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OPTIMAL DISTRIBUTED CONTROL OF A NONLOCAL CAHN–HILLIARD/NAVIER–STOKES SYSTEM IN TWO DIMENSIONS*

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Abstract. We study a diffuse interface model for incompressible isothermal mixtures of two immiscible fluids coupling the Navier–Stokes system with a convective nonlocal Cahn–Hilliard equation in two dimensions of space. We apply recently proved well-posedness and regularity results in order to establish existence of optimal controls as well as first-order necessary optimality conditions for an associated optimal control problem in which a distributed control is applied to the fluid flow.

Key words. distributed optimal control, first-order necessary optimality conditions, nonlocal models, integrodifferential equations, Navier–Stokes system, Cahn–Hilliard equation, phase separation

AMS subject classifications. 49J20, 49J50, 35R09, 45K05, 74N99

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1. Introduction.

In this paper, we consider the nonlocal Cahn–Hilliard/Navier–Stokes system

$$\boldsymbol{u}_t - 2\,\mathrm{div}\big(\nu(\varphi)\,D\boldsymbol{u}\big) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\,\nabla\varphi + \boldsymbol{v}, \tag{1.1}$$

$$\mathrm{div}(\boldsymbol{u}) = 0, \tag{1.2}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu, \tag{1.3}$$

$$\mu = a\,\varphi - K*\varphi + F'(\varphi) \tag{1.4}$$

in $Q := \Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^2$ is a bounded smooth domain with boundary $\partial\Omega$, and where $T>0$ is a prescribed final time. Moreover, D denotes the symmetric gradient, which is defined by $D\boldsymbol{u} := \big(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\big)/2$.

This system models the flow and phase separation of an isothermal mixture of two incompressible immiscible fluids with matched densities (normalized to unity), where nonlocal interactions between the molecules are taken into account. In this connection, $\boldsymbol{u}$ is the (averaged) velocity field, φ is the order parameter (relative concentration of one of the species), π is the pressure, and $\boldsymbol{v}$ is the external volume force density. The mobility in (1.3) is assumed to be constant equal to 1, while in (1.1) we allow the viscosity ν to be φ-dependent. The case of nonconstant mobility is still open and it

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will be the subject of a forthcoming contribution since it requires different techniques. The chemical potential μ contains the spatial convolution $K * \varphi$ over Ω, defined by

$$(K * \varphi)(x) := \int_\Omega K(x-y)\varphi(y)\,dy, \quad x \in \Omega,$$

of the order parameter φ with a sufficiently smooth interaction kernel K (in particular, satisfying hypothesis (H4) below) such that $K(z) = K(-z)$. Moreover, a is given by

$$a(x) := \int_\Omega K(x-y)\,dy,$$

for $x \in \Omega$, and F is a double-well potential, which, in general, may be regular or singular (e.g., of logarithmic or double obstacle type); in this paper, we have to confine ourselves to the regular case (in particular, a polynomial growth is assumed).

The system (1.3)–(1.2) is complemented by the boundary and initial conditions

$$\frac{\partial \mu}{\partial \boldsymbol{n}} = 0, \qquad \boldsymbol{u} = 0, \qquad \text{on } \Sigma := \partial\Omega \times (0,T), \tag{1.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \varphi(0) = \varphi_0, \qquad \text{in } \Omega, \tag{1.6}$$

where, as usual, $\boldsymbol{n}$ is the outward unit normal field to the boundary $\partial\Omega$ and $\partial\mu/\partial\boldsymbol{n}$ denotes the directional derivative of μ in the direction of $\boldsymbol{n}$.

Problem (1.3)–(1.6) is the nonlocal version of the so-called Model H, which is known from the literature (cf., e.g., [5, 28, 29, 34, 35, 36, 40]). The main difference between local and nonlocal models is given by the choice of the interaction potential. Typically, the nonlocal contribution to the free energy has the form $\int_\Omega \widetilde{K}(x,y)\,|\varphi(x) - \varphi(y)|^2\,dy$ with a given symmetric kernel $\widetilde{K}$ defined on $\Omega \times \Omega$; its local Ginzburg–Landau counterpart is given by $(\sigma/2)|\nabla\varphi(x)|^2$, where the positive parameter σ is a measure for the thickness of the interface. More precisely, the total (i.e., including also the kinetic contribution) nonlocal free energy is given by

$$\mathcal{E}_{nloc}(\boldsymbol{u},\varphi) = \frac{1}{2}\int_\Omega \boldsymbol{u}^2 dx + \int_\Omega\int_\Omega \widetilde{K}(x,y)\,|\varphi(x)-\varphi(y)|^2\,dx\,dy + \eta\int_\Omega F(\varphi)dx,$$

while the total local Ginzburg–Landau free energy is

$$\mathcal{E}_{loc}(\boldsymbol{u},\varphi) = \frac{1}{2}\int_\Omega \boldsymbol{u}^2 dx + \int_\Omega \frac{\sigma}{2}|\nabla\varphi(x)|^2 dx + \eta\int_\Omega F(\varphi)dx,$$

where η is proportional to σ^{-1}.

Although the physical relevance of nonlocal interactions was already pointed out in the pioneering paper [44] (see also [14, section 4.2] and the references therein) and studied (in case of constant velocity) in, e.g., [6, 13, 23, 24, 25, 26, 38, 39], and while the classical (local) Model H has been investigated by several authors (see, e.g., [1, 2, 10, 11, 20, 21, 22, 32, 42, 45, 49, 52] and also [3, 9, 27, 37] for models with shear dependent viscosity), its nonlocal version has been tackled (from the analytical viewpoint concerning well-posedness and related questions) only more recently (cf., e.g., [12, 15, 16, 17, 18, 19]).

In particular, the following cases have been studied: regular potential F associated with constant mobility in [12, 15, 16, 18]; singular potential associated with constant mobility in [17]; singular potential and degenerate mobility in [19]; and the case of nonconstant viscosity in [15]. In the two-dimensional case it was shown in [18] that for regular potentials and constant mobilities the problem (1.3)–(1.6) enjoys a unique strong solution. Recently, uniqueness was proved also for weak solutions (see [15]).

With the well-posedness results of [18] and in [15] at hand, the road is paved for studying optimal control problems associated with (1.3)–(1.6) at least in the two-dimensional case. This is the purpose of this paper. To our best knowledge, this has never been done before in the literature; in fact, while there exist recent contributions to associated optimal control problems for the time-discretized local version of the system (cf. [30, 31]) and to numerical aspects of the control problem (see [33]), it seems that a rigorous analysis for the full problem without time discretization has never been performed before. Even for the much simpler case of the convective Cahn–Hilliard equation, that is, if the velocity is prescribed so that the Navier–Stokes equation (1.1) is not present, only very few contributions exist that deal with optimal control problems; in this connection, we refer to [50, 51] for local models in one and two space dimensions and to the recent paper [43], in which first-order necessary optimality conditions were derived for the nonlocal convective Cahn–Hilliard system in three dimensions in the case of degenerate mobilities and singular potentials.

More precisely, the control problem under investigation in this paper reads as follows:

(CP) Minimize the tracking type cost functional

$$\mathcal{J}(y,\boldsymbol{v}) := \frac{\beta_1}{2}\|\boldsymbol{u}-\boldsymbol{u}_Q\|^2_{L^2(Q)^2} + \frac{\beta_2}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{\beta_3}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_\Omega\|^2_{L^2(\Omega)^2} + \frac{\beta_4}{2}\|\varphi(T)-\varphi_\Omega\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}, \tag{1.7}$$

where $y := [\boldsymbol{u},\varphi]$ solves problem (1.3)–(1.6). We assume throughout the paper without further reference that in the cost functional (1.7) the quantities $\boldsymbol{u}_Q \in L^2(0,T;G_{div})$, $\varphi_Q \in L^2(Q)$, $\boldsymbol{u}_\Omega \in G_{div}$, and $\varphi_\Omega \in L^2(\Omega)$, are given target functions, while β_i, $i = 1\ldots 4$, and γ are some fixed nonnegative constants that do not vanish simultaneously. Moreover, the external body force density $\boldsymbol{v}$, which plays the role of the control, is postulated to belong to a suitable closed, bounded, and convex subset (which will be specified later) of the space of controls

$$\mathcal{V} := L^2(0,T;G_{div}),$$

where

$$G_{div} := \overline{\left\{\boldsymbol{u} \in C_0^\infty(\Omega)^2 : \operatorname{div}(\boldsymbol{u}) = 0\right\}}^{L^2(\Omega)^2}.$$

We recall that the spaces G_{div} and

$$V_{div} := \left\{\boldsymbol{u} \in H_0^1(\Omega)^2 : \operatorname{div}(\boldsymbol{u}) = 0\right\}$$

are the classical Hilbert spaces for the incompressible Navier–Stokes equations with no-slip boundary conditions (see, e.g., [48]).

We remark that controls in the form of volume force densities can occur in many technical applications. For instance, they may be induced in the fluid flow from stirring devices, from the application of acoustic fields (ultrasound, say), or, in the case of electrically conducting fluids, from the application of magnetic fields.

The plan of the paper is as follows. In section 2, we collect some preliminary results concerning the well-posedness of system (1.3)–(1.6), and we prove some stability estimates which are necessary for the analysis of the control problem. In section 3, we prove the main results of this paper, namely, the existence of a solution to the optimal control problem (CP), the Fréchet differentiability of the control-to-state operator, as well as the first-order necessary optimality conditions for (CP).

2. Preliminary results. In this section, we first summarize some results from [12, 15, 18] concerning the well-posedness of solutions to the system (1.3)–(1.6). We also establish a stability estimate that later will turn out to be crucial for showing the differentiability of the associated control-to-state mapping.

Before going into this, we introduce some notation.

Throughout the paper, we set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and we denote by $\|\cdot\|$ and $(\cdot\,,\cdot)$ the standard norm and the scalar product, respectively, in H and G_{div}, as well as in $L^2(\Omega)^2$ and $L^2(\Omega)^{2\times 2}$. The notation $\langle\cdot\,,\cdot\rangle_X$ and $\|\cdot\|_X$ will stand for the duality pairing between a Banach space X and its dual X' and for the norm of X, respectively. Moreover, the space V_{div} is endowed with the scalar product

$$(\boldsymbol{u}_1,\boldsymbol{u}_2)_{V_{div}} := (\nabla\boldsymbol{u}_1,\nabla\boldsymbol{u}_2) = 2\big(D\boldsymbol{u}_1, D\boldsymbol{u}_2\big) \qquad \forall\,\boldsymbol{u}_1,\boldsymbol{u}_2\in V_{div}.$$

We also introduce the Stokes operator A with no-slip boundary condition (see, e.g., [48]). Recall that $A: D(A)\subset G_{div}\to G_{div}$ is defined as $A := -P\Delta$, with domain $D(A) = H^2(\Omega)^2\cap V_{div}$, where $P: L^2(\Omega)^2\to G_{div}$ is the Leray projector. Moreover, $A^{-1}: G_{div}\to G_{div}$ is a self-adjoint compact operator in G_{div}. Therefore, according to classical results, A possesses a sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ with $0<\lambda_1\le\lambda_2\le\cdots$ and $\lambda_j\to\infty$ and a family $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}\subset D(A)$ of associated eigenfunctions which is orthonormal in G_{div}. We also recall Poincaré's inequality

$$\lambda_1\,\|\boldsymbol{u}\|^2\le\|\nabla\boldsymbol{u}\|^2 \qquad \forall\,\boldsymbol{u}\in V_{div}$$

and two other inequalities, which are valid in two dimensions of space and will be used repeatedly in the course of our analysis, namely, the particular case of the Gagliardo–Nirenberg inequality (see, e.g., [8])

$$\|v\|_{L^4(\Omega)}\,\le\,\widehat{C}_2\,\|v\|^{1/2}\,\|v\|_V^{1/2} \qquad \forall\,v\in V, \tag{2.1}$$

as well as Agmon's inequality (see [4])

$$\|v\|_{L^\infty(\Omega)}\,\le\,\widehat{C}_3\,\|v\|^{1/2}\,\|v\|_{H^2(\Omega)}^{1/2} \qquad \forall\,v\in H^2(\Omega). \tag{2.2}$$

In these inequalities, the positive constants $\widehat{C}_2,\widehat{C}_3$ depend only on $\Omega\subset\mathbb{R}^2$.

The trilinear form b appearing in the weak formulation of the Navier–Stokes equations is defined as usual, namely,

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,dx \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div}\,.$$

We recall that we have

$$b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\,=\,-\,b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div}$$

and that in two dimensions of space there holds the estimate

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|\,\le\,\widehat{C}_1\,\|\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{v}\|\,\|\boldsymbol{w}\|^{1/2}\,\|\nabla\boldsymbol{w}\|^{1/2} \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div}$$

with a constant $\widehat{C}_1>0$ that only depends on Ω.

Finally, we will also need to use the operator $B := -\Delta + I$ with homogeneous Neumann boundary condition. It is well known that $B : D(B) \subset H \to H$ is an unbounded linear operator in H with the domain

$$D(B) = \left\{ \varphi \in H^2(\Omega) : \ \partial\varphi/\partial\boldsymbol{n} = 0 \ \text{ on } \partial\Omega \right\}$$

and that $B^{-1} : H \to H$ is a self-adjoint compact operator on H. By a classical spectral theorem there exist a sequence of eigenvalues μ_j with $0 < \mu_1 \le \mu_2 \le \cdots$ and $\mu_j \to \infty$ and a family of associated eigenfunctions $w_j \in D(B)$ such that $Bw_j = \mu_j \, w_j$ for all $j \in \mathbb{N}$. The family $\{w_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis in H and is also orthogonal in V and $D(B)$.

We are ready now to state the general assumptions on the data of the state system. We remark that for the well-posedness results cited below not all of these assumptions are always needed in every case; however, they seem to be indispensable for the analysis of the control problem. Since we focus on the control aspects here, we confine ourselves to these assumptions and refer the interested reader to [12, 15, 18] for further details. We postulate as follows:

(H1) It holds that $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in H^2(\Omega)$.

(H2) $F \in C^4(\mathbb{R})$ satisfies the following conditions:

$$\exists\, \hat{c}_1 > 0 : \quad F''(s) + a(x) \ge \hat{c}_1 \ \ \forall\, s \in \mathbb{R} \ \text{ and a.e. } \ x \in \Omega. \tag{2.3}$$

$$\exists\, \hat{c}_2 > 0,\, \hat{c}_3 > 0,\, p > 2 : \quad F''(s) + a(x) \ge \hat{c}_2\, |s|^{p-2} - \hat{c}_3 \quad \forall\, s \in \mathbb{R} \ \text{ and a.e. } \ x \in \Omega. \tag{2.4}$$

$$\exists\, \hat{c}_4 > 0,\, \hat{c}_5 \ge 0,\, r \in (1,2] : \quad |F'(s)|^r \le \hat{c}_4\, |F(s)| + \hat{c}_5 \ \ \forall\, s \in \mathbb{R}. \tag{2.5}$$

(H3) $\nu \in C^2(\mathbb{R})$, and there are constants $\hat{\nu}_1 > 0$, $\hat{\nu}_2 > 0$ such that

$$\hat{\nu}_1 \ \le\ \nu(s) \ \le\ \hat{\nu}_2 \quad \forall\, s \in \mathbb{R}. \tag{2.6}$$

(H4) The kernel K satisfies $K(x) = K(-x)$ for all x in its domain, as well as $a(x) = \int_\Omega K(x-y)\, dy \ \ge\ 0$ for a.e. $x \in \Omega$. Moreover, one of the following two conditions is fulfilled:

 (i) It holds $K \in W^{2,1}(B_\rho)$, where $\rho := \operatorname{diam}\Omega$ and $B_\rho := \{z \in \mathbb{R}^2 : \ |z| < \rho\}$.

 (ii) K is a so-called admissible kernel, which (cf. [7, Definition 1]) for the two-dimensional case means that we have

$$K \in W^{1,1}_{loc}(\mathbb{R}^2) \cap C^3(\mathbb{R}^2 \setminus \{0\}); \tag{2.7}$$

$$K \text{ is radially symmetric, } \ K(x) = \widetilde{K}(|x|), \ \text{ and } \ \widetilde{K} \text{ is nonincreasing;} \tag{2.8}$$

$$\widetilde{K}''(r) \ \text{ and } \ \widetilde{K}'(r)/r \ \text{ are monotone functions on } \ (0, r_0) \ \text{ for some } \ r_0 > 0; \tag{2.9}$$

$$\Big| \frac{\partial^3 K}{\partial x_i \partial x_j \partial x_k}(x) \Big| \ \le\ \hat{c}_6\, |x|^{-3} \quad \text{for } 1 \le i,j,k \le 2 \text{ and for some } \hat{c}_6 > 0. \tag{2.10}$$

Remark 2.1. Since F is bounded from below, it is easy to see that (2.5) implies that F has polynomial growth of order r', where $r' \in [2, \infty)$ is the conjugate index to r. Namely, there exist $\hat{c}_7 > 0$ and $\hat{c}_8 \geq 0$ such that

$$|F(s)| \leq \hat{c}_7 |s|^{r'} + \hat{c}_8 \qquad \forall s \in \mathbb{R}.$$

Observe that assumption (H2) is fulfilled by a potential of arbitrary polynomial growth. For example, (H2) is satisfied for the case of the well-known double well potential $F(s) = (s^2 - 1)^2$.

