regions. This can be done, for instance, by multiplying a given sequence of invariants by a smooth function on the base B vanishing on a prescribed region. In joint work in progress [2], the authors use these and other techniques to give a construction of Lagrangian 3-torus fibrations of compact symplectic 6-manifolds starting from the information encoded in suitable integral affine manifolds, such as those arising from toric degenerations [7]. Such affine structures are expected to appear as Gromov-Hausdorff limits of degenerating families of Calabi-Yau manifolds (in the sense of [8], [12]).

2. Action-angle coordinates

We review the classical theory of action-angle coordinates for C^∞ Lagrangian fibrations. For the details we refer the reader to [1]. Assume we are given a $2n$-dimensional symplectic manifold X with symplectic structure ω, a smooth n-dimensional manifold B and a proper submersion $f : X \to B$ whose fibres are connected Lagrangian submanifolds.

Let F_b be the fibre of f over $b \in B$. We can define an action of $T^*_b B$ on F_b as follows. For every $\alpha \in T^*_b B$ we can associate a vector field v_α on F_b determined by

$$\iota_{v_\alpha}\omega = f^*\alpha. \tag{1}$$

Let ϕ^t_α be the flow of v_α with time $t \in \mathbb{R}$. Define θ_α as $\theta_\alpha(p) = \phi^1_\alpha(p)$ where $p \in F_b$. One can check that θ_α is well defined and that it induces an action $(\alpha, p) \mapsto \theta_\alpha(p)$. Furthermore, the action is transitive. Then, Λ_b defined as

$$\Lambda_b = \{\lambda \in T^*_b B \mid \theta_\lambda(p) = p, \text{ for all } p \in F_b\}$$

is a closed discrete subgroup of $T^*_b B$, i.e. a lattice. From the properness of f it follows that Λ_b is maximal (in particular homomorphic to $\mathbb{Z}^n$) and that F_b is diffeomorphic to $T^*_b B/\Lambda_b$ and therefore F_b is an n-torus.

Let $\Lambda = \cup_{b\in B}\Lambda_b$. One can compute Λ as follows. Given a point $b_0 \in B$ and a contractible neighbourhood U of b_0, for every $b \in U$, $H_1(F_b, \mathbb{Z})$ is naturally identified with $H_1(F_{b_0}, \mathbb{Z})$.

Choose a basis $\gamma_1, \dots, \gamma_n$ of $H_1(F_{b_0}, \mathbb{Z})$. Given a vector field v on U, denote by $\tilde{v}$ a lift of v on $f^{-1}(U)$. We can define the following 1-forms $\lambda_1, \dots, \lambda_n$ on B:

$$\lambda_j(v) = -\int_{\gamma_j} \iota_{\tilde{v}}\omega. \tag{2}$$

It is well known that the 1-forms λ_j are closed and they generate Λ. If $\sigma : B \to X$ is a smooth section of f we can define the map

$$\Theta : T^*B/\Lambda \to X$$

by $\Theta(b, \alpha) = \theta_\alpha(\sigma(b))$. This map is a diffeomorphism and it is a symplectomorphism if $\sigma(B) \subseteq X$ is Lagrangian. A choice of functions a_j such that $da_j = \lambda_j$ defines coordinates $a = (a_1, \dots a_n)$ on U called *action coordinates*. In particular, a covering $\{U_i\}$ of B by small enough contractible open sets and a choice of action coordinates a_i on each U_i defines an integral affine structure on B, i.e. an atlas whose change of coordinates maps are transformations in $\mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{Z})$.

A less invariant approach –but useful for explicit computations– can be described as follows. Let $(b_1, \dots b_n)$ be local coordinates on $U \subseteq B$ and let $f_j = b_j \circ f$. Then $f_1, \dots, f_n$ define an integrable Hamiltonian system. Let $\Phi^t_{\eta_j}$ be the flow of the Hamiltonian vector field η_j of f_j. Let σ be a Lagrangian section of f over U. Then the map Θ above can be expressed as:

$$\Theta : (b, t_1 db_1 + \dots + t_n db_n) \mapsto \Phi^{t_1}_{\eta_1} \circ \cdots \Phi^{t_n}_{\eta_n}(\sigma(b)).$$

One may verify that

$$\Lambda_b = \{(b, t_1 db_1 + \dots + t_n db_n) \in T^*_b U \mid \Phi^{t_1}_{\eta_1} \circ \cdots \Phi^{t_n}_{\eta_n}(\sigma(b)) = \sigma(b)\}.$$

When $(b_1, \dots, b_n)$ are action coordinates, $(b_1, \dots, b_n, t_1, \dots, t_n)$ are *action-angle coordinates*. These coordinates always exist on a fibred neighbourhood $f^{-1}(U)$ of a fibre F_b with $U \subseteq \mathbb{R}^n$ a small neighbourhood of b, thus they can be regarded as *semi-global* canonical coordinates. In particular, we have the following classical result:

Theorem 2.1 (Arnold, Liouville)**.** A proper Lagrangian submersion with connected fibres and a Lagrangian section has no semi-global symplectic invariants.

The *global* existence of action-angle coordinates is obstructed. For the details concerning this issue we refer the reader to Duistermaat [4].

In the next section we consider a larger class of Lagrangian submersions which include some Lagrangian fibrations which fail to be given by C^∞ maps.

3. Stitched Lagrangian fibrations: definitions and examples

Definition 3.1. Let (X, ω) be a smooth $2n$-dimensional symplectic manifold. Suppose there is a free Hamiltonian S^1 action on X with moment map $\mu : X \to \mathbb{R}$. Let $X^+ = \{\mu \geq 0\}$ and $X^- = \{\mu \leq 0\}$. Given a smooth $(n-1)$-dimensional manifold M, a map $f : X \to \mathbb{R} \times M$ is said to be a **stitched Lagrangian fibration** if there is a continuous S^1 invariant function $G : X \to M$, such that the following holds:

(i) Let $G^\pm = G|_{X^\pm}$. Then G^+ and G^- are restrictions of C^∞ maps on X;

(ii) f can be written as

$$f = (\mu, G)$$

and f restricted to $X^\pm$ is a proper submersion with connected Lagrangian fibres. We denote

$$f^\pm = f|_{X^\pm}.$$

We call $Z = \mu^{-1}(0)$ the **seam**.

We warn the reader that throughout the paper the superscript $\pm$ appearing in a sentence means that the sentence is true if read separately with the $+$ superscript and with the $-$ superscript. Notice that a stitched Lagrangian fibration may be non-smooth. In general it will be only piecewise C^∞, however all its fibres are smooth Lagrangian tori. Observe also that f^+ and f^- are restrictions of C^∞ maps, they are not a priori required to extend to smooth Lagrangian fibrations beyond X^+ and X^-, respectively. Later we show, however, that for any stitched fibration, f^+ and f^- are indeed restrictions of some locally defined smooth Lagrangian fibrations (cf. §6).

Let $\pi_{\mathbb{R}}$ be the projection of $\mathbb{R} \times M$ onto $\mathbb{R}$. Given a point $m \in M$ we study the geometry of a stitched Lagrangian fibration f in a neighbourhood of the fibre over $(0, m)$. For this purpose it is convenient to allow a more general set of coordinates on $\mathbb{R} \times M$ than just the smooth ones.

Definition 3.2. Let B be a neighbourhood of $(0, m) \in \mathbb{R} \times M$, let $B^+ = B \cap (\mathbb{R}_{\geq 0} \times M)$ and $B^- = B \cap (\mathbb{R}_{\leq 0} \times M)$. A continuous coordinate chart (B, ϕ) around $(0, m)$ is said to be **admissible** if the components of $\phi = (\phi_1, \ldots, \phi_n)$ satisfy the following properties:

(i) $\phi_1 = \pi_{\mathbb{R}}$;
(ii) for $j = 2, \ldots, n$ the restrictions of ϕ_j to B^+ and B^- are locally restrictions of smooth functions on B.

Lemma 3.3. Let $f : X \to \mathbb{R} \times M$ be a stitched Lagrangian fibration and let (B, ϕ) be an admissible coordinate chart around $(0, m) \in \mathbb{R} \times M$. For $j = 2, \ldots, n$, the function $G_j^{\pm} = (\phi_j \circ f)|_{f^{-1}(B^{\pm})}$ is the restriction of a C^∞ function on X to $X^{\pm}$. Let η_1 and $\eta_j^{\pm}$ be the Hamiltonian vector fields of μ and $G_j^{\pm}$ respectively. Then there are S^1 invariant functions a_j, $j = 2, \ldots, n$ on $Z \cap f^{-1}(B)$ such that

$$(\eta_j^+ - \eta_j^-)|_{Z \cap f^{-1}(B)} = a_j \, \eta_1|_{Z \cap f^{-1}(B)}. \tag{3}$$

Proof. Let $\bar{Z} = Z/S^1$, with projection $p : Z \to \bar{Z}$ and let ω_r be the Marsden-Weinstein reduced symplectic form on $\bar{Z}$. Given a vector field v on $\bar{Z}$, let $\tilde{v}$ be a lift of v on Z. Then we have

$$\begin{aligned} \omega_r(p_*(\eta_j^+ - \eta_j^-), v) &= \omega(\eta_j^+ - \eta_j^-, \tilde{v}) \\ &= (dG_j^+ - dG_j^-)(\tilde{v}) = 0, \end{aligned}$$

where the last equality comes from the fact that, being G continuous, $G_j^+|_{Z \cap f^{-1}(B)} = G_j^-|_{Z \cap f^{-1}(B)}$. Since ω_r is non-degenerate on Z, it follows that

$$p_*(\eta_j^+ - \eta_j^-) = 0.$$

Therefore (3) must hold for some function a_j, which must be S^1 invariant since the left-hand side of (3) is S^1 invariant. $\square$

Clearly, when f and the coordinate map ϕ are smooth, all the a_j's vanish, so equation (3) measures how far f and ϕ are from being smooth. We will say more about this in the coming sections.

We now recall some of the examples which we already introduced and discussed extensively in [3]. Consider the following S^1 action on $\mathbb{C}^3$:

$$e^{i\theta}(z_1, z_2, z_3) = (e^{i\theta} z_1, e^{-i\theta} z_2, z_3). \tag{4}$$

This action is Hamiltonian with respect to the standard symplectic form $\omega_{\mathbb{C}^3}$. Clearly, it is singular along the surface $\Sigma = \{z_1 = z_2 = 0\}$. The corresponding moment map is:

$$\mu(z_1, z_2, z_3) = \frac{|z_1|^2 - |z_2|^2}{2}. \tag{5}$$

The only critical value of μ is $t = 0$ and $\mathrm{Crit}(\mu) = \Sigma \subset \mu^{-1}(0)$.

Let $\gamma : \mathbb{C}^2 \to \mathbb{C}$ be the following piecewise smooth map

$$\gamma(z_1, z_2) = \begin{cases} \frac{z_1 z_2}{|z_1|}, & \text{when } \mu \geq 0 \\ \\ \frac{z_1 z_2}{|z_2|}, & \text{when } \mu < 0. \end{cases} \tag{6}$$

In two dimensions we have the following:

Example 3.4 (**Stitched focus-focus**)**.** Consider the map

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \, \log|\gamma(z_1, z_2) + 1| \right). \tag{7}$$

It is clearly well defined on $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid \gamma(z_1, z_2) + 1 \neq 0\}$ and it has Lagrangian fibres. We showed in [3] that f has the same topology of a smooth focus-focus fibration. The only singular fibre, $f^{-1}(0)$, is a (once) pinched torus. One can easily see that, when restricted to $X - f^{-1}(0)$, f is a stitched Lagrangian fibration. The seam is $Z = \mu^{-1}(0) - f^{-1}(0)$. Notice that Z has two connected components. Let η_1 and $\eta_2^{\pm}$ be the Hamiltonian vector fields defined as in Lemma 3.3. After some computation one can verify that

$$(\eta_2^+ - \eta_2^-)|_Z = a\, \eta_1|_Z,$$

where

$$a = \mathrm{Re}\left(\frac{z_1 z_2}{|z_1|^2 z_1 z_2 - |z_1|^3} \right)|_Z \, .$$

There is an analogous model in three dimensions:

Example 3.5. Consider the map

$$f(z_1, z_2, z_3) = (\mu, \, \log|z_3|, \, \log|\gamma(z_1, z_2) - 1|) \, . \tag{8}$$

The discriminant locus of f is $\Delta = \{0\} \times \mathbb{R} \times \{0\} \subset \mathbb{R}^3$. Again, f restricted to $X - f^{-1}(\Delta)$ defines a stitched Lagrangian fibration.

Example 3.6. Consider the map

$$f(z_1, z_2, z_3) = (\mu, \log \frac{1}{\sqrt{2}}|\gamma - z_3|, \log \frac{1}{\sqrt{2}}|\gamma + z_3 - \sqrt{2}|). \tag{9}$$

Let X be the dense open subset of $\mathbb{C}^3$ where f is well defined. The general construction discussed in §5 of [3] shows that f is a piecewise smooth Lagrangian fibration. It contains singular fibres, in fact the discriminant locus Δ of f is depicted in Figure 1. One easily

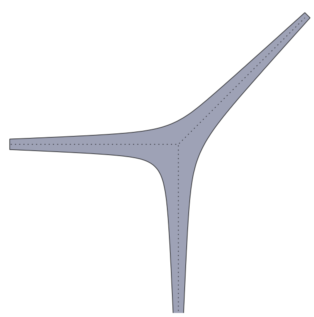

FIGURE 1. Amoeba of $v_1 + v_2 + 1 = 0$

checks that f restricted to $X - f^{-1}(\Delta)$ is a stitched Lagrangian fibration. The seam is $Z = \mu^{-1}(0) - f^{-1}(\Delta)$, notice that Z has three connected components. Let η_1 and $\eta_j^{\pm}$ be the Hamiltonian vector fields defined as in Lemma 3.3. A computation shows that, for $j = 2, 3$

$$(\eta_j^+ - \eta_j^-)|_Z = a_j\, \eta_1|_Z,$$

where

$$a_2 = -\frac{\mathrm{Re}\left((\gamma - z_3) \frac{\overline{z}_1 \overline{z}_2}{|z_1|^3} \right)}{|\gamma - z_3|^2}$$

and

$$a_3 = -\frac{\mathrm{Re}\left((\gamma + z_3 - \sqrt{2}) \frac{\overline{z}_1 \overline{z}_2}{|z_1|^3} \right)}{|\gamma + z_3 - \sqrt{2}|^2}.$$

In [3] we describe the topology of the singular fibres of f and discuss the relevance of this fibration in the context of Gross' topological mirror symmetry construction. We also show how this example can be perturbed to obtain other interesting stitched Lagrangian fibrations with discriminant locus of mixed codimension one and two.

4. The first order invariant

Our goal in this paper is to give a semi-global classification of stitched Lagrangian fibrations up to *smooth* fibre-preserving symplectomorphism. For this purpose in this section we restrict our attention to stitched Lagrangian fibrations $f : X \to \mathbb{R} \times M$, where $M = \mathbb{R}^{n-1}$. We assume that $f(X) \subseteq \mathbb{R}^n$ is a contractible open neighbourhood U of $0 \in \mathbb{R}^n$ and we denote coordinates on U by $b = (b_1, \ldots, b_n)$. Clearly (X, f) is a topologically trivial torus bundle over U. Let $U^+ := U \cap \{b_1 \geq 0\}$, $U^- := U \cap \{b_1 \leq 0\}$ and $\Gamma := U \cap \{b_1 = 0\}$. We assume for simplicity that the pair (U, Γ) is homeomorphic to the pair (D^n, D^{n-1}), where $D^n \subset \mathbb{R}^n$ is an n-dimensional ball centred at 0 and $D^{n-1} \subset D^n$ is the intersection of D^n with an $n-1$ dimensional subspace. We have that $X^{\pm} = f^{-1}(U^{\pm})$ and $Z = f^{-1}(\Gamma)$. If $f_1, \ldots, f_n$ are the components of f, then $f_1 = \mu$ is the moment map of the S^1 action. When $j = 2, \ldots, n$, we denote $f_j^{\pm} = f_j|_{X^{\pm}}$. As in Lemma 3.3 we let η_1 and $\eta_j^{\pm}$ denote the Hamiltonian vector fields of f_1 and $f_j^{\pm}$ respectively. Let us recall the notation used in the proof of Lemma 3.3. Since $Z = \mu^{-1}(0)$, the S^1 action on X induces an S^1-bundle $p : Z \to \bar{Z}$, where $\bar{Z} = Z/S^1$. There exists $\bar{f} : \bar{Z} \to \Gamma$ such that the following diagram commutes

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "f"'] & & \bar{Z} \arrow[dl, "\bar{f}"] \\ & \Gamma & \end{tikzcd}$$

In fact $\bar{f} = (f_2, \ldots, f_n)$, where each component of $\bar{f}$ is thought of as a function on $\bar{Z}$. For $b \in \Gamma$ denote by F_b the fibre over b and $\bar{F}_b = F_b/S^1$, clearly $\bar{F}_b = \bar{f}^{-1}(b)$. Denote by

$$\mathfrak{L} = \ker \bar{f}_*,$$

the bundle over $\bar{Z}$ whose fibre at a point $y \in \bar{F}_b$ is $T_y \bar{F}_b$. From Lemma 3.3 it follows that $p_* \eta_j^+ = p_* \eta_j^-$, so we can define $\bar{\eta} = (\bar{\eta}_2, \ldots, \bar{\eta}_n)$ to be the frame of $\mathfrak{L}$ where $\bar{\eta}_j = p_* \eta_j^{\pm}$. We say that a section of $\Lambda^k \mathfrak{L}^*$ is fibrewise closed (exact) if it is closed (exact) when viewed as a k-form on each fibre $\bar{F}_b$. We have the following:

Proposition 4.1. Let (X, f) be a stitched Lagrangian fibration. If ℓ_1 is the section of $\mathfrak{L}^*$ defined by

$$\ell_1(\bar{\eta}_j) = a_j,$$

where a_j is the S^1-invariant function appearing in (3), then ℓ_1 is fibrewise closed.

