J. Math. Anal. Appl. 413 (2014) 169–184

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Conditionally evenly convex sets and evenly quasi-convex maps

Marco Frittelli, Marco Maggis *

Dipartimento di Matematica, Università degli Studi di Milano, Italy

ARTICLE INFO

Article history:
Received 22 April 2013
Available online 25 November 2013
Submitted by M. Quincampoix

Keywords:
Evenly convex set
Separation theorem
Bipolar theorem
L^0-modules
Nonlinear conditional expectation
Quasi-convex risk measures

ABSTRACT

Evenly convex sets in a topological vector space are defined as the intersection of a family of open half spaces. We introduce a generalization of this concept in the conditional framework and provide a generalized version of the bipolar theorem. This notion is then applied to obtain the dual representation of conditionally evenly quasi-convex maps, which turns out to be a fundamental ingredient in the study of quasi-convex dynamic risk measures.

1. Introduction

A subset C of a topological vector space is *evenly convex* if it is the intersection of a family of open half spaces, or equivalently, if every $x \notin C$ can be openly separated from C by a continuous linear functional. Obviously an evenly convex set is necessarily convex. This idea was firstly introduced by Fenchel [8] aimed to determine the largest family of convex sets C for which the polarity $C = C^{\circ\circ}$ holds true. More recent studies in this area led to a detailed analysis of evenly convex sets and evenly convex functions for the application in quasi-convex programming. Contributions to this branch of recent literature can be found in Daniilidis and Martínez-Legaz [5], Klee et al. [17], Martínez-Legaz and Vicente-Pérez [18] and Rodríguez and Vicente-Pérez [21].

It is well known that in the framework of incomplete financial markets the bipolar theorem is a key ingredient when we represent the super replication price of a contingent claim in terms of the class of martingale measures. Recently evenly convex sets and in particular evenly quasi-concave real valued functions have been considered by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio in the context of decision theory [3] and risk measures [4]. Evenly quasi-concavity is the weakest notion that enables, in the static setting, a *complete* quasi-convex duality: the idea is to prove a *one to one* relationship between quasi-convex monotone functionals ρ and the function R in the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. In decision theory the function R can be interpreted as the decision maker's index of uncertainty aversion: the uniqueness of R becomes crucial (see [3] and [6]) if we want to guarantee a robust dual representation of ρ characterized in terms of the unique R. The results in the present paper are meant to determine the mathematical background to deduce a dynamic version of this complete duality and are applied in [14].

In a conditional framework, as for example when $\mathcal{F}$ is a sigma algebra containing the sigma algebra $\mathcal{G}$ and we deal with $\mathcal{G}$-conditional expectation, $\mathcal{G}$-conditional sublinear expectation, $\mathcal{G}$-conditional risk measure, the analysis of the duality

* Corresponding author.
E-mail addresses: marco.frittelli@unimi.it (M. Frittelli), marco.maggis@unimi.it (M. Maggis).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.11.044

theory is more delicate. We may consider conditional maps $\rho : E \to L^0(\Omega, \mathcal{G}, \mathbb{P})$ defined either on vector spaces (i.e. $E = L^p(\Omega, \mathcal{F}, \mathbb{P})$) or on L^0-modules (i.e. $E = L^p_{\mathcal{G}}(\mathcal{F}) := \{yx \mid y \in L^0(\Omega, \mathcal{G}, \mathbb{P}) \text{ and } x \in L^p(\Omega, \mathcal{F}, \mathbb{P})\}$).

As described in detail by Filipovic, Kupper and Vogelpoth [9,10] and by Guo [16] the L^0-modules approach (see also Section 3 for more details) is a very powerful tool for the analysis of conditional maps and their dual representation.

In this paper we show that in order to achieve a conditional version of the representation of evenly quasi-convex maps a good notion of evenly convexity is crucial. We introduce the concept of a *conditionally evenly convex set*, which is tailor made for the conditional setting, in a framework that exceeds the module setting alone, so that will be applicable in many different context. We emphasize that, differently from the static case where the main tool is functional analysis, in the conditional setting this study involves substantial techniques from conditional probability.

In Section 2 we provide the characterization of evenly convexity (Theorem 1 and Proposition 9) and state the conditional version of the bipolar theorem (Theorem 2). Under additional topological assumptions, we show that conditionally convex sets that are closed or open are conditionally evenly convex (see Section 4, Proposition 4). As a consequence, the conditional evenly quasi-convexity of a function, i.e. the property that the conditional lower level sets are evenly convex, is a weaker assumption than quasi-convexity and lower (or upper) semicontinuity.

In Section 3 we apply the notion of conditionally evenly convex set to the *dual representation of evenly quasi-convex maps*, i.e. conditional maps $\rho : E \to L^0(\Omega, \mathcal{G}, \mathbb{P})$ with the property that the conditional lower level sets are evenly convex. Let $\bar{L}^0(\mathcal{G})$ be the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. We prove in Theorem 3 that an evenly quasi-convex regular map $\pi : E \to \bar{L}^0(\mathcal{G})$ can be represented as

$$\pi(X) = \sup_{\mu \in \mathcal{L}(E, L^0(\mathcal{G}))} \mathcal{R}\big(\mu(X), \mu\big), \tag{1}$$

where

$$\mathcal{R}(Y, \mu) := \inf_{\xi \in E} \big\{\pi(\xi) \bigm| \mu(\xi) \geqslant Y\big\}, \quad Y \in L^0(\mathcal{G}),$$

E is a topological L^0-module and $\mathcal{L}(E, L^0(\mathcal{G}))$ is the module of continuous L^0-linear functionals over E.

The proof of this result is based on a version of the hyperplane separation theorem and not on some approximation or scalarization arguments, as it happened in the vector space setting (see [13]). By carefully analyzing the proof one may appreciate many similarities with the original demonstration in the static setting by Penot and Volle [19]. One key difference with [19], in addition to the conditional setting, is the continuity assumption needed to obtain the representation (1). We work, as in [3], with evenly quasi-convex functions, an assumption weaker than quasi-convexity and lower (or upper) semicontinuity.

1.1. Dynamic risk measures and the L^0-module approach

As explained in [13] the representation of the type (1) is a cornerstone in order to reach a robust representation of quasi-convex risk measures or acceptability indexes.

At the end of the Nineties in the seminal paper by Artzner, Delbaen, Eber and Heath [1], a rigorous axiomatic formalization of coherent risk measures was developed with a normative intent. The regulating agencies asked for computational methods to estimate the capital requirements, exceeding the unmistakable lacks showed by the extremely popular $V@R$. Risk measures are real valued functionals ρ defined on a space of random variables which encloses every possible financial position. The risk of a financial position was originally defined in [1] as the minimal amount of money that an institution will have to sum up to a position X in order to make it acceptable with respect to some *criterium* modeled by an acceptance set $\mathcal{A}$.

The class of coherent risk measures was later extended to the class of convex risk measures, independently introduced by Föllmer and Schied (2002, [11]) and Frittelli and Rosazza Gianin (2002, [15]). Since then, the interest on this subject enormously expanded and the vast literature can be found in [12] 3rd edition, as well as in Ruszczyinski and Shapiro [22], Pflug [20], Bot, Lorenz and Wanka [2].

One key axiom in the class of convex risk measures – the cash additivity property – was relaxed by El Karoui and Ravanelli (2009, [7]) in markets with stochastic discount factors; finally Cerreia-Vioglio et al. (2010, [4]) showed that quasi-convexity describes better than convexity the principle of diversification, whenever cash additivity does not hold true. Following this trajectory we may conclude that the largest class of feasible risk measure is the following.

Definition 1. Let E be any vector space of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the $\mathbb{P}$ almost sure partial order. A quasi-convex risk measure is a functional $\rho : E \to \mathbb{R}$ which satisfies

(i) *monotonicity*, i.e. $X_1 \leqslant X_2$ implies $\rho(X_1) \geqslant \rho(X_2)$ for every $X_1, X_2 \in E$,
(ii) *quasi-convexity*, i.e. $\rho(tX_1 + (1-t)X_2) \leqslant \max\{\rho(X_1), \rho(X_2)\}$ for all $t \in [0, 1]$.

In the dynamic description of risk, we have the following situation: let $0 \leqslant t \leqslant T$ and fix a non-empty convex set $C_T \in E \subset L^0(\mathcal{F})$ such that $C_T + L^0_+ \subseteq C_T$. The set C_T represents the future positions considered acceptable by the supervising agency. For all $m \in \mathbb{R}$ denote by $v_t(m,\omega)$ the price at time t of m euros at time T. The function $v_t(m,\cdot)$ will be in general $\mathcal{G}$-measurable as in the case of stochastic discount factor where $v_t(m,\omega) = D_t(\omega)m$. By adapting the definitions in the static framework of [4] we set:

$$\rho_{C_T,v_t}(X) := \inf_{Y \in L^0(\mathcal{G})} \left\{ v_t(Y) \;\middle|\; X + Y \in C_T \right\}. \tag{2}$$

Notice that the previous definition is well posed only if the sum $X + Y \in E$ for any $X \in E$ and any $Y \in L^0(\mathcal{G})$ and for this reason we need to introduce the more complex structure of module over the ring $L^0(\mathcal{G})$ (see Examples 1 and 8 for details). The variable $Y \in L^0(\mathcal{G})$ plays the role of the (random) minimal capital requirement that the agent will have to save at time t in order to recover possible losses related to X at time T. Under opportune hypothesis the map ρ_{C_T,v_t} defined in (2) is an evenly quasi-convex map. Further details can be found in [14] where the results of the present paper are applied to obtain a complete dual characterization of evenly quasi-convex conditional risk measure $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ via the quasi-convex representation

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} R\big(E_Q[-X|\mathcal{G}], Q\big) \tag{3}$$

where $\mathcal{P}^q = \{Q \ll \mathbb{P} \mid \frac{dQ}{d\mathbb{P}} \in \mathcal{L}(L^p_{\mathcal{G}}(\mathcal{F}), L^0(\mathcal{G}))\}$.

Notice that in this case the dual module $\mathcal{L}(L^p_{\mathcal{G}}(\mathcal{F}), L^0(\mathcal{G}))$ can be identified with $L^q_{\mathcal{G}}(\mathcal{F})$.

2. On conditionally evenly convex sets

The probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is fixed throughout this paper. Whenever we will discuss conditional properties we will always make reference – even without explicitly mentioning it in the notations – to conditioning with respect to the sigma algebra $\mathcal{G}$.

We denote by $L^0 =: L^0(\Omega, \mathcal{G}, \mathbb{P})$ the space of $\mathcal{G}$-measurable random variables that are $\mathbb{P}$-a.s. finite, whereas by $\bar{L}^0$ the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. We remind that all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s. As the expected value $E_{\mathbb{P}}[\cdot]$ is mostly computed w.r.t. the reference probability $\mathbb{P}$, we will often omit $\mathbb{P}$ in the notation. For any $A \in \mathcal{G}$ the element $\mathbf{1}_A \in L^0$ is the random variable a.s. equal to 1 on A and 0 elsewhere. In general since $(\Omega, \mathcal{G}, \mathbb{P})$ are fixed we will always omit them. We define $L^0_+ = \{X \in L^0 \mid X \geqslant 0\}$ and $L^0_{++} = \{X \in L^0 \mid X > 0\}$.

The essential ($\mathbb{P}$ almost surely) *supremum* $\text{ess.sup}_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum* (see [12, Appendix A.5] for reference).