Remark 2.2. Notice that both the physically relevant two-dimensional Newtonian and Bessel kernels do not fulfill condition (i) in (H4); they are, however, known to be admissible in the sense of (ii). The advantage of dealing with admissible kernels is due to the fact that such kernels have the property (cf. [7, Lemma 2]) that for all $p \in (1, +\infty)$ there exists some constant $C_p > 0$ such that

$$\|\nabla(\nabla K * \psi)\|_{L^p(\Omega)^{2\times 2}} \leq C_p \|\psi\|_{L^p(\Omega)} \quad \forall \psi \in L^p(\Omega). \tag{2.11}$$

We also observe that under the hypothesis (H4) we have $a \in W^{1,\infty}(\Omega)$.

The following result combines results that have been shown in the papers [12, 15, 18]; in particular, we refer to [15, Theorems 5 and 6] and [18, Theorem 2, Remarks 2 and 5].

THEOREM 2.3. *Suppose that* (H1)–(H4) *are fulfilled. Then the state system* (1.3)–(1.6) *has for every* $\boldsymbol{v} \in L^2(0,T;G_{div})$ *a unique strong solution* $[\boldsymbol{u}, \varphi]$ *with the regularity properties*

$$\boldsymbol{u} \in C^0([0,T];V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \quad \boldsymbol{u}_t \in L^2(0,T;G_{div}), \tag{2.12}$$

$$\varphi \in C^0([0,T];H^2(\Omega)), \quad \varphi_t \in C^0([0,T];H) \cap L^2(0,T;V), \tag{2.13}$$

$$\mu := a\,\varphi - K * \varphi + F'(\varphi) \in C^0([0,T];H^2(\Omega)). \tag{2.14}$$

Moreover, there exists a continuous and nondecreasing function $\mathbb{Q}_1 : [0,+\infty) \to [0,+\infty)$, *which only depends on the data* F, K, ν, Ω, T, $\boldsymbol{u}_0$ *and* φ_0, *such that*

$$\begin{aligned} \|\boldsymbol{u}\|_{C^0([0,T];V_{div})\cap L^2(0,T;H^2(\Omega)^2)} + \|\boldsymbol{u}_t\|_{L^2(0,T;G_{div})} + \|\varphi\|_{C^0([0,T];H^2(\Omega))} \\ + \|\varphi_t\|_{C^0([0,T];H)\cap L^2(0,T;V)} \leq \mathbb{Q}_1\big(\|\boldsymbol{v}\|_{L^2(0,T;G_{div})}\big). \end{aligned} \tag{2.15}$$

Remark 2.4. As far as the regularity of the pressure π is concerned, from (1.1), (1.2), and the second of (1.5) we first deduce that $\boldsymbol{u}$ satisfies the following inhomogeneous elliptic system in nondivergence form:

$$\begin{cases} -\nu(\varphi)\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \\ \mathrm{div}\,(\boldsymbol{u}) = 0, \\ \boldsymbol{u}|_{\partial\Omega} = 0, \end{cases} \tag{2.16}$$

where

$$\boldsymbol{f} := \mu\nabla\varphi + \boldsymbol{v} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{u}_t + 2\nu'(\varphi)\nabla\varphi \cdot D\boldsymbol{u}.$$

On the other hand it can be proved that φ is also Hölder continuous, i.e., it satisfies $\varphi \in C^{\alpha,\alpha/2}\big(\overline{\Omega}\times[0,T]\big)$ for some $\alpha > 0$ (see [15, Lemma 2]). Therefore, since we have also $\nu(\varphi) \in C^{\beta,\beta/2}\big(\overline{\Omega}\times[0,T]\big)$ for some $\beta > 0$, we can apply [46, Proposition 2.1] to (2.16) and obtain the bound

$$\|\boldsymbol{u}\|_{H^2(\Omega)} + \|\pi\|_V \leq C\|\boldsymbol{f}\|,$$

where $C = C(\hat{\nu}_1, \hat{\nu}_2, R, T, \Omega) > 0$ and $R > 0$ is such that $\|\varphi\|_{L^\infty(\Omega\times(0,T))} \le R$. For this last estimate, the regularity properties of the solution $[\boldsymbol{u}, \varphi]$ (cf. (2.12)–(2.14)) and of the control $\boldsymbol{v}$ immediately yield

$$\pi \in L^2(0,T;V).$$

From Theorem 2.3, it follows that the *control-to-state operator* $\mathcal{S} : \boldsymbol{v} \mapsto \mathcal{S}(\boldsymbol{v}) := [\boldsymbol{u}, \varphi]$, is well defined as a mapping from $L^2(0,T;G_{div})$ into the Banach space defined by the regularity properties of $[\boldsymbol{u}, \varphi]$ as given by (2.12) and (2.13).

We now establish some global stability estimates for the strong solutions to problem (1.3)–(1.6). Let us begin with the following result (see [15, Theorem 6 and Lemma 2]).

LEMMA 2.5. *Suppose that* (H1)–(H4) *are fulfilled, and assume that controls* $\boldsymbol{v}_i \in L^2(0,T;G_{div})$, $i = 1,2$, *are given and that* $[\boldsymbol{u}_i, \varphi_i] := \mathcal{S}(\boldsymbol{v}_i)$, $i = 1,2$, *are the associated solutions to* (1.3)–(1.6). *Then there is a continuous function* $\mathbb{Q}_2 : [0,+\infty)^2 \to [0,+\infty)$, *which is nondecreasing in both its arguments and only depends on the data* F, K, ν, Ω, T, $\boldsymbol{u}_0$, *and* φ_0, *such that we have for every* $t \in (0,T]$ *the estimate*

(2.17)
$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{C^0([0,t];G_{div})} + \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})} + \|\varphi_2 - \varphi_1\|^2_{C^0([0,t];H)}$$
$$+ \|\nabla(\varphi_2 - \varphi_1)\|^2_{L^2(0,t;H)} \le \mathbb{Q}_2\big(\|\boldsymbol{v}_1\|_{L^2(0,T;G_{div})}, \|\boldsymbol{v}_2\|_{L^2(0,T;G_{div})}\big)\|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;(V_{div})')}.$$

Proof. We follow the lines of the proof of [15, Theorem 6] (see also [15, Lemma 2]), just sketching the main steps. We test the difference between (1.1), written for each of the two solutions, by $\boldsymbol{u} := \boldsymbol{u}_2 - \boldsymbol{u}_1$ in G_{div}, and the difference between (1.3), (1.4), written for each solution, by $\varphi := \varphi_2 - \varphi_1$ in H. Adding the resulting identities, and arguing exactly as in the proof of [15, Theorem 6], we are led to a differential inequality of the form

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\right) + \frac{\hat{\nu}_1}{4}\|\nabla \boldsymbol{u}(t)\|^2 + \frac{\hat{c}_1}{4}\|\nabla\varphi(t)\|^2$$
$$\le \gamma(t)\left(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\right) + \frac{1}{\hat{\nu}_1}\|\boldsymbol{v}(t)\|^2_{(V_{div})'} \quad \text{for a.e. } t \in (0,T),$$

where $\gamma \in L^1(0,T)$ is given by

$$\gamma(t) = c\big(1 + \|\nabla \boldsymbol{u}_1(t)\|^2\|\boldsymbol{u}_1(t)\|^2_{H^2(\Omega)} + \|\nabla \boldsymbol{u}_2(t)\|^2 + \|\varphi_1(t)\|^2_{L^4(\Omega)} + \|\varphi_2(t)\|^2_{L^4(\Omega)}$$
$$+ \|\varphi_1(t)\|^2_{H^2(\Omega)} + \|\nabla\varphi_1(t)\|^2\|\varphi_1(t)\|^2_{H^2(\Omega)}\big).$$

The desired stability estimate then follows from applying Gronwall's lemma to the above differential inequality. ∎

Lemma 2.5 already implies that the control-to-state mapping $\mathcal{S}$ is locally Lipschitz continuous as a mapping from $L^2(0,T;(V_{div})')$ (and, a fortiori, also from $L^2(0,T;V_{div})$) into the space $[C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})] \times [C^0([0,T];H) \cap L^2(0,T;V)]$. Since this result is not yet sufficient to establish differentiability, we need to improve the stability estimate. The following higher-order stability estimate for the solution component φ will turn out to be the key tool for the proof of differentiability of the control-to-state mapping.

LEMMA 2.6. *Suppose that the assumptions of Lemma* 2.5 *are fulfilled. Then there is a continuous function* $\mathbb{Q}_3 : [0,+\infty)^2 \to [0,+\infty)$, *which is nondecreasing in both its* *arguments and only depends on the data F, K, ν, Ω, T, $\boldsymbol{u}_0$, and φ_0, such that we have for every $t \in (0,T]$ the estimate*

$$\begin{aligned}(2.18)\qquad &\|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{C^0([0,t];G_{div})}+\|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})}+\|\varphi_2-\varphi_1\|^2_{C^0([0,t];V)}\\ &\quad+\|\varphi_2-\varphi_1\|^2_{L^2(0,t;H^2(\Omega))}+\|\varphi_2-\varphi_1\|^2_{H^1(0,t;H)}\\ &\quad\le \mathbb{Q}_3\big(\|\boldsymbol{v}_1\|_{L^2(0,T;G_{div})},\|\boldsymbol{v}_2\|_{L^2(0,T;G_{div})}\big)\|\boldsymbol{v}_2-\boldsymbol{v}_1\|^2_{L^2(0,T;(V_{div})')}\,.\end{aligned}$$

Proof. For the sake of a shorter exposition, we will in the following always avoid writing the time variable t as an argument of the involved functions; no confusion will arise from this notational convention.

Set $\boldsymbol{u} := \boldsymbol{u}_2 - \boldsymbol{u}_1$ and $\varphi := \varphi_2 - \varphi_1$. Then it follows from (1.3), (1.4) that

$$(2.19)\qquad \varphi_t = \Delta\widetilde{\mu} - \boldsymbol{u}\cdot\nabla\varphi_1 - \boldsymbol{u}_2\cdot\nabla\varphi,$$

$$(2.20)\qquad \widetilde{\mu} := a\,\varphi - K*\varphi + F'(\varphi_2) - F'(\varphi_1).$$

We multiply (2.19) by $\widetilde{\mu}_t$ in H and integrate by parts, using the first boundary condition of (1.5) (which holds also for $\widetilde{\mu}$). Notice that Theorem 2.3 ensures that $\varphi_{i,t} \in C^0([0,T];H)\cap L^2(0,T;V)$, $i=1,2$. Moreover, we have $\mu_{i,t} = a\varphi_{i,t} - K*\varphi_{i,t} + F''(\varphi_i)\varphi_{i,t}$, $i=1,2$, and using also the fact that φ_1, φ_2 satisfy the first of (2.13) and that F is regular, we easily obtain $\widetilde{\mu}_t \in C^0([0,T];H)\cap L^2(0,T;V)$. Therefore, $\widetilde{\mu}_t$ is allowed as a test function for (2.19), and the following differential identity is obtained:

$$(2.21)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\mu}\|^2 + (\varphi_t,\widetilde{\mu}_t) = -(\boldsymbol{u}\cdot\nabla\varphi_1,\widetilde{\mu}_t) - (\boldsymbol{u}_2\cdot\nabla\varphi,\widetilde{\mu}_t).$$

Thanks to (2.20), we can first rewrite the second term on the left-hand side of (2.21) as follows:

(2.22)

$$\begin{aligned}(\varphi_t,\widetilde{\mu}_t) &= \big(\varphi_t, a\,\varphi_t - K*\varphi_t + (F''(\varphi_2)-F''(\varphi_1))\varphi_{2,t} + F''(\varphi_1)\varphi_t\big)\\ &= \int_\Omega \big(a+F''(\varphi_1)\big)\varphi_t^2\,dx + \big(\Delta\widetilde{\mu} - \boldsymbol{u}\cdot\nabla\varphi_1 - \boldsymbol{u}_2\cdot\nabla\varphi, -K*\varphi_t\big)\\ &\quad + \big(\varphi_t,(F''(\varphi_2)-F''(\varphi_1))\varphi_{2,t}\big)\\ &= \int_\Omega \big(a+F''(\varphi_1)\big)\varphi_t^2\,dx + (\nabla\widetilde{\mu},\nabla K*\varphi_t) - (\boldsymbol{u}\varphi_1,\nabla K*\varphi_t) - (\boldsymbol{u}_2\varphi,\nabla K*\varphi_t)\\ &\quad + \big(\varphi_t,(F''(\varphi_2)-F''(\varphi_1))\varphi_{2,t}\big).\end{aligned}$$

Here we have employed (2.19) in the second identity of (2.22), while in the third identity integrations by parts have been performed using the boundary conditions $\partial\widetilde{\mu}/\partial\boldsymbol{n} = 0$ and $\boldsymbol{u}_i = 0$ on Σ, as well as the incompressibility conditions for $\boldsymbol{u}_i$, $i=1,2$.

We now estimate the last four terms on the right-hand side of (2.22). Using Young's inequality for convolution integrals, we have, for every $\epsilon > 0$,

(2.23)

$$|(\nabla\widetilde{\mu},\nabla K*\varphi_t)| \le \|\nabla\widetilde{\mu}\|\|\nabla K*\varphi_t\| \le \|\nabla\widetilde{\mu}\|\|\nabla K\|_{L^1(B_\rho)}\|\varphi_t\| \le \epsilon\|\varphi_t\|^2 + C_{\epsilon,K}\|\nabla\widetilde{\mu}\|^2.$$

Here, and throughout this proof, we use the following notational convention: by C_σ we denote positive constants that may depend on the global data and on the quantities indicated by the index σ; however, C_σ does not depend on the norms of the data of the two solutions. The actual value of C_σ may change from line to line or even within lines. On the other hand, Γ_σ will denote positive constants that may depend not only on the global data and on the quantities indicated by the index σ but also on $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. More precisely, we have

$$\Gamma_\sigma = \widehat{\Gamma}\big(\|\boldsymbol{v}_1\|_{L^2(0,T;G_{div})}, \|\boldsymbol{v}_2\|_{L^2(0,T;G_{div})}\big)$$

with a continuous function $\widehat{\Gamma} : [0,+\infty)^2 \to [0,+\infty)$ which is nondecreasing in both its variables. Also the actual value of Γ_σ may change even within the same line. Now, again using Young's inequality for convolution integrals, as well as Hölder's inequality, we have

$$|(\boldsymbol{u}\,\varphi_1, \nabla K * \varphi_t)| \le C_K\, \|\boldsymbol{u}\|_{L^4(\Omega)^2}\, \|\varphi_1\|_{L^4(\Omega)}\, \|\varphi_t\| \le \epsilon\, \|\varphi_t\|^2 + \Gamma_{\epsilon,K}\, \|\nabla \boldsymbol{u}\|^2, \tag{2.24}$$

$$|(\boldsymbol{u}_2\,\varphi, \nabla K * \varphi_t)| \le C_K\, \|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\, \|\varphi\|_{L^4(\Omega)}\, \|\varphi_t\| \le \epsilon\, \|\varphi_t\|^2 + \Gamma_{\epsilon,K}\, \|\varphi\|_V^2\,. \tag{2.25}$$

Moreover, invoking (H2), (2.15), and the Gagliardo–Nirenberg inequality (2.1), we infer that

$$\begin{aligned}\big|\big(\varphi_t, (F''(\varphi_2) - F''(\varphi_1))\,\varphi_{2,t}\big)\big| &\le \|\varphi_t\|\, \|F''(\varphi_2) - F''(\varphi_1)\|_{L^4(\Omega)}\, \|\varphi_{2,t}\|_{L^4(\Omega)} \\ &\le \Gamma_F\, \|\varphi_t\|\, \|\varphi\|_{L^4(\Omega)}\, \|\varphi_{2,t}\|_{L^4(\Omega)} \le \Gamma_F\, \|\varphi_t\|\, \|\varphi\|^{1/2}\, \|\varphi\|_V^{1/2}\, \|\varphi_{2,t}\|^{1/2}\, \|\varphi_{2,t}\|_V^{1/2} \\ &\le \epsilon\, \|\varphi_t\|^2 + \Gamma_{\epsilon,F}\, \|\varphi_{2,t}\|_V^2\, \|\varphi\|^2 + \Gamma_{\epsilon,F}\, \|\varphi\|_V^2\,.\end{aligned} \tag{2.26}$$