Proof. Since f is a Lagrangian submersion, the Hamiltonian vector fields $\eta_1, \eta_2^{\pm}, \ldots, \eta_2^{\pm}$ commute and are linearly independent. Therefore, for every fixed $b \in \Gamma$, the vector fields $\eta_j^{\pm}|_{F_b}$ span $(n-1)$-dimensional integrable distributions $H_b^{\pm}$, which are horizontal with respect to the S^1-bundle $p_b : F_b \to \bar{F}_b$. From the S^1 invariance of $f_2, \ldots, f_n$, it also follows that $H_b^{\pm}$ are S^1 invariant. Thus they define flat connections $\theta_b^{\pm}$ of the bundle $p_b : F_b \to \bar{F}_b$. From the properties of flat connections, it follows that $\theta_b^- - \theta_b^+$ is the pull back of a closed one form on $\bar{F}_b$. From (3) we obtain

$$(\theta_b^- - \theta_b^+)(\eta_j^{\pm}) = a_j|_{\bar{F}_b},$$

i.e. that

$$\theta_b^- - \theta_b^+ = p_b^*(\ell_1|_{\bar{F}_b}).$$

Therefore ℓ_1 is fibrewise closed. □

Clearly, the definition of ℓ_1 depends on a choice of coordinates on U. Let ℓ_1' be a fibrewise closed section of $\mathfrak{L}^*$. We say that ℓ_1' is equivalent to ℓ_1 up to a change of coordinates on the base if there exists a neighbourhood $W \subseteq U$ of Γ and an admissible coordinate map $\phi : W \to \mathbb{R}^n$ such that ℓ_1' is the section associated to $(f^{-1}(W), \phi \circ f)$ via Proposition 4.1. Denote by $[\ell_1]$ the class represented by ℓ_1 modulo this equivalence relation. We say that a section δ of $\mathfrak{L}^*$ is fibrewise constant if

$$\mathcal{L}_{\bar{\eta}_j} \delta = 0,$$

for all $j = 2, \dots, n$, here $\mathcal{L}_{\bar{\eta}_j}$ denotes the Lie derivative. One can easily check that the latter definition is independent of the admissible coordinates on the base used to define $\bar{\eta}_j$. We have the following

Proposition 4.2. A section ℓ_1' of $\mathfrak{L}^*$ is equivalent to ℓ_1 up to a change of coordinates on the base if and only if

$$\ell_1' = \ell_1 + \delta,$$

where δ is fibrewise constant. In particular, the class of ℓ_1 may be written as

$$[\ell_1] = \{\ell_1 + \delta \mid \delta \text{ is fibrewise constant}\}.$$

Proof. Given an admissible change of coordinates $\phi : W \to \mathbb{R}^n$, we must have $\phi_1 = b_1$. Moreover the partial derivatives $\partial_k \phi_j$ are defined and continuous on W for all $k, j = 2, \dots, n$. As far as derivatives with respect to b_1 are concerned, only left and right derivatives are defined and smooth on Γ, i.e. only $\partial_1 \phi_j^+$ and $\partial_1 \phi_j^-$, which may a priori differ. Let $(\eta_j')^{\pm}$ be the Hamiltonian vector fields on Z corresponding to $\phi_j \circ f$, with $j = 2, \dots, n$, and let $\bar{\eta}_j' = p_*(\eta_j')^{\pm}$. An easy calculation shows that

$$(\eta_j')^{\pm} = \partial_1 \phi_j^{\pm}\, \eta_1 + \sum_{k=2}^{n} \partial_k \phi_j\, \eta_k^{\pm}.$$

In particular this implies

$$\bar{\eta}_j' = \sum_{k=2}^{n} \partial_k \phi_j\, \bar{\eta}_k. \tag{10}$$

and

$$\begin{aligned} (\eta_j')^{+} - (\eta_j')^{-} &= (\partial_1 \phi_j^{+} - \partial_1 \phi_j^{-})\, \eta_1 + \sum_{k=2}^{n} \partial_k \phi_j\, (\eta_k^{+} - \eta_k^{-}) \\ &= (\partial_1 \phi_j^{+} - \partial_1 \phi_j^{-} + \sum_{k=2}^{n} a_k \partial_k \phi_j)\, \eta_1. \end{aligned}$$

If ℓ_1' is the 1-form associated to $\phi \circ f$ via Proposition 4.1, then by definition we must have

$$\ell_1'(\bar{\eta}_j') = \partial_1 \phi_j^{+} - \partial_1 \phi_j^{-} + \sum_{k=2}^{n} a_k \partial_k \phi_j.$$

Let δ be the section of $\mathfrak{L}^*$ defined by

$$\delta(\bar{\eta}_j') = \partial_1 \phi_j^{+} - \partial_1 \phi_j^{-}. \tag{11}$$

Then, also using (10), we see that

$$(\ell_1 + \delta)(\bar{\eta}_j') = \ell_1'(\bar{\eta}_j').$$

Moreover, from (11) one can see that $\delta(\bar{\eta}_j')$ descends to a function on Γ and therefore δ is fibrewise constant.

Now suppose that δ is a fibrewise constant section of $\mathfrak{L}^*$. Let

$$\delta(\bar{\eta}_j) = d_j.$$

Since δ is fibrewise constant, the d_j's are fibrewise constant functions on $\bar{Z}$, i.e. they descend to functions on Γ. Define the following map

$$\phi(b_1, \dots, b_n) = \begin{cases} (b_1,\, b_2 + d_2(b_2, \dots, b_n)\, b_1, \dots,\, b_n + d_n(b_2, \dots, b_n)\, b_1), & \text{when } b_1 \geq 0 \\ \\ \mathrm{Id}, & \text{when } b_1 < 0. \end{cases}$$

It is a well defined admissible coordinate map on some open neighbourhood of Γ. It is also clear that (11) holds. ∎

Definition 4.3. We call ℓ_1 the **first order invariant** of the stitched fibration (X, ω, f).

The name "invariant" in the above Definition will be fully justified later on.

It is clear from the proof of Proposition 4.2 that δ is a first order measure of how far the change of coordinates on the base is from being smooth, in particular if it is smooth then $\delta = 0$.

We also have the following:

Corollary 4.4. If there exists an admissible change of coordinates on the base which makes the stitched Lagrangian fibration smooth, then ℓ_1 is fibrewise constant.

Proof. It is clear that if $\phi \circ f$ is smooth then we must have that its first order invariant ℓ_1' is zero. It then follows from Proposition 4.2 that ℓ_1 must be fibrewise constant. □

We now describe action-angle coordinates of a stitched Lagrangian fibration $f : X \to U$. Let α be a 1-form on U. Since $f^{\pm}$ is the restriction of a smooth map, α pulls back to an honest smooth 1-form $\alpha^{\pm}$ defined on a neighbourhood of Z. The latter defines a smooth vector field $v_{\alpha}^{\pm}$ determined by the equation (1). The flow of $v_{\alpha}^{\pm}$, when restricted to $X^{\pm}$, is fibre-preserving. This induces an action of T_b^*U on the fibre $(f^{\pm})^{-1}(b)$ for all $b \in U^{\pm}$. Let $\sigma : U \to X$ be a continuous section which is smooth and Lagrangian when restricted to $U^{\pm}$. Then, as explained in §2, there is a maximal smooth lattice $\Lambda_{\pm} \subset T^*U^{\pm}$ and a diagram

$$\begin{tikzcd} T^*U^{\pm}/\Lambda_{\pm} \arrow[rr, "\Theta^{\pm}"] \arrow[dr, "\pi^{\pm}"'] & & X^{\pm} \arrow[dl, "f^{\pm}"] \\ & U^{\pm} & \end{tikzcd}$$

where $\Theta^{\pm}$ is a symplectomorphism and $\pi^{\pm}$ is the standard projection. Let $\Phi_{\eta_1}^t, \Phi_{\eta_2^{\pm}}^t \ldots, \Phi_{\eta_n^{\pm}}^t$ denote the flow of $\eta_1, \eta_2^{\pm}, \ldots, \eta_n^{\pm}$ respectively. Then

$$\Theta^{\pm} : (b, \sum_j t_j\, db_j) \mapsto \Phi_{\eta_1}^{t_1} \circ \Phi_{\eta_2^{\pm}}^{t_2} \circ \ldots \circ \Phi_{\eta_n^{\pm}}^{t_n}(\sigma(b)), \tag{12}$$

and

$$\Lambda_{\pm} = \{(b, \sum_j T_j\, db_j) \in T^*U^{\pm} \mid \Phi_{\eta_1}^{T_1} \circ \Phi_{\eta_2^{\pm}}^{T_2} \circ \ldots \circ \Phi_{\eta_n^{\pm}}^{T_n}(\sigma(b)) = \sigma(b)\}$$

Now let

$$\lambda_1 = db_1.$$

The S^1 action implies $db_1 \in \Lambda_{\pm}$. Let us denote a basis for $\Lambda_{\pm}$ by $\{\lambda_1, \lambda_2^{\pm}, \ldots, \lambda_n^{\pm}\}$, where

$$\lambda_j^{\pm} = \sum_{k=1}^{n} T_{jk}^{\pm} db_k.$$

The S^1 action on X corresponds to translations along the λ_1 direction. Let

$$Z^{\pm} = (\pi^{\pm})^{-1}(\Gamma).$$

If we denote

$$\bar{\lambda}_j^{\pm} = \lambda_j^{\pm} \mod db_1$$

and let $\bar{\Lambda}^{\pm} = \operatorname{span}\langle \bar{\lambda}_2^{\pm}, , \ldots, \bar{\lambda}_n^{\pm} \rangle_{\mathbb{Z}}$, then $\Theta^{\pm}$ identifies Z/S^1 with

$$\bar{Z}^{\pm} = T^*\Gamma/\bar{\Lambda}_{\pm}.$$

Denote by $\bar{t} = (t_2, \ldots, t_n)$ the coordinates on the fibres of $\bar{Z}^{-}$.

Now observe that, due to the discrepancy (3) between η_j^+ and η_j^- along Z, Θ^+ and Θ^- behave differently on fibres lying over Γ. We have the diagram:

$$\begin{tikzcd} Z^{-} \arrow[dr, "\Theta^{-}"'] & & Z^{+} \arrow[dl, "\Theta^{+}"] \\ & Z & \end{tikzcd}$$

and the difference between the two maps is measured by

$$(\Theta^+)^{-1} \circ \Theta^- : Z^- \to Z^+.$$

We have the following characterisation of this map:

Proposition 4.5. The discrepancy (3) between the Hamiltonian vector fields of the stitched Lagrangian fibration $f : X \to U$ induces the map

$$Q = (\Theta^+)^{-1} \circ \Theta^-$$

between Z^- and Z^+. Let

$$\ell_1^- = (\Theta^-)^* \ell_1.$$

Then, computed explicitly in the canonical coordinates on T^*U^- and T^*U^+, Q is given by

$$Q : (b, t_1, \bar{t}) \mapsto \left(b,\ t_1 - \int_0^{\bar{t}} \ell_1^-,\ \bar{t} \right), \tag{13}$$

where $(b, \bar{t})$ are the canonical coordinates on $\bar{Z}^-$ and the integral is a line integral in $T_b^*\Gamma$ along a path joining $(b, 0)$ and $(b, \bar{t})$.

Proof. Let $(b_1, \ldots, b_n, t_1, \ldots, t_n)$ and $(b_1, \ldots, b_n, y_1, \ldots, y_n)$ be the canonical coordinates on T^*U^- and T^*U^+ respectively. From its definition, we see that Θ^+ identifies $\eta_1, \eta_2^+, \ldots, \eta_n^+$ with $\partial_{y_1}, \ldots, \partial_{y_n}$ and w.l.o.g. we can assume that it sends σ to the zero section of T^*U. Therefore (3) becomes

$$\eta_j^- = \partial_{y_j} - (a_j \circ \Theta^+)\, \partial_{y_1}. \tag{14}$$

Notice that $a_j \circ \Theta^+$ is independent of y_1. Computing the flows of $\eta_1, \eta_2^-, \ldots, \eta_n^-$ in these coordinates is not difficult and it turns out that Q is given by

$$Q : (b, t_1, \ldots, t_n) \mapsto \left(b,\ t_1 - \sum_{j=2}^{n} \int_0^{t_j} a_j \circ \Theta^-(b, t_2, \ldots, t_{j-1}, t, 0, \ldots, 0) dt,\ t_2, \ldots,\ t_n \right),$$

which is equivalent to (13), since ℓ_1 is fibrewise closed[2]. □

We now explain how the map Q matches the periods in Λ^- with those in Λ^+. The maps $\Theta^\pm$ naturally identify $\Lambda_\pm$ with $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^n$, but in general Θ^- does it differently from Θ^+. Let γ_1 be the cycle represented by the orbit of the S^1 action. We know that γ_1 always corresponds to the period db_1.

We have the following

Corollary 4.6. Suppose we choose bases $\{\lambda_1, \lambda_2^\pm, \ldots, \lambda_n^\pm\}$ of $\Lambda_\pm$ corresponding to two bases $\gamma^\pm = \{\gamma_1, \gamma_2^\pm, \ldots, \gamma_n^\pm\}$ of $H_1(X, \mathbb{Z})$, such that

(i) γ_1 is represented by an orbit of the S^1 action,
(ii) $\gamma_j^+ = \gamma_j^- + m_j \gamma_1$, for some $m_2, \ldots, m_n \in \mathbb{Z}$.

Then at a point $b \in \Gamma$ we have

$$\lambda_j^+(b) = \lambda_j^-(b) + \left(m_j - \int_{\bar{\lambda}_j^-} \ell_1^- \right) \lambda_1, \tag{15}$$

where the integral of ℓ_1^- is taken along the cycle represented by $\bar{\lambda}_j^-$. In particular

$$\bar{\lambda}_j^+(b) = \bar{\lambda}_j^-(b). \tag{16}$$

Proof. To obtain (15) it suffices to observe that since $m_j \lambda_1 + \lambda_j^-$ and λ_j^+ have to represent the same 1-cycle in $f^{-1}(b)$, they must be mapped one to the other by Q. The result is therefore obtained by applying (13) to $m_j \lambda_1 + \lambda_j^-$. □

Remark 4.7. Condition *(ii)* means that under the map $p_* : H_1(X, \mathbb{Z}) \to H_1(X/S^1, \mathbb{Z})$, bases γ^+ and γ^- are mapped to the same base of $H_1(X/S^1, \mathbb{Z})$. We will need to consider condition *(ii)* in § 7 where we discuss stitched Lagrangian fibrations over non simply connected bases, for which non-trivial monodromy may occur.

[2] To verify that the above expression of Q is correct, it is enough to check that $\partial_{t_j} Q = \eta_j^- \circ Q$.

Remark 4.8. From Proposition 4.5 and Corollary 4.6 it follows that $\bar{\Lambda}_- = \bar{\Lambda}_+$ and that on the quotients $\bar{Z}^+$ and $\bar{Z}^-$, Q acts as the identity. Therefore, if we let

$$\ell_1^+ = (\Theta^+)^* \ell_1,$$

then we have $\bar{Z}^+ = \bar{Z}^-$ and $\ell_1^+ = \ell_1^-$. Thus we can remove the $+$ and $-$ signs and denote

$$\bar{\lambda}_j = \bar{\lambda}_j^+ = \bar{\lambda}_j^-$$

$$\bar{\Lambda} = \bar{\Lambda}_- = \bar{\Lambda}_+$$

and, with slight abuse of notation, identify $\bar{Z}$ with $T^*\Gamma/\bar{\Lambda}$ and $\bar{f}$ with the projection $\bar{\pi} : \bar{Z} \to \Gamma$. Notice then that $\mathfrak{L}$ is identified with $\ker \bar{\pi}_*$ and ℓ_1 with $\ell_1^{\pm}$.

It is natural to consider bases of $H_1(X, \mathbb{Z})$ satisfying conditions (i) and (ii) also because of the following

Lemma 4.9. Let $\{\gamma_1, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm}\}$ be bases of $H_1(X, \mathbb{Z})$ satisfying conditions (i) and (ii) of Corollary 4.6 and let $\alpha^{\pm} : U^{\pm} \to \mathbb{R}^n$ be the corresponding action coordinates satisfying $\alpha^{\pm}(0) = 0$. Then the map

$$\alpha = \begin{cases} \alpha^+ & \text{on } U^+, \\ \alpha^- & \text{on } U^-, \end{cases} \tag{17}$$

is an admissible change of coordinates.