Definition 2 *(Dual pair).* A dual pair $(E, E', \langle\cdot,\cdot\rangle)$ consists of:

1. $(E, +)$ (resp. $(E', +)$) is any structure such that the formal sum $x\mathbf{1}_A + y\mathbf{1}_{A^C}$ belongs to E (resp. $x'\mathbf{1}_A + y'\mathbf{1}_{A^C} \in E'$) for any $x, y \in E$ (resp. $x', y' \in E'$) and $A \in \mathcal{G}$ with $\mathbb{P}(A) > 0$ and there exists a null element $0 \in E$ (resp. $0 \in E'$) such that $x + 0 = x$ for all $x \in E$ (resp. $x' + 0 = x'$ for all $x' \in E'$).
2. A map $\langle\cdot,\cdot\rangle : E \times E' \to L^0$ such that

$$\begin{aligned}
&\big\langle x\mathbf{1}_A + y\mathbf{1}_{A^C}, x'\big\rangle = \big\langle x, x'\big\rangle\mathbf{1}_A + \big\langle y, x'\big\rangle\mathbf{1}_{A^C},\\
&\big\langle x, x'\mathbf{1}_A + y'\mathbf{1}_{A^C}\big\rangle = \big\langle x, x'\big\rangle\mathbf{1}_A + \big\langle x, y'\big\rangle\mathbf{1}_{A^C},\\
&\big\langle 0, x'\big\rangle = 0 \quad \text{and} \quad \langle x, 0\rangle = 0
\end{aligned}$$

for every $A \in \mathcal{G}$, $\mathbb{P}(A) > 0$ and $x, y \in E$, $x', y' \in E'$.

Clearly in many applications E will be a class of random variables (as vector lattices, or L^0-modules as in Examples 1 and 8) and E' is a selection of conditional maps, for example conditional expectations, sublinear conditional expectations, conditional risk measures.

We recall from [9] an important type of concatenation:

Definition 3 *(Countable concatenation hull).*

(CSet) A subset $\mathcal{C} \subset E$ has the countable concatenation property if for every countable partition $\{A_n\}_n \subseteq \mathcal{G}$ and for every countable collection of elements $\{x_n\}_n \subset \mathcal{C}$ we have $\sum_n \mathbf{1}_{A_n} x_n \in \mathcal{C}$.

Given $\mathcal{C} \subseteq E$, we denote by $\mathcal{C}^{cc}$ the countable concatenation hull of $\mathcal{C}$, namely the smallest set $\mathcal{C}^{cc} \supseteq \mathcal{C}$ which satisfies (CSet):

$$\mathcal{C}^{cc} = \left\{ \sum_n \mathbf{1}_{A_n} x_n \,\middle|\, x_n \in \mathcal{C},\ \{A_n\}_n \subseteq \mathcal{G} \text{ is a partition of } \Omega \right\}.$$

These definitions can be plainly adapted to subsets of E'.

The action of an element $\xi' = \sum_m \mathbf{1}_{B_m} x'_m \in (E')^{cc}$ over $\xi = \sum_n \mathbf{1}_{A_n} x_n \in E^{cc}$ is defined as

$$\langle \xi, \xi' \rangle = \left\langle \sum_n \mathbf{1}_{A_n} x_n, \sum_m \mathbf{1}_{B_m} x'_m \right\rangle = \sum_n \sum_m \langle x_n, x'_m \rangle \mathbf{1}_{A_n \cap B_m} \tag{4}$$

and does not depend on the representation of $\xi' \in (E')^{cc}$ and $\xi \in \mathcal{C}^{cc}$.

Example 1. Let $\mathcal{F}$ be a sigma algebra containing $\mathcal{G}$. Consider the vector space $E := L^p(\mathcal{F}) := L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $p \geqslant 1$. If we compute the countable concatenation hull of $L^p(\mathcal{F})$ we obtain exactly the L^0-module

$$L^p_{\mathcal{G}}(\mathcal{F}) := \left\{ yx \,\middle|\, y \in L^0(\mathcal{G}) \text{ and } x \in L^p(\mathcal{F}) \right\}$$

as introduced in [9] and [10] (see Example 8 for more details).

Similarly, the class of conditional expectations $\mathcal{E} = \{E[\cdot Z|\mathcal{G}] \mid Z \in L^q(\Omega, \mathcal{F}, \mathbb{P})\}$ and $\frac{1}{p} + \frac{1}{q} = 1$ can be identified with the space $L^q(\mathcal{F})$. Hence the countable concatenation hull $\mathcal{E}^{cc}$ will be exactly $L^q_{\mathcal{G}}(\mathcal{F})$, the dual L^0-module of $L^p_{\mathcal{G}}(\mathcal{F})$.

If E (or E') does not fulfill (CSet) we can always embed the theory in its concatenation hull and henceforth we make the following:

Assumption. In the sequel of this paper we always suppose that both E and E' satisfies (CSet).

We recall that a subset C of a locally convex topological vector space V is *evenly convex* if it is the intersection of a family of open half spaces, or equivalently, if every $x \notin C$ can be openly separated from C by a continuous real valued linear functional. As the intersection of an empty family of half spaces is the entire space V, the whole space V itself is evenly convex.

However, in order to introduce the concept of conditional evenly convex set (with respect to $\mathcal{G}$) we need to take care of the fact that the set C may present some components which degenerate to the entire E. Basically it might occur that for some $A \in \mathcal{G}$

$$C\mathbf{1}_A = E\mathbf{1}_A,$$

i.e., for each $x \in E$ there exists $\xi \in C$ such that $\xi \mathbf{1}_A = x\mathbf{1}_A$. In this case there are no chances of finding an $x \in E$ satisfying $\mathbf{1}_A C \cap \mathbf{1}_A\{x\} = \varnothing$ and consequently no conditional separation may occur. It is clear that the evenly convexity property of a set C is meaningful only on the set where C does not coincide with the entire E. Thus we need to determine the maximal $\mathcal{G}$-measurable set on which C reduces to E. To this end, we set the following notation that will be employed many times.

Notation 2. Fix a set $\mathcal{C} \subseteq E$. As the class $\mathcal{A}(\mathcal{C}) := \{A \in \mathcal{G} \mid \mathcal{C}\mathbf{1}_A = E\mathbf{1}_A\}$ is closed with respect to countable union, we denote with $A_{\mathcal{C}}$ the $\mathcal{G}$-measurable maximal element of the class $\mathcal{A}(\mathcal{C})$ and with $D_{\mathcal{C}}$ the ($\mathbb{P}$-a.s. unique) complement of $A_{\mathcal{C}}$ (see also Remark 10 in Section 6). Hence $\mathcal{C}\mathbf{1}_{A_{\mathcal{C}}} = E\mathbf{1}_{A_{\mathcal{C}}}$.

We now give the formal definition of conditionally evenly convex set in terms of intersections of hyperplanes in the same spirit of [8].

Definition 4. A set $\mathcal{C} \subseteq E$ is conditionally evenly convex if there exists $\mathcal{L} \subseteq E'$ (in general non-unique and empty if $\mathcal{C} = E$) such that

$$\mathcal{C} = \bigcap_{x' \in \mathcal{L}} \left\{ x \in E \,\middle|\, \langle x, x' \rangle < Y_{x'} \text{ on } D_{\mathcal{C}} \right\} \quad \text{for some } Y_{x'} \in L^0. \tag{5}$$

Remark 3. Notice that for any arbitrary $D \in \mathcal{G}$, $\mathcal{L} \subseteq E'$ the set

$$\mathcal{C} = \bigcap_{x' \in \mathcal{L}} \left\{ x \in E \,\middle|\, \langle x, x' \rangle < Y_{x'} \text{ on } D \right\} \quad \text{for some } Y_{x'} \in L^0$$

is evenly convex, even though in general $D_{\mathcal{C}} \subseteq D$.

Remark 4. We observe that since E satisfies (CSet) then automatically any conditionally evenly convex set satisfies (CSet). As a consequence there might exist a set $\mathcal{C}$ which fails to be conditionally evenly convex, since it does not satisfy (CSet), but $\mathcal{C}^{cc}$ is conditionally evenly convex. Consider for instance $E = L^1_{\mathcal{G}}(\mathcal{F})$, $E' = L^\infty_{\mathcal{G}}(\mathcal{F})$, endowed with the pairing $\langle x, x'\rangle = E[xx'|\mathcal{G}]$. Fix $x' \in L^\infty(\mathcal{F})$, $Y \in L^0(\mathcal{G})$ and the set

$$\mathcal{C} = \{x \in L^1(\mathcal{F}) \mid E[xx'|\mathcal{G}] < Y\}.$$

Clearly $\mathcal{C}$ is not conditionally evenly convex since $\mathcal{C} \subsetneq \mathcal{C}^{cc}$; on the other hand

$$\mathcal{C}^{cc} = \{x \in L^1_{\mathcal{G}}(\mathcal{F}) \mid E[xx'|\mathcal{G}] < Y\}$$

which is by definition evenly convex.

Remark 5. Recall that a set $\mathcal{C} \subseteq E$ is L^0-*convex* if $\Lambda x + (1-\Lambda)y \in \mathcal{C}$ for any $x, y \in \mathcal{C}$ and $\Lambda \in L^0$ with $0 \leqslant \Lambda \leqslant 1$.

Suppose that all the elements $x' \in E'$ satisfy:

$$\langle \Lambda x + (1-\Lambda)y, x'\rangle \leqslant \Lambda\langle x, x'\rangle + (1-\Lambda)\langle y, x'\rangle \quad \text{for all } x, y \in E,\ \Lambda \in L^0\text{: } 0 \leqslant \Lambda \leqslant 1.$$

If E is L^0-convex then every conditionally evenly convex set is also L^0-convex.

In order to separate one point $x \in E$ from a set $\mathcal{C} \subseteq E$ in a conditional way we need the following definition:

Definition 5. For $x \in E$ and a subset $\mathcal{C}$ of E, we say that x is *outside* $\mathcal{C}$ if $\mathbf{1}_A\{x\} \cap \mathbf{1}_A\mathcal{C} = \varnothing$ for every $A \in \mathcal{G}$ with $A \subseteq D_{\mathcal{C}}$ and $\mathbb{P}(A) > 0$.

This is of course a much stronger requirement than $x \notin \mathcal{C}$.

Definition 6. For $\mathcal{C} \subseteq E$ we define the polar and bipolar sets as follows

$$\begin{aligned}
\mathcal{C}^\circ &:= \{x' \in E' \mid \langle x, x'\rangle < 1 \text{ on } D_{\mathcal{C}} \text{ for all } x \in \mathcal{C}\},\\
\mathcal{C}^{\circ\circ} &:= \{x \in E \mid \langle x, x'\rangle < 1 \text{ on } D_{\mathcal{C}} \text{ for all } x' \in \mathcal{C}^\circ\}\\
&= \bigcap_{x' \in \mathcal{C}^\circ} \{x \in E \mid \langle x, x'\rangle < 1 \text{ on } D_{\mathcal{C}}\}.
\end{aligned}$$

We now state the main results of this note about the characterization of evenly convex sets and the bipolar theorem. Their proofs are postponed to Section 6.

Theorem 1. *Let $(E, E', \langle\cdot,\cdot\rangle)$ be a dual pairing introduced in Definition 2 and let $\mathcal{C} \subseteq E$. The following statements are equivalent*:

(1) *$\mathcal{C}$ is conditionally evenly convex.*
(2) *$\mathcal{C}$ satisfies* (CSet) *and for every x* outside *$\mathcal{C}$ there exists $x' \in E'$ such that*

$$\langle \xi, x'\rangle < \langle x, x'\rangle \quad \text{on } D_{\mathcal{C}}\ \forall \xi \in \mathcal{C}.$$

Theorem 2 *(Bipolar theorem). Let $(E, E', \langle\cdot,\cdot\rangle)$ be a dual pairing introduced in Definition 2 and assume in addition that the pairing $\langle\cdot,\cdot\rangle$ is L^0-linear in the first component, i.e.*

$$\langle \alpha x + \beta y, x'\rangle = \alpha\langle x, x'\rangle + \beta\langle x, x'\rangle$$

for every $x' \in E'$, $x, y \in E$, $\alpha, \beta \in L^0$. For any $\mathcal{C} \subseteq E$ such that $0 \in \mathcal{C}$ we have:

(1) *$\mathcal{C}^\circ = \{x' \in E' \mid \langle x, x'\rangle < 1$ on $D_{\mathcal{C}}$ for all $x \in \mathcal{C}^{cc}\}$.*
(2) *The bipolar $\mathcal{C}^{\circ\circ}$ is a conditionally evenly convex set containing $\mathcal{C}$.*
(3) *The set $\mathcal{C}$ is conditionally evenly convex if and only if $\mathcal{C} = \mathcal{C}^{\circ\circ}$.*

Suppose that the set $\mathcal{C} \subseteq E$ is an L^0-cone, i.e. $\alpha x \in \mathcal{C}$ for every $x \in \mathcal{C}$ and $\alpha \in L^0_{++}$. In this case, it is immediate to verify that the polar and bipolar can be rewritten as:

$$\begin{aligned}
\mathcal{C}^\circ &= \{x' \in E' \mid \langle x, x'\rangle \leqslant 0 \text{ on } D_{\mathcal{C}} \text{ for all } x \in \mathcal{C}\},\\
\mathcal{C}^{\circ\circ} &= \{x \in E \mid \langle x, x'\rangle \leqslant 0 \text{ on } D_{\mathcal{C}} \text{ for all } x' \in \mathcal{C}^\circ\}.
\end{aligned} \tag{6}$$

3. On conditionally evenly quasi-convex maps

Here we state the dual representation of conditional evenly quasi-convex maps of the Penot–Volle type which extends the results obtained in [13] for topological vector spaces. We work in the general setting outlined in Section 2. The additional basic property that is needed is regularity.