As far as the terms on the right-hand side of (2.21) are concerned, we can in view of (2.20) write

$$(\boldsymbol{u}\cdot\nabla\varphi_1, \widetilde{\mu}_t) = \big(\boldsymbol{u}\cdot\nabla\varphi_1, a\,\varphi_t - K * \varphi_t + (F''(\varphi_2) - F''(\varphi_1))\,\varphi_{2,t} + F''(\varphi_1)\,\varphi_t\big), \tag{2.27}$$

$$(\boldsymbol{u}_2\cdot\nabla\varphi, \widetilde{\mu}_t) = \big(\boldsymbol{u}_2\cdot\nabla\varphi, a\,\varphi_t - K * \varphi_t + (F''(\varphi_2) - F''(\varphi_1))\,\varphi_{2,t} + F''(\varphi_1)\,\varphi_t\big), \tag{2.28}$$

where the terms on the right-hand side of (2.27), (2.28) can be estimated in the following way:

$$\big|\big(\boldsymbol{u}\cdot\nabla\varphi_1, a\,\varphi_t - K * \varphi_t\big)\big| \le C_K\, \|\boldsymbol{u}\|_{L^4(\Omega)^2} \|\varphi_1\|_{H^2(\Omega)} \|\varphi_t\| \le \epsilon\, \|\varphi_t\|^2 + \Gamma_{\epsilon,K}\, \|\nabla \boldsymbol{u}\|^2\,, \tag{2.29}$$

$$\begin{aligned}&\big|\big(\boldsymbol{u}\cdot\nabla\varphi_1, (F''(\varphi_2) - F''(\varphi_1))\,\varphi_{2,t}\big)\big| \le \Gamma_F\, \|\boldsymbol{u}\|\, \|\varphi_1\|_{H^2(\Omega)}\, \|\varphi\|_{L^6(\Omega)}\, \|\varphi_{2,t}\|_{L^6(\Omega)} \\ &\le \Gamma_F\, \|\boldsymbol{u}\|\, \|\varphi\|_V\, \|\varphi_{2,t}\|_V \le \Gamma_F\, \|\varphi_{2,t}\|_V^2\, \|\boldsymbol{u}\|^2 + \Gamma_F\, \|\varphi\|_V^2\,,\end{aligned} \tag{2.30}$$

$$\big|\big(\boldsymbol{u}\cdot\nabla\varphi_1, F''(\varphi_1)\,\varphi_t\big)\big| \le \Gamma_F \|\boldsymbol{u}\|_{L^4(\Omega)^2} \|\varphi_1\|_{H^2(\Omega)} \|\varphi_t\| \le \epsilon \|\varphi_t\|^2 + \Gamma_{\epsilon,F} \|\nabla \boldsymbol{u}\|^2, \tag{2.31}$$

$$\begin{aligned}&\big|\big(\boldsymbol{u}_2\cdot\nabla\varphi, a\varphi_t - K * \varphi_t\big)\big| \le C_K \|\boldsymbol{u}_2\|_{L^4(\Omega)^2} \|\nabla\varphi\|_{L^4(\Omega)^2} \|\varphi_t\| \le \Gamma_K \|\nabla\varphi\|^{1/2} \|\nabla\varphi\|_V^{1/2} \|\varphi_t\| \\ &\le \epsilon\, \|\varphi_t\|^2 + \Gamma_{\epsilon,K}\, \|\nabla\varphi\|\, \|\varphi\|_{H^2(\Omega)} \le \epsilon\, \|\varphi_t\|^2 + \epsilon\, \|\varphi\|_{H^2(\Omega)}^2 + \Gamma_{\epsilon,K}\, \|\nabla\varphi\|^2\,,\end{aligned} \tag{2.32}$$

(2.33)
$$\begin{aligned}&\big|\big(\boldsymbol{u}_2\cdot\nabla\varphi,(F''(\varphi_2)-F''(\varphi_1))\varphi_{2,t}\big)\big| \le \Gamma_F\,\|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\,\|\nabla\varphi\|_{L^4(\Omega)^2}\,\|\varphi\|_{L^4(\Omega)}\,\|\varphi_{2,t}\|_{L^4(\Omega)}\\&\le \Gamma_F\|\varphi\|_{H^2(\Omega)}\|\varphi\|^{1/2}\|\varphi\|_V^{1/2}\|\varphi_{2,t}\|^{1/2}\|\varphi_{2,t}\|_V^{1/2} \le \epsilon\|\varphi\|^2_{H^2(\Omega)}+\Gamma_{\epsilon,F}\,\|\varphi\|\,\|\varphi\|_V\,\|\varphi_{2,t}\|_V\\&\le\,\epsilon\,\|\varphi\|^2_{H^2(\Omega)}\,+\,\Gamma_{\epsilon,F}\,\|\varphi\|_V^2\,+\,\Gamma_{\epsilon,F}\,\|\varphi_{2,t}\|_V^2\,\|\varphi\|^2\,,\end{aligned}$$

(2.34)
$$\begin{aligned}&\big|\big(\boldsymbol{u}_2\cdot\nabla\varphi,F''(\varphi_1)\varphi_t\big)\big| \le \Gamma_F\|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\|\nabla\varphi\|_{L^4(\Omega)^2}\|\varphi_t\| \le \Gamma_F\|\nabla\varphi\|^{1/2}\|\nabla\varphi\|_V^{1/2}\,\|\varphi_t\|\\&\le\,\epsilon\,\|\varphi_t\|^2\,+\,\Gamma_{\epsilon,F}\,\|\nabla\varphi\|\,\|\varphi\|_{H^2(\Omega)}\,\le\,\epsilon\|\varphi_t\|^2\,+\,\epsilon\,\|\varphi\|^2_{H^2(\Omega)}\,+\,\Gamma_{\epsilon,F}\,\|\nabla\varphi\|^2\,,\end{aligned}$$

where we have used the Hölder and Gagliardo–Nirenberg inequalities and (2.15) again.

We now insert the estimates (2.23)–(2.26) and (2.29)–(2.34) in (2.21), taking (2.22), (2.27), and (2.28) into account. By the assumption (2.3) in hypothesis (H2), and choosing $\epsilon>0$ small enough (i. e., $\epsilon\le\hat c_1/16$), we obtain the estimate

$$\begin{aligned}\frac{d}{dt}\,\|\nabla\widetilde\mu\|^2+\hat c_1\,\|\varphi_t\|^2\,\le\,C_{\epsilon,K}\,\|\nabla\widetilde\mu\|^2\,+\,\Gamma_{\epsilon,K,F}\,\big(\|\nabla\boldsymbol{u}\|^2+\|\varphi\|_V^2\big)\\+\,\Gamma_{\epsilon,F}\,\|\varphi_{2,t}\|_V^2\,\big(\|\boldsymbol{u}\|^2+\|\varphi\|^2\big)\,+\,6\,\epsilon\,\|\varphi\|^2_{H^2(\Omega)}\,.\end{aligned}\tag{2.35}$$

Next, we aim to show that the H^2 norm of φ can be controlled by the H^2 norm of $\widetilde\mu$. To this end, we take the second-order derivatives of (2.20) to find that

$$\begin{aligned}\partial^2_{ij}\widetilde\mu\,=\,&a\,\partial^2_{ij}\varphi+\partial_i a\,\partial_j\varphi+\partial_j a\,\partial_i\varphi+\varphi\,\partial_i(\partial_j a)-\partial_i\big(\partial_j K*\varphi\big)\\&+\big(F''(\varphi_2)-F''(\varphi_1)\big)\partial^2_{ij}\varphi_2+F''(\varphi_1)\,\partial^2_{ij}\varphi\\&+\big(F'''(\varphi_2)-F'''(\varphi_1)\big)\,\partial_i\varphi_2\,\partial_j\varphi_2+F'''(\varphi_1)\,(\partial_i\varphi_2\,\partial_j\varphi+\partial_i\varphi\,\partial_j\varphi_1)\,.\end{aligned}\tag{2.36}$$

Let us we multiply (2.36) by $\partial^2_{ij}\varphi$ in H and then estimate the terms on the right-hand side of the resulting equality. We have, invoking (2.3),

$$\Big(\big(a+F''(\varphi_1)\big)\,\partial^2_{ij}\varphi,\partial^2_{ij}\varphi\Big)\,\ge\,\hat c_1\,\|\partial^2_{ij}\varphi\|^2,\tag{2.37}$$

and, for every $\delta>0$ (to be fixed later),

(2.38)
$$\big(\partial_i a\,\partial_j\varphi+\partial_j a\,\partial_i\varphi,\partial^2_{ij}\varphi\big)\,\le\,C_K\,\|\nabla\varphi\|\,\|\partial^2_{ij}\varphi\|\,\le\,\delta\,\|\partial^2_{ij}\varphi\|^2\,+\,C_{\delta,K}\,\|\nabla\varphi\|^2,$$

(2.39)
$$\big(\varphi\,\partial_i(\partial_j a)-\partial_i(\partial_j K*\varphi),\partial^2_{ij}\varphi\big)\,\le\,C_K\,\|\varphi\|\,\|\partial^2_{ij}\varphi\|\,\le\,\delta\,\|\partial^2_{ij}\varphi\|^2\,+\,C_{\delta,K}\,\|\varphi\|^2,$$

where the first inequality in the estimate (2.39) follows from (2.11) if K is admissible, while in the case $K\in W^{2,1}(B_\rho)$ the first term in the product on the left-hand side of (2.39) can be rewritten as $\varphi\,\partial^2_{ij}a-\partial^2_{ij}K*\varphi$ so that (2.39) follows immediately from Young's inequality for convolution integrals. Moreover, invoking Agmon's inequality (2.2) and (2.15), we have

(2.40)
$$\begin{aligned}&\Big(\big(F''(\varphi_2)-F''(\varphi_1)\big)\,\partial^2_{ij}\varphi_2,\partial^2_{ij}\varphi\Big)\,\le\,\Gamma_F\,\|\varphi\|_{L^\infty(\Omega)}\,\|\varphi_2\|_{H^2(\Omega)}\,\|\partial^2_{ij}\varphi\|\\&\le\,\Gamma_F\,\|\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)}\,\|\partial^2_{ij}\varphi\|\,\le\,\Gamma_F\|\varphi\|^{1/2}\,\|\varphi\|^{3/2}_{H^2(\Omega)}\,\le\,\delta\,\|\varphi\|^2_{H^2(\Omega)}\,+\,\Gamma_{\delta,F}\,\|\varphi\|^2.\end{aligned}$$

In addition, by virtue of Hölder's inequality and (2.15), we have

(2.41)
$$\Big(\big(F'''(\varphi_2) - F'''(\varphi_1)\big)\partial_i\varphi_2\,\partial_j\varphi_2, \partial^2_{ij}\varphi\Big) \le \Gamma_F\,\|\varphi\|_{L^6(\Omega)}\|\partial_i\varphi_2\|_{L^6(\Omega)}\,\|\partial_j\varphi_2\|_{L^6(\Omega)}\|\partial^2_{ij}\varphi\|$$
$$\le\,\Gamma_F\,\|\varphi\|_V\,\|\varphi_2\|^2_{H^2(\Omega)}\,\|\partial^2_{ij}\varphi\|\,\le\,\delta\,\|\partial^2_{ij}\varphi\|^2\,+\,\Gamma_{\delta,F}\,\|\varphi\|^2_V,$$

and, invoking the Gagliardo–Nirenberg inequality (2.1) and (2.15),

(2.42)
$$\Big(F'''(\varphi_1)\,(\partial_i\varphi_2\,\partial_j\varphi + \partial_i\varphi\,\partial_j\varphi_1), \partial^2_{ij}\varphi\Big)$$
$$\le\,\Gamma_F\,\big(\|\partial_i\varphi_2\|_{L^4(\Omega)}\,\|\partial_j\varphi\|_{L^4(\Omega)}\,+\,\|\partial_i\varphi\|_{L^4(\Omega)}\,\|\partial_j\varphi_1\|_{L^4(\Omega)}\big)\,\|\partial^2_{ij}\varphi\|$$
$$\le\,\Gamma_F\,\big(\|\varphi_1\|_{H^2(\Omega)} + \|\varphi_2\|_{H^2(\Omega)}\big)\,\|\nabla\varphi\|_{L^4(\Omega)^2}\,\|\partial^2_{ij}\varphi\|\,\le\,\Gamma_F\,\|\nabla\varphi\|^{1/2}\,\|\nabla\varphi\|^{1/2}_V\,\|\partial^2_{ij}\varphi\|$$
$$\le\,\Gamma_F\,\|\nabla\varphi\|^{1/2}\,\|\varphi\|^{3/2}_{H^2(\Omega)}\,\le\,\delta\|\varphi\|^2_{H^2(\Omega)}\,+\,\Gamma_{\delta,F}\,\|\nabla\varphi\|^2.$$

Hence, by means of (2.37)–(2.42), we obtain that

$$\big(\partial^2_{ij}\widetilde{\mu}, \partial^2_{ij}\varphi\big)\,\ge\,\frac{\hat{c}_1}{2}\,\|\partial^2_{ij}\varphi\|^2\,-\,2\,\delta\,\|\varphi\|^2_{H^2(\Omega)} - \Gamma_{\delta,K}\,\|\varphi\|^2_V,$$

provided we choose $0 < \delta \le \hat{c}_1/6$. On the other hand, we have

$$\big(\partial^2_{ij}\widetilde{\mu}, \partial^2_{ij}\varphi\big)\,\le\,\frac{\hat{c}_1}{4}\,\|\partial^2_{ij}\varphi\|^2\,+\,\frac{1}{\hat{c}_1}\,\|\partial^2_{ij}\widetilde{\mu}\|^2,$$

and, by combining the last two estimates, we find that

$$\|\partial^2_{ij}\widetilde{\mu}\|^2\,\ge\,\frac{\hat{c}_1^2}{4}\,\|\partial^2_{ij}\varphi\|^2 - 2\,\hat{c}_1\,\delta\,\|\varphi\|^2_{H^2(\Omega)} - \Gamma_{\delta,K,F}\,\|\varphi\|^2_V,$$

where the factor $\hat{c}_1$ is absorbed in the constant $\Gamma_{\delta,K,F}$. From this, taking the sum over $i, j = 1, 2$, and fixing $0 < \delta \le \hat{c}_1/64$, we get the desired control,

$$\|\widetilde{\mu}\|^2_{H^2(\Omega)}\,\ge\,\frac{\hat{c}_1^2}{8}\,\|\varphi\|^2_{H^2(\Omega)} - \Gamma_{K,F}\|\varphi\|^2_V. \tag{2.43}$$

Let us now prove that the H^2 norm of $\widetilde{\mu}$ can be controlled in terms of the L^2 norm of φ_t. Indeed, from (2.19) we obtain, invoking the Hölder and Gagliardo–Nirenberg inequalities,

$$\begin{aligned}\|\Delta\widetilde{\mu}\|\,&\le\,\|\varphi_t\| + \|\boldsymbol{u}\|_{L^4(\Omega)^2}\,\|\nabla\varphi_1\|_{L^4(\Omega)^2} + \|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\,\|\nabla\varphi\|_{L^4(\Omega)^2}\\ &\le\,\|\varphi_t\| + C\,\|\nabla\boldsymbol{u}\|\|\varphi_1\|_{H^2(\Omega)} + C\,\|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}.\end{aligned} \tag{2.44}$$

Thanks to a classical elliptic regularity result (notice that $\partial\widetilde{\mu}/\partial\boldsymbol{n} = 0$ on $\partial\Omega$), we can infer from (2.20), (2.44), and (2.1) the estimate

$$\begin{aligned}\|\widetilde{\mu}\|_{H^2(\Omega)}\,&\le\,c_e\| - \Delta\widetilde{\mu} + \widetilde{\mu}\|\,\le\,c_e\|\Delta\widetilde{\mu}\| + \Gamma_{K,F}\,\|\varphi\|\\ &\le\,c_e\,\|\varphi_t\| + \Gamma\,\|\nabla\boldsymbol{u}\| + \Gamma\,\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)} + \Gamma_{K,F}\,\|\varphi\|\,,\end{aligned} \tag{2.45}$$

where $c_e > 0$ depends only on Ω. Combining (2.43) with (2.45), we then deduce that

$$\frac{\hat{c}_1}{4}\|\varphi\|_{H^2(\Omega)}\,\le\,c_e\,\|\varphi_t\| + \Gamma_{K,F}\,\big(\|\nabla\boldsymbol{u}\| + \|\varphi\|_V\big). \tag{2.46}$$

With (2.46) now available, we can now go back to (2.35) and fix $\epsilon > 0$ small enough (i.e., $\epsilon \le \epsilon_*$, where $\epsilon_* > 0$ depends only on $\hat{c}_1$ and c_e) to arrive at the differential inequality