Proof. Action coordinates $\alpha^{\pm} = (\alpha_1^{\pm}, \ldots, \alpha_n^{\pm})$ are defined by the integral

$$\alpha_j^{\pm}(b) = \int_0^b \lambda_j^{\pm}.$$

along a curve in $U^{\pm}$ joining 0 and b. When $j = 1$, this gives $\alpha_1^+ = \alpha_1^- = b_1$. Clearly α is a diffeomorphism when restricted to U^+ or U^-. Moreover α is injective. The fact that α^+ and α^- coincide along Γ follows from (16) and the connectedness of Γ. In fact (16) implies that when $b \in \Gamma$ the above integral gives

$$\alpha_j^+(b) = \int_0^b \bar{\lambda}_j^+ = \int_0^b \bar{\lambda}_j^- = \alpha_j^-(b).$$

This concludes the proof. $\square$

Remark 4.10. The upshot of Lemma 4.9 is that after a change of coordinates as in (17) we can always assume that the coordinates on the base U, when restricted to $U^{\pm}$, are action coordinates corresponding to bases $\{\gamma_1, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm}\}$ of $H_1(X, \mathbb{Z})$ satisfying (i) and (ii) of Corollary 4.6. Then $\{db_1, db_2, \ldots, db_n\}$ form a basis of Λ_+ and Λ_-. From (15) it also follows that, in view of the identifications of Remark 4.8, ℓ_1 must satisfy

$$\int_{\bar{\lambda}_j} \ell_1 = m_j.$$

The reader should be warned at this point that, although the map α as in (17) allows us to find action coordinates on both U^+ and U^-, we still have two different sets of action-angle coordinates, $(b_1, \ldots, b_n, y_1, \ldots, y_n)$ on X^+ and $(b_1, \ldots, b_n, t_1, \ldots, t_n)$ on X^-. This is due to the discrepancy between Θ^+ and Θ^-, which makes the map:

$$\Theta = \begin{cases} (\Theta^+)^{-1} & \text{on } X^+ \\ (\Theta^-)^{-1} & \text{on } X^- \end{cases}$$

discontinuous along the seam Z. As pointed out before, this discrepancy is measured by ℓ_1.

In the next theorem we show that any fibrewise closed section $\ell_1 \in \mathfrak{L}^*$ can be the first order invariant of a stitched Lagrangian fibration.

Theorem 4.11. Let U be an open contractible neighbourhood of $0 \in \mathbb{R}^n$ such that $\Gamma = U \cap \{b_1 = 0\}$ is contractible. Let $\bar{\Lambda} \subseteq T^*\Gamma$ be the lattice spanned by $\{db_2, \ldots, db_n\}$, and let $\bar{Z} = T^*\Gamma/\bar{\Lambda}$, with projection $\bar{\pi} : \bar{Z} \to \Gamma$ and bundle $\mathfrak{L} = \ker \bar{\pi}_*$. Given integers $m_2, \ldots, m_n$ and a smooth, fibrewise closed section ℓ_1 of $\mathfrak{L}^*$ such that

$$\int_{db_j} \ell_1 = m_j \quad \text{for all} \quad j = 2, \ldots, n, \tag{18}$$

there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ satisfying the following properties:

(i) the coordinates $(b_1, \ldots, b_n)$ on U are action coordinates of f with $\mu = f^* b_1$;
(ii) the periods $\{db_1, \ldots, db_n\}$, restricted to $U^\pm$ correspond to basis $\{\gamma_1, \gamma_2^\pm, \ldots, \gamma_n^\pm\}$ of $H_1(X, \mathbb{Z})$ satisfying *(i)* and *(ii)* of Corollary 4.6;
(iii) ℓ_1 is the first order invariant of (X, f).

Proof. We regard the two halves of U, U^+ and U^- defined as before, as disjoint sets. Let $\Lambda_\pm$ be the lattices in $T^*U^\pm$ spanned by $\{db_1, db_2, \ldots, db_n\}$ and define $X^\pm = T^*U^\pm/\Lambda_\pm$, with corresponding projections $\pi^\pm$. Let $Z^\pm = \partial X^\pm = (\pi^\pm)^{-1}(\Gamma)$. Translations along the db_1 direction define an S^1 action on $Z^\pm$ such that $\bar{Z} = Z^\pm/S^1$. On T^*U^+ and T^*U^-, we consider canonical coordinates $(b_1, \ldots, b_n, y_1, \ldots, y_n)$ and $(b_1, \ldots, b_n, t_1, \ldots, t_n)$ respectively (or (b, y) and (b, t) for short). Coordinates on $\bar{Z}$ are given by $(b, \bar{t})$ (or $(b, \bar{y})$), where $b = (0, b_2, \ldots, b_n) \in \Gamma$ and $\bar{t} = (t_2, \ldots, t_n)$ (or $\bar{y} = (y_2, \ldots, y_n)$). For $j = 2, \ldots, n$, let

$$a_j = \ell_1(\partial_{t_j}) = \ell_1(\partial_{y_j})$$

On X^+, let $\eta_1 = \partial_{y_1}$ and $\eta_j^+ = \partial_{y_j}$ then on Z^+ we can define vector fields

$$\eta_j^- = \eta_j^+ - a_j\ \eta_1,$$

which is coherent with (14). We can define a map $Q : Z^- \to Z^+$ by composition of the flows of $\eta_1, \eta_2^-, \ldots, \eta_n^-$, i.e.

$$Q : (b, t_1, \ldots, t_n) \mapsto \Phi_{\eta_1}^{t_1} \circ \Phi_{\eta_2^-}^{t_2} \circ \ldots \circ \Phi_{\eta_n^-}^{t_n}(b, 0).$$

Clearly, Q can be written as in (13). One can easily see that the properties of ℓ_1 ensure that Q is a well defined fibre-preserving diffeomorphism which sends the cycles represented by db_1 and db_j in $H_1(Z^-, \mathbb{Z})$ to the cycles represented by db_1 and $db_j - m_j db_1$ in $H_1(Z^+, \mathbb{Z})$, $j = 2, \ldots, n$, respectively. Intuitively, Q identifies fibres of π^- inside Z^- with fibres of π^+ inside Z^+ after the latter ones have been twisted by iteratively flowing in the direction of η_j^-, $j = 2, \ldots, n$. Topologically we define

$$X = X^+ \cup_Q X^-.$$

To give X smooth and symplectic structures we have to extend the gluing map Q to open neighbourhoods of Z^+ and Z^-. Let open sets $\tilde{U}^+$ and $\tilde{U}^-$ be small enlargements of U^+ and U^- respectively, obtained by joining small open neighbourhoods of Γ to U^+ and U^-. Extend $\Lambda_\pm$ to lattices of $T^*\tilde{U}^\pm$ in a constant way. We look for neighbourhoods $V^\pm$ of $Z^\pm$ inside $T^*\tilde{U}^\pm/\Lambda_\pm$ and a symplectomorphism $\tilde{Q} : V^- \to V^+$ extending Q. One can achieve this by considering an "auxiliary" fibration. Suppose for now that we could find a neighbourhood V^+ of Z^+ and a smooth, proper S^1-invariant Lagrangian fibration $u : V^+ \to \mathbb{R}^n$, with components u_j such that:

$$\begin{aligned} &u_1 = b_1, \\ &u|_{Z^+} = \pi^+, \\ &\eta_{u_j}|_{Z^+} = \eta_j^-, \quad \text{when } j = 2, \ldots, n. \end{aligned} \tag{19}$$

This amounts to prescribing zero and first order terms of u along Z^+ in the Taylor expansion of u with respect to b_1. Now inside $\tilde{U}^-$ there will be a small open neighbourhood W of Γ and a symplectomorphism:

$$\tilde{Q} : V^- \to V^+,$$

where $V^- := (\pi^-)^{-1}(W)$ and

$$\tilde{Q} : (b, t_1, \ldots, t_n) \mapsto \Phi_{\eta_1}^{t_1} \circ \Phi_{\eta_{u_2}}^{t_2} \circ \ldots \circ \Phi_{\eta_{u_n}}^{t_n}(b, 0).$$

In other words, $\tilde{Q}$ is the action-angle coordinate map associated to the fibration $u : V^+ \to \mathbb{R}^n$, computed with respect to the cycles $\{db_1, -m_2 db_1 + db_2, \ldots, -m_n db_1 + db_n\}$ (it may be necessary, for this purpose, to restrict to a smaller V^+). From (19) it follows that $\tilde{Q}$ extends Q. We define

$$X = (X^+ \cup V^+) \cup_{\tilde{Q}} (X^- \cup V^-).$$

and the stitched Lagrangian fibration to be

$$f = \begin{cases} \pi^+ & \text{on } X^+ \\ \pi^- & \text{on } X^-. \end{cases}$$

Due to the non-triviality of the gluing map $\tilde{Q}$ used to define X, f is in general piecewise smooth. In fact if we pull back f via the inclusion $X^+ \cup V^+ \hookrightarrow X$, then we obtain

$$f|_{X^+ \cup V^+} = \begin{cases} \pi^+ & \text{on } b_1 \geq 0 \\ u & \text{on } b_1 \leq 0. \end{cases}$$

This is because $\pi^- = \tilde{Q}^* u$. By construction (X, ω) and f satisfy the conditions $(i) - (iii)$.

Now we prove that a fibration $u : V^+ \to \mathbb{R}^n$ satisfying (19) exists. For every $b \in \Gamma$, consider the following one-parameter family of closed 1-forms on the fibre $F_b = (\pi^+)^{-1}(b)$

$$\ell(r) = r(dy_1 + \ell_1),$$

where $r \in \mathbb{R}$. For every r, the graph of $\ell(r)$ defines a Lagrangian submanifold inside T^*F_b. For r sufficiently small, let $L_{r,b}$ be the image of the graph of $\ell(r)$ under the symplectomorphism

$$(y_1, \ldots, y_n, \sum_{k=1}^{n} x_k dy_k) \mapsto (x_1, b_2 + x_2, \ldots, b_n + x_n, y_1, \ldots, y_n),$$

between a neighbourhood of the zero section of T^*F_b and a neighbourhood of F_b inside $T^*\tilde{U}^+/\Lambda_+$. Then there will be a sufficiently small neighbourhood V^+ of Z^+ which is fibred by the submanifolds $L_{r,b}$, i.e. on which the manifolds $L_{r,b}$ are the fibres of a Lagrangian fibration $u : V^+ \to \mathbb{R}^n$. This is due to the fact that the map

$$(r, b_2, \ldots, b_n, y_1, \ldots, y_n) \mapsto (r, b_2 + ra_2(b, \bar{y}), \ldots, b_n + ra_n(b, \bar{y}), y_1, \ldots, y_n)$$

is a diffeomorphism when restricted to a neighbourhood of $\{0\} \times Z^+$ inside $\mathbb{R} \times Z^+$. We now show that a possible choice of u also satisfies (19). Notice that u will be S^1-invariant since its fibres $L_{r,b}$ are S^1-invariant. Given $(b', y') \in V^+$, there exists a unique $(r, b) \in \mathbb{R} \times Z^+$ such that $L_{r,b} \subset V^+$ and $(b', y') \in L_{r,b}$. In fact (r, b) can be determined as a function of (b', y') by solving the non linear system

$$\begin{cases} r = b'_1 \\ b_j + ra_j(b, y') = b'_j & \text{when } j = 2, \ldots, n \end{cases} \tag{20}$$

using the implicit function theorem. Now define

$$u_1(b', y') = b'_1$$

and, when $j = 2, \ldots, n$

$$u_j(b', y') = b_j, \tag{21}$$

where b_j (and thus b) are functions of (b', y'). Notice that S^1-invariance of u_j can also be seen from the fact that u_j is independent of y_1. It is clear that, when $j = 2, \ldots, n$

$$\begin{cases} \partial_{y'_k} u_j|_{Z^+} = 0 & \text{for all } k = 1, \ldots, n \\ \partial_{b'_k} u_j|_{Z^+} = \delta_{kj} & \text{for all } k = 2, \ldots, n. \end{cases}$$

Therefore

$$\eta_{u_j}|_{Z^+} = \partial_{b'_1} b_j \, \partial_{y_1} + \partial_{y_j}.$$

Using (20) we compute that

$$\partial_{b'_1} b_j|_{Z^+} = -a_j,$$

which proves that conditions (19) are satisfied. $\square$

5. Higher order terms

In Theorem 4.11 we provided a (local) construction of stitched Lagrangian fibrations with any given first order invariant satisfying integrality conditions (18). It involved the choice of a Poisson commuting set of functions $u_1, \dots, u_n$ (producing a Lagrangian fibration u) defined on a neighbourhood of Z and with prescribed 0-th and 1-st order terms (cf. (19)). In general there may be many choices of such functions giving stitched Lagrangian fibrations which are not fibrewise symplectomorphic. It is necessary to look at higher order terms. In this Section we give a description of these higher order terms and prove an existence result of stitched Lagrangian fibrations with prescribed higher order terms..

We fix here some basic notation. Let $(b_1, \dots, b_n)$ be standard coordinates on $\mathbb{R}^n$. Let $\mathbb{R}^{n-1}$ be embedded in $\mathbb{R}^n$ as the subset $\{b_1 = 0\}$ and let $\Gamma \subset \mathbb{R}^{n-1}$ be an open neighbourhood of $0 \in \mathbb{R}^{n-1}$. We will denote by U an open neighbourhood of Γ in $\mathbb{R}^n$. We assume that the pair (U, Γ) is diffeomorphic to the pair (D^n, D^{n-1}) where $D^k \subset \mathbb{R}^k$ is a unit ball centred at 0. Denote $U^+ = U \cap \{b_1 \geq 0\}$ and $U^- = U \cap \{b_1 \leq 0\}$. Then $\Gamma = U^+ \cap U^-$. Let Λ be the lattice in T^*U generated by $\{db_1, \dots, db_n\}$ and consider T^*U/Λ with the standard symplectic form and with projection onto U denoted by π. We assume S^1 acts on T^*U/Λ via translations along the db_1 direction. Let $Z = \pi^{-1}(\Gamma)$ and $\bar{Z} = Z/S^1$. If $\bar{\Lambda}$ denotes the lattice in $T^*\Gamma$ spanned by $\{db_2, \dots, db_n\}$, we have $\bar{Z} = T^*\Gamma/\bar{\Lambda}$ with projection $\bar{\pi}$. Given $b \in \Gamma$, we denote $F_b = \pi^{-1}(b)$ and $\bar{F}_b = \bar{\pi}^{-1}(b) = F_b/S^1$. Canonical coordinates on T^*U are denoted by $(b, y) = (b_1, \dots, b_n, y_1, \dots, y_n)$. We also have the bundle $\mathfrak{L} = \ker \bar{\pi}_*$.

Throughout this section we will study the set defined in the following

Definition 5.1. We define $\mathscr{U}_{\bar{Z}}$ to be the set of pairs (V, u) where V is a neighbourhood of Z and $u : V \to \mathbb{R}^n$ is a C^∞, proper, S^1-invariant, Lagrangian submersion, with components $(u_1, \dots, u_n)$, such that $u|_Z = \pi$ and $u_1 = b_1$.

Given $(V, u) \in \mathscr{U}_{\bar{Z}}$, let $Y^+ := \pi^{-1}(U^+)$, $Y := Y^+ \cup V$, $Y^- := Y \cap \pi^{-1}(U^-)$ and define the map $f_u : Y \to \mathbb{R}^n$ by

$$f_u = \begin{cases} u & \text{on } Y^-, \\ \pi & \text{on } Y^+. \end{cases} \tag{22}$$

Clearly (Y, f_u) is a stitched Lagrangian fibration. We study the aforementioned higher order terms of such fibrations.

