FINITE STRAIN SOUZA-AURICCHIO MODEL FOR SHAPE-MEMORY ALLOYS

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Abstract. We address the analysis of a constitutive model for the evolution of a shape-memory alloys at finite strains. The model has been presented in [20] and corresponds to a suitable finite-strain version of the celebrated Souza-Auricchio model for SMAs [4, 46]. We reformulate the model in purely variational fashion under the form of a rate-independent process. Existence of suitably weak (energetic) solutions to the model is obtained by passing to the limit within a constructive time-discretization procedure.

1. Introduction

Shape-memory alloys (SMAs) show an abrupt diffusionless stress and temperature driven martensitic phase change. At the macroscopic level, this results in an amazing thermomechanical behavior. At a certain (suitably high) temperature regimes these materials are *superelastic*, namely, they recover comparably large deformations during mechanical loading-unloading cycles. At lower temperatures severely deformed specimens regain their original shape after a thermal treatment. This is the so-called *shape-memory effect* [23].

The SMAs specific thermomechanical behavior is not present (at least to a comparable extent) in materials traditionally used in Engineering and is at the basis of a variety of innovative applications to aeronautical, structural, earthquake, and biomedical technologies [15, 16], just to mention a few hot topics. This fact motivated the recent intense research toward a comprehensive and effective the description of the complex thermomechanical behavior of SMAs. The relevant Engineering literature on SMAs modeling is vast and a whole menagerie of models have been advanced having ambitions for different ranges of applicability (from lab single-crystal experiments to commercially exploitable tools) and different abilities to fit particular experiments and to explain microstructures, stress/strain relations, or hysteresis. In the macroscopic-phenomenologial setting, the reader shall be minimally referred to [10, 21, 22, 24, 26, 29, 30, 32, 42, 43, 44, 48, 49] and to the survey [45].

We are here concerned here with a phenomenological, internal-variable-type model for polycrystalline shape-memory bosy originally advanced in the *small-strain regime* by Souza, Mamiya, & Zouain [46] and then combined with finite elements by Auricchio & Petrini [4, 5, 6], hence referred to as *Souza-Auricchio* in the following. This model, describe very efficiently both the superelastic and the shape-memory behavior. Moreover it shows a remarkable robustness with respect to parameters and discretizations despite its *simplicity*: the constitutive behavior of the material is determined by 8 easy-fitted material parameters (note that linearized thermo-plasticity with linear hardening already requires 5 material parameters). The Souza-Auricchio model has a sound mathematical foundation: it has been analyzed from the viewpoint of existence and approximation of solutions of the isothermal three-dimensional quasi-static evolution problem in [3] and convergence rates for space-time discretizations [38, 39]. Results in the non-isothermal case have also been obtained [18, 27, 28, 40, 37]. Moreover,

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extensions of the original model to residual plasticity [7, 8, 17], more realistic non-symmetric behaviors and transformation-dependent material parameters [9], and the ferromagnetic shape-memory effect [1, 2, 11, 12, 47] are also available.

This note is focused on the *finite-strain* version of the Souza-Auricchio model advanced by Evangelista, Marfia, & Sacco [20]. Note that shape-memory devices are generally engineered in order to exploit activation strains up to 8%. This sets clearly beyond the possible reach of linearized elasticity theory calling for a finite-strain analysis instead. In the paper [20] the Authors propose a finite-strain choice for the free energy of the material, a specific flow rule for the internal variable, and formally verify the recovery of the small-strain Souza-Auricchio model upon strain linearization. Moreover, some comments on algorithmical aspects of a possible fully-implicit time-discretization of the constitutive relation and the integration of the latter in some finite elements test is reported.

A first issue of this paper is that of reformulating the finite-strain model of [20] in a purely variational fashion. This complements the analysis of [20] where material relations (in particular, the flow rule) are presented via the classical complementary formalism. The variational reformulation of the model bears a clear mathematical advantage as it provides the possibility of recasting the constitutive relation in terms of symmetric right Cauchy-Green tensors only. In particular, this amounts in reproducing at the finite-strain level the remarkable agreement of the original small-strain Souza-Auricchio model with the structure of generalized plasticity theory.

Secondly, the variational reformulation of the model allows us to settle the material constitutive problem within the now quite developed theory of rate-independent processes. This in turn provides the possibility of proving that a time-discretization of the material constitutive relation is actually convergent with respect to the time-step size. In particular, the material consitutive relation is proved to admit a suitaby defined variational solution, the so-called *energetic solution* [41]. Note that no convergence nor existence proof is reported in [20].

We directly develop our analysis in the non-isothermal regime, by however letting the temperature of the body being given a-priori. This assumption seems justified in case of a shape-memory body which is thin in at least one direction and for slowly varying loads. In this case, one could assume that the heat which is mechanically produced within the sample gets immediately transfered to the surrounding heat bath. This perspective is followed in [18, 40, 37] in the small-strain regime. On the contrary, a fully thermomechanically coupled problem for the small-strain model is solved in one spatial dimension in [27, 28]. Note that the smoothness on the given teperature here assumed is weaker than the corresponding one from [40, 37] exactly in the same spirit as in [18].

The paper is organized as follows. In Section 2 we recall the basics of the model from [20] and comment on dissipativity whereas Section 3 is devoted to the *variational reformulation* of the model. Then, Section 4 brings to the convergence proof of time-incremental approximations and, in particular, to the existence of energetic solutions.

2. The model

The aim of this section is that of recalling the model from [20]. We shall limit ourselves to mention the essential modeling features and notation. The Reader is instead referred to [20] for additional details and comments as well as some numerical evidence of the performance of the model.

2.1. **Tensors.** We shall use bold capital letters for 2-tensors in $\mathbb{R}^{d\times d}$ ($d = 2, 3$) and double capitals (e.g., $\mathbb{T}$) for 4-tensors. Let $\mathbb{R}^{d\times d}$ be the space of 2-tensors endowed with the natural scalar product $\boldsymbol{A}{:}\boldsymbol{B} \doteq A_{ij}B_{ij}$ (summation convention) and the corresponding norm $|\boldsymbol{A}|^2 \doteq \boldsymbol{A}{:}\boldsymbol{A}$ for all $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{R}^{d\times d}$.The symbol $\mathbb{R}^{d\times d}_{\rm sym}$ denotes the subspace of symmetric tensors. The space $\mathbb{R}^{d\times d}_{\rm sym}$ is orthogonally decomposed as $\mathbb{R}^{d\times d}_{\rm sym} = \mathbb{R}^{d\times d}_{\rm dev} \oplus \mathbb{R}\,\mathbf{1}$, where $\mathbb{R}\,\mathbf{1}$ is the subspace spanned by the identity 2-tensor $\mathbf{1}$ and $\mathbb{R}^{d\times d}_{\rm dev}$ is the subspace of symmetric and deviatoric tensors. In particular, for all $\boldsymbol{A} \in \mathbb{R}^{d\times d}_{\rm sym}$, we let $\operatorname{tr}\boldsymbol{A} \doteq \boldsymbol{A}{:}\mathbf{1}$ and $\operatorname{dev}\boldsymbol{A} \doteq \boldsymbol{A} - (\operatorname{tr}\boldsymbol{A})\mathbf{1}/d$ so that $\boldsymbol{A} = \operatorname{dev}\boldsymbol{A} + (\operatorname{tr}\boldsymbol{A})\mathbf{1}/d$. On the other hand, the identity 4-tensor $\mathbb{I}$ is given by $\mathbb{I}_{ij\ell k} \doteq \delta_{i\ell}\delta_{jk}$ and we have $\mathbb{I}{:}\boldsymbol{A} = \boldsymbol{A}{:}\mathbb{I} = \boldsymbol{A}$. We shall use the classical notation

$$\begin{aligned}
&\mathrm{GL}_+(d) \doteq \{\boldsymbol{A} \in \mathbb{R}^{d\times d} \mid \det\boldsymbol{A} > 0\},\\
&\mathrm{SL}(d) \doteq \{\boldsymbol{A} \in \mathbb{R}^{d\times d} \mid \det\boldsymbol{A} = 1\}, \quad \mathrm{SO}(d) \doteq \{\boldsymbol{A} \in \mathrm{SL}(d) \mid \boldsymbol{A}^\top\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^\top = \mathbf{1}_2\}.
\end{aligned}$$

The 2-tensors $\mathbb{T}{:}\boldsymbol{B}$ and $\boldsymbol{B}{:}\mathbb{T}$ are classically defined by $(\boldsymbol{B}{:}\mathbb{T})_{ik} := B_{jl}T_{j\ell ik}$ and $(\mathbb{T}{:}\boldsymbol{B})_{ik} := T_{ik\ell j}B_{\ell j}$, respectively. Given the smooth function $\boldsymbol{B} \mapsto \boldsymbol{A}(\boldsymbol{B})$ we shall denote by $\partial_{\boldsymbol{B}}\boldsymbol{A}$ the 4-tensor defined by $(\partial_{\boldsymbol{B}}\boldsymbol{A})_{ikj\ell} := \partial_{B_{j\ell}}A_{ik}$ or, equivalently,

$$\partial_{\boldsymbol{B}}\boldsymbol{A}{:}\boldsymbol{C} := \frac{\mathrm{d}}{\mathrm{d}\alpha}\boldsymbol{A}(\boldsymbol{B}+\alpha\boldsymbol{C})\Big|_{\alpha=0} \qquad \forall \boldsymbol{C} \in \mathbb{R}^{d\times d}.$$