Definition 7. A map $\pi : E \to \bar{L}^0$ is

(REG) regular if for every $x_1, x_2 \in E$ and $A \in \mathcal{G}$,

$$\pi(x_1\mathbf{1}_A + x_2\mathbf{1}_{A^C}) = \pi(x_1)\mathbf{1}_A + \pi(x_2)\mathbf{1}_{A^C}.$$

Remark 6 *(On REG).* It is well known that (REG) is equivalent to:

$$\pi(x\mathbf{1}_A)\mathbf{1}_A = \pi(x)\mathbf{1}_A \quad \forall A \in \mathcal{G},\ \forall x \in E.$$

Under the countable concatenation property it is even true that (REG) is equivalent to countably regularity, i.e.

$$\pi\left(\sum_{i=1}^{\infty} x_i\mathbf{1}_{A_i}\right) = \sum_{i=1}^{\infty}\pi(x_i)\mathbf{1}_{A_i} \quad \text{on } \bigcup_{i=1}^{\infty} A_i$$

if $x_i \in E$ and $\{A_i\}_i$ is a sequence of disjoint $\mathcal{G}$-measurable sets. Indeed $x := \sum_{i=1}^{\infty} x_i\mathbf{1}_{A_i} \in E$ and $\sum_{i=1}^{\infty}\pi(x_i)\mathbf{1}_{A_i} \in \bar{L}^0$; (REG) then implies $\pi(x)\mathbf{1}_{A_i} = \pi(x\mathbf{1}_{A_i})\mathbf{1}_{A_i} = \pi(x_i\mathbf{1}_{A_i})\mathbf{1}_{A_i} = \pi(x_i)\mathbf{1}_{A_i}$.

Let $\pi : E \to \bar{L}^0$ be (REG). There might exist a set $A \in \mathcal{G}$ on which the map π is infinite, in the sense that $\pi(\xi)\mathbf{1}_A = +\infty\mathbf{1}_A$ for every $\xi \in E$. For this reason we introduce

$$\mathcal{M} := \{A \in \mathcal{G} \mid \pi(\xi)\mathbf{1}_A = +\infty\mathbf{1}_A\ \forall\xi \in E\}.$$

Applying Lemma 18 in Appendix A with $F := \{\pi(\xi) \mid \xi \in E\}$ and $Y_0 = +\infty$ we can deduce the existence of two maximal sets $T_\pi \in \mathcal{G}$ and $\Upsilon_\pi \in \mathcal{G}$ for which $P(T_\pi \cap \Upsilon_\pi) = 0$, $P(T_\pi \cup \Upsilon_\pi) = 1$ and

$$\begin{aligned} &\pi(\xi) = +\infty \quad \text{on } \Upsilon_\pi \text{ for every } \xi \in E, \\ &\pi(\zeta) < +\infty \quad \text{on } T_\pi \text{ for some } \zeta \in E. \end{aligned} \tag{7}$$

Definition 8. A map $\pi : E \to \bar{L}^0(\mathcal{G})$ is

(QCO) conditionally quasi-convex if $U_Y = \{\xi \in E \mid \pi(\xi)\mathbf{1}_{T_\pi} \leqslant Y\}$ are L^0-convex (according to Remark 5) for every $Y \in L^0(\mathcal{G})$;
(EQC) conditionally evenly quasi-convex if $U_Y = \{\xi \in E \mid \pi(\xi)\mathbf{1}_{T_\pi} \leqslant Y\}$ are conditionally evenly convex for every $Y \in L^0(\mathcal{G})$.

Remark 7. For $\pi : E \to \bar{L}^0(\mathcal{G})$ the quasi-convexity of π is equivalent to the condition

$$\pi(\Lambda x_1 + (1-\Lambda)x_2) \leqslant \pi(x_1) \vee \pi(x_2) \tag{8}$$

for every $x_1, x_2 \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leqslant \Lambda \leqslant 1$. In this case the sets $\{\xi \in E \mid \pi(\xi)1_D < Y\}$ are $L^0(\mathcal{G})$-convex for every $Y \in \bar{L}^0(\mathcal{G})$ and $D \in \mathcal{G}$ (this follows immediately from (8)).

Moreover under the further structural property of Remark 5 we have that (EQC) implies (QCO). We will see in the L^0-modules framework that if the map π is either lower semicontinuous or upper semicontinuous then the reverse implication holds true (see Proposition 4, Corollary 6 and Proposition 7).

We now state the main result of this section.

Theorem 3. *Let $(E, E', \langle\cdot,\cdot\rangle)$ be a dual pairing introduced in Definition 2. If $\pi : E \to \bar{L}^0(\mathcal{G})$ is* (REG) *and* (EQC) *then*

$$\pi(x) = \sup_{x' \in E'} \mathcal{R}(\langle x, x'\rangle, x'), \tag{9}$$

where for $Y \in L^0(\mathcal{G})$ and $x' \in E'$,

$$\mathcal{R}(Y, x') := \inf_{\xi \in E}\{\pi(\xi) \mid \langle\xi, x'\rangle \geqslant Y\}. \tag{10}$$

4. Conditional evenly convexity in L^0-modules

This section is inspired by the contribution given to the theory of L^0-modules by Filipovic et al. [9] on one hand and on the other to the extended research provided by Guo from 1992 until today (see the references in [16]).

The following Proposition 4 shows that the definition of a conditionally evenly convex set is the appropriate generalization, in the context of topological L^0-module, of the notion of an evenly convex subset of a topological vector space, as in both setting convex (resp. L^0-convex) sets that are either closed or open are evenly (resp. conditionally evenly) convex. This is a key result that allows to show that the assumption (EQC) is the weakest that allows to reach a dual representation of the map π.

We will consider L^0, with the usual operations among random variables, as a partially ordered ring and we will always assume in the sequel that τ_0 is a topology on L^0 such that (L^0, τ_0) is a topological ring. We do not require that τ_0 is a linear topology on L^0 (so that (L^0, τ_0) may not be a topological vector space) nor that τ_0 is locally convex.

Definition 9 *(Topological L^0-module).* We say that (E, τ) is a topological L^0-module if E is an L^0-module and τ is a topology on E such that the module operations

(i) $(E, \tau) \times (E, \tau) \to (E, \tau)$, $(x_1, x_2) \mapsto x_1 + x_2$,
(ii) $(L^0, \tau_0) \times (E, \tau) \to (E, \tau)$, $(\gamma, x_2) \mapsto \gamma x_2$

are continuous w.r.t. the corresponding product topology.

Definition 10 *(Duality for L^0-modules).* For a topological L^0-module (E, τ), we denote

$$E^* := \{x^* : (E, \tau) \to (L^0, \tau_0) \mid x^* \text{ is a continuous module homomorphism}\}. \tag{11}$$

It is easy to check that $(E, E^*, \langle \cdot, \cdot \rangle)$ is a dual pair, where the pairing is given by $\langle x, x^* \rangle = x^*(x)$. Every $x^* \in E^*$ is L^0-linear in the following sense: for all $\alpha, \beta \in L^0$ and $x_1, x_2 \in E$

$$x^*(\alpha x_1 + \beta x_2) = \alpha x^*(x_1) + \beta x^*(x_2).$$

In particular, $x^*(x_1 \mathbf{1}_A + x_2 \mathbf{1}_{A^C}) = x^*(x_1)\mathbf{1}_A + x^*(x_2)\mathbf{1}_{A^C}$, for any $A \in \mathcal{G}$.

Definition 11. A map $\|\cdot\| : E \to L^0_+$ is an L^0-seminorm on E if

(i) $\|\gamma x\| = |\gamma| \|x\|$ for all $\gamma \in L^0$ and $x \in E$,
(ii) $\|x_1 + x_2\| \leqslant \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in E$.

The L^0-seminorm $\|\cdot\|$ becomes an L^0-norm if in addition

(iii) $\|x\| = 0$ implies $x = 0$.

We will consider families of L^0-seminorms $\mathcal{Z}$ satisfying in addition the property:

$$\sup\{\|x\| \mid \|x\| \in \mathcal{Z}\} = 0 \quad \text{iff } x = 0. \tag{12}$$

As clearly pointed out in [16], one family $\mathcal{Z}$ of L^0-seminorms on E may induce on E more than one topology τ such that $\{x_\alpha\}$ converges to x in (E, τ) iff $\|x_\alpha - x\|$ converges to 0 in (L^0, τ_0) for each $\|\cdot\| \in \mathcal{Z}$. Indeed, also the topology τ_0 on L^0 plays a role in the convergence.

Definition 12 *(L^0-module associated to $\mathcal{Z}$).* We say that $(E, \mathcal{Z}, \tau)$ is an L^0-module associated to $\mathcal{Z}$ if

1. $\mathcal{Z}$ is a family of L^0-seminorms satisfying (12),
2. (E, τ) is a topological L^0-module,
3. A net $\{x_\alpha\}$ converges to x in (E, τ) iff $\|x_\alpha - x\|$ converges to 0 in (L^0, τ_0) for each $\|\cdot\| \in \mathcal{Z}$.

Remark 2.2 in [16] shows that any random locally convex module over $\mathbb{R}$ with base $(\Omega, \mathcal{G}, \mathbb{P})$, according to Definition 2.1 in [16], is an L^0-module $(E, \mathcal{Z}, \tau)$ associated to a family $\mathcal{Z}$ of L^0-seminorms, according to the previous definition.

Proposition 4 holds if the topological structure of $(E, \mathcal{Z}, \tau)$ allows for appropriate separation theorems. We now introduce two assumptions that are tailor made for the statements in Proposition 4, but in the following subsection we provide interesting and general examples of L^0-module associated to $\mathcal{Z}$ that fulfill these assumptions.

Separation assumptions. Let E be a topological L^0-module, let E^* be defined in (11) and let $\mathcal{C}_0 \subseteq E$ be non-empty, L^0-convex and satisfy (CSet).

S-Open If $\mathcal{C}_0$ is also open and $\{x\}\mathbf{1}_A \cap \mathcal{C}_0\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$, then there exists $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi)$ $\forall \xi \in \mathcal{C}_0$.

S-Closed If $\mathcal{C}_0$ is also closed and $\{x\}\mathbf{1}_A \cap \mathcal{C}_0\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$, then there exists $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi)$ $\forall \xi \in \mathcal{C}_0$.

Lemma 13.

1. *Let E be a topological L^0-module. If $\mathcal{C}_i \subseteq E$, $i = 1, 2$, are open and non-empty and $A \in \mathcal{G}$, then the set $\mathcal{C}_1\mathbf{1}_A + \mathcal{C}_2\mathbf{1}_{A^C}$ is open.*
2. *Let $(E, \mathcal{Z}, \tau)$ be L^0-module associated to $\mathcal{Z}$. Then for any net $\{\xi_\alpha\} \subseteq E$, $\xi \in E$, $\eta \in E$ and $A \in \mathcal{G}$*

$$\xi_\alpha \xrightarrow{\tau} \xi \quad \Rightarrow \quad (\xi_\alpha 1_A + \eta 1_{A^C}) \xrightarrow{\tau} (\xi 1_A + \eta 1_{A^C}).$$

Proof. 1. To show this claim let $x := x_1\mathbf{1}_A + x_2\mathbf{1}_{A^C}$ with $x_i \in \mathcal{C}_i$ and let U_0 be a neighborhood of 0 satisfying $x_i + U_0 \subseteq \mathcal{C}_i$. Then the set $U := (x_1 + U_0)\mathbf{1}_A + (x_2 + U_0)\mathbf{1}_{A^C} = x + U_0\mathbf{1}_A + U_0\mathbf{1}_{A^C}$ is contained in $\mathcal{C}_1\mathbf{1}_A + \mathcal{C}_2\mathbf{1}_{A^C}$ and it is a neighborhood of x, since $U_0\mathbf{1}_A + U_0\mathbf{1}_{A^C}$ contains U_0 and is therefore a neighborhood of 0.