Proposition 5.2. Let $(V, u) \in \mathscr{U}_{\bar{Z}}$. For every $N \in \mathbb{N}$ and $j = 2, \dots, n$, consider the N-th order Taylor series expansion of u_j in the variable b_1, evaluated at $b_1 = 0$:

$$u_j = \sum_{k=0}^{N} S_{j,k} b_1^k + o(b_1^N), \tag{23}$$

where $S_{j,k}$ are smooth functions on Z which are S^1 invariant (i.e. independent of y_1). For every $m \in \mathbb{N}$, define the following sections of $\mathfrak{L}^*$ and $\Lambda^2\, \mathfrak{L}^*$ respectively

$$S_m = \sum_{j=2}^{n} S_{j,m}\, dy_j \tag{24}$$

and

$$\begin{cases} P_1 = 0 \\ P_m = \sum_{j<l}^{n} \left(\sum_{k=1}^{m-1} \{S_{j,k}, S_{l,m-k}\} \right) dy_j \wedge dy_l \quad \text{when } m \geq 2. \end{cases} \tag{25}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket on $\bar{Z}$. Then on every fibre $\bar{F}_b$, S_m and P_m satisfy the following equations

$$d\, S_m|_{\bar{F}_b} = P_m|_{\bar{F}_b}. \tag{26}$$

Proof. We recall that the Poisson bracket on T^*U/Λ can be written as

$$\{f, g\} = \sum_{k=1}^{n} \partial_{y_k} f\, \partial_{b_k} g - \partial_{b_k} f\, \partial_{y_k} g.$$

Since the functions $S_{j,m}$ do not depend on y_1, one can easily see that the following holds

$$\{S_{j,k}\, b_1^k,\, S_{l,m}\, b_1^m\} = \{S_{j,k},\, S_{l,m}\}\, b_1^{k+m},$$

where the bracket on the right hand side reduces to the bracket on $\bar{Z}$. Also we have

$$\{S_{j,k}\, b_1^k,\, o(b_1^N)\} = o(b_1^{N+k}).$$

Thus we have

$$\begin{aligned}\{u_j, u_l\} &= \sum_{0\leq m+k\leq N} \{S_{j,k},\, S_{l,m}\}\, b_1^{m+k} + o(b_1^N) \\ &= \sum_{m=0}^{N} \left(\sum_{k=0}^{m} \{S_{j,k},\, S_{l,m-k}\} \right) b_1^m + o(b_1^N).\end{aligned}$$

Therefore if u_j and u_l commute then we must have that for all $m \in \mathbb{N}$

$$\sum_{k=0}^{m} \{S_{j,k},\, S_{l,m-k}\} = 0,$$

or that

$$\{S_{j,m}, S_{l,0}\} + \{S_{j,0}, S_{l,m}\} = -\sum_{k=1}^{m-1} \{S_{j,k},\, S_{l,m-k}\}. \tag{27}$$

The condition that $u|_Z = \pi$ implies

$$S_{j,0} = b_j.$$

Therefore

$$\{S_{j,m}, S_{l,0}\} = \{S_{j,m}, b_l\} = \partial_{y_l} S_{j,m}.$$

We then see that (27) becomes

$$\partial_{y_l} S_{j,m} - \partial_{y_j} S_{l,m} = -\sum_{k=1}^{m-1} \{S_{j,k},\, S_{l,m-k}\}$$

which is exactly what we get by expanding (26). □

Remark 5.3. Notice that (26) are a set of partial differential equations satisfied by the sequence $\{S_m\}_{m\in\mathbb{N}}$. Moreover the definition of P_m depends only on the S_k's with $k \leq m-1$, therefore one may think of solving the equations recursively. From each solution S_m of the m-th equation, we may determine another by adding to S_m a fibrewise closed section of $\mathfrak{L}^*$.

Now we provide a method to construct and characterise sequences $\{S_m\}_{m\in\mathbb{N}}$ of solutions to (26). Suppose $(V, u) \in \mathscr{U}_{\bar{Z}}$ and let $W \subseteq u(V)$ be a neighbourhood of Γ. Let $r \in \mathbb{R}$ be a parameter. For $b = (0, b_2, \ldots, b_n) \in \Gamma$, let (r, b) denote the point $(r, b_2, \ldots, b_n) \in \mathbb{R}^n$. Given $(r, b) \in W$, denote by $L_{r,b}$ the fibre $u^{-1}((r, b))$. For every fibre $F_b \subset Z$ of π, consider the symplectomorphism

$$(y_1, \ldots, y_n, \sum_{k=1}^{n} x_k dy_k) \mapsto (x_1, b_2 + x_2, \ldots, b_n + x_n, y_1, \ldots, y_n), \tag{28}$$

between a neighbourhood of the zero section of T^*F_b and a neighbourhood of F_b in V. If W is sufficiently small, for every $(r, b) \in W$, the Lagrangian submanifold $L_{r,b}$ will be the image of the graph of a closed 1-form on F_b. Due to the S^1 invariance of u and the fact that $u_1 = b_1$, this 1-form has to be of the type

$$r dy_1 + \ell(r, b),$$

where $\ell(r, b)$ is the pull back to F_b of a closed one form on $\bar{F}_b$. Denote by $\ell(r)$ the smooth one parameter family of sections of $\mathfrak{L}^*$ such that $\ell(r)|_{\bar{F}_b} = \ell(r, b)$. The condition $u|_Z = \pi$ implies that $\ell(0, b) = 0$. Furthermore, the N-th order Taylor series expansion of $\ell(r)$ in the parameter r can be written as

$$\ell(r) = \sum_{k=1}^{N} \ell_k\, r^k + o(r^N), \tag{29}$$

where the ℓ_k's are fibrewise closed sections of $\mathfrak{L}^*$. We can write

$$\ell_k = \sum_{j=2}^{n} a_{j,k}\, dy_j. \tag{30}$$

The following Lemma is rather technical but straightforward, thus its proof may be skipped on first reading.

Lemma 5.4. Given $(V, u) \in \mathscr{U}_{\bar{Z}}$, let $\{S_m\}_{m\in\mathbb{N}}$ be the sequence (24) of sections of $\mathcal{L}^*$ encoding the Taylor coefficients of u and let $\{\ell_m\}_{m\in\mathbb{N}}$ be the sequence of fibrewise closed sections of $\mathfrak{L}^*$ constructed from u as above. Then for every $m \in \mathbb{N}$, there exist formulae

$$a_{j,m} = -S_{j,m} + R_{j,m}, \tag{31}$$

where $R_{j,m}$ is an explicit polynomial expression depending on the $S_{l,k}$'s and their derivatives in the b_i's up to order $m-1$ and with $0 \le k \le m-1$. In particular $R_{j,1} = 0$. Thus the sequence $\{\ell_m\}_{m\in\mathbb{N}}$ uniquely determines the sequence $\{S_m\}_{m\in\mathbb{N}}$ recursively and viceversa.

Proof. First of all let us write

$$\ell(r) = r\, dy_1 + \sum_{j=2}^{n} a_j(r) dy_j.$$

Then by definition

$$a_j(r) = \sum_{k=1}^{N} a_{j,k}\, r^k + o(r^N). \tag{32}$$

The a_j's are functions of (r, b, y), with $(b, y) \in \bar{Z}$, satisfying by construction

$$\begin{cases} u_1(r, b_2 + a_2, \dots, b_n + a_n, y) = r, \\ u_j(r, b_2 + a_2, \dots, b_n + a_n, y) = b_j \quad \text{for all } j = 2, \dots, n. \end{cases} \tag{33}$$

When W is sufficiently small and $(r, b) \in W$, this system can be solved using the implicit function theorem to determine the a_j's uniquely. We will now use it to compute the $a_{j,m}$'s and determine the formulae (31).

Let $j = 2, \dots, n$, then from the system and the conditions on u we obtain

$$a_j|_{r=0} = 0$$

and

$$\partial_{b_1} u_j + \sum_{k=2}^{n} \partial_{b_k} u_j\, \partial_r a_k = 0. \tag{34}$$

When evaluating at $r = 0$, using $u|_Z = \pi$, we get

$$\partial_{b_1} u_j|_{r=0} + \partial_r a_j|_{r=0} = 0,$$

i.e. that

$$a_{j,1} = -S_{j,1}. \tag{35}$$

Now we do the second order terms. Derivating (34) we obtain

$$\partial^2_{b_1} u_j + \sum_{k=2}^{n} \partial_{b_1}\partial_{b_k} u_j\, \partial_r a_k + \sum_{k,l=2}^{n} \partial_{b_l}\partial_{b_k} u_j\, \partial_r a_l\, \partial_r a_k + \sum_{k=2}^{n} \partial_{b_k} u_j\, \partial^2_r a_k = 0.$$

Evaluating at $r = 0$ we get

$$\partial^2_{b_1} u_j|_{r=0} + \left(\sum_{k=2}^{n} \partial_{b_1}\partial_{b_k} u_j\, \partial_r a_k \right)|_{r=0} + \partial^2_r a_j|_{r=0} = 0,$$

i.e. we obtain

$$a_{j,2} = -S_{j,2} + \sum_{k=2}^{n} \partial_{b_k} S_{j,1}\, S_{k,1}. \tag{36}$$

So we have that (31) holds for $m = 2$, where

$$R_{j,2} = \sum_{k=2}^{n} \partial_{b_k} S_{j,1}\, S_{k,1}.$$

For the terms of order greater that two we refer the reader to the Appendix in §9. $\square$

Remark 5.5. We point out that (35) shows that the definition of ℓ_1 given in this section coincides with the first order invariant defined in the previous section.

One good reason to work with the sequence $\{\ell_k\}_{k\in\mathbb{N}}$ rather than with the sequence $\{S_m\}_{m\in\mathbb{N}}$ is that we can easily prove the following

Proposition 5.6. Given any sequence $\{\ell_k\}_{k\in\mathbb{N}}$ of fibrewise closed sections of $\mathfrak{L}^*$, there exists a smooth 1-parameter family $\ell(r)$ of fibrewise closed sections of $\mathfrak{L}^*$ such that (29) holds for every $N \in \mathbb{N}$.

The proof of this is based on the following general:

Lemma 5.7. For any sequence of C^∞ functions $\{\alpha_k : \mathbb{R}^p \to \mathbb{R}\}$, there is a C^∞ function $f : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$, such that $\alpha_k(x) = \partial_r^k f(r,x)|_{r=0}$, for all $k \in \mathbb{N}$.

A proof of this Lemma in the case when $\{\alpha_k\}$ is a sequence of real numbers is hinted in [15] Exercise 13, page 384. It is an exercise to show that the method proposed there can be adapted to the case when α_k depends smoothly on a parameter $x \in \mathbb{R}^p$.

Proof of Proposition 5.6. Let us first prove the statement assuming that all the ℓ_k's are fibrewise exact, i.e. there exists a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ on $\bar{Z}$ such that

$$\ell_k|_{\bar{F}_b} = d\, f_k|_{\bar{F}_b}.$$

We have $\bar{Z} \cong \mathbb{R}^{n-1} \times T^{n-1}$, where T^{n-1} is the $(n-1)$-torus. Let $\{U_\alpha, \phi_\alpha\}_{\alpha\in J}$ be a partition of unity on T^{n-1}. Define

$$f_{k,\alpha} = \sqrt{\phi_\alpha} f_k.$$

We apply Lemma 5.7, for every $\alpha \in J$, to the sequence $\{f_{k,\alpha}\}_{k\in\mathbb{N}}$ lifted to the covering $\mathbb{R}^{n-1}$ of T^{n-1}. So there exists a C^∞ function $f_\alpha = f_\alpha(r)$ such that

$$f_\alpha(r) = \sum_{k=1}^{N} f_{k,\alpha}\, r^k + o(r^N),$$

for every $N \in \mathbb{N}$. Let

$$f(r) = \sum_{\alpha\in J} \sqrt{\phi_\alpha}\, f_\alpha(r).$$

Then $f(r)$ descends to a smooth 1-parameter family of functions on $\bar{Z}$. Moreover

$$\begin{aligned} f(r) &= \sum_{k=1}^{N} \left(\sum_{\alpha\in J} \sqrt{\phi_\alpha} f_{k,\alpha} \right) r^k + o(r^N) \\ &= \sum_{k=1}^{N} \left(\sum_{\alpha\in J} \phi_\alpha f_k \right) r^k + o(r^N) \\ &= \sum_{k=1}^{N} f_k\, r^k + o(r^N). \end{aligned}$$

If we let $\ell(r)$ be the 1-parameter family of sections of $\mathfrak{L}^*$ such that

$$\ell(r)|_{\bar{F}_b} = d\, f(r)|_{\bar{F}_b},$$

then we clearly have

$$\ell(r) = \sum_{k=1}^{N} \ell_k\, r^k + o(r^N).$$

We now do the general case. There certainly is a sequence $\{l_k\}_{k\in\mathbb{N}}$ of fibrewise constant sections of $\mathfrak{L}^*$ such that for every $k \in \mathbb{N}$, $\ell_k - l_k$ is fibrewise exact. Since l_k is fibrewise constant we can write

$$l_k = \sum_{j=2}^{n} q_{j,k}\, dy_j,$$

where the $q_{j,k}$'s are fibrewise constant functions. Invoking Lemma 5.7, for every $j = 2,\ldots,n$, there exists a family of fibrewise constant functions $q_j(r)$ such that

$$q_j(r) = \sum_{k=1}^{N} q_{j,k}\, r^k + o(r^N).$$

Let

$$l(r) = \sum_{j=2}^{n} q_j(r) dy_j,$$

and let $\tilde{\ell}(r)$ be the fibrewise exact family of forms such that

$$\tilde{\ell}(r) = \sum_{k=1}^{N} (\ell_k - l_k)\, r^k + o(r^N),$$

which exists from the previous step. Define

$$\ell(r) = \tilde{\ell}(r) + l(r).$$

One easily checks that (29) holds. □

The following is an existence result

Theorem 5.8. Let $\{\ell_m\}_{m\in\mathbb{N}}$ be a sequence of fibrewise closed sections of $\mathfrak{L}^*$ and let $\{S_m\}_{m\in\mathbb{N}}$ be the sequence of sections of $\mathfrak{L}^*$ obtained recursively from $\{\ell_m\}_{m\in\mathbb{N}}$ using formulae (31) in Proposition 5.4, then there exists $(V,u) \in \mathscr{U}_{\bar{Z}}$ such that for every $N \in \mathbb{N}$

$$u_j = \sum_{k=0}^{N} S_{j,k}\, b_1^k + o(b_1^N).$$

Proof. Following Proposition 5.6, given the sequence $\{\ell_m\}_{m\in\mathbb{N}}$, we can construct $\ell(r)$, a smooth 1-parameter family of fibrewise closed sections of $\mathfrak{L}^*$ satisfying (29). We show that $\ell(r)$ can be used to construct the pair (V,u). In fact the process is the inverse of the one which led us to the construction of a family $\ell(r)$ from a fibration u. The construction is identical to the one in the proof of Theorem 4.11. Denote $\ell(r,b) = \ell(r)|_{\bar{F}_b}$ and write

$$\ell(r,b) = \sum_{j=2}^{n} a_j(r,b) dy_j,$$

where the $a_j(r,b)$'s are functions depending on y and they satisfy

$$a_j(r,b) = \sum_{k=1}^{N} a_{j,k}(b)\, r^k + o(r^N). \tag{37}$$

Let $L_{r,b}$ be the Lagrangian submanifold of T^*U/Λ which is the image of the closed one form

$$r dy_1 + \ell(r,b)$$

under the symplectomorphism (28). When $W \subseteq U$ is sufficiently small and $(r,b) \in W$, then the submanifolds $L_{r,b}$ are the fibres of a Lagrangian fibration $u : V \to \mathbb{R}^n$. We describe u explicitly and show that $(V,u) \in \mathscr{U}_{\bar{Z}}$. Given $(b',y') \in V$, there exists a unique $(r,b) \in W$ such that $L_{r,b} \subset V$ and $(b',y') \in L_{r,b}$, in fact (r,b) can be determined as functions of (b',y') by solving the non linear system

$$\begin{cases} r = b_1' \\ b_j + a_j(r,b,y') = b_j' \quad \text{when } j = 2,\ldots,n \end{cases} \tag{38}$$

using the implicit function theorem. We define

$$u_1(b', y') = b'_1$$

and, when $j = 2, \dots, n$

$$u_j(b', y') = b_j(b', y'). \tag{39}$$

We claim that the coefficients of the Taylor series expansion of u_j in b'_1 are exactly the coefficients $S_{j,m}$ obtained from the sequence $\{\ell_m\}_{m\in\mathbb{N}}$ through formulae (31). In fact notice that, by construction of u_j, the functions a_j satisfy

$$u_j(r, b_2 + a_2(r, b, y'), \dots, b_n + a_n(r, b, y')) = b_j,$$

i.e. they are obtained from u_j as the unique solution to system (33) and therefore the claim follows from the proof of Lemma 5.4. □

Corollary 5.9. Let $\{\ell_m\}_{m\in\mathbb{N}}$ and $\{S_m\}_{m\in\mathbb{N}}$ be sequences of sections of $\mathfrak{L}^*$ with coefficients $a_{j,k}$ and $S_{j,k}$, respectively, related by formulae (31). Then all the ℓ_m's are fibrewise closed if and only if the sequence $\{S_m\}_{m\in\mathbb{N}}$ satisfies equations (26).

Proof. If all the ℓ_m's are fibrewise closed, then Theorem 5.8 shows that there is a Lagrangian fibration $u : V \to \mathbb{R}^n$ whose Taylor coefficients are given by the sequence $\{S_m\}_{m\in\mathbb{N}}$. Being u Lagrangian, the claim follows from Proposition 5.2.