In particular, we have that

$$(\partial_{\boldsymbol{B}}\boldsymbol{A}{:}\boldsymbol{C})_{ik} = \partial_{B_{j\ell}}A_{ik}C_{j\ell}, \qquad (\boldsymbol{C}{:}\partial_{\boldsymbol{B}}\boldsymbol{A})_{ik} = C_{j\ell}{:}\partial_{B_{ik}}A_{j\ell},$$

for all $\boldsymbol{C} \in \mathbb{R}^{d\times d}$. Note that the associative property $(\boldsymbol{C}{:}\partial_{\boldsymbol{B}}\boldsymbol{A}){:}\boldsymbol{D} = \boldsymbol{C}{:}(\partial_{\boldsymbol{B}}\boldsymbol{A}{:}\boldsymbol{D})$ holds for every $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{d\times d}$, and hence we can write $\boldsymbol{C}{:}\partial_{\boldsymbol{B}}\boldsymbol{A}{:}\boldsymbol{D}$ without ambiguity. Note also that $\partial_{\boldsymbol{A}}\boldsymbol{A} = \mathbb{I}$. The 4-tensor $\mathbb{T}_1{:}\mathbb{T}_2$ is defined by $(\mathbb{T}_1{:}\mathbb{T}_2)_{ijk\ell} := (\mathbb{T}_1)_{ijmn}(\mathbb{T}_2)_{mnk\ell}$. In particular, if $\boldsymbol{A}$ is a function of $\boldsymbol{B}$ and $\boldsymbol{C}$ is a function of $\boldsymbol{D}$ (all 2-tensors), we have

$$(\partial_{\boldsymbol{B}}\boldsymbol{A}{:}\partial_{\boldsymbol{D}}\boldsymbol{C})_{ikj\ell} = \partial_{B_{mn}}A_{ik}\partial_{D_{j\ell}}C_{mn}.$$

We have that $\boldsymbol{A}{:}(\mathbb{T}_1{:}\mathbb{T}_2) = (\boldsymbol{A}{:}\mathbb{T}_1){:}\mathbb{T}_2$ and $(\mathbb{T}_1{:}\mathbb{T}_2){:}\boldsymbol{B} = \mathbb{T}_1{:}(\mathbb{T}_2{:}\boldsymbol{B})$ for every $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}$ so that brackets can be omitted. Finally, the product derivative formulas

$$\partial_{\boldsymbol{A}}(\boldsymbol{B}\boldsymbol{C}){:}\boldsymbol{D} = (\partial_{\boldsymbol{A}}\boldsymbol{B}{:}\boldsymbol{D})\boldsymbol{C} + \boldsymbol{B}(\partial_{\boldsymbol{A}}\boldsymbol{C}{:}\boldsymbol{D}) \tag{2.1}$$
eq:2

$$\boldsymbol{B}{:}\partial_{\boldsymbol{A}}(\boldsymbol{C}\boldsymbol{D}) = \boldsymbol{B}\boldsymbol{D}^\top{:}\partial_{\boldsymbol{A}}\boldsymbol{C} + \boldsymbol{C}^\top\boldsymbol{B}{:}\partial_{\boldsymbol{A}}\boldsymbol{D} \tag{2.2}$$
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will turn out to be useful in the following.

2.2. **Deformation gradient.** Let $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^d$ denote the deformation of the SMA sample from the reference configuration $\Omega \subset \mathbb{R}^d$. We shall classically decompose the deformation gradient $\mathrm{D}\boldsymbol{\varphi}$ as [31] Lee69

$$\mathrm{D}\boldsymbol{\varphi} = \boldsymbol{F} = \boldsymbol{F}_{\rm el}\boldsymbol{F}_{\rm tr}$$

where $\boldsymbol{F}_{\rm el}$ stands for the *elastic* deformation gradient whereas $\boldsymbol{F}_{\rm tr}$ denotes the inelastic (or *transformation*) part and is constrained to $\boldsymbol{F}_{\rm tr} \in \mathrm{SL}(d)$ in order to encode classical *plastic incompressibility.*

In order to formulate the model, we will rely on the *right Cauchy-Green* deformation tensor

$$\boldsymbol{C} \doteq \boldsymbol{F}^\top\boldsymbol{F}$$

and the corresponding elastic and inelastic analogues

$$\boldsymbol{C}_{\rm el} \doteq \boldsymbol{F}_{\rm el}^\top\boldsymbol{F}_{\rm el}, \quad \boldsymbol{C}_{\rm tr} \doteq \boldsymbol{F}_{\rm tr}^\top\boldsymbol{F}_{\rm tr}.$$

In particular, one shall note that $\boldsymbol{C}_{\rm tr} \in \mathrm{SL}_{\rm sym}(d) \doteq \mathrm{SL}(d) \cap \mathbb{R}^{d\times d}_{\rm sym}$.

2.3. **Free energy.** The free energy of the body is assumed to be decomposed as

$$\psi \doteq \psi_{\rm el}(\boldsymbol{C}_{\rm el}) + \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta). \tag{2.3}$$

Here, θ denotes the absolute temperature of the sample, $\psi_{\rm el}$ is the *elastic part* of the free energy whereas $\psi_{\rm tr}$ stands for the *inelastic part*.

As for the elastic part of the energy, we shall simply assume it to be smooth far from the set where $\det \boldsymbol{F} = \det \boldsymbol{F}_{\rm el} \leq 0$, frame indifferent, and isotropic. In particular, $\psi_{\rm el}$ is derived from an elastic potential $W \in C^1({\rm GL}_+(d))$ via $\psi_{\rm el}(\boldsymbol{C}_{\rm el}) \doteq W(\boldsymbol{F}_{\rm el})$ with $W : \mathbb{R}^{d\times d} \to [0,\infty]$ such that

$$W(\boldsymbol{R}\boldsymbol{F}_{\rm el}) = W(\boldsymbol{F}_{\rm el}) \quad \forall \boldsymbol{R} \in {\rm SO}(d),\ \boldsymbol{F}_{\rm el} \in \mathbb{R}^{d\times d}, \tag{2.4}$$

$$W(\boldsymbol{F}_{\rm el}\widehat{\boldsymbol{R}}) = W(\boldsymbol{F}_{\rm el}) \quad \forall \widehat{\boldsymbol{R}} \in {\rm SO}(d),\ \boldsymbol{F}_{\rm el} \in \mathbb{R}^{d\times d}, \tag{2.5}$$

$$|\partial_{\boldsymbol{F}_{\rm el}} W(\boldsymbol{F}_{\rm el})\boldsymbol{F}_{\rm el}^\top| \leq c_0(W(\boldsymbol{F}_{\rm el}) + 1). \tag{2.6}$$

The above assumptions are nothing but frame-indifference (2.4), isotropy (2.5), and the controllability of the Kirchhoff tensor (2.6). In particular, they are completely compatible with the standard polyconvexity frame. Note that the functional W can be additionally asked to fulfill

$$W(\boldsymbol{F}_{\rm el}) \to \infty \quad \text{for} \quad \det \boldsymbol{F}_{\rm el} \to 0^+.$$

The specific form for the inelastic energy $\psi_{\rm tr}$ is chosen as

$$\psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta) \doteq b(\theta - \theta_M)^+ |\boldsymbol{E}_{\rm tr}| + \frac{h}{2}|\boldsymbol{E}_{\rm tr}|^2 + I_{\epsilon_L}(\boldsymbol{E}_{\rm tr}) \tag{2.7}$$

where $\boldsymbol{E}_{\rm tr}$ is the inelastic *Green-St. Venant* tensor

$$\boldsymbol{E}_{\rm tr} \doteq \frac{1}{2}(\boldsymbol{C}_{\rm tr} - \boldsymbol{1}_2),$$

$b > 0$, $\theta_M > 0$ is the critical martensite-austenite transition temperature at zero stress, $h > 0$ is a kinematic hardening modulus, and I_{ϵ_L} is the indicator function of the set $\{\boldsymbol{E}_{\rm tr} \in \mathbb{R}^{d\times d} \mid |\boldsymbol{E}_{\rm tr}| \leq \epsilon_L\}$ with $\epsilon_L > 0$ representing a measure of the maximal strain which is obtainable via martensitic variants reorientation. In particular, we have that

$$I_{\epsilon_L}(\boldsymbol{E}_{\rm tr}) \doteq \begin{cases} 0 & \text{if } |\boldsymbol{E}_{\rm tr}| \leq \epsilon_L \\ \infty & \text{else.} \end{cases}$$

We shall stick to the specific form (2.7) for the inelastic energy for the sake of reference with the original model in [20]. Still, let us mention that the considerations developed hereafter apply to a much broader class of energies, possibly including kinematic-hardening finite-strain plasticity.