2. Observe that a seminorm satisfies $\|1_A(\xi_\alpha - \xi)\| = 1_A\|\xi_\alpha - \xi\| \leqslant \|\xi_\alpha - \xi\|$ and therefore, by condition 3. in Definition 12 the claim follows. In particular, $\xi_\alpha \xrightarrow{\tau} \xi \Rightarrow (\xi_\alpha 1_A) \xrightarrow{\tau} (\xi 1_A)$. □

Proposition 4. *Let $(E, \mathcal{Z}, \tau)$ be L^0-module associated to $\mathcal{Z}$ and suppose that $\mathcal{C} \subseteq E$ satisfies* (CSet)*.*

1. *Suppose that the strictly positive cone L^0_{++} is τ_0-open and that there exist $x'_0 \in E^*$ and $x_0 \in E$ such that $x'_0(x_0) > 0$. Under assumption S-Open, if $\mathcal{C}$ is open and L^0-convex then $\mathcal{C}$ is conditionally evenly convex.*
2. *Under assumption S-Closed, if $\mathcal{C}$ is closed and L^0-convex then it is conditionally evenly convex.*

Proof. 1. Let $\mathcal{C} \subseteq E$ be open, L^0-convex, $\mathcal{C} \neq \varnothing$ and let $A_\mathcal{C} \in \mathcal{G}$ be the maximal set given in Notation 2, being $D_\mathcal{C}$ its complement. Suppose that x is outside $\mathcal{C}$, i.e. $x \in E$ satisfies $\{x\}\mathbf{1}_A \cap \mathcal{C}\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$, $A \subseteq D_\mathcal{C}$, $P(A) > 0$. Define the L^0-convex set

$$\mathcal{E} := \{\xi \in E \mid x'_0(\xi) > x'_0(x)\} = (x'_0)^{-1}(x'_0(x) + L^0_{++})$$

and notice that $\{x\}\mathbf{1}_A \cap \mathcal{E}\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$. As L^0_{++} is τ_0-open, $\mathcal{E}$ is open in E. As $x'_0(x_0) > 0$, then $(x + x_0) \in \mathcal{E}$ and $\mathcal{E}$ is non-empty.

Then the set $\mathcal{C}_0 = \mathcal{C}\mathbf{1}_{D_\mathcal{C}} + \mathcal{E}\mathbf{1}_{A_\mathcal{C}}$ is L^0-convex, open (by Lemma 13) and satisfies $\{x\}\mathbf{1}_A \cap \mathcal{C}_0\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$ s.t. $P(A) > 0$. Assumption S-Open guarantees the existence of $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi)$ $\forall \xi \in \mathcal{C}_0$, which implies $x^*(x) > x^*(\xi)$ on $D_\mathcal{C}$ $\forall \xi \in \mathcal{C}$. Hence, by Theorem 1, $\mathcal{C}$ is conditionally evenly convex.

2. Let $\mathcal{C} \subset E$ be closed, L^0-convex, $\mathcal{C} \neq \varnothing$ and suppose that $x \in E$ satisfies $\{x\}\mathbf{1}_A \cap \mathcal{C}\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$, $A \subseteq D_\mathcal{C}$, $\mathbb{P}(A) > 0$. Let $\mathcal{C}_0 = \mathcal{C}\mathbf{1}_{D_\mathcal{C}} + \{x + \varepsilon\}\mathbf{1}_{A_\mathcal{C}}$ where $\varepsilon \in L^0_{++}$. Clearly $\mathcal{C}_0$ is L^0-convex. In order to prove that $\mathcal{C}_0$ is closed consider any net $\xi_\alpha \xrightarrow{\tau} \xi$, $\{\xi_\alpha\} \subset \mathcal{C}_0$. Then $\xi_\alpha = Z_\alpha\mathbf{1}_{D_\mathcal{C}} + \{x + \varepsilon\}\mathbf{1}_{A_\mathcal{C}}$, with $Z_\alpha \in \mathcal{C}$, and $(x + \varepsilon)\mathbf{1}_{A_\mathcal{C}} = \xi 1_{A_\mathcal{C}}$. Take any $\eta \in \mathcal{C}$. As $\mathcal{C}$ is L^0-convex, $\xi_\alpha\mathbf{1}_{D_\mathcal{C}} + \eta\mathbf{1}_{A_\mathcal{C}} = Z_\alpha\mathbf{1}_{D_\mathcal{C}} + \eta\mathbf{1}_{A_\mathcal{C}} \in \mathcal{C}$ and, by Lemma 13, $\xi_\alpha\mathbf{1}_{D_\mathcal{C}} + \eta\mathbf{1}_{A_\mathcal{C}} \xrightarrow{\tau} \xi\mathbf{1}_{D_\mathcal{C}} + \eta\mathbf{1}_{A_\mathcal{C}} := Z \in \mathcal{C}$, as $\mathcal{C}$ is closed. Therefore, $\xi = Z\mathbf{1}_{D_\mathcal{C}} + \{x + \varepsilon\}\mathbf{1}_{A_\mathcal{C}} \in \mathcal{C}_0$. Since $\mathcal{C}_0$ is closed, L^0-convex and $\{x\}\mathbf{1}_A \cap \mathcal{C}_0\mathbf{1}_A = \varnothing$ for every $A \in \mathcal{G}$, assumption S-Closed guarantees the existence of $x^* \in E^*$ s.t. $x^*(x) > x^*(\xi)$ $\forall \xi \in \mathcal{C}_0$, which implies $x^*(x) > x^*(\xi)$ on $D_\mathcal{C}$ $\forall \xi \in \mathcal{C}$. Hence, by Theorem 1, $\mathcal{C}$ is conditionally evenly convex. □

Proposition 5. *Let $(E, \mathcal{Z}, \tau)$ and E^* be respectively as in Definitions 10 and 12, and let τ_0 be a topology on L^0 such that the positive cone L^0_+ is closed. Then any conditionally evenly convex L^0-cone containing the origin is closed.*

Proof. From (6) and the Bipolar Theorem 2 we know that

$$\mathcal{C} = \mathcal{C}^{\circ\circ} = \bigcap_{x' \in \mathcal{C}^\circ} \{x \in E \mid \langle x, x' \rangle \leqslant 0 \text{ on } D_\mathcal{C}\}.$$

We only need to prove that $\mathcal{S}_{x'} = \{x \in E \mid \langle x, x' \rangle \leqslant 0 \text{ on } D_\mathcal{C}\}$ is closed for any $x' \in \mathcal{C}^\circ$. Let $x_\alpha \in \mathcal{S}_{x'}$ be a net such that $x_\alpha \xrightarrow{\tau} x$. Since $x' \in E^*$ is continuous we have $Y_\alpha =: \langle x_\alpha, x' \rangle \xrightarrow{\tau_0} Y =: \langle x, x' \rangle$, with $Y_\alpha \leqslant 0$ on $D_\mathcal{C}$. We surely have that $x_\alpha\mathbf{1}_{D_\mathcal{C}} \xrightarrow{\tau} x\mathbf{1}_{D_\mathcal{C}}$ which implies that $Y_\alpha\mathbf{1}_{D_\mathcal{C}} \xrightarrow{\tau_0} Y\mathbf{1}_{D_\mathcal{C}}$. Since $-Y_\alpha\mathbf{1}_{D_\mathcal{C}} \in L^0_+$ for every α and L^0_+ is closed we conclude that $Y = \langle x, x' \rangle \leqslant 0$ on $D_\mathcal{C}$. □

4.1. On L^0-module associated to $\mathcal{Z}$ satisfying S-Open and S-Closed

Based on the results of Guo [16] and Filipovic et al. [9], we show that a family of seminorms on E may induce more than one topology on the L^0-module E and that these topologies satisfy the assumptions S-Open and S-Closed.

These examples are quite general and therefore support the claim made in the previous section about the relevance of conditional evenly convex sets. A concrete and significant example, already introduced in Section 2, is provided next. To help the reader in finding further details we use the same notations and definitions given in [9] and [16].

Example 8. (See [10].) Let $\mathcal{F}$ be a sigma algebra containing $\mathcal{G}$ and consider the generalized conditional expectation of $\mathcal{F}$-measurable non-negative random variables: $E[\cdot|\mathcal{G}] : L^0_+(\Omega, \mathcal{F}, \mathbb{P}) \to \bar{L}^0_+ := \bar{L}^0_+(\Omega, \mathcal{G}, \mathbb{P})$,

$$E[x|\mathcal{G}] =: \lim_{n\to+\infty} E[x \wedge n|\mathcal{G}].$$

Let $p \in [1, \infty]$ and consider the L^0-module defined as

$$L^p_{\mathcal{G}}(\mathcal{F}) =: \{x \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|x|\mathcal{G}\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P})\}$$

where $\|\cdot|\mathcal{G}\|_p$ is the L^0-norm assigned by

$$\|x|\mathcal{G}\|_p =: \begin{cases} E[|x|^p|\mathcal{G}]^{\frac{1}{p}} & \text{if } p < +\infty, \\ \inf\{y \in \bar{L}^0(\mathcal{G}) \mid y \geqslant |x|\} & \text{if } p = +\infty. \end{cases} \tag{13}$$

Then $L^p_{\mathcal{G}}(\mathcal{F})$ becomes an L^0-normed module associated to the norm $\|\cdot|\mathcal{G}\|_p$ having the product structure:

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{yx \mid y \in L^0(\mathcal{G}),\ x \in L^p(\mathcal{F})\}.$$

For $p < \infty$, any L^0-linear continuous functional $\mu : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0$ can be identified with a random variable $z \in L^q_{\mathcal{G}}(\mathcal{F})$ as $\mu(\cdot) = E[z \cdot |\mathcal{G}]$ where $\frac{1}{p} + \frac{1}{q} = 1$. So we can identify E^* with $L^q_{\mathcal{G}}(\mathcal{F})$.

The two different topologies on E depend on which topology is selected on L^0: either the uniform topology or the topology of convergence in probability.

The two topologies on E will collapse to the same one whenever $\mathcal{G} = \sigma(\emptyset)$ is the trivial sigma algebra, but in general present different structural properties.

We set

$$\|x\|_{\mathcal{S}} := \sup\{\|x\| \mid \|x\| \in \mathcal{S}\}$$

for any finite subfamily $\mathcal{S} \subset \mathcal{Z}$ of L^0-seminorms. Recall from the assumption given in Eq. (12) that $\|x\|_{\mathcal{S}} = 0$ if and only if $x = 0$.

4.1.1. The uniform topology τ_c [9]

In this case, L^0 is equipped with the following uniform topology. For every $\varepsilon \in L^0_{++}$, the ball $B_\varepsilon := \{Y \in L^0 \mid |Y| \leqslant \varepsilon\}$ centered in $0 \in L^0$ gives the neighborhood basis of 0. A set $V \subset L^0$ is a neighborhood of $Y \in L^0$ if there exists $\varepsilon \in L^0_{++}$ such that $Y + B_\varepsilon \subset V$. A set V is open if it is a neighborhood of all $Y \in V$. A net converges in this topology, namely $Y_N \xrightarrow{|\cdot|} Y$ if for every $\varepsilon \in L^0_{++}$ there exists $\bar{N}$ such that $|Y - Y_N| < \varepsilon$ for every $N > \bar{N}$. In this case the space $(L^0, |\cdot|)$ looses the property of being a topological vector space. In this topology the positive cone L^0_+ is closed and the strictly positive cone L^0_{++} is open.