Suppose now that $\{S_m\}_{m\in\mathbb{N}}$ satisfies equations (26). We prove the claim by induction. First of all notice that when $m = 1$, $S_1 = -\ell_1$ and equation (26) implies that ℓ_1 is fibrewise closed. Now suppose we have proved that ℓ_m is fibrewise closed for all $m \leq N$. Consider the sequence $\{\tilde{\ell}_m\}_{m\in\mathbb{N}}$, where $\tilde{\ell}_m = \ell_m$ when $m \leq N$ and 0 otherwise. Using formulae (31), we construct the associated sequence $\{\tilde{S}_m\}_{m\in\mathbb{N}}$. Since all the $\tilde{\ell}_m$'s are fibrewise closed, from the first part of this Corollary it follows that $\{\tilde{S}_m\}_{m\in\mathbb{N}}$ satisfies equations (26). Denote by $\tilde{P}_m$ the 2-forms in (25) constructed from $\{\tilde{S}_m\}_{m\in\mathbb{N}}$. Now notice that

$$\tilde{S}_m = S_m,$$

when $m \leq N$ and

$$\tilde{P}_m = P_m$$

when $m \leq N + 1$. Moreover, if we denote by $\tilde{R}_{j,k}$ the expressions $R_{j,k}$ appearing in (31) applied to $\{\tilde{S}_m\}_{m\in\mathbb{N}}$, then

$$\tilde{R}_{j,N+1} = R_{j,N+1},$$

where the right hand side denotes the same expression obtained using $\{S_m\}_{m\in\mathbb{N}}$. Therefore formula (31) with $m = N + 1$ and the fact that $\tilde{\ell}_{N+1} = 0$, implies

$$\tilde{S}_{j,N+1} = \tilde{R}_{j,N+1} = R_{j,N+1}. \tag{40}$$

Define the one form

$$R_{N+1} = \sum_{j=2}^{n} R_{j,N+1} dy_j.$$

Clearly (40) says that

$$\tilde{S}_{N+1} = R_{N+1}$$

and that equation (26) for $\{\tilde{S}_m\}_{m\in\mathbb{N}}$ when $m = N + 1$ becomes

$$dR_{N+1}|_{\bar{F}_b} = \tilde{P}_{N+1}|_{\bar{F}_b} = P_{N+1}|_{\bar{F}_b}. \tag{41}$$

Using (31) for $\{S_m\}_{m\in\mathbb{N}}$ when $m = N + 1$, we obtain

$$d\ell_{N+1}|_{\bar{F}_b} = -dS_{N+1}|_{\bar{F}_b} + dR_{N+1}|_{\bar{F}_b}.$$

Now substituting (41) and using the fact that (26) holds for $\{S_m\}_{m\in\mathbb{N}}$ when $m = N + 1$ we obtain

$$d\ell_{N+1}|_{\bar{F}_b} = -dS_{N+1}|_{\bar{F}_b} + P_{N+1}|_{\bar{F}_b} = 0,$$

which completes the proof. □

Finally we have the most general existence result

Theorem 5.10. Let $\{S_m\}_{m\in\mathbb{N}}$ be a sequence of sections of $\mathfrak{L}^*$, satisfying (26), then there exists $(V,u) \in \mathscr{U}_{\bar{Z}}$ such that for every $N \in \mathbb{N}$

$$u_j = \sum_{k=0}^{N} S_{j,k}\, b_1^k + o(b_1^N).$$

Proof. It is just a matter of applying the previous results. In fact given the sequence $\{S_m\}_{m\in\mathbb{N}}$ satisfying (26), using Proposition 5.4 we construct the sequence $\{\ell_m\}_{m\in\mathbb{N}}$, whose terms are all fibrewise closed thanks to Corollary 5.9. Finally we apply Theorem 5.8 to obtain u. □

Define the following sets:

$$\mathscr{L}_{\bar{Z}} = \{\{\ell_m\}_{m\in\mathbb{N}} \mid \ell_m \text{ is a } C^\infty,\ \text{fibrewise closed section of } \mathfrak{L}^*\};$$

$$\mathscr{S}_{\bar{Z}} = \{\{S_m\}_{m\in\mathbb{N}} \mid S_m \text{ is a } C^\infty \text{ section of } \mathfrak{L}^* \text{ satisfying (26)}\}.$$

Clearly Proposition 5.2 gives a map $T : \mathscr{U}_{\bar{Z}} \to \mathscr{S}_{\bar{Z}}$, assigning to (V,u) the Taylor coefficients of u. We summarise the previous results in the following:

Theorem 5.11. There is a one to one correspondence between the sets $\mathscr{L}_{\bar{Z}}$, $\mathscr{S}_{\bar{Z}}$ and $T(\mathscr{U}_{\bar{Z}})$. In particular from every sequence $\ell \in \mathscr{L}_{\bar{Z}}$ we can construct a unique element $S(\ell) \in \mathscr{S}_{\bar{Z}}$ and an element $(V,u) \in \mathscr{U}_{\bar{Z}}$ such that $T(V,u) = S(\ell)$.

Given two elements (V,u) and $(\tilde{V},\tilde{u}) \in \mathscr{U}_{\bar{Z}}$, we can construct two stitched Lagrangian fibrations (Y,f_u) and $(\tilde{Y},f_{\tilde{u}})$ as in (22). We recall that f_u and $f_{\tilde{u}}$ are equivalent up to a change of coordinates on the base if $f_{\tilde{u}} = \phi \circ f_u$, where $\phi : W \to \phi(W) \subseteq \mathbb{R}^n$ is an admissible change of coordinates on the base. If we write $\phi = (\phi_1,\ldots,\phi_n)$, then ϕ must satisfy $\phi_1 = b_1$ and $\phi|_{U^+} = \mathrm{Id}$.

Similarly, we say that two sequences $\ell, \tilde{\ell} \in \mathscr{L}_{\bar{Z}}$ are equivalent up to a change of coordinates in the base if they define fibrations f_u and $f_{\tilde{u}}$ respectively which are equivalent up to a change of coordinates in the base. We now describe this equivalence relation in terms of a group action. Given a change of coordinate map ϕ on the base satisfying the above properties, we can consider its Taylor expansion in b_1 from the left, i.e. where the coefficients are given by left derivatives. For each component ϕ_j, $j = 2,\ldots,n$, it can be written as

$$\phi_j(b_1,\ldots,b_n)|_{W\cap U^-} = b_j + \sum_{k=1}^{N} \Phi_{j,k}(b_2,\ldots,b_n) b_1^k + o(b_1^N).$$

The left Taylor coefficients of ϕ thus define a sequence $\{\Phi_m\}_{m\in\mathbb{N}}$, where $\Phi_m : \Gamma \to \mathbb{R}^{n-1}$ is a C^∞ map whose components are $\Phi_m = (\Phi_{2,m},\ldots,\Phi_{n,m})$.

Lemma 5.12. Given any sequence $\{\Phi_m\}_{m\in\mathbb{N}}$ of smooth maps $\Phi_m : \Gamma \to \mathbb{R}^{n-1}$ with components $\Phi_m = (\Phi_{2,m},\ldots,\Phi_{n,m})$ there exists an admissible change of coordinate map $\phi = (\phi_1,\ldots,\phi_n)$ defined on some neighbourhood W of Γ such that $\phi_1 = b_1$, $\phi|_{U^+\cap W} = \mathrm{Id}$ and

$$\phi_j(b_1,\ldots,b_n)|_{U^-\cap W} = b_j + \sum_{k=1}^{N} \Phi_{j,k}(b_2,\ldots,b_n) b_1^k + o(b_1^N).$$

for all $N \in \mathbb{N}$.

Proof. It follows from Lemma 5.7. □

Define the following set

$$\mathscr{D}_\Gamma = \{\{\Phi_m\}_{m\in\mathbb{N}} \,|\, \Phi_m \in C^\infty(\Gamma,\mathbb{R}^{n-1})\}.$$

We say that two admissible change of coordinate maps ϕ and ϕ' are equivalent if their corresponding left Taylor coefficients define the same element in $\mathscr{D}_\Gamma$. We call $\mathscr{D}_\Gamma$ the set of **germs of admissible change of coordinates**. Given a germ $\Phi \in \mathscr{D}_\Gamma$ we say that an admissible change of coordinates ϕ is a representative of Φ, if ϕ satisfies Lemma 5.12.

Composition of germs of admissible maps induces a group structure on $\mathscr{D}_\Gamma$, i.e. given $\Phi, \Phi' \in \mathscr{D}_\Gamma$, we define $\Phi \cdot \Phi'$ to be the germ of the map $\phi \circ \phi'$, where ϕ and ϕ' are representatives of Φ and Φ' respectively. It is easy to see that this product on $\mathscr{D}_\Gamma$ does not depend on the choice of representatives.

The group $\mathscr{D}_\Gamma$ acts on the set $\mathscr{L}_{\bar{Z}}$ as follows. Given $\ell \in \mathscr{L}_{\bar{Z}}$ and $\Phi \in \mathscr{D}_\Gamma$, we define $\Phi \cdot \ell$ to be the sequence $\tilde{\ell} \in \mathscr{L}_{\bar{Z}}$ associated to the Lagrangian fibration $\tilde{u} = \phi \circ u$, where ϕ is a representative of Φ and u is a Lagrangian fibration obtained from ℓ via Theorem 5.8.

Lemma 5.13. The above action is well-defined.

Proof. We need to show that the action does not depend on the choices made. The sequence $\tilde{\ell} \in \mathscr{L}_{\bar{Z}}$ determines a unique sequence $\{\tilde{S}_m\}_{m\in\mathbb{N}} \in \mathscr{S}_{\bar{Z}}$, where each $\tilde{S}_m$ is defined in terms of the Taylor coefficients of $\tilde{u} = \phi \circ u$. These coefficients, in turn, are expressed in terms of the Taylor coefficients of ϕ and u. If we take a different representative ϕ' of Φ, clearly, the Taylor coefficients of $\phi' \circ u$ and $\phi \circ u$ coincide. Now let $S \in \mathscr{S}_{\bar{Z}}$ be the sequence determined by ℓ. If $u' \in \mathscr{U}_{\bar{Z}}$ is a different realisation of ℓ then, by construction, u' defines a sequence $S' \in \mathscr{S}_{\bar{Z}}$ such that $S = S'$. Therefore the Taylor coefficients of $\phi \circ u$ and $\phi \circ u'$ coincide. $\square$

Proposition 5.14. Let f_u and $f_{\tilde{u}}$ be two stitched Lagrangian fibrations constructed as in (22). If locally $f_{\tilde{u}} = \phi \circ f_u$ for some admissible change of coordinates, then the sequences $\ell, \tilde{\ell} \in \mathscr{L}_{\bar{Z}}$ associated to f_u and $f_{\tilde{u}}$ are in the same orbit of $\mathscr{D}_\Gamma$. Moreover f_u is equivalent to a smooth fibration up to a change of coordinates on the base if and only if ℓ is a sequence of fibrewise constant sections of $\mathfrak{L}^*$.

Proof. The first part of the statement is obvious. If $f_{\tilde{u}} = \phi \circ f_u$ is smooth, then $\tilde{\ell}$ is the zero sequence 0. It is easy to verify that $\ell = \Phi^{-1} \cdot 0$ is a sequence of fibrewise constant sections of $\mathfrak{L}^*$. Suppose, viceversa, that ℓ is a sequence of fibrewise constant sections. Consider the associated sequence $S \in \mathscr{S}_{\bar{Z}}$. The coefficients $S_{j,m}$ of each element $S_m \in S$ can be regarded as functions on the base Γ, therefore S also defines a sequence $\Phi \in \mathscr{D}_\Gamma$ by setting $\Phi_m = (S_{2,m}, \ldots, S_{n,m})$. Let ϕ be an admissible change of coordinates representing Φ. It is clear that $\phi^{-1} \circ f_u$ is smooth and that $\Phi^{-1} \cdot \ell = 0$. $\square$

In §6 we will consider equivalences up to smooth fibre preserving symplectomorphism.

6. The semiglobal classification

Let (X, ω) be a symplectic manifold and $f : X \to B$ be a stitched fibration as in Definition 3.1. Let $Z \subset X$ be the seam of f and let $\Gamma := f(Z) \subset B$. Since we are interested in a semiglobal classification, throughout this section we will consider stitched Lagrangian fibrations satisfying the following assumption

Assumption 6.1. The stitched Lagrangian fibration $f : X \to B$ satisfies the following condition

(1) the pair (B, Γ) is diffeomorphic to the pair (D^n, D^{n-1}) where $D^n \subset \mathbb{R}^n$ is a ball centred at the origin and $D^{n-1} \subset D^n$ is the intersection of D^n with an $n-1$ dimensional subspace;

Also, the following data is specified

(2) a basis $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of $H_1(X, \mathbb{Z})$ so that γ_1 is represented by the orbit of the S^1 action;
(3) a continuous section σ of f defined on a neighbourhood of Γ, such that $\sigma|_{f(X^+)}$ and $\sigma|_{f(X^-)}$ are restrictions of smooth maps on B and the image of σ is a smooth Lagrangian submanifold of X.

We denote a stitched Lagrangian fibration together with this data by $(X, B, f, \gamma, \sigma)$.

We will soon show that a section σ as in (3) always exists.

Definition 6.2. We say that two stitched fibrations $(X, B, f, \gamma, \sigma)$ and $(X', B', f', \gamma', \sigma')$, with seams Z and Z' respectively are **fibrewise symplectically equivalent** (or just equivalent) if there are neighbourhoods $W \subseteq B$ of $\Gamma := f(Z)$ and $W' \subseteq B'$ of $\Gamma' := f'(Z')$ and a commutative diagram:

$$\begin{array}{ccc} (f')^{-1}(W') & \xrightarrow{\Psi} & f^{-1}(W) \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W' & \xrightarrow{\phi} & W \end{array}$$

where Ψ is an S^1 equivariant C^∞ symplectomorphism sending Z' to Z and ϕ is a C^∞ diffeomorphism such that $\Psi \circ \sigma' = \sigma \circ \phi$ and $\Psi_* \gamma' = \gamma$. The set of equivalence classes under this relation will be denoted by $\mathscr{F}$ and elements therein will be called **germs of stitched fibrations**.

Now we show that any stitched fibration, satisfying Assumption 6.1, is fibrewise symplectically equivalent to a stitched fibration of the type (Y, f_u) studied in §5. Before doing this we need some preliminary results.

Recall we can write a stitched fibration as:

$$f = \begin{cases} f^+ & \text{on } X^+; \\ f^- & \text{on } X^-, \end{cases}$$

where $f^\pm$ is the restriction of a C^∞, S^1 invariant map to $X^\pm$ whose fibres are Lagrangian when restricted to $X^\pm$. As pointed out in §3, the fibres of such a map are not a priori required to be Lagrangian beyond $X^\pm$. Nevertheless we have the following:

Proposition 6.3. Let (X, B, f) be a stitched fibration with seam $Z \subset X$, satisfying condition (1) of Assumption 6.1. Then there are neighbourhoods $V \subseteq X$ of Z and $W \subseteq B$ of $\Gamma := f(Z)$ and a C^∞, proper Lagrangian fibration $\tilde{f}^+ : V \to W$ such that $\tilde{f}^+|_{X^+ \cap V} = f^+|_{X^+ \cap V}$. The same is true for f^-.

Proof. Define $f_0 : Z \to \Gamma$ to be $f_0 = f|_Z$. Consider the reduced space $\bar{Z} = Z/S^1$ with its reduced symplectic form ω_{red}. On $\mathbb{R} \times S^1 \times \bar{Z}$ we define the symplectic form:

$$\omega_{red} + ds \wedge dt$$

where (t, s) are coordinates on $\mathbb{R} \times S^1$. From the coisotropic neighbourhood theorem (cf. [13] §3.3) there exists a function $\epsilon : \Gamma \to \mathbb{R}_{>0}$, a neighbourhood $V \subset X$ of Z and a S^1-equivariant symplectomorphism between V and

$$\{(t, s, p) \in \mathbb{R} \times S^1 \times \bar{Z} \mid -\epsilon(\bar{f}(p)) < t < \epsilon(\bar{f}(p))\}. \tag{42}$$

In particular, the projection onto $\mathbb{R}$ corresponds to the moment map μ on V. Now, on the set in (42), we can define an "auxiliary" smooth Lagrangian fibration given by

$$\tilde{\pi}(t, s, p) = (t, f_0(s, p)).$$

Fix a basis γ of $H_1(V, \mathbb{Z}) \cong H_1(S^1 \times \bar{Z}, \mathbb{Z})$, satisfying condition (2) of Assumption 6.1 and a smooth Lagrangian section of $\tilde{\pi}$. The action-angle coordinates map Θ associated to $\tilde{\pi}$, with respect to γ and σ, together with (42), induces a C^∞ symplectomorphism

$$\tilde{V} := T^*U/\Lambda \cong V \tag{43}$$

for some open neighbourhood U of $0 \in \mathbb{R}^n$ with coordinates $(b_1, \ldots, b_n)$, which are the action coordinates of $\tilde{\pi}$. The pull back to $\tilde{V}$ of the S^1 action on V is given by translations along db_1 and the corresponding moment map is b_1. Pulling back $f|_V$ to $\tilde{V}$ via the latter identification we obtain a stitched fibration –with abuse of notation– defined by:

$$f = \begin{cases} u^+ & \text{on } \tilde{V}^+; \\ u^- & \text{on } \tilde{V}^-, \end{cases} \tag{44}$$

where $u^\pm$ is the pull back of $f^\pm$. It follows that $u^+|_Z = u^-|_Z = \pi|_Z$.