(2.47)
$$\frac{d}{dt}\|\nabla\widetilde{\mu}\|^2 + \frac{\hat{c}_1}{2}\|\varphi_t\|^2 \le C_K\|\nabla\widetilde{\mu}\|^2 + \Gamma_{K,F}\big(\|\nabla\boldsymbol{u}\|^2 + \|\varphi\|_V^2\big) + \Gamma_F\|\varphi_{2,t}\|_V^2\big(\|\boldsymbol{u}\|^2 + \|\varphi\|^2\big).$$

Now observe that $\widetilde{\mu}(0) = 0$. Thus, applying Gronwall's lemma to (2.47), and using (2.15) for $\varphi_{2,t}$, we obtain, for every $t \in [0,T]$,

$$\|\nabla\widetilde{\mu}(t)\|^2 \le \Gamma\Big(\int_0^t \big(\|\nabla\boldsymbol{u}(\tau)\|^2 + \|\varphi(\tau)\|_V^2\big)d\tau + \big(\|\boldsymbol{u}\|^2_{C^0([0,t];G_{div})} + \|\varphi\|^2_{C^0([0,t];H)}\big)\int_0^t \|\varphi_{2,t}(\tau)\|_V^2 d\tau\Big),$$

where, for the sake of shorter notation, we have omitted the indexes K and F in the constant Γ. Hence, using the stability estimate of Lemma 2.5, we obtain from the last two inequalities that

$$\|\nabla\widetilde{\mu}(t)\|^2 \le \Gamma\,\|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;(V_{div})')}. \tag{2.48}$$

Now, taking the gradient of (2.20), and arguing as in the proof of [15, Lemma 2], it is not difficult to see that we have

$$(\nabla\widetilde{\mu}, \nabla\varphi) \ge \frac{\hat{c}_1}{4}\,\|\nabla\varphi\|^2 - \Gamma\,\|\varphi\|^2,$$

and this estimate, together with

$$(\nabla\widetilde{\mu}, \nabla\varphi) \le \frac{\hat{c}_1}{8}\,\|\nabla\varphi\|^2 + \frac{2}{\hat{c}_1}\,\|\nabla\widetilde{\mu}\|^2,$$

yields

$$\|\nabla\widetilde{\mu}\|^2 \ge \frac{\hat{c}_1^2}{16}\,\|\nabla\varphi\|^2 - \Gamma\,\|\varphi\|^2\,,$$

where the factor $\hat{c}_1/2$ is again absorbed in the constant Γ. This last estimate, combined with (2.48), gives

$$\|\varphi(t)\|_V^2 \le \Gamma\,\|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;(V_{div})')}. \tag{2.49}$$

By integrating (2.47) in time over $[0,t]$ and using (2.48) and the stability estimate of Lemma 2.5 again, we also get

$$\hat{c}_1\int_0^t \|\varphi_t(\tau)\|^2\,d\tau \le \Gamma\,\|\boldsymbol{v}_2 - \boldsymbol{v}_1\|^2_{L^2(0,T;(V_{div})')}. \tag{2.50}$$

The stability estimate (2.18) now follows from (2.49), (2.50), (2.46), and Lemma 2.5. □

3. Optimal control. We now study the optimal control problem (CP), where throughout this section we assume that the cost functional $\mathcal{J}$ is given by (1.7) and that the general hypotheses (H1)–(H4) are fulfilled. Moreover, we assume that the set of admissible controls $\mathcal{V}_{ad}$ is given by

(3.1)
$$\mathcal{V}_{ad} := \big\{ \boldsymbol{v} \in L^2(0,T;G_{div}) : v_{a,i}(x,t) \le v_i(x,t) \le v_{b,i}(x,t), \text{a.e. } (x,t) \in Q,\ \ i = 1,2 \big\},$$

with prescribed functions $\boldsymbol{v}_a, \boldsymbol{v}_b \in L^2(0,T;G_{div}) \cap L^\infty(Q)^2$. According to Theorem 2.3, the control-to-state mapping

$$\mathcal{S} : \mathcal{V} \to \mathcal{H}, \quad \boldsymbol{v} \in \mathcal{V} \mapsto \mathcal{S}(\boldsymbol{v}) := [\boldsymbol{u}, \varphi] \in \mathcal{H}, \tag{3.2}$$

where the space $\mathcal{H}$ is given by

$$\begin{aligned} \mathcal{H} \ := \ & \big[H^1(0,T;G_{div}) \cap C^0([0,T];V_{div}) \cap L^2(0,T;H^2(\Omega)^2)\big] \\ & \times \big[C^1([0,T];H) \cap H^1(0,T;V) \cap C^0([0,T];H^2(\Omega))\big], \end{aligned} \tag{3.3}$$

is well defined and locally bounded. Moreover, it follows from Lemma 2.6 that $\mathcal{S}$ is locally Lipschitz continuous from $\mathcal{V}$ into the space

(3.4)
$$\mathcal{W} := \big[C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})\big] \times \big[H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega))\big].$$

Notice also that problem (CP) is equivalent to the minimization problem

$$\min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v}),$$

for the reduced cost functional defined by $f(\boldsymbol{v}) := \mathcal{J}\big(\mathcal{S}(\boldsymbol{v}), \boldsymbol{v}\big)$, for every $\boldsymbol{v} \in \mathcal{V}$.

We have the following existence result.

THEOREM 3.1. *Assume that the hypotheses* (H1)–(H4) *are satisfied and that* $\mathcal{V}_{ad}$ *is given by* (3.1). *Then the optimal control problem* (CP) *admits a solution.*

Proof. Take a minimizing sequence $\{\boldsymbol{v}_n\} \subset \mathcal{V}_{ad}$ for (CP). Since $\mathcal{V}_{ad}$ is bounded in $\mathcal{V}$, we may assume without loss of generality that

$$\boldsymbol{v}_n \to \overline{\boldsymbol{v}} \quad \text{weakly in } \ L^2(0,T;G_{div})$$

for some $\overline{\boldsymbol{v}} \in \mathcal{V}$. Since $\mathcal{V}_{ad}$ is convex and closed in $\mathcal{V}$, and thus weakly sequentially closed, we have $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$.

Moreover, since $\mathcal{S}$ is a locally bounded mapping from $\mathcal{V}$ into $\mathcal{H}$, we may without loss of generality assume that the sequence $[\boldsymbol{u}_n, \varphi_n] = \mathcal{S}(\boldsymbol{v}_n)$, $n \in \mathbb{N}$, satisfies with appropriate limit points $[\overline{\boldsymbol{u}}, \overline{\varphi}]$ the convergences

(3.5)
$$\boldsymbol{u}_n \to \overline{\boldsymbol{u}}, \ \text{ weakly}^* \text{ in } L^\infty(0,T;V_{div}) \text{ and weakly in } H^1(0,T;G_{div}) \cap L^2(0,T;H^2(\Omega)^2),$$

$$\begin{aligned} & \varphi_n \to \overline{\varphi}, \ \text{weakly}^* \text{ in } L^\infty(0,T;H^2(\Omega)) \\ & \quad \text{and in } W^{1,\infty}(0,T;H), \text{ and weakly in } H^1(0,T;V). \end{aligned} \tag{3.6}$$

In particular, it follows from the compactness of the embedding $H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \subset C^0([0,T];H^s(\Omega))$ for $0 \le s < 2$, given by Aubin–Lions lemma (cf. [41]), that $\varphi_n \to \overline{\varphi}$ strongly in $C^0(\overline{Q})$, whence we conclude that also

(3.7)
$$\begin{aligned} & \mu_n := a\,\varphi_n - K * \varphi_n + F'(\varphi_n) \to \overline{\mu} := a\,\overline{\varphi} - K * \overline{\varphi} + F'(\overline{\varphi}) \quad \text{strongly in } \ C^0(\overline{Q}), \\ & \nu(\varphi_n) \to \nu(\overline{\varphi}) \quad \text{strongly in } \ C^0(\overline{Q}). \end{aligned}$$

We also have, by compact embedding,

$$\boldsymbol{u}_n \to \overline{\boldsymbol{u}} \quad \text{strongly in } L^2(0,T;G_{div}),$$

and it obviously holds that

$$\boldsymbol{u}_n(t) \to \overline{\boldsymbol{u}}(t) \quad \text{weakly in } G_{div} \quad \forall\, t \in [0,T]. \tag{3.8}$$

Now, by passing to the limit in the weak formulation of problem (1.1)–(1.6), written for each solution $[\boldsymbol{u}_n, \varphi_n] = \mathcal{S}(\boldsymbol{v}_n)$, $n \in \mathbb{N}$, and using the above weak and strong convergences (in particular, we can use [12, Lemma 1] in order to pass to the limit in the nonlinear term $-2\,\mathrm{div}(\nu(\varphi_n)D\boldsymbol{u}_n)$), it is not difficult to see that $[\overline{\boldsymbol{u}}, \overline{\varphi}]$ satisfies the weak formulation corresponding to $\overline{\boldsymbol{v}}$. Hence, we have $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$, that is, the pair $([\overline{\boldsymbol{u}}, \overline{\varphi}], \overline{\boldsymbol{v}})$ is admissible for (CP).

Finally, thanks to the weak sequential lower semicontinuity of $\mathcal{J}$ and to the weak convergences (3.5), (3.6), (3.8), we infer that the state $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$ is a solution to (CP). ∎

The linearized system. Suppose that the general hypotheses (H1)–(H4) are fulfilled. We assume that a fixed $\overline{\boldsymbol{v}} \in \mathcal{V}$ is given, that $[\overline{\boldsymbol{u}}, \overline{\varphi}] := \mathcal{S}(\overline{\boldsymbol{v}}) \in \mathcal{H}$ is the associated solution to the state system (1.3)–(1.6) according to Theorem 1, and that $\boldsymbol{h} \in \mathcal{V}$ is given. In order to show that the control-to-state operator is differentiable at $\overline{\boldsymbol{v}}$, we first consider the following system, which is obtained by linearizing the state system (1.3)–(1.6) at $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$:

$$\begin{aligned} \boldsymbol{\xi}_t \,-\, 2\,\mathrm{div}\big(\nu(\overline{\varphi})D\boldsymbol{\xi}\big) - 2\,\mathrm{div}\big(\nu'(\overline{\varphi})\,\eta\, D\overline{\boldsymbol{u}}\big) + (\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\overline{\boldsymbol{u}} + \nabla\widetilde{\pi} \\ =\, \big(a\,\eta - K*\eta + F''(\overline{\varphi})\,\eta\big)\nabla\overline{\varphi} + \overline{\mu}\,\nabla\eta + \boldsymbol{h} \quad \text{in } Q, \end{aligned} \tag{3.9}$$

$$\eta_t + \overline{\boldsymbol{u}}\cdot\nabla\eta \,=\, -\boldsymbol{\xi}\cdot\nabla\overline{\varphi} + \Delta\big(a\,\eta - K*\eta + F''(\overline{\varphi})\,\eta\big) \quad \text{in } Q, \tag{3.10}$$

$$\mathrm{div}(\boldsymbol{\xi}) = 0 \quad \text{in } Q, \tag{3.11}$$

$$\boldsymbol{\xi} = [0,0]^T, \quad \frac{\partial}{\partial \boldsymbol{n}}\big(a\,\eta - K*\eta + F''(\overline{\varphi})\,\eta\big) = 0 \quad \text{on } \Sigma, \tag{3.12}$$

$$\boldsymbol{\xi}(0) = [0,0]^T, \quad \eta(0) = 0, \quad \text{in } \Omega, \tag{3.13}$$

where

$$\overline{\mu} = a\overline{\varphi} - K*\overline{\varphi} + F'(\overline{\varphi}). \tag{3.14}$$

We first prove that (3.9)–(3.13) has a unique weak solution.

Proposition 3.2. *Suppose that the hypotheses* (H1)–(H4) *are satisfied. Then problem* (3.9)–(3.13) *has for every* $\boldsymbol{h} \in \mathcal{V}$ *a unique weak solution* $[\boldsymbol{\xi}, \eta]$ *such that*

$$\begin{aligned} &\boldsymbol{\xi} \in H^1(0,T;(V_{div})') \cap C^0([0,T];G_{div}) \cap L^2(0,T;V_{div}), \\ &\eta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V). \end{aligned} \tag{3.15}$$

Proof. We will make use of a Faedo–Galerkin approximating scheme. Following the lines of [12], we introduce the family $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ of the eigenfunctions to the Stokes operator A as a Galerkin basis in V_{div} and the family $\{\psi_j\}_{j\in\mathbb{N}}$ of the eigenfunctions to the Neumann operator $B := -\Delta + I$ as a Galerkin basis in V. Both these eigenfunction families $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ are assumed to be suitably ordered and normalized.

Moreover, recall that, since $\boldsymbol{w}_j \in D(A)$, we have $\mathrm{div}(\boldsymbol{w}_j) = 0$. Then we look for two functions of the form

$$\boldsymbol{\xi}_n(t) := \sum_{j=1}^{n} a_j^{(n)}(t)\boldsymbol{w}_j\,, \qquad \eta_n(t) := \sum_{j=1}^{n} b_j^{(n)}(t)\psi_j$$

that solve the following approximating problem:

(3.16)
$$\langle \partial_t \boldsymbol{\xi}_n(t), \boldsymbol{w}_i \rangle_{V_{div}} + 2\left(\nu(\overline{\varphi}(t))\, D\boldsymbol{\xi}_n(t), D\boldsymbol{w}_i\right) + 2\left(\nu'(\overline{\varphi}(t))\,\eta_n(t)\, D\overline{\boldsymbol{u}}(t), D\boldsymbol{w}_i\right)$$
$$+ b(\overline{\boldsymbol{u}}(t), \boldsymbol{\xi}_n(t), \boldsymbol{w}_i) + b(\boldsymbol{\xi}_n(t), \overline{\boldsymbol{u}}(t), \boldsymbol{w}_i)$$
$$= \left((a\,\eta_n(t) - K * \eta_n(t) + F''(\overline{\varphi}(t))\eta_n(t))\nabla\overline{\varphi}(t), \boldsymbol{w}_i\right) + (\overline{\mu}(t)\,\nabla\eta_n(t), \boldsymbol{w}_i) + (\boldsymbol{h}(t), \boldsymbol{w}),$$

(3.17)
$$\langle \partial_t \eta_n(t), \psi_i \rangle_V = -\left(\nabla(a\,\eta_n - K * \eta_n + F''(\overline{\varphi})\,\eta_n)(t), \nabla\psi_i\right) + (\overline{\boldsymbol{u}}(t)\,\eta_n(t), \nabla\psi_i)$$
$$+ (\boldsymbol{\xi}_n(t)\,\overline{\varphi}(t), \nabla\psi_i),$$

$$\boldsymbol{\xi}_n(0) = [0,0]^T, \quad \eta_n(0) = 0, \tag{3.18}$$

for $i = 1, \ldots, n$, and for almost every $t \in (0,T)$. Apparently, this is nothing but a Cauchy problem for a system of $2n$ linear ordinary differential equations in the $2n$ unknowns $a_i^{(n)}$, $b_i^{(n)}$, in which, owing to the regularity properties of $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, all of the coefficient functions belong to $L^2(0,T)$. Thanks to Carathéodory's theorem, we can conclude that this problem enjoys a unique solution $\boldsymbol{a}^{(n)} := (a_1^{(n)}, \ldots, a_n^{(n)})^T$, $\boldsymbol{b}^{(n)} := (b_1^{(n)}, \ldots, b_n^{(n)})^T$ such that $\boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)} \in H^1(0,T;\mathbb{R}^n)$.

We now aim to derive a priori estimates for $\boldsymbol{\xi}_n$ and η_n that are uniform in $n \in \mathbb{N}$. For the sake of keeping the exposition at a reasonable length, we will always omit the argument t. To begin with, let us multiply (3.16) by $a_i^{(n)}$, (3.17) by $b_i^{(n)}$, sum over $i = 1, \ldots, n$, and add the resulting identities. We then obtain, almost everywhere in $(0,T)$,

(3.19)
$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{\xi}_n\|^2 + \|\eta_n\|^2\right) + 2\left(\nu(\overline{\varphi})\,D\boldsymbol{\xi}_n, D\boldsymbol{\xi}_n\right) + \left((a + F''(\overline{\varphi}))\,\nabla\eta_n, \nabla\eta_n\right)$$
$$= -b(\boldsymbol{\xi}_n, \overline{\boldsymbol{u}}, \boldsymbol{\xi}_n) - 2\left(\nu'(\overline{\varphi})\,\eta_n\, D\overline{\boldsymbol{u}}, D\boldsymbol{\xi}_n\right) + \left((a\,\eta_n - K * \eta_n + F''(\overline{\varphi})\,\eta_n)\,\nabla\overline{\varphi}, \boldsymbol{\xi}_n\right)$$
$$+ (\overline{\mu}\,\nabla\eta_n, \boldsymbol{\xi}_n) + (\boldsymbol{h}, \boldsymbol{\xi}_n) - \left(\eta_n \nabla a - \nabla K * \eta_n, \nabla\eta_n\right) - \left(\eta_n\, F'''(\overline{\varphi})\,\nabla\overline{\varphi}, \nabla\eta_n\right)$$
$$+ (\overline{\boldsymbol{u}}\,\eta_n, \nabla\eta_n) + (\boldsymbol{\xi}_n\,\overline{\varphi}, \nabla\eta_n).$$

Let us now estimate the terms on the right-hand side of this equation individually. In the remainder of this proof, we use the following abbreviating notation: the letter C will stand for positive constants that depend only on the global data of the system (1.1)–(1.6), on $\overline{\boldsymbol{v}}$, and on $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, but not on $n \in \mathbb{N}$; moreover, by C_σ we denote constants that in addition depend on the quantities indicated by the index σ, but not on $n \in \mathbb{N}$. Both C and C_σ may change within formulas and even within lines.