Under the assumptions that there exists an $x \in E$ such that $x\mathbf{1}_A \neq 0$ for every $A \in \mathcal{G}$ and that the topology τ on E is Hausdorff, Theorem 2.8 in [9] guarantees the existence of $x_0 \in E$ and $x'_0 \in E^*$ such that $x'_0(x_0) > 0$. This and the next item 2 allow the application of Proposition 4.

A family $\mathcal{Z}$ of L^0-seminorms on E induces a topology on E in the following way. For any finite $\mathcal{S} \subset \mathcal{Z}$ and $\varepsilon \in L^0_{++}$ define

$$U_{\mathcal{S},\varepsilon} := \{x \in E \mid \|x\|_{\mathcal{S}} \leqslant \varepsilon\},$$
$$\mathcal{U} := \{U_{\mathcal{S},\varepsilon} \mid \mathcal{S} \subset \mathcal{Z} \text{ finite and } \varepsilon \in L^0_{++}\}.$$

$\mathcal{U}$ gives a convex neighborhood base of 0 and it induces a topology on E denoted by τ_c. We have the following properties:

1. $(E, \mathcal{Z}, \tau_c)$ is an $(L^0, |\cdot|)$-module associated to $\mathcal{Z}$, which is also a locally convex topological L^0-module (see Proposition 2.7 [16]),
2. $(E, \mathcal{Z}, \tau_c)$ satisfies S-Open and S-Closed (see Theorems 2.6 and 2.8 [9]),
3. Any topological $(L^0, |\cdot|)$-module (E, τ) is locally convex if and only if τ is induced by a family of L^0-seminorms, i.e. $\tau \equiv \tau_c$ (see Theorem 2.4 [9]).

4.1.2. A probabilistic topology $\tau_{\epsilon,\lambda}$ [16]

The second topology on the L^0-module E is a topology of a more probabilistic nature and originated in the theory of probabilistic metric spaces (see [23]).

Here L^0 is endowed with the topology $\tau_{\epsilon,\lambda}$ of convergence in probability and so the positive cone L^0_+ is τ_0-closed. According to [16], for every $\epsilon, \lambda \in \mathbb{R}$ and a finite subfamily $\mathcal{S} \subset \mathcal{Z}$ of L^0-seminorms we let

$$\mathcal{V}_{\mathcal{S},\epsilon,\lambda} := \left\{x \in E \,\middle|\, \mathbb{P}\left(\|x\|_{\mathcal{S}} < \epsilon\right) > 1 - \lambda\right\},$$

$$\mathcal{V} := \{\mathcal{U}_{\mathcal{S},\epsilon,\lambda} \mid \mathcal{S} \subset \mathcal{Z} \text{ finite}, \ \epsilon > 0, \ 0 < \lambda < 1\}.$$

$\mathcal{V}$ gives a neighborhood base of 0 and it induces a linear topology on E, also denoted by $\tau_{\epsilon,\lambda}$ (indeed if $E = L^0$ then this is exactly the topology of convergence in probability). This topology may not be locally convex, but has the following properties:

1. $(E, \mathcal{Z}, \tau_{\epsilon,\lambda})$ becomes an $(L^0, \tau_{\epsilon,\lambda})$-module associated to $\mathcal{Z}$ (see Proposition 2.6 [16]),
2. $(E, \mathcal{Z}, \tau_{\epsilon,\lambda})$ satisfies S-Closed (see Theorems 3.6 and 3.9 [16]).

Therefore Proposition 4 can be applied.

5. On conditionally evenly quasi-convex maps on L^0-module

As an immediate consequence of Proposition 4 we have that lower (resp. upper) semicontinuity and quasi-convexity imply evenly quasi-convexity of ρ. From Theorem 3 we then deduce the representation for lower (resp. upper) semicontinuous quasi-convex maps.

(LSC) A map $\pi : E \to \bar{L}^0(\mathcal{G})$ is lower semicontinuous if for every $Y \in L^0$ the lower level sets $U_Y = \{\xi \in E \mid \pi(\xi)\mathbf{1}_{T_\pi} \leqslant Y\}$ are τ-closed.

Corollary 6. *Let $(E, \mathcal{Z}, \tau)$ and $E' = E^*$ be respectively as in Definitions 10 and 12, satisfying S-Closed.*
If $\pi : E \to \bar{L}^0(\mathcal{G})$ is (REG), (QCO) *and* (LSC) *then (9) holds true.*

In the upper semicontinuous case we can say more (the proof is postponed to Section 6).

(USC) A map $\pi : E \to \bar{L}^0(\mathcal{G})$ is upper semicontinuous if for every $Y \in L^0$ the lower level sets $\{\xi \in E \mid \pi(\xi)\mathbf{1}_{T_\pi} < Y\}$ are τ-open.

Proposition 7. *Let $(E, \mathcal{Z}, \tau)$ and $E' = E^*$ be respectively as in Proposition 4 statement* 1*, satisfying S-Open.*
If $\pi : E \to \bar{L}^0(\mathcal{G})$ is (REG), (QCO) *and* (USC) *then*

$$\pi(x) = \max_{x^* \in E^*} \mathcal{R}\left(\langle x, x^*\rangle, x^*\right). \tag{14}$$

In Theorem 3, π can be represented as a supremum but not as a maximum. The following corollary shows that nevertheless we can find an $\mathcal{R}(\langle x, x^*\rangle, x^*)$ arbitrary close to $\pi(x)$.

Corollary 8. *Under the same assumption of Theorem 3 or Corollary 6, for every $\varepsilon \in L^0_{++}$ there exists $x^*_\varepsilon \in E^*$ such that*

$$\pi(x) - \mathcal{R}\left(\langle x, x^*_\varepsilon\rangle, x^*_\varepsilon\right) < \varepsilon \quad \text{on the set } \left\{\pi(x) < +\infty\right\}. \tag{15}$$

Proof. The statement is a direct consequence of the inequalities (30) through (31) of Step 3 in the proof of Theorem 3. $\square$

6. Proofs

Notation 9. The condition $\mathbf{1}_A\{\eta\} \cap \mathbf{1}_A\mathcal{C} \neq \varnothing$ is equivalent to: $\exists \xi \in \mathcal{C}$ s.t. $\mathbf{1}_A\eta = \mathbf{1}_A\xi$.
For $\eta \in E$, $B \in \mathcal{G}$ and $\mathcal{C} \subseteq E$ we say that

$$\eta \text{ is outside}|_B\, \mathcal{C} \text{ if } \forall A \subseteq B, \ A \in \mathcal{G}, \ \mathbb{P}(A) > 0, \ \mathbf{1}_A\{\eta\} \cap \mathbf{1}_A\mathcal{C} = \varnothing.$$

If $\mathbb{P}(B) = 0$ then η is outside$|_B$ $\mathcal{C}$ is equivalent to $\eta \in \mathcal{C}$. Recall that $A_\mathcal{C}$ is the maximal set of $\mathcal{A}(\mathcal{C}) = \{B \in \mathcal{G} \mid \mathbf{1}_A E = \mathbf{1}_A\mathcal{C}\}$, $D_\mathcal{C}$ is the complement of $A_\mathcal{C}$ and that η is outside $\mathcal{C}$ if η is outside$|_{D_\mathcal{C}}$ $\mathcal{C}$.

Remark 10. By Lemma 2.9 in [9], we know that any non-empty class $\mathcal{A}$ of subsets of a sigma algebra $\mathcal{G}$ has a supremum $\operatorname{ess.sup}\{\mathcal{A}\} \in \mathcal{G}$ and that if $\mathcal{A}$ is closed with respect to finite union (i.e. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$) then there is a sequence $A_n \in \mathcal{A}$ such that $\operatorname{ess.sup}\{\mathcal{A}\} = \bigcup_{n\in\mathbb{N}} A_n$. Obviously, if $\mathcal{A}$ is closed with respect to countable union then $\operatorname{ess.sup}\{\mathcal{A}\} = \bigcup_{n\in\mathbb{N}} A_n := A_M \in \mathcal{A}$ is the maximal element in $\mathcal{A}$.

For our proofs we need a simplified version of a result proved by Guo (Theorem 3.13, [16]) concerning hereditarily disjoint stratification of two subsets. We reformulate his result in the following

Lemma 14. *Suppose that $\mathcal{C} \subset E$ satisfies $\mathbf{1}_A\mathcal{C} + \mathbf{1}_{A^C}\mathcal{C} \subseteq \mathcal{C}$ for every $A \in \mathcal{G}$. If there exists $x \in E$ with $x \notin \mathcal{C}$ then there exists a set $H := H_{\mathcal{C},x} \in \mathcal{G}$ such that $\mathbb{P}(H) > 0$ and*

$$\mathbf{1}_{\Omega\setminus H}\{x\} \cap \mathbf{1}_{\Omega\setminus H}\mathcal{C} \neq \varnothing, \tag{16}$$

$$x \text{ is outside}|_H\, \mathcal{C}. \tag{17}$$

The two above conditions guarantee that $H_{\mathcal{C},x}$ is the largest set $D \in \mathcal{G}$ such that x is outside$|_D$ $\mathcal{C}$.

Lemma 15. *Suppose that $\mathcal{C}$ satisfies* (CSet).

1. *If $x \notin \mathcal{C}$ then the set $H_{\mathcal{C},x}$ defined in Lemma 14 satisfies $H_{\mathcal{C},x} \subseteq D_{\mathcal{C}}$ and so $\mathbb{P}(D_{\mathcal{C}}) \geqslant \mathbb{P}(H_{\mathcal{C},x}) > 0$.*
2. *If x is outside $\mathcal{C}$ then $\mathbb{P}(H_{\mathcal{C},x}) > 0$ and $H_{\mathcal{C},x} = D_{\mathcal{C}}$.*
3. *If $x \notin \mathcal{C}$ then*

$$\chi := \{y \in E \mid y \text{ is outside } \mathcal{C}\} \neq \varnothing. \tag{18}$$

Proof. 1. Lemma 14 shows that $\mathbb{P}(H_{\mathcal{C},x}) > 0$. Since $\mathbf{1}_{A_{\mathcal{C}}}E = \mathbf{1}_{A_{\mathcal{C}}}\mathcal{C}$, if $x \notin \mathcal{C}$ we necessarily have that $\mathbb{P}(H_{\mathcal{C},x} \cap A_{\mathcal{C}}) = 0$ and therefore $H_{\mathcal{C},x} \subseteq D_{\mathcal{C}}$.

2. If x is outside| $\mathcal{C}$ then x is outside$|_{D_{\mathcal{C}}}$ $\mathcal{C}$ and $x \notin \mathcal{C}$. The thesis follows from $H_{\mathcal{C},x} \subseteq D_{\mathcal{C}}$ and the fact that $H_{\mathcal{C},x}$ is the largest set $D \in \mathcal{G}$ for which x is outside$|_D$ $\mathcal{C}$.

3 is a consequence of Lemma 17 (see Appendix A). □

Proof of Theorem 1. (1) ⇒ (2): Let $\mathcal{L} \subset E'$, $Y_{x'} \in L^0$ and let

$$\mathcal{C} =: \bigcap_{x'\in\mathcal{L}} \bigl\{\xi \in E \bigm| \langle \xi, x'\rangle < Y_{x'} \text{ on } D_{\mathcal{C}}\bigr\},$$

which clearly satisfies $\mathcal{C}^{cc} = \mathcal{C}$. By definition, if there exists $x \in E$ s.t. x is outside $\mathcal{C}$ then $\mathbf{1}_A\{x\} \cap \mathbf{1}_A\mathcal{C} = \varnothing$ $\forall A \subseteq D_{\mathcal{C}}$, $A \in \mathcal{G}$, $\mathbb{P}(A) > 0$, and therefore by the definition of $\mathcal{C}$ there exists $x' \in \mathcal{L}$ s.t. $\langle x, x'\rangle \geqslant Y_{x'}$ on $D_{\mathcal{C}}$. Hence $\langle x, x'\rangle \geqslant Y_{x'} > \langle \xi, x'\rangle$ on $D_{\mathcal{C}}$ for all $\xi \in \mathcal{C}$.