What we gained so far is an identification which allows us to view $f|_V$ as a stitched fibration on the smooth symplectic manifold $\tilde{V}$, where global canonical coordinates exist. Now we can use the results of §5 to show that u^+ (equivalently, u^-) can be extended as required. This can be done as follows. Since u^+ is the restriction of a C^∞ map to $\tilde{V}^+$, all the derivatives of its function components with respect to b_1 exist. Evaluating them at $b_1 = 0$

produces a unique sequence in $\mathscr{S}_{\bar{Z}}$ which in turn induces a unique sequence in $\mathscr{L}_{\bar{Z}}$ and a smooth Lagrangian fibration $(\tilde{V}, w) \in \mathscr{U}_{\bar{Z}}$ (where eventually we restricted to a smaller $\tilde{V}$) whose Taylor coefficients in b_1 evaluated at $b_1 = 0$ coincide with those of u^+ (cf. Theorem 5.11). In particular, this allows us to define

$$\tilde{u}^+ = \begin{cases} u^+ & \text{on } \tilde{V}^+; \\ w & \text{on } \tilde{V}^-, \end{cases} \tag{45}$$

obtaining an element $(\tilde{V}, \tilde{u}^+) \in \mathscr{U}_{\bar{Z}}$, where $\tilde{u}^+$ extends u^+. Observe that different choices of w induce different smooth extensions of u^+, however, all such choices are obtained starting from the same sequence in $\mathscr{S}_{\bar{Z}}$ determined by the derivatives of u^+. Finally, pulling back $\tilde{u}^+$ to V under the identification (43), and perhaps shrinking V, gives us the required $\tilde{f}^+$. One can use the same arguments to find a suitable smooth extension of f^-. □

Corollary 6.4. A section σ of (X, B, f) satisfying condition (3) of Assumption 6.1 exists.

Proof. Perhaps after an admissible change of coordinates on the base, a smooth Lagrangian section of $\tilde{f}^+$ is also a section of f. □

Let $(b_1, \dots b_n)$ be coordinates on $U \subseteq \mathbb{R}^n$ and let Λ be the integral lattice inside T^*U generated by $db_1, \dots, db_n$. Consider T^*U/Λ with its standard symplectic structure and let $\pi : T^*U/\Lambda \to U$ be the standard projection. For convenience we change slightly the usual notation. Let $\Gamma_{\text{nor}} = \{b_1 = 0\} \cap U$, $U^+ := \{b_1 \geq 0\} \cap U$, $U^- := \{b_1 \leq 0\} \cap U$, $Z_{\text{nor}} := \pi^{-1}(\Gamma)$ and $\bar{Z}_{\text{nor}} := Z_{\text{nor}}/S^1$. We assume that (U, Γ_{nor}) is diffeomorphic to the pair (D^n, D^{n-1}). Given $(V, u) \in \mathscr{U}_{\bar{Z}_{\text{nor}}}$, we can construct a stitched Lagrangian fibration (Y, f_u) as in (22). The zero section of π also defines a section of f_u, after perhaps an admissible change of coordinates on the base. We denote this section by σ_0. As a basis of $H_1(Y, \mathbb{Z})$ we take the basis $(db_1, \dots, db_2)$ of Λ. We denote it by γ_0. Then $(Y, f_u(Y), f_u, \sigma_0, \gamma_0)$ satisfies Assumption 6.1.

Definition 6.5. Let $F := (X, B, f, \sigma, \gamma)$ be a stitched Lagrangian fibration with seam Z, satisfying Assumption 6.1. A stitched fibration $F_u := (Y, f_u(Y), f_u, \sigma_0, \gamma_0)$ of the type above is a **normal form** of F if F_u and F define the same germ of a stitched fibration, according to Definition 6.2.

Observe that the above is a normalisation of a T^n-fibred neighbourhood of the seam of F. In this sense, F_u is a *semi-global* normal form.

If $F_u = (Y, f_u(Y), f_u, \sigma_0, \gamma_0)$ is a normal form of $F := (X, B, f, \sigma, \gamma)$ and Z is the seam of F, then Z_{nor} of F_u is nothing else but Z expressed in action angle coordinates, and thus it is a normalisation of Z. Since σ_0 and γ_0 are chosen canonically, we will from now on omit to specify them and just denote the normal form by $F_u = (Y, f_u)$.

Proposition 6.6. Every stitched Lagrangian fibration (X, B, f) satisfying (1) of Assumption 6.1 has a section σ and a basis γ as in (2) and (3) of Assumption 6.1 such that $(X, B, f, \sigma, \gamma)$ has a normal form (Y, f_u) .

Proof. From Proposition 6.3 we can assume there exist open neighbourhoods $V \subset X$ of Z and $W \subseteq B$ of Γ and a proper smooth Lagrangian fibration $\tilde{f}^+ : V \to W$ extending f^+. Now, fixing a basis γ of $H_1(V, \mathbb{Z})$ as in (3) of Assumption 6.1 and a smooth Lagrangian section σ of $\tilde{f}^+$, we obtain a unique symplectomorphism

$$\Theta^+ : T^*U/\Lambda \to V$$

given by the action-angle coordinates associated to $\tilde{f}^+$. Then by defining u to be the pull back of $\tilde{f}^-$ under Θ^+ one readily sees that f transforms into a fibration of the type (Y, f_u). □

Definition 6.7. Let $(X, B, f, \sigma, \gamma)$ be a stitched fibration with a normal form (Y, f_u). Let $\ell \in \mathscr{L}_{\bar{Z}_{\text{nor}}}$ be the unique sequence determined by u. We denote $\text{inv}(f_u) := (\bar{Z}_{\text{nor}}, \ell)$ and we call it the **invariants** of (Y, f_u). The invariants of $F = (X, B, f, \sigma, \gamma)$ are defined to be $\text{inv}(F) := \text{inv}(f_u)$.

Proposition 6.8. Let $F = (X, B, f, \sigma, \gamma)$ and $F' = (X', B', f', \sigma', \gamma')$ be stitched fibrations with normal forms (Y, f_u) and $(Y', f_{u'})$ defining invariants $\text{inv}(F)$ and $\text{inv}(F')$ respectively. If F and F' are fibrewise symplectically equivalent, then $\text{inv}(F) = \text{inv}(F')$.

Proof. Assume there is a commutative diagram as in Definition 6.2. To keep notation simple, let us assume $W = B$ and $W' = B'$. We have the diagram with commutative squares:

$$
\begin{tikzcd}
Y \arrow[d, "f_u"'] & X \arrow[l, "\Theta"'] \arrow[r, "\Psi"] \arrow[d, "f"'] & X' \arrow[r, "\Theta'"] \arrow[d, "f'"'] & Y' \arrow[d, "f_{u'}"] \\
U & B \arrow[l, "a"'] \arrow[r, "\phi"] & B' \arrow[r, "a'"] & U'.
\end{tikzcd}
\tag{46}
$$

Let us concentrate on the outermost square of (46) and define $\tilde{\Psi} = \Theta' \circ \Psi \circ \Theta^{-1}$ and $\tilde{\phi} = a' \circ \phi \circ a^{-1}$. We claim that:

(i) $\mathrm{inv}(f_{\tilde{\phi}\circ u}) = \mathrm{inv}(f_u)$; and

(ii) $\mathrm{inv}(f_{u'\circ\tilde{\Psi}}) = \mathrm{inv}(f_{u'})$.

Since $\tilde{\phi} \circ f_u = f_{u'} \circ \tilde{\Psi}$, *(i)* and *(ii)* would imply that $\mathrm{inv}(F) = \mathrm{inv}(F')$. It is clear that $\bar{Z}_{\mathrm{nor}}$ and $\bar{Z}'_{\mathrm{nor}}$ must coincide. Observe that $\tilde{\Psi}$, restricted to Y^+, is a symplectomorphism onto $(Y')^+$ which commutes with the projections π and π' on T^*U and T^*U' and sends the zero section to the zero section. Therefore we must have $\tilde{\Psi}|_{Y^+} = \tilde{\phi}^*$.

To prove *(i)* observe that $\tilde{\phi}^*|_{T^*U^+}$ must send the lattice Λ' defining Y' to the lattice Λ defining Y. From this it follows that $\tilde{\phi}|_{U^+}$ is the identity map and the restriction of $\tilde{\Psi}$ to Y^+ is also the identity map. Then $\tilde{\phi}\circ u|_{V^+} = u|_{V^+}$. From this and the smoothness of $\tilde{\phi}\circ u$ it follows that the sequences in $\mathscr{S}_{\bar{Z}_{\mathrm{nor}}}$ defined by $\tilde{\phi} \circ u$ and u coincide. Hence $\mathrm{inv}(f_u) = \mathrm{inv}(\tilde{\phi} \circ f_u)$. Similarly, to prove *(ii)*, observe that $u' \circ \tilde{\Psi}|_{V^+} = u'|_{V^+}$. Since u' and $\tilde{\Psi}$ are smooth it follows that $\mathrm{inv}(f_{u'\circ\tilde{\Psi}}) = \mathrm{inv}(f_{u'})$. □

Corollary 6.9. The definition of the invariants of $F = (X, B, f, \sigma, \gamma)$ is independent on the choice of normalisation.

Proof. Suppose we have two normalisations (Y, f_u) and $(Y', f_{u'})$. Clearly $\bar{Z}_{\mathrm{nor}}$ and $\bar{Z}'_{\mathrm{nor}}$ must coincide. We can also assume, w.l.o.g. that $Y = Y'$. What may be different are the maps u and u' such that f_u and $f_{u'}$ are two different normalisations of f induced from different extensions $\tilde{f}^+$ of f^+. Consider the invariants $\mathrm{inv}(f_u)$ and $\mathrm{inv}(f_{u'})$, respectively. Since f_u and $f_{u'}$ are symplectically equivalent via $\tilde{\Psi} = \Theta' \circ \Theta^{-1}$ and $\tilde{\phi} = a' \circ a^{-1}$, it follows that $\mathrm{inv}(f_u) = \mathrm{inv}(f_{u'})$. □

Proposition 6.10. Let $F = (X, B, f, \sigma, \gamma)$ and $F' = (X', B', f', \sigma', \gamma')$ be stitched Lagrangian fibrations satisfying Assumption 6.1. If $\mathrm{inv}(F) = \mathrm{inv}(F')$ then F is fibrewise symplectically equivalent to F' .

Proof. Let (Y, f_u) and $(Y', f_{u'})$ be normal forms of F and F', respectively. We can assume, w.l.o.g., $Y = Y'$. Let S_u and $S_{u'}$ be the series in $\mathscr{S}_{\bar{Z}_{\mathrm{nor}}}$ defined by u and u' respectively. By assumption $S_u = S_{u'}$. This allows us to find Lagrangian fibrations $(\bar{V}, \bar{u}), (\tilde{V}, \tilde{u}), (\tilde{V}', \tilde{u}') \in \mathscr{U}_{\bar{Z}_{\mathrm{nor}}}$ such that

$$
\tilde{u} = \begin{cases} u & \text{on } \tilde{V}^- \\ \bar{u} & \text{on } \tilde{V}^+ \end{cases} \quad \text{and} \quad \tilde{u}' = \begin{cases} u' & \text{on } (\tilde{V}')^- \\ \bar{u} & \text{on } (\tilde{V}')^+ \end{cases}
$$

where $S_{\bar{u}} = S_{\tilde{u}} = S_{\tilde{u}'}$. Now there is a neighbourhood W of Γ_{nor} and smooth symplectomorphisms $\Theta : T^*W/\Lambda \to \tilde{V}$ and $\Theta' : T^*W/\Lambda \to \tilde{V}'$ which are the action-angle coordinate map of the fibrations $\tilde{u}$ and $\tilde{u}'$, respectively. Defining $\Psi = \Theta' \circ \Theta^{-1}$, it is clear that $\Psi|_{\tilde{V}^+}$ is the identity. Furthermore, when restricted to $\tilde{V}^-$, Ψ sends the fibres of $\tilde{u}|_{\tilde{V}^-} = u|_{\tilde{V}^-}$ to the fibres of $\tilde{u}'|_{(\tilde{V}')^-} = u'|_{(\tilde{V}')^-}$. Therefore Ψ is fibre preserving with respect to f_u and $f_{u'}$. It follows that f and f' are symplectically equivalent. □

We summarise the previous Propositions in the following:

Theorem 6.11. Let $F = (X, B, f, \sigma, \gamma)$ and $F' = (X', B', f', \sigma', \gamma')$ be stitched Lagrangian fibrations satisfying Assumption 6.1, with invariants $\mathrm{inv}(F)$ and $\mathrm{inv}(F')$, respectively. Then F and F' define the same germ if and only if $\mathrm{inv}(F) = \mathrm{inv}(F')$. In other words, the set of germs of stitched fibrations $\mathscr{F}$ is classified by the pairs $(\bar{Z}_{\mathrm{nor}}, \ell)$, where $\ell \in \mathscr{L}_{\bar{Z}_{\mathrm{nor}}}$.

The above provides a semi-global classification of stitched Lagrangian fibrations. In contrast to what happens for smooth Lagrangian submersions where no semi-global symplectic invariants exist, stitched fibrations in general do give rise to non trivial semi-global invariants.

We can now also state a more precise version of Theorem 4.11:

Theorem 6.12. Let (U, Γ) be a pair, where U is an open neighbourhood of $0 \in \mathbb{R}^n$ and $\Gamma = U \cap \{b_1 = 0\}$. Assume (U, Γ) is diffeomorphic to the pair (D^n, D^{n-1}). Let $\bar{\Lambda} \subseteq T^*\Gamma$ be the lattice spanned by $\{db_2, \ldots, db_n\}$, and let $\bar{Z} = T^*\Gamma/\bar{\Lambda}$, with projection $\bar{\pi} : \bar{Z} \to \Gamma$ and bundle $\mathfrak{L} = \ker \bar{\pi}_*$. Given integers $m_2, \ldots, m_n$ and a sequence $\ell = \{\ell_k\}_{k \in \mathbb{N}} \in \mathscr{L}_{\bar{Z}}$ such that

$$\int_{db_j} \ell_1 = m_j \quad \text{for all} \quad j = 2, \ldots, n, \tag{47}$$

there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ satisfying the following properties:

(i) the coordinates $(b_1, \ldots, b_n)$ on U are action coordinates of f with $\mu = f^*b_1$ the moment map of the S^1 action;
(ii) the periods $\{db_1, \ldots, db_n\}$, restricted to $U^\pm$ correspond to bases $\gamma^\pm = \{\gamma_1, \gamma_2^\pm, \ldots, \gamma_n^\pm\}$ of $H_1(X, \mathbb{Z})$ satisfying (i) and (ii) of Corollary 4.6;
(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}, \ell)$ are the invariants of $(X, f, U, \sigma, \gamma^+)$.

The fibration (X, f, U) satisfying the above properties is unique up to fibre preserving symplectomorphism.

Proof. The construction of (X, ω) is like in the proof of Theorem 4.11, i.e.

$$X = (X^+ \cup V^+) \cup_{\tilde{Q}} (X^- \cup V^-).$$

But now the map u, used to construct $\tilde{Q}$, is chosen so that the fibration $f_u : X^+ \cup V^+ \to \mathbb{R}^n$, defined by

$$f_u = \begin{cases} \pi^+ & \text{on } b_1 \geq 0 \\ u & \text{on } b_1 \leq 0. \end{cases}$$

satisfies $\operatorname{inv}(f_u) = (\bar{Z}, \ell)$. Such a u exists thanks to Theorem 5.11. The fibration f is again defined by

$$f = \begin{cases} \pi^+ & \text{on } X^+ \\ \pi^- & \text{on } X^-. \end{cases}$$

It is clear that by construction (X, f, U) satisfies $(i) - (iii)$. Notice that $\tilde{Q}$ matches the zero section of π^+ to the zero section of π^-. Therefore the section σ is just given by the zero section of π^+ on U^+ and by the zero section of π^- on U^-.

It is clear from the results proved in this Section (in particular from the existence of a normal form) that any stitched Lagrangian fibration (X, f, U) satisfying $(i) - (iii)$ can be constructed in this way.

Uniqueness of (X, f, U) is proved as follows. The only choice involved in the construction is the function u. Any other choice u' must still satisfy $\operatorname{inv}(f_{u'}) = (\bar{Z}, \ell)$. Denote by X and X' the manifolds obtained from choices u and u' respectively. Let $\Psi : X \to X'$ be the map defined to be the identity on X^+ and on X^-. One can see that Ψ is well defined since the first order invariants of f_u and $f_{u'}$ coincide. It is clearly a smooth symplectomorphism away from Z. We need to show that it is smooth on Z. To see this we can use an argument similar to the one used in Proposition 6.10. If we think of Ψ in the coordinates on $X^+ \cup V^+$, Ψ is a symplectomorphism sending the fibres of f_u to the fibres of $f_{u'}$ and the zero section to the zero section. In a neighbourhood of Z and in these coordinates, we can describe Ψ, as follows. Since $\operatorname{inv}(f_u) = \operatorname{inv}(f_{u'})$, we can replace u and u' with $\tilde{u}$ and $\tilde{u}'$ as in Proposition 6.10. Let $\Theta : T^*W/\Lambda \to V^+$ and $\Theta' : T^*W/\Lambda \to V^+$ be action angle coordinates of $\tilde{u}$ and $\tilde{u}'$ respectively, associated to the zero section and to the basis γ^-. Then, in these coordinates, Ψ coincides with $\Theta' \circ \Theta^{-1}$. It is therefore smooth. $\square$

7. Stitched Lagrangian fibrations with monodromy

We now study stitched Lagrangian fibrations defined over a non simply connected open set U. In this case it may be that the fibration has non-trivial monodromy. When the fibration is smooth, this monodromy is usually detected by the behaviour of the periods of the fibration expressed in terms of smooth coordinates on the base. In the case of stitched Lagrangian fibrations there may not exist smooth coordinates on U, i.e. coordinates with respect to which the fibration is smooth. We will see how to detect monodromy from the behaviour of the first order invariant ℓ_1. This will be done mainly through the discussion of examples.