Throughout this work we shall assume the absolute temperature $t \mapsto \theta(t)$ to be given. This in particular motivates the absence of the classical purely caloric term $c_V\theta(1 - \ln\theta)$ in the expression of the free energy (2.3) (c_V being the specific heat). Note that, even by including such term into the free energy, the model is not directly suited for describing fully coupled thermomechanical evolutions as the choice (2.7) (which is motivated in the isothermal frame) would prescribe the non-monotone temperature-entropy relation

$$s \doteq -\partial_\theta \psi = c_V \ln\theta - bH(\theta - \theta_M)|\boldsymbol{E}_{\rm tr}|$$

where H stands for the classical Heaviside function. Further details on this issue are to be found in [27, 28].

2.4. **Clausius-Duhem inequality.** Let us complement the analysis of [20] and provide some detail in the direction of the Clausius-Duhem inequality for the body. In particular, this reduces to check that, at least for smooth evolutions,

$$-\dot\psi - s\dot\theta + \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}} - \boldsymbol{q}\cdot\frac{\nabla\theta}{\theta} \geq 0 \tag{2.8}$$

where s is the *entropy*, $\boldsymbol{S}$ is the *second Piola-Kirchhoff stress tensor*, and $\boldsymbol{q}$ is the *heat flux*. With the aid of (2.3), we manipulate the term $-\dot\psi + \boldsymbol{S}{:}\dot{\boldsymbol{C}}/2$ in the above left-hand side as follows

$$\begin{aligned}
-\dot\psi + \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}} &= -\frac{\mathrm{d}}{\mathrm{d}t}\big(\psi_{\rm el}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}) + \psi_{\rm tr}(\boldsymbol{C}_{\rm tr},\theta)\big) + \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}}\\
&= -\partial_\theta\psi\dot\theta - \partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\big(-\boldsymbol{F}_{\rm tr}^{-\top}\dot{\boldsymbol{F}}_{\rm tr}^{\top}\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1} + \boldsymbol{F}_{\rm tr}^{-\top}\dot{\boldsymbol{C}}\boldsymbol{F}_{\rm tr}^{-1} - \boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}\dot{\boldsymbol{F}}_{\rm tr}\boldsymbol{F}_{\rm tr}^{-1}\big)\\
&\quad - \partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}{:}\dot{\boldsymbol{C}}_{\rm tr} + \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}}\\
&= -\partial_\theta\psi\dot\theta + \Big(\boldsymbol{S} - 2\boldsymbol{F}_{\rm tr}^{-1}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top}\Big){:}\frac{1}{2}\dot{\boldsymbol{C}}\\
&\quad + \partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{C}_{\rm el}{:}\boldsymbol{F}_{\rm tr}^{-\top}\dot{\boldsymbol{F}}_{\rm tr}^{\top} + \boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\dot{\boldsymbol{F}}_{\rm tr}\boldsymbol{F}_{\rm tr}^{-1} - \partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}{:}\dot{\boldsymbol{C}}_{\rm tr}\\
&= -\partial_\theta\psi\dot\theta + \Big(\boldsymbol{S} - 2\boldsymbol{F}_{\rm tr}^{-1}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top}\Big){:}\frac{1}{2}\dot{\boldsymbol{C}}\\
&\quad + \Big(2\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top} - 2\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}\Big){:}\frac{1}{2}\dot{\boldsymbol{C}}_{\rm tr}
\end{aligned}\tag{2.9}$$

where we used the coaxiality of $\boldsymbol{C}_{\rm el}$ and $\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}$ (which in turn follows from isotropy (2.5)), the identity $\dot{\boldsymbol{C}}_{\rm tr} = \dot{\boldsymbol{F}}_{\rm tr}^{\top}\boldsymbol{F}_{\rm tr} + \boldsymbol{F}_{\rm tr}^{\top}\dot{\boldsymbol{F}}_{\rm tr}$, and the short-hand notation $\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el} = \partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1})$ and $\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr} = \partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}(\boldsymbol{C}_{\rm tr},\theta)$ (here and in the following). We shall systematically make use of the symbol ∂ for the *subdifferential* of a convex function [13]. In particular, given any $\phi:\mathbb{R}^{d\times d}\to(-\infty,\infty]$ convex, we denote its *subdifferential* $\partial\phi$ as

$$\boldsymbol{B}\in\partial\phi(\boldsymbol{A}) \quad\Longleftrightarrow\quad \phi(\boldsymbol{A})<\infty \text{ and } \boldsymbol{B}{:}(\boldsymbol{T}-\boldsymbol{A}) \leq \phi(\boldsymbol{T}) - \phi(\boldsymbol{A}) \quad \forall \boldsymbol{T}\in\mathbb{R}^{d\times d}.$$

Note incidentally that, as $\psi_{\rm tr}$ is non-smooth, the symbol $\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}$ represents *a selection* in the convex set $\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}(\boldsymbol{C}_{\rm tr},\theta)$.

2.5. **Constitutive relations and flow rule.** We let the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ be given by

$$\boldsymbol{S} \doteq 2\boldsymbol{F}_{\rm tr}^{-1}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top}$$

and define

$$\boldsymbol{T} \doteq \boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{M}\boldsymbol{F}_{\rm tr}^{-\top} - \boldsymbol{A} \tag{2.10}$$

where

$$\boldsymbol{M} \doteq 2\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}, \quad \boldsymbol{A} \doteq 2\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr} \tag{2.11}$$

are the so-called *Mandel stress* and *back stress*, respectively [44]. Owing to the computation in (2.9), the tensor $\boldsymbol{T}$ is the thermodynamic force associated with the internal variable $\boldsymbol{C}_{\rm tr}$ for we have that

$$2\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top} - 2\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr} = \boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{M}\boldsymbol{F}_{\rm tr}^{-\top} - \boldsymbol{A} = \boldsymbol{T}.$$

Let us stress that both the Mandel stress $\boldsymbol{M}$ and the back stress $\boldsymbol{A}$ are symmetric ($\boldsymbol{M}$ by the coaxiality of $\boldsymbol{C}_{\rm el}$ and $\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}$ and $\boldsymbol{A}$ by the current choice of $\psi_{\rm tr}$) so that $\boldsymbol{T} = \boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{M}\boldsymbol{F}_{\rm tr}^{-\top} - \boldsymbol{A}$ is symmetric as well.

The dissipative evolution of the material is prescribed by the associative *flow rule*

$$\dot{\boldsymbol{C}}_{\rm tr} = \dot\zeta \partial_{\boldsymbol{T}} f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T}). \tag{2.12}$$

for some *yield function* $f : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}$, such that for all $\boldsymbol{F}_{\rm tr} \in \mathbb{R}^{d\times d}$ the function $f(\boldsymbol{F}_{\rm tr}, \cdot)$ is convex and positively 1-homogeneous ($f(\boldsymbol{F}_{\rm tr}, \lambda \boldsymbol{T}) = \lambda f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T})$ for all $\lambda \geq 0$), along with the classical Kuhn-Tucker complementarity conditions

$$\dot\zeta \geq 0, \quad f \leq 0, \quad \dot\zeta f = 0. \tag{2.13}$$

Following [19, 20], we focus on the specific choice for yield function f given by

$$f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T}) = |{\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top)| - r \tag{2.14}$$

where $r > 0$ plays the role of a given *yield stress*.

Along with this choice, the flow rule (2.12)-(2.13) reads

$$\dot{\boldsymbol{C}}_{\rm tr} \in \begin{cases} \dot\zeta \boldsymbol{F}_{\rm tr}^\top \dfrac{{\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top)}{|{\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top)|}\boldsymbol{F}_{\rm tr} & \text{if } {\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top) \neq 0, \\ \dot\zeta \boldsymbol{F}_{\rm tr}^\top \{\boldsymbol{B} \in \mathbb{R}^{d\times d}_{\rm dev} \cap \mathbb{R}^{d\times d}_{\rm sym} \ : \ |\boldsymbol{B}| \leq 1\}\boldsymbol{F}_{\rm tr} & \text{if } {\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top) = 0. \end{cases} \tag{2.15}$$

Note that all evolutions $t \mapsto \boldsymbol{C}_{\rm tr}(t)$ fulfilling the latter flow rule and starting from a symmetric initial datum will stay symmetric for all times.