We have, using Hölder's inequality, the elementary Young's inequality, and the global bounds (2.15) as main tools, the following series of estimates:

(3.20)
$$|b(\boldsymbol{\xi}_n, \overline{\boldsymbol{u}}, \boldsymbol{\xi}_n)| \le \|\boldsymbol{\xi}_n\|_{L^4(\Omega)^2}\, \|\nabla\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}}\, \|\boldsymbol{\xi}_n\| \le C\, \|\nabla\boldsymbol{\xi}_n\|\, \|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}\, \|\boldsymbol{\xi}_n\|$$
$$\le \epsilon\, \|\nabla\boldsymbol{\xi}_n\|^2 + C_\epsilon\, \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\, \|\boldsymbol{\xi}_n\|^2,$$

(3.21)
$$\begin{aligned}&\big|2\left(\nu'(\overline{\varphi})\,\eta_n\,D\overline{\boldsymbol{u}},D\boldsymbol{\xi}_n\right)\big| \,\le\, C\,\|\eta_n\|_{L^4(\Omega)}\,\|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}}\,\|\nabla\boldsymbol{\xi}_n\|\\ &\le\,\epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2\,+\,C_\epsilon\left(\|\eta_n\|^2+\|\eta_n\|\,\|\nabla\eta_n\|\right)\|D\overline{\boldsymbol{u}}\|^2_{L^4(\Omega)^{2\times 2}}\\ &\le\,\epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2\,+\,C_\epsilon\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\eta_n\|^2\,+\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon,\epsilon'}\,\|\nabla\overline{\boldsymbol{u}}\|^2\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\eta_n\|^2\\ &\le\,\epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2\,+\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon,\epsilon'}\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\eta_n\|^2,\end{aligned}$$

(3.22)
$$\begin{aligned}&\big|\big((a\,\eta_n-K*\eta_n+F''(\overline{\varphi})\,\eta_n)\,\nabla\overline{\varphi},\boldsymbol{\xi}_n\big)\big|\,\le\,C\,\|\eta_n\|\,\|\overline{\varphi}\|_{H^2(\Omega)}\,\|\boldsymbol{\xi}_n\|_{L^4(\Omega)^2}\\ &\le\,C\,\|\eta_n\|\,\|\nabla\boldsymbol{\xi}_n\|\,\le\,\epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2\,+\,C_\epsilon\,\|\eta_n\|^2,\end{aligned}$$

(3.23)
$$\begin{aligned}&|(\overline{\mu}\,\nabla\eta_n,\boldsymbol{\xi}_n)|\,=\,|(\eta_n\,\nabla\overline{\mu},\boldsymbol{\xi}_n)|\,\le\,\|\nabla\overline{\mu}\|_{L^4(\Omega)^{2\times 2}}\,\|\eta_n\|\,\|\boldsymbol{\xi}_n\|_{L^4(\Omega)^2}\,\le\,C\,\|\nabla\boldsymbol{\xi}_n\|\,\|\eta_n\|\\ &\le\,\epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2\,+\,C_\epsilon\,\|\eta_n\|^2,\end{aligned}$$

$$|(\boldsymbol{h},\boldsymbol{\xi}_n)|\,\le\,C\,\|\boldsymbol{\xi}_n\|^2\,+\,C\,\|\boldsymbol{h}\|^2, \tag{3.24}$$

$$\big|\big(\eta_n\nabla a-\nabla K*\eta_n,\nabla\eta_n\big)\big|\,\le\,C\,\|\eta_n\|\,\|\nabla\eta_n\|\,\le\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon'}\,\|\eta_n\|^2. \tag{3.25}$$

Moreover, also employing the Gagliardo–Nirenberg inequality (2.1), we find that

(3.26)
$$\begin{aligned}&|(\eta_n\,F'''(\overline{\varphi})\,\nabla\overline{\varphi},\nabla\eta_n)|\,\le\,C\,\|\eta_n\|_{L^4(\Omega)}\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|\nabla\eta_n\|\\ &\le\,C\,(\|\eta_n\|\,+\,\|\eta_n\|^{1/2}\,\|\nabla\eta_n\|^{1/2})\,\|\nabla\eta_n\|\,\le\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon'}\,\|\eta_n\|^2,\end{aligned}$$

(3.27)
$$\begin{aligned}&|(\overline{\boldsymbol{u}}\,\eta_n,\nabla\eta_n)|\,\le\,\|\overline{\boldsymbol{u}}\|_{L^4(\Omega)^2}\,\|\eta_n\|_{L^4(\Omega)}\,\|\nabla\eta_n\|\,\le\,C\left(\|\eta_n\|\,+\,\|\eta_n\|^{1/2}\,\|\nabla\eta_n\|^{1/2}\right)\|\nabla\eta_n\|\\ &\le\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon'}\,\|\eta_n\|^2,\end{aligned}$$

$$|(\boldsymbol{\xi}_n\,\overline{\varphi},\nabla\eta_n)|\,\le\,C\,\|\overline{\varphi}\|_{H^2(\Omega)}\,\|\boldsymbol{\xi}_n\|\,\|\nabla\eta_n\|\,\le\,\epsilon'\,\|\nabla\eta_n\|^2\,+\,C_{\epsilon'}\,\|\boldsymbol{\xi}_n\|^2. \tag{3.28}$$

Hence, inserting the estimates (3.20)–(3.28) in (3.19), applying the conditions (2.3) in (H2) and (2.6) in (H3), respectively, to the second and third terms on the left-hand side of (3.19), and choosing $\epsilon>0$ and $\epsilon'>0$ small enough, we obtain the estimate

$$\begin{aligned}&\frac{d}{dt}\big(\|\boldsymbol{\xi}_n\|^2+\|\eta_n\|^2\big)+\hat{\nu}_1\|\nabla\boldsymbol{\xi}_n\|^2+\hat{c}_1\|\nabla\eta_n\|^2\\ &\quad\le C\big(1+\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\big)\big(\|\boldsymbol{\xi}_n\|^2+\|\eta_n\|^2\big)+C\|\boldsymbol{h}\|^2.\end{aligned} \tag{3.29}$$

Since, owing to (2.15), the mapping $t\mapsto\|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}$ belongs to $L^1(0,T)$, we may employ Gronwall's lemma to conclude the estimate

$$\begin{aligned}&\|\boldsymbol{\xi}_n\|_{L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div})}\,\le\,C\,\|\boldsymbol{h}\|_{\mathcal{V}},\\ &\|\eta_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\,\le\,C\,\|\boldsymbol{h}\|_{\mathcal{V}}\quad\forall\;n\in\mathbb{N}\,.\end{aligned} \tag{3.30}$$

Moreover, by comparison in (3.16), (3.17), we can easily deduce also the estimates for the time derivatives $\partial_t \boldsymbol{\xi}_n$ and $\partial_t \eta_n$. Indeed, we have

$$(3.31) \qquad \|\partial_t \boldsymbol{\xi}_n\|_{L^2(0,T;(V_{div})')} \le C\,\|\boldsymbol{h}\|_{\mathcal{V}}, \quad \|\partial_t \eta_n\|_{L^2(0,T;V')} \le C\,\|\boldsymbol{h}\|_{\mathcal{V}} \quad \forall\ n \in \mathbb{N}.$$

From (3.30), (3.31) we deduce the existence of two functions $\boldsymbol{\xi}$, η satisfying (3.15) and of two (not relabeled) subsequences $\{\boldsymbol{\xi}_n\}$, $\{\eta_n\}$ (and $\{\partial_t \boldsymbol{\xi}_n\}$, $\{\partial_t \eta_n\}$) converging weakly respectively to $\boldsymbol{\xi}$, η (and to $\boldsymbol{\xi}_t$, η_t) in the spaces where the bounds given by (3.30) (and by (3.31)) hold.

Then, by means of standard arguments, we can pass to the limit as $n \to \infty$ in (3.16)–(3.18) and prove that $\boldsymbol{\xi}$, η satisfy the weak formulation of problem (3.9)–(3.13). Notice that we actually have the regularity (3.15), since the space $H^1(0,T;(V_{div})') \cap L^2(0,T;V_{div})$ is continuously embedded in $C^0([0,T];G_{div})$; similarly we obtain that $\eta \in C^0([0,T];H)$.

Finally, in order to prove that the solution $\boldsymbol{\xi}, \eta$ is unique, we can test the difference between (3.9), (3.10), written for two solutions $\boldsymbol{\xi}_1, \eta_1$ and $\boldsymbol{\xi}_2, \eta_2$, by $\boldsymbol{\xi} := \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ and by $\eta := \eta_1 - \eta_2$, respectively. Since the problem is linear, the argument is straightforward, and we may leave the details to the reader. ∎

Remark 3.3. By virtue of the weak sequential lower semicontinuity of norms, we can conclude from the estimates (3.30) and (3.31) that the linear mapping $\boldsymbol{h} \mapsto [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$, which assigns to each $\boldsymbol{h} \in \mathcal{V}$ the corresponding unique weak solution pair $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}] := [\boldsymbol{\xi}, \eta]$ to the linearized system (3.9)–(3.13), is continuous as a mapping between the spaces $\mathcal{V}$ and $\left[H^1(0,T;(V_{div})') \cap C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})\right] \times \left[H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)\right]$.

Differentiability of the control-to-state operator. We now prove the following result.

THEOREM 3.4. *Suppose that the hypotheses* (H1)–(H4) *are fulfilled. Then the control-to-state operator $\mathcal{S} : \mathcal{V} \to \mathcal{H}$ is Fréchet differentiable on $\mathcal{V}$ when viewed as a mapping between the spaces $\mathcal{V}$ and $\mathcal{Z}$, where*

$$\mathcal{Z} := \left[C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})\right] \times \left[C^0([0,T];H) \cap L^2(0,T;V)\right].$$

Moreover, for any $\overline{\boldsymbol{v}} \in \mathcal{V}$ the Fréchet derivative $\mathcal{S}'(\overline{\boldsymbol{v}}) \in \mathcal{L}(\mathcal{V},\mathcal{Z})$ is given by $\mathcal{S}'(\overline{\boldsymbol{v}})\boldsymbol{h} = [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ for all $\boldsymbol{h} \in \mathcal{V}$, where $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ is the unique weak solution to the linearized system (3.9)–(3.13) *at $[\overline{\boldsymbol{u}}, \overline{\varphi}] = S(\overline{\boldsymbol{v}})$ that corresponds to $\boldsymbol{h} \in \mathcal{V}$.*

Proof. Let $\overline{\boldsymbol{v}} \in \mathcal{V}$ be fixed and $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$. Recalling Remark 3.3, we first note that the linear mapping $\boldsymbol{h} \mapsto [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ belongs to $\mathcal{L}(\mathcal{V},\mathcal{Z})$.

Now let $\Lambda > 0$ be fixed. In the following, we consider perturbations $\boldsymbol{h} \in \mathcal{V}$ such that $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$. For any such perturbation $\boldsymbol{h}$, we put

$$[\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}] := S(\overline{\boldsymbol{v}} + \boldsymbol{h}), \qquad \boldsymbol{p}^{\boldsymbol{h}} := \boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}} - \boldsymbol{\xi}^{\boldsymbol{h}}, \qquad q^{\boldsymbol{h}} := \varphi^{\boldsymbol{h}} - \overline{\varphi} - \eta^{\boldsymbol{h}}.$$

Notice that we have the regularity

$$(3.32) \qquad \begin{aligned} &\boldsymbol{p}^{\boldsymbol{h}} \in H^1(0,T;V_{div}') \cap C^0([0,T];G_{div}) \cap L^2(0,T;V_{div}),\\ &q^{\boldsymbol{h}} \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)\,. \end{aligned}$$

By virtue of (2.15) in Theorem 2.3 and of (2.18) in Lemma 2.6, there is a constant $C_1^* > 0$, which may depend on the data of the problem and on Λ, such that we have that for every $\boldsymbol{h} \in \mathcal{V}$ with $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$ it holds that

$$(3.33) \qquad \left\|[\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}]\right\|_{\mathcal{H}} \le C_1^*\,, \quad \|\varphi^{\boldsymbol{h}}\|_{C^0(\overline{Q})} \le C_1^*\,,$$

(3.34)
$$\|\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}\|^2_{C^0([0,t];G_{div})\cap L^2(0,t;V_{div})}+\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|^2_{H^1(0,t;H)\cap C^0([0,t];V)\cap L^2(0,t;H^2(\Omega))} \leq C_1^*\|\boldsymbol{h}\|^2_{\mathcal{V}} \quad \text{for every } t\in(0,T].$$

Now, after some easy computations, we can see that $\boldsymbol{p}^{\boldsymbol{h}}, q^{\boldsymbol{h}}$ (which, for simplicity, shall henceforth be denoted by $\boldsymbol{p}, q$) is a solution to the weak analogue of the following problem:

(3.35)
$$\begin{aligned}&\boldsymbol{p}_t-2\mathrm{div}\big(\nu(\overline{\varphi})D\boldsymbol{p}\big)-2\mathrm{div}\big((\nu(\varphi^{\boldsymbol{h}})-\nu(\overline{\varphi}))D(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\big)-2\mathrm{div}\big((\nu(\varphi^{\boldsymbol{h}})-\nu(\overline{\varphi})-\nu'(\overline{\varphi})\eta^{\boldsymbol{h}})D\overline{\boldsymbol{u}}\big)\\&+(\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}}+(\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{p}+\big((\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})+\nabla\pi^{\boldsymbol{h}}\\&=a\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})\,\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})-\big(K*(\varphi^{\boldsymbol{h}}-\overline{\varphi})\big)\,\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})+(a\,q-K*q)\,\nabla\overline{\varphi}\\&\quad+(a\,\overline{\varphi}-K*\overline{\varphi})\,\nabla q+\big(F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})+F'(\overline{\varphi})\,\nabla q\\&\quad+\big(F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})-F''(\overline{\varphi})\,\eta^{\boldsymbol{h}}\big)\,\nabla\overline{\varphi}\quad\text{in } Q,\end{aligned}$$

(3.36)
$$\begin{aligned}&q_t+(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})+\boldsymbol{p}\cdot\nabla\overline{\varphi}+\overline{\boldsymbol{u}}\cdot\nabla q=\Delta\big(a\,q-K*q+F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})\\&\quad-F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\ \text{in } Q,\end{aligned}$$

$$\mathrm{div}(\boldsymbol{p})=0\quad\text{in } Q, \tag{3.37}$$

(3.38)
$$\boldsymbol{p}=[0,0]^T,\qquad \frac{\partial}{\partial\boldsymbol{n}}\big(aq-K*q+F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})-F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)=0,\qquad\text{on }\Sigma,$$

$$\boldsymbol{p}(0)=[0,0]^T,\quad q(0)=0,\quad\text{in }\Omega, \tag{3.39}$$

where the pressure $\pi^{\boldsymbol{h}}$ is given by $\pi^{\boldsymbol{h}}:=\hat{\pi}^{\boldsymbol{h}}-\overline{\pi}-\widetilde{\pi}$, with $\hat{\pi}^{\boldsymbol{h}}$ and $\overline{\pi}$ representing the pressure terms appearing in (1.1) written for $\overline{\boldsymbol{v}}+\boldsymbol{h}$ and for $\overline{\boldsymbol{v}}$, respectively, and $\widetilde{\pi}$ the pressure term appearing in (3.9) (cf. Remark 2.4, as far as the regularity of these terms is concerned).