In Example 3.4, the fibration is topologically isomorphic to a focus-focus fibration. The singular fibre is over $0 \in \mathbb{R}^2$. Restricted to $X - f^{-1}(0)$, f is a stitched Lagrangian fibration onto $U = \mathbb{R}^2 - \{0\}$. We know that the locally constant presheaf on U given by

$$W \mapsto H_1(f^{-1}(W), \mathbb{Z})$$

has monodromy around 0, i.e. the monodromy map

$$\mathcal{M}_b : \pi_1(U) \to H_1(F_b, \mathbb{Z})$$

at a fibre over $b \in U$ is non-trivial. In fact, if e is a generator of $\pi_1(U)$, $\mathcal{M}_b(e)$ is conjugate to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We now look at a more general 2-dimensional case.

Example 7.1. Let $U \subset \mathbb{R}^2$ be an open annulus in $\mathbb{R}^2$ centred at the origin. As usual denote $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. This time Γ is disconnected. We let $\Gamma_u = \Gamma \cap \{b_2 \geq 0\}$ and $\Gamma_d = \Gamma \cap \{b_2 \leq 0\}$ be the upper and lower parts of Γ respectively. Now let $f : X \to \mathbb{R}^2$ be a stitched Lagrangian fibration such that $f(X) = U$. Observe that the seam Z has two connected components: $Z_u = f^{-1}(\Gamma_u)$ and $Z_d = f^{-1}(\Gamma_d)$. Denote by $\bar{Z}_u$ and $\bar{Z}_d$ the respective S^1 quotients, i.e. the connected components of $\bar{Z}$. Given $b \in \Gamma_u$ and choosing a curve going anticlock-wise once around 0 as generator $e \in \pi_1(U)$, suppose that with respect to a basis $\{\gamma_1, \gamma_2\}$ of $H_1(F_b, \mathbb{Z})$ the monodromy is

$$\mathcal{M}_b(e) = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}, \tag{48}$$

for some integer $m \neq 0$. In this case we must have that γ_1 is represented by the orbits of the S^1 action. As usual let $X^{\pm} = f^{-1}(U^{\pm})$. Since $U - \Gamma_d$ is contractible we can think of $\{\gamma_1, \gamma_2\}$ as a basis of $H_1(f^{-1}(U - \Gamma_d), \mathbb{Z})$. Consider the diagrams:

$$\begin{tikzcd}
 & H_1(X^+, \mathbb{Z}) \arrow[dr] & \\
H_1(f^{-1}(U - \Gamma_d), \mathbb{Z}) \arrow[ur] \arrow[rr, "j_+"] & & H_1(f^{-1}(U - \Gamma_u), \mathbb{Z})
\end{tikzcd}$$

or

$$\begin{tikzcd}
H_1(f^{-1}(U - \Gamma_d), \mathbb{Z}) \arrow[dr] \arrow[rr, "j_-"] & & H_1(f^{-1}(U - \Gamma_u), \mathbb{Z}) \\
 & H_1(X^-, \mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

induced by inclusions and restrictions. The map j_+ identifies $\{\gamma_1, \gamma_2\}$ with a basis $\{\gamma_1, \gamma_2^+\}$ of $H_1(f^{-1}(U - \Gamma_u), \mathbb{Z})$, whereas j_- with a basis $\{\gamma_1, \gamma_2^-\}$. Notice that monodromy is given by $j_+^{-1} \circ j_-$. Therefore we must have $\gamma_2^+ = m\gamma_1 + \gamma_2^-$. Hence $\{\gamma_1, \gamma_2^+\}$ and $\{\gamma_1, \gamma_2^-\}$ satisfy conditions (i) and (ii) of Corollary 4.6. Applying Lemma 4.9 to f restricted to $f^{-1}(U - \Gamma_u)$ we can consider the action coordinates map α constructed by taking action coordinates with respect to $\{\gamma_1, \gamma_2^+\}$ on U^+ and with respect to $\{\gamma_1, \gamma_2^-\}$ on U^-. Denote by (b_1^d, b_2^d) such coordinates. Similarly on $U - \Gamma_d$ we can consider action angle coordinates with respect to the basis $\{\gamma_1, \gamma_2\}$. Denote by (b_1^u, b_2^u) these coordinates. In particular we can identify

$$\bar{Z}_d = T^*\Gamma_d \,/\, \langle db_2^d \rangle_{\mathbb{Z}}$$

and

$$\bar{Z}_u = T^*\Gamma_u \,/\, \langle db_2^u \rangle_{\mathbb{Z}}$$

With respect to this choice of coordinates we can construct the first order invariants ℓ_1^u and ℓ_1^d of f on $\bar{Z}_u$ and $\bar{Z}_d$ respectively, then by applying Remark 4.10 we obtain

$$\int_{db_2^u} \ell_1^u = 0 \quad \text{and} \quad \int_{db_2^d} \ell_1^d = m.$$

This tells us that monodromy can be read from a jump in cohomology class of the first order invariant associated to action coordinates.

Using the methods of Theorem 4.11 we can also construct stitched Lagrangian fibrations with prescribed monodromy and and invariants. In fact we have

Theorem 7.2. Let $U \subset \mathbb{R}^2$ be an annulus as above with coordinates (b_1, b_2). Let $\bar{Z}_d = T^*\Gamma_d \,/\, \langle db_2 \rangle_{\mathbb{Z}}$ and $\bar{Z}_u = T^*\Gamma_u \,/\, \langle db_2 \rangle_{\mathbb{Z}}$ with projections $\bar{\pi}^d$ and $\bar{\pi}^u$ and bundles $\mathfrak{L}_d = \ker \bar{\pi}^d_*$ and $\mathfrak{L}_u = \ker \bar{\pi}^u_*$ respectively. Given an integer m and sequences $\ell^d = \{\ell^d_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_d}$ and $\ell^u = \{\ell^u_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_u}$ such that

$$\int_{db_2} \ell_1^u = 0 \quad \text{and} \quad \int_{db_2} \ell_1^d = m,$$

there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ having monodromy (48) with respect to some basis $\gamma = \{\gamma_1, \gamma_2\}$ of $H_1(f^{-1}(U - \Gamma_d), \mathbb{Z})$ and satisfying the following properties:

(i) the coordinates (b_1, b_2) are action coordinates of f with moment map f^*b_1;
(ii) the periods $\{db_1, db_2\}$, restricted to $U^\pm$ correspond to the basis $\{\gamma_1, \gamma_2\}$;
(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}_u, \ell^u)$ and $(\bar{Z}_d, \ell^d)$ are the invariants of $(f^{-1}(U-\Gamma_d),\, f,\, U-\Gamma_d,\, \sigma,\, \gamma)$ and $(f^{-1}(U-\Gamma_u),\, f,\, U-\Gamma_u,\, \sigma,\, j_+(\gamma))$ respectively.

The fibration (X, f, U) satisfying the above properties is unique up to fibre preserving symplectomorphism.

Proof. We let Λ_+ and Λ_- be the lattices generated by db_1 and db_2 in T^*U^+ and T^*U^- respectively. Define $X^\pm = T^*U^\pm/\Lambda_\pm$, $Z_u^\pm = (\pi^\pm)^{-1}(\Gamma_u)$ and $Z_d^\pm = (\pi^\pm)^{-1}(\Gamma_d)$. Then, using ℓ_1^u and ℓ_1^d, we construct maps

$$Q_u : Z_u^- \to Z_u^+$$

and

$$Q_d : Z_d^- \to Z_d^+$$

like in Theorem 4.11. We use these maps to glue X^+ and X^- topologically along their boundary and thus form X. For the smooth and symplectic gluing we follow the same method as in Theorem 6.12, where higher order invariants are used. From the discussion of Example 7.1 it follows that the fibration has the prescribed monodromy. Uniqueness is proved like in Theorem 6.12. $\square$

We now discuss a three dimensional example.

Example 7.3. In $\mathbb{R}^3$ consider the three-valent graph

$$\Delta = \{(0,0,-t),\ t \geq 0\} \cup \{(0,-t,0),\ t \geq 0\} \cup \{(0,t,t),\ t \geq 0\}$$

and let D be a tubular neighbourhood of Δ. Take $U = \mathbb{R}^3 - D$ and assume we have a stitched Lagrangian fibration $f : X \to \mathbb{R}^3$ such that $U = f(X)$. The seam is $Z = f^{-1}(\{b_1 = 0\} \cap U)$. Again we let $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. Also let $X^\pm = f^{-1}(U^\pm)$. This time Γ (and thus Z) has three connected components

$$\begin{aligned} \Gamma_c &= \{(0,t,s),\ t,s < 0\} \cap U, \\ \Gamma_d &= \{(0,t,s),\ t > 0, s < t\} \cap U, \\ \Gamma_e &= \{(0,t,s),\ s > 0, t < s\} \cap U. \end{aligned}$$

Also denote by Z_c, Z_d and Z_e the corresponding connected components of Z and by $\bar{Z}_c$, $\bar{Z}_d$ and $\bar{Z}_e$ their S^1 quotients.

Fix $b \in \Gamma_c$ and suppose that there is a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(F_b, \mathbb{Z})$ and generators e_0, e_1, e_2 of $\pi_1(U)$, satisfying $e_0 e_1 e_2 = 1$, with respect to which the monodromy transformations are

$$\mathcal{M}_b(e_1) = T_1 = \begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_b(e_2) = T_2 = \begin{pmatrix} 1 & -m_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

and $M_b(e_0) = T_0 = T_1^{-1} T_2^{-1}$, for non zero integers m_1 and m_2. We have that γ_1 is represented by the orbits of the S^1 action, since it is the only monodromy invariant cycle. Now, since $U - (\Gamma_d \cup \Gamma_e)$ is contractible, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$. Consider the diagrams:

$$\begin{tikzcd}
 & H_1(X^+, \mathbb{Z}) \arrow[dr] & \\
H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z}) \arrow[ur] \arrow[rr, "j_+"] & & H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z})
\end{tikzcd}$$

or

$$\begin{tikzcd}
H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z}) \arrow[dr] \arrow[rr, "j_-"] & & H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z}) \\
 & H_1(X^-, \mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

induced by inclusions and restrictions. The map j_+ identifies $\{\gamma_1, \gamma_2, \gamma_3\}$ with a basis of $H_1(f^{-1}(U - (\Gamma_c \cup \Gamma_d)), \mathbb{Z})$, which we call $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$, while j_- identifies it with another basis, which we call $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$. Notice that the monodromy map $\mathcal{M}_b(e_1) = j_+^{-1} \circ j_-$. We must have

$$\begin{cases} \gamma_2^+ = \gamma_2^-, \\ \gamma_3^+ = m_1 \gamma_1 + \gamma_3^-. \end{cases} \tag{50}$$

Therefore $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$ and $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$ satisfy conditions (i) and (ii) of Corollary 4.6. Applying Lemma 4.9 to f restricted to $f^{-1}(U - (\Gamma_c \cup \Gamma_d))$, we can consider the action coordinates map α on $U - (\Gamma_c \cup \Gamma_d)$ constructed by taking action coordinates with respect to $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$ on U^+ and with respect to $\{\gamma_1, \gamma_2^-, \gamma_3^-\}\}$ on U^-. Let us denote these coordinates by (b_1^e, b_2^e, b_3^e). Similarly we can consider action coordinates on $U - (\Gamma_d \cup \Gamma_e)$ with respect to the basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$. We denote them by (b_1^c, b_2^c, b_3^c). We have the identifications

$$\bar{Z}_e = T^*\Gamma_e \,/\, \langle db_2^e, db_3^e \rangle_{\mathbb{Z}}$$

and

$$\bar{Z}_c = T^*\Gamma_c \,/\, \langle db_2^c, db_3^c \rangle_{\mathbb{Z}}.$$

With respect to these coordinates we can compute the first order invariants ℓ_1^e and ℓ_1^c on $\bar{Z}_e$ and $\bar{Z}_c$ respectively. From Remark 4.10 and identities (50) applied to ℓ_1^c and ℓ_1^e we obtain

$$\int_{db_2^c} \ell_1^c = \int_{db_3^c} \ell_1^c = 0$$

and

$$\int_{db_2^e} \ell_1^e = 0 \quad \text{and} \quad \int_{db_3^e} \ell_1^e = m_1.$$

Similarly we construct the first order invariant ℓ_1^d on $\bar{Z}_d$. It will satisfy

$$\int_{db_2^d} \ell_1^d = m_2 \quad \text{and} \quad \int_{db_3^d} \ell_1^d = 0.$$

Again, monodromy is understood in terms of the difference in the cohomology class of the first order invariant. Example 3.6 is a special case of this situation, where $m_1 = m_2 = 1$.

Again, one can produce stitched Lagrangian fibrations of the type described in this example with the gluing method Theorem 4.11. In fact we can prove

Theorem 7.4. Let $U \subset \mathbb{R}^3$, Γ_c, Γ_d and Γ_e be as in Example 7.3 and let (b_1, b_2, b_3) be coordinates on U. Define $\bar{Z}_c = T^*\Gamma_c \,/\, \langle db_2, db_3\rangle_{\mathbb{Z}}$, $\bar{Z}_d = T^*\Gamma_d \,/\, \langle db_2, db_3\rangle_{\mathbb{Z}}$ and $\bar{Z}_e = T^*\Gamma_e \,/\, \langle db_2, db_3\rangle_{\mathbb{Z}}$ with projections $\bar{\pi}^c$, $\bar{\pi}^d$, $\bar{\pi}^e$ and bundles $\mathfrak{L}_c = \ker \bar{\pi}^c_*$, $\mathfrak{L}_d = \ker \bar{\pi}^d_*$, $\mathfrak{L}_e = \ker \bar{\pi}^e_*$. Suppose we are given integers m_1, m_2 and sequences $\ell^c = \{\ell^c_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_c}$, $\ell^d = \{\ell^d_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_d}$ and $\ell^e = \{\ell^e_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_e}$ satisfying

$$\begin{aligned} \int_{db_2} \ell^c_1 &= \int_{db_3} \ell^c_1 = 0, \\ \int_{db_2} \ell^e_1 &= 0 \ \text{ and } \int_{db_3} \ell^e_1 = m_1, \\ \int_{db_2} \ell^d_1 &= m_2 \ \text{ and } \int_{db_3} \ell^d_1 = 0. \end{aligned}$$

Then there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ having the same monodromy of Example 7.3 with respect to some basis $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U - (\Gamma_d \cup \Gamma_e)), \mathbb{Z})$ and satisfying the following properties:

(i) the coordinates (b_1, b_2, b_3) are action coordinates of f with moment map f^*b_1;

(ii) the periods $\{db_1, db_2, db_3\}$, restricted to $U^\pm$ correspond to the basis γ;

(iii) there is a Lagrangian section σ of f, such that $(\bar{Z}_c, \ell^c)$, $(\bar{Z}_d,\, \ell^d)$ and $(\bar{Z}_e,\, \ell^e)$ are the invariants of $(f^{-1}(U - (\Gamma_d \cup \Gamma_e)),\, f,\, U - (\Gamma_d \cup \Gamma_e),\, \sigma,\, \gamma)$, $(f^{-1}(U - (\Gamma_c \cup \Gamma_e)),\, f,\, U - (\Gamma_c \cup \Gamma_e),\, \sigma,\, j_+(\gamma))$ and $(f^{-1}(U - (\Gamma_c \cup \Gamma_d)),\, f,\, U - (\Gamma_c \cup \Gamma_d),\, \sigma,\, j_+(\gamma))$ respectively.

The fibration (X, f, U) satisfying the above properties is unique up to fibre preserving symplectomorphism.

We omit the proof which is simply a repetition of the usual gluing method from Theorems 4.11 and 6.12.

8. More examples?

In this section we would like to propose a conjectural construction generalising the one, described in [3], which led us to Example 3.6. In [9], Guillemin and Sternberg make the following observation. Let $N = n + m$, with n, m positive integers. Consider $\mathbb{C}^{N+1}$ with its standard symplectic structure, then S^1 acts on it, in a Hamiltonian way, via the action given by,

$$\theta : (z_1, \ldots, z_{N+1}) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, \ldots, e^{-i\theta} z_{n+1}, z_{n+2}, \ldots, z_{N+1}) \tag{51}$$

with moment map

$$\mu = \frac{|z_1|^2 - |z_2|^2 - \ldots - |z_{n+1}|^2}{2}.$$

The action is singular along $\Sigma = \{z_1 = \ldots = z_{n+1} = 0\}$, which can be identified with $\mathbb{C}^m$. The observation is that for any $\epsilon \in \mathbb{R}_{\geq 0}$ the reduced spaces $(M_\epsilon, \omega_r(\epsilon))$ can be identified with $(\mathbb{C}^N, \omega_{\mathbb{C}^N})$ with standard symplectic form, (this includes the case of the critical value $\epsilon = 0$). While when $\epsilon \in \mathbb{R}_{<0}$, $(M_\epsilon, \omega_r(\epsilon))$ can be identified with the ϵ-blow up of $(\mathbb{C}^N, \omega_{\mathbb{C}^N})$ along the symplectic submanifold Σ.