(2) ⇒ (1): We are assuming that $\mathcal{C}$ is (CSet), and there exists $x \in E$ s.t. $x \notin \mathcal{C}$ (otherwise $\mathcal{C} = E$). From (18) we know that $\chi = \{y \in E \mid y \text{ is outside } \mathcal{C}\}$ is non-empty. By assumption, for all $y \in \chi$ there exists $\xi'_y \in E'$ such that $\langle \xi, \xi'_y\rangle < \langle y, \xi'_y\rangle$ on $D_{\mathcal{C}}$ $\forall \xi \in \mathcal{C}$. Define

$$B_y := \bigl\{\xi \in E \bigm| \langle \xi, \xi'_y\rangle < \langle y, \xi'_y\rangle \text{ on } D_{\mathcal{C}}\bigr\}.$$

B_y clearly depends also on the selection of the $\xi'_y \in E'$ associated to y and on $\mathcal{C}$, but this notation will not cause any ambiguity. We have $\mathcal{C} \subseteq B_y$ for all $y \in \chi$, and $\mathcal{C} \subseteq \bigcap_{y\in\chi} B_y$. We now claim that $x \notin \mathcal{C}$ implies $x \notin \bigcap_{y\in\chi} B_y$, thus showing

$$\mathcal{C} = \bigcap_{y\in\chi} B_y = \bigcap_{\xi'_y\in\mathcal{L}} \bigl\{\xi \in E \bigm| \langle \xi, \xi'_y\rangle < Y_{\xi'_y} \text{ on } D_{\mathcal{C}}\bigr\}, \tag{19}$$

where $\mathcal{L} := \{\xi'_y \in E' \mid y \in \chi\}$, $Y_{\xi'_y} := \langle y, \xi'_y\rangle \in L^0$, and the thesis is proved.

Suppose that $x \notin \mathcal{C}$, then, by Lemma 14, x is outside$|_H$ $\mathcal{C}$, where we set for simplicity $H = H_{\mathcal{C},x}$. Take any $y \in \chi \neq \varnothing$ and define $y_0 := x\mathbf{1}_H + y\mathbf{1}_{\Omega\setminus H} \in \chi$. Take $B_{y_0} = \{\xi \in E \mid \langle \xi, \xi'_{y_0}\rangle < \langle y_0, \xi'_{y_0}\rangle \text{ on } D_{\mathcal{C}}\}$ where $\xi'_{y_0} \in E'$ is the element associated to y_0. If $x \in B_{y_0}$ then we would have $\langle x, \xi'_{y_0}\rangle < \langle y_0, \xi'_{y_0}\rangle = \langle x, \xi'_{y_0}\rangle$ on $H \subseteq D_{\mathcal{C}}$, by Lemma 15 item 1, which is a contradiction, since $\mathbb{P}(H) > 0$. Hence $x \notin B_{y_0} \supseteq \bigcap_{y\in\chi} B_y$. □

Proposition 9. *Under the same assumptions of Theorem 1, the following are equivalent*:

(1) *$\mathcal{C}$ is conditionally evenly convex.*

(2) *For every $x \in E$, $x \notin \mathcal{C}$, there exists $x' \in E'$ such that*

$$\langle \xi, x' \rangle < \langle x, x' \rangle \quad \text{on } H_{\mathcal{C},x} \ \forall \xi \in \mathcal{C},$$

where $H_{\mathcal{C},x}$ is defined in Lemma 14.

Proof. (1) $\Rightarrow$ (2): We know that $\mathcal{C}$ satisfies (CSet). As $x \notin \mathcal{C}$, from (18) and Lemma 14 we know that there exists $y \in E$ s.t. y is outside C and that $H =: H_{\mathcal{C},x}$ satisfies $\mathbb{P}(H) > 0$. Define $\tilde{x} = x\mathbf{1}_H + y\mathbf{1}_{\Omega \setminus H}$. Then $\tilde{x}$ is outside $\mathcal{C}$ and by Theorem 1 item (2) there exists $x' \in E'$

$$\langle \xi, x' \rangle < \langle \tilde{x}, x' \rangle \quad \text{on } D_{\mathcal{C}} \ \forall \xi \in \mathcal{C}.$$

This implies the thesis since $\langle \tilde{x}, x' \rangle = \langle x, x' \rangle \mathbf{1}_H + \langle y, x' \rangle \mathbf{1}_{\Omega \setminus H}$ and $H \subseteq D_{\mathcal{C}}$.

(2) $\Rightarrow$ (1): We show that item (2) of Theorem 1 holds true. This is trivial since if x is outside $\mathcal{C}$ then $x \notin \mathcal{C}$ and $H_{\mathcal{C},x} = D_{\mathcal{C}}$. $\square$

Proof of Theorem 2. Item (1) is straightforward; the fact that $\mathcal{C}^{\circ\circ}$ is conditionally evenly convex follows from the definition; the proof of $\mathcal{C} \subseteq \mathcal{C}^{\circ\circ}$ is also obvious. We now suppose that $\mathcal{C}$ is conditionally evenly convex and show the reverse inequality $\mathcal{C}^{\circ\circ} \subseteq \mathcal{C}$. By contradiction let $x \in \mathcal{C}^{\circ\circ}$ and $x \notin \mathcal{C}$. As $\mathcal{C}$ is conditionally evenly convex we apply Proposition 9 and find $x' \in E'$ such that

$$\langle \xi, x' \rangle < \langle x, x' \rangle \quad \text{on } H_{\mathcal{C},x} \text{ for all } \xi \in \mathcal{C}.$$

Since $0 \in \mathcal{C}$, $0 = \langle 0, x' \rangle < \langle x, x' \rangle$ on $H =: H_{\mathcal{C},x}$. Take any $x_1' \in \mathcal{C}^{\circ}$ (which is clearly not empty) and set $y' := \frac{x'}{\langle x, x' \rangle}\mathbf{1}_H + x_1'\mathbf{1}_{\Omega \setminus H}$. Then $y' \in E'$ and $\langle \xi, y' \rangle < 1$ on $D_{\mathcal{C}}$ for all $\xi \in \mathcal{C}$. This implies $y' \in \mathcal{C}^{\circ}$. In addition, $\langle x, y' \rangle = 1$ on $H \subseteq D_{\mathcal{C}}$ which is in contradiction with $x \in \mathcal{C}^{\circ\circ}$. $\square$

6.1. General properties of $\mathcal{R}(Y, \mu)$

Following the path traced in [13], we adapt to the module framework the proofs of the foremost properties holding for the function $\mathcal{R} : L^0(\mathcal{G}) \times E^* \to \bar{L}^0(\mathcal{G})$ defined in (10). Let the effective domain of the function $\mathcal{R}$ be:

$$\Sigma_{\mathcal{R}} := \{(Y, \mu) \in L^0(\mathcal{G}) \times E^* \mid \exists \xi \in E \text{ s.t. } \mu(\xi) \geqslant Y\}. \tag{20}$$

Lemma 16. *Let $\mu \in E^*$, $X \in E$ and $\pi : E \to \bar{L}^0(\mathcal{G})$ satisfy (REG).*

(i) *$\mathcal{R}(\cdot, \mu)$ is monotone non-decreasing.*
(ii) *$\mathcal{R}(\Lambda\mu(X), \Lambda\mu) = \mathcal{R}(\mu(X), \mu)$ for every $\Lambda \in L^0(\mathcal{G})$.*
(iii) *For every $Y \in L^0(\mathcal{G})$ and $\mu \in E^*$, the set*

$$\mathcal{A}_\mu(Y) \doteq \{\pi(\xi) \mid \xi \in E,\ \mu(\xi) \geqslant Y\}$$

is downward directed in the sense that for every $\pi(\xi_1), \pi(\xi_2) \in \mathcal{A}_\mu(Y)$ there exists $\pi(\xi^*) \in \mathcal{A}_\mu(Y)$ such that $\pi(\xi^*) \leqslant \min\{\pi(\xi_1), \pi(\xi_2)\}$.
In addition, if $\mathcal{R}(Y, \mu) < \alpha$ for some $\alpha \in L^0(\mathcal{G})$ then there exists ξ such that $\mu(\xi) \geqslant Y$ and $\pi(\xi) < \alpha$.
(iv) *For every $A \in \mathcal{G}$, $(Y, \mu) \in \Sigma_{\mathcal{R}}$*

$$\mathcal{R}(Y, \mu)\mathbf{1}_A = \inf_{\xi \in E}\{\pi(\xi)\mathbf{1}_A \mid Y \geqslant \mu(X)\} \tag{21}$$

$$= \inf_{\xi \in E}\{\pi(\xi)\mathbf{1}_A \mid Y\mathbf{1}_A \geqslant \mu(X\mathbf{1}_A)\} = \mathcal{R}(Y\mathbf{1}_A, \mu)\mathbf{1}_A. \tag{22}$$

(v) *For every $X_1, X_2 \in E$*

$$\text{(a)} \quad \mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \wedge \mu(X_2), \mu),$$
$$\text{(b)} \quad \mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu) = \mathcal{R}(\mu(X_1) \vee \mu(X_2), \mu).$$

(vi) *The map $\mathcal{R}(\mu(X), \mu)$ is quasi-affine with respect to X in the sense that for every $X_1, X_2 \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leqslant \Lambda \leqslant 1$, we have*

$$\mathcal{R}(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \geqslant \mathcal{R}(\mu(X_1), \mu) \wedge \mathcal{R}(\mu(X_2), \mu) \quad (\textit{quasi-concavity}),$$
$$\mathcal{R}(\mu(\Lambda X_1 + (1-\Lambda)X_2), \mu) \leqslant \mathcal{R}(\mu(X_1), \mu) \vee \mathcal{R}(\mu(X_2), \mu) \quad (\textit{quasi-convexity}).$$

(vii) $\inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_1) = \inf_{Y \in L^0(\mathcal{G})} \mathcal{R}(Y, \mu_2)$ *for every $\mu_1, \mu_2 \in E^*$.*

Proof. (i) and (ii) follow trivially from the definition.

(iii) The set $\{\pi(\xi) \mid \xi \in E,\ \mu(\xi) \geqslant Y\}$ is clearly downward directed. Thus there exists a sequence $\{\xi_m^\mu\}_{m=1}^\infty \in E$ such that

$$\mu\left(\xi_m^\mu\right) \geqslant Y \quad \forall m \geqslant 1, \qquad \pi\left(\xi_m^\mu\right) \downarrow \mathcal{R}(Y,\mu) \quad \text{as } m \uparrow \infty.$$

Now let $\mathcal{R}(Y,\mu) < \alpha$. Consider the sets $F_m = \{\pi(\xi_m^\mu) < \alpha\}$ and the partition of Ω given by $G_1 = F_1$ and $G_m = F_m \setminus G_{m-1}$. Since we assume that E satisfies (CSet) from the property (REG) we get:

$$\xi = \sum_{m=1}^\infty \xi_m^\mu \mathbf{1}_{G_m} \in E, \quad \mu(\xi) \geqslant Y \text{ and } \pi(\xi) < \alpha.$$

(iv), (v) and (vi) follow as in [13].

(vii) Notice that $\mathcal{R}(Y,\mu) \geqslant \inf_{\xi\in E} \pi(\xi)$, $\forall Y \in L^0_{\mathcal{F}}$, implies $\inf_{Y\in L^0(\mathcal{G})} \mathcal{R}(Y,\mu) \geqslant \inf_{\xi\in E}\pi(\xi)$. On the other hand, $\pi(\xi) \geqslant \mathcal{R}(\mu(\xi),\mu) \geqslant \inf_{Y\in L^0(\mathcal{G})} \mathcal{R}(Y,\mu)$, $\forall \xi \in E$, implies $\inf_{Y\in L^0(\mathcal{G})} \mathcal{R}(Y,\mu) \leqslant \inf_{\xi\in E}\pi(\xi)$. $\square$

Proof of Theorem 3. Let $\pi : E \to \bar{L}^0(\mathcal{G})$. There might exist a set $A \in \mathcal{G}$ on which the map π is constant, in the sense that $\pi(\xi)\mathbf{1}_A = \pi(\eta)\mathbf{1}_A$ for every $\xi, \eta \in E$. For this reason we introduce

$$\mathcal{A} := \left\{B \in \mathcal{G} \;\middle|\; \pi(\xi)\mathbf{1}_B = \pi(\eta)\mathbf{1}_B \ \forall \xi,\eta \in E\right\}.$$

Applying Lemma 18 in Appendix A with $F := \{\pi(\xi) - \pi(\eta) \mid \xi,\eta \in E\}$ (we consider the convention $+\infty - \infty = 0$) and $Y_0 = 0$ we can deduce the existence of two maximal sets $A \in \mathcal{G}$ and $A^\vdash \in \mathcal{G}$ for which $P(A \cap A^\vdash) = 0$, $P(A \cup A^\vdash) = 1$ and

$$\begin{aligned} &\pi(\xi) = \pi(\eta) \quad \text{on } A \text{ for every } \xi,\eta \in E, \\ &\pi(\zeta_1) < \pi(\zeta_2) \quad \text{on } A^\vdash \text{ for some } \zeta_1,\zeta_2 \in E. \end{aligned} \tag{23}$$

Recall that $\Upsilon_\pi \in \mathcal{G}$ is the maximal set on which $\pi(\xi)\mathbf{1}_{\Upsilon_\pi} = +\infty\mathbf{1}_{\Upsilon_\pi}$ for every $\xi \in E$ and T_π its complement. Notice that $\Upsilon_\pi \subset A$.