That is, $\boldsymbol{p}$ and q solve the following variational problem (where we avoid writing the argument t of the involved functions):

(3.40)
$$\begin{aligned}&\langle\boldsymbol{p}_t,\boldsymbol{w}\rangle_{V_{div}}+2\left(\nu(\overline{\varphi})D\boldsymbol{p},D\boldsymbol{w}\right)+2\left((\nu(\varphi^{\boldsymbol{h}})-\nu(\overline{\varphi}))D(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}),D\boldsymbol{w}\right)\\&+2\left((\nu(\varphi^{\boldsymbol{h}})-\nu(\overline{\varphi})-\nu'(\overline{\varphi})\eta^{\boldsymbol{h}})D\overline{\boldsymbol{u}},D\boldsymbol{w}\right)+b(\boldsymbol{p},\overline{\boldsymbol{u}},\boldsymbol{w})+b(\overline{\boldsymbol{u}},\boldsymbol{p},\boldsymbol{w})\\&+b(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}},\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}},\boldsymbol{w})\\&=\left(a\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi}),\boldsymbol{w}\right)-\left(\big(K*(\varphi^{\boldsymbol{h}}-\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi}),\boldsymbol{w}\right)\\&\quad+\left((a\,q-K*q)\nabla\overline{\varphi},\boldsymbol{w}\right)+\left((a\overline{\varphi}-K*\overline{\varphi})\nabla q,\boldsymbol{w}\right)+\left(\big(F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi}),\boldsymbol{w}\right)\\&\quad+\left(F'(\overline{\varphi})\nabla q,\boldsymbol{w}\right)+\left(\big(F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})-F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\nabla\overline{\varphi},\boldsymbol{w}\right),\end{aligned}$$

(3.41)
$$\begin{aligned}&\langle q_t,\psi\rangle_V+\left((\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi}),\psi\right)+\left(\boldsymbol{p}\cdot\nabla\overline{\varphi},\psi\right)+\left(\overline{\boldsymbol{u}}\cdot\nabla q,\psi\right)\\&=-\left(\nabla\big(a\,q-K*q+F'(\varphi^{\boldsymbol{h}})-F'(\overline{\varphi})-F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big),\nabla\psi\right)\end{aligned}$$

for every $\boldsymbol{w}\in V_{div}$, every $\psi\in V$, and almost every $t\in(0,T)$.

We choose $\boldsymbol{w} = \boldsymbol{p}(t) \in V_{div}$ and $\psi = q(t) \in V$ as test functions in (3.40) and (3.41), respectively, to obtain the equations (where we will again always suppress the argument t of the involved functions)

(3.42)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{p}\|^2 + 2\int_\Omega \nu(\overline{\varphi})D\boldsymbol{p} : D\boldsymbol{p}\,dx + 2\int_\Omega (\nu(\varphi^{\boldsymbol{h}}) - \nu(\overline{\varphi}))\,D(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) : D\boldsymbol{p}\,dx \\
&+ 2\int_\Omega \nu'(\overline{\varphi})\,q\,D\overline{\boldsymbol{u}} : D\boldsymbol{p}\,dx + \int_\Omega \nu''(\sigma_1^{\boldsymbol{h}})\,(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2\,D\overline{\boldsymbol{u}} : D\boldsymbol{p}\,dx + \int_\Omega (\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}}\cdot\boldsymbol{p}\,dx \\
&+ \int_\Omega \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\boldsymbol{p}\,dx \\
&= \int_\Omega a\,(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\cdot\boldsymbol{p}\,dx - \int_\Omega \big(K*(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\cdot\boldsymbol{p}\,dx \\
&\quad + \int_\Omega (a\,q - K*q)\nabla\overline{\varphi}\cdot\boldsymbol{p}\,dx + \int_\Omega (a\,\overline{\varphi} - K*\overline{\varphi})\nabla q\cdot\boldsymbol{p}\,dx \\
&\quad + \int_\Omega \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\cdot\boldsymbol{p}\,dx + \int_\Omega F'(\overline{\varphi})\nabla q\cdot\boldsymbol{p}\,dx \\
&\quad + \int_\Omega F''(\overline{\varphi})q\nabla\overline{\varphi}\cdot\boldsymbol{p}\,dx + \frac{1}{2}\int_\Omega F'''(\sigma_2^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2\,\nabla\overline{\varphi}\cdot\boldsymbol{p}\,dx\,,
\end{aligned}$$

(3.43)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q\|^2 + \int_\Omega \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\,q\,dx + \int_\Omega (\boldsymbol{p}\cdot\nabla\overline{\varphi})\,q\,dx \\
&\qquad = -\int_\Omega \nabla q\cdot\nabla\big(a\,q - K*q + F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}}\big)\,dx\,.
\end{aligned}$$

In (3.42), we have used Taylor's formula

$$\begin{aligned}
\nu(\varphi^{\boldsymbol{h}}) &= \nu(\overline{\varphi}) + \nu'(\overline{\varphi})(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \frac{1}{2}\nu''(\sigma_1^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2, \\
F'(\varphi^{\boldsymbol{h}}) &= F'(\overline{\varphi}) + F''(\overline{\varphi})(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \frac{1}{2}F'''(\sigma_2^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2,
\end{aligned}$$

where

$$\sigma_i^{\boldsymbol{h}} = \theta_i^{\boldsymbol{h}}\varphi^{\boldsymbol{h}} + (1 - \theta_i^{\boldsymbol{h}})\overline{\varphi}, \quad \theta_i^{\boldsymbol{h}} = \theta_i^{\boldsymbol{h}}(x,t) \in (0,1) \quad \text{for } i = 1, 2.$$

Moreover, in the integration by parts on the right-hand side of (3.43) we employed the second boundary condition in (3.38), which is a consequence of $\partial\mu^{\boldsymbol{h}}/\partial\boldsymbol{n} = \partial\overline{\mu}/\partial\boldsymbol{n} = 0$ on Σ and of (3.12) (where $\mu^{\boldsymbol{h}} := a\,\varphi^{\boldsymbol{h}} - K*\varphi^{\boldsymbol{h}} + F'(\varphi^{\boldsymbol{h}})$).

We now begin to estimate all the terms in (3.42). In this process, we will make repeated use of the global estimates (3.33), (3.34) and of the Gagliardo–Nirenberg inequality (2.1). Again, we denote by C positive constants that may depend on the data of the system but not on the choice of $\boldsymbol{h} \in \mathcal{V}$ with $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$, while C_σ denotes a positive constant that also depends on the quantity indicated by the index σ. We have, with constants $\epsilon > 0$ and $\epsilon' > 0$ that will be fixed later, the following series of estimates:

(3.44)
$$\begin{aligned}&\Big|2\int_\Omega(\nu(\varphi^{\boldsymbol{h}})-\nu(\overline{\varphi}))D(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}):D\boldsymbol{p}\,dx\Big| = \Big|2\int_\Omega \nu'(\sigma_3^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}}-\overline{\varphi})D(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}):D\boldsymbol{p}\,dx\Big|\\
&\quad\le C\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_{L^4(\Omega)}\,\|D(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\|_{L^4(\Omega)^{2\times2}}\,\|D\boldsymbol{p}\|\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^2\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\|\,\big(\|\boldsymbol{u}^{\boldsymbol{h}}\|_{H^2(\Omega)^2}+\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}\big)\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\|\,\big(\|\boldsymbol{u}^{\boldsymbol{h}}\|_{H^2(\Omega)^2}+\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}\big)\,\|\boldsymbol{h}\|_{\mathcal{V}}^2,\end{aligned}$$

as well as

$$\begin{aligned}&\Big|2\int_\Omega \nu'(\overline{\varphi})\,q\,D\overline{\boldsymbol{u}}:D\boldsymbol{p}\,dx\Big| \le C\,\|q\|_{L^4(\Omega)}\,\|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times2}}\,\|\nabla\boldsymbol{p}\|\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|q\|\,\|q\|_V\,\|\nabla\overline{\boldsymbol{u}}\|\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + \epsilon'\,\|\nabla q\|^2 + C_{\epsilon,\epsilon'}\,\big(1+\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\big)\,\|q\|^2.\end{aligned}\tag{3.45}$$

Moreover, by similar reasoning,

(3.46)
$$\begin{aligned}&\Big|\int_\Omega \nu''(\sigma_1^{\boldsymbol{h}})\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})^2\,D\overline{\boldsymbol{u}}:D\boldsymbol{p}\,dx\Big| \le C\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_{L^8(\Omega)}^2\,\|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times2}}\,\|\nabla\boldsymbol{p}\|\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^4\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2 \le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\,\|\boldsymbol{h}\|_{\mathcal{V}}^4,\end{aligned}$$

(3.47)
$$\Big|\int_\Omega(\boldsymbol{p}\cdot\nabla)\overline{\boldsymbol{u}}\cdot\boldsymbol{p}\,dx\Big| \le \|\boldsymbol{p}\|_{L^4(\Omega)^2}\,\|\nabla\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times2}}\,\|\boldsymbol{p}\| \le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\,\|\boldsymbol{p}\|^2,$$

(3.48)
$$\begin{aligned}&\Big|\int_\Omega\big((\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\nabla\big)(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\boldsymbol{p}\,dx\Big| = \Big|\int_\Omega\big((\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\cdot\nabla\big)\boldsymbol{p}\cdot(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\,dx\Big|\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}\|_{L^4(\Omega)^2}^4 \le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}}\|^2\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\|^2\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})\|^2\,\|\boldsymbol{h}\|_{\mathcal{V}}^2,\end{aligned}$$

(3.49)
$$\begin{aligned}&\int_\Omega a\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})\cdot\boldsymbol{p}\,dx = -\int_\Omega\frac{(\varphi^{\boldsymbol{h}}-\overline{\varphi})^2}{2}\,\nabla a\cdot\boldsymbol{p}\,dx\\
&\quad\le C\,\|\boldsymbol{p}\|\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_{L^4(\Omega)}^2 \le \|\boldsymbol{p}\|^2 + C\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^4 \le \|\boldsymbol{p}\|^2 + C\,\|\boldsymbol{h}\|_{\mathcal{V}}^4,\end{aligned}$$

(3.50)
$$\begin{aligned}&-\int_\Omega\big(K*(\varphi^{\boldsymbol{h}}-\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})\cdot\boldsymbol{p}\,dx = \int_\Omega\big(\nabla K*(\varphi^{\boldsymbol{h}}-\overline{\varphi})\big)(\varphi^{\boldsymbol{h}}-\overline{\varphi})\cdot\boldsymbol{p}\,dx\\
&\quad\le C\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_{L^4(\Omega)}\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|\,\|\boldsymbol{p}\|_{L^4(\Omega)^2} \le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|^2\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^2\\
&\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 + C_\epsilon\,\|\boldsymbol{h}\|_{\mathcal{V}}^4,\end{aligned}$$

(3.51)
$$\int_\Omega (a\,q - K * q)\,\nabla\overline{\varphi}\cdot \boldsymbol{p}\,dx \,\le\, C\,\|q\|\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)}\,\|\boldsymbol{p}\|_{L^4(\Omega)^2} \,\le\, \epsilon\,\|\nabla \boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|q\|^2\,,$$

(3.52)
$$\int_\Omega (a\,\overline{\varphi} - K * \overline{\varphi})\,\nabla q\cdot \boldsymbol{p}\,dx \,\le\, C\,\|\overline{\varphi}\|_{H^2(\Omega)}\,\|\nabla q\|\,\|\boldsymbol{p}\| \,\le\, \epsilon'\,\|\nabla q\|^2 \,+\, C_{\epsilon'}\,\|\boldsymbol{p}\|^2\,,$$

(3.53)
$$\begin{aligned}&\int_\Omega \big(F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\cdot \boldsymbol{p}\,dx \,=\, \int_\Omega F''(\sigma_4^{\boldsymbol{h}})\,(\varphi^{\boldsymbol{h}} - \overline{\varphi})\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\cdot\boldsymbol{p}\,dx\\ &\quad\le\, C\,\|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{L^4(\Omega)}\,\|\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\|\,\|\boldsymbol{p}\|_{L^4(\Omega)^2} \,\le\, \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^4\\ &\quad\le\, \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|\boldsymbol{h}\|_{\mathcal{V}}^4\,,\end{aligned}$$

(3.54)
$$\int_\Omega F'(\overline{\varphi})\nabla q\cdot\boldsymbol{p}\,dx \,\le\, C\,\|\nabla q\|\,\|\boldsymbol{p}\| \,\le\, \epsilon'\,\|\nabla q\|^2 \,+\, C_{\epsilon'}\,\|\boldsymbol{p}\|^2\,,$$

(3.55)
$$\int_\Omega F''(\overline{\varphi})\,q\nabla\overline{\varphi}\cdot\boldsymbol{p}\,dx \,\le\, C\,\|q\|\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|\boldsymbol{p}\|_{L^4(\Omega)^2} \,\le\, \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|q\|^2\,,$$

(3.56)
$$\begin{aligned}&\frac{1}{2}\int_\Omega F'''(\sigma_2^{\boldsymbol{h}})\,(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2\,\nabla\overline{\varphi}\cdot\boldsymbol{p}\,dx \,\le\, C\,\|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{L^4(\Omega)}^2\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|\boldsymbol{p}\|_{L^4(\Omega)^2}\\ &\quad\le\, \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^4 \,\le\, \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|\boldsymbol{h}\|_{\mathcal{V}}^4\,.\end{aligned}$$

Observe that in the derivation of (3.48), (3.49), and (3.50), we have used (3.37) and the first boundary condition in (3.38), while in (3.44), (3.53), and (3.56), we have set $\sigma_j^{\boldsymbol{h}} := \theta_j^{\boldsymbol{h}}\varphi^{\boldsymbol{h}} + (1-\theta_j^{\boldsymbol{h}})\overline{\varphi}$, where $\theta_j^{\boldsymbol{h}} = \theta_j^{\boldsymbol{h}}(x,t) \in (0,1)$, for $j = 3, 4$.

Let us now estimate all the terms in (3.43). At first, we have

(3.57)
$$\begin{aligned}&\Big|\int_\Omega \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\cdot\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\,q\,dx\Big| \le \|\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}\|_{L^4(\Omega)^2}\,\|\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\|\,\|q\|_{L^4(\Omega)}\\ &\quad\le\, C\|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|\|\boldsymbol{h}\|_{\mathcal{V}}\big(\|\nabla q\| + \|q\|\big) \le\, \epsilon'\|\nabla q\|^2 + \|q\|^2 + C_{\epsilon'}\|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|^2\|\boldsymbol{h}\|_{\mathcal{V}}^2,\end{aligned}$$

(3.58)
$$\begin{aligned}&\Big|\int_\Omega (\boldsymbol{p}\cdot\nabla\overline{\varphi})\,q\,dx\Big| \,\le\, \|\boldsymbol{p}\|_{L^4(\Omega)^2}\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|q\| \,\le\, C\,\|\boldsymbol{p}\|_{L^4(\Omega)^2}\,\|\overline{\varphi}\|_{H^2(\Omega)}\,\|q\|\\ &\quad\le \epsilon\,\|\nabla\boldsymbol{p}\|^2 \,+\, C_\epsilon\,\|q\|^2\,.\end{aligned}$$

As far as the term on the right-hand side of (3.43) is concerned, we first observe that we can write

$$F'(\varphi^{\boldsymbol{h}}) - F'(\overline{\varphi}) - F''(\overline{\varphi})\eta^{\boldsymbol{h}} = (\varphi^{\boldsymbol{h}} - \overline{\varphi})\int_0^1 \big[F''(\tau\varphi^{\boldsymbol{h}} + (1-\tau)\overline{\varphi}) - F''(\overline{\varphi})\big]d\tau + F''(\overline{\varphi})q.$$

Therefore, we have

$$
\begin{aligned}
&\nabla\big(F'(\varphi^{\boldsymbol h})-F'(\overline{\varphi})-F''(\overline{\varphi})\eta^{\boldsymbol h}\big)=\nabla(\varphi^{\boldsymbol h}-\overline{\varphi})\int_0^1\big[F''(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})-F''(\overline{\varphi})\big]d\tau\\
&+(\varphi^{\boldsymbol h}-\overline{\varphi})\int_0^1\big[F'''(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})(\tau\nabla\varphi^{\boldsymbol h}+(1-\tau)\nabla\overline{\varphi})-F'''(\overline{\varphi})\nabla\overline{\varphi}\big]d\tau\\
&+\,F''(\overline{\varphi})\nabla q\,+\,q\,F'''(\overline{\varphi})\nabla\overline{\varphi}\\
&=\nabla(\varphi^{\boldsymbol h}-\overline{\varphi})\int_0^1\!\!\int_0^1F'''\big(s(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})+(1-s)\overline{\varphi}\big)(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi}-\overline{\varphi})ds\,d\tau\\
&+(\varphi^{\boldsymbol h}-\overline{\varphi})\int_0^1\Big[F'''(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})\tau\nabla(\varphi^{\boldsymbol h}-\overline{\varphi})\\
&+\nabla\overline{\varphi}\int_0^1F^{(4)}\big(s(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})+(1-s)\overline{\varphi}\big)(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi}-\overline{\varphi})\,ds\Big]d\tau\\
&+\,F''(\overline{\varphi})\nabla q\,+\,q\,F'''(\overline{\varphi})\nabla\overline{\varphi}\\
&=\overline{A}_{\boldsymbol h}\,(\varphi^{\boldsymbol h}-\overline{\varphi})\nabla(\varphi^{\boldsymbol h}-\overline{\varphi})+\overline{B}_{\boldsymbol h}\,(\varphi^{\boldsymbol h}-\overline{\varphi})^2\nabla\overline{\varphi}+F''(\overline{\varphi})\nabla q+qF'''(\overline{\varphi})\nabla\overline{\varphi},
\end{aligned}
\tag{3.59}
$$

where we have set

$$
\begin{aligned}
\overline{A}_{\boldsymbol h}&:=\int_0^1\tau\int_0^1F'''(s\tau\varphi^{\boldsymbol h}+(1-s\tau)\overline{\varphi})\,ds\,d\tau\,+\int_0^1\tau\,F'''(\tau\varphi^{\boldsymbol h}+(1-\tau)\overline{\varphi})\,d\tau\,,\\
\overline{B}_{\boldsymbol h}&:=\int_0^1\tau\int_0^1F^{(4)}(s\tau\varphi^{\boldsymbol h}+(1-s\tau)\overline{\varphi})\,ds\,d\tau\,.
\end{aligned}
$$

Observe that in view of the global bounds (3.33) we have

$$
\|\overline{A}_{\boldsymbol h}\|_{L^\infty(Q)}\,+\,\|\overline{B}_{\boldsymbol h}\|_{L^\infty(Q)}\,\le\,C_2^* \tag{3.60}
$$

with a constant $C_2^*>0$ that does not depend on the choice of $\boldsymbol h\in\mathcal V$ with $\|\boldsymbol h\|_{\mathcal V}\le\Lambda$.