2.6. **Dissipativity.** The above choices for constitutive relations and flow rule entail the validity of the Clausius-Duhem inequality (2.8), at least for smooth evolutions. Indeed, we may equivalently rewrite (2.12)-(2.13) or (2.15) as

$$\boldsymbol{T} \in \partial_{\dot{\boldsymbol{C}}_{\rm tr}} R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \tag{2.16}$$

where $R(\boldsymbol{F}_{\rm tr}, \cdot) \doteq f^*(\boldsymbol{F}_{\rm tr}, \cdot)$ is the classical Legendre conjugate of $f(\boldsymbol{F}_{\rm tr}, \cdot)$. Namely, by the 1-homogeneity of f, the function $R(\boldsymbol{F}_{\rm tr}, \cdot)$ is the *support function* of the convex set $\{\boldsymbol{T} \in \mathbb{R}^{d\times d}_{\rm sym} \mid f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T}) \leq 0\}$. In particular,

$$R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \doteq \sup\{\boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm tr} \mid f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T}) \leq 0\}.$$

Namely, $R(\boldsymbol{F}_{\rm tr}, \boldsymbol{0}) = 0$ and we have that, for all $(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr})$ with $R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) < \infty$,

$$\partial_{\dot{\boldsymbol{C}}_{\rm tr}} R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \doteq \Big\{\boldsymbol{T} \in \mathbb{R}^{d\times d}_{\rm sym} \mid \boldsymbol{T}{:}(\boldsymbol{B} - \dot{\boldsymbol{C}}_{\rm tr}) \leq R(\boldsymbol{F}_{\rm tr}, \boldsymbol{B}) - R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \ \ \forall \boldsymbol{B} \in \mathbb{R}^{d\times d}\Big\}.$$

Since the right-hand side of (2.9) equals $s\dot\theta + \boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm tr}/2$ by definition of the entropy $s \doteq -\partial_\theta \psi$ and we have

$$\boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm tr} \overset{(2.16)}{\geq} R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) - R(\boldsymbol{F}_{\rm tr}, \boldsymbol{0}) = R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \geq 0,$$

by choosing the classical *Fourier* flux $\boldsymbol{q} \doteq -\kappa\nabla\theta, \kappa > 0$ we have that

$$-\dot\psi - s\dot\theta + \boldsymbol{S}{:}\frac{1}{2}\dot{\boldsymbol{C}} - \boldsymbol{q}\cdot\frac{\nabla\theta}{\theta} \geq \frac{1}{2}R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) + \kappa\nabla\theta\cdot\frac{\nabla\theta}{\theta} \geq 0.$$

In particular, the Clausius-Duhem inequality holds true.

3. Variational reformulation

As already commented, one of the main issues of this paper is that of reformulating the constitutive problem in a purely variational fashion. This yields a clear advantage with respect to the original formulation of [19, 20]. In particular, owing to the variational setting we are able to prove the convergence of the time-discretization scheme. Moreover, we shall be using the variables $\theta, \boldsymbol{C}$ (given) and $\boldsymbol{C}_{\rm tr}$ (unknown) only and the model with turn out to be not directly depending on $\boldsymbol{F}_{\rm tr}$. This possibility is indeed the effect of the overall isotropy and frame-indifference assumptions and, although not new in the setting of finite strain plasticity, it was not yet pointed out in connection with shape-memory alloys. Eventually, as an effect of this variational reformulation we will be able to exhibit an existence theory in Section 4 below.

The first step in the direction of reformulating variationally the model consists in avoiding the use of the tensor $\boldsymbol{C}_{\rm el}$ in the definition of the free energy ψ (2.3). By recalling that

$$\boldsymbol{F}_{\rm tr} = \boldsymbol{R}\boldsymbol{C}_{\rm tr}^{1/2} = \boldsymbol{C}_{\rm tr}^{1/2}\widehat{\boldsymbol{R}} \quad \text{for some} \quad \boldsymbol{R},\, \widehat{\boldsymbol{R}} \in {\rm SO}(d) \tag{3.1}$$

RR

we have that

$$\psi_{\rm el}(\boldsymbol{C}_{\rm el}) = \psi_{\rm el}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}) = \psi_{\rm el}(\boldsymbol{R}^{-\top}\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{R}^{-1}) = \psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}). \tag{3.2}$$

revised

The last equality above is a consequence of (2.4)-(2.5). Indeed, let for brevity $\boldsymbol{S} = \boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2} = \boldsymbol{O}^\top\boldsymbol{D}\boldsymbol{O}$ with $\boldsymbol{O} \in {\rm SO}(d)$ and $\boldsymbol{D}$ diagonal and positive definite. Hence

$$\psi_{\rm el}(\boldsymbol{R}^{-\top}\boldsymbol{S}\boldsymbol{R}^{-1}) = \psi_{\rm el}(\boldsymbol{R}^{-\top}\boldsymbol{O}^\top\boldsymbol{D}\boldsymbol{O}\boldsymbol{R}^{-1}) = W(\boldsymbol{R}^{-\top}\boldsymbol{O}^\top\boldsymbol{D}^{1/2}\boldsymbol{O}\boldsymbol{R}^{-1}) \stackrel{(2.4)+(2.5)}{=} W(\boldsymbol{O}^\top\boldsymbol{D}^{1/2}\boldsymbol{O}) = \psi_{\rm el}(\boldsymbol{S})$$

whence (3.2) follows. Let us now define

$$\widehat{\psi}(\boldsymbol{C}, \boldsymbol{C}_{\rm tr}, \theta) \doteq \psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}) + \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta).$$

Let us now compute $R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr})$ from the specific form of $f(\boldsymbol{F}_{\rm tr}, \boldsymbol{T})$ in (2.14) as

$$\begin{aligned}
R(\boldsymbol{F}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) &= \sup\Big\{\boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm tr} \ \Big|\ \boldsymbol{T} \in \mathbb{R}^{d\times d}_{\rm sym},\ |{\rm dev}\,(\boldsymbol{F}_{\rm tr}\boldsymbol{T}\boldsymbol{F}_{\rm tr}^\top)| \le r\Big\}\\
&= \sup\Big\{\boldsymbol{T}{:}\dot{\boldsymbol{C}}_{\rm tr} \ \Big|\ \boldsymbol{T} \in \mathbb{R}^{d\times d}_{\rm sym},\ |{\rm dev}\,(\boldsymbol{C}_{\rm tr}^{1/2}\boldsymbol{T}\boldsymbol{C}_{\rm tr}^{1/2})| \le r\Big\}\\
&= \sup\Big\{\boldsymbol{B}{:}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2} \ \Big|\ \boldsymbol{B} \in \mathbb{R}^{d\times d}_{\rm sym},\ |{\rm dev}\,\boldsymbol{B}| \le r\Big\}\\
&= \begin{cases} r|{\rm dev}\,(\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2})| & \text{if } {\rm tr}\,(\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2}) = 0\\ \infty & \text{else.}\end{cases}\\
&= \begin{cases} r|\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2}| & \text{if } {\rm tr}\,(\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2}) = 0\\ \infty & \text{else.}\end{cases}
\end{aligned}$$

We hence define the *dissipation* $\widehat{R} : \mathbb{R}^{d\times d}_{\rm sym} \times \mathbb{R}^{d\times d}_{\rm sym} \to [0,\infty]$ as (the meaning of the factor $1/2$ will become clear in the proof of Theorem 3.1 below)

$$\widehat{R}(\boldsymbol{C}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \doteq \begin{cases} \dfrac{r}{2}|\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2}| & \text{if } {\rm tr}\,(\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}_{\rm tr}\boldsymbol{C}_{\rm tr}^{-1/2}) = 0\\ \infty & \text{else.}\end{cases}$$

The core of this section resides in the following result where we prove that one can rewrite the constitutive material relation from (2.10)-(2.13) for the internal variable $\boldsymbol{C}_{\rm tr}$ in terms of $\widehat{R}$ and $\widehat{\psi}$ only.

represent **Theorem 3.1** (Constitutive relation)**.** *The constitutive material relation from (2.10)-(2.13) can be equivalently restated as*

$$\partial_{\dot{\boldsymbol{C}}_{\rm tr}} \widehat{R}(\boldsymbol{C}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) + \partial_{\boldsymbol{C}_{\rm tr}} \widehat{\psi}(\boldsymbol{C}, \boldsymbol{C}_{\rm tr}, \theta) \ni \boldsymbol{0}. \tag{3.3}$$ const

Proof. Indeed, relations (2.12) and (2.13) are equivalent to (2.16) and hence to

$$2\partial_{\dot{\boldsymbol{C}}_{\rm tr}} \widehat{R}(\boldsymbol{C}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \ni \boldsymbol{T}.$$

Hence, the statement follows by proving directly that

$$-\frac{1}{2}\boldsymbol{T} \in \partial_{\boldsymbol{C}_{\rm tr}} \widehat{\psi}(\boldsymbol{C}, \boldsymbol{C}_{\rm tr}, \theta) = \partial_{\boldsymbol{C}_{\rm tr}} \psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}) + \partial_{\boldsymbol{C}_{\rm tr}} \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta). \tag{3.4}$$ Tnew