Fix $x \in E$ and $G = \{\pi(x) < +\infty\}$. For every $\varepsilon \in L^0_{++}(\mathcal{G})$ we set

$$Y_\varepsilon =: 0\mathbf{1}_{\Upsilon_\pi} + \pi(x)\mathbf{1}_{A\setminus\Upsilon_\pi} + \big(\pi(x) - \varepsilon\big)\mathbf{1}_{G\cap A^\vdash} + \varepsilon\mathbf{1}_{G^C\cap A^\vdash} \tag{24}$$

and for every $\varepsilon \in L^0(\mathcal{G})_{++}$ we set

$$\mathcal{C}_\varepsilon = \left\{\xi \in E \;\middle|\; \pi(\xi)\mathbf{1}_{T_\varepsilon} \leqslant Y_\varepsilon\right\}. \tag{25}$$

Step 1. On the set A, $\pi(x) = \mathcal{R}(\langle x,x'\rangle, x')$ for any $x' \in E'$ and the representation

$$\pi(x)\mathbf{1}_A = \max_{x'\in E'} \mathcal{R}\big(\langle x,x'\rangle, x'\big)\mathbf{1}_A \tag{26}$$

trivially holds true on A.

Step 2. By the definition of Y_ε we deduce that if $\mathcal{C}_\varepsilon = \varnothing$ for every $\varepsilon \in L^0_{++}$ then $\pi(x) \leqslant \pi(\xi)$ on the set $A^\vdash$ for every $\xi \in E$ and $\pi(x)\mathbf{1}_{A^\vdash} = \mathcal{R}(\langle x,x'\rangle, x')\mathbf{1}_{A^\vdash}$ for any x'. The representation

$$\pi(x)\mathbf{1}_{A^\vdash} = \max_{x'\in E'} \mathcal{R}\big(\langle x,x'\rangle, x'\big)\mathbf{1}_{A^\vdash} \tag{27}$$

trivially holds true on $A^\vdash$. The thesis follows pasting together Eqs. (26) and (27).

Step 3. We now suppose that there exists $\varepsilon \in L^0_{++}$ such that $\mathcal{C}_\varepsilon \neq \varnothing$. The definition of Y_ε implies that $\mathcal{C}_\varepsilon\mathbf{1}_A = E\mathbf{1}_A$ and A is the maximal element, i.e. $A = A_{\mathcal{C}_\varepsilon}$ (given by Definition 2). Moreover this set is conditionally evenly convex and x is outside C_ε. The definition of evenly convex set guarantees that there exists $x'_\varepsilon \in E'$ such that

$$\langle x, x'_\varepsilon\rangle > \langle \xi, x'_\varepsilon\rangle \quad \text{on } D_{\mathcal{C}_\varepsilon} = A^\vdash \ \forall \xi \in \mathcal{C}_\varepsilon. \tag{28}$$

Claim.

$$\left\{\xi \in E \;\middle|\; \langle x, x'_\varepsilon\rangle\mathbf{1}_{A^\vdash} \leqslant \langle \xi, x'_\varepsilon\rangle\mathbf{1}_{A^\vdash}\right\} \subseteq \left\{\xi \in E \;\middle|\; \pi(\xi) > \big(\pi(x) - \varepsilon\big)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C} \textit{ on } A^\vdash\right\}. \tag{29}$$

In order to prove the claim take $\xi \in E$ such that $\langle x, x'_\varepsilon\rangle \mathbf{1}_{A^\vdash} \leqslant \langle \xi, x'_\varepsilon\rangle \mathbf{1}_{A^\vdash}$. By contra we suppose that there exists an $F \subset A^\vdash$, $F \in \mathcal{G}$ and $\mathbb{P}(F) > 0$ such that $\pi(\xi)\mathbf{1}_F \leqslant (\pi(x) - \varepsilon)\mathbf{1}_{G\cap F} + \varepsilon \mathbf{1}_{G^c \cap F}$. Take $\eta \in \mathcal{C}_\varepsilon$ and define $\bar{\xi} = \eta \mathbf{1}_{F^C} + \xi \mathbf{1}_F \in \mathcal{C}_\varepsilon$ so that we conclude that $\langle x, x'_\varepsilon\rangle > \langle \bar{\xi}, x'_\varepsilon\rangle$ on $A^\vdash$. Since $\langle \bar{\xi}, x'_\varepsilon\rangle = \langle \xi, x'_\varepsilon\rangle$ on F we reach a contradiction.

Once the claim is proved we end the argument observing that

$$\begin{aligned}
\pi(x)\mathbf{1}_{A^\vdash} &\geqslant \sup_{x' \in E'} \mathcal{R}\big(\langle x, x'\rangle, x'\big)\mathbf{1}_{A^\vdash} = \mathcal{R}\big(\langle x, x'_\varepsilon\rangle, x'_\varepsilon\big)\mathbf{1}_{A^\vdash} &(30)\\
&= \inf_{\xi \in E}\big\{\pi(\xi)\mathbf{1}_{A^\vdash} \;\big|\; \langle x, x'_\varepsilon\rangle \mathbf{1}_A \leqslant \langle \xi, x'_\varepsilon\rangle \mathbf{1}_{A^\vdash}\big\}\\
&\geqslant \inf_{\xi \in E}\big\{\pi(\xi)\mathbf{1}_{A^\vdash} \;\big|\; \pi(\xi) > \big(\pi(x) - \varepsilon\big)\mathbf{1}_G + \varepsilon \mathbf{1}_{G^c} \text{ on } A^\vdash\big\}\\
&\geqslant \big(\pi(x) - \varepsilon\big)\mathbf{1}_{G\cap A^\vdash} + \varepsilon \mathbf{1}_{G^c \cap A^\vdash}. &(31)
\end{aligned}$$

The representation (9) follows by taking ε arbitrary small on $G \cap A^\vdash$ and arbitrary big on $G^C \cap A^\vdash$ and pasting together the result with Eq. (26). □

Proof of Proposition 7. Fix $X \in E$ and consider the classes of sets

$$\begin{aligned}
&\mathcal{A} := \big\{B \in \mathcal{G} \;\big|\; \forall \xi \in E \;\pi(\xi) \geqslant \pi(X) \text{ on } B\big\},\\
&\mathcal{A}^\vdash := \big\{B \in \mathcal{G} \;\big|\; \exists \xi \in E \text{ s.t. } \pi(\xi) < \pi(X) \text{ on } B\big\}.
\end{aligned}$$

Then $\mathcal{A} = \{B \in \mathcal{G} \mid \forall Y \in F \;\; Y \geqslant Y_0 \text{ on } B\}$, where $F := \{\pi(\xi) \mid \xi \in E\}$ and $Y_0 = \pi(X)$. Applying Lemma 18, there exist two maximal elements $A \in \mathcal{A}$ and $A^\vdash \in \mathcal{A}^\vdash$ so that $P(A \cup A^\vdash) = 1$, $P(A \cap A^\vdash) = 0$,

$$\pi(\xi) \geqslant \pi(X) \quad \text{on } A \text{ for every } \xi \in E \quad \text{and} \quad \exists \bar{\xi} \in E \text{ s.t.} \quad \pi(\bar{\xi}) < \pi(X) \quad \text{on } A^\vdash.$$

Clearly for every $\mu \in E^*$.

$$\pi(X)\mathbf{1}_A \geqslant \mathcal{R}\big(\mu(X), \mu\big)\mathbf{1}_A \geqslant \pi(X)\mathbf{1}_A. \tag{32}$$

Consider $\delta \in L^0_{++}(\mathcal{G})$. The set

$$\mathcal{O} := \big\{\xi \in E \;\big|\; \pi(\xi)\mathbf{1}_{T_\pi} < \pi(X)\mathbf{1}_{A^\vdash} + \big(\pi(X) + \delta\big)\mathbf{1}_A\big\}$$

is open, $L^0(\mathcal{G})$-convex (from Remark 7) and not empty. Clearly $X \notin \mathcal{O}$ and $\mathcal{O}$ satisfies (CSet). We thus can apply Theorem 3.15 in [16] and find $\mu_* \in E^*$ so that

$$\mu_*(X) > \mu_*(\xi) \quad \text{on } H\big(\{X\}, \mathcal{O}\big) \; \forall \xi \in \mathcal{O}.$$

Notice that $\mathbb{P}(H(\{X\}, \mathcal{O}) \setminus A^\vdash) = 0$. We apply the argument in Step 3 of the proof of Theorem 3 to find that

$$\big\{\xi \in E \;\big|\; \mu_*(X)\mathbf{1}_{A^\vdash} \leqslant \mu_*(\xi)\mathbf{1}_{A^\vdash}\big\} \subseteq \big\{\xi \in E \;\big|\; \pi(\xi)\mathbf{1}_{A^\vdash} \geqslant \pi(X)\mathbf{1}_{A^\vdash}\big\}.$$

From (21)–(22) we derive

$$\begin{aligned}
\pi(X)\mathbf{1}_{A^\vdash} &\geqslant \mathcal{R}\big(\mu_*(X), \mu_*\big)\mathbf{1}_{A^\vdash} = \inf_{\xi \in E}\big\{\pi(\xi)\mathbf{1}_{A^\vdash} \;\big|\; \mu_*(X)\mathbf{1}_{A^\vdash} \leqslant \mu_*(\xi)\mathbf{1}_{A^\vdash}\big\}\\
&\geqslant \inf_{\xi \in E}\big\{\pi(\xi)\mathbf{1}_{A^\vdash} \;\big|\; \pi(\xi)\mathbf{1}_{A^\vdash} \geqslant \pi(X)\mathbf{1}_{A^\vdash_M}\big\} \geqslant \pi(X)\mathbf{1}_{A^\vdash_M}.
\end{aligned}$$

The thesis then follows from (32). □

Appendix A

Lemma 17. *For any sets $\mathcal{C} \subseteq E$ and $\mathcal{D} \subseteq E$ set*:

$$\begin{aligned}
&\mathcal{A} = \big\{B \in \mathcal{G} \;\big|\; \forall y \in \mathcal{D}^{cc} \; \exists \xi \in \mathcal{C}^{cc} \text{ s.t. } \mathbf{1}_B y = \mathbf{1}_B \xi\big\},\\
&\mathcal{A}^\vdash = \big\{B \in \mathcal{G} \;\big|\; \exists y \in \mathcal{D}^{cc} \text{ s.t. } y \text{ is outside}|_B \; \mathcal{C}^{cc}\big\}.
\end{aligned}$$

Then there exist the maximal set $A_M \in \mathcal{A}$ of $\mathcal{A}$ and the maximal set $A^\vdash_M \in \mathcal{A}^\vdash$ of $\mathcal{A}^\vdash$, one of which may have zero probability, that satisfy

$$\forall y \in \mathcal{D}^{cc}\ \exists \xi \in \mathcal{C}^{cc} \text{ s.t. } \quad \mathbf{1}_{A_M} y = \mathbf{1}_{A_M} \xi,$$
$$\exists y \in \mathcal{D}^{cc} \text{ s.t. } \quad y \text{ is outside}|_{A_M^{\vdash}}\ \mathcal{C}^{cc},$$

and $A_M^{\vdash}$ is the $\mathbb{P}$-a.s. unique complement of A_M.