Now, on account of (3.59), the expression on the right-hand side of (3.43) takes the form

$$
\begin{aligned}
&-\int_\Omega\nabla q\cdot\nabla\big(a\,q-K*q+F'(\varphi^{\boldsymbol h})-F'(\overline{\varphi})-F''(\overline{\varphi})\eta^{\boldsymbol h}\big)\,dx\\
&\quad=-\big(\nabla q,(a+F''(\overline{\varphi}))\nabla q\big)-\big(\nabla q,q\,F'''(\overline{\varphi})\nabla\overline{\varphi}\big)-\big(\nabla q,q\,\nabla a-\nabla K*q\big)\\
&\quad\ -\big(\nabla q,\overline{A}_{\boldsymbol h}\,(\varphi^{\boldsymbol h}-\overline{\varphi})\nabla(\varphi^{\boldsymbol h}-\overline{\varphi})\big)-\big(\nabla q,\overline{B}_{\boldsymbol h}\,(\varphi^{\boldsymbol h}-\overline{\varphi})^2\,\nabla\overline{\varphi}\big),
\end{aligned}
\tag{3.61}
$$

and the last four terms in (3.61) can be estimated in the following way:

$$
\begin{aligned}
\big|\big(\nabla q,qF'''(\overline{\varphi})\nabla\overline{\varphi}\big)\big|&\le\,C\,\|\nabla q\|\,\|q\|_{L^4(\Omega)}\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\\
&\le\,C\,\|\nabla q\|\,\big(\|q\|+\|q\|^{1/2}\,\|\nabla q\|^{1/2}\big)\,\le\,\epsilon'\,\|\nabla q\|^2\,+\,C_{\epsilon'}\,\|q\|^2\,,
\end{aligned}
\tag{3.62}
$$

$$\big|\big(\nabla q, q\,\nabla a - \nabla K * q\big)\big| \,\le C\,\|\nabla q\|\,\|q\| \,\le\, \epsilon'\,\|\nabla q\|^2 \,+\, C_{\epsilon'}\,\|q\|^2\,, \tag{3.63}$$

(3.64)

$$\begin{aligned}\big|\big(\nabla q, \overline{A}_{\boldsymbol{h}}\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})\big)\big| \,&\le\, C\,\|\nabla(\varphi^{\boldsymbol{h}}-\overline{\varphi})\|_{L^4(\Omega)^2}\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_{L^4(\Omega)}\,\|\nabla q\| \\ &\le \epsilon'\|\nabla q\|^2 + C_{\epsilon'}\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^2\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|^2_{H^2(\Omega)} \le \epsilon'\|\nabla q\|^2 + C_{\epsilon'}\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|^2_{H^2(\Omega)}\|\boldsymbol{h}\|^2_{\mathcal{V}},\end{aligned}$$

$$\begin{aligned}\big|\big(\nabla q, \overline{B}_{\boldsymbol{h}}\,(\varphi^{\boldsymbol{h}}-\overline{\varphi})^2\,\nabla\overline{\varphi}\big)\big| \,&\le\, C\,\|\nabla q\|\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|^2_{L^8(\Omega)}\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2} \\ &\le\, \epsilon'\,\|\nabla q\|^2 \,+\, C_{\epsilon'}\,\|\varphi^{\boldsymbol{h}}-\overline{\varphi}\|_V^4 \,\le\, \epsilon'\,\|\nabla q\|^2 \,+\, C_{\epsilon'}\,\|\boldsymbol{h}\|^4_{\mathcal{V}}\,.\end{aligned} \tag{3.65}$$

We now insert the estimates (3.44)–(3.56) in (3.42) and the estimates (3.57), (3.58) and (3.62)–(3.65) in (3.43) and recall (3.61) and the conditions (2.3) and (2.6). Adding the resulting inequalities, and fixing $\epsilon > 0$ and $\epsilon' > 0$ small enough (i.e., $\epsilon \le \hat{\nu}_1/22$ and $\epsilon' \le \hat{c}_1/16$), we obtain that almost everywhere in $(0,T)$ we have the inequality

$$\frac{d}{dt}\big(\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \|q^{\boldsymbol{h}}\|^2\big) + \hat{\nu}_1\,\|\nabla \boldsymbol{p}^{\boldsymbol{h}}\|^2 + \hat{c}_1\,\|\nabla q^{\boldsymbol{h}}\|^2 \le\, \alpha\,\big(\|\boldsymbol{p}^{\boldsymbol{h}}\|^2 + \|q^{\boldsymbol{h}}\|^2\big) \,+\, \beta_{\boldsymbol{h}}, \tag{3.66}$$

where the functions $\alpha, \beta_{\boldsymbol{h}} \in L^1(0,T)$ are given by

$$\alpha(t) := C\,\big(1 + \|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}\big),$$

$$\begin{aligned}\beta_{\boldsymbol{h}}(t) := C\|\boldsymbol{h}\|^4_{\mathcal{V}}\big(1 + \|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}\big) + C\|\boldsymbol{h}\|^2_{\mathcal{V}}\Big(&\|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})(t)\|^2 + \|(\varphi^{\boldsymbol{h}}-\overline{\varphi})(t)\|^2_{H^2(\Omega)} \\ &+\, \|\nabla(\boldsymbol{u}^{\boldsymbol{h}}-\overline{\boldsymbol{u}})(t)\|\,\big(\|\boldsymbol{u}^{\boldsymbol{h}}(t)\|_{H^2(\Omega)^2} + \|\overline{\boldsymbol{u}}(t)\|_{H^2(\Omega)^2}\big)\Big)\,.\end{aligned}$$

Now, since $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$, it follows from the global bounds (3.33) and (3.34) that

$$\int_0^T \beta_{\boldsymbol{h}}(t)\,dt \,\le\, C\,\|\boldsymbol{h}\|^3_{\mathcal{V}}.$$

Taking (3.39) into account, we therefore can infer from Gronwall's lemma that

$$\|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{C^0([0,T];G_{div})} \,+\, \|\boldsymbol{p}^{\boldsymbol{h}}\|^2_{L^2(0,T;V_{div})} \,+\, \|q^{\boldsymbol{h}}\|^2_{C^0([0,T];H)} \,+\, \|q^{\boldsymbol{h}}\|^2_{L^2(0,T;V)} \,\le\, C\,\|\boldsymbol{h}\|^3_{\mathcal{V}}.$$

Hence, it holds that

$$\frac{\|S(\overline{\boldsymbol{v}}+\boldsymbol{h}) - S(\overline{\boldsymbol{v}}) - [\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \,=\, \frac{\|[\boldsymbol{p}^{\boldsymbol{h}},q^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \,\le\, C\,\|\boldsymbol{h}\|^{1/2}_{\mathcal{V}} \to 0,$$

as $\|\boldsymbol{h}\|_{\mathcal{V}} \to 0$, which concludes the proof of the theorem. ∎

First-order necessary optimality conditions. From Theorem 3.4 we can deduce the following necessary optimality condition:

COROLLARY 3.5. *Suppose that the general hypotheses* (H1)–(H4) *are fulfilled, and assume that* $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ *is an optimal control for* (CP) *with associated state* $[\overline{\boldsymbol{u}},\overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$. *Then it holds that*

(3.67)
$$\beta_1 \int_0^T\!\!\int_\Omega (\overline{\boldsymbol{u}}-\boldsymbol{u}_Q)\cdot \boldsymbol{\xi}^{\boldsymbol{h}} dxdt+\beta_2 \int_0^T\!\!\int_\Omega (\overline{\varphi}-\varphi_Q)\,\eta^{\boldsymbol{h}} dx\,dt+\beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T)-\boldsymbol{u}_\Omega)\cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T)\,dx$$
$$+\ \beta_4 \int_\Omega (\overline{\varphi}(T)-\varphi_\Omega)\,\eta^{\boldsymbol{h}}(T)\,dx\ +\ \gamma \int_0^T\!\!\int_\Omega \overline{\boldsymbol{v}}\cdot(\boldsymbol{v}-\overline{\boldsymbol{v}})\,dx\,dt\ \geq\ 0 \qquad \forall\, \boldsymbol{v}\in\mathcal{V}_{ad},$$

where $[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]$ *is the unique solution to the linearized system* (3.9)–(3.13) *corresponding to* $\boldsymbol{h}=\boldsymbol{v}-\overline{\boldsymbol{v}}$.

Proof. Introducing the reduced cost functional $f:\mathcal{V}\to[0,\infty)$ given by $f(\boldsymbol{v}):=\mathcal{J}\big(\mathcal{S}(\boldsymbol{v}),\boldsymbol{v}\big)$, for all $\boldsymbol{v}\in\mathcal{V}$, where $\mathcal{J}:\mathcal{Z}\times\mathcal{V}\to[0,\infty)$ is given by (1.7), and invoking the convexity of $\mathcal{V}_{ad}$, we have (see, e.g., [47, Lemma 2.21])

$$(f'(\overline{\boldsymbol{v}}),\boldsymbol{v}-\overline{\boldsymbol{v}})\geq 0 \qquad \forall\,\boldsymbol{v}\in\mathcal{V}_{ad}. \tag{3.68}$$

Obviously, by the chain rule,

$$f'(\boldsymbol{v})=\mathcal{J}'_y\big(\mathcal{S}(\boldsymbol{v}),\boldsymbol{v}\big)\circ\mathcal{S}'(\boldsymbol{v})+\mathcal{J}'_{\boldsymbol{v}}\big(\mathcal{S}(\boldsymbol{v}),\boldsymbol{v}\big), \tag{3.69}$$

where, for every fixed $\boldsymbol{v}\in\mathcal{V}$, $\mathcal{J}'_y\big(y,\boldsymbol{v}\big)\in\mathcal{Z}'$ is the Fréchet derivative of $\mathcal{J}=\mathcal{J}(y,\boldsymbol{v})$ with respect to y at $y\in\mathcal{Z}$ and, for every fixed $y\in\mathcal{Z}$, $\mathcal{J}'_{\boldsymbol{v}}\big(y,\boldsymbol{v}\big)\in\mathcal{V}'$ is the Fréchet derivative of $\mathcal{J}=\mathcal{J}(y,\boldsymbol{v})$ with respect to $\boldsymbol{v}$ at $\boldsymbol{v}\in\mathcal{V}$. We have

(3.70)
$$\mathcal{J}'_y\big(y,\boldsymbol{v}\big)(\zeta)=\beta_1\int_0^T\!\!\int_\Omega(\boldsymbol{u}-\boldsymbol{u}_Q)\cdot\boldsymbol{\zeta}_1\,dx\,dt\ +\beta_2\int_0^T\!\!\int_\Omega(\varphi-\varphi_Q)\,\zeta_2\,dx\,dt$$
$$+\beta_3\int_\Omega(\boldsymbol{u}(T)-\boldsymbol{u}_\Omega)\cdot\boldsymbol{\zeta}_1(T)\,dx\ +\beta_4\int_\Omega(\varphi(T)-\varphi_\Omega)\,\zeta_2(T)\,dx \quad \forall\,\zeta=[\boldsymbol{\zeta}_1,\zeta_2]\in\mathcal{Z},$$

where $y=[\boldsymbol{u},\varphi]$. Moreover,

$$\mathcal{J}'_{\boldsymbol{v}}\big(y,\boldsymbol{v}\big)(\boldsymbol{w})=\gamma\int_0^T\!\!\int_\Omega\boldsymbol{v}\cdot\boldsymbol{w}\,dx\,dt \qquad \forall\,\boldsymbol{w}\in\mathcal{V}. \tag{3.71}$$

Hence, (3.67) follows from (3.68)–(3.71) on account of the fact that, thanks to Theorem 3.4, we have

$$\mathcal{S}'(\overline{\boldsymbol{v}})(\boldsymbol{v}-\overline{\boldsymbol{v}})=[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}],$$

where $[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]$ is the unique solution to the linearized system (3.9)–(3.13) corresponding to $\boldsymbol{h}=\boldsymbol{v}-\overline{\boldsymbol{v}}$. ∎

The adjoint system and first-order necessary optimality conditions. We now aim to eliminate the variables $[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]$ from the variational inequality (3.67). To this end, let us introduce the following *adjoint system*:

$$\widetilde{\boldsymbol{p}}_t\ =\ -\ 2\,\mathrm{div}\big(\nu(\overline{\varphi})\,D\widetilde{\boldsymbol{p}}\big)-(\overline{\boldsymbol{u}}\cdot\nabla)\,\widetilde{\boldsymbol{p}}+(\widetilde{\boldsymbol{p}}\cdot\nabla^T)\,\overline{\boldsymbol{u}}\ +\ \widetilde{q}\,\nabla\overline{\varphi}-\beta_1(\overline{\boldsymbol{u}}-\boldsymbol{u}_Q)\,, \tag{3.72}$$
$$\widetilde{q}_t\ =\ -\,(a\,\Delta\widetilde{q}\ +\ \nabla K\overset{*}{\cdot}\nabla\widetilde{q}\ +\ F''(\overline{\varphi})\,\Delta\widetilde{q})-\overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}\ +\ 2\,\nu'(\overline{\varphi})\,D\overline{\boldsymbol{u}}:D\widetilde{\boldsymbol{p}}$$
$$-\big(a\,\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}-K*(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi})+F''(\overline{\varphi})\,\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}\big)+\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu}-\beta_2(\overline{\varphi}-\varphi_Q)\,, \tag{3.73}$$
$$\mathrm{div}(\widetilde{\boldsymbol{p}})=0, \tag{3.74}$$
$$\widetilde{\boldsymbol{p}}=0, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}}=0 \quad \text{on } \Sigma, \tag{3.75}$$

(3.76) $\qquad \widetilde{\boldsymbol{p}}(T) = \beta_3(\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega), \qquad \widetilde{q}(T) = \beta_4(\overline{\varphi}(T) - \varphi_\Omega).$

Here, we have set

$$(\nabla K \dot{*} \nabla \widetilde{q})(x) := \int_\Omega \nabla K(x-y) \cdot \nabla \widetilde{q}(y)\, dy \quad \text{for a.e. } x \in \Omega\,.$$

Since $\boldsymbol{u}_\Omega \in G_{div}$, $\varphi_\Omega \in H$, the solution to (3.72)–(3.76) can only be expected to enjoy the regularity

$$\begin{aligned}(3.77)\qquad &\widetilde{\boldsymbol{p}} \in H^1(0,T;(V_{div})') \cap C^0([0,T];G_{div}) \cap L^2(0,T;V_{div}),\\ &\widetilde{q} \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V).\end{aligned}$$

Hence, the pair $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$ must be understood as a solution to the following weak formulation of the system (3.72)–(3.75) (where the argument t is again omitted):

(3.78)
$$\langle \widetilde{\boldsymbol{p}}_t, \boldsymbol{z}\rangle_{V_{div}} - 2\big(\nu(\overline{\varphi}) D\widetilde{\boldsymbol{p}}, D\boldsymbol{z}\big) = -b(\overline{\boldsymbol{u}}, \widetilde{\boldsymbol{p}}, \boldsymbol{z}) + b(\boldsymbol{z}, \overline{\boldsymbol{u}}, \widetilde{\boldsymbol{p}}) + \big(\widetilde{q}\nabla\overline{\varphi}, \boldsymbol{z}\big) - \beta_1\big((\overline{\boldsymbol{u}} - \boldsymbol{u}_Q), \boldsymbol{z}\big),$$
(3.79)
$$\begin{aligned}\langle \widetilde{q}_t, \chi\rangle_V - \big((a + F''(\overline{\varphi}))\nabla\widetilde{q}, \nabla\chi\big) &= \big(\nabla a + F'''(\overline{\varphi})\nabla\overline{\varphi}, \chi\nabla\widetilde{q}\big) - \big(\nabla K \dot{*} \nabla\widetilde{q}, \chi\big) - \big(\overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}, \chi\big)\\ &+ 2\left(\nu'(\overline{\varphi}) D\overline{\boldsymbol{u}} : D\widetilde{\boldsymbol{p}}, \chi\right) - \big((a\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi} - K * (\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}) + F''(\overline{\varphi})\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\varphi}), \chi\big)\\ &+ \big(\widetilde{\boldsymbol{p}}\cdot\nabla\overline{\mu}, \chi\big) - \beta_2\big((\overline{\varphi} - \varphi_Q), \chi\big)\end{aligned}$$

for every $\boldsymbol{z} \in V_{div}$, every $\chi \in V$, and almost every $t \in (0,T)$. We have the following result.