The ϵ-blow up can be described as follows. Let L be the total space of the tautological line bundle on $\mathbb{P}^{n-1}$. The incidence relation gives L as

$$L = \{(v, l) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \,|\, v \in l\}.$$

There are two natural projections: $\pi : L \to \mathbb{P}^{n-1}$, which is the bundle projection, and $\beta : L \to \mathbb{C}^n$ which is the blow-up map. The latter is a biholomorphism onto $\mathbb{C}^n - \{0\}$ once the zero section is removed from L. Let ω_{FS} be the standard Fubini-Study symplectic form on $\mathbb{P}^{n-1}$. The ϵ-blow up of $\mathbb{C}^n$ at 0 is L together with the symplectic form given by

$$\omega_\epsilon = \beta^*\omega_{\mathbb{C}^n} + \epsilon\, \pi^*\omega_{FS}.$$

The ϵ-blow up of $\mathbb{C}^N$ along $\Sigma = \mathbb{C}^m$ can be identified with $L \times \mathbb{C}^m$ with symplectic form $\omega_\epsilon + \omega_{\mathbb{C}^m}$.

In the case $n = 1$ the blow-up is topologically (and holomorphically) trivial, i.e. blowing up does not do anything. In fact one can also show, by following Guillemin and Sternberg's argument, that the reduced spaces can all be identified with $(\mathbb{C}^{m+1}, \omega_{\mathbb{C}^{m+1}})$ for all values of ϵ. This identification can also be explained as follows. Consider the map γ given in (6) and define the map $p : \mathbb{C}^{m+2} \to \mathbb{C}^{m+1}$ given by

$$p : (z_1, z_2, z_3, \dots, z_{m+2}) \mapsto (\gamma(z_1, z_2), z_3, \dots, z_{m+2}). \tag{52}$$

Restricted to $\mu^{-1}(\epsilon)$, this map can be regarded as the quotient map $\mu^{-1}(\epsilon) \to M_\epsilon$. It can be shown that the reduced symplectic form with respect to this map is precisely $\omega_{\mathbb{C}^{m+1}}$.

Example 3.6 comes from this construction in the case $m = n = 1$. In fact the fibration f is of the type $\mathrm{Log} \circ \Phi \circ p$, where Φ is a symplectomorphism of $\mathbb{C}^2$ and $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ is the map $(v_1, v_2) \mapsto (\log |v_1|, \log |v_2|)$. The fact that f is not smooth is due to the non-smoothness of p, i.e. the reduced spaces are not identified with $\mathbb{C}^2$ in a smooth way.

We think that it may be possible to generalize this construction. The idea is to use another result of Guillemin and Sternberg proved in the same paper. The result is as follows. Let $\bar{X}$ be a compact $2N$ dimensional symplectic manifold with symplectic form ω and $2m$-dimensional symplectic submanifold Y. Consider now a principal S^1 bundle $p_0 : P \to \bar{X}$ with a connection one form α. Given an interval $I = (-\epsilon, \epsilon)$, Guillemin and Sternberg [9]§12 construct a $2(N + 1)$ symplectic manifold X with the following properties.

(1) There exists a Hamiltonian S^1 action on X with proper, surjective moment map $\mu : X \to I$.
(2) For positive $t \in I$, $\mu^{-1}(t)$ is equivalent, as an S^1 bundle, to P and the reduced symplectic space $(X_t, \omega_r(t))$ is symplectomorphic to $(\bar{X}, \omega)$.
(3) The only critical value of Φ is $t = 0$. If $\Sigma := \mathrm{Crit}(\mu) \subset \mu^{-1}(0)$, i.e. the set of critical points of μ, then Σ is a smooth symplectic, $2m$ dimensional submanifold of X and the S^1 action is locally modelled on (51) (in this case 0 is also called a simple critical value). If X_0 denotes the reduced symplectic space at 0, with reduced symplectic form $\omega_r(0)$ and quotient map $\pi_0 : \mu^{-1}(0) \to X_0$, then the triple $(X_0, \pi_0(\Sigma), \omega_r(0))$ can be identified with $(\bar{X}, Y, \omega)$.
(4) When $t \in I$ is negative, then the reduced space $(X_t, \omega_r(t))$ can be identified with the blow-up $\tilde{X}$ of $\bar{X}$ along Y with symplectic form $\omega_{Y,t} + \beta^* t d\alpha$, where $\omega_{Y,t}$ is the t-blow-up form along Y on $\tilde{X}$ and $\beta : \tilde{X} \to \bar{X}$ is the blow down map.

We are interested in Guillemin-Sternberg's construction in the case $N = m + 1$, i.e. in the case Y is a codimension 2 symplectic manifold. For simplicity we also assume that $P = \bar{X} \times S^1$ and $\alpha = 0$. We can make the following observations.

(a) Topologically $\tilde{X}$ is equivalent to $\bar{X}$, but symplectically $(\tilde{X}, \omega_{Y,t})$ and $(\bar{X}, \omega)$ differ since the latter one has less area (blowing up removes the area of a small tubular neighbourhood of Y).
(b) Consider the quotient $p : X \to X/S^1$, then X/S^1 can be identified with $\bar{X} \times I$. If we restrict p to $X - \Sigma$ then it becomes an S^1 bundle onto $(\bar{X} \times I) - (Y \times \{0\})$. Let c_1 be the first Chern class of this bundle. If S is a small 2-sphere centred at the origin in a fibre of the normal bundle of $Y \times \{0\}$ inside $(\bar{X} \times I)$, then $c_1(S) = 1$.

As we saw in the beginning of this section, in the non-compact case $(\bar{X}, \omega) = (\mathbb{C}^{m+1}, \omega_{\mathbb{C}^{m+1}})$ and $Y = \mathbb{C}^m$, the observation in (a) was not true, in the sense that the identification could be made also symplectically. This is because, although blowing up locally reduces area, in this non-compact case the area is infinite so it does not constitute a symplectic invariant. So the idea is to try to generalize Guillemin and Sternberg's construction to other non-compact cases. One interesting situation is if we take $(\bar{X}, \omega)$ with $\bar{X} = (\mathbb{C}^*)^N$ and

$$\omega = \sum_{k=1}^{N} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2}.$$

As symplectic submanifold Y we can take some smooth algebraic hypersurface.

We think it may be possible to generalize Guillemin and Sternberg's construction to this case. The hypothesis of compactness was made in order to be able to use the coisotropic embedding theorem in symplectic topology, but this theorem holds also in non-compact situations. The question is whether the reduced spaces can all be identified with $((\mathbb{C}^*)^N, \omega)$. Since the space is non-compact, area is not an obstruction.

Why would such a construction be useful? We could use it to construct interesting examples of piecewise smooth Lagrangian fibrations with singular fibres. In fact suppose the conjectured symplectic manifold X exists with the above properties and such that all reduced spaces can be identified with $((\mathbb{C}^*)^N, \omega)$. Then, on X we could define a piecewise smooth Lagrangian fibration as follows. On $(\mathbb{C}^*)^N \times I$ define the T^N fibration given by

$$F : (z_1, \ldots, z_N, t) \to (\log|z_1|, \ldots, \log|z_N|, t).$$

Clearly $F_t = F|_{(\mathbb{C}^*)^N \times \{t\}}$ is Lagrangian. Now suppose there exists a map $p : X \to (\mathbb{C}^*)^N \times I$, equivalent to the quotient $X \to X/S^1$ and with respect to which the reduced spaces are all $((\mathbb{C}^*)^N, \omega)$. Presumably this map would be locally modelled on (52), in particular it would fail to be smooth on $\mu^{-1}(0)$. The piecewise smooth Lagrangian fibration would be

$$f = F \circ p. \tag{53}$$

We expect f to be a stitched Lagrangian fibration when restricted to $X - f^{-1}(\Delta)$. The interesting aspect of this map is the structure of the singular fibres. In fact its discriminant locus is $\Delta = F(Y \times \{0\})$, which is $\mathrm{Log}(Y) \times \{0\}$. Images of algebraic hypersurfaces of $(\mathbb{C}^*)^N$ by Log are called amoebas and they have shapes of the type pictured in Figure 2

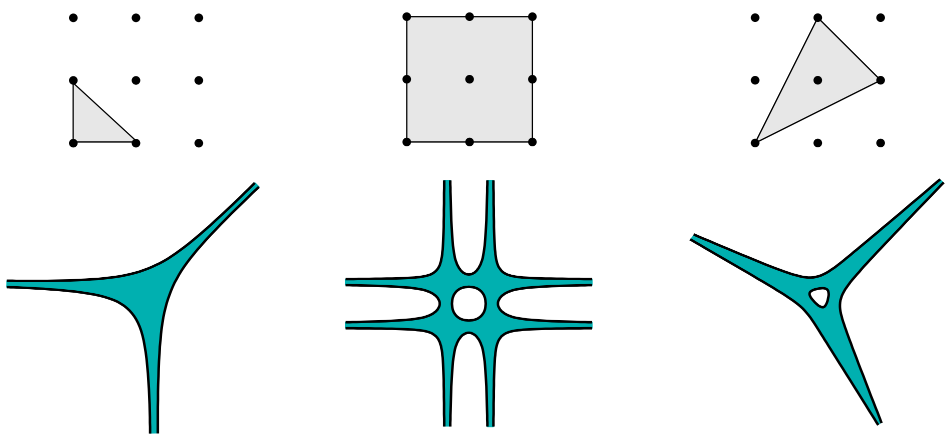

Figure 2. Amoebas with their respective Newton polygons.

The topological property, discussed in the observation (b), of the bundle $p : X - \Sigma \to (\bar{X} \times I) - (Y \times \{0\})$, ensures that the fibration f, restricted to $X - f^{-1}(\Delta)$ has non-trivial monodromy. In fact one can find examples where monodromy would be of the types discussed in (7.3). These examples, and the calculation of monodromy, generalize the construction in [5] of the negative fibre, also called the fibre of type $(2, 1)$, where a circle bundle with the topological property (b) is used.

In a work in progress [2] the authors use the piecewise smooth Lagrangian fibration in Example 3.6 as one of the building blocks for the construction of Lagrangian fibrations of 6-dimensional compact Calabi-Yau manifolds. One of the ideas involved is that the invariants we have defined for stitched Lagrangian fibrations can be used to perturb the fibration in Example 3.6 away from the singular fibres in order to glue it to other pieces of fibration. In fact the sequence $\ell = \{\ell_k\}_{k\in\mathbb{N}}$ of fibrewise closed sections of $\mathcal{L}^*$ on $\bar{Z}$ can be easily perturbed, for example by multiplying each element by cut-off functions on the base Γ or by summing to each element other fibrewise closed section and so on.

We believe that the more general construction proposed in this section is interesting because, if it can be carried through, then these Lagrangian fibrations could be used as building blocks of more general Lagrangian fibrations of compact symplectic manifolds.

9. Appendix to Lemma 5.4

We give here a proof of Lemma 5.4 for all $m \in \mathbb{N}$. Recall we can write

$$a_j(r) = \sum_{k=1}^{N} a_{j,k}\, r^k + o(r^N). \tag{54}$$

The a_j's are functions of (r, b, y), with $(b, y) \in \bar{Z}$, satisfying

$$\begin{cases} u_1(r, b_2 + a_2, \dots, b_n + a_n, y) = r, \\ u_j(r, b_2 + a_2, \dots, b_n + a_n, y) = b_j \quad \text{for all } j = 2, \dots, n. \end{cases}$$

When W is sufficiently small and $(r, b) \in W$, the functions $a_{j,m}$'s can be uniquely determined using the implicit function theorem. We will now use it to compute the $a_{j,m}$'s and obtain formulae (31). We can rewrite the second equation of the above system by applying

$$u_j = \sum_{k=0}^{N} S_{j,k} b_1^k + o(b_1^N). \tag{55}$$

We obtain

$$b_j + a_j + \sum_{k=1}^{N} S_{j,k}(b_2 + a_2, \dots, b_n + a_n, y) r^k + o(r^N) = b_j$$

which implies

$$a_j + \sum_{k=1}^{N} S_{j,k}(b_2 + a_2, \dots, b_n + a_n, y) r^k + o(r^N) = 0. \tag{56}$$

To express everything as a power series in r we use the Taylor expansion up to a certain order N' of the $S_{j,k}$'s, which in the multi-index notation is given by:

$$S_{j,k}(b_2 + a_2, \dots, b_n + a_n, y) = \sum_{l=0}^{N'} \sum_{|I|=l} C_I\, \partial_I^l S_{j,k}(b_2, \dots, b_n)\, a_2^{i_2} \cdot \ldots \cdot a_n^{i_n} + \dots,$$

where $I = (i_2, \dots, i_n)$ is a multi-index and the C_I's are suitable constants.

Let us introduce the following notation. For every multi-index $I = (i_2, \dots, i_n)$, let us define the following set

$$\mathcal{H}_I = \{(H_2, \dots, H_n) \,|\, H_k \in (\mathbb{Z}_{>0})^{i_k} \text{ if } i_k \geq 1 \text{ and } H_k = 0 \in \mathbb{Z} \text{ if } i_k = 0\}.$$

When $i_k \geq 1$, we also write $H_k = (h_{k,1}, \dots, h_{k,i_k})$. For every $m \in \mathbb{N}$, we denote

$$\mathcal{H}_{I,m} = \left\{ (H_2, \dots, H_n) \in \mathcal{H}_I \;|\; \sum_{i_k \neq 0} \sum_{j=1}^{i_k} h_{k,j} = m \right\}.$$

Clearly if $|I| = 0$ and $m \geq 1$ or if $0 \leq m < |I|$ then $\mathcal{H}_{I,m}$ is empty. When $i_k \neq 0$ for all $k = 2, \dots, n$, substituting (32) we compute that

$$a_1^{i_1} \cdot \ldots \cdot a_n^{i_n} = \sum_{m=1}^{N'} \left(\sum_{H \in \mathcal{H}_{I,m}} a_{2,h_{2,1}} \cdot \ldots \cdot a_{2,h_{2,i_2}} \cdot \ldots \cdot a_{n,h_{n,1}} \cdot \ldots \cdot a_{2,h_{n,i_n}} \right) r^m + o(r^{N'}).$$

Let us introduce another bit of notation. When $|I| \neq 0$, for all $H \in \mathcal{H}_I$, let

$$A_H = \prod_{i_k \neq 0} \prod_{j=1}^{i_k} a_{k,h_{k,j}}.$$

When $|I| = 0$, the only element in $\mathcal{H}_I$ is $0 \in \mathbb{Z}^n$, so we set

$$A_0 = 1.$$

Thus for all multi-indices I, we have

$$a_1^{i_1} \cdot \ldots \cdot a_n^{i_n} = \sum_{m=0}^{N'} \left(\sum_{H \in \mathcal{H}_{I,m}} A_H \right) r^m + o(r^{N'}).$$

Therefore $S_{j,k}(b_2 + a_2, \ldots, b_n + a_n, y)$ written as a power series in r becomes

$$S_{j,k}(b_2 + a_2, \ldots, b_n + a_n, y) = \sum_{m=0}^{N'} \left(\sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,k}(b, y) \, A_H \right) r^m + o(r^{N'}).$$

Substituting this into (56) we obtain

$$a_j + \sum_{l=1}^{N} \left(\sum_{m=0}^{l-1} \sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,l-m}(b, y) \, A_H \right) r^l + o(r^N) = 0.$$

Substituting also (54) we have

$$\sum_{l=1}^{N} \left(a_{j,l} + \sum_{m=0}^{l-1} \sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,l-m}(b, y) \, A_H \right) r^l + o(r^N) = 0.$$

Therefore, for every $l \in \mathbb{Z}_{>0}$, we have

$$a_{j,l} = - \sum_{m=0}^{l-1} \sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,l-m}(b, y) \, A_H.$$

When $l = 1$, this becomes

$$a_{j,1} = -S_{j,1},$$

when $l \geq 2$ it can also be written as

$$a_{j,l} = -S_{j,l} - \sum_{m=1}^{l-1} \sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,l-m} \, A_H.$$

Now notice that when $1 \leq m \leq l - 1$ and $H \in \mathcal{H}_{I,m}$, then A_H only depends on the $a_{j,k}$'s with $1 \leq k \leq l - 1$. Therefore if we define

$$R_{j,l} = - \sum_{m=1}^{l-1} \sum_{|I| \leq m} \sum_{H \in \mathcal{H}_{I,m}} C_I \, \partial_I^{|I|} S_{j,l-m}(b, y) \, A_H,$$

when $l \geq 2$ and $R_{j,1} = 0$, then (31) holds with $R_{j,m}$ satisfying the required properties.

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