Suppose in addition that $\mathcal{D} = E$ and $\mathcal{C} = \mathcal{C}^{cc}$. Then the class $\mathcal{A}$ coincides with the class $\mathcal{A}(\mathcal{C}) = \{B \in \mathcal{G} \mid \mathbf{1}_A E = \mathbf{1}_A \mathcal{C}\}$ introduced in Notation 2. Henceforth: the maximal set of $\mathcal{A}(\mathcal{C})$ is $A_{\mathcal{C}} = A_M$; $D_{\mathcal{C}} = A_M^{\vdash}$; $\mathbf{1}_{A_{\mathcal{C}}} E = \mathbf{1}_{A_{\mathcal{C}}} \mathcal{C}$; and there exists $y \in E$ s.t. y is outside $\mathcal{C}$. If $x \notin \mathcal{C}$ then $\mathbb{P}(D_{\mathcal{C}}) > 0$ and $\chi = \{y \in E \mid y \text{ is outside } \mathcal{C}\} \neq \varnothing$.

Proof. The two classes $\mathcal{A}$ and $\mathcal{A}^{\vdash}$ are closed with respect to *countable* union. Indeed, for the family $\mathcal{A}^{\vdash}$, suppose that $B_i \in \mathcal{A}^{\vdash}$, $y_i \in \mathcal{D}^{cc}$ s.t. y_i is outside$|_{B_i}$ $\mathcal{C}^{cc}$. Define $\widetilde{B}_1 := B_1$, $\widetilde{B}_i := B_i \setminus B_{i-1}$, $B := \bigcup_{i=1}^{\infty} \widetilde{B}_i = \bigcup_{i=1}^{\infty} B_i$. Then y_i is outside$|_{\widetilde{B}_i}$ $\mathcal{C}^{cc}$, $\widetilde{B}_i$ are disjoint elements of $\mathcal{A}^{\vdash}$ and $y^* := \sum_1^{\infty} y_i 1_{\widetilde{B}_i} \in \mathcal{D}^{cc}$. Since $y_i 1_{\widetilde{B}_i} = y^* 1_{\widetilde{B}_i}$, y is outside$|_{\widetilde{B}_i}$ $\mathcal{C}^{cc}$ for all i and so y is outside$|_B$ $\mathcal{C}^{cc}$. Thus $B \in \mathcal{A}^{\vdash}$. Similarly for the class $\mathcal{A}$.

Remark 10 guarantees the existence of the two maximal sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$, so that: $B \in \mathcal{A}$ implies $B \subseteq A_M$; $B^{\vdash} \in \mathcal{A}^{\vdash}$ implies $B^{\vdash} \subseteq A_M^{\vdash}$.

Obviously, $P(A_M \cap A_M^{\vdash}) = 0$, as $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$. We claim that

$$P\big(A_M \cup A_M^{\vdash}\big) = 1. \tag{33}$$

To show (33) let $D := \Omega \setminus \{A_M \cup A_M^{\vdash}\} \in \mathcal{G}$. By contradiction suppose that $\mathbb{P}(D) > 0$. From $D \subseteq (A_M)^C$ and the maximality of A_M we get $D \notin \mathcal{A}$. This implies that there exists $y \in \mathcal{D}^{cc}$ such that

$$\mathbf{1}_D\{y\} \cap \mathbf{1}_D \mathcal{C}^{cc} = \varnothing \tag{34}$$

and obviously $y \notin \mathcal{C}^{cc}$, as $\mathbb{P}(D) > 0$. By Lemma 14 there exists a set $H_{\mathcal{C}^{cc},y} := H \in \mathcal{G}$ satisfying $\mathbb{P}(H) > 0$, (16) and (17) with $\mathcal{C}$ replaced by $\mathcal{C}^{cc}$.

Condition (17) implies that $H \in \mathcal{A}^{\vdash}$ and then $H \subseteq A_M^{\vdash}$. From (16) we deduce that there exists $\xi \in \mathcal{C}^{cc}$ s.t. $\mathbf{1}_A y = \mathbf{1}_A \xi$ for all $A \subseteq \Omega \setminus H$. Then (34) implies that D is not contained in $\Omega \setminus H$, so that $\mathbb{P}(D \cap H) > 0$. This is a contradiction since $D \cap H \subseteq D \subseteq (A_M^{\vdash})^C$ and $D \cap H \subseteq H \subseteq A_M^{\vdash}$.

The second part of the statement is a trivial consequence of the definitions. □

Lemma 18. *With the symbol $\trianglerighteq$ denote any one of the binary relations $\geqslant$, $\leqslant$, $=$, $>$, $<$ and with $\triangleleft$ its negation. Consider a class $F \subseteq \bar{L}^0(\mathcal{G})$ of random variables, $Y_0 \in \bar{L}^0(\mathcal{G})$ and the classes of sets*

$$\mathcal{A} := \{A \in \mathcal{G} \mid \forall Y \in F\ Y \trianglerighteq Y_0 \text{ on } A\},$$
$$\mathcal{A}^{\vdash} := \big\{A^{\vdash} \in \mathcal{G} \bigm| \exists Y \in F \text{ s.t. } Y \triangleleft Y_0 \text{ on } A^{\vdash}\big\}.$$

Suppose that for any sequence of disjoint sets $A_i^{\vdash} \in \mathcal{A}^{\vdash}$ and the associated r.v. $Y_i \in F$ we have $\sum_1^{\infty} Y_i 1_{A_i^{\vdash}} \in F$. Then there exist two maximal sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$ such that $P(A_M \cap A_M^{\vdash}) = 0$, $P(A_M \cup A_M^{\vdash}) = 1$ and

$$Y \trianglerighteq Y_0 \quad \text{on } A_M\ \forall Y \in F,$$
$$\bar{Y} \triangleleft Y_0 \quad \text{on } A_M^{\vdash} \text{ for some } \bar{Y} \in F.$$

Proof. Notice that $\mathcal{A}$ and $\mathcal{A}^{\vdash}$ are closed with respect to *countable* union. This claim is obvious for $\mathcal{A}$. For $\mathcal{A}^{\vdash}$, suppose that $A_i^{\vdash} \in \mathcal{A}^{\vdash}$ and that $Y_i \in F$ satisfies $P(\{Y_i \triangleleft Y_0\} \cap A_i^{\vdash}) = P(A_i^{\vdash})$. Defining $B_1 := A_i^{\vdash}$, $B_i := A_i^{\vdash} \setminus B_{i-1}$, $A_{\infty}^{\vdash} := \bigcup_{i=1}^{\infty} A_i^{\vdash} = \bigcup_{i=1}^{\infty} B_i$ we see that B_i are disjoint elements of $\mathcal{A}^{\vdash}$ and that $Y^* := \sum_1^{\infty} Y_i 1_{B_i} \in F$ satisfies $P(\{Y^* \triangleleft Y_0\} \cap A_{\infty}^{\vdash}) = P(A_{\infty}^{\vdash})$ and so $A_{\infty}^{\vdash} \in \mathcal{A}^{\vdash}$.

Remark 10 guarantees the existence of two sets $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$ such that:

(a) $P(A \cap (A_M)^C) = 0$ for all $A \in \mathcal{A}$,
(b) $P(A^{\vdash} \cap (A_M^{\vdash})^C) = 0$ for all $A^{\vdash} \in \mathcal{A}^{\vdash}$.

Obviously, $P(A_M \cap A_M^{\vdash}) = 0$, as $A_M \in \mathcal{A}$ and $A_M^{\vdash} \in \mathcal{A}^{\vdash}$. To show that $P(A_M \cup A_M^{\vdash}) = 1$, let $D := \Omega \setminus \{A_M \cup A_M^{\vdash}\} \in \mathcal{G}$. By contradiction suppose that $P(D) > 0$. As $D \subseteq (A_M)^C$, from condition (a) we get $D \notin \mathcal{A}$. Therefore, $\exists \bar{Y} \in F$ s.t. $P(\{\bar{Y} \trianglerighteq Y_0\} \cap D) < P(D)$, i.e. $P(\{\bar{Y} \triangleleft Y_0\} \cap D) > 0$. If we set $B := \{\bar{Y} \triangleleft Y_0\} \cap D$ then it satisfies $P(\{\bar{Y} \triangleleft Y_0\} \cap B) = P(B) > 0$ and, by definition of $\mathcal{A}^{\vdash}$, B belongs to $\mathcal{A}^{\vdash}$. On the other hand, as $B \subseteq D \subseteq (A_M^{\vdash})^C$, $P(B) = P(B \cap (A_M^{\vdash})^C)$, and from condition (b) $P(B \cap (A_M^{\vdash})^C) = 0$, which contradicts $P(B) > 0$. □

References

[1] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, Coherent measures of risk, Math. Finance 4 (1999) 203–228.

[2] R.I. Bot, N. Lorenz, G. Wanka, Dual representation for convex risk measures via conjugate duality, J. Optim. Theory Appl. 144 (2) (2010) 185–203.
[3] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, L. Montrucchio, Complete monotone quasiconcave duality, Math. Oper. Res. 36 (2009) 321–339.
[4] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, L. Montrucchio, Risk measures: rationality and diversification, Math. Finance 21 (2010) 743–774.
[5] A. Daniilidis, J.-E. Martínez-Legaz, Characterizations of evenly convex sets and evenly quasiconvex functions, J. Math. Anal. Appl. 273 (2002) 58–66.
[6] S. Drapeau, M. Kupper, Risk preferences and their robust representation, Math. Oper. Res. 38 (1) (2013).
[7] N. El Karoui, C. Ravanelli, Cash sub-additive risk measures and interest rate ambiguity, Math. Finance 19 (4) (2009) 561–590.
[8] W. Fenchel, A remark on convex sets and polarity, Medd. Lunds Univ. Mat. Sem. (1952) 82–89.
[9] D. Filipovic, M. Kupper, N. Vogelpoth, Separation and duality in locally L^0-convex modules, J. Funct. Anal. 256 (12) (2009) 3996–4029.
[10] D. Filipovic, M. Kupper, N. Vogelpoth, Approaches to conditional risk, SIAM J. Financial Math. 3 (1) (2013) 402–432.
[11] H. Föllmer, A. Shied, Convex measures of risk and trading constraints, Finance Stoch. 6 (2002) 429–447.
[12] H. Föllmer, A. Shied, Stochastic Finance. An Introduction in Discrete Time, 3rd ed., de Gruyter Stud. Math., vol. 27, 2011.
[13] M. Frittelli, M. Maggis, Dual representation of quasiconvex conditional maps, SIAM J. Financial Math. 2 (2011) 357–382.
[14] M. Frittelli, M. Maggis, Complete duality for quasiconvex dynamic risk measures on modules of the L^p-type, Stat. Risk Model. (2013), forthcoming.
[15] M. Frittelli, E. Rosazza Gianin, Putting order in risk measures, J. Bank. Financ. 26 (7) (2002) 1473–1486.
[16] T.X. Guo, Relations between some basic results derived from two kinds of topologies for a random locally convex module, J. Funct. Anal. 258 (2010) 3024–3047.
[17] V. Klee, E. Maluta, C. Zanco, Basic properties of evenly convex sets, J. Convex Anal. 14 (1) (2007) 137–148.
[18] J.E. Martínez-Legaz, J. Vicente-Pérez, The e-support function of an e-convex set and conjugacy for e-convex functions, J. Math. Anal. Appl. 376 (2) (2011) 602–612.
[19] J.P. Penot, M. Volle, On quasi-convex duality, Math. Oper. Res. 15 (4) (1990) 597–625.
[20] G.Ch. Pflug, Subdifferential representation of risk measures, Math. Program. 108 (2–3) (2007) 339–354.
[21] M.M.L. Rodríguez, J. Vicente-Pérez, On evenly convex functions, J. Convex Anal. 18 (3) (2011) 721–736.
[22] A. Ruszczyinski, A. Shapiro, Optimization of convex risk functions, Math. Oper. Res. 31 (3) (2006) 433–452.
[23] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, Elsevier/North-Holland, New York, 1983.