Let us work out the two terms in the above right-hand side separately. We shall preliminarily observe that, for all $\boldsymbol{B} \in \mathbb{R}^{d\times d}$, we have the following

$$\partial_{\boldsymbol{F}_{\rm tr}} \boldsymbol{F}_{\rm tr}^{-T}{:}\boldsymbol{B} = -\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{B}^\top\boldsymbol{F}_{\rm tr}^{-\top} \qquad \partial_{\boldsymbol{F}_{\rm tr}} \boldsymbol{F}_{\rm tr}^{-1}{:}\boldsymbol{B} = -\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{B}\boldsymbol{F}_{\rm tr}^{-1}, \tag{3.5}$$ T00

$$(\partial_{\boldsymbol{C}_{\rm tr}} \boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B})^\top = \partial_{\boldsymbol{C}_{\rm tr}} \boldsymbol{F}_{\rm tr}{:}\boldsymbol{B}, \qquad (\partial_{\boldsymbol{C}_{\rm tr}} \boldsymbol{F}_{\rm tr}{:}\boldsymbol{B})^\top = \partial_{\boldsymbol{C}_{\rm tr}} \boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B}, \tag{3.6}$$ T00

$$\begin{aligned}\partial_{\boldsymbol{F}_{\rm tr}}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}){:}\boldsymbol{B} &= -\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{B}^\top\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1} - \boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{B}\boldsymbol{F}_{\rm tr}^{-1}\\ &= -\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{B}^\top\boldsymbol{C}_{\rm el} - \boldsymbol{C}_{\rm el}\boldsymbol{B}\boldsymbol{F}_{\rm tr}^{-1} = \partial_{\boldsymbol{F}_{\rm tr}^\top}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}){:}\boldsymbol{B}^\top, \end{aligned} \tag{3.7}$$ T3

$$\boldsymbol{B} = \partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{C}_{\rm tr}{:}\boldsymbol{B} = \partial_{\boldsymbol{C}_{\rm tr}}(\boldsymbol{F}_{\rm tr}^\top\boldsymbol{F}_{\rm tr}){:}\boldsymbol{B} = (\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B})\boldsymbol{F}_{\rm tr} + \boldsymbol{F}_{\rm tr}^\top(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B}). \tag{3.8}$$ T02

In particular, relation (3.8) entails that, for all $\boldsymbol{B} \in \mathbb{R}^{d\times d}$,

$$\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{B}\boldsymbol{F}_{\rm tr}^{-1} = \boldsymbol{F}_{\rm tr}^{-\top}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}^\top : \boldsymbol{B}) + (\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr} : \boldsymbol{B})\boldsymbol{F}_{\rm tr}^{-1}. \tag{3.9}$$ T4

Hence, given $\boldsymbol{B} \in \mathbb{R}^{d\times d}$ one has that

$$\begin{aligned}&\partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}){:}\boldsymbol{B} = \partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\partial_{\boldsymbol{F}_{\rm tr}}(\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}){:}\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B}\\ &= -\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\boldsymbol{F}_{\rm tr}^{-\top}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B})\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1} - \partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\boldsymbol{F}_{\rm tr}^{-\top}\boldsymbol{C}\boldsymbol{F}_{\rm tr}^{-1}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B})\boldsymbol{F}_{\rm tr}^{-1}\\ &= -\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{C}_{\rm el}{:}\boldsymbol{F}_{\rm tr}^{-\top}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B}) - \boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B})\boldsymbol{F}_{\rm tr}^{-1}\\ &\overset{(3.9)}{=} -\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}[\boldsymbol{F}_{\rm tr}^{-T}(\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}^\top{:}\boldsymbol{B}) + (\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B})\boldsymbol{F}_{\rm tr}^{-1}]\\ &= -\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}{:}\boldsymbol{F}_{\rm tr}^{-T}\boldsymbol{B}\boldsymbol{F}_{\rm tr}^{-1} = -\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{C}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}\boldsymbol{F}_{\rm tr}^{-\top}{:}\boldsymbol{B}\\ &= -\frac{1}{2}\boldsymbol{F}_{\rm tr}^{-1}\boldsymbol{M}\boldsymbol{F}_{\rm tr}^{-\top}{:}\boldsymbol{B}. \end{aligned} \tag{3.10}$$ quattro

In the second identity above we have used (3.7) by replacing $\boldsymbol{B}$ by $\partial_{\boldsymbol{C}_{\rm tr}}\boldsymbol{F}_{\rm tr}{:}\boldsymbol{B}$.

Namely, we have finally checked that

$$\partial_{\boldsymbol{C}_{\rm tr}}\widehat{\psi} = \partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2}) + \partial_{\boldsymbol{C}_{\rm tr}}\psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta) = -\frac{1}{2}(\boldsymbol{M} - \boldsymbol{A}) \equiv -\frac{1}{2}\boldsymbol{T}$$

and (3.4) holds. □

3.1. **Dissipation distance.** Let us close this Section by observing that the dissipation functional $\widehat{R}$ defines a left-invariant Finsler metric. Hence, one can coordinate to $\widehat{R}$ a global metric by letting $\widehat{D} : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0, \infty]$ be defined by

$$\begin{aligned}&\widehat{D}(\boldsymbol{C}_{\rm tr,0}, \boldsymbol{C}_{\rm tr,1})\\ &\doteq \inf\left\{ \int_0^1 \widehat{R}(\boldsymbol{C}_{\rm tr}, \dot{\boldsymbol{C}}_{\rm tr}) \;\Big|\; \boldsymbol{C}_{\rm tr} \in C^1([0,1]; \mathbb{R}^{d\times d}),\ \boldsymbol{C}_{\rm tr}(0) = \boldsymbol{C}_{\rm tr,0},\ \boldsymbol{C}_{\rm tr}(1) = \boldsymbol{C}_{\rm tr,1} \right\}.\end{aligned}$$

Note in particular that

$$\widehat{D}(\boldsymbol{C}_{\rm tr,0}, \boldsymbol{C}_{\rm tr,1}) = \widehat{D}(\mathbf{1}, \boldsymbol{C}_{\rm tr,1}\boldsymbol{C}_{\rm tr,0}^{-1}) \quad \forall \boldsymbol{C}_{\rm tr,0},\, \boldsymbol{C}_{\rm tr,1} \in \mathbb{R}^{d\times d},$$
$$\widehat{D}(\mathbf{1}, \boldsymbol{C}_{\rm tr,1}) < \infty \implies \boldsymbol{C}_{\rm tr,1} \in {\rm SL}(d).$$

The interested Reader is referred to the papers Carstensen02,Mielke02,Mielke03 [14, 33, 34] for additional details.

4. Existence of energetic solutions

energetic

We shall now turn our attention to the existence of suitably weak solutions to the constitutive material relation (const 3.3). To this aim, assume to be given $t \mapsto \theta(t) > 0$ and $t \mapsto \boldsymbol{C}(t) \in \mathbb{R}^{d\times d}_{\rm sym}$ and set for the sake of notational simplicity

$$e(t, \boldsymbol{C}_{\rm tr}) \doteq \widehat{\psi}(\boldsymbol{C}(t), \boldsymbol{C}_{\rm tr}, \theta(t))$$

throughout. In order to make precise the solution notion we shall be dealing with we need to introduce the *stable states* $\mathcal{S}(t)$ at time $t \in [0,T]$ as

$$\mathcal{S}(t) \doteq \{\boldsymbol{C}_{\rm tr} \in {\rm SL}_{\rm sym}(d) \mid e(t,\boldsymbol{C}_{\rm tr}) < \infty \ \text{ and } \ e(t,\boldsymbol{C}_{\rm tr}) \le e(t,\boldsymbol{B}) + \widehat{D}(\boldsymbol{C}_{\rm tr},\boldsymbol{B}) \quad \forall \boldsymbol{B} \in {\rm SL}_{\rm sym}(d)\}. \tag{4.1}$$

stable

An *energetic solution* Mielke05,Mielke-Theil04 [35, 41] corresponds to a trajectory $\boldsymbol{C}_{\rm tr} : [0,T] \to \mathbb{R}^{d\times d}$ such that $\boldsymbol{C}_{\rm tr}(0) = \boldsymbol{C}_{\rm tr,0}$, where $\boldsymbol{C}_{\rm tr,0}$ is some given initial datum, and

$$\boldsymbol{C}_{\rm tr}(t) \in \mathcal{S}(t) \quad \forall t \in [0,T], \tag{4.2}$$

stability

$$e(t,\boldsymbol{C}_{\rm tr}(t)) + {\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr}) = e(0,\boldsymbol{C}_{\rm tr}(0)) + \int_0^t \partial_t e(s,\boldsymbol{C}_{\rm tr}(s)){\rm d}s \quad \forall t \in [0,T] \tag{4.3}$$

energy

where the *total dissipation* ${\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr})$ in the time interval $[0,t]$ is given by

$${\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr}) \doteq \sup\left\{\sum_{i=1}^N \widehat{D}(\boldsymbol{C}_{\rm tr}(t^{i-1}), \boldsymbol{C}_{\rm tr}(t^i)) \;\middle|\; 0 = t^0 < t^1 < \cdots < t^N = t\right\}$$

the supremum being taken among all partitions of the interval $[0,t]$. Note that ${\rm Diss}_{[0,T]}(\boldsymbol{C}_{\rm tr}) < \infty$ implies that $\boldsymbol{C}_{\rm tr}(t) \in {\rm SL}(d)$ for all times. Let us now formulate the assumptions on $\boldsymbol{C}$ and θ as

$$\boldsymbol{C} \in W^{1,1}(0,T;\mathbb{R}^{d\times d}_{\rm sym}), \qquad \det \boldsymbol{C}(t) \ge \alpha_0 > 0 \quad \forall t \in [0,T], \tag{4.4}$$

assp1

$$\theta \in W^{1,1}(0,T). \tag{4.5}$$

assp2

Theorem 4.1 (Existence of energetic solutions)**.** *Let (assp1 4.4)-(assp2 4.5) hold and $\boldsymbol{C}_{\rm tr,0} \in \mathcal{S}(0)$. Then, there exist an energetic solution of (const 3.3).*

The proof of this theorem is developed in the remainder of this section and follows by adapting the by-now classical argument of Mielke-Theil04 [41]. In particular, we shall proceed by time-discretization. An interesting by-product of this procedure relies in the possibility of approximating the limit problem constructively. Note that some numerical computation for this model has been provided in Evangelista09 [20]. Here, we somehow complement the theory by rigorously proving convergence of the numerical scheme.