PROPOSITION 3.6. *Suppose that the hypotheses* (H1)–(H4) *are fulfilled. Then the adjoint system* (3.72)–(3.76) *has a unique weak solution* $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$ *satisfying* (3.77).

Proof. We only give a sketch of the proof, which can be carried out in a similar way as the proof of Proposition 3.2. In particular, we omit the implementation of the Faedo–Galerkin scheme and only derive the basic estimates that weak solutions must satisfy. To this end, we insert $\widetilde{\boldsymbol{p}}(t) \in V_{div}$ in (3.78) and $\widetilde{q}(t) \in H$ in (3.79) and add the resulting equations, observing that we have $b(\overline{u}(t), p(t), p(t)) = (\overline{u}(t) \cdot \nabla\widetilde{q}(t), \widetilde{q}(t)) = 0$. Omitting the argument t again, we now estimate the resulting terms on the right-hand side individually. We denote by C positive constants that only depend on the global data and on $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, while C_σ stands for positive constants that also depend on the quantity indicated by the index σ. Using the elementary Young's inequality, the Hölder and Gagliardo–Nirenberg inequalities, Young's inequality for convolution integrals, as well as the hypotheses (H1)–(H4) and the global bound (2.15), we obtain (with postive constants ϵ and ϵ' that will be fixed later) the following series of estimates:

$$\Big|\int_\Omega (\widetilde{\boldsymbol{p}}\cdot\nabla^T)\overline{\boldsymbol{u}}\cdot\widetilde{\boldsymbol{p}}\, dx\Big| \le \|\widetilde{\boldsymbol{p}}\|\, \|\nabla\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}} \|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \le \epsilon\, \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon\, \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\, \|\widetilde{\boldsymbol{p}}\|^2,$$

$$\Big|\int_\Omega \widetilde{q}\,\nabla\overline{\varphi}\cdot\widetilde{\boldsymbol{p}}\, dx\Big| \le \|\widetilde{q}\|\, \|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\, \|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \le \epsilon\, \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon\, \|\widetilde{q}\|^2,$$

$$\Big|\beta_1 \int_\Omega (\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_Q)\cdot\widetilde{\boldsymbol{p}}\, dx\Big| \le \beta_1\, \|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_Q\|\, \|\widetilde{\boldsymbol{p}}\| \le \|\widetilde{\boldsymbol{p}}\|^2 + \frac{\beta_1^2}{4}\, \|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}}_Q\|^2,$$

$$\Big|\int_\Omega \widetilde{q}\,\nabla\widetilde{q}\cdot\big(\nabla a + F'''(\overline{\varphi})\nabla\overline{\varphi}\big)\, dx\Big| \le C_K\, \|\widetilde{q}\|\, \|\nabla\widetilde{q}\| + C\, \|\widetilde{q}\|_{L^4(\Omega)}\, \|\nabla\widetilde{q}\|\, \|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}$$

$$\leq C_K \|\widetilde{q}\| \|\nabla\widetilde{q}\| + C\left(\|\widetilde{q}\| + \|\widetilde{q}\|^{1/2} \|\nabla\widetilde{q}\|^{1/2}\right) \|\nabla\widetilde{q}\| \|\overline{\varphi}\|_{H^2(\Omega)}$$
$$\leq \epsilon' \|\nabla\widetilde{q}\|^2 + C_{\epsilon'} \|\widetilde{q}\|^2,$$

$$\Big|\int_\Omega \left(\nabla K \ast \nabla\widetilde{q}\right) \widetilde{q}\, dx\Big| \leq C_K \|\nabla\widetilde{q}\| \|\widetilde{q}\| \leq \epsilon' \|\nabla\widetilde{q}\|^2 + C_{\epsilon'} \|\widetilde{q}\|^2,$$

$$\Big|2\int_\Omega \left(\nu'(\overline{\varphi})\, D\overline{\boldsymbol{u}} : D\widetilde{\boldsymbol{p}}\right) \widetilde{q}\, dx\Big| \leq C \|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}} \|D\widetilde{\boldsymbol{p}}\| \|\widetilde{q}\|_{L^4(\Omega)}$$
$$\leq C \|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}} \|D\widetilde{\boldsymbol{p}}\| \left(\|\widetilde{q}\| + \|\widetilde{q}\|^{1/2} \|\nabla\widetilde{q}\|^{1/2}\right)$$
$$\leq \epsilon \|\nabla\widetilde{\boldsymbol{p}}\|^2 + \epsilon' \|\nabla\widetilde{q}\|^2 + C_{\epsilon,\epsilon'} \left(1 + \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\right) \|\widetilde{q}\|^2,$$

$$\Big|\int_\Omega (a\, \widetilde{\boldsymbol{p}} \cdot \nabla\overline{\varphi})\, \widetilde{q}\, dx\Big| \leq C_K \|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \|\nabla\overline{\varphi}\|_{L^4(\Omega)^2} \|\widetilde{q}\| \leq C \|\nabla\widetilde{\boldsymbol{p}}\| \|\widetilde{q}\|$$
$$\leq \epsilon \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon \|\widetilde{q}\|^2,$$

$$\Big|\int_\Omega K \ast (\widetilde{\boldsymbol{p}} \cdot \nabla\overline{\varphi})\, \widetilde{q}\, dx\Big| \leq C_K \|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \|\nabla\overline{\varphi}\|_{L^4(\Omega)^2} \|\widetilde{q}\| \leq C \|\nabla\widetilde{\boldsymbol{p}}\| \|\widetilde{q}\|$$
$$\leq \epsilon \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon \|\widetilde{q}\|^2,$$

$$\Big|\int_\Omega F''(\overline{\varphi})(\widetilde{\boldsymbol{p}} \cdot \nabla\overline{\varphi})\, \widetilde{q}\, dx\Big| \leq C \|\nabla\widetilde{\boldsymbol{p}}\| \|\widetilde{q}\| \leq \epsilon \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon \|\widetilde{q}\|^2,$$

$$\Big|\int_\Omega (\widetilde{\boldsymbol{p}} \cdot \nabla\overline{\mu})\widetilde{q} dx\Big| \leq \|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \|\nabla\overline{\mu}\|_{L^4(\Omega)^2} \|\widetilde{q}\| \leq C \|\overline{\mu}\|_{H^2(\Omega)} \|\nabla\widetilde{\boldsymbol{p}}\| \|\widetilde{q}\| \leq \epsilon \|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon \|\widetilde{q}\|^2,$$

$$\Big|\beta_2 \int_\Omega (\overline{\varphi} - \varphi_Q)\, \widetilde{q}\, dx\Big| \leq \beta_2 \|\overline{\varphi} - \varphi_Q\| \|\widetilde{q}\| \leq \|\widetilde{q}\|^2 + \frac{\beta_2^2}{4} \|\overline{\varphi} - \varphi_Q\|^2 .$$

Fixing now $\epsilon > 0$ and $\epsilon' > 0$ small enough (in particular, $7\epsilon \leq \hat{\nu}_1/2$ and $3\epsilon' \leq \hat{c}_1/2$), and using (2.3) and (2.6), we arrive at the following differential inequality:

$$\frac{d}{dt}\left(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2\right) + \sigma\left(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2\right) + \theta \geq \hat{\nu}_1 \|\nabla\widetilde{\boldsymbol{p}}\|^2 + \hat{c}_1 \|\nabla\widetilde{q}\|^2, \tag{3.80}$$

where the functions $\sigma, \theta \in L^1(0,T)$ are given by

$$\sigma(t) := C\left(1 + \|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}\right), \quad \theta(t) := C\left(\beta_1^2 \|(\overline{\boldsymbol{u}} - \boldsymbol{u}_Q)(t)\|^2 + \beta_2^2 \|(\overline{\varphi} - \varphi_Q)(t)\|^2\right).$$

By applying the (backward) Gronwall lemma to (3.80), we obtain

$$\|\widetilde{\boldsymbol{p}}(t)\|^2 + \|\widetilde{q}(t)\|^2 \leq \Big[\|\widetilde{\boldsymbol{p}}(T)\|^2 + \|\widetilde{q}(T)\|^2 + \int_t^T \theta(\tau)d\tau\Big] e^{\int_t^T \sigma(\tau)d\tau}$$
$$\leq C\Big[\|\widetilde{\boldsymbol{p}}(T)\|^2 + \|\widetilde{q}(T)\|^2 + \beta_1^2 \|\overline{\boldsymbol{u}} - \boldsymbol{u}_Q\|^2_{L^2(0,T;G_{div})} + \beta_2^2 \|\overline{\varphi} - \varphi_Q\|^2_{L^2(Q)}\Big]$$

for all $t \in [0,T]$. From this estimate, and by integrating (3.80) over $[t,T]$, we can deduce the estimates for $\widetilde{\boldsymbol{p}}$ and $\widetilde{q}$ in $C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})$ and in $C^0([0,T];H) \cap L^2(0,T;V)$, respectively. By a comparison argument in (3.72), (3.73), we also obtain the estimates for $\widetilde{\boldsymbol{p}}_t$ and $\widetilde{q}_t$ in $L^2(0,T;V'_{div})$ and in $L^2(0,T;V')$, respectively. Therefore, we deduce the existence of a weak solution to system (3.72)–(3.76) satisfying

(3.77). The proof of uniqueness is rather straightforward, and we therefore may omit the details here. ∎

Using the adjoint system, we can now eliminate $\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}$ from (3.67). Indeed, we have the following result.

THEOREM 3.7. *Suppose that the hypotheses* (H1)–(H4) *are fulfilled. Let* $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ *be an optimal control for the control problem* (CP) *with associated state* $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$ *and adjoint state* $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$. *Then the following variational inequality holds:*

$$\gamma \int_0^T\!\!\int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}})\, dx\, dt + \int_0^T\!\!\int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}})\, dx\, dt \,\geq\, 0 \quad \forall\, \boldsymbol{v} \in \mathcal{V}_{ad}. \tag{3.81}$$

Proof. Note that, thanks to (3.76), we have for the sum (that we denote by $\mathcal{I}$) of the first four terms on the left-hand side of (3.67)

$$\begin{aligned}
\mathcal{I} :=\, & \beta_1 \int_0^T\!\!\int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}} dxdt + \beta_2 \int_0^T\!\!\int_\Omega (\overline{\varphi} - \varphi_Q)\eta^{\boldsymbol{h}} dx\, dt + \beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T) dx \\
& + \beta_4 \int_\Omega (\overline{\varphi}(T) - \varphi_\Omega)\eta^{\boldsymbol{h}}(T) dx = \beta_1 \int_0^T\!\!\int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}} dxdt + \beta_2 \int_0^T\!\!\int_\Omega (\overline{\varphi} - \varphi_Q)\eta^{\boldsymbol{h}} dxdt \\
& + \int_0^T \big(\langle \widetilde{\boldsymbol{p}}_t(t), \boldsymbol{\xi}^{\boldsymbol{h}}(t) \rangle_{V_{div}} + \langle \boldsymbol{\xi}^{\boldsymbol{h}}_t(t), \widetilde{\boldsymbol{p}}(t) \rangle_{V_{div}} \big) dt + \int_0^T \big(\langle \widetilde{q}_t(t), \eta^{\boldsymbol{h}}(t) \rangle_V + \langle \eta^{\boldsymbol{h}}_t(t), \widetilde{q}(t) \rangle_V \big) dt.
\end{aligned} \tag{3.82}$$

Now, recalling the weak formulation of the linearized system (3.9)–(3.13) for $\boldsymbol{h} = \boldsymbol{v} - \overline{\boldsymbol{v}}$, we obtain, omitting the argument t,

$$\begin{aligned}
\langle \boldsymbol{\xi}^{\boldsymbol{h}}_t, \widetilde{\boldsymbol{p}} \rangle_{V_{div}} \,=\, & -2 \left(\nu(\overline{\varphi})\, D\boldsymbol{\xi}^{\boldsymbol{h}}, D\widetilde{\boldsymbol{p}} \right) - 2 \left(\nu'(\overline{\varphi})\, \eta^{\boldsymbol{h}}\, D\overline{\boldsymbol{u}}, D\widetilde{\boldsymbol{p}} \right) - b(\overline{\boldsymbol{u}}, \boldsymbol{\xi}^{\boldsymbol{h}}, \widetilde{\boldsymbol{p}}) \\
& - b(\boldsymbol{\xi}^{\boldsymbol{h}}, \overline{\boldsymbol{u}}, \widetilde{\boldsymbol{p}}) + \left((a\, \eta^{\boldsymbol{h}} - K * \eta^{\boldsymbol{h}} + F''(\overline{\varphi})\, \eta^{\boldsymbol{h}})\, \nabla\overline{\varphi}, \widetilde{\boldsymbol{p}} \right) \\
& + \left(\overline{\mu}\, \nabla \eta^{\boldsymbol{h}}, \widetilde{\boldsymbol{p}} \right) + (\boldsymbol{v} - \overline{\boldsymbol{v}}, \widetilde{\boldsymbol{p}})\,,
\end{aligned} \tag{3.83}$$

$$\begin{aligned}
\langle \eta^{\boldsymbol{h}}_t, \widetilde{q} \rangle_V \,=\, & - \left(\nabla \big(a\, \eta^{\boldsymbol{h}} - K * \eta^{\boldsymbol{h}} + F''(\overline{\varphi})\, \eta^{\boldsymbol{h}} \big), \nabla \widetilde{q} \right) + (\overline{\boldsymbol{u}}\, \eta^{\boldsymbol{h}}, \nabla \widetilde{q}) \\
& + (\boldsymbol{\xi}^{\boldsymbol{h}}\, \overline{\varphi}, \nabla \widetilde{q})\,.
\end{aligned} \tag{3.84}$$

Now, we insert these two equalities as well as (3.78) and (3.79) into (3.82). Integration by parts, using the boundary conditions for the involved quantities and the fact that $\boldsymbol{\xi}^{\boldsymbol{h}}$ and $\widetilde{\boldsymbol{p}}$ are divergence-free vector fields, and observing that the symmetry of the kernel K implies the identity

$$\int_\Omega (K * \eta)\, \omega\, dx \,=\, \int_\Omega (K * \omega)\, \eta\, dx \quad \forall\, \eta, \omega \in H,$$

we arrive after a straightforward standard calculation (which can be omitted here) at the conclusion that $\mathcal{I}$ can be rewritten as

$$\mathcal{I} := \int_0^T\!\!\int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}})\, dx\, dt\,.$$

Therefore, (3.81) follows from this identity and (3.67). ∎

Remark 3.8. The system (1.3)–(1.6), written for $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, the adjoint system (3.72)–(3.76), and the variational inequality (3.81) form together the first-order necessary optimality conditions. Moreover, since $\mathcal{V}_{ad}$ is a nonempty, closed, and convex subset of $L^2(Q)^2$, then (3.81) is in the case $\gamma > 0$ equivalent to the following condition for the optimal control $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$:

$$\overline{\boldsymbol{v}} = \mathbb{P}_{\mathcal{V}_{ad}}\Big(- \frac{\widetilde{\boldsymbol{p}}}{\gamma}\Big),$$

where $\mathbb{P}_{\mathcal{V}_{ad}}$ is the orthogonal projector in $L^2(Q)^2$ onto $\mathcal{V}_{ad}$. From standard arguments it follows from this projection property the pointwise condition

$$\overline{v}_i(x,t) = \max\left\{v_{a,i}(x,t), \min\left\{-\gamma^{-1}\widetilde{p}_i(x,t),\, v_{b,i}(x,t)\right\}\right\}, i = 1, 2, \quad \text{for a. e. } (x,t) \in Q,$$

where $\widetilde{p}_i = \widetilde{\boldsymbol{p}}_i$, $i = 1, 2$.

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