4.1. **Incremental minimization.** Assume to be given a partition of $[0,T]$ which we identify with the corresponding vector $\boldsymbol{\tau} = (\tau^1, \ldots, \tau^{N_{\boldsymbol{\tau}}})$ of strictly positive time-steps. Note that we indicate with superscripts the elements of a generic vector. In particular τ^j represents the j-th component of the vector $\boldsymbol{\tau}$ (and not the j-th power of the scalar τ).

We let $t^0_{\boldsymbol{\tau}} \doteq 0$ and recursively define

$$t^i_{\boldsymbol{\tau}} \doteq t^{i-1}_{\boldsymbol{\tau}} + \tau^i, \quad I^i_{\boldsymbol{\tau}} \doteq [t^{i-1}_{\boldsymbol{\tau}}, t^i_{\boldsymbol{\tau}}) \quad \text{for} \quad i = 1, \dots, N_{\boldsymbol{\tau}},$$

and we will use the symbol $\overline{\boldsymbol{\tau}} = \max_{i=1,\dots,N_{\boldsymbol{\tau}}} \tau^i$, for the maximum time-step (fineness of the partition). Moreover, we will make use of the notation $(\boldsymbol{B}^0_{\boldsymbol{\tau}}, \dots, \boldsymbol{B}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}})$ for elements in $(\mathbb{R}^{d\times d})^{N_{\boldsymbol{\tau}}+1}$ and let $\boldsymbol{B}_{\boldsymbol{\tau}} : [0,T] \to \mathbb{R}^{d\times d}$ be the corresponding piecewise constant interpolant on the intervals $I^i_{\boldsymbol{\tau}}$, namely $\boldsymbol{B}_{\boldsymbol{\tau}}(t) \doteq \boldsymbol{B}^{i-1}_{\boldsymbol{\tau}}$ for all $t \in I^i_{\boldsymbol{\tau}}$, $i = 1, \dots, N_{\boldsymbol{\tau}}$ and $\boldsymbol{B}_{\boldsymbol{\tau}}(T) \doteq \boldsymbol{B}^{N_{\boldsymbol{\tau}}}_{\boldsymbol{\tau}}$.

As we are given $(\boldsymbol{C}, \theta) : [0,T] \to \mathbb{R}^{d\times d}_{\rm sym} \times (0,\infty)$, we define $(\boldsymbol{C}^i_{\boldsymbol{\tau}}, \theta^i_{\boldsymbol{\tau}}) \doteq (\boldsymbol{C}(t^i_{\boldsymbol{\tau}}), \theta(t^i_{\boldsymbol{\tau}}))$. Then, we look for a vector $(\boldsymbol{C}^0_{{\rm tr},\boldsymbol{\tau}}, \dots, \boldsymbol{C}^{N_{\boldsymbol{\tau}}}_{{\rm tr},\boldsymbol{\tau}}) \in ({\rm SL}_{\rm sym}(d))^{N_{\boldsymbol{\tau}}+1}$ such that

$$\boldsymbol{C}^0_{{\rm tr},\boldsymbol{\tau}} = \boldsymbol{C}_{{\rm tr},0}, \tag{4.6}$$

$$\boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}} \in \operatorname{Arg\,min}\Big\{ {\rm e}({\rm t}^{\rm i}_{\boldsymbol{\tau}}, \boldsymbol{C}_{\rm tr}) + \widehat{\rm D}(\boldsymbol{C}^{\rm i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{C}_{\rm tr}) \ \Big|\ \boldsymbol{C}_{\rm tr} \in {\rm SL}_{\rm sym}({\rm d}) \Big\} \quad \text{for} \quad {\rm i} = 1, \dots, {\rm N}_{\boldsymbol{\tau}}. \tag{4.7}$$

Hence, the solution $\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}$ is obtained by *sequentially* minimizing on ${\rm SL}_{\rm sym}(d)$

$$\boldsymbol{C}_{\rm tr} \mapsto \psi_{\rm el}(\boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C}^i_{\boldsymbol{\tau}} \boldsymbol{C}^{-1/2}_{\rm tr}) + \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta^i_{\boldsymbol{\tau}}) + \widehat{D}(\boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{C}_{\rm tr}).$$

Such incremental solution clearly exists as the above functional turns out to be coercive and lower semicontinuous and ${\rm SL}_{\rm sym}(d)$ is closed.

4.2. **Discrete stability and estimates.** Let us show that $\boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}} \in \mathcal{S}(t^i_{\boldsymbol{\tau}})$ for $i = 1, \dots, N_{\boldsymbol{\tau}}$. Indeed, by using the minimality (4.7) and the triangle inequality for $\widehat{D}$, we have that, for all $\boldsymbol{B} \in {\rm SL}_{\rm sym}(d)$,

$$\begin{aligned} &e(t^i_{\boldsymbol{\tau}}, \boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}) + \widehat{D}(\boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}) \le e(t^i_{\boldsymbol{\tau}}, \boldsymbol{B}) + \widehat{D}(\boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{B}) \\ &\le e(t^i_{\boldsymbol{\tau}}, \boldsymbol{B}) + \widehat{D}(\boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}) + \widehat{D}(\boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{B}) \end{aligned}$$

whence stability follows. Furthermore, again by minimality (4.7) the following inequality holds

$$e(t^i_{\boldsymbol{\tau}}, \boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}) + \widehat{D}(\boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}, \boldsymbol{C}^i_{{\rm tr},\boldsymbol{\tau}}) \le e(t^{i-1}_{\boldsymbol{\tau}}, \boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}) + \int_{t^{i-1}_{\boldsymbol{\tau}}}^{t^i_{\boldsymbol{\tau}}} \partial_s e(s, \boldsymbol{C}^{i-1}_{{\rm tr},\boldsymbol{\tau}}) {\rm d}s.$$

By summing up we get the *discrete upper energy estimate* for the piecewise constant $\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}$

$$\begin{aligned} &e(t^m_{\boldsymbol{\tau}}, \boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}(t^m_{\boldsymbol{\tau}})) + {\rm Diss}_{[t^i_{\boldsymbol{\tau}}, t^m_{\boldsymbol{\tau}}]}(\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}) \\ &\le e(t^i_{\boldsymbol{\tau}}, \boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}(t^i_{\boldsymbol{\tau}})) + \int_{t^i_{\boldsymbol{\tau}}}^{t^m_{\boldsymbol{\tau}}} \partial_s e(s, \boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}(s)) {\rm d}s \quad \text{for } i \le m. \end{aligned} \tag{4.8}$$

Our next aim is that of showing a control on the *power* of external actions. In particular, for all given $\boldsymbol{C}_{\rm tr} \in K \doteq \{\boldsymbol{C}_{\rm tr} \in {\rm SL}_{\rm sym}(d) \mid |\boldsymbol{C}_{\rm tr} - \boldsymbol{1}_2| \le 2\epsilon_L\}$ (recall (2.7)), we have

$$\partial_t e(t, \boldsymbol{C}_{\rm tr}) = \partial_{\boldsymbol{C}} \psi_{\rm el}(\boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C}(t) \boldsymbol{C}^{-1/2}_{\rm tr}) : \dot{\boldsymbol{C}}(t) + \partial_t \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta(t)). \tag{4.9}$$

Due to assumption (4.5) and to the explicit form (2.7) of the inelastic energy $\psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta)$, we have that $t \mapsto \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta(t)) \in W^{1,1}(0,T)$ and

$$\partial_t \psi_{\rm tr}(\boldsymbol{C}_{\rm tr}, \theta(t)) = bH(\theta(t) - \theta_M)|\boldsymbol{E}_{\rm tr}|\dot{\theta}(t) \quad \text{for a.e.} \ \ t \in (0,T) \tag{4.10}$$

where H is the Heaviside function. As for the first term in the right-hand side of (4.9) we compute by (2.2)

$$\begin{aligned} &\partial_{\boldsymbol{C}} \psi_{\rm el}(\boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C}(t) \boldsymbol{C}^{-1/2}_{\rm tr}) = \partial_{\boldsymbol{A}} \psi_{\rm el}(\boldsymbol{A})|_{\boldsymbol{A} = \boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C} \boldsymbol{C}^{-1/2}_{\rm tr}} : \partial_{\boldsymbol{C}} (\boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C}(t) \boldsymbol{C}^{-1/2}_{\rm tr}) \\ &= \boldsymbol{C}^{-1/2}_{\rm tr} \partial_{\boldsymbol{A}} \psi_{\rm el}(\boldsymbol{A})|_{\boldsymbol{A} = \boldsymbol{C}^{-1/2}_{\rm tr} \boldsymbol{C}(t) \boldsymbol{C}^{-1/2}_{\rm tr}} \boldsymbol{C}^{-1/2}_{\rm tr}. \end{aligned} \tag{4.11}$$

Let us observe that assumption (4.4) entails in particular that $\boldsymbol{C} \in L^\infty(0,T;\mathbb{R}^{d\times d}_{\rm sym})$. Moreover, we have that

$$\boldsymbol{C}_{\rm tr} \in K \ \Rightarrow\ |\boldsymbol{C}_{\rm tr}| + |\boldsymbol{C}_{\rm tr}^{-1}| \leq c_K$$

where $c_K > 0$ depends on data only. From the smoothness of the square root [25, pag. 23] an analogous bound holds true for $|\boldsymbol{C}_{\rm tr}^{1/2}| + |\boldsymbol{C}_{\rm tr}^{-1/2}|$. In particular, given the above bounds, one can check that $|\boldsymbol{F}_{\rm el}|+|\boldsymbol{F}_{\rm el}^{-1}|$ is also a priori bounded in terms of data and $|\boldsymbol{C}(t)|$. Now, we remark that, for all $\boldsymbol{B} \in \mathbb{R}^{d\times d}$, one has

$$\partial_{\boldsymbol{F}_{\rm el}} W(\boldsymbol{F}_{\rm el})\boldsymbol{F}_{\rm el}^\top{:}B = \boldsymbol{F}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}(\boldsymbol{C}_{\rm el})\boldsymbol{F}_{\rm el}^\top{:}(\boldsymbol{B}+\boldsymbol{B}^\top).$$

Hence, we can control the first term in the right-hand side of (4.9) as follows (recall (3.1), which implies $\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}\boldsymbol{C}_{\rm tr}^{-1/2} = \boldsymbol{R}^\top\boldsymbol{C}_{\rm el}\boldsymbol{R}$)

$$\begin{aligned}
&\big|\partial_{\boldsymbol{C}}\psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}(t)\boldsymbol{C}_{\rm tr}^{-1/2}){:}\dot{\boldsymbol{C}}(t)\big|\\
&= \big|\boldsymbol{F}_{\rm el}\partial_{\boldsymbol{A}}\psi_{\rm el}(\boldsymbol{A})\big|_{\boldsymbol{A}=\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}(t)\boldsymbol{C}_{\rm tr}^{-1/2}}\boldsymbol{F}_{\rm el}^\top{:}\boldsymbol{F}_{\rm el}^{-\top}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{F}_{\rm el}^{-1}\big|\\
&= |\boldsymbol{F}_{\rm el}\boldsymbol{R}^\top\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}(\boldsymbol{C}_{\rm el})\boldsymbol{R}\boldsymbol{F}_{\rm el}^\top : \boldsymbol{F}_{\rm el}^{-\top}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{F}_{\rm el}^{-1}|\\
&= |\boldsymbol{F}_{\rm el}\partial_{\boldsymbol{C}_{\rm el}}\psi_{\rm el}(\boldsymbol{C}_{\rm el})\boldsymbol{F}_{\rm el}^\top : \boldsymbol{F}_{\rm el}^{-\top}\boldsymbol{R}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{R}^T\boldsymbol{F}_{\rm el}^{-1}|\\
&= \frac{1}{2}|\partial_{\boldsymbol{F}_{\rm el}}W(\boldsymbol{F}_{\rm el})\boldsymbol{F}_{\rm el}^\top : \boldsymbol{F}_{\rm el}^{-T}\boldsymbol{R}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{R}^T\boldsymbol{F}_{\rm el}^{-1}|\\
&\leq \frac{1}{2}c_0(W(\boldsymbol{F}_{\rm el})+1)|\boldsymbol{F}_{\rm el}^{-\top}\boldsymbol{R}\boldsymbol{C}_{\rm tr}^{-1/2}\dot{\boldsymbol{C}}(t)\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{R}^T\boldsymbol{F}_{\rm el}^{-1}|\\
&\leq c(\psi_{\rm el}(\boldsymbol{C}_{\rm tr}^{-1/2}\boldsymbol{C}(t)\boldsymbol{C}_{\rm tr}^{-1/2})+1)|\dot{\boldsymbol{C}}(t)|.
\end{aligned}$$

where c depends on data only. Hence, we can conclude that

$$|e(t,\boldsymbol{C}_{\rm tr})| \leq \gamma(t)(1+e(t,\boldsymbol{C}_{\rm tr})) \tag{4.12}$$

where $t \mapsto \gamma(t) \doteq c_e(|\dot{\boldsymbol{C}}(t)| + |\dot\theta(t)|) \in L^1(0,T)$ and $c_e > 0$ depends on data only. Moreover, note that for almost all times $t \in [0,T]$ the power $\boldsymbol{C}_{\rm tr} \mapsto \partial_t e(t,\boldsymbol{C}_{\rm tr})$ turns out to be continuous on K as a consequence of the smoothness of $\psi_{\rm el}$ and the continuity of the inverse on ${\rm SL}(d)$ and of the square root in $\mathbb{R}^{d\times d}_{\rm sym}$.

Now, from the minimality (4.7) we deduce that the piecewise constant $\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}$ satisfies $\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}(t) \in K$ for every $t \in [0,T]$. Moreover, owing to the bound (4.12), the discrete upper energy estimate (4.8) entails via Gronwall (see [35]) that

$$\sup_{\boldsymbol{\tau}}\Big(\sup_{t\in[0,T]} e(t,\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}}(t)) + {\rm Diss}_{[0,T]}(\boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}})\Big) < \infty \tag{4.13}$$

Let us now take a sequence of partitions $\boldsymbol{\tau}_k$ such that $\overline{\tau}_k \to 0$ as $k \to \infty$. Let $\boldsymbol{C}_{\rm tr}^k \doteq \boldsymbol{C}_{{\rm tr},\boldsymbol{\tau}_k}$ be the corresponding discrete solutions. From the bound (4.13) and Helly's selection principle (see, e.g., [36]) we can deduce the existence of a (not relabeled) subsequence $\boldsymbol{C}_{\rm tr}^k$ and of a function $\boldsymbol{C}_{\rm tr} \in BV_{\widehat{D}}([0,T];{\rm SL}_{\rm sym}(d)) \doteq \{t \in [0,T] \mapsto \boldsymbol{C}_{\rm tr}(t) \in {\rm SL}_{\rm sym}(d) \mid {\rm Diss}_{[0,T]}(\boldsymbol{C}_{\rm tr}) < \infty\}$ such that

$$\boldsymbol{C}_{\rm tr}^k(t) \to \boldsymbol{C}_{\rm tr}(t) \quad \text{as } k\to\infty \quad \forall t \in [0,T].$$

We now aim at showing that the limit function $\boldsymbol{C}_{\rm tr}$ is an energetic solution of (3.3), namely that it satisfies the stability condition (4.2) and the energy balance (4.3).

Stability of the limit function. We first show that the set $\mathcal{S}_{[0,T]} \doteq \{(t, \boldsymbol{C}_{\rm tr}) \in [0,T] \times {\rm SL}_{\rm sym}(d) : \boldsymbol{C}_{\rm tr} \in \mathcal{S}(t)\}$ is closed in $[0,T] \times {\rm SL}_{\rm sym}(d)$. Let us take $(t_n, \boldsymbol{C}_{\rm tr}^{(n)}) \in \mathcal{S}_{[0,T]}$ such that $t_n \to t^*$ in $[0,T]$ and $\boldsymbol{C}_{\rm tr}^{(n)} \to \boldsymbol{C}_{\rm tr}^*$ in ${\rm SL}_{\rm sym}(d)$. Since $\boldsymbol{C}_{\rm tr}^{(n)} \in \mathcal{S}(t_n)$, we have $e(t_n, \boldsymbol{C}_{\rm tr}^{(n)}) < \infty$ and

$$e(t_n, \boldsymbol{C}_{\rm tr}^{(n)}) \le e(t_n, \widehat{\boldsymbol{C}}_{\rm tr}) + \widehat{D}(\boldsymbol{C}_{\rm tr}^{(n)}, \widehat{\boldsymbol{C}}_{\rm tr}) \quad \forall \widehat{\boldsymbol{C}}_{\rm tr} \in {\rm SL}_{\rm sym}(d).$$

By passing to the limit for $n \to \infty$ and using the continuity of $\widehat{\psi}$ (restricted to $\mathbb{R}^{d\times d}_{\rm sym} \times K \times \mathbb{R}$), we immediately get that $(t^*, \boldsymbol{C}_{\rm tr}^*) \in \mathcal{S}_{[0,T]}$. Let us now fix $t \in [0,T]$ and define $s_k^t \doteq \max\{t^i_{\boldsymbol{\tau}_k} \mid t^i_{\boldsymbol{\tau}_k} \le t\}$. We have that $s_k^t \to t$ and $\boldsymbol{C}^k_{\rm tr}(s_k^t) = \boldsymbol{C}^k_{\rm tr}(t) \to \boldsymbol{C}_{\rm tr}(t)$ as $k \to \infty$. Since $\boldsymbol{C}^k_{\rm tr}(s^t_k) \in \mathcal{S}(s^t_k)$, by the closedness of $\mathcal{S}_{[0,T]}$ we have that $\boldsymbol{C}_{\rm tr}(t) \in \mathcal{S}(t)$.

Upper energy estimate. Along with the above notation. the discrete upper energy estimate (4.8) can be rewritten as

$$e(s_k^t, \boldsymbol{C}^k_{\rm tr}(s^t_k)) + {\rm Diss}_{[0,s^t_k]}(\boldsymbol{C}^k_{\rm tr}) \le e(0, \boldsymbol{C}_{{\rm tr},0}) + \int_0^{s^t_k} \partial_s e(s, \boldsymbol{C}^k_{\rm tr}(s)){\rm d}s.$$

By observing that ${\rm Diss}_{[0,s^t_k]}(\boldsymbol{C}^k_{\rm tr}) = {\rm Diss}_{[0,t]}(\boldsymbol{C}^k_{\rm tr})$, passing to the limit as $k \to \infty$, and exploiting the bound (4.12), relation (4.13), the lower semicontinuity of the functional ${\rm Diss}_{[0,t]}$, and the Lebesgue's Dominated Convergence Theorem, we deduce the *upper energy estimate*

$$e(t, \boldsymbol{C}_{\rm tr}(t)) + {\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr}) \le e(0, \boldsymbol{C}_{{\rm tr},0}) + \int_0^t \partial_s e(s, \boldsymbol{C}_{\rm tr}(s)){\rm d}s \quad \forall t \in [0,T].$$

Lower energy estimate. Let us take a sequence of partitions $\boldsymbol{\tau}_m$ of the interval $[0,t]$, with $\overline{\boldsymbol{\tau}}_m \to 0$ as $m \to \infty$. Since $\boldsymbol{C}_{\rm tr}(t) \in \mathcal{S}(t)$, for every $t \in [0,T]$, we have (setting for brevity $N_m \doteq N_{\boldsymbol{\tau}_m}$ and $t^j_m \doteq t^j_{\boldsymbol{\tau}_m}$)

$$\begin{aligned} & e(t^j_m, \boldsymbol{C}_{\rm tr}(t^j_m)) + \widehat{D}(\boldsymbol{C}_{\rm tr}(t^{j-1}_m), \boldsymbol{C}_{\rm tr}(t^j_m)) \\ & = e(t^{j-1}_m, \boldsymbol{C}_{\rm tr}(t^j_m)) + \int_{t^{j-1}_m}^{t^j_m} \partial_s e(s, \boldsymbol{C}_{\rm tr}(t^j_m)){\rm d}s + \widehat{D}(\boldsymbol{C}_{\rm tr}(t^{j-1}_m), \boldsymbol{C}_{\rm tr}(t^j_m)) \\ & \ge e(t^{j-1}_m, \boldsymbol{C}_{\rm tr}(t^{j-1}_m)) + \int_{t^{j-1}_m}^{t^j_m} \partial_s e(s, \boldsymbol{C}_{\rm tr}(t^j_m)){\rm d}s. \end{aligned}$$

Summing up for $j = 1, \dots, N_m$ we deduce that

$$\begin{aligned} & e(t, \boldsymbol{C}_{\rm tr}(t)) - e(0, \boldsymbol{C}_{{\rm tr},0}) + {\rm Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr}) \\ & \ge \sum_{j=1}^{N_m} \Big[e(t^j_m, \boldsymbol{C}_{\rm tr}(t^j_m)) - e(t^{j-1}_m, \boldsymbol{C}_{\rm tr}(t^{j-1}_m)) \Big] + \sum_{j=1}^{N_m} \widehat{D}(\boldsymbol{C}_{\rm tr}(t^{j-1}_m), \boldsymbol{C}_{\rm tr}(t^j_m)) \\ & \ge \int_0^t \partial_s e(s, \overline{\boldsymbol{C}}^{(m)}_{\rm tr}(s)){\rm d}s, \end{aligned} \tag{4.14}$$

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where $\overline{\boldsymbol{C}}^{(m)}_{\rm tr}$ is defined as $\overline{\boldsymbol{C}}^{(m)}_{\rm tr}(t) = \boldsymbol{C}_{\rm tr}(t^j_m)$ for every $t \in (t^{j-1}_m, t^j_m]$. By recalling(4.9)-(4.10), the integral on the right-hand side is given by the sum of two contributions, namely

$$\begin{aligned} \int_0^t \partial_s e(s, \overline{\boldsymbol{C}}^{(m)}_{\rm tr}(s)){\rm d}s = & \int_0^t bH(\theta(s) - \theta_M)|\overline{\boldsymbol{E}}^{(m)}_{\rm tr}(s)|\dot\theta(s){\rm d}s \\ & + \int_0^t \partial_{\boldsymbol{C}} \psi_{\rm el}((\overline{\boldsymbol{C}}^{(m)}_{\rm tr}(s))^{-1/2} \boldsymbol{C} (\overline{\boldsymbol{C}}^{(m)}_{\rm tr}(s))^{-1/2}) \Big|_{\boldsymbol{C}=\boldsymbol{C}(s)} : \dot{\boldsymbol{C}}(s){\rm d}s \end{aligned}$$

where $\overline{\boldsymbol{E}}_{\rm tr}^{(m)} = (\overline{\boldsymbol{C}}_{\rm tr}^{(m)} - \mathbf{1}_2)/2$. As $\boldsymbol{C}_{\rm tr} \in BV_{\widehat{D}}([0,T]; \mathrm{SL}_{\rm sym}(d))$, we have that $\boldsymbol{C}_{\rm tr}$ is continuous on $[0,T]$ with the exception of at most a countable set of points [36, Thm. 3.3]. Hence, we have $\overline{\boldsymbol{C}}_{\rm tr}^{(m)}(t) \to \boldsymbol{C}_{\rm tr}(t)$ for a.e. $t \in (0,T)$. We can then apply Lebesgue's Dominated Convergence Theorem to conclude that

$$\int_0^t bH(\theta(s)-\theta_M)|\overline{\boldsymbol{E}}_{\rm tr}^{(m)}(s)|\dot\theta(s)\mathrm{d}s \longrightarrow \int_0^t bH(\theta(s)-\theta_M)|\boldsymbol{E}_{\rm tr}(s)|\dot\theta(s)\mathrm{d}s$$

Next, by recalling the continuity of the power term and assumption (4.4), we may use again Lebesgue's theorem and show that

$$\int_0^t \partial_{\boldsymbol{C}}\psi_{\rm el}((\overline{\boldsymbol{C}}_{\rm tr}^{(m)}(s))^{-1/2}\boldsymbol{C}(s)(\overline{\boldsymbol{C}}_{\rm tr}^{(m)}(s))^{-1/2}){:}\dot{\boldsymbol{C}}(s)\mathrm{d}s$$
$$\to \int_0^t \partial_{\boldsymbol{C}}\psi_{\rm el}((\boldsymbol{C}_{\rm tr}(s))^{-1/2}\boldsymbol{C}(s)(\boldsymbol{C}_{\rm tr}(s))^{-1/2}){:}\dot{\boldsymbol{C}}(s)\mathrm{d}s.$$

Hence, the integral on the right-hand side of (4.14) converges to

$$\int_0^t bH(\theta(s)-\theta_M)|\boldsymbol{E}_{\rm tr}(s)|\dot\theta(s)\mathrm{d}s + \int_0^t \partial_{\boldsymbol{C}}\psi_{\rm el}((\boldsymbol{C}_{\rm tr}(s))^{-1/2}\boldsymbol{C}(s)(\boldsymbol{C}_{\rm tr}(s))^{-1/2}){:}\dot{\boldsymbol{C}}(s)\mathrm{d}s$$

and, by taking the limit $m \to \infty$, we get the desired lower energy estimate

$$e(t,\boldsymbol{C}_{\rm tr}(t)) + \mathrm{Diss}_{[0,t]}(\boldsymbol{C}_{\rm tr}) \geq e(0,\boldsymbol{C}_{{\rm tr},0}) + \int_0^t \partial_s e(s,\boldsymbol{C}_{\rm tr}(s))\mathrm{d}s \quad \forall t \in [0,T].$$

In particular, we have proved that $\boldsymbol{C}_{\rm tr}$ fulfills both the stability (4.2) and the energy equality (4.3). Namely, $\boldsymbol{C}_{\rm tr}$ is an energetic solution.

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